

# Screening Under Fixed Cost of Misrepresentation\*

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## Abstract

The paper studies optimal screening problem in which an agent incurs a fixed cost of lying when she misrepresents her private information. In this environment, local incentive constraints are not binding in the optimal mechanism, and standard techniques for solving screening problems are not applicable. Significantly, the problem can no longer be dichotomized into two parts solved sequentially: an implementability part which involves the envelope condition and monotonicity of the allocation, and an optimization part. We develop a new methodology to tackle this problem and use it to characterize the optimal mechanism and compute it in special cases. Our method involves a procedure that jointly solves for the binding incentive constraints and the optimal allocation. The optimal mechanism has a number of interesting qualitative properties, such as lack of exclusion and first-best efficient allocation being offered to high- and low-value types. Also, bunching never arises, as the optimal quantity allocation is always increasing in type independently of type distribution,

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# 1 Introduction

This paper studies a screening problem in which an uninformed principal interacts with a privately informed agent who incurs a fixed cost of “lying” when she misrepresents her private information to the principal. The analysis of the fixed cost of lying is novel and, as we argue below, well-motivated, and produces qualitatively new and interesting results.

Whereas most literature on incentives and mechanism design assumes that a privately informed party is unconstrained in her ability to misrepresent and manipulate her information, several strands in this literature have explored alternative frameworks in which misrepresentation is costly. A notable direction in this research, which originated in the work of Lacker and Weinberg (1989) and has been further developed by Maggi and Rodriguez-Clare (1995) and Crocker and Morgan (1998) consider settings in which an agent incurs a cost of misrepresentation increasing in the size of her “lie” or type misrepresentation.

Another strand of literature on honesty in mechanisms, which includes Alger and Ma (2003), Alger and Renault (2006, 2007), and Severinov and Deneckere (2006) has explored situations in which a principal has to deal with a population of agents some of whom are “honest” and are not able to misrepresent their private information, whereas a complementary fraction consists of fully “strategic” agents who behave in a standard fashion.

Thus, this paper differs from both of this literatures in studying a setting in which the cost of misrepresentation or lying is finite and does not depend on the magnitude of a “lie.”

Misrepresentation costs may exist for several reasons. First, misrepresenting the truth may require costly effort or actions either to manufacture evidence or, conversely, to hide evidence that reveals the true state of the world and conceal one’s information. For example, a firm seeking a loan or a contract or an individual applying for a grant or promotion may need to be perceived as productive, highly competent and/or creditworthy. This goal may be attained by manufacturing “evidence” exaggerating prior performance and concealing the risk of default or non-performance. It is plausible that the cost of misrepresentation or the effort required for the production of such favorable but inaccurate “evidence” is independent of the magnitude of misrepresentation. For instance, the cost of misrepresentation or concealment could involve the loss of future business, benefits or reputation that may have “once and for all” nature

making it unrelated to the size of misrepresentation.

Second, the cost of misrepresenting the truth may have psychological or ethical nature. A moral barrier, a feeling of shame or discomfort, or stress may prevent people from lying.<sup>1</sup> Since being honest or not is often a binary decision, the size of a lie would not affect such psychological costs.

Third, studies in cognitive science and neuroscience indicate that lying is costly because it requires more cognitive resources (Christ et al. (2008)). Therefore, if the potential benefit of lying is small, people tend not to think about it and stay honest as a default choice. On the other hand, if temptation to lie is high enough, the individuals tend to take full advantage of it regardless of the extent of the lie.

In a general sense, the effort or cost required for lying may not be largely affected by the size of a lie. For example, for a consumer with a high valuation for a product, pretending to be mildly interested in the product is not necessarily easier than pretending to be not interested at all.

There is substantial experimental evidence supporting the hypothesis that individuals are averse to lying and incur a cost when doing so. In particular, Abeler, Becker and Falk (2014) measure intrinsic cost of lying in a setup where other motives such as reputational concerns, altruism, efficiency concerns and conditional cooperation can be ruled out, and find that lying costs are significant and widespread. Kajackaite and Gneezy (2017) report experimental data indicating that intrinsic costs of lying are positive and finite. They conclude that “the evidence suggests that lying is a “normal good” in which people compare the intrinsic cost and benefit of the lie, and when the benefit from lying is higher than the intrinsic cost of lying, they lie.” Abeler, Nosenzo and Raymond (2016) provide a meta-analysis of 72 experimental studies with 32503 subjects and find that subjects obtain only about a quarter of the maximal payoff they could obtain by making payoff maximizing reports. They examine a range of popular explanations and conclude that the data is explained by a combination of lying cost and reputational concern.

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<sup>1</sup>Behavioral psychologists have identified and studied a number of physical symptoms associated with emotional discomfort that people experience when lying, such as blushing, “feeling wrong”, gaze aversion, elevated eye-blink rate, etc. See, for example, Ekman(1973, 1988, 2003), Porter and Ten Brinke (2008).

While most experimental studies indicate that lying costs exist, the exact shape and nature of these costs remains unclear. Gneezy, Kajackaite and Sobel (2018) study the effect of the relation between the size of a lie and three different factors: payoffs, outcomes and the likelihood. They find that while social identity (the likelihood factor) has an important impact on lying costs, the other two factors have smaller effects on lying behavior, which indicates that the distance between the report and the truth itself plays little role in the cost of lying. On the other hand, Hilbig and Hessler (2013) find that people’s willingness to lie decreases with the degree of the required distortion of the truth, which suggests that the cost of lying is increasing in the size of lie. It is likely that in reality the cost of lying includes both fixed and variable cost elements.

In this paper, we adopt the fixed cost of lying hypothesis as our working assumption. From a theoretical perspective, it is important to understand the effect of the fixed cost of lying on the optimal mechanism and pricing. As we show below, the introduction of a non-zero fixed cost of lying reshapes the landscape of the optimal screening problem and produces qualitatively new results.

The first significant difference with the standard screening problem is that local incentive constraints are no longer binding where there is a fixed cost of lying. Indeed, imitating a close-by type invariably yields a lower payoff than telling the truth. Therefore, we can no longer use standard Mirrlees’ method and the Envelope Theorem to derive the agent’s surplus from the first-order condition and use it to replace all incentive constraints.

Instead, we need to identify binding non-local incentive constraints in an optimal mechanism. To describe them, we introduce a concept of a “targeted type”  $\tau(\theta)$  - a type or a set of types to which type  $\theta$  has a binding incentive constraint. Significantly, which constraints are binding are endogenous, and so characterizing the targeted type  $\tau(\cdot)$  in the optimal mechanism is one of the major challenges that we face.

Further, targeted types form endogenous “chains.” Specifically, if type  $\theta$  targets some type  $\theta'$  i.e.,  $\tau(\theta) = \theta'$ , and type  $\theta'$  targets some type  $\theta''$  i.e.,  $\tau(\theta') = \theta''$ , then the types  $\theta, \theta', \theta''$  are part of a single chain. The optimal quantity allocation of any type in a chain is then determined jointly with all other types in this chain.

There are several properties of the solution that are worth mentioning. First, monotonicity of the quantity allocation in type is no longer a necessary condition for implementation in our set-up. Non-monotone allocations are implementable, even though the single-crossing property (SCP) holds. However, we show that only quantity allocations increasing in type can be optimal.

Second, the assigned quantities are strictly increasing in type without any additional restrictions on the parameters of the model. In the standard problem, ironing (flat segments of the quantity profile) could happen if the probability distribution from which the types are drawn does not satisfy monotone hazard rate condition. Here, however, the assigned quantity is always strictly increasing in the optimal mechanism, regardless of the type distribution. The reason behind this is two-fold. For one thing, optimality implies that increasing quantity schedule is optimal. Further, fixed costs imply that one can always make the quantity schedule at least slightly increasing, without violating any incentive constraints.

Third, in the optimal mechanism full allocative efficiency is achieved for an interval of low types and an interval of high types who are assigned their first-best quantity, while downward quantity distortion occurs for medium types. This result is in contrast to the “sacrifice efficiency of low types to extract more rent from the high types” intuition from the standard screening problem. Given a positive fixed cost, it is not worth for any type to imitate a low type even if the latter is assigned her first-best quantity. Therefore, no distortion is needed for low types. The intuition behind the efficiency of the allocation for the high-value types is somewhat similar: it is not worth for anyone to imitate those types because, despite their high information rent, they also pay a large transfer to the mechanism designer. So, with the addition of fixed costs, the surplus from imitating those high types is negative.

Fourth, the efficiency of the allocation for the low types also means that there is no exclusion in the sense that every type with positive valuation receive a positive quantity. Severinov and Deneckere (2006) establish a no exclusion property when there is a positive fraction of completely honest agent. This paper shows that this property also holds when there are intermediate barriers to the agents’ opportunism in the form of a fixed cost.

Establishing these properties allows us to develop our methodology for characterizing the

optimal mechanism, to formulate our problem as an optimal control one and derive the necessary and sufficient conditions describing the optimal mechanism for general utility function and type distribution. These first-order conditions come in the form of ordinary differential equations for the optimal quantity  $q(\theta)$  and the targeted type  $\tau(\theta)$ . In the case of a quadratic utility function and uniform type distribution we are able to derive a closed form solution and exhibit the optimal mechanism explicitly.

The overall structure of the optimal mechanism involves an endogenous partition of the type space into intervals such that any type in an interval targets some type in the adjacent lower interval. As the fixed cost of lying decreases, the number of intervals in this partition increases, binding incentive converge to the local ones i.e.,  $\tau(\theta) \rightarrow \theta$ , and the optimal quantity allocation profile and transfers converge to the standard second-best. Conversely, the number of intervals decreases as the fixed cost becomes large. In particular, for a range of costs this partition contains only two elements. As the fixed costs increases further, binding incentive constraint disappear and the quantity allocation becomes the first-best. While not being particularly surprising, this limiting result provides an insight that second-best and first-best can be viewed as the two extreme cases as lying costs vary. Our model provides a generalization which is compatible with both cases, and also allows us to make predictions under limited honesty.

Thus the contribution of this paper is two-fold. First, we characterize the optimal screening mechanism offered by a principal to an agent who has limited ability to misrepresent her information in the form of a fixed cost of lying. We highlight important qualitative properties of this mechanism, which include the first-best efficient allocation assigned to intervals of types near the “top” and the “bottom.” Thus, the mechanism exhibits no exclusion in the sense that all types whose marginal utility of the good exceeds the marginal cost of production consume a positive quantity. So the standard exclusion property of optimal screening mechanisms is not robust to a small cost of lying. We provide a closed form solution for the optimal mechanism in a special but common case of linear-quadratic utility under uniform type distribution.

The second contribution of this paper is methodological and involves development of new techniques to solve a class of principle-agent problems in which local incentive constraints are not binding and which, in contrast to standard ones, cannot be dichotomized into two parts, an

implementability part which involves an envelope condition and monotonicity of the allocation requirement, and the second part involving an optimization. We believe that key elements of our approach, such as the characterization of binding non-local incentive constraints and the “targeted types,” as well as the techniques of solving for them, could also be useful for solving other problems with binding non-local incentive constraints, potentially providing an important analytical instrument for various applications.

The remainder of the paper is organized as follows. Section presents the formal model. Section 3 exhibits important properties of optimal screening mechanism. Section 4 formulates the screening problem as an optimal control and derives the optimality conditions for the solution. Section 5 presents an example of uniform-quadratic case with closed form solution of the optimal mechanism and comparative statics. Section 6 Concludes. All proofs are relegated to an Appendix.

## 2 Model and Preliminaries

We will cast our model in the context of a relationship between a monopolistic seller, who acts as a principal, and a privately informed buyer, who acts as an agent. However, our results apply in any other principal/agent setting.

Thus, we consider a monopolistic profit-maximizing firm facing a consumer with privately known preference parameter (value)  $\theta$  distributed according to cdf  $F(\theta)$  over the interval  $[0, 1]$ . A consumer with value  $\theta$  gets utility  $u(q, \theta) - t$  from consuming quantity/quality  $q$  of the good in exchange for payment  $t$ . The consumer’s reservation utility is assumed to be zero. We also make the following standard assumption on  $u(q, \theta)$ :

**Assumption 1** (i) *The function  $u(q, \theta)$  is twice continuously differentiable, strictly increasing in  $\theta$  when  $q > 0$ , is strictly concave in  $q$  and satisfies  $u(0, \theta) = 0$  for all  $\theta \in [0, 1]$ ;*

(ii)  *$u_q(0, \theta) > 0$  for all  $\theta > 0$ ,  $u_{qq}(q, \theta) < 0$ .*

(iii) *There exists  $q^m$  s.t.  $u_q(q^m, \theta) < 0$  for all  $\theta \in [0, 1]$ .*

(iv) *There exist  $\underline{K} > 0$  and  $\overline{K} > 0$  such that  $\underline{K} < u_{q\theta} < \overline{K}$  for all  $q > 0$  and  $\theta \in [0, 1]$ .*

Assumption 1 implies that  $q^{fb}(\theta) \equiv \arg \max_q u(q, \theta)$  is well-defined, finite, strictly positive for

$\theta > 0$  and increasing in  $\theta$ .

We also assume that the firm has zero cost of production. This assumption is without loss of generality. Indeed, if the firm instead faced a cost of production  $c(q)$ , the model would be equivalent to one in which the firm's cost is identically zero, while the consumer's utility function is  $u(q, \theta) - c(q)$ .

The firm has all the bargaining power and designs a mechanism to maximize its expected payoff. The consumer can either accept or reject the mechanism offered by the firm. In the latter case she earns her reservation utility level of 0.

To this standard screening environment we now add a new element in the form of the fixed cost of misrepresentation, or lying about one's type. Specifically, we assume that the consumer of type  $\theta \in [0, 1]$  incurs a cost  $C$  if, being asked to report her value to the firm, she reports some  $\hat{\theta} \neq \theta$ . Our goal is to characterize the firm's optimal mechanism in this environment.

It is immediate to see that, under the fixed cost of lying assumption, the Revelation Principle still applies provided that type announcement is considered to be a part of the allocation. So the mechanism designed by the firm is now described as a menu  $(q(\theta), t(\theta), A(\theta)) \in \mathbb{R}^+ \times \mathbb{R} \times [0, 1]$  for  $\theta \in [0, 1]$  where  $q(\theta)$  is the quantity assigned to type  $\theta$ ,  $t(\theta)$  is her payment to the firm and  $A(\theta)$  is the type announcement recommended by the mechanism to the consumer of type  $\theta$ .

Let  $1(A(\theta') \neq \theta)$  denote an indicator function equal to 1 when  $A(\theta') \neq \theta$  and equal to zero otherwise. Then the firm's optimal mechanism solves the following problem:

$$\max_{q(\theta), t(\theta), A(\theta)} \int_0^1 t(\theta) f(\theta) d\theta$$

subject to the following incentive and individual rationality constraints:

$$\begin{aligned} u(q(\theta), \theta) - t(\theta) - C \times 1(A(\theta) \neq \theta) &\geq u(q(\theta'), \theta) - t(\theta') - C \times 1(A(\theta') \neq \theta) && \forall \theta, \theta' \in [0, 1] \\ u(q(\theta), \theta) - t(\theta) - C \times 1(A(\theta) \neq \theta) &\geq 0 && \forall \theta \in [0, 1] \end{aligned}$$

Let us first show that without loss of generality we can restrict considerations to mechanisms in which there is no lying.

**Theorem 1** *Consider an incentive compatible, individually rational mechanism  $(q(\theta), t(\theta), A(\theta))$  such that for a set of types  $\theta$  of a positive measure we have  $A(\theta) \neq \theta$ . Then there exists an al-*



ternative mechanism  $(\hat{q}(\theta), \hat{t}(\theta), \hat{A}(\theta))$  such that  $\hat{A}(\theta) = \theta$  for almost all  $\theta$  and which is strictly more profitable for the firm than the original mechanism.

**Proof of Theorem 1:**

Given incentive compatible, individually rational mechanism  $(q(\theta), t(\theta), A(\theta))$  which satisfies  $A(\theta) \neq \theta$  for all  $\theta \in \Theta^l$ , where  $\Theta^l$  has a positive measure, consider an alternative mechanism  $(\hat{q}(\theta), \hat{t}(\theta), \hat{A}(\theta))$  such that  $(\hat{q}(\theta), \hat{t}(\theta), \hat{A}(\theta)) = (q(\theta), t(\theta), A(\theta))$  for all  $\theta$  such that  $A(\theta) = \theta$  and  $(\hat{q}(\theta), \hat{t}(\theta), \hat{A}(\theta)) = (q(\theta), t(\theta) + C, \theta)$  for  $\theta$  such that  $A(\theta) \neq \theta$ . Clearly,  $(\hat{q}(\theta), \hat{t}(\theta), \hat{A}(\theta))$  is strictly more profitable for the firm, provided that it is incentive compatible and individually rational. The individual rationality of the new mechanism follows immediately from the individual rationality of the original mechanism. So to complete the proof we only need to show that the new mechanism is incentive compatible. Indeed, for all  $\theta, \theta' \in [0, 1]$  we have:

$$\begin{aligned} u(\hat{q}(\theta), \theta) - \hat{t}(\theta) - C \times 1(\hat{A}(\theta) \neq \theta) &= u(\hat{q}(\theta), \theta) - \hat{t}(\theta) = u(q(\theta), \theta) - t(\theta) - C \times 1(A(\theta) \neq \theta) \\ &\geq u(q(\theta'), \theta) - t(\theta') - C \times 1(A(\theta') \neq \theta) \geq u(\hat{q}(\theta'), \theta) - \hat{t}(\theta') - C \times 1(\hat{A}(\theta') \neq \theta) \end{aligned}$$

where the first equality holds because  $\hat{A}(\theta) = \theta$  for all  $\theta \in [0, 1]$ , the second equality holds by definition of  $(\hat{q}(\theta), \hat{t}(\theta), \hat{y}(\theta))$ , the first inequality holds because  $(q(\theta), t(\theta), A(\theta))$  is incentive compatible, and the second inequality holds because  $\hat{t}(\theta') \geq t(\theta')$  and  $\hat{A}(\theta') \neq \theta'$  for  $\theta' \neq \theta$ .

*Q.E.D.*

Theorem 1 implies that the firm's problem can be stated as follows:

$$\max_{q(\theta) \geq 0, t(\theta)} \int_0^1 t(\theta) f(\theta) d\theta \tag{1}$$

subject to

$$u(q(\theta), \theta) - t(\theta) \geq u(q(\theta'), \theta) - t(\theta') - C \quad \forall \theta, \theta' \in [0, 1] \quad (IC) \tag{2}$$

$$u(q(\theta), \theta) - t(\theta) \geq 0 \quad \forall \theta \in [0, 1] \quad (IR) \tag{3}$$

We call  $(q(\theta), t(\theta))$  an optimal mechanism if it solves the principal's maximization problem (1) subject to (2) and (3). We now have:

**Theorem 2** *An optimal mechanism exists. It is unique if  $u_{\theta qq}(q, \theta) \geq 0$  for all  $(q, \theta)$ .*

### 3 General Structure of the Optimal Mechanism

In this section we will establish a number of important general properties of an optimal mechanism. First, we need some additional notation. So, let  $q^{fb}(\theta)$  be the first-best quantity which maximizes  $u(q, \theta)$  and thus solves  $u_q(q, \theta) = 0$ . Given our assumptions  $q^{fb}(\theta)$  is uniquely defined for all  $\theta$  and is strictly positive and increasing for  $\theta > 0$ . Also, let  $(q(\cdot), t(\cdot))$  denote an incentive compatible individually rational mechanism and let  $V(\theta) = u(q(\theta), \theta) - t(\theta)$  denote the associated net payoff of the agent-type  $\theta$  in this mechanism.

The first set of properties allow us to focus without loss of generality on the set of mechanism that have regularity properties which simplify the task of deriving the optimal mechanism.

**Theorem 3** *There exists an optimal mechanism  $(q(\cdot), t(\cdot))$  such that for all  $\theta \in [0, 1]$ :*

1.  $V(\theta)$ ,  $q(\theta)$  and  $t(\theta)$  are continuous in  $\theta$ , with  $t(\theta) \geq 0$ , for all  $\theta \in [0, 1]$ .
2.  $V(\theta)$  is non-decreasing;
3.  $q(\theta)$  is strictly increasing;
4.  $0 < q(\theta) \leq q^{fb}(\theta)$  for all  $\theta > 0$ ;

The continuity and monotonicity results of Theorem 3 are standard in screening models without lying costs. In particular, the continuity and monotonicity of  $V(\cdot)$  and the monotonicity of  $q(\cdot)$  are usually a direct consequences of incentive compatibility. In fact, it is well known that the monotonicity of  $q(\cdot)$  is a necessary and sufficient condition for implementability under single-crossing assumption. This means, in particular, that “ironing” is used sometimes in the optimal mechanism setting  $q(\cdot)$  to be constant on certain intervals.

Yet, the nature of these monotonicity and continuity results and their significance are different in our case. Particularly, the presence of fixed costs creates a gap between the payoff that an agent gets by acting truthfully and the payoff that she gets by imitating a close-by type, which makes it possible to implement non-monotone and discontinuous quantity schedules  $q(\cdot)$  and a discontinuous associated payoff function  $V(\cdot)$ .

To show this, suppose  $q(\cdot)$  and  $V(\cdot)$  are continuous and monotone. Then if type  $\theta$  imitated type  $\theta' + \epsilon$  for some small  $\epsilon$ , she would get a payoff equal to  $V(\theta + \epsilon) + u(q(\theta + \epsilon), \theta) - u(q(\theta + \epsilon), \theta + \epsilon) - C$  which is strictly less than her payoff  $V(\theta)$ . So, local incentive constraints are not binding for any type  $\theta$ , and we can change  $q(\cdot)$  and  $V(\cdot)$  slightly in each neighborhood and, in particular, choose them to be decreasing on some interval and choose  $V(\cdot)$  to be discontinuous without violating local incentive constraint. To see that this can be done without violating any global incentive and individual rationality constraints, consider a standard second-best mechanism  $(q^{sb}(\theta), t^{sb}(\theta))$  and associated net payoff function  $V^{sb}(\theta)$  which is optimal under zero fixed costs. Now suppose that  $C > 0$ . Then no incentive constraints would be binding in the mechanism  $(q^{sb}(\theta), t^{sb}(\theta))$ . So we can modify it slightly and, in particular, introduce intervals of non-monotonicity of  $q(\cdot)$  and  $V(\cdot)$  and discontinuity in  $V(\cdot)$ .

So, instead of relying on the analysis of incentive and/or individual rationality constraints, the proof of Theorem 3 uses optimality arguments to rule out the non-monotonicities and discontinuities in question. As an illustration, consider two close-by types  $\theta_1$  and  $\theta_2$ ,  $\theta_1 < \theta_2$  and suppose that the allocation of  $\theta_i$ ,  $i \in \{1, 2\}$  is  $(q(\theta_i), t(\theta_i))$  such that  $q(\theta_2) < q(\theta_1)$ . It is fairly easy to establish that  $q(\theta_i) \leq q^{fb}(\theta_i)$  by ruling out binding upwards incentive constraints in an optimal mechanism. Then  $\theta_2$  can be assigned the allocation  $(q(\theta_1), t')$  such that  $t' > t(\theta_2)$ . The transfer  $t'$  can be chosen so that all incentive constraints of type  $\theta_2$  hold and no type would want to imitate  $\theta_2$ . In fact, we can then increase the quantity allocation of  $\theta_2$  to  $q(\theta_1) + \delta$ , where  $\delta$  is sufficiently small and further increase the transfer paid by  $\theta_2$  by some small amount without violating any incentive constraints. This modification increases the seller's profits.

The no-exclusion property  $q(\theta) > 0$  is also due to the presence of fixed cost. Indeed, for every  $\theta > 0$ , there exists a sufficiently small  $q > 0$  such that  $u(q, 1) - u(q, \theta) < C$ . Then assigning  $q$  to type  $\theta$  in exchange for transfer  $u(q, \theta)$  increases the seller's expected profit without violating any other type's incentive constraint.

As we argued above, as implication of the continuity of the optimal allocation and payoff in type is that local incentive constraints are not binding for any type. Yet, of course, some incentive constraints must be binding, for otherwise the optimal mechanism would involve first-best quantities and full surplus extraction by the principal. Thus, identifying and char-

acterizing the set of binding incentive constraints is an important part in our analysis, and it is especially challenging since those binding constraints are non-local. We deal with this issue by, at first, establishing general properties of the binding incentive constraint correspondence in the following two Theorems. Building on these results in later sections, we derive necessary and sufficient conditions characterizing these constraints.

First let us define the targeted type correspondence  $\tau(\theta)$  in the mechanism  $(q(\cdot), t(\cdot))$  as follows:

$$\tau(\theta) = \{\theta' | u(q(\theta), \theta) - t(\theta) = u(q(\theta'), \theta) - t(\theta') - C\} \quad (4)$$

In words,  $\tau(\theta)$  is the set of all such types  $\theta'$  that incentive constraint  $IC(\theta, \theta')$  of type  $\theta$  is binding. Note that  $\tau(\theta)$  may be empty. If  $\tau(\theta)$  is non-empty which as we establish below, is so when  $\theta$  is sufficiently large, then we will call the types in  $\tau(\theta)$  as targeted types of type  $\theta$ .

Also, for any set  $\Theta \subseteq [0, 1]$ , we let  $\tau(\Theta) = \cup_{\theta \in \Theta} \tau(\theta)$ . The following Theorem provides basic key properties of the  $\tau(\cdot)$  correspondence.

- Theorem 4**
1. For any fixed cost  $C$ ,  $0 < C < \bar{C} \equiv \max_{\theta \in [0, 1]} u(q^{fb}(\theta), 1) - u(q^{fb}(\theta), \theta)$  there exists  $\hat{\theta}^c \in (0, 1)$  such that  $\tau(\theta) \neq \emptyset$  iff  $\theta \in [\hat{\theta}^c, 1]$ .
  2. The correspondence  $\tau(\theta)$  is strictly increasing, upper hemicontinuous and compact-valued on  $[\hat{\theta}^c, 1]$ , with  $\max \tau(\theta) < \theta$  and  $\min \tau(\theta) > 0$ .
  3. For all  $\theta \in [0, \max \tau(\hat{\theta}^c)] \cup [\min \tau(1), 1]$ , we have  $q(\theta) = q^{fb}(\theta)$ .
  4. If  $\theta_1, \theta_2 \in \tau(\theta)$  for some  $\theta$  and  $\theta_1 < \theta_2$ , then  $q(\theta') = q^{fb}(\theta')$  for all  $\theta' \in [\theta_1, \theta_2]$ .
  5. Let  $\bar{W} = \max_{q, \theta} u_{\theta q}(q, \theta) \times q^{fb}(1)$ . Then for any  $\theta \in [\hat{\theta}^c, 1]$ ,  $\theta - \max \tau(\theta) \geq \frac{C}{\bar{W}}$ .
  6.  $V(\theta) = 0$  for all  $\theta \in [0, \hat{\theta}^c]$ ,  $V(\theta) > 0$  for all  $\theta \in (\hat{\theta}^c, 1]$ .

By Theorem 4 the screening problem is non-trivial iff  $C < \bar{C} = \max_{\theta \in [0, 1]} u(q^{fb}(\theta), 1) - u(q^{fb}(\theta), \theta)$ . For each positive fixed cost below  $\bar{C}$  only sufficiently high types have binding incentive constraints pointing to some strictly lower types, and earn positive surpluses. Moreover, not all types are “targeted” i.e. have binding incentive constraints pointing to them.

**Figure 1: Chains of incentive constraint.**



In particular, high and low types are not targeted by any other types. This is intuitive, as imitating a low type given any other type too little surplus that is not sufficient to cover the fixed cost  $C$ . Likewise, imitating (or targeting) a high type does not given enough surplus for another type to cover the fixed cost because high types are paying sufficiently high transfer to the mechanism in exchange for a high quantity that they obtain. Figure ?? illustrates the relationship between targeted type correspondence  $\tau$  and quantity  $q$  in the optimal mechanism. Since types below  $\tau(\hat{\theta})$  and above  $\tau(1)$  are not targeted by any type, there is no reason for principal to distort allocation of those types. As a result, they receive first best quantities.

Significantly, Theorem 4 shows that targeted type correspondence is strictly increasing and compact-valued and that low and high types are not in its image, and are therefore assigned the first-best quantities.

To illustrate the implication of an increasing targeted type correspondence, let us formally define the high order targeted type recursively as follow: for any  $\theta$  and integer  $k$ ,  $\tau^0(\theta) = \theta$ ,  $\tau^k(\theta) = \tau(\tau^{k-1}(\theta))$ . As illustrated in Figure 1,  $(\tau(\theta), \dots, \tau^k(\theta), \dots)$  is a chain of targeted types originating from  $\theta$ . Let  $M = \max\{k : \tau^k(1) \neq \emptyset\}$  be the length of such chain originating from the highest type  $\theta = 1$ . Since  $\tau(\cdot)$  is continuous and increasing in  $\theta$ , it maps the interval  $[\tau^k(1), \tau^{k-1}(1)]$  onto the interval  $[\tau^{k+1}(1), \tau^k(1)]$  or, equivalently,  $\tau^{k-1}(\cdot)$  maps the interval  $[\tau(1), 1]$  onto the interval  $[\tau^k(1), \tau^{k-1}(1)]$  for all  $k \in \{1, \dots, M-1\}$ . Thus the type space  $[\hat{\theta}, 1]$  can be partitioned into a collection of adjacent intervals  $[\tau^k(1), \tau^{k-1}(1)]$  for  $k \in \{1, \dots, M-1\}$  such that for every  $\theta \in [\tau^{k+1}(1), \tau^k(1)]$ , we have  $\tau(\theta) \in [\tau^{k+2}(1), \tau^{k+1}(1)]$ , and the residual interval  $[\hat{\theta}, \tau^{M-1}(1)]$ .

In the next Lemma we show how the optimal mechanism changes as  $C$  goes to zero. For the purposes of this Lemma we slightly modify the notation and let  $q(\theta|C)$ ,  $V(\theta|C)$  be the

quantity, the net payoff of the type  $\theta$ , respectively, and let  $M(C)$  be the maximal length of a chain of targeted types in the unique optimal mechanism under fixed cost  $C$ . Also, let  $q^{sb}(\theta)$  and  $V^{sb}(\theta)$  be the optimal quantity and the net payoff of type  $\theta$  in the solution to the standard screening problem with zero cost of misrepresentation.

**Lemma 1** *We have  $\lim_{C \downarrow 0} q(\theta|C) = q^{sb}(\theta)$ ,  $\lim_{C \downarrow 0} V(\theta|C) = V^{sb}(\theta)$  for all  $\theta \in [0, 1]$ , and  $\lim_{C \downarrow 0} M(C) = \infty$ .*

## 4 Deriving the Optimal Mechanism

### 4.1 Reformulation of the Problem

In this section we reformulate our mechanism design problem of choosing the quantity/transfer profile  $(q(\cdot), t(\cdot))$  to solve (1) - (3) as a problem of choosing an optimal profile  $(q(\cdot), \tau(\cdot), \hat{\theta})$ , where  $\tau(\theta)$  is a “targeted type” which is defined in (4) and which in an incentive compatible mechanism solves  $\max_{\theta'} u(q(\theta'), \theta) - t(\theta') - C$  and  $\hat{\theta}$  is the lowest type for which  $\tau$  is non-empty, so that  $V(\theta) > 0$  iff  $\theta > \hat{\theta}$ . Lemma 2 states a condition under which  $\tau(\theta)$  is single-valued in the optimal mechanism.

**Lemma 2** *Let  $(q(\theta), t(\theta))$  be an optimal mechanism and  $(V(\theta), \tau(\theta))$  be the corresponding net payoff function and targeted type correspondence. Suppose that  $G(\theta, \theta') + V(\theta') \equiv u(q^{fb}(\theta'), \theta) - u(q^{fb}(\theta'), \theta') + V(\theta')$  is strictly quasi-concave in  $\theta'$ , then  $\tau(\theta)$  is single-valued.*

By Theorem 3, we may without loss of generality assume that  $q(\cdot)$ ,  $t(\cdot)$  and  $V(\cdot)$  are increasing and continuous, and hence almost everywhere differentiable. By Theorem 4 the targeted type in  $\tau(\theta)$  is increasing and, therefore, satisfies the following first-order condition for all  $\theta \in [\hat{\theta}, 1]$ :

$$u_q(q(\tau(\theta)), \theta) \dot{q}(\tau(\theta)) - \dot{t}(\tau(\theta)) = 0 \quad (5)$$

Then differentiating  $V(\theta) = u(q(\tau(\theta)), \theta) - t(\tau(\theta)) - C$  at  $\theta \in [\hat{\theta}, 1]$  and using (5) yields:

$$\dot{V}(\theta) = u_\theta(q(\tau(\theta)), \theta) + \dot{\tau}(\theta)[u_q(q(\tau(\theta)), \theta) \dot{q}(\tau(\theta)) - \dot{t}(\tau(\theta))] = u_\theta(q(\tau(\theta)), \theta) \quad (6)$$

In combination with  $V(\theta) = 0$  for  $\theta \in [0, \hat{\theta}]$ , (6) implies that for all  $\theta \in [0, 1]$  we have:

$$V(\theta) = \int_{\hat{\theta}}^{\max\{\theta, \hat{\theta}\}} u_{\theta}(q(\tau(s)), s) ds \quad (7)$$

So, for all  $\theta \in [0, 1]$  we have:

$$t(\theta) = u(q(\theta), \theta) - V(\theta) = u(q(\theta), \theta) - \int_{\hat{\theta}}^{\max\{\theta, \hat{\theta}\}} u_{\theta}(q(\tau(s)), s) ds. \quad (8)$$

Using (8) yields the following expression for the seller's expected profits:

$$\int_0^1 [u(q(\theta), \theta) - \int_{\hat{\theta}}^{\max\{\theta, \hat{\theta}\}} u_{\theta}(q(\tau(s)), s) ds] f(\theta) d\theta = \int_0^1 u(q(\theta), \theta) f(\theta) - \int_{\hat{\theta}}^1 (1 - F(\theta)) u_{\theta}(q(\tau(\theta)), \theta) d\theta \quad (9)$$

Since  $q(\theta) = q^{fb}(\theta)$  for all  $\theta \in [0, \tau(\hat{\theta})] \cup [\tau(1), 1]$  we can rewrite (9) as follows:

$$\begin{aligned} & \int_{\tau(1)}^1 u(q^{fb}(\theta), \theta) f(\theta) d\theta + \int_{\tau(\hat{\theta})}^{\tau(1)} u(q(\theta), \theta) f(\theta) d\theta - \int_{\hat{\theta}}^1 (1 - F(\theta)) u_{\theta}(q(\tau(\theta)), \theta) d\theta + \int_0^{\tau(\hat{\theta})} u(q^{fb}(\theta), \theta) f(\theta) d\theta \\ &= \int_{\tau(1)}^1 u(q^{fb}(\theta), \theta) f(\theta) d\theta + \int_{\hat{\theta}}^1 u(q(\tau(\theta)), \tau(\theta)) f(\tau(\theta)) \dot{\tau}(\theta) - (1 - F(\theta)) u_{\theta}(q(\tau(\theta)), \theta) d\theta \\ &+ \int_0^{\tau(\hat{\theta})} u(q^{fb}(\theta), \theta) f(\theta) d\theta \end{aligned} \quad (10)$$

where the equality is obtained by making a change of variables in the second integral before the equality from  $\theta \in [\tau(\hat{\theta}), \tau(1)]$  to  $\tau(\theta)$ .

Next, we use (5), the first-order condition (11) for  $\tau$ , to derive the law of motion of  $q(\cdot)$ . First, differentiate (8) to get:

$$\dot{t}(\theta) = u_q(q(\theta), \theta) \dot{q}(\theta) + u_{\theta}(q(\theta), \theta) - 1(\theta \geq \hat{\theta}) u_{\theta}(q(\tau(\theta)), \theta) \quad (11)$$

Combining (5) and (11) we obtain that for all  $\theta \in [\hat{\theta}, 1]$ ,

$$[u_q(q(\tau(\theta)), \theta) - u_q(q(\tau(\theta)), \tau(\theta))] \dot{q}(\tau(\theta)) - u_{\theta}(q(\tau(\theta)), \tau(\theta)) + 1(\tau(\theta) \geq \hat{\theta}) u_{\theta}(q(\tau(\tau(\theta))), \tau(\theta)) = 0 \quad (12)$$

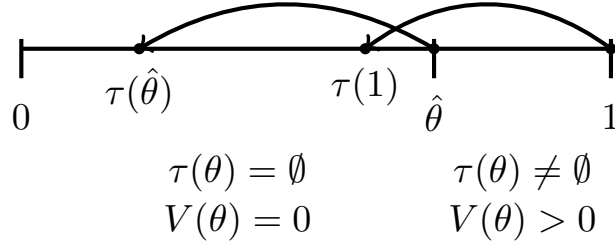
Finally, recall that the following boundary conditions must hold:

$$q(\tau(1)) = q^{fb}(\tau(1)) \quad (13)$$

$$q(\tau(\hat{\theta})) = q^{fb}(\tau(\hat{\theta})) \quad (14)$$

$$V(\hat{\theta}) = u(q(\tau(\hat{\theta})), \hat{\theta}) - u(q(\tau(\hat{\theta})), \tau(\hat{\theta})) - C = 0 \quad (15)$$

**Figure 2: Structure of targeted types and informational rents under intermediate costs of lying**



We will refer to the problem of maximizing (10) with respect to choice variables  $(q(\theta), \tau(\theta), \hat{\theta})$  subject to the differential equation (12) and the boundary conditions (13)-(15) as a relaxed program. It is a relaxed program, because we have not imposed all incentive constraints on it directly. Neither have we required  $q(\cdot)$  and  $\tau(\cdot)$  to be increasing and  $\tau(\theta) < \theta$  whenever  $\tau(\theta) \neq \emptyset$  which must be the case in the optimal mechanism. At the same time, the individual rationality of the solution to the relaxed program follows directly from (8).

Furthermore, in the sequel we will solve the relaxed program and then establish that it satisfies the omitted constraints and its solution uniquely defines a solution  $(q(\cdot), t(\cdot))$  to our original problem (1) - (3).

## 4.2 Solving for the Optimal Mechanisms: Intermediate Costs

In this section we solve the relaxed program formulated in the previous subsection and characterize the optimal mechanism for a range of intermediate values of the fixed cost  $C$  under which the optimal mechanism has a particularly simple structure, as shown in the next Theorem.

**Theorem 5** *There exists  $\underline{C} \in (0, \bar{C})$  such that if  $C \in (\underline{C}, \bar{C})$ , then in the optimal mechanism  $\tau(1) < \hat{\theta}$ .*

According to this Theorem, when  $C \in (\underline{C}, \bar{C})$ , then  $\tau([0, 1]) \subseteq [0, \hat{\theta}]$ . Since  $\tau(\theta) = \emptyset$  and  $V(\theta) = 0$  for all  $\theta < \hat{\theta}$ , it follows that  $\tau(\tau(\theta)) = \emptyset$  for all  $\theta$ . Thus, the maximal length of the chain of targeted types is 1, as illustrated in Figure 2, and the last term in (12) is



zero. Moreover, all types within the image of  $\tau$  get zero net payoff, i.e.  $V(\theta) = 0$  for all  $\theta \in [\tau(\hat{\theta}), \tau(1)]$ . Therefore, Lemma 2 ensures that  $\tau(\cdot)$  is a single-valued function given the following assumption.

**Assumption 2**  $G(\theta, \theta') \equiv u(q^{fb}(\theta'), \theta) - u(q^{fb}(\theta'), \theta')$  is strictly strictly quasi-concave in  $\theta'$ .

Assumption 2 holds for many commonly specified utility functions, for example, a linear quadratic one,  $\theta q - \frac{q^2}{2}$ . A sufficient condition for it is concavity of the first-best quantity  $q^{fb}(\theta)$  in  $\theta$  together with  $u_{\theta\theta} \geq 0$ .

Our next step is to derive a solution to the relaxed program- maximizing (10) subject to (12) and (13)-(15)- via optimal control method. To this end, we will first make a change of variables. Specifically, let  $Q(\theta) = q(\tau(\theta))$  be the quantity assigned to the targeted type of  $\theta$ . Note that finding a solution  $(q(\theta), \tau(\theta), \hat{\theta})$  to the relaxed program is equivalent to finding a solution  $(Q(\theta), \tau(\theta), \hat{\theta})$ . In particular, since  $\dot{Q}(\theta) = \dot{q}(\tau(\theta))\dot{\tau}(\theta)$ , we can rewrite (12) as follows:

$$\dot{Q}(\theta) = \frac{u_{\theta}(Q(\theta), \tau(\theta))}{u_q(Q(\theta), \theta) - u_q(Q(\theta), \tau(\theta))} \dot{\tau}(\theta) \text{ for all } \theta \in [\hat{\theta}, 1] \quad (16)$$

Next, let us define scrap values  $S_0(\hat{\theta}, \tau(\hat{\theta}))$  and  $S_1(\tau(1))$ :

$$S_0(\hat{\theta}, \tau(\hat{\theta})) = \int_0^{\tau(\hat{\theta})} u(q^{fb}(\theta), \theta) f(\theta) d\theta \quad (17)$$

$$S_1(\tau(1)) = \int_{\tau(1)}^1 u(q^{fb}(\theta), \theta) f(\theta) d\theta \quad (18)$$

Now we can rewrite our relaxed program as follows:

$$\max_{Q(\theta), \tau(\theta), \hat{\theta}} \int_{\hat{\theta}}^1 u(Q(\theta), \tau(\theta)) f(\tau(\theta)) \dot{\tau}(\theta) - (1 - F(\theta)) u_{\theta}(Q(\theta), \theta) d\theta + S_0(\hat{\theta}, \tau(\hat{\theta})) + S_1(\tau(1)) \quad (19)$$

subject to (16) and the boundary conditions:

$$R_1 \equiv Q(1) - q^{fb}(\tau(1)) = 0, \quad (20)$$

$$R_2 \equiv Q(\hat{\theta}) - q^{fb}(\tau(\hat{\theta})) = 0, \quad (21)$$

$$R_3 \equiv u(Q(\hat{\theta}), \hat{\theta}) - u(Q(\hat{\theta}), \tau(\hat{\theta})) - C = 0. \quad (22)$$

This problem is amenable to optimal control approach, with state variables  $Q(\cdot)$  and  $\tau(\cdot)$ , control variable  $\alpha$  satisfying  $\dot{\tau}(\theta) = \alpha$ , and a free boundary  $\hat{\theta}$ . Introducing the notation  $h(\theta, Q, \tau) = \frac{u_\theta(Q, \tau)}{u_q(Q, \theta) - u_q(Q, \tau)}$ , the Hamiltonian of this optimal control problem is given by:

$$H = u(Q, \tau)f(\tau)\alpha - (1 - F(\theta))u_\theta(Q, \theta) + \lambda_Q h(\theta, Q, \tau)\alpha + \lambda_\tau \alpha \quad (23)$$

The linearity of the Hamiltonian (23) in the control variable  $\alpha$  creates certain technical difficulties, as it implies that  $\alpha$  cannot be solved for directly from the standard first-order conditions. However, Pontryagin's Maximum principle still applies and requires that the optimal control  $\alpha$  maximizes the Hamiltonian (23).

Particularly, let us introduce the following *switching function*  $J(\theta, Q(\theta), \tau(\theta), \lambda_Q(\theta), \lambda_\tau(\theta))$ :

$$J(\theta, Q(\theta), \tau(\theta), \lambda_Q(\theta), \lambda_\tau(\theta)) = u(Q, \tau)f(\tau) + \lambda_Q h(\theta, Q, \tau) + \lambda_\tau \quad (24)$$

Note that the switching function  $J$  can never be strictly positive, since then the optimal value of  $\alpha$  is infinity and, correspondingly, the value of the objective would be infinite. Optimality requires the following "switching conditions" to hold:

$$\begin{aligned} J(\theta, Q, \tau, \lambda_Q, \lambda_\tau) < 0 &\Rightarrow \alpha = 0 \\ J(\theta, Q, \tau, \lambda_Q, \lambda_\tau) = 0 &\Rightarrow \alpha \geq 0 \end{aligned}$$

An interval of  $\theta$  on which  $J < 0$  is called a *nonsingular arc*. The optimal solution involves setting  $\alpha(\theta) = 0$  for all  $\theta$  on a non-singular arc.

An interval of  $\theta$  on which  $J$  vanishes ( $J = 0$ ) is called a *singular arc*. On a singular arc, the optimality conditions do not pin down the value of the optimal control  $\alpha$ . As a consequence, such problems of singular control are quite difficult to solve. Only a few solutions have been developed up to now, most notably Merton (1969)'s celebrated portfolio choice problem in finance, and trajectory optimization in aeronautics (see e.g. Bryson and Ho (1975) Ch. 8). The approach we follow here is to recover the optimal control  $\alpha$  along a singular arc by differentiating the identity  $J = 0$  with respect to  $\theta$  until the control variable appears in a non-trivial way, and then solve for it. Significantly, our solution is simplified by finding that the whole domain in our case constitutes a singular arc, so that we do not have to characterize the juncture points between singular and non-singular arcs.

In addition, by Pontryagin's Maximum principle the solution has to satisfy the following costate equations:

$$-\dot{\lambda}_Q = \frac{\partial H}{\partial Q} = u_q(Q, \tau)f(\tau)\alpha - [1 - F(\theta)]u_{\theta q}(Q, \theta) + \lambda_Q \frac{\partial h}{\partial Q}\alpha = \quad (25)$$

$$u_q(Q, \tau)f(\tau)\alpha - [1 - F(\theta)]u_{\theta q}(Q, \theta) + \lambda_Q \frac{u_{\theta q}(Q, \tau)[u_q(Q, \theta) - u_q(Q, \tau)] - u_{\theta}(Q, \tau)[u_{qq}(Q, \theta) - u_{qq}(Q, \tau)]}{[u_q(Q, \theta) - u_q(Q, \tau)]^2}\alpha$$

$$-\dot{\lambda}_\tau = \frac{\partial H}{\partial \tau} = u_\theta(Q, \tau)f(\tau)\alpha + u(Q, \tau)f'(\tau)\alpha + \lambda_Q \frac{\partial h}{\partial \tau}\alpha = \quad (26)$$

$$u_\theta(Q, \tau)f(\tau)\alpha + u(Q, \tau)f'(\tau)\alpha + \lambda_Q \frac{u_{\theta\theta}(Q, \tau)[u_q(Q, \theta) - u_q(Q, \tau)] + u_\theta(Q, \tau)u_{\theta q}(Q, \tau)}{[u_q(Q, \theta) - u_q(Q, \tau)]^2}\alpha$$

In addition, the following transversality conditions have to hold for some  $\gamma_1, \gamma_2, \gamma_3$ :

$$\lambda_Q(1) = \gamma_1 \frac{\partial R_1}{\partial Q(1)} = \gamma_1 \quad (27)$$

$$\lambda_\tau(1) = \gamma_1 \frac{\partial R_1}{\partial \tau(1)} + \frac{\partial S_1}{\partial \tau(1)} = -\gamma_1 \dot{q}^{fb}(\tau(1)) - u(Q(1), \tau(1))f(\tau(1)) \quad (28)$$

$$-\lambda_Q(\hat{\theta}) = \gamma_2 \frac{\partial R_2}{\partial Q(\hat{\theta})} + \gamma_3 \frac{\partial R_3}{\partial Q(\hat{\theta})} = \gamma_2 + \gamma_3 [u_q(Q(\hat{\theta}), \hat{\theta}) - u_q(Q(\hat{\theta}), \tau(\hat{\theta}))] \quad (29)$$

$$-\lambda_\tau(\hat{\theta}) = \gamma_2 \frac{\partial R_2}{\partial \tau(\hat{\theta})} + \gamma_3 \frac{\partial R_3}{\partial \tau(\hat{\theta})} + \frac{\partial S_0}{\partial \tau(\hat{\theta})} = -\gamma_2 \dot{q}^{fb}(\tau(\hat{\theta})) - \gamma_3 u_\theta(Q(\hat{\theta}), \tau(\hat{\theta})) + u(Q(\hat{\theta}), \tau(\hat{\theta}))f(\tau(\hat{\theta})) \quad (30)$$

$$H(\hat{\theta}) = \gamma_3 \frac{\partial R_3}{\partial \hat{\theta}} = \gamma_3 u_\theta(Q(\hat{\theta}), \hat{\theta}) \quad (31)$$

Now, consider a singular arc where we have  $J = u(Q, \tau)f(\tau) + \lambda_Q h(\theta, Q, \tau) + \lambda_\tau = 0$ . Then by (23)  $H(\hat{\theta}) = J(\hat{\theta})\alpha(\hat{\theta}) - [1 - F(\hat{\theta})]u_\theta(Q(\hat{\theta}), \hat{\theta}) = -[1 - F(\hat{\theta})]u_\theta(Q(\hat{\theta}), \hat{\theta})$ . It can be easily verified that with  $\gamma_3 = F(\hat{\theta}) - 1$  and  $\gamma_1 = \gamma_2 = 0$ , transversality conditions (27)-(31) are satisfied.

Differentiating the switching function  $J$  on a singular arc we get:

$$\frac{dJ}{d\theta} = \dot{\lambda}_Q h(\theta, Q, \tau) + \lambda_Q \left( \frac{\partial h}{\partial \theta} + \frac{\partial h}{\partial Q} \dot{Q} + \frac{\partial h}{\partial \tau} \dot{\tau} \right) + \dot{\lambda}_\tau + u_q(Q, \tau)f(\tau)\dot{Q} + u_\theta(Q, \tau)f(\tau)\dot{\tau} + u(Q, \tau)f'(\tau)\dot{\tau} = 0 \quad (32)$$

Totally differentiating (32) yields:

$$-\dot{\lambda}_\tau = \dot{\lambda}_Q h + \lambda_Q \left( \frac{\partial h}{\partial Q} \dot{Q} + \frac{\partial h}{\partial \tau} \dot{\tau} + \frac{\partial h}{\partial \theta} \right) + u_q(Q, \tau) f(\tau) \dot{Q} + u_\theta(Q, \tau) f(\tau) \dot{\tau} + u(Q, \tau) f'(\tau) \dot{\tau} \quad (33)$$

From (25), (33) and  $\dot{\tau} = \alpha$ ,

$$\begin{aligned} -\dot{\lambda}_\tau &= -u_q(Q, \tau) f(\tau) \dot{\tau} h + (1 - F(\theta)) u_{\theta q}(Q, \theta) h - \lambda_Q \frac{\partial h}{\partial Q} \dot{\tau} h \\ &\quad + \lambda_Q \left( \frac{\partial h}{\partial Q} \dot{Q} + \frac{\partial h}{\partial \tau} \dot{\tau} + \frac{\partial h}{\partial \theta} \right) + u_q(Q, \tau) f(\tau) \dot{Q} + u_\theta(Q, \tau) f(\tau) \dot{\tau} + u(Q, \tau) f'(\tau) \dot{\tau} \\ &= (1 - F(\theta)) u_{\theta q}(Q, \theta) h + \lambda_Q \left( \frac{\partial h}{\partial \tau} \dot{\tau} + \frac{\partial h}{\partial \theta} \right) + u_\theta(Q, \tau) f(\tau) \dot{\tau} + u(Q, \tau) f'(\tau) \dot{\tau} \end{aligned} \quad (34)$$

Substituting (26) for  $\dot{\lambda}_\tau$  on the left-hand side of (34) and using  $\dot{\tau} = \alpha$  yields:

$$\begin{aligned} u_\theta(Q, \tau) f(\tau) \dot{\tau} + u(Q, \tau) f'(\tau) \dot{\tau} + \lambda_Q \frac{\partial h}{\partial \tau} \dot{\tau} &= \\ &= (1 - F(\theta)) u_{\theta q}(Q, \theta) h + \lambda_Q \left( \frac{\partial h}{\partial \tau} \dot{\tau} + \frac{\partial h}{\partial \theta} \right) + u_\theta(Q, \tau) f(\tau) \dot{\tau} + u(Q, \tau) f'(\tau) \dot{\tau} \end{aligned}$$

which, after collecting terms and using  $\dot{Q} = h\dot{\tau}$ , simplifies to:

$$\lambda_Q \frac{\partial h}{\partial \theta} = -(1 - F(\theta)) u_{\theta q}(Q, \theta) h \quad (35)$$

Using  $\frac{\partial h}{\partial \theta} = \frac{-u_{\theta q}(Q, \theta)}{u_q(Q, \theta) - u_q(Q, \tau)} h$  in (35) yields:

$$\lambda_Q = (1 - F(\theta)) (u_q(Q, \theta) - u_q(Q, \tau)) \quad (36)$$

Next totally differentiate (36) to obtain:

$$\dot{\lambda}_Q = (1 - F(\theta)) [u_{qq}(Q, \theta) \dot{Q} - u_{qq}(Q, \tau) \dot{Q} + u_{\theta q}(Q, \theta) - u_{\theta q}(Q, \tau) \dot{\tau}] - f(\theta) [u_q(Q, \theta) - u_q(Q, \tau)] \quad (37)$$

Now we can substitute (25) for  $\dot{\lambda}_Q$  in (37) to obtain:

$$\begin{aligned} u_q(Q, \tau) f(\tau) \dot{\tau} - [1 - F(\theta)] u_{\theta q}(Q, \theta) + \lambda_Q \frac{\partial h}{\partial Q} \dot{\tau} &= \\ - (1 - F(\theta)) [u_{qq}(Q, \theta) \dot{Q} - u_{qq}(Q, \tau) \dot{Q} + u_{\theta q}(Q, \theta) - u_{\theta q}(Q, \tau) \dot{\tau}] + f(\theta) [u_q(Q, \theta) - u_q(Q, \tau)] \end{aligned} \quad (38)$$

Using (36),  $\frac{\partial h}{\partial Q} = \frac{u_{\theta q}(Q, \tau) - [u_{qq}(Q, \theta) - u_{qq}(Q, \tau)]h}{u_q(Q, \theta) - u_q(Q, \tau)}$  and  $\dot{Q} = h\dot{\tau}$  and cancelling terms in the previous equation yields the following differential equation:

$$\dot{\tau} = \frac{f(\theta)(u_q(Q, \theta) - u_q(Q, \tau))}{f(\tau)u_q(Q, \tau)} \quad (39)$$

Finally, using  $\dot{Q} = h\dot{\tau} = \frac{u_{\theta}(Q, \tau)}{u_q(Q, \theta) - u_q(Q, \tau)}\dot{\tau}$  we obtain:

$$\dot{Q} = \frac{f(\theta)u_{\theta}(Q, \tau)}{f(\tau)u_q(Q, \tau)} \quad (40)$$

The system of ordinary differential equations (39) and (40) describes the dynamics of  $Q$  and  $\tau$  in the optimal mechanism. The following Theorem shows that (39) and (40) with boundary conditions (20)-(22) uniquely characterize the optimal mechanism.

**Theorem 6** *Suppose that  $u_{\theta qq}(q, \theta) \geq 0$  for all  $(q, \theta) \in \mathbf{R}_+ \times [0, 1]$ . For any  $C \in (\underline{C}, \overline{C})$ , there is a unique triple  $(\tau(\theta), Q(\theta), \hat{\theta})$  such that  $(\tau(\theta), Q(\theta))$  is an increasing solution to the system of ordinary differential equations (39) and (40) with boundary conditions (20)-(22), where in particular  $\tau(\hat{\theta})$  is the smallest solution to (22).*

*This triple  $(\tau(\theta), Q(\theta), \hat{\theta})$  uniquely defines the optimal mechanism  $(q(\cdot), t(\cdot))$  as follows:  $q(\theta) = q^{fb}(\theta)$  for all  $\theta \in [0, \tau(\hat{\theta})] \cup [\tau(1), 1]$ ,  $q(\theta) = Q(\tau^{-1}(\theta))$  for all  $\theta \in [\tau(\hat{\theta}), \tau(1)]$ , and  $t(\cdot)$  is given by (8).*

The next Theorem shows comparative statics of the optimal mechanism. Further discussion on comparative statics are given in the next subsection.

**Theorem 7** *Suppose that  $u_{\theta qq}(q, \theta) \geq 0$  for all  $(q, \theta) \in \mathbf{R}_+ \times [0, 1]$ . Given any  $C_i \in (\underline{C}, \overline{C})$ ,  $i = 1, 2$ , let  $(q_i(\theta), t_i(\theta))$  be the optimal mechanism and  $(\tau_i(\theta), Q_i(\theta), \hat{\theta}_i)$  be the corresponding triple. If  $C_2 > C_1$ , then:*

- (1)  $\hat{\theta}_2 > \hat{\theta}_1$ ; (2)  $\tau_2(\hat{\theta}_2) > \tau_1(\hat{\theta}_1)$ ;
- (3)  $\tau_2(\theta) < \tau_1(\theta)$  for  $\theta \in [\hat{\theta}_2, 1]$ ; (4)  $q_2(\theta) > q_1(\theta)$  for  $\theta \in [\tau_2(\hat{\theta}_2), \tau_2(1)]$ .

### 4.3 Quadratic-Uniform Example

In this section we derive a closed-form solution to our problem for the case when  $u(q, \theta) = \theta q - \frac{q^2}{2}$ ,  $\theta$  is uniformly distributed on  $[0, 1]$ , and length of chain  $M = 1$ . Details of derivation are provided in Appendix C.

Given the quadratic-uniform assumptions and  $M = 1$ , differential equations (39)-(40) and boundary conditions (20)-(22) imply that the optimal  $\tau(\theta)$  and  $Q(\theta)$  satisfy the following differential equations:

$$\dot{\tau} = \frac{\theta - \tau}{\tau - Q} \quad (41)$$

$$\dot{Q} = \frac{Q}{\tau - Q} \quad (42)$$

with boundary conditions

$$Q(1) = \tau(1) \quad (43)$$

$$Q(\hat{\theta}) = \tau(\hat{\theta}) \quad (44)$$

$$Q(\hat{\theta})[\hat{\theta} - \tau(\hat{\theta})] = C \quad (45)$$

Ordinary differential equation system (41)-(45) has the following parametric solution defined for  $t \in [\hat{t}, 1]$ :

$$\theta(t) = b_1 \left( t - \frac{1 + 3\sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} + \frac{3\sqrt{\frac{1}{5}} - 1}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{\sqrt{5} + 1}{2\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5} - 1}{2\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \quad (46)$$

$$Q(t) = -\frac{b_1}{2} t \quad (47)$$

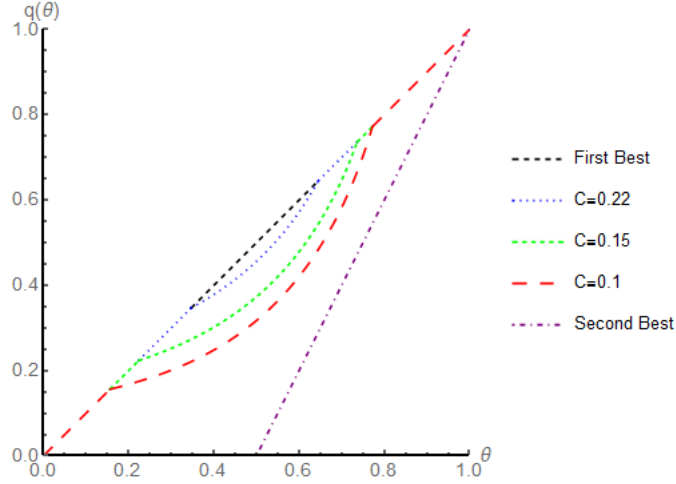
$$\tau(t) = b_1 \left( \frac{t}{2} - \frac{1 + \sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} - \frac{1 - \sqrt{\frac{1}{5}}}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \quad (48)$$

$$b_1 = -\frac{\frac{1}{\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+1}{2}}}{\hat{t} - \frac{1 + \sqrt{\frac{1}{5}}}{2} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1 - \sqrt{\frac{1}{5}}}{2} \hat{t}^{-\frac{\sqrt{5}+1}{2}}} \quad (49)$$

$$C = -\frac{b_1}{2} \left( b_1 \left( \frac{\hat{t}^2}{2} - \frac{1}{\sqrt{5}} \hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{1}{\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}-1}{2}} \right) + \frac{\sqrt{5} - 1}{2\sqrt{5}} \hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{\sqrt{5} + 1}{2\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}-1}{2}} \right) \quad (50)$$

where  $(\theta(t), Q(t), \tau(t))$  characterize the implicit functions  $(Q(\theta), \tau(\theta))$  defined on  $\theta \in [\hat{\theta}, 1]$ ,

**Figure 3: Optimal quantities, quadratic-uniform case.**



with  $\theta(\hat{t}) = \hat{\theta}$  and  $\theta(1) = 1$ . The optimal quantity  $q(\theta)$  for  $\theta \in [\tau(\hat{\theta}), \tau(1)]$  can be computed via the following relation:  $q(\tau(t)) = Q(t)$ .

The admissible cost range for this example (where  $\tau(1) < \hat{\theta}$ ) is  $(\underline{C}, \overline{C})$ , where  $\overline{C} = 0.25$  and  $\underline{C} \approx 0.09$ . For  $C$  in this range,  $(Q(\theta), \tau(\theta), q(\theta))$  and scalars  $(b_1, \hat{t})$  are uniquely determined by (46)-(50).

This solution exhibits several properties. The optimal quantity  $q(\theta)$  is strictly increasing in  $\theta$ , which is consistent with the general property given by Theorem 3. In this particular example,  $q(\theta)$  is also strictly convex for  $\theta \in [\tau(\hat{\theta}), \tau(1)]$ .

For comparative statics, an increase in cost of lying create potential slackness of incentive compatibility, which is filled by two forces to generate extra profit. First, principal generate higher revenue by improving efficiency of the mechanism. As illustrated in Figure 3, optimal quantities increase for the medium types. The interval of types with distorted quantity  $[\tau(\hat{\theta}), \tau(1)]$ , becomes narrower, and quantities converge to first best level as the cost goes to  $\overline{C}$ . Second, principal extracts more surplus from the agent. Note from Figure 4 that for a higher  $C$ , the targeted type  $\tau(\theta)$  is lower for any given type. It reduces agent's surplus  $V(\theta) = \int_{\hat{\theta}}^{\theta} u_{\theta}(q(\tau(\theta')), \theta') d\theta'$  in intensive margin. In addition, the cutoff type  $\hat{\theta}$  is increasing in  $C$ , which reduces agent's surplus in extensive margin. As  $C$  goes to  $\overline{C}$ ,  $\hat{\theta}$  converges to 1 and

Figure 4: Optimal targeted types, quadratic-uniform case.

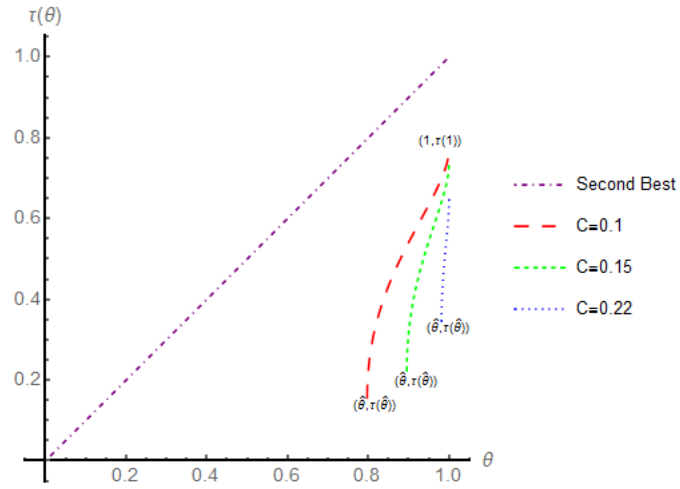


Figure 5: Optimal values of  $\hat{\theta}$ ,  $\tau(1)$  and  $\tau(\hat{\theta})$ , quadratic-uniform case.

$C$	$\hat{\theta}$	$\tau(1)$	$\tau(\hat{\theta})$
0.09	0.78	0.78	0.15
0.1	0.80	0.77	0.16
0.15	0.90	0.74	0.23
0.22	0.98	0.65	0.35
0.25	1	0.5	0.5



all surplus is extracted.

Finally, this solution applies under the condition that any type in the image of  $\tau$  gets zero surplus in the optimal mechanism, i.e.  $\tau(1) \leq \hat{\theta}$ . The lower bound of cost of lying that satisfies this condition is  $\underline{C} \approx 0.09$ . Figure 5 shows that  $\hat{\theta}$  and  $\tau(1)$  converge to each other as  $C$  approaches the lower bound.

## 5 Optimal Mechanisms: low costs

The goal of this subsection is to reformulate our problem as an optimal control for low values of fixed cost  $C$ . The objective on the right-hand side of (10) cannot be used directly for this purpose because its associated law of motion in (12) contains a composite function  $\tau(\tau(\theta))$ . However, by introducing new variables and performing a change of variable of integration in (10) we can obtain a formulation amenable to optimal control methods.

Recall from Section 3 that the interval  $[\hat{\theta}, 1]$  can be partitioned into a set of intervals  $[\tau^k(1), \tau^{k-1}(1)]$  for  $k \in \{1, \dots, M-1\}$  where  $\tau([\tau^k(1), \tau^{k-1}(1)]) = [\tau^{k+1}(1), \tau^k(1)]$ , and the residual interval  $[\hat{\theta}, \tau^{M-1}(1)]$ .

To handle the residual interval, note that there exists  $\hat{\theta}_M \in [\tau(1), 1]$  such that  $\tau^{M-1}(\hat{\theta}_M) = \hat{\theta}$ , and  $\max\{k : \tau^k(\theta) \neq \emptyset | \theta \in [\hat{\theta}_M, 1]\} = M$ ;  $\max\{k : \tau^k(\theta) \neq \emptyset | \theta \in [\tau(1), \hat{\theta}_M]\} = M-1$ . Let  $M(\theta)$  be the length of the chain of targeted types originating at  $\theta \in [\tau(1), 1]$ . We have:

$$M(\theta) = \begin{cases} M & \text{if } \theta \in [\hat{\theta}_M, 1] \\ M-1 & \text{if } \theta \in [\tau(1), \hat{\theta}_M] \end{cases}$$

Now, consider the range of integration  $[\hat{\theta}, 1]$  in the middle term on the right-hand side of (10). It can be represented as follows:  $[\hat{\theta}, 1] = \cup_{k \in \{1, \dots, M-1\}} [\tau^k(1), \tau^{k-1}(1)] \cup [\tau^{M-1}(\hat{\theta}_M), \tau^{M-1}(1)]$ . Any type in  $[\tau(\hat{\theta}), \tau(1)]$  is represented as an element of a chain of targeted types originating from some type in  $[\tau(1), 1]$ .

Therefore, we can first rewrite the integral over  $[\hat{\theta}, 1]$  on the right-hand side of (10) as a sum of integrals on a disjoint collection of intervals  $[\tau^k(1), \tau^{k-1}(1)]$ ,  $k \in \{1, \dots, M-1\}$  and  $[\hat{\theta}, \tau^{M-1}(1)]$ , and then use a change of variables in each interval using the functions  $\tau^{k-1}(\cdot)$ ,

$k \in \{1, \dots, M-1\}$ , and  $\tau^{M-1}(\cdot)$ , respectively, to rewrite it as a sum of integrals over the single range  $[\tau(1), 1]$  as follows:

$$\begin{aligned}
& \int_{\hat{\theta}}^1 u(q(\tau(\theta)), \tau(\theta))f(\tau(\theta))\dot{\tau}(\theta) - (1 - F(\theta))u_{\theta}(q(\tau(\theta)), \theta)d\theta \\
&= \sum_{k=1}^{M-1} \int_{\tau^k(1)}^{\tau^{k-1}(1)} u(q(\tau(\theta)), \tau(\theta))f(\tau(\theta))\dot{\tau}(\theta) - (1 - F(\theta))u_{\theta}(q(\tau(\theta)), \theta)d\theta \\
&\quad + \int_{\hat{\theta}}^{\tau^{M-1}(1)} u(q(\tau(\theta)), \tau(\theta))f(\tau(\theta))\dot{\tau}(\theta) - (1 - F(\theta))u_{\theta}(q(\tau(\theta)), \theta)d\theta \\
&= \sum_{k=1}^{M-1} \int_{\tau(1)}^1 u(q(\tau^k(\theta)), \tau^k(\theta))f(\tau^k(\theta))\dot{\tau}^k(\theta) - (1 - F(\tau^{k-1}(\theta)))u_{\theta}(q(\tau^k(\theta)), \tau^{k-1}(\theta))\dot{\tau}^{k-1}(\theta)d\theta \\
&\quad + \int_{\hat{\theta}_M}^1 u(q(\tau^M(\theta)), \tau^M(\theta))f(\tau^M(\theta))\dot{\tau}^M(\theta) - (1 - F(\tau^{M-1}(\theta)))u_{\theta}(q(\tau^M(\theta)), \tau^{M-1}(\theta))\dot{\tau}^{M-1}(\theta)d\theta \\
&= \int_{\tau(1)}^1 \sum_{k=1}^{M(\theta)} u(q(\tau^k(\theta)), \tau^k(\theta))f(\tau^k(\theta))\dot{\tau}^k(\theta) - (1 - F(\tau^{k-1}(\theta)))u_{\theta}(q(\tau^k(\theta)), \tau^{k-1}(\theta))\dot{\tau}^{k-1}(\theta)d\theta
\end{aligned} \tag{51}$$

Next, let  $Q^k(\theta) = q(\tau^k(\theta))$  for  $k = 1, \dots, M(\theta)$ . Substituting this into (51) and using the result in (10) yields the following reformulated objective if our problem.

$$\begin{aligned}
& \int_{\tau(1)}^1 \sum_{k=1}^M u(Q^k(\theta), \tau^k(\theta))f(\tau^k(\theta))\dot{\tau}^k(\theta) - (1 - F(\tau^{k-1}(\theta)))u_{\theta}(Q^k(\theta), \tau^{k-1}(\theta))\dot{\tau}^{k-1}(\theta)d\theta \\
&+ \int_{\hat{\theta}_M}^1 u(Q^M(\theta), \tau^M(\theta))f(\tau^M(\theta))\dot{\tau}^M(\theta) - (1 - F(\tau^{M-1}(\theta)))u_{\theta}(Q^M(\theta), \tau^{M-1}(\theta))\dot{\tau}^{M-1}(\theta)d\theta \\
&+ S_0(\tau^M(\hat{\theta}_M)) + S_1(\tau(1))
\end{aligned} \tag{52}$$

where  $S_0(\tau^M(\hat{\theta}_M))$  and  $S_1(\tau(1))$  are the scrap values of our problem given by:

$$S_0(\tau^M(\hat{\theta}_M)) = \int_0^{\tau^M(\hat{\theta}_M)} u(q^{fb}(\theta), \theta)f(\theta)d\theta \tag{53}$$

$$S_1(\tau(1)) = \int_{\tau(1)}^1 u(q^{fb}(\theta), \theta)f(\theta)d\theta \tag{54}$$

Next, differentiating  $Q^k(\theta) = q(\tau^k(\theta))$  and using (12) yields:

$$\dot{Q}^k(\theta) = h^k(\theta, Q^k(\theta), \tau^k(\theta), \tau^{k-1}(\theta))\dot{\tau}^k(\theta) \tag{55}$$

where  $h^k(\theta) \equiv \frac{u_{\theta}(Q^k(\theta), \tau^k(\theta)) - 1(\tau^k(\theta) \geq \hat{\theta})u_{\theta}(Q^{k+1}(\theta), \tau^k(\theta))}{u_q(Q^k(\theta), \tau^{k-1}(\theta)) - u_q(Q^k(\theta), \tau^k(\theta))}$ .

Now let  $\alpha^k(\theta) = \dot{\tau}^k(\theta)$ ,  $k = 1, \dots, M$ , be our control variables. So we finally obtain an optimal control problem with objective (52),  $2M$  state variables  $Q^1, \dots, Q^M$ ,  $\tau^1, \dots, \tau^M$ , and the “laws of motion:”

$$\dot{Q}^k(\theta) = h^k(\theta, Q^k(\theta), \tau^k(\theta), \tau^{k-1}(\theta))\alpha^k(\theta) \quad (56)$$

$$\dot{\tau}^k(\theta) = \alpha^k(\theta) \quad (57)$$

and boundary conditions

$$\tau^{k+1}(1) = \tau^k(\tau(1)) \quad \text{for } k = 0, \dots, M-1 \quad (58)$$

$$Q^{k+1}(1) = Q^k(\tau(1)) \quad \text{for } k = 1, \dots, M-1 \quad (59)$$

$$Q^1(1) = q^{fb}(\tau(1)) \quad (60)$$

$$Q^M(\hat{\theta}_M) = q^{fb}(\tau^M(\hat{\theta}_M)) \quad (61)$$

$$u(Q^M(\hat{\theta}_M), \tau^{M-1}(\hat{\theta}_M)) - u(Q^M(\hat{\theta}_M), \tau^M(\hat{\theta}_M)) - C = 0 \quad (62)$$

The boundary conditions (58) and (59) connect the upper bound of  $(\tau^{k+1}, Q^{k+1})$  and the lower bound of  $(\tau^k, Q^k)$ . Conditions (60) and (61) ensure that the optimal quantity is first-best at the end-points of the intervals of types which have no binding constraints pointing to them. Condition (62) ensures that type  $\hat{\theta}$  receive zero surplus.

The maximization problem (52)-(62) is an optimal control with state variables  $Q^k(\cdot)$  and  $\tau^k(\cdot)$ , control variable  $\alpha^k(\theta)$ , free boundaries  $\theta_M$ ,  $\tau(1)$ , and scrap values (53) and (54). The Hamiltonian of the optimal control is as follows:

$$H = \sum_{k=1}^{M(\theta)} u(Q^k, \tau^k) f(\tau^k) \alpha^k - (1 - F(\tau^{k-1})) u_\theta(Q^k, \tau^{k-1}) \alpha^{k-1} + [\lambda_{Q^k} h^k \alpha^k + \lambda_{\tau^k} \alpha^k] \quad (63)$$

The following Theorem states the general solution of the optimal control problem.

**Theorem 8** *The solution of optimal control problem (63) subject to (53)-(62) satisfies the following system of differential equations of  $\tau^1, \dots, \tau^M; Q^1, \dots, Q^M$ :*

$$\dot{\tau}^k = \frac{f(\theta)[u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)]}{f(\tau^k)u_q(Q^k, \tau^k)} \prod_{s=1}^{k-1} \frac{u_q(Q^s, \tau^{s-1})}{u_q(Q^s, \tau^s)} \quad (64)$$

$$\dot{Q}^k = \begin{cases} \frac{f(\theta)[u_\theta(Q^k, \tau^k) - u_\theta(Q^{k+1}, \tau^k)]}{f(\tau^k)u_q(Q^k, \tau^k)} \prod_{s=1}^{k-1} \frac{u_q(Q^s, \tau^{s-1})}{u_q(Q^s, \tau^s)} & \text{if } k < M(\theta) \\ \frac{f(\theta)u_\theta(Q^k, \tau^k)}{f(\tau^k)u_q(Q^k, \tau^k)} \prod_{s=1}^{k-1} \frac{u_q(Q^s, \tau^{s-1})}{u_q(Q^s, \tau^s)} & \text{if } k = M(\theta) \end{cases} \quad (65)$$

with boundary conditions (58)-(62).

Conditions (64) and (65) is a system of first order differential equations with  $2M$  variables, and with  $2M + 2$  boundary conditions from (58)-(62), we can solve it along with the free boundaries  $\theta_M$  and  $\tau(1)$ .

The following intermediate step of reaching (64) and (65) provides some intuition of these optimality conditions.

$$u_q(Q^k, \tau^k) f(\tau^k) \dot{\tau}^k = [u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)] \sum_{s=1}^k f(\tau^{k-s}) \dot{\tau}^{k-s} \quad (66)$$

The left hand side of (66) is the marginal gain on efficiency for increasing  $Q^k(\theta)$  given the relative density of  $\tau^k(\theta)$ . The right hand side of (66) is the marginal cost on informational rent for increasing  $Q^k(\theta)$ . It reflects the nature of the trade-off: for an additional unit of quantity assigned to  $\tau^k(\theta)$ , informational rent has to be given to the type who targets  $\tau^k(\theta)$ , i.e.  $\tau^{k-1}(\theta)$ , in order to prevent  $\tau^{k-1}(\theta)$  from imitating  $\tau^k(\theta)$ . Furthermore, increase in informational rent makes the contract of  $\tau^{k-1}(\theta)$  more attractive, so the same amount of informational rent has to be given to  $\tau^{k-2}(\theta)$ , and thus every preceding types in the chain, to prevent imitation.

Note that the number of partitions  $M$  is endogenous. To find it practically, start with assuming  $M = 1$  and solve the optimal control and the optimal  $\theta_M$ . If  $\tau^M(1) \leq \tau^{M-1}(\theta_M)$ , then  $M$  is the optimal number of partitions; otherwise, do the same process with  $M + 1$ .

## 6 Conclusion

The paper sheds light on the role of fixed cost of lying in screening frameworks. The introduction of a fixed cost of lying reshape the screening problem into a new class of principle-agent problem with non-locally binding incentive constraints. We develop a method to represent the problem as an optimal control, in which the binding non-local constraints, "targeted types", and the physical allocations are jointly solved. We derive the optimality condition of the

problem, which can be interpreted as an endogenous discretization of the standard optimal screening. The model produces several qualitatively novel results. We show that the standard exclusion property is not robust to a small fixed cost of lying. On the contrary, full efficiency is achieved for low types. We provide an example for the optimal mechanism given linear-quadratic utility under uniform type distribution.

While this paper only characterizes the optimal screening mechanism given type-independent fixed cost of lying, we believe that the important properties of our methodological approach, such as the characterization of binding non-local incentive constraints and the targeted type concept, also apply under more general cost of lying with non-zero fixed cost. In future work, I hope to analyze a screening problem with more general structures of lying cost. Such a model will be compatible with other screening problems in the literature, and provides a richer set of testable predictions.

## 7 Appendix A

In this Appendix we provide proof to Theorems 2, 3, 4 and 5 through a series of Lemmas.

Theorem 2 follows from Lemmas 3 and 15.

Theorem 3 follows from Lemmas 4, 5, 10, 14, 19 and 23.

Theorem 4 follows from Lemmas 6, 8, 9, 16, 18, 22, 26 and Corollaries 1 and 3.

Theorem 5 follows from Lemmas 16 and 25.

**Lemma 3** *There exists an optimal mechanism solving the principal's maximization problem (1) subject to (2) and (3).*

**Proof of Lemma 3:** First, we can restrict  $q(\theta)$  to  $[0, \bar{Q}]$  where  $\bar{Q} = \max\{Q, u(Q, 1) \geq 0\}$ . Indeed, if some type  $\theta$  is assigned an allocation  $(q(\theta), t(\theta))$  such that  $q(\theta) > Q$ , then  $t(\theta) < 0$  by individual rationality. So the principal would be better off by assigning the allocation  $(t = 0, q = 0)$  to  $\theta$ . This modification would not violate the incentive constraints of any other type  $\theta'$  by the individual rationality of the mechanism. By the same argument we can restrict  $t(\theta) \geq 0$ . Finally, individual rationality requires that  $t(\theta) \leq u(q^{fb}(1), 1)$  where  $q^{fb}(1) = \arg \max_q u(q, 1)$ .

So, consider a space of bounded measurable, and hence integrable, functions  $(t(\theta), q(\theta)) : [0, 1]^2 \mapsto [0, u(q^{fb}(1), 1)] \times [0, \bar{Q}]$ . endowed with pointwise convergence topology. This space is compact by Tychonoff Theorem. Note that the objective (1) is continuous in this space. Furthermore, the subset of this space satisfying the constraints (2) and (3) is compact. So by Weierstrass Theorem, there exists a solution  $(q^*(.), t^*(.))$  maximizing (1) subject to (2) and (3). *Q.E.D.*

The next Lemma shows that the payment  $t$  is non-negative for almost every type.

**Lemma 4** *There exists an optimal mechanism  $(q(.), t(.))$  such that  $t(\theta) \geq 0$  for all  $\theta \in [0, 1]$ .*

**Proof of lemma 4:** Suppose that  $(q(.), t(.))$  is an incentive compatible individually rational mechanism such that  $t(\theta) < 0$  iff  $\theta \in \Theta^-$  where  $\Theta^-$  is some non-empty subset of  $[0, 1]$ . Then consider an alternative mechanism  $(\tilde{q}(.), \tilde{t}(.))$  such that  $(\tilde{q}(\theta), \tilde{t}(\theta)) = (q(\theta), t(\theta))$  for any  $\theta \notin \Theta^-$ . and  $(\tilde{q}(\theta), \tilde{t}(\theta)) = (0, 0)$  for any  $\theta \in \Theta^-$ . So  $\tilde{t}(\theta) \geq 0$  for all  $\theta \in [0, 1]$ . Then mechanism  $(\tilde{q}(.), \tilde{t}(.))$  is individually rational and is incentive compatible for all  $\theta \notin \Theta^-$ . If this mechanism is not incentive compatible for some  $\theta \in \Theta^-$  i.e.,  $\theta$  prefers to imitate some  $\theta' \in [0, 1]$  in it, then  $\theta$  ends up making a positive transfer to the firm instead of a negative transfer in the mechanism  $(q(.), t(.))$ . So  $(\tilde{q}(.), \tilde{t}(.))$  is more profitable (at least weakly) for the firm than  $(q(.), t(.))$ . *Q.E.D*

The next Lemma establishes continuity of  $V(.), t(.)$  and  $q(.)$ . in an optimal mechanism.

**Lemma 5** *There exists an optimal mechanism  $(q(.), t(.))$  such that  $V(.), q(.)$  and  $t(.)$  are continuous at any  $\theta \in [0, 1]$ .*

**Proof of Lemma 5:** Suppose that  $(q(.), t(.))$  is an optimal mechanism such that  $V(.)$  is not continuous at some  $\theta^*$ . Then there exists a sequence  $\theta_n \rightarrow \theta^*$  such that  $\lim_{\theta_n \rightarrow \theta^*} V(\theta_n) = V^* \neq V(\theta^*)$ . The limit  $V^*$  exists without loss of generality because, since  $t(\theta) \geq 0$ ,  $V(\theta)$  is bounded above by  $u(q^{fb}(\theta), \theta)$ , and so we can always pass on to a converging subsequence. Furthermore, since  $IR(\theta)$  holds for all  $\theta$  in  $(q(.), t(.))$ , we have  $V(\theta) \geq 0$  for all  $\theta$  and so  $V^* \geq 0$ .

If  $V(\theta^*) > V^*$ , then let us replace the original mechanism  $(q(.), t(.))$  with a modified mechanism  $(\tilde{q}(.), \tilde{t}(.))$  which differs from the original mechanism only in the transfer paid by

type  $\theta^*$ . Particularly, we set  $\tilde{t}(\theta^*) = t(\theta^*) + \epsilon$  for some small  $\epsilon \in (0, \frac{V(\theta^*) - V^*}{2})$ . The seller's expected profit under  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  is weakly greater than her payoff under the original mechanism  $(q(\cdot), t(\cdot))$ .

Let us show that the mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  is incentive compatible and individually rational. First, it satisfies  $IR(\theta)$  and  $IC(\theta, \theta^*)$  for all  $\theta \in [0, 1]$ , the latter -because  $\tilde{t}(\theta^*) > t(\theta^*)$ .  $IR(\theta^*)$  holds since  $\theta^*$ 's payoff in it is  $V(\theta^*) - \epsilon > V^* \geq 0$ .

It remains to show that  $IC(\theta^*, \theta)$  hold in the modified mechanism for all  $\theta \in [0, 1]$ . First,  $IC(\theta^*, \theta_n)$  holds when  $n$  is sufficiently large since for such  $n$ ,  $V(\theta_n) < V^* + \epsilon < V(\theta^*) - \epsilon$  and  $u(\cdot)$  is continuous in  $\theta$ . Now, fix some  $\theta \in [0, 1]$ .  $IC(\theta_n, \theta)$  holds for all  $\theta_n$  i.e.:

$$V^* - \epsilon > u(q(\theta_n), \theta_n) - t(\theta_n) \geq u(q(\theta), \theta_n) - t(\theta) - C$$

Since  $u(\cdot)$  is continuous, we have  $\lim_{\theta_n \rightarrow \theta^*} u(q(\theta), \theta_n) = u(q(\theta), \theta^*)$ . It follows that

$$V(\theta^*) - \epsilon = u(q(\theta^*), \theta_n) - \tilde{t}(\theta_n) > u(q(\theta), \theta^*) - t(\theta) - C$$

So,  $IC(\theta^*, \theta)$  also holds.

Now suppose, instead, that  $\lim_{\theta_n \rightarrow \theta^*} V(\theta_n) = V^* > V(\theta^*)$ . Fix some  $\delta \in (0, \frac{V^* - V(\theta^*)}{2})$ . Then there exists  $N$  large enough such that for all  $n \geq N$ ,  $V(\theta_n) > V^* - \delta$ . Now, define a new mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  which differs from  $(q(\cdot), t(\cdot))$  only in transfers paid by types  $\theta_n$ ,  $n \geq N$ . Particularly, set  $\tilde{t}(\theta_n) = t(\theta_n) + \delta$  for some large  $n$ . The seller gets weakly higher profits under  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  than under  $(q(\cdot), t(\cdot))$ . So we only need to show that the former mechanism is incentive compatible and individually rational. In fact, the only constraints that we need to check in  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  are those of types  $\theta_n$ ,  $n \geq N$ , who pay  $\tilde{t}(\theta_n) = t(\theta_n) + \delta$  in this mechanism.

$IR(\theta_n)$  holds because the payoff of type  $\theta_n$  in  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  is equal to  $\tilde{V}(\theta_n) > V^* - 2\delta > V(\theta^*) > 0$ . Furthermore,  $IC(\theta_n, \theta)$  holds for all  $\theta \in [0, 1]$  since  $IC(\theta^*, \theta)$  holds,  $\tilde{V}(\theta_n) > V(\theta^*)$  and  $u(\cdot)$  is continuous in  $\theta$ .

Now let us show that  $t(\cdot)$  is continuous in  $\theta$ . The proof is by contradiction, so suppose  $t(\cdot)$  is discontinuous at some  $\theta^* \in [0, 1]$ . Since  $V(\cdot)$  is continuous everywhere, there exists a sequence  $\theta_n$  converging to  $\theta^*$  such that  $\lim_{\theta_n \rightarrow \theta^*} t(\theta_n) = t^* \neq t(\theta^*)$  and  $\lim_{\theta \rightarrow \theta^*} q(\theta_n) = q^* \neq q(\theta^*)$ .

We need to consider two cases. First, suppose that  $t(\theta^*) < t^*$ . Then define a new mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  which only differs from the original mechanism  $(q(\cdot), t(\cdot))$  at  $\theta^*$ . Precisely,

set  $\tilde{t}(\theta^*) = t^*$  and  $\tilde{q}(\theta^*) = q^*$ . The seller's profits from  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  is at least as large as from  $(q(\cdot), t(\cdot))$ . Moreover, since  $IR(\theta_n)$ ,  $IC(\theta_n, \theta)$  and  $IC(\theta, \theta_n)$  hold for all  $n$  and all  $\theta \in [0, 1]$ ,  $\lim_{\theta_n \rightarrow \theta^*} t(\theta_n) = t^* = \tilde{t}(\theta^*)$ ,  $\lim_{\theta_n \rightarrow \theta^*} q(\theta_n) = q^* = \tilde{q}(\theta^*)$ , and  $u(\cdot)$  is continuous it follows that  $IR(\theta^*)$ ,  $IC(\theta^*, \theta)$  and  $IC(\theta, \theta^*)$  hold for all  $\theta \in [0, 1]$  in the modified mechanism.

Now suppose that  $t(\theta^*) > t^*$ . Then fix some  $\tilde{t}$  such that  $t^* < \tilde{t} < t(\theta^*)$ , and set  $\tilde{q}$  so that  $V(\theta^*) = u(\tilde{q}, \theta^*) - \tilde{t}$ . Because  $\tilde{t} < t(\theta^*)$ , such  $\tilde{q}$  exists with  $\tilde{q} \in (\min\{q^*, q(\theta^*)\}, \max\{q^*, q(\theta^*)\})$ .

Next, define a new mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  which differs from the original mechanism  $(q(\cdot), t(\cdot))$  only at  $\theta_n$  for  $n \geq N$  where  $N$  is sufficiently large. Particularly, set  $\tilde{q}(\theta_n) = \tilde{q}$  and  $\tilde{t}(\theta_n) = u(\tilde{q}, \theta_n) - V(\theta_n)$  for  $n \geq N$ . Note that  $\tilde{t}(\theta_n) \approx \tilde{t} > t^* \approx t(\theta_n)$ . So the new mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  is more profitable for the seller than  $(q(\cdot), t(\cdot))$ .

We need to check that the new mechanism is individually rational and incentive compatible. First, the net payoff of any type  $\theta \in [0, 1]$  in the new mechanism,  $\tilde{V}(\theta)$ , satisfies  $\tilde{V}(\theta) = V(\theta)$ . So,  $IR(\theta)$  and  $IC(\theta, \theta')$  hold for all  $\theta \in [0, 1]$  and  $\theta' \neq \theta_n$ ,  $n \geq N$ , because the allocation  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  differs from  $(q(\cdot), t(\cdot))$  only for  $\theta_n$  such that  $n \geq N$ .

It remains to consider  $IC(\theta, \theta_n)$ ,  $n \geq N$ . Since  $IC(\theta, \theta^*)$  holds for all  $\theta \in [0, 1]$  in both mechanisms, we have  $V(\theta) \geq u(q(\theta^*), \theta) - t(\theta^*) - C$ . Also, since  $IC(\theta, \theta_n)$  holds for all  $\theta, \theta_n \in [0, 1]$  in the original mechanism,  $\lim_{\theta \rightarrow \theta^*} t(\theta_n) = t^*$ ,  $\lim_{\theta \rightarrow \theta^*} q(\theta_n) = q^*$ , and  $u(\cdot)$  is continuous, it follows that for any  $\theta$ ,  $V(\theta) \geq u(q^*, \theta) - t^* - C$ . So,  $V(\theta) = \max\{u(q(\theta^*), \theta) - u(q(\theta^*), \theta^*) + V(\theta^*) - C, u(q^*, \theta) - u(q^*, \theta^*) + V(\theta^*) - C\} > u(\tilde{q}, \theta) - u(\tilde{q}, \theta^*) + V(\theta^*) - C$  where the inequality holds because  $u_{q\theta}(q, \theta) > 0$  and  $\tilde{q} \in (\min\{q^*, q(\theta^*)\}, \max\{q^*, q(\theta^*)\})$ .

Finally,  $u(\tilde{q}, \theta) - u(\tilde{q}, \theta^*) + V(\theta^*) - C \approx u(\tilde{q}(\theta_n), \theta) - \tilde{t}(\theta_n) - C$  by continuity. Therefore,  $IC(\theta, \theta_n)$  hold for all  $\theta \in [0, 1]$ ,  $\theta_n$ ,  $n \geq N$  in the mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$ . *Q.E.D*

Recall that, as defined in (4),  $\tau(\theta)$  is the set of all such types  $\theta'$  that incentive constraint  $IC(\theta, \theta')$  is binding. Also, for any  $\Theta \subseteq [0, 1]$ ,  $\tau(\Theta) = \cup_{\theta \in \Theta} \tau(\theta)$ .

By Lemma 5, by imitating type  $\theta'$  type  $\theta$  gets a payoff  $u(q(\theta'), \theta) - t(\theta') - C$ , which is continuous in  $\theta'$ . So  $u(q(\theta'), \theta) - t(\theta') - C$  attains a maximum at some  $\theta' \in [0, 1]$ . Therefore,  $\tau(\theta) = \emptyset$  implies that there exists some  $\epsilon > 0$  such that  $V(\theta) - u(q(\theta'), \theta) - t(\theta') - C > \epsilon$  for all  $\theta' \in [0, 1]$ .

Lemma 6 establishes the upper hemicontinuity of  $\tau(\cdot)$ .



**Lemma 6** *The correspondence  $\tau(\theta)$  is upper hemicontinuous in an optimal mechanism.*

**Proof of Lemma 6:** Let  $(\theta_n, \theta'_n)$  be a sequence of vectors such that  $\theta'_n \in \tau(\theta_n)$  for all  $n = 1, 2, \dots$  and  $\lim_{n \rightarrow \infty} (\theta_n, \theta'_n) = (\tilde{\theta}, \tilde{\theta}')$ . We want to show that  $\tilde{\theta}' \in \tau(\tilde{\theta})$ . Define  $\beta((\theta, \theta')) = V(\theta) - u(q(\theta'), \theta) + t(\theta') + C$ . Since  $\theta'_n \in \tau(\theta_n)$ ,  $\beta(\theta_n, \theta'_n) = 0$  for all  $n$ . Assumption 1 and Lemma 5 implies that  $\beta(\cdot)$  is continuous. Therefore we have  $\beta(\tilde{\theta}, \tilde{\theta}') = \lim_{n \rightarrow \infty} \beta((\theta_n, \theta'_n)) = 0$ , which means  $\tilde{\theta}' \in \tau(\tilde{\theta})$ . *Q.E.D.*

Lemma 7 shows that if  $\theta$  has no binding incentive constraint then her surplus must be zero.

**Lemma 7** *If  $\tau(\theta) = \emptyset$ , then  $V(\theta) = 0$ .*

**Proof of Lemma 7:** The proof is by contradiction. So suppose that for some  $\theta \in [0, 1]$ ,  $\tau(\theta) = \emptyset$  and  $V(\theta) > 0$ . The continuity of  $V(\cdot)$ ,  $q(\cdot)$  and  $t(\cdot)$  established in Lemma 5 and  $\tau(\theta) = \emptyset$  imply that there exists  $\epsilon > 0$  such that  $V(\theta) - u(q(\theta'), \theta) - t(\theta') - C < \epsilon$  for all  $\theta' \in [0, 1]$ . Therefore, the seller can increase her profit by raising  $t(\theta)$  by  $\min\{\epsilon, V(\theta)\}$ . This modification clearly does not violate and *IR* or *IC* constraints. *Q.E.D.*

The converse of Lemma 7 is as follows:

**Lemma 8** *If  $V(\theta) = 0$ , then  $\tau(\theta') = \emptyset$  for all  $\theta' \in [0, \theta)$ .*

**Proof of Lemma 8:** The proof is by contradiction. Suppose that  $\tau(\theta') = \theta''$  for some  $\theta' \in [0, \theta)$  and  $\theta''$ , so  $V(\theta') = u(q(\theta''), \theta') - t(\theta'') - C \geq 0$ . But then  $V(\theta) \geq u(q(\theta''), \theta) - t(\theta'') - C > 0$  because  $u_\theta > 0$ , which contradicts  $V(\theta) = 0$ . *Q.E.D.*

Lemma 9 shows that for any positive cost of lying, there is a threshold  $\hat{\theta}$  such that all types below have slack incentive constraints and hence get zero surplus.

**Lemma 9** *For any  $C > 0$ , there exists  $\check{\theta} > 0$  such that for any  $\theta \in [0, \check{\theta}]$ ,  $\tau(\theta) = \emptyset$  and so  $V(\theta) = 0$ .*

**Proof of Lemma 9:** Incentive compatibility of the mechanism for a given  $\theta$  requires that

$$u(q(\theta), \theta) - t(\theta) \geq u(q(\theta'), \theta) - t(\theta') - C \quad \forall \theta' \in [0, 1]$$

The left-hand side of this inequality is non-negative by individual rationality. Consider now its right-hand side. Since  $u$  is continuous with  $u(q, 0) \leq 0$ ,  $u(q(\theta'), \check{\theta}) \leq u(q^{fb}(\check{\theta}), \check{\theta}) \rightarrow 0$  as

$\check{\theta} \rightarrow 0$ . Since  $t(\theta') \geq 0$  by Lemma 4, we have  $u(q(\theta'), \check{\theta}) - t(\theta') - C < 0$  for small enough  $\check{\theta}$  and any  $\theta' \in [0, 1]$ . Since  $u_\theta > 0$ , it is also true that  $u(q(\theta'), \theta) - t(\theta') - C < 0$  for  $\theta \leq \check{\theta}$  and any  $\theta' \in [0, 1]$ . So  $\tau(\theta) = \emptyset$  and, by Lemma 7,  $V(\theta) = 0$  for any  $\theta \in [0, \check{\theta}]$ . *Q.E.D*

Lemma 10 shows that the value function is non-decreasing

**Lemma 10**  *$V(\theta)$  is non-decreasing in  $\theta$ .*

**Proof of Lemma 10:** Suppose that  $\theta_1 > \theta_2$ . If  $V(\theta_2) = 0$ , then obviously  $V(\theta_1) \geq V(\theta_2)$  by *IR*.

Now suppose that  $V(\theta_2) > 0$ . Then by Lemma 7 there exists  $\theta'_2 \in \tau(\theta_2)$ . Then  $V(\theta_1) \geq u(q(\theta'_2), \theta_1) - t(\theta'_2) - C > u(q(\theta'_2), \theta_2) - t(\theta'_2) - C = V(\theta_2)$ , where the first inequality holds by *IC*( $\theta_1, \theta'_2$ ), and the second inequality holds because  $u_\theta > 0$ , and the equality holds because  $\theta'_2 \in \tau(\theta_2)$ . *Q.E.D*

Lemma 11 shows that higher types wish to imitate types who are assigned higher quantities.

**Lemma 11** *Let  $\theta_1 > \theta_2$  and suppose that there exist  $\theta'_1 \in \tau(\theta_1)$  and  $\theta'_2 \in \tau(\theta_2)$ . Then  $q(\theta'_1) \geq q(\theta'_2)$ .*

**Proof of Lemma 11:** Since  $\theta'_1 \in \tau(\theta_1)$ ,  $V(\theta_1) = u(q(\theta'_1), \theta_1) - t(\theta'_1) - C \geq u(q(\theta'_2), \theta_1) - t(\theta'_2) - C$ . Similarly,  $V(\theta_2) = u(q(\theta'_2), \theta_2) - t(\theta'_2) - C \geq u(q(\theta'_1), \theta_2) - t(\theta'_1) - C$ . Combining these two inequalities yields:  $u(q(\theta'_1), \theta_1) - u(q(\theta'_2), \theta_1) \geq t(\theta'_1) - t(\theta'_2) \geq u(q(\theta'_1), \theta_2) - u(q(\theta'_2), \theta_2)$ . Since  $\theta_1 > \theta_2$  and  $u_{q\theta} > 0$ , it must be that  $q(\theta'_1) \geq q(\theta'_2)$ . *Q.E.D*

Lemma 12 shows that for any  $\theta$  all binding incentive constraints pointing to  $\theta$  must come either from higher types or lower types, but not both.

**Lemma 12** *There do not exist  $\theta, \theta_1, \theta_2 \in [0, 1]$  such that  $\theta_1 < \theta < \theta_2$  and  $\theta \in \tau(\theta_1) \cap \tau(\theta_2)$ .*

**Proof of Lemma 12:** The proof is by contradiction, so suppose that  $\theta \in \tau(\theta_1) \cap \tau(\theta_2)$  for some  $\theta_1 < \theta < \theta_2$ . Then  $V(\theta) = u(q(\theta), \theta) - t(\theta) > u(q(\theta), \theta_1) - t(\theta) - C = V(\theta_1) \geq 0$ . So,  $V(\theta) > 0$  and, by Lemma 7, there exists  $\theta' \in \tau(\theta)$ . By Lemma 11 we have: (i) since  $\theta > \theta_1$ ,  $q(\theta') \geq q(\theta)$ ; (ii) since  $\theta < \theta_2$ ,  $q(\theta') \leq q(\theta)$ . So,  $q(\theta) = q(\theta')$ ,

Since  $IC(\theta, \theta')$  is binding, we have  $u(q(\theta), \theta) - t(\theta) = u(q(\theta'), \theta) - t(\theta') - C$ . Since  $q(\theta) = q(\theta')$ , it follows that  $t(\theta') = t(\theta) - C$ . But then  $IC(\theta_1, \theta)$  and  $IC(\theta_2, \theta)$  cannot be binding because types  $\theta_1$  and  $\theta_2$  get strictly higher payoff by imitating  $\theta'$  rather than  $\theta$ . *Q.E.D*

The next Lemma is as follows:

**Lemma 13** *If  $q(\theta') < q^{fb}(\theta')$  for some  $\theta' \in [0, 1]$ , then there exists  $\theta \in (\theta', 1]$  such that  $\theta' \in \tau(\theta)$ , and there is no  $\theta'' \in [0, \theta')$  such that  $\theta' \in \tau(\theta'')$ .*

*If  $q(\theta') > q^{fb}(\theta')$  for some  $\theta' \in [0, 1]$ , then there exists  $\theta \in (0, \theta')$  such that  $\theta' \in \tau(\theta)$ , and there is no  $\theta'' \in (\theta', 1]$  such that  $\theta' \in \tau(\theta'')$ .*

**Proof of Lemma 13:** To prove the first claim, suppose that  $\theta' \notin \tau([\theta', 1])$ . We claim that there exists  $\delta > 0$  such that for all  $\theta > \theta'$ ,  $V(\theta) > u(q(\theta'), \theta) - t(\theta') - C + \delta$ . Suppose not, then there exists a sequence  $\theta_n \in (\theta', 1]$  and  $\theta^* \in [\theta', 1]$  such that  $\theta_n \rightarrow \theta^*$  and  $\lim_{n \rightarrow \infty} V(\theta_n) - [u(q(\theta'), \theta_n) - t(\theta') - C] = 0$ . Since both  $V(\cdot)$  and  $u(\cdot, \cdot)$  are continuous,  $V(\theta^*) = u(q(\theta'), \theta^*) - t(\theta') - C$  which implies  $\theta' \in \tau(\theta^*)$ . Straightforwardly,  $\theta^* \neq \theta'$ . But that contradicts that  $\theta^* \in (\theta', 1]$  and  $\theta' \notin \tau([\theta', 1])$ .

Now let  $\tilde{q}(\theta')$  be the solution to  $u(\tilde{q}(\theta'), 1) - u(q(\theta'), 1) = \delta$  if such exists and satisfies  $\tilde{q}(\theta') \leq q^{fb}(\theta')$  and otherwise let  $\tilde{q}(\theta') = q^{fb}(\theta')$ .

Then the firm could generate more profit by offering an alternative mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  in which the allocation of type  $\theta'$  is given by  $\tilde{q}(\theta')$ ,  $\tilde{t}(\theta') = t(\theta') + u(\tilde{q}(\theta'), \theta') - u(q(\theta'), \theta') > t(\theta')$  and all other elements remain the same as in the original mechanism  $(q(\cdot), t(\cdot))$ .

This modification does not affect the net payoff  $V(\theta)$  of any type, so  $IR(\theta)$  still hold for all  $\theta$ . Also,  $IC(\theta, \theta')$  hold for any  $\theta \in [0, \theta')$  in the mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  because  $u(\tilde{q}(\theta'), \theta) - \tilde{t}(\theta') = u(\tilde{q}(\theta'), \theta) - u(\tilde{q}(\theta'), \theta') + u(q(\theta'), \theta') - t(\theta') < u(q(\theta'), \theta) - t(\theta')$ . The last inequality holds since  $\tilde{q}(\theta') > q(\theta')$ ,  $\theta < \theta'$  and  $u_{q\theta} > 0$ . For  $\theta > \theta'$ ,  $V(\theta) > u(q(\theta'), \theta) - t(\theta') - C + \delta \geq u(q(\theta'), \theta) - t(\theta') - C + u(q(\theta') + \tilde{\epsilon}, \theta) - u(q(\theta'), \theta) \geq u(q(\theta') + \epsilon, \theta) - t(\theta') - C > u(\tilde{q}(\theta'), \theta) - \tilde{t}(\theta') - C$ , which implies  $IC(\theta, \theta')$  is still slack for  $\theta > \theta'$ . Lemma 12 then implies that there is no  $\theta'' \in [0, \theta')$  such that  $\theta' \in \tau(\theta'')$ .

A symmetric argument shows that if  $q(\theta') > q^{fb}(\theta')$  for some  $\theta'$ , then there exists  $\theta \in (0, \theta')$  such that  $\theta' \in \tau(\theta)$  and that there is no  $\theta'' \in (\theta', 1]$  such that  $\theta' \in \tau(\theta'')$ . *Q.E.D*

Lemma 13 implies that  $q(\theta') = q^{fb}(\theta')$  if  $\theta' \notin \tau([0, 1])$ .

Lemma 14 shows that optimal quantities never exceed the first-best level. So by Lemmas 12 and 13 only downward incentive constraints can be binding.

**Lemma 14** *For any  $\theta' \in [0, 1]$ ,  $q(\theta') \leq q^{fb}(\theta')$ .*

**Proof of Lemma 14:** Suppose that  $q(\theta_1) > q^{fb}(\theta_1)$  for some  $\theta_1$ . Then by Lemma 13,  $\theta_1 \in \tau(\theta_0)$  for some  $\theta_0 \in [0, \theta_1)$ . Therefore,

$$V(\theta_0) = u(q(\theta_1), \theta_0) - t(\theta_1) - C$$

Combining this with  $V(\theta_1) = u(q(\theta_1), \theta_1) - t(\theta_1)$  yields:

$$V(\theta_1) = V(\theta_0) + u(q(\theta_1), \theta_1) - u(q(\theta_1), \theta_0) + C > C.$$

Next we will show that there exists an infinite sequence  $\{\theta_n\}_{n=0}^{\infty}$  such that for any  $n \geq 1$ ,  $\theta_n \in \tau(\theta_{n-1})$ ,  $\theta_n > \theta_{n-1}$ ,  $q(\theta_n) \geq q^{fb}(\theta_n)$  and  $V(\theta_n) \geq nC$ . We have established this for  $n = 1$ , so it suffices to establish the following inductive step: if for some fixed  $k \geq 1$  there exists  $\theta_k$  satisfying these conditions, then there exists  $\theta_{k+1}$  for which these conditions also hold.

Indeed, since  $V(\theta_k) \geq kC$ , Lemma 7 implies that there exists some  $\theta_{k+1} \in \tau(\theta_k)$ . Since  $\theta_k \in \tau(\theta_{k-1})$  and  $\theta_k > \theta_{k-1}$ , Lemma 11 then implies that  $q(\theta_{k+1}) \geq q(\theta_k)$ . If  $\theta_{k+1} < \theta_k$ , then  $q(\theta_{k+1}) \geq q(\theta_k) > q^{fb}(\theta_k) > q^{fb}(\theta_{k+1})$ , which contradicts Lemma 13. Therefore  $\theta_{k+1} > \theta_k$ . Then  $q(\theta_{k+1}) \geq q^{fb}(\theta_{k+1})$  by Lemma 13.

Since  $\theta_{k+1} \in \tau(\theta_k)$ , we have  $V(\theta_k) = u(q(\theta_{k+1}), \theta_k) - t(\theta_{k+1}) - C$ . Combining this with  $V(\theta_{k+1}) = u(q(\theta_{k+1}), \theta_{k+1}) - t(\theta_{k+1})$ , we get:

$$V(\theta_{k+1}) = V(\theta_k) + u(q(\theta_{k+1}), \theta_{k+1}) - u(q(\theta_{k+1}), \theta_k) + C > V(\theta_k) + C > (k+1)C.$$

This completes the proof of the existence of the sequence  $\{\theta_n\}_{n=0}^{\infty}$ .

However,  $u(q(\theta^n), \theta^n)$  is bounded from above, and so  $t(\theta^n) < 0$  for sufficiently large  $n$ , contradicts to Lemma 4. *Q.E.D*

Combining Lemmas 13 and 14 we get the following important Corollary:

**Corollary 1** *If  $\theta' \in \tau(\theta)$  for some  $\theta$ , then  $\theta' < \theta$ .*

Relying on Corollary 1 we can now establish the uniqueness of the optimal mechanism.

**Lemma 15** *Suppose that  $u_{\theta qq}(q, \theta) \geq 0$  for all  $(q, \theta)$ . Then the optimal mechanism is unique.*

**Proof of Lemma 15:** By Corollary 1 only downwards incentive constraints may be binding. So it is sufficient to establish the uniqueness of the solution to the relaxed problem in which (1) is maximized subject to the individual rationality constraints (3) and downwards incentive constraints i.e., (2) holding for all  $\theta, \theta' \in [0, 1]$  s.t.  $\theta \geq \theta'$ . The proof is by contradiction. So suppose that there exist two solutions to this problem,  $(q_1(\cdot), t_1(\cdot))$  and  $(q_2(\cdot), t_2(\cdot))$ . Then let  $V_i(\theta) \equiv u(q_i(\theta), \theta) - t_i(\theta)$  be the agent's net payoff function in the solution  $i \in \{1, 2\}$ .

Next, fix some  $\lambda \in (0, 1)$  and consider an allocation function  $\lambda q_1(\cdot) + (1 - \lambda)q_2(\cdot)$  and a net payoff function  $\lambda V_1(\theta) + (1 - \lambda)V_2(\theta)$ . Let us demonstrate that this allocation and payoff functions define a mechanism which is associated with a strictly higher payoff for the principal and which satisfies (2) for all  $\theta, \theta' \in [0, 1]$  s.t.  $\theta \geq \theta'$ . The individual rationality of every type  $\theta$  in (3) is trivially satisfied since  $V_i(\theta) \geq 0$  for all  $\theta$  and  $i \in \{1, 2\}$ . Further, the transfer of type  $\theta$  in this mechanism,  $t^\lambda(\theta)$ , is equal to

$$\begin{aligned} t^\lambda(\theta) &= u(\lambda q_1(\theta) + (1 - \lambda)q_2(\theta), \theta) - (\lambda V_1(\theta) + (1 - \lambda)V_2(\theta)) > \\ &\lambda u(q_1(\theta), \theta) + (1 - \lambda)u(q_2(\theta), \theta) - (\lambda V_1(\theta) + (1 - \lambda)V_2(\theta)) = \lambda t_1(\theta) + (1 - \lambda)t_2(\theta). \end{aligned}$$

Since this inequality holds for all  $\theta \in [0, 1]$ , the principal gets a strictly higher payoff in this mechanism.

Incentive compatibility condition in this mechanism is  $\lambda V_1(\theta) + (1 - \lambda)V_2(\theta) \geq$

$$u(\lambda q_1(\theta') + (1 - \lambda)q_2(\theta'), \theta) - u(\lambda q_1(\theta') + (1 - \lambda)q_2(\theta'), \theta') + (\lambda V_1(\theta') + (1 - \lambda)V_2(\theta')) - C \quad (67)$$

Now, note that

$$\begin{aligned} u(\lambda q_1(\theta') + (1 - \lambda)q_2(\theta'), \theta) - u(\lambda q_1(\theta') + (1 - \lambda)q_2(\theta'), \theta') &= \int_{\theta'}^{\theta} u_{\theta}(\lambda q_1(\theta') + (1 - \lambda)q_2(\theta'), t) dt \leq \\ \int_{\theta'}^{\theta} \lambda u_{\theta}(q_1(\theta'), t) + (1 - \lambda)u_{\theta}(q_2(\theta'), t) dt &= \lambda(u(q_1(\theta'), \theta) - u(q_1(\theta'), \theta')) + (1 - \lambda)(u(q_2(\theta'), \theta) - u(q_2(\theta'), \theta')) \end{aligned}$$

where the equalities hold by integration and the inequality holds because  $u_{\theta qq} \geq 0$ . Combining this inequality with the fact that incentive constraints (2) holds in the mechanisms  $(q_1(\cdot), t_1(\cdot))$  and  $(q_2(\cdot), t_2(\cdot))$  implies that the incentive constraints (67) also hold. *Q.E.D.*

The next Lemma establishes that the set of binding incentive constraints is non-empty when  $C$  is not too high. First, define

$$G(\theta, \theta') = u(q^{fb}(\theta'), \theta) - u(q^{fb}(\theta'), \theta'), \quad (68)$$

We have:

**Lemma 16**  $\tau([0, 1]) \neq \emptyset$  if  $C < \bar{C}$  and  $\tau([0, 1]) = \emptyset$  if  $C > \bar{C}$ , where

$$\bar{C} = \max_{\theta, \theta' \in [0, 1]} G(\theta, \theta') = \max_{\theta' \in [0, 1]} G(1, \theta') \quad (69)$$

Furthermore, if  $\tau(\theta) \neq \emptyset$ , then  $\tau(\theta)$  is compact-valued and  $\tau(\theta') \neq \emptyset$  for all  $\theta' > \theta$ .

If  $C = \bar{C}$ , then  $V(\theta) = 0$  for all  $\theta \in [0, 1]$ , and  $\tau(\theta) \neq \emptyset$  only if  $\theta = 1$ .

**Proof of Lemma 16:**

(i) To prove the first claim of the Lemma for  $C < \bar{C}$  we argue by contradiction. So suppose that  $\tau([0, 1]) = \emptyset$ . Then for all  $\theta \in [0, 1]$   $V(\theta) = 0$  by Lemma 7, and  $q(\theta) = q^{fb}(\theta)$  by Lemma 13. But then  $IC(1, \theta)$  fails for some  $\theta$  because  $C < \bar{C} = \max_{\theta, \theta'} u(q^{fb}(\theta'), \theta) - u(q^{fb}(\theta'), \theta')$ .

(ii) The proof that  $\tau([0, 1]) = \emptyset$  if  $C > \bar{C}$  is straightforward and is therefore omitted.

(iii) The compact-valuedness of  $\tau(\theta) \neq \emptyset$  follows because  $\theta'' \in \tau(\theta)$  iff

$$\theta'' \in \arg \max u(q(\theta''), \theta) - t(\theta'') - C$$

The set of such maximizers is compact by Berge's Maximum Theorem because  $q(\cdot)$  and  $t(\cdot)$  are continuous functions.

(iv) Now, suppose that  $\theta'' \in \tau(\theta)$ . Further, if  $\tau(\theta') = \emptyset$  for some  $\theta' > \theta$ , then by Lemma 7  $V(\theta') = 0$ . However,  $V(\theta') \geq u(q(\theta''), \theta') - t(\theta'') - C > u(q(\theta''), \theta) - t(\theta'') - C \geq 0$ . Contradiction.

(iv) Finally, if  $C = \bar{C}$ , then with  $q(\theta) = q^{fb}(\theta)$  and  $V(\theta) = 0$  for all  $\theta \in [0, 1]$ , all  $IC$  constraints slack except  $IC(1, \theta')$ , which is binding for some  $\theta'$ . This mechanism achieves the first-best profits for the principal so it is optimal. *Q.E.D.*

Lemma 17 shows that the binding IC correspondence is non-decreasing.

**Lemma 17** Let  $\theta_1 > \theta_2$ . Suppose  $\theta'_1 \in \tau(\theta_1)$ ,  $\theta'_2 \in \tau(\theta_2)$ , then  $\theta'_1 \geq \theta'_2$ .

**Proof of Lemma 17:** Suppose that contrary to our claim,  $\theta'_1 < \theta'_2$ .

Since  $\theta'_2 \in \tau(\theta_2)$ , Corollary 1 implies that  $\theta'_2 < \theta_2$ .

Since  $\theta_1 > \theta_2$ , Lemma 11 implies that  $q(\theta'_1) \geq q(\theta'_2)$ .

Let us consider two cases. First suppose that  $q(\theta'_1) > q(\theta'_2)$ . Then let us define an alternative mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  which differs from  $(q(\cdot), t(\cdot))$  only in the allocation of type  $\theta'_2$ . Precisely, set  $\tilde{q}(\theta'_2) = q(\theta'_1)$  and  $\tilde{t}(\theta'_2) = t(\theta'_2) + u(q(\theta'_1), \theta'_2) - u(q(\theta'_2), \theta'_2) = u(q(\theta'_1), \theta'_2) - V(\theta'_2)$ . So, the net payoff of  $\theta'_2$  in the modified mechanism is still equal to  $V(\theta'_2)$ .

By Lemma 14,  $q^{fb}(\theta'_1) \geq q(\theta'_1)$ . Since  $\theta'_2 > \theta'_1$ , it follows that  $q^{fb}(\theta'_2) > q^{fb}(\theta'_1) \geq q(\theta'_1) = \tilde{q}(\theta'_2) > q(\theta'_2)$ . So,  $u(q(\theta'_1), \theta'_2) - u(q(\theta'_2), \theta'_2) > 0$ , and hence  $\tilde{t}(\theta'_2) > t(\theta'_2)$ . Thus, the modified mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  is weakly more profitable for the principal than  $(q(\cdot), t(\cdot))$ .

To complete the proof, we need to check that no type has an incentive to imitate  $\theta'_2$  in the mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$ .

If  $V(\theta'_2) = 0$ , then  $V(\theta'_1) = 0$  by monotonicity of  $V(\cdot)$ , and so  $\tilde{t}(\theta'_2) = u(q(\theta'_1), \theta'_2) > u(q(\theta'_1), \theta'_1) = t(\theta'_1)$ . Therefore, since  $\tilde{q}(\theta_2) = q(\theta'_1)$  and  $IC(\theta, \theta'_1)$  holds in the original mechanism,  $\tilde{IC}(\theta, \theta'_2)$  holds in the mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$ .

If  $V(\theta'_2) > 0$ , then there exists  $\theta''_2 \in \tau(\theta'_2)$ . So,

$$u(q(\theta'_2), \theta'_2) - t(\theta'_2) = u(q(\theta''_2), \theta'_2) - t(\theta''_2) - C$$

Combining this with  $IC(\theta'_1, \theta''_2)$  i.e.,  $u(q(\theta'_1), \theta'_1) - t(\theta'_1) \geq u(q(\theta''_2), \theta'_1) - t(\theta''_2) - C$ , we get:

$$t(\theta'_2) - t(\theta'_1) \geq u(q(\theta'_2), \theta'_2) - u(q(\theta'_1), \theta'_1) + u(q(\theta''_2), \theta'_1) - u(q(\theta''_2), \theta'_2).$$

Substituting  $\tilde{t}(\theta'_2) = t(\theta'_2) + u(q(\theta'_1), \theta'_2) - u(q(\theta'_2), \theta'_2)$  into the last inequality we get:

$$\tilde{t}(\theta'_2) - t(\theta'_1) \geq [u(q(\theta'_1), \theta'_2) - u(q(\theta'_1), \theta'_1)] - [u(q(\theta''_2), \theta'_2) - u(q(\theta''_2), \theta'_1)] > 0$$

The last inequality holds because  $u_{q\theta} > 0$ ,  $\theta'_2 > \theta'_1$ , and  $q(\theta'_1) > q(\theta''_2)$ . The latter inequality holds because  $q(\theta'_1) > q(\theta'_2)$  by assumption and also  $q(\theta'_2) \geq q(\theta''_2)$ , which follows from Lemma 11 because  $\theta_2 > \theta'_2$ .

Thus, since  $\tilde{t}(\theta'_2) > t(\theta'_1)$  and  $\tilde{q}(\theta'_2) = q(\theta'_1)$ , no type has incentive to deviate to  $\theta'_2$  under the new contract.

Now, suppose that  $q(\theta'_1) = q(\theta'_2)$ . Then a similar arguments shows that  $t(\theta'_2) > t(\theta'_1)$ , contradicts to the fact that  $\theta'_2 \in \tau(\theta_2)$ . (The only difference in this case is that we have to show that  $q(\theta'_2) > q(\theta''_2)$ , which must be true because otherwise  $q(\theta'_2) = q(\theta''_2)$  and  $t(\theta'_2) = t(\theta''_2)$ , and this would contradict  $\theta''_2 \in \tau(\theta_2)$ ). *Q.E.D.*

Lemma 17 has the following important Corollary upon which we rely below:

**Corollary 2** *For any two types  $\theta_1, \theta_2$  such that  $\theta_1 > \theta_2$ ,  $\tau(\theta_1) \cap \tau(\theta_2)$  is either empty or consists of a single type  $\theta'$  such that  $\theta' = \min \tau(\theta_1) = \max \tau(\theta_2)$ .*

Lemma 18 shows that for any positive cost of lying, there is a threshold  $\underline{\theta}$  such that all types below have zero surplus, are assigned their first-best quantities, and no type has binding incentive towards them.

**Lemma 18** *For any  $C > 0$ , in the optimal mechanism  $(q(\cdot), t(\cdot))$  there exists  $\underline{\theta}, \bar{\theta}$ ,  $0 < \underline{\theta} \leq \bar{\theta} < 1$ , such that  $\theta \notin \tau([0, 1])$  for any  $\theta \in [0, \underline{\theta}] \cap [\bar{\theta}, 1]$  and  $V(\theta) = 0$  for any  $\theta \in [0, \underline{\theta}]$ .*

**Proof of Lemma 18:** Recall from (68) that  $G(\theta', \theta) = u(q^{fb}(\theta), \theta') - u(q^{fb}(\theta), \theta)$ . Note that  $G(1, \theta)$  is continuous in  $\theta$  and  $G(1, 0) = 0$  because  $q^{fb}(0) = 0$ . Therefore, there exists  $\tilde{\theta} > 0$  such that  $G(1, \theta) - C < 0$  for any  $\theta \in [0, \tilde{\theta}]$ . Note that, since  $u_\theta > 0$ ,  $G(\theta', \theta) < G(1, \theta)$  for all  $\theta' \in [0, 1)$ .

Let  $\underline{\theta} = \min\{\tilde{\theta}, \check{\theta}\}$  where, as defined in Lemma 9,  $V(\theta) = 0$  for any  $\theta \in [0, \check{\theta}]$ . Now, let us replace the original mechanism  $(q(\cdot), t(\cdot))$  with an alternative mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  in which for all  $\theta \in [0, \underline{\theta}]$   $\tilde{q}(\theta) = q^{fb}(\theta)$  and  $\tilde{t}(\theta) = u(q^{fb}(\theta), \theta)$ , and the rest of the elements are the same as in  $(q(\cdot), t(\cdot))$ . So, the mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  is at least as profitable for the seller as  $(q(\cdot), t(\cdot))$ .

In  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  all *IR* constraints hold by construction. Let us now check that all *IC* hold. First, no type has an incentive to imitate any type  $\theta \in [0, \underline{\theta}]$  because  $G(\theta', \theta) < 0$  for all  $\theta', \theta \in [0, 1] \times [0, \underline{\theta}]$ . So,  $\theta \notin \tau([0, 1])$  if  $\theta \in [0, \underline{\theta}]$ .

Now fix any pair  $(\theta, \theta') \in [0, \underline{\theta}] \times (\underline{\theta}, 1]$  Since  $V(\theta) = 0$  in both mechanisms  $(q(\cdot), t(\cdot))$  and  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$ ,  $q(\theta') = \tilde{q}(\theta')$  and  $t(\theta') = \tilde{t}(\theta')$ , and *IC*( $\theta, \theta'$ ) holds in the mechanism  $(q(\cdot), t(\cdot))$  it follows that *IC*( $\theta, \theta'$ ) also holds in  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$ .



For the upper bound  $\bar{\theta}$ , let  $\bar{\theta} = \max\{\theta' : \theta' \in \tau(1)\}$ , note that this maximum exists as  $q$  and  $t$  are continuous. Now for any  $\theta' \in \tau([0, 1])$ ,  $\theta' \leq \bar{\theta} < 1$ , where the first inequality holds because of Lemma 17 and the second inequality holds because of Corollary 1. Therefore,  $\theta' \notin \tau([0, 1])$  for any  $\theta' \in [\bar{\theta}, 1]$ . *Q.E.D.*

Lemma 19 establishes that there is no exclusion in the optimal mechanism.

**Lemma 19** *In an optimal mechanism,  $q(\theta) > 0$  for any  $\theta > 0$ .*

**Proof of Lemma 19:** Suppose that  $q(\theta) = 0$  for some  $\theta > 0$ . Then  $t(\theta) = 0$ . Lemma 18 implies that there exists  $\theta' \in (0, \theta)$  such that  $q(\theta') = q^{fb}(\theta') > 0$ . Now let us replace the original mechanism  $(q(\cdot), t(\cdot))$  with an alternative mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  in which  $\tilde{q}(\theta) = q(\theta')$  and  $\tilde{t}(\theta) = t(\theta') = u(q(\theta'), \theta')$ , and the rest of the elements are the same as in  $(q(\cdot), t(\cdot))$ . The mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  is at least as profitable for the seller as  $(q(\cdot), t(\cdot))$ . It is obvious that  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  satisfies all *IR* and *IC* constraints because  $(q(\cdot), t(\cdot))$  does. *Q.E.D.*

From now on, assume that the type distribution  $F$  is continuous with full support on  $[0, 1]$ .

For any  $\theta' \in [0, 1]$ , define  $\tau^{-1}(\theta') = \{\theta \in [0, 1] : \theta' \in \tau(\theta)\}$ . The next Lemma shows that the inverse set of  $\tau$  is a singleton.

**Lemma 20** *In an optimal mechanism, for any  $\theta^\dagger \in [0, 1]$ ,  $\tau^{-1}(\theta^\dagger)$  is either empty or consists of a single type.*

**Proof of Lemma 20:** The proof is by contradiction, so suppose that there exist  $\theta_1, \theta_2$  such that  $\theta_1 < \theta_2$  and  $\theta^\dagger \in \tau(\theta_1) \cap \tau(\theta_2)$ .

Lemmas 16 and 17 and Corollary 2 imply that  $\theta^\dagger = \tau(\theta)$  for all  $\theta \in (\theta_1, \theta_2)$ . Also,  $q(\theta^\dagger) > 0$  and  $V(\theta) > 0$  for all  $\theta \in (\theta_1, \theta_2]$  because  $\theta^\dagger = \tau(\theta_1)$ .

Next, choose some  $\check{\theta}_1, \check{\theta}_2$  such that  $\theta_1 < \check{\theta}_1 < \check{\theta}_2 < \theta_2$ .

Lemmas 16 and 17 and Corollary 2 imply that  $\check{\theta} = \tau(\theta)$  for all  $\theta \in (\theta_1, \theta_2)$ . Also,  $q(\check{\theta}) > 0$  and  $V(\theta) > 0$  for all  $\theta \in (\theta_1, \theta_2]$  because  $\check{\theta} = \tau(\theta_1)$ .

Next, choose some  $\check{\theta}_1, \check{\theta}_2$  such that  $\theta_1 < \check{\theta}_1 < \check{\theta}_2 < \theta_2$ . Let us prove the following claim.

**Claim 1.** For any  $\lambda > 0$  there exists  $\delta > 0$  such that for all  $(\theta, \theta') \in [\check{\theta}_1, \check{\theta}_2] \times [0, 1] \setminus [\theta^\dagger - \lambda, \theta^\dagger + \lambda]$ , we have  $V(\theta) > u(q(\theta'), \theta) - t(\theta') - C + \delta$ .

Suppose not, then there exist  $\underline{\lambda} > 0$  such that for any  $\delta_n > 0$ ,  $\delta_n \rightarrow 0$ , there exists  $(\theta_n, \theta'_n) \in [\check{\theta}_1, \check{\theta}_2] \times [0, 1] \setminus [\theta^\dagger - \underline{\lambda}, \theta^\dagger + \underline{\lambda}]$  such that  $V(\theta_n) \leq u(q(\theta'_n), \theta_n) - t(\theta'_n) - C + \delta_n$ . Since  $[\theta^\dagger - \underline{\lambda}, \theta^\dagger + \underline{\lambda}]$  is compact  $(\theta_n, \theta'_n)$  has a subsequence converging to some  $(\theta^*, \theta'^*) \in [\check{\theta}_1, \check{\theta}_2] \times [0, 1] \setminus (\theta^\dagger - \underline{\lambda}, \theta^\dagger + \underline{\lambda})$ . By continuity of  $V(\cdot)$ ,  $q(\cdot)$  and  $t(\cdot)$ ,  $V(\theta^*) \leq u(q(\theta'^*), \theta^*) - t(\theta'^*) - C$ . Thus,  $\theta'^* \in \tau(\theta^*)$  which contradicts that  $\tau(\theta) = \theta^\dagger$  for all  $\theta \in [\check{\theta}_1, \check{\theta}_2]$ . This completes the proof of Claim 1.

Next fix some  $\lambda > 0$  and let  $\delta(\lambda)$  satisfy the conditions of the above claim. Note that  $\lim_{\lambda \rightarrow 0} \delta(\lambda) = 0$ .

For  $\theta \in [\theta^\dagger - \lambda, \theta^\dagger + \lambda]$ , define  $\epsilon(\theta, \lambda)$  such that

$$[u(q(\theta), \check{\theta}_1) - u(q(\theta) - \epsilon(\theta, \delta), \check{\theta}_1)] - [u(q(\theta), \theta) - u(q(\theta) - \epsilon(\theta, \delta), \theta)] = \delta(\lambda)$$

Given some small  $\delta$ , consider an alternative mechanism  $(\tilde{q}, \tilde{t})$  such that for  $\theta' \in [\theta^\dagger - \lambda(\delta), \theta^\dagger + \lambda(\delta)]$ ,  $\tilde{q}(\theta') = q(\theta') - \epsilon(\theta', \delta)$  and  $\tilde{t}(\theta') = t(\theta') - u(q(\theta'), \theta') + u(q(\theta') - \epsilon(\theta', \delta), \theta')$ ; for  $\theta \in [\check{\theta}_1, \check{\theta}_2]$ ,  $\tilde{t}(\theta) = t(\theta) + \delta$ . We will show that all IC and IR are satisfied in the new contract. For  $\theta' \in [\theta^\dagger - \lambda(\delta), \theta^\dagger + \lambda(\delta)]$ ,  $\tilde{V}(\theta') = V(\theta')$ , therefore IR are satisfied. For  $\theta \in [\check{\theta}_1, \check{\theta}_2]$ , since  $V(\theta) > 0$ , we have  $\tilde{V}(\theta) = V(\theta) - \delta > 0$  for small  $\delta$ , therefore IR are satisfied.

Now we have to check that  $\tilde{IC}(\theta, \theta')$  are satisfied for  $\theta \in [0, 1]$  and  $\theta' \in [\theta^\dagger - \lambda(\delta), \theta^\dagger + \lambda(\delta)]$ . First note that  $\lim_{\delta \rightarrow 0} \epsilon(\theta', \delta) = 0$ . For  $\theta \leq \theta'$ , given Corollary 1  $IC(\theta, \theta')$  is slack, by continuity  $\tilde{IC}(\theta, \theta')$  is still slack for small  $\delta$ . For  $\theta > \theta'$  and  $\theta \notin [\check{\theta}_1, \check{\theta}_2]$ ,  $\tilde{IC}(\theta, \theta')$  improves. For  $\theta \in [\check{\theta}_1, \check{\theta}_2]$ ,  $\tilde{IC}(\theta, \theta')$  holds because:

$$\begin{aligned} \tilde{V}(\theta) &= V(\theta) - \delta = V(\theta) - [u(q(\theta'), \check{\theta}_1) - u(q(\theta') - \epsilon(\theta', \delta), \check{\theta}_1)] + [u(q(\theta'), \theta') - u(q(\theta') - \epsilon(\theta', \delta), \theta')] \\ &\geq u(q(\theta'), \theta) - t(\theta') - C - [u(q(\theta'), \check{\theta}_1) - u(q(\theta') - \epsilon(\theta', \delta), \check{\theta}_1)] + [u(q(\theta'), \theta') - u(q(\theta') - \epsilon(\theta', \delta), \theta')] \\ &= u(q(\theta'), \theta) - \tilde{t}(\theta') - C - u(q(\theta'), \check{\theta}_1) + u(q(\theta') - \epsilon(\theta', \delta), \check{\theta}_1) \\ &\geq u(q(\theta'), \theta) - \tilde{t}(\theta') - C - u(q(\theta'), \theta) + u(q(\theta') - \epsilon(\theta', \delta), \theta) \geq u(\tilde{q}(\theta'), \theta) - \tilde{t}(\theta') - C \end{aligned}$$

where the first equality holds because  $\tilde{t}(\theta) = t(\theta) + \delta$ , the second equality holds by definition of  $\delta$ , the first inequality holds by  $IC(\theta, \theta')$ , the third equality holds by definition of  $\tilde{t}(\theta')$ , and the second inequality holds because  $\theta \geq \check{\theta}_1$ , and the last inequality holds because  $-u(q(\theta'), \theta) + u(q(\theta') - \epsilon(\theta', \delta), \theta) < 0$ .

For  $\theta \in [0, 1]$  and  $\theta' \notin [\theta^\dagger - \lambda(\delta), \theta^\dagger + \lambda(\delta)]$ ,  $\tilde{IC}(\theta, \theta')$  since  $\tilde{V}(\theta) = V(\theta) - \delta > u(q(\theta'), \theta) - t(\theta') - C$ . The last inequality holds by definition of  $\lambda(\delta)$ .

The change in seller's profits from switching to the new mechanism is equal to  $[F(\check{\theta}_2) - F(\check{\theta}_1)]\delta - \int_{\theta^\dagger - \lambda(\delta)}^{\theta^\dagger + \lambda(\delta)} [u(q(\theta'), \theta') - u(q(\theta') - \epsilon(\theta', \delta), \theta')] f(\theta') d\theta' \geq [F(\check{\theta}_2) - F(\check{\theta}_1)]\delta - [F(\theta^\dagger + \lambda(\delta)) - F(\theta^\dagger - \lambda(\delta))] \max_{\theta' \in [\theta^\dagger - \lambda(\delta), \theta^\dagger + \lambda(\delta)]} [u(q(\theta'), \theta'') - u(q(\theta') - \epsilon(\theta', \delta), \theta')]$ . Since  $\lim_{\delta \rightarrow 0} \lambda(\delta) = 0$ ,  $\lim_{\delta \rightarrow 0} [F(\theta^\dagger + \Delta(\delta)) - F(\theta^\dagger - \Delta(\delta))] = 0$  and  $F(\check{\theta}_2) - F(\check{\theta}_1) > 0$  as  $F$  is continuous with full support. Finally,

$$\lim_{\delta \rightarrow 0} \frac{\delta}{\max_{\theta' \in [\theta^\dagger - \lambda(\delta), \theta^\dagger + \lambda(\delta)]} [u(q(\theta'), \theta') - u(q(\theta') - \epsilon(\theta', \delta), \theta')]} = \lim_{\delta \rightarrow 0} \frac{\delta}{u_q(q(\theta^\dagger), \theta^\dagger) \epsilon(\theta', \delta)}$$

$$= \frac{u_q(q(\theta^\dagger), \theta_1) - u_q(q(\theta^\dagger), \theta^\dagger)}{u_q(q(\theta^\dagger), \theta^\dagger)} > 0. \text{ So } \delta \text{ and } \max_{\theta' \in [\theta^\dagger - \lambda(\delta), \theta^\dagger + \lambda(\delta)]} [u(q(\theta'), \theta') - u(q(\theta') - \epsilon(\theta', \delta), \theta')]$$

converge to zero at the same rate. Therefore, our alternative mechanism generates higher profit while satisfying all IC and IR for small enough  $\delta$ , contradiction. *Q.E.D.*

Lemmas 17 and 20 imply that the correspondence  $\tau(\cdot)$  is strictly increasing.

**Corollary 3** *Let  $\theta_1 > \theta_2$ . Suppose  $\theta'_1 \in \tau(\theta_1)$ ,  $\theta'_2 \in \tau(\theta_2)$ , then  $\theta'_1 > \theta'_2$ .*

**Lemma 21** *Define  $\tau^{-k}(\cdot) = \tau^{-1}(\tau^{-(k-1)}(\cdot))$  for  $k = 1, 2, \dots$ . There exists  $\bar{K} < \infty$  such that for any  $\theta$   $\tau^{-k}(\theta) = \emptyset$  for some  $k \leq \bar{K}$ .*

**Proof of Lemma 21:** Since  $\tau(\cdot)$  is increasing in its argument it is sufficient to establish the claim of the Lemma for  $\theta_1 = \tau(\hat{\theta})$ . We argue by contradiction, so suppose that the claim of the Lemma is not true for  $\theta_1$ . Note that by Lemma 20  $\tau^{-k}(\theta)$  is either an empty set or a singleton for any  $\theta \in [0, 1]$  and any positive integer  $k$ . So there exists a sequence  $\theta_k$ ,  $k = 1, \dots, \infty$  such that  $\theta_{k+1} = \tau^{-1}(\theta_k)$  for all  $k \geq 1$  i.e.,  $u(q(\theta_{k+1}), \theta_{k+1}) - t(\theta_{k+1}) = u(q(\theta_k), \theta_{k+1}) - t(\theta_k) - C$ . Corollary 1 implies that  $\theta_k < \theta_{k+1}$ . Since  $\theta_k \in [0, 1]$  for all  $k$ , it follows that  $\lim_{k \rightarrow \infty} \theta_k - \theta_{k+1} = 0$ . But then by continuity of  $q$  and  $t$ ,  $\lim_{k \rightarrow \infty} [u(q(\theta_{k+1}), \theta_{k+1}) - t(\theta_{k+1})] - [u(q(\theta_k), \theta_{k+1}) - t(\theta_k)] = 0 > -C$ , a contradiction. *Q.E.D.*

**Lemma 22** *If  $\theta'_1, \theta'_2 \in \tau(\check{\theta}^0)$  for some  $\check{\theta}^0$ , with  $\theta'_1 < \theta'_2$ , then  $q(\theta') = q^{fb}(\theta')$  for any  $\theta' \in [\theta'_1, \theta'_2]$ .*

**Proof of Lemma 22:**

Suppose that contrary to our claim,  $q(\theta) < q^{fb}(\theta)$  for some  $\theta \in [\theta'_1, \theta'_2]$ . Then by continuity of  $q(\cdot)$  there exist  $\check{\theta}'_1, \check{\theta}'_2$  such that  $\theta'_1 < \check{\theta}'_1 < \check{\theta}'_2 < \theta'_2$  and  $q(\theta') < q^{fb}(\theta')$  for any  $\theta' \in [\check{\theta}'_1, \check{\theta}'_2]$ . Then by Lemmas 13 and 17,  $\tau^{-1}(\theta) = \{\check{\theta}^0\}$  for all  $\theta \in [\check{\theta}'_1, \check{\theta}'_2]$ .

Recall that  $\tau^{-k}(\cdot) = \tau^{-1}(\tau^{-(k-1)}(\cdot))$  where  $k$  is a positive integer  $k$ . By Lemma 21 there exists  $M \geq 0$  such that  $\tau^{-k}(\check{\theta}^0)$  is singleton for  $k \leq M$  and empty for  $k > M$ . So, if  $M \geq 1$ , then for  $k \in \{1, \dots, M\}$  let us define  $\check{\theta}^k = \tau^{-k}(\check{\theta}^0)$ . Next, we prove three claims.

**Claim 1.** For any  $\lambda^0 > 0$  there exists  $\delta^0 > 0$  such that for any  $(\theta, \theta') \in [0, 1] \setminus [\check{\theta}^0 - \lambda^0, \check{\theta}^0 + \lambda^0] \times [\check{\theta}'_1, \check{\theta}'_2]$ , we have  $V(\theta) > u(q(\theta'), \theta) - t(\theta') - C + \delta$ .

Suppose not, then for some  $\underline{\lambda}$  there exists sequences  $\delta_n^0 > 0 : \delta_n^0 \rightarrow 0$  and  $(\theta_n, \theta'_n) \in [0, 1] \setminus [\check{\theta}^0 - \underline{\lambda}, \check{\theta}^0 + \underline{\lambda}] \times [\check{\theta}'_1, \check{\theta}'_2]$  such that  $V(\theta_n) \leq u(q(\theta'_n), \theta_n) - t(\theta'_n) - C + \delta_n$  for all  $n$ .

Without loss of generality, suppose that  $(\theta_n, \theta'_n)$  converges to some  $(\theta^*, \theta'^*) \in [0, 1] \setminus (\check{\theta}^0 - \underline{\lambda}, \check{\theta}^0 + \underline{\lambda}) \times [\check{\theta}'_1, \check{\theta}'_2]$ . Then by continuity of  $V(\cdot)$ ,  $q(\cdot)$  and  $t(\cdot)$  we have  $V(\theta^*) \leq u(q(\theta'^*), \theta^*) - t(\theta'^*) - C$ , which contradicts the assumption that  $\tau^{-1}([\check{\theta}'_1, \check{\theta}'_2]) = \{\check{\theta}^0\}$ . This completes the proof of Claim 1.

**Claim 2.** If  $M \geq 1$ , then for any  $k \in \{1, \dots, M\}$ , and any  $\lambda^k > 0$  there exists  $\lambda^{k-1} > 0$  and  $\delta^k > 0$  such that for any  $(\theta, \theta') \in [0, 1] \setminus [\check{\theta}^k - \lambda^k, \check{\theta}^k + \lambda^k] \times [\check{\theta}^{k-1} - \lambda^{k-1}, \check{\theta}^{k-1} + \lambda^{k-1}]$ , we have  $V(\theta) > u(q(\theta'), \theta) - t(\theta') - C + \delta^k$ .

Suppose not, then there exists  $\underline{\lambda}^k > 0$  such that for any sequence  $\delta_n^k > 0 : \delta_n^k \rightarrow 0$ , there exists a sequence  $\theta_n \rightarrow \theta^* \in [0, 1] \setminus (\check{\theta}^k - \underline{\lambda}^k, \check{\theta}^k + \underline{\lambda}^k)$  such that  $V(\theta_n) \leq u(q(\check{\theta}^{k-1}), \theta_n) - t(\check{\theta}^{k-1}) - C + \delta_n^k$ . Then by continuity of  $V(\cdot)$ ,  $q(\cdot)$  and  $t(\cdot)$  we have  $V(\theta^*) \leq u(q(\check{\theta}^{k-1}), \theta^*) - t(\check{\theta}^{k-1}) - C$ . So,  $\check{\theta}^{k-1} \in \tau(\theta^*)$  which contradicts the fact that  $\tau^{-1}(\check{\theta}^{k-1}) = \{\check{\theta}^k\}$ .

**Claim 3.** There exists  $\delta^{M+1} > 0$  and  $\lambda^M > 0$  such that for any  $(\theta, \theta') \in [0, 1] \times [\check{\theta}^M - \lambda^M, \check{\theta}^M + \lambda^M]$ , we have  $V(\theta) > u(q(\theta'), \theta) - t(\theta') - C + \delta^{M+1}$ .

Suppose not, then there exists a sequence  $\delta_n^{M+1} > 0 : \delta_n^{M+1} \rightarrow 0$  and a sequence  $\theta_n \rightarrow \theta^{*(M+1)} \in [0, 1]$  such that  $V(\theta_n) \leq u(q(\check{\theta}^M), \theta_n) - t(\check{\theta}^M) - C + \delta_n^{M+1}$ . Then by continuity of  $V(\cdot)$ ,  $q(\cdot)$  and  $t(\cdot)$  we have  $V(\theta^{*(M+1)}) \leq u(q(\check{\theta}^M), \theta^{*(M+1)}) - t(\check{\theta}^M) - C$ . So,  $\check{\theta}^M \in \tau(\theta^{*(M+1)})$  which contradicts the fact that  $\tau^{-1}(\check{\theta}^M) = \emptyset$ . This completes the proof of Claim 3.

Note that if Claim 3 holds for some pair  $(\delta^{M+1}, \lambda^M)$ , it also holds for  $(\delta^{(M+1)'}, \lambda^{M'}) \leq (\delta^{M+1}, \lambda^M)$ . Also, by Claims 1 and 2, for any  $\lambda^M$  there exist  $(\delta^M, \lambda^{M-1})$  such that Claim 2 holds.

Now let  $\delta = \min_{k \in \{1, \dots, M\}} \delta^k$ . Note that if let  $\lambda_M$  converge to zero then  $\delta$  also converges to zero.

Next, let us use the following iterative procedure to construct a strictly positive sequence  $\{\delta_j, \lambda_j^0, \dots, \lambda_j^M\}$ ,  $j \in \{1, \dots, \infty\}$ , which converges to zero, and such that for all  $j$ : (i)  $\delta_j \leq \lambda_j^0 \leq \dots \leq \lambda_j^M$ ; (ii)  $(\delta_j, \lambda_j^M)$  satisfy the condition of Claim 3; (iii) Given  $\lambda_j^k$ ,  $(\delta_j, \lambda_j^{k-1})$  satisfy the condition of Claim 2; (iv) Given  $\lambda_j^0$ ,  $\delta_j$  satisfies the condition of Claim 1.

**Step 1.** Let  $\Omega^M$  be the non-empty set of  $(\lambda^M, \delta^{M+1})$  that satisfy the condition of Claim 3. Define  $\bar{\lambda}^M = \sup\{\lambda^M : (\lambda^M, \delta^{M+1}) \in \Omega^M, \lambda^M = \delta^{M+1}\}$ . Then let  $\lambda_j^M = \frac{\bar{\lambda}^M}{j+1}$  for  $j \in \{1, \dots, \infty\}$ . Note that, if  $(\lambda^M, \delta^{M+1}) \in \Omega^M$ , then  $(\tilde{\lambda}^M, \tilde{\delta}^{M+1}) \in \Omega^M$  if  $0 \ll (\tilde{\lambda}^M, \tilde{\delta}^{M+1}) \leq (\lambda^M, \delta^{M+1})$ . Therefore,  $\lambda_j^M$  is well-defined and is strictly positive and  $(\lambda_j^M, \delta) \in \Omega^M$  for all  $\delta$  s.t.  $\delta \leq \lambda_j^M$ .

**Step 2.** Suppose now that we have constructed  $\{\lambda_j^{k+1} > 0\}_{j=1}^\infty$  for  $k \leq M-1$ . Given  $\lambda_j^{k+1}$  let  $\Omega^k(\lambda_j^{k+1})$  be the set of  $(\lambda^k, \delta^{k+1})$  that satisfy the conditions of claim 2. Then set  $\bar{\lambda}_j^k = \sup\{\lambda^k : (\lambda^k, \delta^{k+1}) \in \Omega^k(\lambda_j^{k+1}), \lambda^k = \delta^{k+1}\}$  and  $\lambda_j^k = \min\{\lambda_j^{k+1}, \frac{\bar{\lambda}_j^k}{2}\}$  for  $j \in \{1, \dots, \infty\}$ .

Analogously to Step 2, if  $(\lambda^k, \delta^{k+1}) \in \Omega^k(\lambda_j^{k+1})$ , then  $(\tilde{\lambda}^k, \tilde{\delta}^{k+1}) \in \Omega^k(\lambda_j^{k+1})$  if  $0 \ll (\tilde{\lambda}^k, \tilde{\delta}^{k+1}) \leq (\lambda^k, \delta^{k+1})$ . Therefore,  $\lambda_j^k$  is well-defined and is strictly positive and  $(\lambda_j^k, \delta) \in \Omega^k(\lambda_j^{k+1})$  for all  $\delta$  s.t.  $\delta \leq \lambda_j^k$ .

**Step 3.** Given  $\lambda_j^0 > 0$ , let  $\Omega(\lambda_j^0)$  be the set of  $\delta^0$  that satisfy the conditions of Claim 1. Then define  $\bar{\delta}_j = \sup\{\delta \in \Delta(\lambda_j^0)\}$  and set  $\delta_j = \min\{\lambda_j^0, \frac{\bar{\delta}_j}{2}\}$  for all  $j \in \{1, \dots, \infty\}$ . Note that if  $\delta_0 \in \Delta(\lambda^0)$ , then  $\tilde{\delta}^0 \in \Delta(\lambda^0)$  for any  $0 < \tilde{\delta}^0 \leq \delta^0$ . Therefore,  $\delta_j$  is well-defined and is strictly positive and  $\delta_j \in \Omega(\lambda_j^0)$ .

Since  $\delta_j \leq \lambda_j^k$  for all  $k \in \{0, \dots, M\}$  we have  $(\lambda_j^k, \delta_j) \in \Omega^k(\lambda_j^{k+1})$  as required.

Now, for  $\delta_j$  and for any  $\theta \in [\check{\theta}_1^j, \check{\theta}_2^j]$  define  $\epsilon(\theta, \delta_j)$  as a solution in  $\epsilon$  to the following equation:

$$[u(q(\theta) + \epsilon, 1) - u(q(\theta) + \epsilon, \theta)] - [u(q(\theta), 1) - u(q(\theta), \theta)] = \delta_j \quad (70)$$

Note that the left-hand side of (70) is equal to zero when  $\epsilon = 0$  and is increasing in  $\epsilon$ . So, there exists  $N$  such that for all  $j \geq N$ ,  $\delta_j$  is small enough and the solution to (70) is well-defined with  $q(\theta) + \epsilon(\theta, \delta_j) < q^{fb}(\theta)$  (since  $q(\theta) < q^{fb}(\theta)$ ).

Now, given  $j : j \geq N$ , consider an alternative mechanism  $(\tilde{q}_j(\cdot), \tilde{t}_j(\cdot))$  which differs from the original mechanism  $(q(\cdot), t(\cdot))$  only as follows: for  $\theta \in [\check{\theta}_1^j, \check{\theta}_2^j]$ ,  $\tilde{q}_j(\theta) = q(\theta) + \epsilon(\theta, \delta_j)$  and  $\tilde{t}_j(\theta) = t(\theta) + u(q(\theta) + \epsilon(\theta, \delta_j), \theta) - u(q(\theta), \theta)$ , and for  $\theta \in \cup_{k=0}^M [\check{\theta}^k - \lambda_j^k, \check{\theta}^k + \lambda_j^k]$ ,  $\tilde{t}_j(\theta) = t(\theta) - \delta_j$ .

Below we show that: (i) the mechanism  $(\tilde{q}_j(\cdot), \tilde{t}_j(\cdot))$  satisfies all *IC* and *IR* constraints; (ii)  $(\tilde{q}_j(\cdot), \tilde{t}_j(\cdot))$  is strictly more profitable for the principal than the original mechanism  $(q(\cdot), t(\cdot))$

when  $j$  is sufficiently large.

First,  $IR$  constraints hold in  $(\tilde{q}_j(\cdot), \tilde{t}_j(\cdot))$  because  $\tilde{V}_j(\theta) > V(\theta)$  for  $\theta \in \cup_{k=0}^M[\check{\theta}^k - \lambda_j^k, \check{\theta}^k + \lambda_j^k]$ , and  $\tilde{V}_j(\theta) = V(\theta)$  for all other types  $\theta$ .

Now let us consider incentive constraints. For  $\theta \notin \cup_{k=0}^M[\check{\theta}^k - \lambda_j^k, \check{\theta}^k + \lambda_j^k]$  and  $\theta' \in [\check{\theta}'_1, \check{\theta}'_2]$ ,

$$\begin{aligned} \tilde{V}_j(\theta) &= V(\theta) > u(q(\theta'), \theta) - t(\theta') - C + \delta_j \\ &= u(q(\theta'), \theta) - t(\theta') - C + [u(q(\theta') + \epsilon(\theta', \delta_j), 1) - u(q(\theta'), 1)] - [u(q(\theta') + \epsilon(\theta', \delta_j), \theta') - u(q(\theta'), \theta')] \\ &= u(q(\theta'), \theta) - \tilde{t}(\theta') - C + u(q(\theta') + \epsilon(\theta', \delta_j), 1) - u(q(\theta'), 1) \\ &\geq u(q(\theta'), \theta) - \tilde{t}(\theta') - C + u(q(\theta') + \epsilon(\theta', \delta_j), \theta) - u(q(\theta'), \theta) = u(\tilde{q}_j(\theta'), \theta) - \tilde{t}(\theta') - C \end{aligned}$$

where the first inequality holds since  $\theta \notin [\check{\theta}^0 - \lambda_j^0, \check{\theta}^0 + \lambda_j^0]$ , the second equality holds by definition of  $\epsilon(\theta', \delta_j)$ , the third equality holds by definition of  $\tilde{t}_j(\theta')$ , the second inequality holds because  $\theta \leq 1$  and the last equality holds by definition of  $\tilde{q}_j(\theta')$ .

For  $\theta \in \cup_{k=0}^M[\check{\theta}^k - \lambda_j^k, \check{\theta}^k + \lambda_j^k]$  and  $\theta' \in [\check{\theta}'_1, \check{\theta}'_2]$ ,

$$\begin{aligned} \tilde{V}_j(\theta) &= V(\theta) + \delta_j \geq u(q(\theta'), \theta) - t(\theta') - C + \delta_j \\ &= u(q(\theta'), \theta) - t(\theta') - C + [u(q(\theta') + \epsilon(\theta', \delta_j), 1) - u(q(\theta'), 1)] - [u(q(\theta') + \epsilon(\theta', \delta_j), \theta') - u(q(\theta'), \theta')] \\ &= u(q(\theta'), \theta) - \tilde{t}_j(\theta') - C + u(q(\theta') + \epsilon(\theta', \delta_j), 1) - u(q(\theta'), 1) \\ &\geq u(q(\theta'), \theta) - \tilde{t}_j(\theta') - C - u(q(\theta'), \theta) + u(q(\theta') - \epsilon(\theta', \delta_j), \theta) = u(\tilde{q}_j(\theta'), \theta) - \tilde{t}_j(\theta') - C \end{aligned}$$

where the first equality holds by definition of  $\tilde{t}_j(\theta)$ , the first inequality holds by  $IC(\theta, \theta')$ , the second equality holds by definition of  $\epsilon(\theta', \delta_j)$ , the third equality holds because by definition of  $\tilde{t}_j(\theta')$ , the second inequality holds because  $\theta \leq 1$  and the last equality holds by definition of  $\tilde{q}_j(\theta')$ .

For  $\theta \notin \cup_{k=0}^M[\check{\theta}^k - \lambda_j^k, \check{\theta}^k + \lambda_j^k]$  and  $\theta' \in \cup_{k=0}^M[\check{\theta}^k - \lambda_j^k, \check{\theta}^k + \lambda_j^k]$ ,

$$\tilde{V}_j(\theta) = V(\theta) > u(q(\theta'), \theta) - t(\theta') - C + \delta_j = u(\tilde{q}_j(\theta'), \theta) - \tilde{t}_j(\theta') - C$$

Where the first inequality holds because  $\theta \notin \cup_{k=0}^M[\check{\theta}^k - \lambda_j^k, \check{\theta}^k + \lambda_j^k]$  and the last equality holds by definition of  $\tilde{t}_j(\theta')$ .

For  $\theta \in \cup_{k=0}^M[\check{\theta}^k - \lambda_j^k, \check{\theta}^k + \lambda_j^k]$  and  $\theta' \in \cup_{k=0}^M[\check{\theta}^k - \lambda_j^k, \check{\theta}^k + \lambda_j^k]$ ,

$$\tilde{V}_j(\theta) = V(\theta) + \delta_j \geq u(q(\theta'), \theta) - t(\theta') - C + \delta_j = u(\tilde{q}_j(\theta'), \theta) - \tilde{t}_j(\theta') - C$$

where the first equality holds by definition of  $\tilde{t}_j(\theta)$ , the first inequality holds by  $IC(\theta, \theta')$ , the last equality holds by definition of  $\tilde{t}_j(\theta')$ . Thus, all  $\tilde{IC}_j(\theta, \theta')$  are satisfied.

The change in profit in the new mechanism is equal to

$$\begin{aligned} & \int_{\check{\theta}'_1}^{\check{\theta}'_2} [u(q(\theta') + \epsilon(\theta', \delta_j), \theta') - u(q(\theta'), \theta')] f(\theta') d\theta' - F(\cup_{k=0}^M [\check{\theta}^k - \lambda_j^k, \check{\theta}^k + \lambda_j^k]) \delta_j \\ & \geq [F(\check{\theta}'_2) - F(\check{\theta}'_1)] [u(q(\underline{\theta}') + \epsilon(\underline{\theta}', \delta_j), \underline{\theta}') - u(\underline{\theta}', \underline{\theta}')] - F(\cup_{k=0}^M [\check{\theta}^k - \lambda_j^k, \check{\theta}^k + \lambda_j^k]) \delta_j \end{aligned}$$

where  $\underline{\theta}' = \arg \min_{\theta' \in [\check{\theta}'_1, \check{\theta}'_2]} u(q(\theta') + \epsilon(\theta', \delta_j), \theta') - u(q(\theta'), \theta')$ .

Since  $\lim_{j \rightarrow \infty} \lambda_j^k = 0$  for  $k = 0, \dots, M$ ,  $\lim_{j \rightarrow \infty} F(\cup_{k=0}^M [\check{\theta}^k - \lambda_j^k, \check{\theta}^k + \lambda_j^k]) = 0$  and  $F(\check{\theta}'_2) - F(\check{\theta}'_1) > 0$  as  $F$  is continuous with full support.

Finally, we have  $\lim_{j \rightarrow \infty} \frac{u(q(\underline{\theta}') + \epsilon(\underline{\theta}', \delta_j), \underline{\theta}') - u(\underline{\theta}', \underline{\theta}')}{\delta_j} = \frac{u_q(q(\underline{\theta}'), \underline{\theta}')}{u_q(q(\underline{\theta}'), 1) - u_q(q(\underline{\theta}'), \underline{\theta}')} > 0$  because  $q(\underline{\theta}') < q^{fb}(\underline{\theta}')$ . Since  $\delta$  and  $u(q(\underline{\theta}') + \epsilon(\underline{\theta}', \delta), \underline{\theta}') - u(\underline{\theta}', \underline{\theta}')$  converge to zero at the same rate, we conclude that when  $j$  is sufficiently large, this alternative mechanism generates a higher profit while satisfying all IC and IR contradiction. *Q.E.D.*

Now we can show the following:

**Lemma 23**  $q(\theta)$  is strictly increasing in  $\theta$

**Proof of Lemma 23:** Let  $\theta'_2 > \theta'_1$ . By Lemma 20  $\tau^{-1}(\theta'_1)$  and  $\tau^{-1}(\theta'_2)$  are either empty or singleton. We need to consider five cases.

- (i) If  $\tau^{-1}(\theta'_1) = \tau^{-1}(\theta'_2) = \emptyset$ , then by Lemma 13,  $q(\theta'_1) = q^{fb}(\theta'_1) < q^{fb}(\theta'_2) = q(\theta'_2)$ .
- (ii) If  $\tau^{-1}(\theta'_1) \neq \emptyset$  and  $\tau^{-1}(\theta'_2) = \emptyset$ , Lemmas 13 and 14 imply that  $q(\theta'_1) \leq q^{fb}(\theta'_1) < q^{fb}(\theta'_2) = q(\theta'_2)$ .
- (iii) If  $\tau^{-1}(\theta'_1) = \emptyset$  and  $\tau^{-1}(\theta'_2) \neq \emptyset$ , by continuity of  $V$ ,  $q$  and  $t$  there exists  $\theta'$  such that  $\theta'_1 < \theta' \leq \theta'_2$ ,  $q(\theta') = q^{fb}(\theta')$  and  $\tau^{-1}(\theta') \neq \emptyset$ . Then using Lemma 11 we have  $q(\theta'_1) = q^{fb}(\theta'_1) < q^{fb}(\theta') = q(\theta') \leq q(\theta'_2)$ .
- (iv) If  $\tau^{-1}(\theta'_1) = \tau^{-1}(\theta'_2) = \{\theta\}$  for some  $\theta$ , then by Lemma 22,  $q(\theta'_1) = q^{fb}(\theta'_2) < q^{fb}(\theta'_1) = q(\theta'_2)$ .
- (v) If  $\tau^{-1}(\theta'_1)$  and  $\tau^{-1}(\theta'_2)$  are non-empty and  $\tau^{-1}(\theta'_1) \neq \tau^{-1}(\theta'_2)$ , then by Corollary 3  $\tau^{-1}(\theta'_1) < \tau^{-1}(\theta'_2)$ , and by Lemma 11  $q(\theta'_2) \geq q(\theta'_1)$ . If  $q(\theta'_2) = q(\theta'_1)$  then we must have  $t(\theta'_2) = t(\theta'_1)$  since  $\tau^{-1}(\theta'_1)$  and  $\tau^{-1}(\theta'_2)$  are non-empty. But this contradicts  $\tau^{-1}(\theta'_1) \neq \tau^{-1}(\theta'_2)$ . So,  $q(\theta'_2) > q(\theta'_1)$ . *Q.E.D.*

**Lemma 24** *Suppose there exist  $\underline{\theta}, \bar{\theta}$  and integer  $M \geq 1$  such that  $\bar{\theta} > \underline{\theta}$ ,  $[\underline{\theta}, \bar{\theta}] \subseteq [\tau(1), 1]$ , and  $\tau^s(\theta)$  is single-valued for all  $\theta \in [\underline{\theta}, \bar{\theta}]$  and  $s \in \{1, \dots, M\}$ . Then for any such  $s$  and  $\theta \in [\underline{\theta}, \bar{\theta}]$ ,  $u_q(q(\tau^s(\theta)), \tau^s(\theta))f(\tau^s(\theta))\dot{\tau}^s(\theta) = [u_q(q(\tau^s(\theta)), \tau^{s-1}(\theta)) - u_q(q(\tau^s(\theta)), \tau^s(\theta))] \sum_{k=1}^s f(\tau^{s-k}(\theta))\dot{\tau}^{s-k}(\theta)$ .*

**Proof of Lemma 24:** Since  $\tau(\cdot)$  is strictly increasing and upper hemicontinuous and  $\tau^s(\cdot)$  is single-valued by assumption,  $\tau^s(\cdot)$  must also be strictly increasing, continuous and differentiable almost everywhere on  $[\underline{\theta}, \bar{\theta}]$ .

The proof is by contradiction. So suppose that for some  $\tilde{\theta} \in (\underline{\theta}, \bar{\theta})$  and  $s \in \{1, \dots, M\}$  :

$$u_q(q(\tau^s(\tilde{\theta})), \tau^s(\tilde{\theta}))f(\tau^s(\tilde{\theta}))\dot{\tau}^s(\tilde{\theta}) > [u_q(q(\tau^s(\tilde{\theta})), \tau^{s-1}(\tilde{\theta})) - u_q(q(\tau^s(\tilde{\theta})), \tau^s(\tilde{\theta}))] \sum_{k=1}^s f(\tau^{s-k}(\tilde{\theta}))\dot{\tau}^{s-k}(\tilde{\theta}). \quad (71)$$

Let us now consider the case where the opposite inequality holds.

Note that the left hand side of (71) is the marginal efficiency gain of raising  $q$  on a neighborhood around  $\tau^s(\tilde{\theta})$ , while its right hand side is the marginal increases in rent that the principal needs to provide to the types in the neighborhoods around every predecessor of  $\tau^s(\tilde{\theta})$  in the chain of targeted types,  $\tau^{s-k}(\tilde{\theta})$  for  $k = 1, \dots, s$ . The multiplier  $f(\tau^{s-k}(\tilde{\theta}))\dot{\tau}^{s-k}(\tilde{\theta})$  for  $k = 0, \dots, s$ , reflects the relative probability weight of the neighborhood around  $\tau^{s-k}(\tilde{\theta})$ . So, when (71) holds, the principal could get higher profits by increasing the quantities assigned to the types around  $\tau^s(\tilde{\theta})$  and collecting the additional revenue generated thereby, while providing increased rents required by types around  $\tau^{s-k}(\tilde{\theta})$ ,  $k = 1, \dots, s$ .

The rest of the proof formalizes this intuition. We proceed through three steps. In Step 1, we construct an alternative mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  reflecting the aforementioned modification. In Steps 2 and 3 we show, respectively, that this mechanism is incentive compatible and is more profitable for the principal than the original one, when the quantity changes for the types around  $\tau^s(\tilde{\theta})$  are sufficiently small.

**Step 1. Constructing an Alternative Mechanism.**

First, (71) and the continuity of  $\tau^s(\cdot)$  and  $q(\cdot)$  imply that there exist  $\sigma > 0$  and  $\mu > 0$  such



that  $\underline{\theta} < \tilde{\theta} - \sigma < \tilde{\theta} + \sigma < \bar{\theta}$  and for all  $\theta \in [\tilde{\theta} - \sigma, \tilde{\theta} + \sigma]$ ,

$$\frac{u_q(q(\tau^s(\theta)), \tau^s(\theta))}{u_q(q(\tau^s(\theta)), \tau^{s-1}(\theta)) - u_q(q(\tau^s(\theta)), \tau^s(\theta))} f(\tau^s(\theta)) \dot{\tau}^s(\theta) - \sum_{k=1}^s f(\tau^{s-k}) \dot{\tau}^{s-k}(\theta) - \mu > 0 \quad (72)$$

Note that  $q(\tau^s(\theta)) < q^{fb}(\tau^s(\theta))$  for all  $\theta \in [\tilde{\theta} - \sigma, \tilde{\theta} + \sigma]$ , for otherwise the first term in (72) is zero while its second term is positive.

Next, for  $j = 1, 2, \dots, \infty$ , let  $\theta_{lj}^0 = \tilde{\theta} - \frac{\sigma}{m\sqrt{j}}$  and  $\theta_{uj}^0 = \tilde{\theta} + \frac{\sigma}{m\sqrt{j}}$  be, respectively, the lower and upper bounds of a  $j$ -neighborhood around  $\tilde{\theta}$ , where  $m > 4$  is a constant. Given the interval  $[\theta_{lj}^0, \theta_{uj}^0]$ , we iteratively define the upper bounds and lower bounds of neighborhoods around the chain of targeted types originating from  $\tilde{\theta}$  as follow: for  $k = 1, \dots, s$ ,  $\theta_{lj}^k = \tau(\theta_{lj}^{k-1} + \frac{1}{4\sqrt{j}})$  and  $\theta_{uj}^k = \tau(\theta_{uj}^{k-1} - \frac{1}{4\sqrt{j}})$ . Note that  $\lim_{j \rightarrow \infty} \theta_{lj}^k = \lim_{j \rightarrow \infty} \theta_{uj}^k = \tau^k(\tilde{\theta})$ , with  $\theta_{uj}^k > \theta_{lj}^k$  for large enough  $j$ .

Let us now introduce quantity perturbations for the types in the set  $[\theta_{lj}^s, \theta_{uj}^s]$ . Specifically, for  $\theta \in [\theta_{lj}^s, \theta_{uj}^s]$ , define a sequence  $\epsilon_j(\theta)$  as a solution in  $\epsilon$  to the following equation:

$$[u(q(\theta) + \epsilon, \theta_{uj}^{s-1}) - u(q(\theta) + \epsilon, \theta)] - [u(q(\theta), \theta_{uj}^{s-1}) - u(q(\theta), \theta)] = \frac{1}{j} \quad (73)$$

Note that the left-hand side of (73) is equal to zero when  $\epsilon = 0$  and is increasing in  $\epsilon$ . So, there exists  $N$  such that for all  $j \geq N$  the solution to (73) is well-defined with  $q(\theta) + \epsilon_j(\theta) < q^{fb}(\theta)$  (since  $q(\theta) < q^{fb}(\theta)$ ).

Next consider alternative mechanism  $(\tilde{q}_j(\cdot), \tilde{t}_j(\cdot))$  which differs from the original mechanism  $(q(\cdot), t(\cdot))$  only as follows: for  $\theta \in [\theta_{lj}^s, \theta_{uj}^s]$ ,  $\tilde{q}_j(\theta) = q(\theta) + \epsilon_j(\theta)$  and  $\tilde{t}_j(\theta) = t(\theta) + u(q(\theta) + \epsilon_j(\theta), \theta) - u(q(\theta), \theta)$ , and for  $\theta \in \cup_{k=0}^{s-1} [\theta_{lj}^k, \theta_{uj}^k]$ ,  $\tilde{t}_j(\theta) = t(\theta) - \frac{1}{j}$ .

Let  $\tilde{V}_j(\theta) = u(\tilde{q}_j(\theta), \theta) - t_j(\theta)$  be the net utility of (truth-telling) type  $\theta$  in the mechanism  $(\tilde{q}_j(\cdot), \tilde{t}_j(\cdot))$ . Note that  $\tilde{V}_j(\theta) = V(\theta) + \frac{1}{j}$  for  $\theta \in \cup_{k=0}^{s-1} [\theta_{lj}^k, \theta_{uj}^k]$ , and  $\tilde{V}_j(\theta) = V(\theta)$  for all other types  $\theta$ . So,  $IR$  constraints hold for all  $\theta \in [0, 1]$  in  $(\tilde{q}_j(\cdot), \tilde{t}_j(\cdot))$ .

We will now show that when  $j$  is sufficiently large the mechanism  $(\tilde{q}_j(\cdot), \tilde{t}_j(\cdot))$  is incentive compatible and strictly more profitable for the principal than the original mechanism  $(q(\cdot), t(\cdot))$ .

### Step 2. Establishing incentive compatibility of the alteranative mechanism.

We will show that incentive constraints in the mechanism  $(\tilde{q}_j(\cdot), \tilde{t}_j(\cdot))$ , denoted by  $IC^j(\theta, \theta')$ , hold for all  $(\theta, \theta') \in [0, 1]^2$ . The argument is given separately for several subsets of  $[0, 1]^2$ .

First, if  $\theta \in [0, 1]$  and  $\theta' \notin \cup_{k=0, \dots, s} [\theta_{lj}^k, \theta_{uj}^k]$ , then  $IC^j(\theta, \theta')$  holds because  $\tilde{V}_j(\theta) \geq V(\theta)$ ,  $\tilde{q}_j(\theta') = q_j(\theta')$ ,  $\tilde{t}_j(\theta') = t_j(\theta')$  and  $IC(\theta, \theta')$  holds.

Second, if  $\theta \in [0, 1]$  and  $\theta' \in [\theta_{lj}^0, \theta_{uj}^0]$ , then  $\tau^{-1}(\theta') = \emptyset$  since  $[\theta_{lj}^0, \theta_{uj}^0] \subset (\theta, \bar{\theta}) \subset [\tau(1), 1]$ . So, in the original mechanism incentive constraints  $IC(\theta, \theta')$  are slack on this set of types, with minimal slack  $\delta > 0$  over all  $\theta \in [0, 1]$  and all  $\theta' \in [\theta_{lj}^0, \theta_{uj}^0]$ . In the mechanism,  $(\tilde{q}_j(\cdot), \tilde{t}_j(\cdot))$ ,  $\tilde{V}_j(\theta) \geq V(\theta)$  for all  $\theta \in [0, 1]$ , and  $\tilde{V}_j(\theta') = V(\theta') + \frac{1}{j}$  for  $\theta' \in [\theta_{lj}^0, \theta_{uj}^0]$ . Therefore,  $IC^j(\theta, \theta')$  holds for sufficiently large  $j$  i.e., when  $\frac{1}{j} \leq \delta$ .

Third, consider  $IC(\theta, \theta')$  where  $\theta \in [\theta_{lj}^{s-1}, \theta_{uj}^{s-1}]$  and  $\theta' \in [\theta_{lj}^s, \theta_{uj}^s]$ . Recall that  $U(\theta, \theta') = u(q(\theta'), \theta) - t(\theta') - C$ . So, when  $j$  is sufficiently large, we have:

$$\begin{aligned} \tilde{V}_j(\theta) &= V(\theta) + \frac{1}{j} \geq U(\theta, \theta') + \frac{1}{j} \\ &= u(q(\theta'), \theta) - t(\theta') - C + [u(\tilde{q}_j(\theta'), \theta_{uj}^{s-1}) - u(\tilde{q}_j(\theta'), \theta')] - [u(q(\theta'), \theta_{uj}^{s-1}) - u(q(\theta'), \theta')] \geq \\ &u(q(\theta'), \theta) - t(\theta') - C + [u(\tilde{q}_j(\theta'), \theta) - u(\tilde{q}_j(\theta'), \theta')] - [u(q(\theta'), \theta) - u(q(\theta'), \theta')] = u(\tilde{q}_j(\theta'), \theta) - \tilde{t}_j(\theta') - C \end{aligned}$$

where the first equality holds by definition of  $\tilde{t}_j(\theta)$ , the first inequality holds by incentive compatibility of the original mechanism, the second equality holds by definition of  $\tilde{q}_j(\theta')$  and  $\epsilon_j(\theta')$ , the second inequality holds because  $\tilde{q}_j(\theta') > q(\theta')$ ,  $\theta \leq \theta_{uj}^{s-1}$  and  $u_{\theta q} > 0$ , and the last inequality holds by definition of  $\tilde{t}_j(\theta')$ .

Fourth, if  $\theta \in \cup_{k=0}^{s-1} [\theta_{lj}^k, \theta_{uj}^k]$  and  $\theta' \in \cup_{k=1}^{s-1} [\theta_{lj}^k, \theta_{uj}^k]$ , then  $\tilde{V}_j(\theta) = V(\theta) + \frac{1}{j}$  and  $\tilde{V}_j(\theta') = V(\theta') + \frac{1}{j}$  since both  $\theta$  and  $\theta'$  get the same quantity as in the original mechanism but their transfer is decreased by  $\frac{1}{j}$  in  $(\tilde{q}_j(\cdot), \tilde{t}_j(\cdot))$ . Therefore,  $IC^j(\theta, \theta')$  holds because  $IC(\theta, \theta')$  holds.

Fifth, to show that  $IC^j(\theta, \theta')$  holds for: (i)  $\theta \notin \cup_{k=0}^{s-1} [\theta_{lj}^k, \theta_{uj}^k]$  and  $\theta' \in \cup_{k=1}^{s-1} [\theta_{lj}^k, \theta_{uj}^k]$ , and (ii)  $\theta \notin [\theta_{lj}^{s-1}, \theta_{uj}^{s-1}]$  and  $\theta' \in [\theta_{lj}^s, \theta_{uj}^s]$ , we need to establish the following Claim.

**Claim 1.** *There exists  $\underline{a} > 0$  such that for any  $j, k = 0, \dots, s-1$ ,  $\theta \in [0, 1] \setminus (\theta_{lj}^k, \theta_{uj}^k)$  and  $\theta' \in [\theta_{lj}^{k+1}, \theta_{uj}^{k+1}]$ , in the original mechanism  $V(\theta) \geq U(\theta, \theta') + \frac{\underline{a}}{\sqrt{j}}$ .*

First, suppose that  $\theta \geq \theta_{uj}^k$ . Then we have:

$$\begin{aligned} V(\theta) - U(\theta, \theta') &\geq U(\theta, \tau(\theta_{uj}^k)) - U(\theta, \theta') = \int_{\theta'}^{\tau(\theta_{uj}^k)} u_q(q(x), \theta) \dot{q}(x) - \dot{t}(x) dx \\ &= \int_{\theta'}^{\tau(\theta_{uj}^k)} [u_q(q(x), \theta) - u_q(q(x), \tau^{-1}(x))] \dot{q}(x) dx = \int_{\theta'}^{\tau(\theta_{uj}^k)} \int_{\tau^{-1}(x)}^{\theta} u_{\theta q}(q(x), r) \dot{q}(x) dr dx \quad (74) \end{aligned}$$

The inequality holds because  $IC(\theta, \tau(\theta_{u_j}^k))$  holds in the original mechanism, the first equality holds by definition, the second equality uses the first-order condition (5) which holds because  $\tau^{-1}(x) \in [\theta_{l_j}^k, \theta_{u_j}^k] \subset [\tau^k(\underline{\theta}), \tau^k(\bar{\theta})]$  where  $\tau$  is single-valued and continuous.

By Lemma 23,  $q$  is strictly increasing on  $(0,1)$ , so there exists  $\underline{q} > 0$  such that  $\dot{q}(x) \geq \underline{q}$  for all  $x \in [\theta', \tau(\theta_{u_j}^k)]$ . By Assumption 1,  $u_{\theta q} > \underline{K} > 0$ . Therefore, using (74) we obtain:

$$\begin{aligned} V(\theta) - U(\theta, \theta') &\geq \underline{K}\underline{q} \int_{\theta'}^{\tau(\theta_{u_j}^k)} (\theta - \tau^{-1}(x)) dx \geq \underline{K}\underline{q} \int_{\tau(\theta_{u_j}^k - \frac{1}{4\sqrt{j}})}^{\tau(\theta_{u_j}^k)} (\theta_{u_j}^k - \tau^{-1}(x)) dx \\ &= \underline{K}\underline{q} \int_{\theta_{u_j}^k - \frac{1}{4\sqrt{j}}}^{\theta_{u_j}^k} \dot{\tau}(y)(\theta_{u_j}^k - y) dy \geq \frac{\underline{K}\underline{q}\dot{\tau}}{\sqrt{j}} = \frac{\underline{a}}{\sqrt{j}}, \end{aligned} \quad (75)$$

where the first inequality follows from (74), the second inequality holds because  $\theta \geq \theta_{u_j}^k$ ,  $\theta' \leq \theta_{u_j}^{k+1} = \tau(\theta_{u_j}^k - \frac{1}{4\sqrt{j}})$ , and  $(\theta_{u_j}^k - \tau^{-1}(x)) > 0$  on the range of integration. The first equality holds by a change of variable of integration  $x = \tau(y)$ . The third inequality holds because  $\tau$  is continuous and strictly increasing on  $[\tau^k(\underline{\theta}), \tau^k(\bar{\theta})]$ , and so  $\dot{\tau}(x) \geq \underline{\dot{\tau}}$  for some  $\underline{\dot{\tau}} > 0$ . Finally, the last equality holds as we set  $\underline{a} = \frac{1}{2}\underline{K}\underline{q}\underline{\dot{\tau}} > 0$ .

A symmetrical argument establishes that the Claim also holds for  $\theta \leq \theta_{l_j}^k$ . ■

Now, we will use Claim 1 to establish that  $IC^j(\theta, \theta')$  holds in the remaining two cases.

First, if  $\theta \notin \cup_{k=0}^{s-1} [\theta_{l_j}^k, \theta_{u_j}^k]$  and  $\theta' \in \cup_{k=1}^{s-1} [\theta_{l_j}^k, \theta_{u_j}^k]$ , we have  $\tilde{V}_j(\theta) = V(\theta)$ ,  $\tilde{t}_j(\theta') = t(\theta') - \frac{1}{j}$ , and  $\tilde{q}_j(\theta') = q(\theta')$ . Combining this with  $V(\theta) \geq U(\theta, \theta') + \frac{\underline{a}}{\sqrt{j}}$  yields  $\tilde{V}(\theta) \geq U(\theta, \theta') - \frac{1}{j} + \frac{\underline{a}}{\sqrt{j}}$  i.e.,  $IC^j(\theta, \theta')$  holds for sufficiently large  $j$ .

Second, for  $\theta \notin [\theta_{l_j}^{s-1}, \theta_{u_j}^{s-1}]$  and  $\theta' \in [\theta_{l_j}^s, \theta_{u_j}^s]$ , note by (73) that there exists some  $\bar{a} < \infty$  such that

$$\begin{aligned} &\lim_{j \rightarrow \infty} j[(u(q(\theta') + \epsilon_j(\theta'), \theta) - u(q(\theta') + \epsilon_j(\theta'), \theta')) - (u(q(\theta'), \theta) - u(q(\theta'), \theta'))] \\ &= \lim_{j \rightarrow \infty} \frac{[(u(q(\theta') + \epsilon_j(\theta'), \theta) - u(q(\theta') + \epsilon_j(\theta'), \theta')) - (u(q(\theta'), \theta) - u(q(\theta'), \theta'))]}{[u(q(\theta') + \epsilon_j(\theta'), \theta_{u_j}^{s-1}) - u(q(\theta') + \epsilon_j(\theta'), \theta')] - [u(q(\theta'), \theta_{u_j}^{s-1}) - u(q(\theta'), \theta')]} \\ &= \frac{u_q(q(\theta'), \theta') - u_q(q(\theta'), \theta')}{u_q(q(\theta'), \theta_{u_j}^{s-1}) - u_q(q(\theta'), \theta')} < \bar{a} \end{aligned} \quad (76)$$

Therefore, for sufficiently large  $j$ ,

$$\begin{aligned}
\tilde{V}_j(\theta) &\geq V(\theta) \geq U(\theta, \theta') + \frac{a}{\sqrt{j}} = \\
u(\tilde{q}_j(\theta'), \theta) - \tilde{t}_j(\theta') - C + \frac{a}{\sqrt{j}} &- [u(q(\theta') + \epsilon_j(\theta'), \theta) - u(q(\theta'), \theta) - u(q(\theta') + \epsilon_j(\theta'), \theta') + u(q(\theta'), \theta')] \\
&\geq u(\tilde{q}_j(\theta'), \theta) - \tilde{t}_j(\theta') - C + \frac{a}{\sqrt{j}} - \frac{\bar{a}}{j} > u(\tilde{q}_{ij}(\theta'), \theta) - \tilde{t}_{ij}(\theta') - C
\end{aligned}$$

where the first inequality holds by construction, the second inequality holds by Claim 1, the first equality hold by definition of  $(\tilde{q}_j(\theta'), \tilde{t}_j(\theta'))$ , the third inequality holds for sufficiently large  $j$  by (76), and the last inequality holds for sufficiently large  $j$ . This completes the proof of the incentive compatibility of the mechanism  $(\tilde{q}_j(\cdot), \tilde{t}_j(\cdot))$  for sufficiently large  $j$ .

**Step 2. Establishing that the the mechanism  $(\tilde{q}_j(\cdot), \tilde{t}_j(\cdot))$  is is more profitable for the principal than the original mechanism.**

We start by proving the following.

**Claim 2.** For any  $k = 0, \dots, s$  and  $i = u, l$ ,  $\lim_{j \rightarrow \infty} \frac{1}{\sqrt{j}} |\tau^{-(s-k)}(\theta_{ij}^s) - \theta_{ij}^k| = 0$ .

We will establish this for  $i = l$ . The other cases follows by symmetric arguments. For large  $j$ , we have

$$\begin{aligned}
\theta_{lj}^s &= \tau(\theta_{lj}^{s-1} + \frac{1}{\sqrt{j}}) \\
&= \tau(\tau(\dots \tau(\theta_{lj}^k + \frac{1}{\sqrt{j}}) \dots) + \frac{1}{\sqrt{j}}) \\
&\approx \tau(\tau(\dots \tau(\theta_{lj}^k) + \frac{1}{\sqrt{j}} \dot{\tau}(\tilde{\theta}^k) \dots) + \frac{1}{\sqrt{j}}) \\
&\approx \tau(\tau(\dots \tau^2(\theta_{lj}^k) + \frac{1}{\sqrt{j}} \dot{\tau}(\tilde{\theta}^{k+1})(\dot{\tau}(\tilde{\theta}^k) + 1)) \dots) + \frac{1}{\sqrt{j}}) \\
&\approx \tau^{s-k}(\theta_{lj}^k) + \frac{1}{\sqrt{j}} [1 + \dot{\tau}(\tilde{\theta}^{s-1}) + \dot{\tau}(\tilde{\theta}^{s-1})\dot{\tau}(\tilde{\theta}^{s-2}) + \dots + \prod_{r=k}^{s-1} \dot{\tau}(\tilde{\theta}^r)]
\end{aligned}$$

Therefore,

$$\begin{aligned}
\tau^{-(s-k)}(\theta_{ij}^s) - \theta_{ij}^k &= \tau^{-(s-k)}(\theta_{ij}^s) - \tau^{-(s-k)}(\tau^{s-k}(\theta_{ij}^k)) \\
&\approx \dot{\tau}^{-(s-k)}(\tilde{\theta}^k)[\theta_{ij}^s - \tau^{s-k}(\theta_{ij}^k)] \\
&\approx \dot{\tau}^{-(s-k)}(\tilde{\theta}^k) \frac{1}{\sqrt[4]{j}} [1 + \dot{\tau}(\tilde{\theta}^{s-1}) + \dot{\tau}(\tilde{\theta}^{s-1})\dot{\tau}(\tilde{\theta}^{s-2}) + \dots + \prod_{r=k}^{s-1} \dot{\tau}(\tilde{\theta}^r)] \\
&= \frac{1}{\sqrt[4]{j}} \frac{1}{\prod_{r=k}^{s-1} \dot{\tau}(\tilde{\theta}^r)} [1 + \dot{\tau}(\tilde{\theta}^{s-1}) + \dot{\tau}(\tilde{\theta}^{s-1})\dot{\tau}(\tilde{\theta}^{s-2}) + \dots + \prod_{r=k}^{s-1} \dot{\tau}(\tilde{\theta}^r)]
\end{aligned}$$

Since for each  $k$ ,  $\tilde{\theta}^k \equiv \tau^k(\tilde{\theta}) \in [\tau^k(\underline{\theta}), \tau^k(\bar{\theta})]$  where  $\tau$  is continuous and strictly increasing, it follows that  $\dot{\tau}(\tilde{\theta}^k)$  is bounded below from 0 and bounded above. Therefore,

$$\lim_{j \rightarrow \infty} \sqrt[4]{j} |\tau^{-(s-k)}(\theta_{ij}^s) - \theta_{ij}^k| = \lim_{j \rightarrow \infty} \frac{j^{(\frac{1}{m} - \frac{1}{4})}}{\prod_{r=k}^{s-1} \dot{\tau}(\tilde{\theta}^r)} [1 + \dot{\tau}(\tilde{\theta}^{s-1}) + \dot{\tau}(\tilde{\theta}^{s-1})\dot{\tau}(\tilde{\theta}^{s-2}) + \dots + \prod_{r=k}^{s-1} \dot{\tau}(\tilde{\theta}^r)] = 0$$

as  $m > 4$ . ■

The change in profit in the new mechanism is equal to

$$\begin{aligned}
\Pi_j &= \int_{\theta_{ij}^s}^{\theta_{uj}^s} [u(q(\theta) + \epsilon_j(\theta), \theta) - u(q(\theta), \theta)] f(\theta) d\theta - \frac{1}{j} \sum_{k=0}^{s-1} \int_{\theta_{ij}^k}^{\theta_{uj}^k} f(\theta) d\theta \\
&= \int_{\tau^{-(s-k)}(\theta_{ij}^s)}^{\tau^{-(s-k)}(\theta_{uj}^s)} [u(q(\tau^s(\theta)) + \epsilon_j(\tau^s(\theta)), \tau^s(\theta)) - u(q(\tau^s(\theta)), \tau^s(\theta))] f(\tau^s(\theta)) \dot{\tau}^s(\theta) - \frac{1}{j} \sum_{k=0}^{s-1} f(\tau^k(\theta)) \dot{\tau}^k(\theta) d\theta \\
&\quad - \frac{1}{j} \Omega_j \\
&\approx \frac{1}{j} \int_{\tau^{-(s-k)}(\theta_{ij}^s)}^{\tau^{-(s-k)}(\theta_{uj}^s)} \frac{u_q(q(\tau^s(\theta)), \tau^s(\theta))}{u_q(q(\tau^s(\theta), \theta_{uj}^{s-1})) - u_q(q(\tau^s(\theta)), \tau^s(\theta))} f(\tau^s(\theta)) \dot{\tau}^s(\theta) - \sum_{k=0}^{s-1} f(\tau^k(\theta)) \dot{\tau}^k(\theta) d\theta \\
&\quad - \frac{1}{j} \Omega_j \tag{77}
\end{aligned}$$

where  $\Omega_j = \sum_{k=0}^{s-1} [F(\theta_{uj}^k) - F(\tau^{-(s-k)}(\theta_{uj}^s))] + [F(\tau^{-(s-k)}(\theta_{ij}^s)) - F(\theta_{ij}^k)]$ , the second equality is derived using change of variables, and the approximation comes from the definition of  $\epsilon_j$  at

(73) and  $\lim_{j \rightarrow \infty} \epsilon_j = 0$ . Given inequality (72), we have

$$\begin{aligned}
& \lim_{j \rightarrow \infty} j^{1+\frac{1}{m}} \Pi_j \\
&= j^{\frac{1}{m}} \int_{\tau^{-s}(\theta_{l_j}^s)}^{\tau^{-s}(\theta_{u_j}^s)} \frac{u_q(q(\tau^s(\theta)), \tau^s(\theta))}{u_q(q(\tau^s(\theta), \theta_{u_j}^{s-1}) - u_q(q(\tau^s(\theta)), \tau^s(\theta)))} f(\tau^s(\theta)) \dot{\tau}^s(\theta) - \sum_{k=0}^{s-1} f(\tau^k(\theta)) \dot{\tau}^k(\theta) d\theta - j^{\frac{1}{m}} \Omega_j \\
&> j^{\frac{1}{m}} \int_{\tau^{-s}(\theta_{l_j}^s)}^{\tau^{-s}(\theta_{u_j}^s)} \left[ \frac{u_q(q(\tau^s(\theta)), \tau^s(\theta))}{u_q(q(\tau^s(\theta), \theta_{u_j}^{s-1}) - u_q(q(\tau^s(\theta)), \tau^s(\theta)))} - \frac{u_q(q(\tau^s(\theta)), \tau^s(\theta))}{u_q(q(\tau^s(\theta), \tau^{s-1}(\theta)) - u_q(q(\tau^s(\theta)), \tau^s(\theta)) + \tau^s(\theta))} \right] \\
&\quad f(\tau^s(\theta)) \dot{\tau}^s(\theta) d\theta \\
&\quad + j^{\frac{1}{m}} [\tau^{-s}(\theta_{u_j}^s) - \tau^{-s}(\theta_{l_j}^s)] \mu - j^{\frac{1}{m}} \Omega_j \tag{78}
\end{aligned}$$

Claim 1 and continuity of  $F(\cdot)$  imply  $\lim_{j \rightarrow \infty} j^{\frac{1}{m}} \Omega_j = 0$ . Claim 1 also implies  $\lim_{j \rightarrow \infty} j^{\frac{1}{m}} [\tau^{-s}(\theta_{u_j}^s) - \tau^{-s}(\theta_{l_j}^s)] = \lim_{j \rightarrow \infty} j^{\frac{1}{m}} [\theta_{u_j}^0 - \theta_{l_j}^0] = 2\sigma$ . Finally, for any  $\theta \in [\tau^{-s}(\theta_{u_j}^s) - \tau^{-s}(\theta_{l_j}^s)]$ ,  $\lim_{j \rightarrow \infty} \tau^{s-1}(\theta) = \lim_{j \rightarrow \infty} \theta_{u_j}^{s-1} = \tilde{\theta}^{s-1}$ , which means the term in the integral of (78) converges to 0. It implies

$$\lim_{j \rightarrow \infty} j^{1+\frac{1}{m}} \Pi_j = 2\sigma\mu > 0$$

change of profit is positive for large enough  $j$ , which contradicts the optimality of the original mechanism.

*Q.E.D.*

Lemma 25 shows that for a range of  $C$ , any type  $\theta \in \tau([0, 1])$  gets zero surplus.

**Lemma 25** *There exists  $\underline{C} \in (0, \bar{C})$ , such that in the optimal mechanism for any  $C \in [\underline{C}, \bar{C}]$  we have: if  $\theta' \in \tau([0, 1])$  then  $V(\theta') = 0$ .*

**Proof of Lemma 25:** Recall from (68) that  $G(\theta, \theta') = u(q^{fb}(\theta'), \theta) - u(q^{fb}(\theta), \theta')$ , and

$$\bar{C} \equiv G^*(\theta) = \max_{\theta'} G(\theta', \theta)$$

Since  $G(\cdot, \cdot)$  is continuous in both arguments and  $u_\theta > 0$ ,  $G^*(\theta)$  is continuous and strictly increasing.

Define

$$\hat{\Theta}(C) = \{\theta \in [0, 1] : G^*(\theta) - C \geq 0\}$$

Then for any  $C \in (0, \bar{C})$  the set  $\hat{\Theta}(C)$  is non-empty. Furthermore, since  $G^*(\theta)$  is continuous and strictly increasing in  $\theta$ , there exists  $\theta^C \in (0, 1)$  such that  $\hat{\Theta}(C) = [\theta^C, 1]$ , with  $\lim_{C \rightarrow \bar{C}} \theta^C \rightarrow 1$ .

Next, let us show that there exists  $\underline{C} \in (0, \bar{C})$  such that whenever  $C \in (\underline{C}, \bar{C})$ :

- (i)  $V(\theta) = 0$  for all  $\theta \notin \hat{\Theta}(C)$ ,
- (ii)  $\hat{\Theta}(C) \cap \tau(\hat{\Theta}(C)) = \emptyset$ .

To establish (i), suppose that  $V(\theta) > 0$  for some  $\theta \notin \hat{\Theta}(C)$ . Then consider an alternative mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  which differs from the original mechanism  $(q(\cdot), t(\cdot))$  only in transfers. Particularly,  $\tilde{t}(\theta) = u(q(\theta), \theta)$  for  $\theta \notin \hat{\Theta}(C)$  and  $\tilde{t}(\theta) = \max\{u(q(\theta), \theta^C) - C, t(\theta)\}$  for  $\theta \in \hat{\Theta}(C)$ .

The mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  is (weakly) more profitable for the seller than  $(q(\cdot), t(\cdot))$ . So we only need to verify that  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  satisfies *IR* and *IC* constraints. For  $\theta \notin \hat{\Theta}(C)$ , *IR*( $\theta$ ) is binding by construction. If  $\theta \in \hat{\Theta}(C)$  and  $\tilde{t}(\theta) = t(\theta)$  then *IR*( $\theta$ ) holds in  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  because it holds in  $(q(\cdot), t(\cdot))$ . If  $\tilde{t}(\theta) = u(q(\theta), \theta^C) - C > t(\theta)$ , then  $\theta$  gets a payoff  $u(q(\theta), \theta) - u(q(\theta), \theta^C) + C > 0$ . So all *IR* constraints hold.

Now consider *IC* constraints. Fix any pair  $(\theta, \theta') \in ([0, 1] \setminus \hat{\Theta}(C)) \times \hat{\Theta}(C)$ . *IC*( $\theta, \theta'$ ) holds in the mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  because by reporting her type truthfully type  $\theta$  gets zero payoff. At the same time,  $\tilde{t}(\theta') \geq u(q(\theta'), \theta^C) - C$ , and so type  $\theta$ 's payoff from imitating  $\theta$  does not exceed  $u(q(\theta'), \theta) - u(q(\theta'), \theta^C) \leq 0$ .

Now fix any pair  $(\theta, \theta') \in \hat{\Theta}(C) \times ([0, 1] \setminus \hat{\Theta}(C))$ . If  $\tilde{t}(\theta) = t(\theta)$ , then *IC*( $\theta, \theta'$ ) holds in  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  because it holds in  $(q(\cdot), t(\cdot))$  and  $\tilde{t}(\theta') \geq t(\theta')$ . Now suppose that  $\tilde{t}(\theta) = u(q(\theta), \theta^C) - C > t(\theta)$ . Then *IC*( $\theta, \theta'$ ) holds iff  $u(q(\theta), \theta) - u(q(\theta), \theta^C) + C \geq u(q(\theta'), \theta) - u(q(\theta'), \theta') - C$ . Note that  $q(\theta') \leq q^{fb}(\theta')$ . So, this inequality holds if  $C \geq \frac{\bar{C}}{2}$ .

Next, *IC*( $\theta, \theta'$ ) holds for any pair  $(\theta, \theta') \in ([0, 1] \setminus \hat{\Theta}(C)) \times ([0, 1] \setminus \hat{\Theta}(C))$ , because  $q(\theta') \leq q^{fb}(\theta')$  and so, by definition of  $C$

$$u(q(\theta'), \theta) - \tilde{t}(\theta') - C = u(q(\theta'), \theta) - u(q(\theta'), \theta') - C \leq u(q^{fb}(\theta'), \theta) - u(q^{fb}(\theta'), \theta') - C \leq 0$$

Finally, consider a pair  $(\theta, \theta') \in \hat{\Theta}(C) \times \hat{\Theta}(C)$ . If  $\tilde{t}(\theta) = t(\theta)$ , then *IC*( $\theta, \theta'$ ) holds in  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$  because it holds in  $(q(\cdot), t(\cdot))$  and  $\tilde{t}(\theta') \geq t(\theta')$ .

Now suppose that  $\tilde{t}(\theta) = u(q(\theta), \theta^C) - C > t(\theta)$ . Since  $\tilde{t}(\theta') \geq u(q(\theta'), \theta^C) - C$ , *IC*( $\theta, \theta'$ ) holds if  $u(q(\theta), \theta) - u(q(\theta), \theta^C) + C \geq u(q(\theta'), \theta) - u(q(\theta'), \theta^C)$ . This inequality clearly holds if  $q(\theta) \geq q(\theta')$ . Now, if  $q(\theta) < q(\theta')$ , let us rewrite the last inequality as follows:

$$C \geq u(q(\theta'), \theta) - u(q(\theta), \theta) - (u(q(\theta'), \theta^C) - u(q(\theta), \theta^C)) \quad (79)$$

Since  $q(\cdot)$  is continuous in  $\theta$  by Lemma 5 The right-hand side of inequality (79) converges to zero as  $C$  increase to  $\bar{C}$ . So the inequality (79) holds strictly when  $C$  is sufficiently close to  $\bar{C}$ . This completes the proof of incentive compatibility of the mechanism  $(\tilde{q}(\cdot), \tilde{t}(\cdot))$ , and hence of statement

Next, let us establish claim (ii):  $\hat{\Theta}(C) \cap \tau(\hat{\Theta}(C)) = \emptyset$ . The proof is by contradiction, so suppose there exists  $\theta' \in \hat{\Theta}(C) \cap \tau(\hat{\Theta}(C))$ . Then there exists  $\theta \in \hat{\Theta}(C)$  such that  $IC(\theta, \theta')$  is binding. But as we have shown above, this does hold in the optimal mechanism when  $C$  is close to  $\bar{C}$ . In particular, in this case (79) holds strictly. A contradiction. *Q.E.D.*

**Lemma 26** *Let  $\bar{W} = \max_{q, \theta} u_{\theta q}(q, \theta) \times q^{fb}(1)$ . Then  $\theta - \tau(\theta) \geq \frac{C}{\bar{W}}$  for all  $\theta$ . So,  $\tau(\theta) = \emptyset$  when  $C \geq \bar{W}$ .*

**Proof of Lemma 26:** Take any  $\theta$  such that  $\tau(\theta) \neq \emptyset$ . By definition of  $\tau(\cdot)$ , we have

$$u(q(\tau(\theta)), \theta) - u(q(\tau(\theta)), \tau(\theta)) = C + V(\theta) - V(\tau(\theta))$$

Using  $V(\theta) = \int_{\hat{\theta}}^{\theta} u_{\theta}(q(\tau(s)), s) ds$  in the above equation and rearranging yields:

$$\int_{\tau(\theta)}^{\theta} u_{\theta}(q(\tau(\theta)), s) ds - \int_{\tau(\theta)}^{\theta} u_{\theta}(q(\tau(s)), s) ds = \int_{\tau(\theta)}^{\theta} \int_{q(\tau(s))}^{q(\tau(\theta))} u_{\theta q}(q, s) dq ds = C \tag{80}$$

Since  $q(\theta) \leq q^{fb}(1)$  for all  $\theta$  and  $u_{\theta q} \leq \bar{K}$ , the previous equation implies that

$$\theta - \tau(\theta) \geq \frac{C}{\bar{K} q^{fb}(1)} \equiv \frac{C}{\bar{W}},$$

which establishes the claim of the Lemma. *Q.E.D.*

## 8 Appendix B

In this Appendix we provide proof to Lemmas 1, 2 , Theorem 6, 7 and 8.

**Proof of Lemma 1:** First, let us rewrite the problem (1)-(3) as the following equivalent problem using the net payoff function  $V(\theta) = u(q(\theta), \theta) - t(\theta)$ :

$$\max_{q(\theta), V(\theta)} \int_0^1 [u(q(\theta), \theta) - V(\theta)] f(\theta) d\theta \tag{81}$$



subject to:

$$V(\theta) - V(\theta') \geq u(q(\theta'), \theta) - u(q(\theta'), \theta') - C \quad \forall \theta, \theta' \in [0, 1] \quad (82)$$

$$V(\theta) \geq 0 \quad \forall \theta \in [0, 1] \quad (83)$$

$$q(\theta) \geq 0 \quad \forall \theta \in [0, 1] \quad (84)$$

By Lemmas 4, 5 and 14, we can without loss of generality restrict  $q(\cdot)$  to belong to the space of continuous functions from  $[0, 1]$  to  $[0, q^{fb}(1)]$  and  $V(\cdot)$  to belong to the space of continuous functions from  $[0, 1]$  to  $[0, u(q^{fb}(1), 1)]$ . Let  $K = \max\{q^{fb}(\theta), u(q^{fb}(1), 1)\}$  and let  $C([0, 1])^{[0, K]}$  be the space of continuous functions from  $[0, 1]$  to  $[0, K]$ .

Endow  $C([0, 1])^{[0, K]}$  with weak-\* topology.<sup>2</sup> By Alaoglu Theorem the space  $C([0, 1])^{[0, K]}$  is compact in the weak\* topology, and by Tychonoff's Theorem the product  $C([0, 1])^{[0, K]} \times C([0, 1])^{[0, K]}$  is compact in the product topology generated by the weak\* topology. Further, for every value of the fixed cost  $C$ , the set of functions  $(q(\cdot), V(\cdot)) \in C([0, 1])^{[0, K]} \times C([0, 1])^{[0, K]}$  that satisfy the constraints (82)-(84) is a closed subset of  $C([0, 1])^{[0, K]} \times C([0, 1])^{[0, K]}$ , and is therefore compact in the product topology generated by the weak\* topology. Also, this set varies continuously with the fixed costs  $C$ . Thus, the correspondence  $\{(q(\cdot), V(\cdot)) \in C([0, 1])^{[0, K]} \times C([0, 1])^{[0, K]} : (q(\cdot), V(\cdot)) \text{ satisfy (82)-(84)}\}$  specifying the set of admissible quantity and surplus functions for fixed cost  $C$  is continuous in  $C$  and compact valued.

Let  $(q(\cdot|C), V(\cdot|C))$  be the solution to problem (81)-(84). By Theorem 2 the solution exists and is unique. Since the objective function (81) is continuous in  $q(\cdot)$ ,  $V(\cdot)$  and  $C$ , by Berge's Maximum Theorem  $(q(\cdot|C), V(\cdot|C))$  is upper hemicontinuous in  $C$ . This implies that  $\lim_{C \downarrow 0} (q(\theta|C), V(\theta|C)) = (q(\theta|0), V(\theta|0)) \equiv (q^{sb}(\theta), V^{sb}(\theta))$  for all  $\theta \in [0, 1]$ .

Further,  $(q^{sb}(\theta), V^{sb}(\theta)) = (q(\theta|0), V(\theta|0))$  is the standard second-best solution to our problem for  $C = 0$ . Note that  $q^{sb}(\theta)$  is continuous and  $q^{sb}(0) = 0 < q^{sb}(1) = q^{fb}(1)$ . Therefore, there exist  $\underline{\theta}, \bar{\theta} \in [0, 1]$ ,  $\underline{\theta} < \bar{\theta}$ , such that  $q^{sb}(\theta)$  is strictly increasing and  $V^{sb}(\theta) > 0$  on  $[\underline{\theta}, \bar{\theta}]$ . Since  $\lim_{C \downarrow 0} V(\theta|C) = V^{sb}(\theta) > 0$  for  $\theta \in [\underline{\theta}, \bar{\theta}]$ , Lemma 7 implies that there exists  $\widehat{C} > 0$  such that  $\tau(\theta|C) \neq \emptyset$  for all  $C \in (0, \widehat{C})$  and  $\theta \in [\underline{\theta}, \bar{\theta}]$ .

---

<sup>2</sup>A sequence  $x_n(\theta)$  converges to  $x(\theta)$  in the weak\* topology iff  $\int_0^1 x_n(\theta)y(\theta)dF(\theta) \rightarrow \int_0^1 x(\theta)y(\theta)dF(\theta)$  for all  $y \in L^2(F)$ .

Now to show that  $\lim_{C \downarrow 0} M(C) = \infty$ , fix any pair  $(\theta, \theta')$  s.t.  $\theta \in (\underline{\theta}, \bar{\theta}]$  and  $\theta' < \theta$ , and consider the corresponding incentive constraint (82). Putting all terms on one side and taking the limit as  $C \rightarrow 0$  we get:

$$\begin{aligned} & \lim_{C \downarrow 0} (V(\theta|C) - V(\theta'|C) + C - u(q(\theta'|C), \theta) + u(q(\theta'|C), \theta')) = \\ & V^{sb}(\theta) - V^{sb}(\theta') - u(q^{sb}(\theta'), \theta) - u(q^{sb}(\theta'), \theta') = \int_{\theta'}^{\theta} u_{\theta}(q^{sb}(s), s) ds - u(q^{sb}(\theta'), \theta) + u(q^{sb}(\theta'), \theta') > 0 \end{aligned}$$

where the last inequality holds because  $q^{sb}(\cdot)$  is increasing, strictly on  $(\underline{\theta}, \bar{\theta})$ . So, for any  $\theta \in (\underline{\theta}, \bar{\theta}]$  and  $\theta' < \theta$  we have  $\tau(\theta|C) > \theta'$  when  $C$  is sufficiently small. Hence,  $\lim_{C \downarrow 0} \tau(\theta|C) = \theta$  for  $\theta \in [\underline{\theta}, \bar{\theta}]$ .

Finally, fix some integer  $M > 0$  and let  $\epsilon_M = \frac{\bar{\theta} - \underline{\theta}}{M}$ . Since  $\lim_{C \downarrow 0} \tau(\theta|C) = \theta$  for  $\theta \in [\underline{\theta}, \bar{\theta}]$ , there exists  $C_M > 0$  such that  $\tau^{k-1}(\bar{\theta}|C) - \tau^k(\bar{\theta}|C) \leq \epsilon_M$  for any  $k = 1, \dots, M$  and hence  $\tau^M(\bar{\theta}|C) \geq \underline{\theta}$  for all  $C \in (0, C_M]$ . By Corollary 3,  $\tau^M(1|C) > \tau^M(\bar{\theta}|C) \geq \underline{\theta}$  for  $C \in (0, C_M]$ . Since  $M$  was chosen arbitrarily, it follows that  $\tau^M(1|C) \neq \emptyset$  for any  $M < \infty$  when  $C$  is sufficiently small i.e.,  $\lim_{C \downarrow 0} M(C) = \infty$ .

*Q.E.D.*

**Proof of Lemma 2:** The proof is by contradiction. So suppose that there exists  $\theta$  and  $\theta_1, \theta_2 \in \tau(\theta)$  such that  $\theta_1 < \theta_2$ . By Lemma 22,  $q(\theta') = q^{fb}(\theta')$  for all  $\theta' \in [\theta_1, \theta_2]$ , while by strict quasi-concavity of  $(G + V)$ ,  $G(\theta, \theta') + V(\theta') > \min\{G(\theta, \theta_1) + V(\theta_1), G(\theta, \theta_2) + V(\theta_1)\}$  for all  $\theta' \in (\theta_1, \theta_2)$ . So,  $u(q^{fb}(\theta'), \theta) - t(\theta') > \min\{u(q^{fb}(\theta_1), \theta) - t(\theta_1), u(q^{fb}(\theta_2), \theta) - t(\theta_2)\}$ , contradicting that  $\theta_1, \theta_2 \in \tau(\theta)$ .

*Q.E.D.*

**Proof of Theorem 6:**

Let  $(q(\theta), t(\theta))$  be an optimal mechanism, which exists and is unique by Theorem 2. Consider the triple  $(\tau(\theta), Q(\theta), \hat{\theta})$  where  $\tau(\theta)$  is defined by (4),  $\hat{\theta} = \max\{\theta : V(\theta) = 0\}$  and  $Q(\theta) = q(\tau(\theta))$  for  $\theta \in [\hat{\theta}, 1]$ . Let us show that the triple  $(\tau(\theta), Q(\theta), \hat{\theta})$  is an increasing solution to the relaxed program.

Since the optimal mechanism is unique,  $\tau(\theta)$  must be strictly increasing by Theorem 4, and  $q(\theta)$  must be strictly increasing by Theorem 3, and so  $Q(\theta) = q(\tau(\theta))$  is also strictly increasing. Since  $C_i \in (\underline{C}, \bar{C})$ , Theorem 5 implies that  $\tau(\theta) < \hat{\theta}$  for all  $\theta \in [\hat{\theta}, 1]$ , and so

$\dot{Q}(\theta) = \frac{u_\theta(Q(\theta), \tau(\theta))}{u_q(Q(\theta), \theta) - u_q(Q(\theta), \tau(\theta))} \dot{\tau}(\theta)$  for all  $\theta \in [\hat{\theta}, 1]$ , which is equivalent to (12).

Consider any  $\theta \in [\hat{\theta}, 1]$ . Then  $\tau(\theta)$  is single-valued by Assumption 2. Also,  $\tau^{-1}(\theta) = \emptyset$  by Theorem 5, and so  $[\tau(\hat{\theta}), \tau(1)] \cup [\hat{\theta}, 1] = \emptyset$ . Therefore, Lemma 24 with  $M = 1$  implies that for all  $\theta \in [\hat{\theta}, 1]$ ,  $u_q(q(\tau(\theta)), \tau(\theta))f(\tau(\theta))\dot{\tau}(\theta) = [u_q(q(\tau(\theta)), \theta) - u_q(q(\tau(\theta)), \tau(\theta))]f(\theta)$ , which is equivalent to equation (39). In combination with  $\dot{Q}(\theta) = \frac{u_\theta(Q(\theta), \tau(\theta))}{u_q(Q(\theta), \theta) - u_q(Q(\theta), \tau(\theta))} \dot{\tau}(\theta)$  this yields (40).

Boundary conditions (20) and (21) hold because by Theorem 4,  $q(\theta) = q^{fb}(\theta)$  for  $\theta \in [0, \tau(\hat{\theta})] \cup [\tau(1), 1]$ . Boundary condition (22) holds because by Theorem 4  $V(\theta) = 0$  for  $\theta \in [0, \hat{\theta}]$  and  $\tau(\hat{\theta}) < \hat{\theta}$ , and therefore  $V(\hat{\theta}) = u(q(\tau(\hat{\theta})), \hat{\theta}) - t(\tau(\hat{\theta})) - C = u(q(\tau(\hat{\theta})), \hat{\theta}) - u(q(\tau(\hat{\theta})), \tau(\hat{\theta})) - C = 0$ .

Finally, let us show that  $\tau(\hat{\theta})$  must be the smallest solution to (22). Conditions (21) and (22) imply  $G(\hat{\theta}, \tau(\hat{\theta})) \equiv u(q^{fb}(\tau(\hat{\theta})), \hat{\theta}) - u(q^{fb}(\tau(\hat{\theta})), \tau(\hat{\theta})) = C$ . By Assumption 2 there are at most two solutions to this equation. If we set  $\tau(\hat{\theta})$  to be equal to the larger solution, then  $G_2(\hat{\theta}, \tau(\hat{\theta})) < 0$ . Therefore, there exists  $\theta' < \tau(\hat{\theta})$ , s.t. if we set  $q(\theta') = q^{fb}(\theta')$  and  $V(\theta') = 0$  it follows that  $u(q(\theta'), \hat{\theta}) - t(\theta') = G(\hat{\theta}, \theta') - C > G(\hat{\theta}, \tau(\hat{\theta})) - C = 0$ , violating  $IC(\hat{\theta}, \theta')$ . Therefore,  $\tau(\hat{\theta})$  must be the smaller solution of (22).

To summarize the above, we have shown that the optimal mechanism induces a triple  $(\tau(\theta), Q(\theta), \hat{\theta})$  which constitutes an increasing solution to the relaxed program. Thus, to complete the proof it is sufficient to show that an increasing solution to the relaxed program is unique. We establish this below via a sequence of Claims.

First, fix some  $\hat{\theta}_i$  and  $C_j$  where  $i, j \in \{1, 2\}$  and let  $\Gamma(\hat{\theta}_i, C_j) = \{\theta' : G(\hat{\theta}_i, \theta') \equiv u(q^{fb}(\theta'), \hat{\theta}_i) - u(q^{fb}(\theta'), \theta') = C_j\}$  i.e.,  $\Gamma(\hat{\theta}_i, C_j)$  is the set of types satisfying the boundary condition (22). Suppose that  $\Gamma(\hat{\theta}_i, C_j) \neq \emptyset$  for  $i, j \in \{1, 2\}$ . Note that  $\Gamma(\hat{\theta}_i, C_j)$  contains at most two elements because  $G(\hat{\theta}_i, \theta') \equiv u(q^{fb}(\theta'), \hat{\theta}_i) - u(q^{fb}(\theta'), \theta')$  is strictly quasi-concave in  $\theta'$  by Assumption 2,

**Claim 1:** *If  $\hat{\theta}_1 > \hat{\theta}_2$  and  $C_1 < C_2$ , then  $\min\Gamma(\hat{\theta}_1, C_j) < \min\Gamma(\hat{\theta}_2, C_j)$  and  $\min\Gamma(\hat{\theta}_i, C_1) < \min\Gamma(\hat{\theta}_i, C_2)$ .*

**Proof of Claim 1:**

Since  $G(\cdot)$  is strictly quasi-concave and  $G(\hat{\theta}_i, \min\Gamma_i) > G(\hat{\theta}_i, 0) = 0$ , it follows that  $G_2(\hat{\theta}_i, \min\Gamma_i) \geq$

0. On the other hand, we have  $G_1(\hat{\theta}_i, \min\Gamma_i) = u_\theta(q^{fb}(\min\Gamma_i), \hat{\theta}_i) > 0$ . The last two inequalities together imply Claim 1. ■

**Claim 2:** *Suppose that there exist  $(\hat{\theta}_1, \hat{\tau}_1)$  and  $(\hat{\theta}_2, \hat{\tau}_2)$  such that for  $i = 1, 2$ ,  $(Q_i(\theta), \tau_i(\theta))$  is an increasing solution to the system of differential equations (39) and (40) on  $[\hat{\theta}_i, 1]$  that satisfies boundary conditions  $\tau_i(\hat{\theta}_i) = \hat{\tau}_i$ ,  $Q_i(\hat{\theta}_i) = q^{fb}(\hat{\tau}_i)$  and  $Q_i(1) = q^{fb}(\tau_i(1))$ . Let  $q_i(\theta) = Q_i(\tau_i^{-1}(\theta))$  for  $\theta \in [\hat{\tau}_i, \tau_i(1)]$ .*

*Then the following “no-crossing” property holds:*

*If there exists  $\theta^\dagger \in [\max\{\hat{\tau}_1, \hat{\tau}_2\}, \min\{\tau_1(1), \tau_2(1)\}]$  such that  $q_2(\theta^\dagger) < q_1(\theta^\dagger)$ , then  $q_2(\theta) < q_1(\theta)$  for all  $\theta \in [\max\{\hat{\tau}_1, \hat{\tau}_2\}, \min\{\tau_1(1), \tau_2(1)\}]$ .*

**Proof of Claim 2:**

The proof is by contradiction, so suppose that there exists  $\theta' \in [\max\{\hat{\tau}_1, \hat{\tau}_2\}, \min\{\tau_1(1), \tau_2(1)\}]$  such that  $q_2(\theta') = q_1(\theta') \equiv q'$  and  $\dot{q}_2(\theta') \neq \dot{q}_1(\theta')$ . Without loss of generality we can assume  $\dot{q}_2(\theta') > \dot{q}_1(\theta')$ . Differential equations (39) and (40) and  $\dot{q}_i = \frac{\dot{Q}_i}{\dot{\tau}_i}$  imply  $\frac{u_\theta(q', \theta')}{u_q(q', \tau_2^{-1}(\theta')) - u_q(q', \theta')} > \frac{u_\theta(q', \theta')}{u_q(q', \tau_1^{-1}(\theta')) - u_q(q', \theta')}$ . Since  $u_{\theta q} > 0$ ,  $\tau_1^{-1}(\theta') > \tau_2^{-1}(\theta')$ . Let  $\tilde{\theta}' = \tau_1^{-1}(\theta')$ .

Next we consider the following two cases:

Case 1:  $q_2(\theta) > q_1(\theta)$  for  $\theta \in (\theta', \min\{\tau_1(1), \tau_2(1)\}]$ .

First note that (40) i.e.,  $\dot{Q}_i(\theta) = \frac{f(\theta)u_\theta(Q_i, \tau)}{f(\tau)u_q(Q_i, \tau)}$  and  $\dot{Q}_i(\theta) > 0$  in combination imply that  $q_i(\theta) \leq q^{fb}(\theta)$  for all  $\theta \in (\hat{\tau}_i, \tau_i(1))$ . It follows that  $\tau_1(1) > \tau_2(1)$ , for otherwise  $q_2(\tau_1(1)) > q_1(\tau_1(1)) = q^{fb}(\tau_1(1))$ , where the inequality hold by case assumption, and the equality holds by boundary condition (20), violating  $q_2(\cdot) \leq q^{fb}(\cdot)$ .

While  $\tau_1(1) > \tau_2(1)$ , we also have  $\tau_1(\tilde{\theta}') = \theta' < \tau_2(\tilde{\theta}')$  since  $\tilde{\theta}' = \tau_1^{-1}(\theta') > \tau_2^{-1}(\theta')$ . Therefore there exists  $\tilde{\theta}'' \in (\tilde{\theta}', 1)$  such that  $\tau_1(\tilde{\theta}'') = \tau_2(\tilde{\theta}'') \equiv \theta''$  and  $\dot{\tau}_1(\tilde{\theta}'') > \dot{\tau}_2(\tilde{\theta}'')$ . By (39) the latter is equivalent to  $\frac{f(\tilde{\theta}'')(u_q(Q_1(\tilde{\theta}''), \tilde{\theta}'') - u_q(Q_1(\tilde{\theta}''), \theta''))}{f(\tau_1(\tilde{\theta}''))u_q(Q_1(\tilde{\theta}''), \theta'')} > \frac{f(\tilde{\theta}'')(u_q(Q_2(\tilde{\theta}''), \tilde{\theta}'') - u_q(Q_2(\tilde{\theta}''), \theta''))}{f(\tau_2(\tilde{\theta}''))u_q(Q_2(\tilde{\theta}''), \theta'')}$ . Then from  $u_{qq} < 0$  and  $u_{\theta qq} \geq 0$  it follows that  $Q_1(\tilde{\theta}'') > Q_2(\tilde{\theta}'')$ , or equivalently  $q_1(\theta'') > q_2(\theta'')$ . However, this contradicts the case assumption since  $\theta'' \in (\theta', \tau_2(1))$ .

Case 2: There exists  $\theta'' \in (\theta', \min\{\tau_1(1), \tau_2(1)\}]$  such that  $q_2(\theta) > q_1(\theta)$  for  $\theta \in (\theta', \theta'')$ ,  $q_2(\theta'') = q_1(\theta'') \equiv q''$  and  $\dot{q}_2(\theta'') < \dot{q}_1(\theta'')$ .

Given  $\dot{q}_2(\theta'') < \dot{q}_1(\theta'')$ , a similar argument to that in Case 1 yields that  $\tau_1^{-1}(\theta'') < \tau_2^{-1}(\theta'')$ , and  $\tau_2(\tau_1^{-1}(\theta'')) < \tau_2(\tau_2^{-1}(\theta'')) = \theta'' = \tau_1(\tau_1^{-1}(\theta''))$ . Let  $\tilde{\theta}'' = \tau_1^{-1}(\theta'')$ . Note that  $\tilde{\theta}'' > \tilde{\theta}'$  as  $\theta'' > \theta'$ .

Since  $\tau_1(\tilde{\theta}') < \tau_2(\tilde{\theta}')$  and  $\tau_2(\tilde{\theta}'') < \tau_1(\tilde{\theta}'')$ , there exists  $\tilde{\theta}''' \in [\tilde{\theta}', \tilde{\theta}'']$  such that  $\tau_1(\tilde{\theta}''') = \tau_2(\tilde{\theta}''') \equiv \theta'''$  and  $\dot{\tau}_1(\tilde{\theta}''') > \dot{\tau}_2(\tilde{\theta}''')$ . A similar argument to the in Case 1 yields  $Q_1(\tilde{\theta}''') > Q_2(\tilde{\theta}''')$ , or equivalently  $q_1(\theta''') > q_2(\theta''')$ . But by the case assumption  $q_1(\theta''') < q_2(\theta''')$ . Contradiction.

**Claim 3:** *If there exists  $\tilde{\theta}' \in [\max\{\hat{\theta}_1, \hat{\theta}_2\}, 1]$  such that  $\tau_2(\tilde{\theta}') < \tau_1(\tilde{\theta}')$ , then  $\tau_2(\theta) < \tau_1(\theta)$  for all  $\theta \in [\max\{\hat{\theta}_1, \hat{\theta}_2\}, 1]$ .*

**Proof of Claim 3:**

The proof is by contradiction, so suppose the Claim is not true. Then there exists a “crossing point”  $\tilde{\theta}' \in [\max\{\hat{\theta}_1, \hat{\theta}_2\}, 1]$  such that  $\tau_2(\tilde{\theta}') = \tau_1(\tilde{\theta}') \equiv \theta'$  and  $\dot{\tau}_1(\tilde{\theta}') \neq \dot{\tau}_2(\tilde{\theta}')$ . Without loss of generality we can assume  $\dot{\tau}_1(\tilde{\theta}') > \dot{\tau}_2(\tilde{\theta}')$ . Then from the differential equation (39) it follows that  $Q_1(\tilde{\theta}') > Q_2(\tilde{\theta}')$ , or equivalently  $q_1(\theta') > q_2(\theta')$ .

Note that  $\tilde{\theta}' < 1$  for otherwise we would have  $\theta' = \tau_1(1) = \tau_2(1)$  and  $q^{fb}(\theta') = q_1(\theta') = q_2(\theta')$  which contradicts  $q_1(\theta') > q_2(\theta')$ .

Now consider the following two cases:

Case 1:  $\tau_1(\theta) > \tau_2(\theta)$  for  $\theta \in (\tilde{\theta}', 1]$ .

Since  $\tau_1(1) > \tau_2(1)$ , we have  $q_1(\tau_2(1)) \leq q^{fb}(\tau_2(1)) = q_2(\tau_2(1))$ , which combined with  $q_1(\theta') > q_2(\theta')$  violates Claim 2, the no-crossing property of  $q$ .

Case 2: There exists  $\tilde{\theta}'' \in (\tilde{\theta}', 1]$  such that  $\tau_1(\theta) > \tau_2(\theta)$  for  $\theta \in (\tilde{\theta}', \tilde{\theta}'')$ ,  $\tau_1(\tilde{\theta}'') = \tau_2(\tilde{\theta}'') \equiv \theta''$  and  $\dot{\tau}_1(\tilde{\theta}'') < \dot{\tau}_2(\tilde{\theta}'')$ .

Using  $\dot{\tau}_1(\tilde{\theta}'') < \dot{\tau}_2(\tilde{\theta}'')$  and  $\tau_1(\tilde{\theta}'') = \tau_2(\tilde{\theta}'')$  in differential equation (39) yields  $Q_1(\tilde{\theta}'') < Q_2(\tilde{\theta}'')$ , or equivalently  $q_1(\theta'') < q_2(\theta'')$ , which combined with  $q_1(\theta') > q_2(\theta')$  violates Claim 2, the no-crossing property of  $q$ .

**Claim 4:** *Suppose there exist  $(\hat{\theta}_1, \hat{\tau}_1) \neq (\hat{\theta}_2, \hat{\tau}_2)$  such that for  $i = 1, 2$ ,  $(Q_i(\cdot), \tau_i(\cdot))$  is an increasing solution to differential equations (39)-(40) with boundary conditions (21)- (22). Then  $\hat{\theta}_2 > \hat{\theta}_1$  if and only if  $\hat{\tau}_2 > \hat{\tau}_1$ .*

**Proof of Claim 4:**

Suppose not, then without loss of generality we have  $\hat{\theta}_2 \geq \hat{\theta}_1$  and  $\hat{\tau}_1 \geq \hat{\tau}_2$  with at least one strict inequality. Then  $\tau_1(\hat{\theta}_2) \geq \tau_1(\hat{\theta}_1)$  and  $\tau_1(\hat{\theta}_1) \geq \tau_2(\hat{\theta}_2)$  with at least one strict inequality, from which it immediately follows that  $\tau_1(\hat{\theta}_2) > \tau_2(\hat{\theta}_2)$ , and so  $q_1(\tau_1(\hat{\theta}_1)) = q^{fb}(\tau_1(\hat{\theta}_1)) \geq q_2(\tau_1(\hat{\theta}_1))$ .

By Claim 3 (the “no-crossing” property of  $\tau$ ),  $\tau_1(1) > \tau_2(1)$ , and therefore  $q_2(\tau_2(1)) = q^{fb}(\tau_2(1)) > q_1(\tau_2(1))$ . The last inequality in combination with  $q_1(\tau_1(\hat{\theta}_1)) \geq q_2(\tau_1(\hat{\theta}_1))$  contradict the no-crossing property of  $q$  in Claim 2.

**Uniqueness.** Now we can establish the uniqueness of the solution to the relaxed program with the help of Claims 1-4. Again the proof is by contradiction, so suppose the solution is not unique. Then there exist  $\hat{\theta}_1$  and  $\hat{\theta}_2$ ,  $\hat{\theta}_1 \neq \hat{\theta}_2$  s.t.  $(Q_1(\theta), \tau_1(\theta))$  and  $(Q_2(\theta), \tau_2(\theta))$  solve the system of differential equations (39) and (40) with corresponding boundary conditions (20)-(22) where  $\hat{\tau}_i \equiv \tau_i(\hat{\theta}_i) = \min \Gamma_i$ . Without loss of generality suppose that  $\hat{\theta}_1 > \hat{\theta}_2$ . Claim 4 implies that  $\hat{\tau}_1 > \hat{\tau}_2$ . However, this contradicts Claim 1. *Q.E.D.*

**Proof of Theorem 7:**

**Part (1) and (2):**

Claim 3 in Proof of Theorem 6 implies that  $\hat{\theta}_2 \leq \hat{\theta}_1$  if and only if  $\tau_2(\hat{\theta}_2) \leq \tau_1(\hat{\theta}_1)$ , but that contradicts to Claim 1 in Proof of Theorem 6 given  $C_2 > C_1$ . Therefore, it must be the case that both  $\hat{\theta}_2 > \hat{\theta}_1$  and  $\tau_2(\hat{\theta}_2) > \tau_1(\hat{\theta}_1)$ .

**Part (3) and (4):**

Part (2) and boundary condition (21) implies  $q_2(\tau_2(\hat{\theta}_2)) = q^{fb}(\tau_2(\hat{\theta}_2)) > q_1(\tau_2(\hat{\theta}_2))$ , therefore part (4) follows from the no-crossing property of  $q$  from Claim 2 in Proof of Theorem 6. Now since  $q_1(\tau_2(1)) < q_2(\tau_2(1)) = q^{fb}(\tau_2(1))$ , it must be the case that  $\tau_1(1) > \tau_2(1)$ , and part (3) follows from the no-crossing property of  $\tau$  from Claim 2 in Proof of Theorem 6. *Q.E.D.*

**Proof of Theorem 8:** By Pontryagin’s Maximum principle the solution has to satisfy the following adjoint equations:

For  $k = 1$ ,

$$-\dot{\lambda}_{Q^1} = u_q(Q^1, \tau^1) f(\tau^1) \alpha^1 - [1 - F(\theta)] u_{\theta q}(Q^1, \theta) + \lambda_{Q^1} \frac{\partial h^1}{\partial Q^1} \alpha^1 \quad (85)$$

For  $k > 1$ ,

$$-\dot{\lambda}_{Q^k} = u_q(Q^k, \tau^k) f(\tau^k) \alpha^k - [1 - F(\tau^{k-1})] u_{\theta q}(Q^k, \tau^{k-1}) \alpha^{k-1} + \lambda_{Q^k} \frac{\partial h^k}{\partial Q^k} \alpha^k + \lambda_{Q^{k-1}} \frac{\partial h^{k-1}}{\partial Q^k} \alpha^{k-1} \quad (86)$$

For  $k = M(\theta)$ ,

$$-\dot{\lambda}_{\tau^k} = u_{\theta}(Q^k, \tau^k) f(\tau^k) \alpha^k + u(Q^k, \tau^k) f'(\tau^k) \alpha^k + \lambda_{Q^k} \frac{\partial h^k}{\partial \tau^k} \alpha^k \quad (87)$$

For  $k < M(\theta)$ ,

$$\begin{aligned} -\dot{\lambda}_{\tau^k} = & u_{\theta}(Q^k, \tau^k) f(\tau^k) \alpha^k + u(Q^k, \tau^k) f'(\tau^k) \alpha^k + \lambda_{Q^k} \frac{\partial h^k}{\partial \tau^k} \alpha^k \\ & - [1 - F(\tau^k)] u_{\theta\theta}(Q^{k+1}, \tau^k) + \lambda_{Q^{k+1}} \frac{\partial h^{k+1}}{\partial \tau^k} \alpha^{k+1} \end{aligned} \quad (88)$$

The linearity of the Hamiltonian (23) in the control variable  $\alpha^k$  creates certain technical difficulties, as it implies that  $\alpha^k$  cannot be solved for directly from the standard first-order conditions of optimality. However, Pontryagin's Maximum principle still applies and requires that the optimal control  $\alpha^k$  maximizes the Hamiltonian (63).

Particularly, let us introduce the following *switching function* :

$$J^k(Q^k, \tau^k, Q^{k+1}, \lambda_{Q^k}, \lambda_{\tau^k}, h^k) = \begin{cases} u(Q^k, \tau^k) f(\tau^k) + \lambda_{Q^k} h^k + \lambda_{\tau^k} - (1 - F(\tau^k)) u_{\theta}(Q^{k+1}, \tau^k) & \text{if } k < M(\theta) \\ u(Q^k, \tau^k) f(\tau^k) + \lambda_{Q^k} h^k + \lambda_{\tau^k} & \text{if } k = M(\theta) \end{cases} \quad (89)$$

Note that  $J^k$  can never be strictly positive, since then the optimal value of  $\alpha^k$  is infinity and, correspondingly, the value of the objective would be infinite. Optimality requires the following "switching conditions" to hold:

$$\begin{aligned} J^k(Q^k, \tau^k, Q^{k+1}, \lambda_{Q^k}, \lambda_{\tau^k}, h^k) < 0 & \Rightarrow \alpha^k = 0 \\ J^k(Q^k, \tau^k, Q^{k+1}, \lambda_{Q^k}, \lambda_{\tau^k}, h^k) = 0 & \Rightarrow \alpha^k \geq 0 \end{aligned}$$

An interval of  $\theta$  on which  $J^k$  vanishes ( $J^k = 0$ ) is called a *singular arc*. On a singular arc, the optimality conditions do not pin down the value of the optimal control  $\alpha^k$ . As a consequence, such problems of singular control are notoriously difficult to solve. Only a few solutions have

been developed up to now, most notably Merton (1969)'s celebrated portfolio choice problem in finance, and trajectory optimization in aeronautics (see e.g. Bryson and Ho (1975) Ch. 8).

An interval of  $\theta$  on which  $J < 0$  is called a *nonsingular arc*. As pointed above,  $\alpha^k(\theta) = 0$  for all  $\theta$  on a non-singular arc. Pontryagin's Maximum principle yields the remaining optimality conditions along such an arc.

The approach we follow here is to recover the optimal control  $\alpha^k$  along a singular arc by differentiating the identity  $J^k = 0$  with respect to  $\theta$  until the control variable appears in a non-trivial way, and then solve for it.

Now, consider a singular arc where we have  $J^k = 0$ . Differentiating the switching function  $J^k$  on a singular arc we get:

For  $k = M(\theta)$ ,

$$\begin{aligned} \frac{dJ^k}{d\theta} &= \dot{\lambda}_{Q^k} h^k + \lambda_{Q^k} \left[ \frac{\partial h^k}{\partial \tau^{k-1}} \dot{\tau}^{k-1} + \frac{\partial h^k}{\partial Q^k} \dot{Q}^k + \frac{\partial h^k}{\partial \tau^k} \dot{\tau}^k \right] + \dot{\lambda}_{\tau^k} \\ &+ [u_q(Q^k, \tau^k) f(\tau^k) \dot{Q}^k + u_\theta(Q^k, \tau^k) f(\tau^k) \dot{\tau}^k + u(Q^k, \tau^k) f'(\tau^k) \dot{\tau}^k] = 0 \end{aligned} \quad (90)$$

For  $k < M(\theta)$ ,

$$\begin{aligned} \frac{dJ^k}{d\theta} &= \dot{\lambda}_{Q^k} h^k + \lambda_{Q^k} \left[ \frac{\partial h^k}{\partial \tau^{k-1}} \dot{\tau}^{k-1} + \frac{\partial h^k}{\partial Q^k} \dot{Q}^k + \frac{\partial h^k}{\partial \tau^k} \dot{\tau}^k + \frac{\partial h^k}{\partial Q^{k+1}} \dot{Q}^{k+1} \right] + \dot{\lambda}_{\tau^k} \\ &+ [u_q(Q^k, \tau^k) f(\tau^k) \dot{Q}^k + u_\theta(Q^k, \tau^k) f(\tau^k) \dot{\tau}^k + u(Q^k, \tau^k) f'(\tau^k) \dot{\tau}^k] \\ &- [1 - F(\tau^k)] [u_{\theta q}(Q^{k+1}, \tau^k) \dot{Q}^{k+1} + u_{\theta\theta}(Q^{k+1}, \tau^k) \dot{\tau}^k] \\ &+ f(\tau^k) u_{\theta q}(Q^{k+1}, \tau^k) \dot{\tau}^k = 0 \end{aligned} \quad (91)$$

Given (88), (91) and  $\alpha^k = \dot{\tau}^k$ , for  $k < M(\theta)$ :

$$\begin{aligned} \dot{\lambda}_{Q^k} h^k + \lambda_{Q^k} \left[ \frac{\partial h^k}{\partial \tau^{k-1}} \dot{\tau}^{k-1} + \frac{\partial h^k}{\partial Q^k} \dot{Q}^k + \frac{\partial h^k}{\partial Q^{k+1}} \dot{Q}^{k+1} \right] - \lambda_{Q^{k+1}} \frac{\partial h^{k+1}}{\partial \tau^k} \alpha^{k+1} \\ - [1 - F(\tau^k)] u_{\theta q}(Q^{k+1}, \tau^k) + \dot{Q}^{k+1} u_q(Q^k, \tau^k) f(\tau^k) \dot{Q}^k = 0 \end{aligned} \quad (92)$$

Given (86), (92) and  $\dot{Q}^k = h^k \dot{\tau}^k$ , for  $k = 2, \dots, M(\theta) - 1$ :

$$\begin{aligned} \lambda_{Q^k} \left[ \frac{\partial h^k}{\partial Q^{k+1}} \dot{Q}^{k+1} + \frac{\partial h^k}{\partial \tau^{k-1}} \dot{\tau}^{k-1} \right] &= \lambda_{Q^{k-1}} h^k \frac{\partial h^{k-1}}{\partial Q^k} \dot{\tau}^{k-1} + \lambda_{Q^{k+1}} \frac{\partial h^{k+1}}{\partial \tau^k} \dot{\tau}^{k+1} \\ &- h^k [1 - F(\tau^{k-1})] u_{\theta q}(Q^k, \tau^{k-1}) \dot{\tau}^{k-1} + [1 - F(\tau^k)] u_{\theta q}(Q^{k+1}, \tau^k) \dot{Q}^{k+1} \end{aligned} \quad (93)$$



Analogously, for  $k = 1$ :

$$\lambda_{Q^1} \frac{\partial h^1}{\partial \theta} = \lambda_{Q^2} \frac{\partial h^2}{\partial \tau^1} \dot{\tau}^2 - h^1 [1 - F(\theta)] u_{\theta q}(Q^1, \theta) + [1 - F(\tau^1)] u_{\theta q}(Q^2, \tau^1) \dot{Q}^2 \quad (94)$$

and for  $k = M(\theta)$ :

$$\lambda_{Q^k} \frac{\partial h^k}{\partial \tau^{k-1}} \dot{\tau}^{k-1} = \lambda_{Q^{k-1}} h^k \frac{\partial h^{k-1}}{\partial Q^k} \dot{\tau}^{k-1} - h^k [1 - F(\tau^{k-1})] u_{\theta q}(Q^k, \tau^{k-1}) \dot{\tau}^{k-1} \quad (95)$$

Given  $\frac{\partial h^k}{\partial \tau^{k-1}} = -h^k \frac{u_{\theta q}(Q^k, \tau^{k-1})}{u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)}$  and  $\frac{\partial h^{k-1}}{\partial Q^k} = \frac{-u_{\theta q}(Q^k, \tau^{k-1})}{u_q(Q^{k-1}, \tau^{k-2}) - u_q(Q^{k-1}, \tau^{k-1})}$ , rewrite (95) so that for  $k = M(\theta)$ :

$$\lambda_{Q^k} = \lambda_{Q^{k-1}} \frac{u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)}{u_q(Q^{k-1}, \tau^{k-2}) - u_q(Q^{k-1}, \tau^{k-1})} + [1 - F(\tau^{k-1})] [u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)] \quad (96)$$

For  $k = M(\theta) - 1$ , (96) implies  $\lambda_{Q^{k+1}} = \lambda_{Q^k} \frac{u_q(Q^{k+1}, \tau^k) - u_q(Q^{k+1}, \tau^{k+1})}{u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)} + [1 - F(\tau^k)] [u_q(Q^{k+1}, \tau^k) - u_q(Q^{k+1}, \tau^{k+1})]$ , combined with (93):

$$\begin{aligned} & \lambda_{Q^k} \left[ \frac{\partial h^k}{\partial Q^{k+1}} \dot{Q}^{k+1} + \frac{\partial h^k}{\partial \tau^{k-1}} \dot{\tau}^{k-1} \right] \\ &= \lambda_{Q^{k-1}} h^k \frac{\partial h^{k-1}}{\partial Q^k} \dot{\tau}^{k-1} \\ &+ \left( \lambda_{Q^k} \frac{u_q(Q^{k+1}, \tau^k) - u_q(Q^{k+1}, \tau^{k+1})}{u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)} + [1 - F(\tau^k)] [u_q(Q^{k+1}, \tau^k) - u_q(Q^{k+1}, \tau^{k+1})] \right) \frac{\partial h^{k+1}}{\partial \tau^k} \dot{\tau}^{k+1} \\ &- h^k [1 - F(\tau^{k-1})] u_{\theta q}(Q^k, \tau^{k-1}) \dot{\tau}^{k-1} + [1 - F(\tau^k)] u_{\theta q}(Q^{k+1}, \tau^k) \dot{Q}^{k+1} \end{aligned} \quad (97)$$

Given  $\frac{\partial h^k}{\partial \tau^{k-1}} = -h^k \frac{u_{\theta q}(Q^k, \tau^{k-1})}{u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)}$ ,  $\frac{\partial h^k}{\partial Q^{k+1}} = \frac{-u_{\theta q}(Q^{k+1}, \tau^k)}{u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)}$  and  $h^k = \frac{u_{\theta}(Q^k(\theta), \tau^k(\theta)) - u_{\theta}(Q^{k+1}(\theta), \tau^k(\theta))}{u_q(Q^k(\theta), \tau^{k-1}(\theta)) - u_q(Q^k(\theta), \tau^k(\theta))}$  rewrite (97) so that for  $k = M(\theta) - 1$ :

$$\lambda_{Q^k} = \lambda_{Q^{k-1}} \frac{u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)}{u_q(Q^{k-1}, \tau^{k-2}) - u_q(Q^{k-1}, \tau^{k-1})} + [1 - F(\tau^{k-1})] [u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)] \quad (98)$$

Condition (98) applies to any  $k = 2, \dots, M(\theta)$  recursively.

Given  $\frac{\partial h^1}{\partial Q^2} = \frac{-u_{\theta q}(Q^2, \tau^1)}{u_q(Q^1, \theta) - u_q(Q^1, \tau^1)}$ ,  $\frac{\partial h^1}{\partial \theta} = -h^1 \frac{u_{\theta q}(Q^1, \theta)}{u_q(Q^1, \theta) - u_q(Q^1, \tau^1)}$  and  $\frac{\partial h^2}{\partial \tau^1} = -h^2 \frac{u_{\theta q}(Q^2, \tau^1)}{u_q(Q^2, \tau^1) - u_q(Q^2, \tau^2)}$ ,

Rewrite (94):

$$\begin{aligned}
& -\lambda_{Q^1} h^1 \left[ \frac{u_{\theta q}(Q^2, \tau^1)}{u_{\theta}(Q^1, \tau^1) - u_{\theta q}(Q^2, \tau^1)} \dot{Q}^2 + \frac{u_{\theta q}(Q^1, \theta)}{u_q(Q^1, \theta) - u_q(Q^1, \tau^1)} \right] \\
& = \lambda_{Q^2} \frac{u_{\theta q}(Q^2, \tau^1)}{u_q(Q^2, \tau^1) - u_q(Q^2, \tau^2)} h^2 \dot{\tau}^2 \\
& \quad - h^1 [1 - f(\theta)] u_{\theta q}(Q^1, \theta) + [1 - F(\tau^1)] u_{\theta q}(Q^2, \tau^1) \dot{Q}^2
\end{aligned} \tag{99}$$

Combining (99) with (98) for  $k = 2$  and rearrange:

$$\lambda_{Q^1} = [1 - F(\theta)] [u_q(Q^1, \theta) - u_q(Q^1, \tau^1)] \tag{100}$$

From (98) and (100), for any  $k = 1, \dots, M(\theta)$ :

$$\lambda_{Q^k} = [u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)] \sum_{s=1}^k [1 - F(\tau^{k-s})] \tag{101}$$

Differentiating (101):

$$\begin{aligned}
\dot{\lambda}_{Q^k} & = - [u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)] \sum_{s=1}^k f(\tau^{k-s}) \dot{\tau}^{k-s} \\
& \quad + [u_{qq}(Q^k, \tau^{k-1}) \dot{Q}^k - u_{qq}(Q^k, \tau^k) \dot{Q}^k + u_{\theta q}(Q^k, \tau^{k-1}) \dot{\tau}^{k-1} - u_{\theta q}(Q^k, \tau^k) \dot{\tau}^k] \sum_{s=1}^k [1 - F(\tau^{k-s})]
\end{aligned} \tag{102}$$

From (86) and (101), for  $k > 1$ :

$$\begin{aligned}
\dot{\lambda}_{Q^k} &= -u_q(Q^k, \tau^k) f(\tau^k) \dot{\tau}^k + [1 - F(\tau^{k-1})] u_{\theta q}(Q^k, \tau^{k-1}) \dot{\tau}^{k-1} \\
&\quad - ([u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)] \sum_{s=1}^k [1 - F(\tau^{k-s})]) \frac{\partial h^k}{\partial Q^k} \dot{\tau}^k \\
&\quad - ([u_q(Q^{k-1}, \tau^{k-2}) - u_q(Q^{k-1}, \tau^{k-1})] \sum_{s=1}^{k-1} [1 - F(\tau^{k-1-s})]) \frac{\partial h^{k-1}}{\partial Q^k} \dot{\tau}^{k-1} \\
&= -u_q(Q^k, \tau^k) f(\tau^k) \dot{\tau}^k + [1 - F(\tau^{k-1})] u_{\theta q}(Q^k, \tau^{k-1}) \dot{\tau}^{k-1} \\
&\quad + [u_q q(Q^k, \tau^{k-1}) h^k \dot{\tau}^k - u_q q(Q^k, \tau^k) h^k \dot{\tau}^k - u_{\theta q}(Q^k, \tau^k) \dot{\tau}^k] \sum_{s=1}^k [1 - F(\tau^{k-s})] \\
&\quad + u_{\theta q}(Q^k, \tau^{k-1}) \dot{\tau}^{k-1} \sum_{s=1}^{k-1} [1 - F(\tau^{k-1-s})] \\
&= -u_q(Q^k, \tau^k) f(\tau^k) \dot{\tau}^k + \\
&\quad + [u_q q(Q^k, \tau^{k-1}) h^k \dot{\tau}^k - u_q q(Q^k, \tau^k) h^k \dot{\tau}^k + u_{\theta q}(Q^k, \tau^{k-1}) \dot{\tau}^{k-1} - u_{\theta q}(Q^k, \tau^k) \dot{\tau}^k] \sum_{s=1}^k [1 - F(\tau^{k-s})]
\end{aligned} \tag{103}$$

From (102) and (103), for  $k > 1$ :

$$u_q(Q^k, \tau^k) f(\tau^k) \dot{\tau}^k = [u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)] \sum_{s=1}^k f(\tau^{k-s}) \dot{\tau}^{k-s} \tag{104}$$

From (85) and (100), for  $k = 1$ :

$$\begin{aligned}
\dot{\lambda}_{Q^1} &= -u_q(Q^1, \tau^1) f(\tau^1) \dot{\tau}^1 - [1 - F(\theta)] u_{\theta q}(Q^1, \theta) + [1 - F(\theta)] [u_q(Q^1, \theta) - u_q(Q^1, \tau^1)] \frac{\partial h^1}{\partial Q^1} \dot{\tau}^1 \\
&= -u_q(Q^1, \tau^1) f(\tau^1) \dot{\tau}^1 - [1 - F(\theta)] [u_q q(Q^1, \theta) h^1 \dot{\tau}^1 - u_q q(Q^1, \tau^1) h^1 \dot{\tau}^1 + u_{\theta q}(Q^1, \theta) - u_{\theta q}(Q^1, \tau^1) \dot{\tau}^1]
\end{aligned} \tag{105}$$

From (102) and (105),

$$u_q(Q^1, \tau^1) f(\tau^1) \dot{\tau}^1 = [u_q(Q^1, \theta) - u_q(Q^1, \tau^1)] f(\theta) \tag{106}$$

which means (104) applies to all  $k = 1, \dots, M(\theta)$ .

Now Let  $A(k) = f(\theta) + \sum_{s=1}^k f(\tau^s)\dot{\tau}^s$ . Then from (104) we have:

$$\dot{\tau}^k = \frac{u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)}{f(\tau^k)u_q(Q^k, \tau^k)} A(k-1) \quad (107)$$

and

$$\begin{aligned} A(k) &= A(k-1) + f(\tau^k)\dot{\tau}^k \\ &= A(k-1) \frac{u_q(Q^k, \tau^{k-1})}{u_q(Q^k, \tau^k)} \\ &= f(\theta) \prod_{s=1}^k \frac{u_q(Q^s, \tau^{s-1})}{u_q(Q^s, \tau^s)} \end{aligned} \quad (108)$$

recursively. From (107) and (108),

$$\dot{\tau}^k = \frac{f(\theta)[u_q(Q^k, \tau^{k-1}) - u_q(Q^k, \tau^k)]}{f(\tau^k)u_q(Q^k, \tau^k)} \prod_{s=1}^{k-1} \frac{u_q(Q^s, \tau^{s-1})}{u_q(Q^s, \tau^s)} \quad (109)$$

and since  $\dot{Q}^k = h^k \dot{\tau}^k$ ,

$$\dot{Q}^k = \begin{cases} \frac{f(\theta)[u_\theta(Q^k, \tau^k) - u_\theta(Q^{k+1}, \tau^k)]}{f(\tau^k)u_q(Q^k, \tau^k)} \prod_{s=1}^{k-1} \frac{u_q(Q^s, \tau^{s-1})}{u_q(Q^s, \tau^s)} & \text{if } k < M(\theta) \\ \frac{f(\theta)u_\theta(Q^k, \tau^k)}{f(\tau^k)u_q(Q^k, \tau^k)} \prod_{s=1}^{k-1} \frac{u_q(Q^s, \tau^{s-1})}{u_q(Q^s, \tau^s)} & \text{if } k = M(\theta) \end{cases} \quad (110)$$

*Q.E.D*

## 9 Appendix C

This Appendix presents derivation of the solution of ordinary differential equation system (41)-(45).

First, let us make a change of variables:

$$y = \tau - Q, \quad z = \tau + Q \quad (111)$$

Then the system (41)-(42) is equivalent to the following system:

$$\dot{y}y = \theta - z \quad (112)$$

$$\dot{z} = \frac{\theta}{y} - 1 \quad (113)$$

Differentiating (112) yields:

$$\ddot{y}y + (\dot{y})^2 = 1 - \dot{z} = 2 - \frac{\theta}{y} \quad (114)$$

Let us make another change of variables:  $w = \frac{y^2}{4}$ . Then (114) becomes:

$$\ddot{w} = 1 - \frac{x}{4\sqrt{w}} \quad (115)$$

The general solution to the differential equation (115) is parametric. Specifically, let  $b_1, b_2$  and  $b_3$  be some constants and  $t \in [0, \infty)$  be a parameter. Then:

$$\theta(t) = b_1 t + b_2 t^{\frac{\sqrt{5}-1}{2}} + b_3 t^{-\frac{\sqrt{5}+1}{2}} \quad (116)$$

$$\frac{y^2(t)}{4} \equiv w(t) = \left( \frac{1}{2} b_1 t + \frac{\sqrt{5}-1}{4} b_2 t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}+1}{4} b_3 t^{-\frac{\sqrt{5}+1}{2}} \right)^2 \quad (117)$$

Note that we must have  $0 \leq y < \theta$ , since  $y = \tau - Q$ ,  $\tau < \theta$ , and the optimal quantity  $Q$  cannot be greater than its first-best level, which in this case is equal to  $\tau$ . So,

$$y(t) = \left| b_1 t + \frac{\sqrt{5}-1}{2} b_2 t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}+1}{2} b_3 t^{-\frac{\sqrt{5}+1}{2}} \right| \quad (118)$$

We can without loss of generality take that  $\theta(1) = 1$ . Indeed, if  $\theta(t_1) = 1$  for some  $t_1 \in (0, \infty)$ ,  $t_1 \neq 1$ , then we can replace the parameter  $t$  with the parameter  $s = \frac{t}{t_1}$ , and replace the constants  $b_1, b_2, b_3$  with constants  $b'_1, b'_2, b'_3$  such that  $b'_1 = b_1 t_1$ ,  $b'_2 = b_2 t_1^{\frac{\sqrt{5}-1}{2}}$  and  $b'_3 = b_3 t_1^{-\frac{\sqrt{5}+1}{2}}$ . Then we would have  $\theta(s) = \theta(t)$  and  $y(s) = y(t)$  for all  $t \in [0, \infty)$ , with  $\theta(s)_{s=1} = 1$ .

Using  $\theta(1) = 1$  in (116) yields  $b_1 + b_2 + b_3 = 1$ . Also,  $\theta(1) = 1$  and the boundary condition  $\tau(1) = Q(1)$  imply that  $y(1) = 0$ . In turn, the latter implies that  $b_1 + \frac{\sqrt{5}-1}{2} b_2 - \frac{\sqrt{5}+1}{2} b_3 = 0$ . Now, we can solve for  $b_2$  and  $b_3$  in terms of  $b_1$  to obtain:

$$b_2 = -b_1 \frac{5 + 3\sqrt{5}}{10} + \frac{\sqrt{5} + 1}{2\sqrt{5}}.$$

$$b_3 = b_1 \frac{3\sqrt{5} - 5}{10} + \frac{\sqrt{5} - 1}{2\sqrt{5}}.$$

Then (116) and (118) become:

$$\theta(t) = b_1 \left( t - \frac{1 + 3\sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} + \frac{3\sqrt{\frac{1}{5}} - 1}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{\sqrt{5} + 1}{2\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5} - 1}{2\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \quad (119)$$

$$y(t) = \left| b_1 \left( t - \frac{1 + \sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} - \frac{1 - \sqrt{\frac{1}{5}}}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \right| \quad (120)$$

At first, let us suppose that the expression under the absolute value sign on the right-hand side of (120) is positive i.e.<sup>3</sup>

$$y(t) = b_1 \left( t - \frac{1 + \sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} - \frac{1 - \sqrt{\frac{1}{5}}}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \quad (121)$$

Next, we solve the differential equation (113) for  $z$ , which we will also parameterize by  $t$ . So, we have  $z'(t) \equiv \frac{dz}{dt} = z'(\theta)\theta'(t)$ . By (119) and (121),  $y(t) = \theta'(t)t$ . Then (113) can be rewritten as:

$$z'(t) = \left( \frac{\theta}{y} - 1 \right) \theta'(t) = \frac{\theta}{\theta'(t)t} \theta'(t) - \theta'(t) = \frac{\theta}{t} - \theta'(t). \quad (122)$$

Substituting (119) for  $\theta(t)$  we obtain:

$$z'(t) = b_1 \left( -\frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-3}{2}} + \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+3}{2}} \right) + \frac{\sqrt{5} - 1}{2\sqrt{5}} t^{\frac{\sqrt{5}-3}{2}} + \frac{\sqrt{5} + 1}{2\sqrt{5}} t^{-\frac{\sqrt{5}+3}{2}} \quad (123)$$

Integrating (123) yields:

$$z(t) = b_1 \left( -\frac{1 + \sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} - \frac{1 - \sqrt{\frac{1}{5}}}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} + k \quad (124)$$

where  $k$  is a constant of integration. Now, let us show that equation (112),  $y(\dot{\theta})y = \theta - z$ , implies that the constant of integration  $k$  is equal to zero. Note that  $y'(t) = y(\dot{\theta})\theta'(t)$ . So we can rewrite (112) as  $y'(t)y = (\theta - z)\theta'(t)$ . Since  $y = \theta'(t)t$ , the previous equation can be rewritten as follows:  $y'(t)t = (\theta - z)$

Next, from (121) we obtain:

$$y'(t)t = b_1 \left( t - \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{1 - \sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} + \frac{1 + \sqrt{\frac{1}{5}}}{2} t^{-\frac{\sqrt{5}+1}{2}} \quad (125)$$

---

<sup>3</sup>Later we will show that this is, indeed, the case since the opposite case when this expression is negative leads to a contradiction.

Also, (119) and (129) yield:

$$\theta(t) - z(t) = b_1 \left( t - \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{\sqrt{5}-1}{2\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} - k \quad (126)$$

Equating (125) and (126) yields  $k = 0$ .

Furthermore, observe that  $z(t) - y(t) = -b_1 t$ . Since  $z(t) - y(t) = 2Q(t)$ , it follows that  $Q(t) = -\frac{b_1}{2} t$  and so  $b_1 < 0$ .

Now, let us confirm that, as claimed, the expression under the absolute value sign on the right-hand side of (120) is positive. The proof is by contradiction, so suppose otherwise i.e.,

$$y(t) = -b_1 \left( t - \frac{1 + \sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} - \frac{1 - \sqrt{\frac{1}{5}}}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) - \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \quad (127)$$

Then (119) and (127) yield  $y(t) = -\theta'(t)t$  and so, instead of (122), we now have:

$$z'(t) = \left( \frac{\theta}{y} - 1 \right) \theta'(t) = \frac{\theta}{-\theta'(t)t} \theta'(t) - \theta'(t) = -\frac{\theta}{t} - \theta'(t) = \frac{\theta}{t} - \theta'(t) - 2\frac{\theta}{t}. \quad (128)$$

Substituting (119) for  $\theta(t)$  in (128) and integrating yields:

$$\begin{aligned} z(t) = & b_1 \left( -\frac{1 + \sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} - \frac{1 - \sqrt{\frac{1}{5}}}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \\ & - 2b_1 \left( t - \frac{1 + 3\sqrt{\frac{1}{5}}}{\sqrt{5}-1} t^{\frac{\sqrt{5}-1}{2}} - \frac{3\sqrt{\frac{1}{5}}-1}{\sqrt{5}+1} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{\sqrt{5}+1}{\sqrt{5}(\sqrt{5}-1)} t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}-1}{\sqrt{5}(\sqrt{5}-1)} t^{-\frac{\sqrt{5}+1}{2}} + k_2 \end{aligned} \quad (129)$$

where  $k_2$  is a constant of integration.

Since in this case  $y = -\theta'(t)t$ , the equation  $y'(t)y = (\theta - z)\theta'(t)$  (i.e., equation (112) parameterized by  $t$ ) can be rewritten as  $-y'(t)t = (\theta - z)$ . Differentiating (127) and combining the results with (119) and (129) the latter equation can be rewritten as:

$$-2b_1 \left( t - \frac{1 + 3\sqrt{\frac{1}{5}}}{\sqrt{5}-1} t^{\frac{\sqrt{5}-1}{2}} - \frac{3\sqrt{\frac{1}{5}}-1}{\sqrt{5}+1} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{\sqrt{5}+1}{\sqrt{5}(\sqrt{5}-1)} t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}-1}{\sqrt{5}(\sqrt{5}-1)} t^{-\frac{\sqrt{5}+1}{2}} + k_2 = 0$$

which cannot hold on any neighborhood of  $t$ .

Thus, we have confirmed that  $y(t)$  is given by (121), and hence  $y(t) = \theta'(t)t$ . Since  $y(t) \geq 0$ , it follows that  $\theta'(t) > 0$ .

So, to complete the solution, it remains to characterize  $b_1$  and  $\hat{t}$  such that  $\hat{t} < 1$  and  $y(\hat{t}) = 0$  and  $y(t) \geq 0$  for all  $t \in [\hat{t}, 1]$ . We will then have  $\hat{\theta} = \theta(\hat{t}) < 1$ . For this, we need to compute  $y'(t)$  and  $y''(t)$ . We have:

$$y'(t) = b_1 + \frac{(\sqrt{5}-1) - 2b_1 \hat{t}^{\frac{\sqrt{5}-3}{2}}}{2\sqrt{5}} + \frac{(\sqrt{5}+1) + 2b_1 \hat{t}^{-\frac{\sqrt{5}+3}{2}}}{2\sqrt{5}} \quad (130)$$

$$y''(t) = -\frac{3 - \sqrt{5}(\sqrt{5}-1) - 2b_1 \hat{t}^{\frac{\sqrt{5}-5}{2}}}{2} \frac{1}{2\sqrt{5}} - \frac{\sqrt{5}+3}{2} \frac{(\sqrt{5}+1) + 2b_1 \hat{t}^{-\frac{\sqrt{5}+5}{2}}}{2\sqrt{5}} \quad (131)$$

Using (121) and (125) we obtain:

$$\begin{aligned} y(t) - ty'(t) &= b_1 \left( t - \frac{1 + \sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} - \frac{1 - \sqrt{\frac{1}{5}}}{2} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \\ &\quad - b_1 \left( t - \frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \right) - \frac{1 - \sqrt{\frac{1}{5}}}{2} t^{\frac{\sqrt{5}-1}{2}} - \frac{1 + \sqrt{\frac{1}{5}}}{2} t^{-\frac{\sqrt{5}+1}{2}} = \\ & b_1 \left( -\frac{\sqrt{5}-1}{2\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}+1}{2\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{3 - \sqrt{5}}{2\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} - \frac{3 + \sqrt{5}}{2\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \end{aligned} \quad (132)$$

As established above,  $b_1 < 0$ . In fact, let us show that  $b_1 \in [-\frac{\sqrt{5}+1}{2}, -1)$ .

First, let us rule out  $b_1 < -\frac{\sqrt{5}+1}{2}$ . Observe that if  $b_1 < -\frac{\sqrt{5}+1}{2}$ , then by (130)  $y'(t) < 0$  for all  $t \geq 1$ . Since  $y(1) = 0$ , it follows that  $y(t) < 0$  for all  $t > 1$  and  $y(1 - \epsilon) > 0$  for sufficiently small  $\epsilon > 0$ . Further, observe from (121) that  $y(t) > 0$  when  $t$  is sufficiently small, with  $\lim_{t \rightarrow 0^+} y(t) = \infty$ . Finally, (132) implies that  $y'(t) < 0$  if  $y(t) = 0$ . So, if  $b_1 < -\frac{\sqrt{5}+1}{2}$  then there does not exist  $\hat{t} \neq 1$  such that  $y(\hat{t}) = 0$ .

Consider now  $b_1 \in [-\frac{\sqrt{5}+1}{2}, 0]$ . Note that in this case: (i) by (131),  $y''(t) < 0$  for all  $t$ ; (ii)  $y(t) < 0$  when  $t$  is sufficiently small, with  $\lim_{t \rightarrow 0^+} y(t) = -\infty$ , (iii)  $y(t) < 0$  when  $t$  is sufficiently large, with  $\lim_{t \rightarrow \infty} y(t) = -\infty$ . (iv) By (130)  $y'(1) = b_1 + 1$ .

So, if  $b_1 \in (-1, 0]$ , then  $y'(1) > 0$ . This, in combination with (i)-(iii) above, implies that if  $b_1 \in (-1, 0]$ , then there exists a unique  $\hat{t}$ ,  $\hat{t} \neq 1$  such that  $y(\hat{t}) = 0$  and, moreover,  $\hat{t} > 1$  and  $y(t) > 0$  for all  $t \in (1, \hat{t})$ . But we also have  $y(t) = \theta'(t)t$  and  $\theta(1) = 1$ . So  $\theta(t) > 1$  for all  $t \in (1, \hat{t})$ . This contradicts the fact that  $\theta(t) \in [0, 1]$ . Hence, we can rule out  $b_1 \in (-1, 0]$ .

Similarly, we can rule out  $b_1 = -1$  because in this case  $y(t) = 1$  only if  $t = 1$ .

Finally, if  $b_1 \in [-\frac{\sqrt{5}+1}{2}, -1)$ , then (i)-(iv) above imply that there exists  $\hat{t}$ ,  $\hat{t} < 1$  such that  $y(\hat{t}) = 0$ , and  $y(t) > 0$  for all  $t \in (\hat{t}, 1)$ . Also, since  $y(t) = \theta'(t)t$  and  $\theta(1) = 1$ , it follows that



$\theta(t) \in [0, 1)$  for all  $t \in (\hat{t}, 1)$ . Moreover,

$$\begin{aligned}\theta(t) - y(t) &= b_1 \left( -\frac{1}{\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{1}{\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} \right) + \frac{\sqrt{5}-1}{2\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}} = \\ &= \frac{\sqrt{5}-1-2b_1}{2\sqrt{5}} t^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}+1+2b_1}{2\sqrt{5}} t^{-\frac{\sqrt{5}+1}{2}}\end{aligned}\quad (133)$$

To summarize,  $\theta(t) - y(t) > 0$  and  $\theta(t) \leq 1$  for all  $t \in [\hat{t}, 1]$  when  $b_1 \in [-\frac{\sqrt{5}+1}{2}, -1]$ , as required for the solution. We conclude that  $b_1 \in [-\frac{\sqrt{5}+1}{2}, -1)$ .

Thus, the two remaining parameters completing the solution are  $\hat{t} \in (0, 1)$  and  $b_1 \in [-\frac{\sqrt{5}+1}{2}, -1)$ . They are jointly determined as the solutions to the two equations:  $y(\hat{t}) = 0$  where  $y(\hat{t})$  is given by (127) and the boundary condition  $Q(\hat{t})(\theta(\hat{t}) - \tau(\hat{t})) = C$ .

Setting (127) to zero at  $\hat{t}$  yields:

$$b_1 = -\frac{\frac{1}{\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+1}{2}}}{\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2} \hat{t}^{-\frac{\sqrt{5}+1}{2}}}\quad (134)$$

Differentiating (134) we obtain for  $\hat{t} \in (0, 1)$ :

$$\begin{aligned}\frac{\partial b_1}{\partial \hat{t}} &= -\frac{\frac{\sqrt{5}-1}{2\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+3}{2}}}{\hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2} \hat{t}^{-\frac{\sqrt{5}+1}{2}}} + \frac{\left( \frac{1}{\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+1}{2}} \right) \left( 1 - \frac{1}{\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{1}{\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+3}{2}} \right)}{\left( \hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2} \hat{t}^{-\frac{\sqrt{5}+1}{2}} \right)^2} \\ &= \frac{\frac{3-\sqrt{5}}{2\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{3+\sqrt{5}}{2\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+1}{2}} + \hat{t}^{-2}}{\left( \hat{t} - \frac{1+\sqrt{\frac{1}{5}}}{2} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1-\sqrt{\frac{1}{5}}}{2} \hat{t}^{-\frac{\sqrt{5}+1}{2}} \right)^2} > 0\end{aligned}\quad (135)$$

where the last inequality follows from the fact that  $\frac{3-\sqrt{5}}{2\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{3+\sqrt{5}}{2\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+1}{2}} + \hat{t}^{-2} = 0$  for  $\hat{t} = 1$  and  $\frac{\partial \left( \frac{3-\sqrt{5}}{2\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{3+\sqrt{5}}{2\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+1}{2}} + \hat{t}^{-2} \right)}{\partial \hat{t}} = \frac{(3-\sqrt{5})(\sqrt{5}-1)}{4\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{(\sqrt{5}+1)(3+\sqrt{5})}{4\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+3}{2}} - 2\hat{t}^{-3} < 0$  for  $\hat{t} \in (0, 1)$ .

Recall that  $Q(\hat{t}) = \tau(\hat{t}) = -\frac{b_1}{2} \hat{t}$ . Also, since  $y(\hat{t}) = 0$ ,  $\theta(\hat{t})$  is given by the right-hand side of (133). Using this, we can rewrite the boundary condition  $Q(\hat{t})(\theta(\hat{t}) - \tau(\hat{t})) = C$  as follows:

$$F(b_1, \hat{t}, C) \equiv -\frac{b_1}{2} \left( b_1 \left( \frac{\hat{t}^2}{2} - \frac{1}{\sqrt{5}} \hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{1}{\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}-1}{2}} \right) + \frac{\sqrt{5}-1}{2\sqrt{5}} \hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}-1}{2}} \right) - C = 0\quad (136)$$

Next, from (135) and (136) we get  $\frac{dF}{dC} = -1 < 0$  and

$$\frac{dF(b_1(\hat{t}), \hat{t}, C)}{d\hat{t}} = -\frac{b_1}{2}y(\hat{t}) - \frac{\partial b_1}{\partial \hat{t}} \left( \frac{b_1}{2}\hat{t}^2 + \frac{\sqrt{5}-1-4b_1}{4\sqrt{5}}\hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{\sqrt{5}+1+4b_1}{4\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}-1}{2}} \right) > 0.$$

The last inequality holds since: (i)  $y(\hat{t}) = 0$ ; (ii)  $\frac{\partial b_1}{\partial \hat{t}} > 0$  as shown in (135); (iii) the multiplier of  $\frac{\partial b_1}{\partial \hat{t}}$ ,  $\frac{b_1}{2}\hat{t}^2 + \frac{\sqrt{5}-1-4b_1}{4\sqrt{5}}\hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{\sqrt{5}+1+4b_1}{4\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}-1}{2}}$ , is negative when  $\hat{t} = 1$  and  $b_1 < -1$  and is increasing in  $\hat{t}$  at any  $\hat{t} \in (0, 1)$  and  $b_1 < -1$ .

Next, applying l'Hospital's rule to (134) we obtain:

$$\lim_{\hat{t} \rightarrow 1} b_1(\hat{t}) = -\frac{\lim_{\hat{t} \rightarrow 1} \left( \frac{\sqrt{5}-1}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+3}{2}} \right)}{\lim_{\hat{t} \rightarrow 1} \left( 1 - \frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+3}{2}} \right)} = -1.$$

So,  $\lim_{\hat{t} \rightarrow 1} F(b_1(\hat{t}), \hat{t}, C) = \frac{1}{4} - C$ .

On the other hand,  $\lim_{\hat{t} \rightarrow 0} b_1(\hat{t}) = -\frac{\sqrt{5}+1}{2}$ , and so  $\lim_{\hat{t} \rightarrow 0} F(b_1(\hat{t}), \hat{t}, C) = -C$ ,

From the above we conclude that for  $C \in (0, \frac{1}{4})$  there exist a unique solution  $\hat{t} \in (0, 1)$  to the equation  $F(b_1(\hat{t}), \hat{t}, C) = 0$  and that  $\frac{d\hat{t}}{dC} > 0$ .

Now let us establish the interval of  $C$  on which our solution applies. The upper bound of  $C$  is equal to  $\frac{1}{4}$ , since for  $C > \frac{1}{4}$  none of the incentive constraints are binding. To establish the lower bound of  $C$ ,  $C_m$ , note that our solution applies when  $\hat{\theta} \geq \tau(1)$ . At  $C_m$  we then have  $\hat{\theta} = \tau(1) = Q(1)$ . Let  $\hat{t}_m$ , and  $b_{1,m}$  denote the parameter values where the latter condition holds. Then we can rewrite the boundary condition  $Q(\hat{\theta})(\hat{\theta} - \tau(\hat{\theta})) = C$  as follows:

$$\begin{aligned} Q(\hat{t}_m)(Q(1) - Q(\hat{t}_m)) &= C_m \\ \frac{(b_{1,m})^2}{4}\hat{t}_m(1 - \hat{t}_m) &= C_m \end{aligned} \quad (137)$$

So,  $C_m$ ,  $\hat{t}_m$ , and  $b_{1,m}$  are determined by (134), (137) and condition  $\theta(\hat{t}_m) = \tau(1) = Q(1)$ . Since  $\tau(1) = Q(1) = -\frac{b_{1,m}}{2}$ , we can equate the latter to  $\theta(\hat{t}_m)$  as given by (133), since  $y(\hat{t}_m) = 0$ , to obtain:

$$b_{1,m} \left( -\frac{1}{2} + \frac{1}{\sqrt{5}}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}_m^{-\frac{\sqrt{5}+1}{2}} \right) = \frac{\sqrt{5}-1}{2\sqrt{5}}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}}\hat{t}_m^{-\frac{\sqrt{5}+1}{2}} \quad (138)$$

Using (134) in (138) and simplifying yields:

$$\begin{aligned}
& - \left( \frac{1}{\sqrt{5}} \hat{t}_m^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} \hat{t}_m^{-\frac{\sqrt{5}+1}{2}} \right) \left( -\frac{1}{2} + \frac{1}{\sqrt{5}} \hat{t}_m^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} \hat{t}_m^{-\frac{\sqrt{5}+1}{2}} \right) = \\
& \left( \frac{\sqrt{5}-1}{2\sqrt{5}} \hat{t}_m^{\frac{\sqrt{5}-1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}} \hat{t}_m^{-\frac{\sqrt{5}+1}{2}} \right) \left( \hat{t}_m - \frac{1 + \sqrt{\frac{1}{5}}}{2} \hat{t}_m^{\frac{\sqrt{5}-1}{2}} - \frac{1 - \sqrt{\frac{1}{5}}}{2} \hat{t}_m^{-\frac{\sqrt{5}+1}{2}} \right) \quad (139)
\end{aligned}$$

The last equation simplifies to:

$$\hat{t}_m^{\sqrt{5}+1}(1 - \sqrt{5}) + \hat{t}_m^{\sqrt{5}} - \hat{t}_m(1 + \sqrt{5}) + 2\sqrt{5}\hat{t}_m^{\frac{\sqrt{5}-1}{2}} - 1 = 0 \quad (140)$$

The approximate root of the last equation in  $(0, 1)$  is  $\hat{t}_m = 0.187169$ . Then from (134) we obtain  $b_{1,m} \approx -1.554$  and from (137),  $C_m \approx 0.0918$ .

Let us now establish some useful comparative statics results. First, we have:

$$\frac{d\hat{\theta}}{dC} = \frac{\partial \hat{\theta}}{\partial b_1} \frac{\partial b_1}{\partial \hat{t}} \frac{d\hat{t}}{dC} + \theta'(\hat{t}) \frac{d\hat{t}}{dC} = \left( \hat{t} - \frac{1 + 3\sqrt{\frac{1}{5}}}{2} \hat{t}^{\frac{\sqrt{5}-1}{2}} + \frac{3\sqrt{\frac{1}{5}} - 1}{2} \hat{t}^{-\frac{\sqrt{5}+1}{2}} \right) \frac{\partial b_1}{\partial \hat{t}} \frac{d\hat{t}}{dC} > 0 \quad (141)$$

The second equality follows from the fact that  $\theta'(\hat{t}) = \frac{y(\hat{t})}{\hat{t}} = 0$  and (119), while the last inequality holds because, as established above,  $\frac{\partial b_1}{\partial \hat{t}} > 0$ ,  $\frac{d\hat{t}}{dC} > 0$ , and  $\hat{t} - \frac{1+3\sqrt{\frac{1}{5}}}{2} \hat{t}^{\frac{\sqrt{5}-1}{2}} + \frac{3\sqrt{\frac{1}{5}}-1}{2} \hat{t}^{-\frac{\sqrt{5}+1}{2}} = 0$  if  $\hat{t} = 1$  and  $\frac{\partial \left( \hat{t} - \frac{1+3\sqrt{\frac{1}{5}}}{2} \hat{t}^{\frac{\sqrt{5}-1}{2}} + \frac{3\sqrt{\frac{1}{5}}-1}{2} \hat{t}^{-\frac{\sqrt{5}+1}{2}} \right)}{\partial \hat{t}} < 0$  for any  $\hat{t} \in (0, 1)$ .

We can now confirm that  $\hat{\theta} > \tau(1)$  for  $C \in (C_m, \frac{1}{4})$ . We have shown above that  $\frac{d\hat{\theta}}{dC} > 0$ . Next, since  $\tau(1) = Q(1) = -\frac{b_1}{2}$ , we have  $\frac{d\tau(1)}{dC} = -\frac{1}{2} \frac{db_1}{dC} < 0$  where  $b_1$  is given by (134). So, since  $\hat{\theta} = \tau(1)$  at  $C = C_m$ , it follows that  $\hat{\theta} > \tau(1)$  when  $C \in (C_m, \frac{1}{4})$ , as required.

To obtain the comparative statics for  $\tau(\hat{\theta})$ , recall that  $\tau(\hat{\theta}) = Q(\hat{\theta}) = -\frac{b_1}{2} \hat{t}$ . Therefore,  $\frac{d\tau(\hat{\theta})}{dC} = \frac{d\tau(\hat{\theta})}{d\hat{t}} \frac{d\hat{t}}{dC} = \left( -\frac{b_1}{2} - \frac{1}{2} \hat{t} \frac{\partial b_1}{\partial \hat{t}} \right) \frac{d\hat{t}}{dC}$ . Using (134) and (135) we obtain:

$$\begin{aligned}
& -\frac{b_1}{2} - \frac{1}{2} \hat{t} \frac{\partial b_1}{\partial \hat{t}} = \frac{1}{2} \frac{\frac{1}{\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+1}{2}}}{\hat{t} - \frac{1 + \sqrt{\frac{1}{5}}}{2} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1 - \sqrt{\frac{1}{5}}}{2} \hat{t}^{-\frac{\sqrt{5}+1}{2}}} - \frac{1}{2} \hat{t} \frac{\frac{3 - \sqrt{5}}{2\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{3 + \sqrt{5}}{2\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+1}{2}} + \hat{t}^{-2}}{\left( \hat{t} - \frac{1 + \sqrt{\frac{1}{5}}}{2} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1 - \sqrt{\frac{1}{5}}}{2} \hat{t}^{-\frac{\sqrt{5}+1}{2}} \right)^2} \\
& = \frac{1}{2} \frac{\left( \frac{1}{\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+1}{2}} \right) \left( \hat{t} - \frac{1 + \sqrt{\frac{1}{5}}}{2} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1 - \sqrt{\frac{1}{5}}}{2} \hat{t}^{-\frac{\sqrt{5}+1}{2}} \right) - \hat{t} \left( \frac{3 - \sqrt{5}}{2\sqrt{5}} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{3 + \sqrt{5}}{2\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}+1}{2}} + \hat{t}^{-2} \right)}{\left( \hat{t} - \frac{1 + \sqrt{\frac{1}{5}}}{2} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1 - \sqrt{\frac{1}{5}}}{2} \hat{t}^{-\frac{\sqrt{5}+1}{2}} \right)^2} \\
& = \frac{\frac{\sqrt{5}-1}{2\sqrt{5}} \hat{t}^{\frac{\sqrt{5}+1}{2}} + \frac{\sqrt{5}+1}{2\sqrt{5}} \hat{t}^{-\frac{\sqrt{5}-1}{2}} - \frac{\sqrt{5}+1}{10} \hat{t}^{\sqrt{5}-1} + \frac{\sqrt{5}-1}{10} \hat{t}^{-(\sqrt{5}+1)} - \frac{4}{5} \hat{t}^{-1}}{\left( \hat{t} - \frac{1 + \sqrt{\frac{1}{5}}}{2} \hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1 - \sqrt{\frac{1}{5}}}{2} \hat{t}^{-\frac{\sqrt{5}+1}{2}} \right)^2} \quad (142)
\end{aligned}$$

Let  $G(\hat{t})$  be the numerator of the last equation in (142). Note that  $G(1) = 0$ , and  $\frac{\partial G}{\partial \hat{t}} = \frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}} - \frac{4}{10}\hat{t}^{\sqrt{5}-2} - \frac{4}{10}\hat{t}^{-(\sqrt{5}+2)} + \frac{4}{5}\hat{t}^{-2} = \frac{1}{\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-1}{2}} - \frac{1}{\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+1}{2}} - \frac{4}{10}\hat{t}^{-2}(\hat{t}^{-\sqrt{5}} - 1)(1 - \hat{t}^{\sqrt{5}})$ . So,  $\frac{\partial G}{\partial \hat{t}} < 0$  for all  $\hat{t} \in (0, 1)$ . Hence,  $G(\hat{t}) > 0$  for all  $\hat{t} \in (0, 1)$ , which by (142) means that  $-\frac{b_1}{2} - \frac{1}{2}\hat{t}\frac{\partial b_1}{\partial \hat{t}} > 0$  for  $\hat{t} \in (0, 1)$ . Since  $\frac{d\hat{t}}{dC} > 0$ , we conclude that  $\frac{d\tau(\hat{\theta})}{dC} > 0$ .

Finally, let us show that  $Q$  is convex in  $\tau$ . Note that  $\tau(t) = y(t) + Q(t) = y(t) - \frac{b_1}{2}t$ . Therefore,  $\frac{dy}{d\tau} = \frac{y'(t)}{\tau'(t)} = \frac{y'(t)}{y'(t) - \frac{b_1}{2}}$  and  $\frac{d^2y}{d\tau^2} = \frac{\frac{d}{dt}\frac{dy}{d\tau}}{\tau'(t)} = \frac{-\frac{b_1}{2}y''(t)}{(y'(t) - \frac{b_1}{2})^3}$ . From (130),  $\tau'(t) = y'(t) - \frac{b_1}{2} = \frac{1}{2}b_1 + \frac{(\sqrt{5}-1)-2b_1}{2\sqrt{5}}\hat{t}^{\frac{\sqrt{5}-3}{2}} + \frac{(\sqrt{5}+1)+2b_1}{2\sqrt{5}}\hat{t}^{-\frac{\sqrt{5}+3}{2}}$  which is equal to  $\frac{b_1}{2} + 1 > 0$  when  $t = 1$ . Since  $y''(t) < 0$  by (131), it follows that  $y'(t) - \frac{b_1}{2} > 0$  for  $t \in (0, 1)$ . So,  $\frac{d^2y}{d\tau^2} < 0$ . Since  $Q = \tau - y$ , we have  $\frac{dQ}{d\tau} = 1 - \frac{dy}{d\tau} = \frac{-\frac{b_1}{2}}{y'(t) - \frac{b_1}{2}} > 0$  and  $\frac{d^2Q}{d\tau^2} = -\frac{d^2y}{d\tau^2} > 0$ .

Also, since  $Q(t) = q(\tau(t))$ , we have  $Q'(t) = q'(\tau(t))\tau'(t)$ . So, since  $Q'(t) > 0$  and  $\tau'(t) > 0$ , it follows that  $q'(\theta) \equiv q'(\tau(t)) > 0$ . Finally, differentiating  $Q'(t) = q'(\tau(t))\tau'(t)$  we get:  $0 = Q''(t) = q''(\tau(t))(\tau'(t))^2 + q'(\tau(t))\tau''(t)$ . Since  $\tau''(t) = y''(t) < 0$ , we conclude that  $q''(\theta) \equiv q''(\tau(t)) > 0$  for  $\theta \in (\tau(\hat{\theta}), \tau(1))$ . So  $q(\theta)$  is strictly increasing and convex for  $\theta \in (\tau(\hat{\theta}), \tau(1))$ .

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