# Persuading part of an audience

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**Abstract** I propose a cheap-talk model in which the sender can use private messages and only cares about persuading a subset of her audience. For example, a candidate only needs to persuade a majority of the electorate in order to win an election. I find that senders can gain credibility by speaking truthfully to some receivers while lying to others. The model always admits information transmission in equilibrium for some prior beliefs, and the sender can approximate her preferred outcome when the fraction of the audience she needs to persuade is sufficiently small. I characterize the sender-optimal equilibrium and the value of not having to persuade your whole audience in separable environments. I also extend the model to allow for full-commitment as in Kamenica and Gentzkow (2011).

**Keywords** Cheap talk · Information transmission · Persuassion

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A politician running for office only needs half plus one of the votes. A seller with a capacity constraint only needs to persuade a certain number of consumers to purchase her product. A person looking for a job may apply to many positions, but she only has to convince a single firm to extend an offer. This paper studies the problem of an informed sender who can engage in private conversation with many homogeneous receivers and cares about the behavior of *some but not all* of them. My main finding is that having to persuade only part of an audience significantly facilitates information transmission and increases persuasion power.

Let us examine the first example in more detail. Suppose a politician (the sender, she) is running for office. All voters (the receivers, he) share the same preferences. The unknown state of the world equals either 0 or 1. Each voter will vote for the politician if his expectation about the state of the world is greater than 1/2. The voters share a common prior expectation in the interval $(1/3, 1/2)$. Suppose that the politician learns the true state of the world, and can engage in private cheap talk with each voter. I claim that, if there are sufficiently many voters, then there exists an equilibrium in which she wins the election for sure regardless of the state.

This is possible because the politician only needs to persuade half plus one of the electorate in order to win the election. She can attain this goal by using the following strategy. If the state is indeed 1, then she will let every single voter know this fact. If the state is 0, then she will randomly choose half plus one of the voters and tell them that the state is 1, despite the fact that it is not.

A voter that receives a message saying that the state equals 1 knows that this could be a lie. However, he also knows that he would be more likely to receive this message if it was actually true. Hence, the message conveys some information. When the population is large enough, it conveys sufficient information to overturn prior beliefs arbitrarily close to 1/3. In that case, every voter who receives this message prefers to vote for the politician.

I study a general cheap talk model with many homogeneous receivers and both public and private communication. I depart from the literature by assuming that there are $n$ receivers, but the sender only cares about the highest $n_0 < n$ actions taken. In such cases, the utility of the sender can be completely determined by strict subsets of the receivers. Thus, she only needs to persuade part of her audience in order to maximize her utility. I call the gap between $n$ and $n_0$ an *excess audience.*

I find that the sender can influence the behavior of receivers in equilibrium in a very wide class of environments as long as there is an excess audience (Proposition 4). In some applications, effective information transmission is *only* possible if there is an excess audience (examples 1 and 2). When the fraction of the audience that the sender cares about is small enough, she can achieve her preferred outcome in equilibrium (Proposition 3).

I characterize the sender's benefit from having an excess audience under a separability assumption (Theorem 5). Lipnowski and Ravid (2018), characterize the sender's maximum equilibrium payoff when the sender cares about her entire audience in terms of her value function. The value function is the highest payoff the sender can obtain when all the receivers behave optimally given their posterior beliefs. Under Lipnowski and Ravids' assumptions, the sender's maximum equilibrium payoff equals the quasiconcave envelope of her value function. This envelope is obtained by "flooding the valleys" of the value function. I find that an additional step is needed in the presence of an excess audience.

This step involves a generalization from the politician's communication strategy described above. The sender starts by randomly and privately splitting her audience into a *target* audience that she wants to persuade, and the rest of the receivers. Receivers in the target audience always receive whichever message induces the behavior most favorable to the sender. The communication strategy for the rest of the receivers is chosen to maximize the credibility of the message sent to the target group. This message conveys information because individual receivers are not told whether they were assigned to the target audience.

This kind of strategy allows the sender to implant a fixed posterior belief in a fixed proportion of the audience regardless of the state. I say that such beliefs are *attainable*. The set of attainable beliefs admits a simple and computationally tractable characterization (Lemmas 1 and 2). When the sender wishes to persuade her entire audience, the only attainable belief is the prior. However, the set of attainable beliefs is strictly increasing in the size of the excess audience. The missing step to characterize the sender's maximum equilibrium payoff is to replace the original value with the maximum value over the set of attainable beliefs. This transformation can be thought of as "widening the hills".

The "wider hills" correspond to the benefit of having an excess audience. It is always non-negative, it is strictly positive for some prior beliefs, and it is monotone in the fraction of the audience that the sender wishes to persuade (Proposition 6). As the fraction of the audience that the sender cares about converges to zero, the set of attainable beliefs totally covers the interior of the simplex. Consequently, the maximum equilibrium payoff approaches the best feasible payoff for the sender.

Section 5 considers two extensions of the model. First, I analyze a model with full commitment as in Kamenica and Gentzkow (2011) and the information design literature (Bergemann and Morris, 2016, Taneva, 2019). I find a partial characterization of the maximum sender equilibrium payoff in the full-commitment game. The characterization is similar to the one I found for the cheap talk game. The only difference is that it uses the concave envelope of the value, instead of the quasiconcave envelope. It is a partial characterization at the moment, because I have only shown it to be a lower bound. Whether it is also an upper bound remains work in progress.

I also analyze an example with the classic quadratic loss functions from Crawford and Sobel (1982). This example does not satisfy the assumption that the sender's preferences are common knowledge. However, it is still possible to use strategies with a random target audience in order to transmit information. When the fraction of the audience that the sender cares about is small, the sender can approximate her preferred outcome and transmit large amounts of information to most of her audience in equilibrium. Unlike the case without an excess audience, information transmission is possible for any level of bias.

Since the seminal work of Crawford and Sobel, different authors have found different mechanisms that can give credibility to an expert. Information transmission is possible via cheap talk when incentives are not too misaligned, or there are multiple senders (Battaglini, 2002), or multiple dimensions of information (Chakraborty and Harbaugh, 2010), or strategic complementarities (Levy and Razin, 2004, Baliga and Sjöström, 2012), or the sender has transparent motives (Lipnowski and Ravid, 2018), among other reasons. An excess audience is a novel mechanism which allows for information transmission in some settings in which none of the aforementioned mechanisms operate.

Some authors have studied cheap talk communication with multiple audiences. However, this literature has focused on situations when the sender cares about the actions of all receivers, either directly or indirectly. Farrell and Gibbons (1989) showed that senders with multiple audiences sometimes prefer public communication and sometimes private communication. Goltsman and Pavlov (2011) show that the sender might be strictly better off by combining both types of messages. Hence, I allow the sender to use both private and public messages.

Basu et al. (2018) study the problem of an informed sender who wants to use cheap talk to prevent an ethnic conflict. Their model has a large audience, and the sender is allowed to use private messages. However, they restrict attention to strategies that are anonymous conditional on observed heterogeneity. This restriction precludes the strategies with random target audiences that I analyze. Instead, they exploit preference complementarities in order to find an equilibrium with effective information transmission.

There is a large body of literature using cheap talk to study information transmission between politicians and electorates, dating at least as far back as Harrington (1992). Some recent work in this area includes Schnakenberg (2015), Panova (2017), Jeong (2019), and Kartik and Van Weelden (2017). Other recent papers analyze the problem from the information design perspective, including Alonso and Câmara (2016), and Chan et al. (2019). Within this literature, the papers that assume talk is cheap focus on public messages. My contributions highlight the importance of private anonymous communication (e.g., through social media).

# 1. Model

There is one sender $s$, and a set of receivers $r \in R = \{1, \ldots, n\}$. The true state $\theta^0$ is drawn form a finite set $\Theta \subseteq \mathbb{R}$ with at least two distinct elements. Let $\Delta^\circ(\Theta)$ be the set of beliefs with full support. The sender and receivers share a common prior $p^0 \in \Delta^\circ(\Theta)$.

The sender learns the state, and then she chooses a public message $m_0 \in M$, and a private message $m_r^p \in M$ for each receiver $r$. $M$ is finite but rich enough as to *not* restrict the set of equilibrium outcomes.[1] Each receiver observes the compound message $m_r = (m_0, m_r^p)$, but observes neither the state nor other receivers' private messages. Then, all receiver observe a public sunspot $\omega^0 \sim \text{unif}[0,1]$. Finally, each receiver $r$ chooses an action $a_r$ from a finite set $A \subseteq \mathbb{R}$. Action profiles are denoted by $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$, and message profiles by $\mathbf{m} = (m_0, m_1^p, \ldots, m_n^p) \in M^{n+1}$.

All receivers have identical preferences. The utility of $r$ depends only on his own action and the state, and is given by $u_R(a_r, \theta)$. The sender's utility $u_S(\mathbf{a})$ does *not* depend on the state, and satisfies $u_S(\mathbf{a}) = u_S(\mathbf{a}')$ whenever $\mathbf{a}$ and $\mathbf{a}'$ have the same empirical distribution.

[1] A sufficient condition is $\|M\| \geq \|A\|^2(\|\Theta\| + 1)$.

### 1.1. Pivotal fraction of the audience

Define the *pivotal number of receivers* to be the smallest number $n^0$ such that the sender's utility only depends on the the highest $n^0$ actions taken by the receivers. Let $a^{(r)}$ denote the $r$-th order statistic of $\mathbf{a}$. The pivotal number of receivers is given by

$$n^0 := \min\left\{n' \,\middle|\, u_S(\mathbf{a}) = u_S(\tilde{\mathbf{a}}) \text{ whenever } a^{(r)} = \tilde{a}^{(r)} \text{ for all } r \geq n_0 + 1 - n'\right\}. \quad (1)$$

The *pivotal fraction of the audience* is $\gamma^0 = n^0/n$. If $n^0 < n$, the number of people sender can talk to is strictly greater than the number of people she wishes to persuade. In that case, I say that there is an *excess audience*.

*Example 1 [Election]* The sender is a candidate in an election. Each receiver $r$ will either vote for the sender ($a_r = 1$), or against her ($a_r = 0$). The state is either 0 or 1. And voters share a common prior with $\pi^0 := p^0(1) \in (0, 1)$. The sender wins the election if and only if she obtains a super-majority of at least $\gamma \in (0, 1)$ of the votes. All receivers share the same preferences. Receiver $r$ prefers $a_r = 1$ if and only if his posterior beliefs $p_r$ satisfy $\pi := p_r(1) \geq \eta^0$, where $\eta^0 \in (0, 1)$ is a fixed parameter. The sender gets a utility of 1 if she wins the election, and 0 otherwise. The pivotal number of receivers is $n^0 = \min\{n' \mid n' \geq n\gamma\}$.

*Example 2 [Concert]* The sender is a musician performing at a venue with maximum capacity of $n^0 < n$. The (common) value that each receiver would get from attending the performance is $\theta^0 \in \{1, 2, 3\}$. Prior beliefs are given by $p^0 = (1/2, 1/3, 1/6)$. Receivers are risk neutral and each receiver $r$ demands at most one ticket. His choice consists of either buying or nor buying a ticket as a function of the price. This individual demand can be summarized by the maximum price (in whole dollars) $a_r$ that would induce him to buy. In equilibrium we must have $a_r = \mathbb{E}[\,\theta^0 | m_r\,]$.

The venue manager anticipates the total demand and set a revenue maximizing price taking into account the capacity constraint. The sender's profits are proportional to the revenue. Because of the capacity constraint, $u_S$ only depends on the $n^0$ receivers with highest individual demands.

### 1.2. Strategies, updating, and equilibrium

*Communication strategies* map states into distributions $x(\theta)$ over message profiles. *Receiver strategies* are measurable maps from messages and sunspot realizations into actions $y_r(m, \omega)$. *Updating rules* map compound messages $m$ into posterior beliefs $q_r(m)$ over the state. With slight abuse of notation, let $x(\mathbf{m}|\theta) := [x(\theta)](\mathbf{m})$ and $q_r(\theta|m) := [q_r(m)](\theta)$. Also, let $x_r(\cdot|\theta)$ denote the marginal distribution over $m_r$ induced by $x(\theta)$. $\mathrm{BR}(p)$ denotes the receivers' best response correspondence. Profiles of receiver strategies and updating rules are denoted by $\mathbf{y} = (y_1, \ldots, y_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$.

**Definition 1** A *(cheap-talk) equilibrium* is a tuple $(x, \mathbf{y}, \mathbf{q})$ such that

(i) For every receiver $r$, $q_r$ is consistent with Bayes' rule given $x$.

(ii) For every receiver $r$, message $m$, and sunspot realization $\omega$, $y_r(m, \omega)$ maximizes $\sum_{\theta \in \Theta} u_R(a, \theta) q_r(\theta|m)$.

(iii) For every message profile $\mathbf{m}$, if there exists a state $\theta$ such that $x(\mathbf{m}|\theta) > 0$, then $\mathbf{m}$ maximizes $\int_0^1 u_S(y_1(m_1, \omega), \ldots, y_n(m_n, \omega))\, d\omega$.

## 2. Attainable posteriors

This section discusses two technical lemmas that drive the rest of the results. Readers interested in the main results can skip to Section 3. A key step in my analysis is to determine the maximum influence that the sender can exert over the beliefs of part of her audience. If the sender wants to guarantee that there are always $n^0$ receivers having certain posterior beliefs, what values can these posteriors take?

**Definition 2** For $\gamma \in [0, 1]$, a belief $p' \in \Delta(\Theta)$ is $\gamma$-*attained* by a communication strategy $x$ and a profile of updating rules $\mathbf{q}$ if

(i) For every receiver $r$, $q_r$ is consistent with Bayes' rule given $x$.

(ii) For every state $\theta$ and every message profile $\mathbf{m}$ in the support of $x(\theta)$, there exists a set $T \subseteq R$ such that $\|T\| \geq n\gamma$ and $q_r(m_r) = p'$ for all $r \in T$.

Say that $p'$ is $\gamma$-attainable of there exist $x$ and $\mathbf{q}$ that $\gamma$-attain it. Let $\hat{P}$ be the set of $\gamma^0$-attainable beliefs. Denote likelihood ratios by $\Lambda(\theta, \theta'; p) = p(\theta)/p(\theta')$. The following lemma asserts that a posterior belief is $\gamma$-attainable if and only if the prior likelihood ratios are not too distorted.

**Lemma 1** *A belief $p' \in \Delta(\Theta)$ is $\gamma$-attainable if and only if for all states $\theta$ and $\theta'$*

$$\Lambda(\theta, \theta'; p') \geq \gamma \Lambda(\theta, \theta'; p^0). \tag{2}$$

It follows that $\hat{P}$ is a nonempty closed and convex polytope. Note that every posterior is 0-attainable, and only $p^0$ is 1-attainable. For $\gamma > 0$, only posteriors with full support are $\gamma$-attainable. All the proofs are in the appendix. The following example illustrates how to reach the bounds in (2).

*Example 1 [Election continued]* Suppose the politician uses the following communication strategy. She first chooses a target audience $T \subset R$ consisting of exactly $n^0$ receivers. Each receiver $r$ only observes one of two possible messages $m_r = H$ or $m_r = L$. The sender always sends message $H$ to all the receivers in $T$. Receivers not in $T$ receive message $H$ if and only if $\theta^0 = 1$. This strategy results in the conditional probabilities $x_r(H|1) = 1$ and $x_r(H|0) = \gamma^0$. Therefore, the posterior belief about $\theta^0 = 1$ after observing $H$ is

$$\pi^H = \frac{\pi^0}{\pi^0 + (1 - \pi^0)\gamma}. \tag{3}$$

It is easy to verify that this posterior satisfies the condition from Lemma 1 with equality. Moreover, at least $n^0$ receive message $H$. Hence, $\pi^H$ is $\gamma^0$-attained.

Lemma 2 asserts that each vertex of $\hat{P}$ corresponds to a partition of states into two blocks. States in one of these blocks have increased likelihoods relative to the prior, and states in the other block have decreased likelihoods. This characterization makes $\hat{P}$ computationally tractable. It is particularly advantageous in monotone environments when the sender would always want to increase the receivers' beliefs about the state.

**Lemma 2** *A belief $p' \in \Delta(\Theta)$ is a vertex of $\hat{P}$, if and only if there exists a partition*

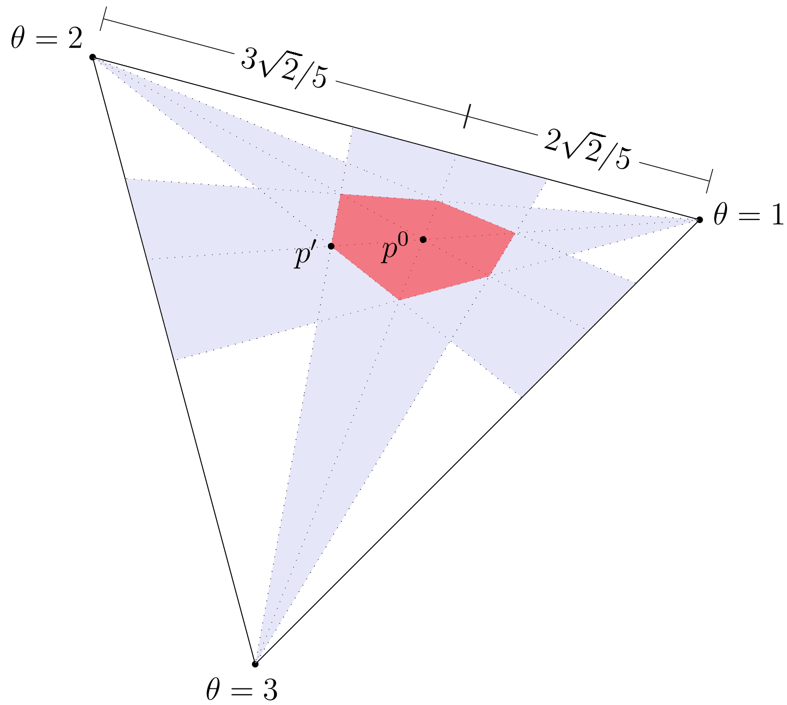


**Figure 1** – $\gamma^0$-attainable set for concert example with $\gamma^0 = 1/2$.

$\{\Theta^+, \Theta^-\}$ *of* $\Theta$ *such that for every two stater* $\theta'$ *and* $\theta''$

$$\Lambda(\theta', \theta''; p') = \frac{\gamma^0 + (1-\gamma^0)\mathbb{1}(\theta' \in \Theta^+)}{\gamma^0 + (1-\gamma^0)\mathbb{1}(\theta'' \in \Theta^+)} \cdot \Lambda(\theta', \theta''; p^0). \tag{4}$$

*Example 2 [Concert continued]* Suppose that the venue capacity equals $n^0 = n/2$, so that $\gamma^0 = 1/2$. The prior likelihood ratios are $\Lambda(1,2;p^0) = 3/2$, $\Lambda(1,3;p^0) = 3$, and $\Lambda(2,3;p^0) = 2$. Hence, it follows from (2) that a posterior $p$ is $\gamma^0$-attainable if and only if $\Lambda(1,2;p) \in [3/4, 3]$, $\Lambda(1,3;p) \in [3/2, 6]$, and $\Lambda(2,3;p) \in [1,4]$. These conditions correspond to the cones spanning from each vertex of the simplex in Figure 1. $\hat{P}$ is the shaded irregular hexagon surrounding $p^0$. The vertex $p' = (1/3, 4/9, 2/9$ is corresponds to the partition of $\Theta$ into $\Theta^+ = \{\theta_3, \theta_2\}$ and $\Theta^- = \{\theta_1\}$. This vertex maximizes $\mathbb{E}_p[\,\theta^0\,]$ subject to $p \in \hat{P}$.

# 3. Information transmission and persuasion

## 3.1. Persuading a small part of an audience

It is possible to excerpt great influence over very small fractions of an audience. Suppose that the sender's utility can be maximized by a constant action profile, and this action profile is a best response a belief with full support. Lemma 1 implies that this belief is $\gamma$-attainable for sufficiently low values of $\gamma$. Hence, when the sender only cares about small fractions of her audience, she can reach her preferred outcome in equilibrium. Formally,

**Assumption 1** There exists $a^* \in A$ and $p^* \in \Delta^\circ(\Theta)$ such that $a^* \in \text{BR}(p^*)$ and $u_S(a^*, \ldots, a^*) \geq u_S(\mathbf{a})$ for every $\mathbf{a} \in A^n$.

**Proposition 3** *For any $\Theta$, $A$, $u_R$, $u_S$, and $p^0$, there exists a threshold $\bar{\gamma} \in (0,1)$ such that if $\gamma^0 \leq \bar{\gamma}$ and Assumption 1 holds, then the game admits an equilibrium in which the sender obtains her preferred outcome.*

*Example 1 [Election continued]* When is victory attainable for the politician? She wins the election when the posterior beliefs of at least $\gamma^0$ of the electorate satisfy $\pi_r \geq \eta_0$. From lemma 1, any such belief can be $\gamma^0$-attained if and only if

$$\frac{\eta^0}{1-\eta^0} \leq \frac{1}{\gamma^0}\frac{\pi^0}{1-\pi^0}. \tag{5}$$

This condition is satisfied whenever $\pi^0$ is high enough, $\eta_0$ is small enough, or $\gamma^0$ is small enough. For the case $\gamma^0 = 1/2$ with $\eta^0 = 1/2$, (5) reduces to the condition $\pi^0 \geq 1/3$ from the introduction.

## 3.2. Effective information transmission

Cheap-talk models often allow for some information transmission in equilibrium. See, for instance, Proposition 1 in Lipnowski and Ravid (2018). The question I address is whether the sender can transmit sufficient information as to influence the behavior of receivers. Say that an equilibrium $(x, \mathbf{y}, \mathbf{q})$ exhibits *effective information transmission* if at least one receiver plays an action $a \notin \text{BR}(p^0)$ with positive probability. Effective information transmission is possible for some prior beliefs under a mild sensitivity assumption that rules out trivial cases. Namely, the receivers' preferred actions should actually depend on the state. For the remainder of this section, fix all the features of the environment except for $p^0$.

**Assumption 2** $\mathrm{BR}(\theta) \cap \mathrm{BR}(\theta') = \emptyset$ for some $\theta, \theta' \in \Theta$

**Proposition 4** *Under assumptions 1 and 2, if there is an excess audience, then there exists an open set of priors $P' \subseteq \Delta(\Theta)$ such that, if $p^0 \in P'$, then the game has an equilibrium with effective information transmission.*

In order to gain some intuition about this result, suppose that $a^* \notin \mathrm{BR}(p^0)$. If $p^0$ was "similar enough" to $p^*$ so that the condition from Lemma 1 is satisfied, then there would exist an equilibrium in which the sender persuades $n^0$ receivers to take action $a^*$. The set of prior distributions that are "similar enough" to $p^*$ has a non-empty interior.

Proposition 4 guarantees that the sender can influence the behavior of the receivers in equilibrium for *some* prior beliefs, but not necessarily for all. For example, in the election example, effective information transmission is only possible in equilibrium when victory is attainable, i.e., when condition (5) holds. If the prior probability of $\theta^0 = 1$ is too low, then no receivers will ever vote for the sender in any equilibrium.

## 4. Sender-optimal equilibrium under separability

### 4.1. Geometric characterization

The sender's *value function* $v^0(p)$ specifies the maximum utility that the sender could obtain if all receivers shared a posterior $p$ and acted optimally,

$$v^0(p) = \max \left\{ u_S(\mathbf{a}) \,\middle|\, a_r \in \mathrm{BR}(p) \text{ for all receivers } r \right\}. \tag{6}$$

The sender's *maximum equilibrium payoff* $v^*$ is the maximum utility that the sender can obtain in any equilibrium. This section characterizes $v^*$ under the following separability assumption.

**Assumption 3** There exist a strictly increasing function $\tilde{u}_S : A \to \mathbb{R}$ such that

$$u_s(\mathbf{a}) = \frac{1}{n_0} \sum_{i=1}^{n_0} \tilde{u}_S \left( a^{(n+1-i)} \right). \tag{7}$$

I use two operators on the set of upper-semicontinuous functions from beliefs into sender payoffs. First, $\mathrm{env}_{\mathrm{q}}(v)$ is the quasiconcave envelope of $v$. That is, $\mathrm{env}_{\mathrm{q}}\, v$ is the pointwise-minimum, quasiconcave function that majorizes $v$. Second, $\mathrm{att}_{\gamma^0}\, v$ gives the maximum of $v$ arising from $\gamma^0$-attainable beliefs. Formally,

$$\mathrm{att}_{\gamma^0}\, v(p) = \max \left\{ v(p') \,\middle|\, p' \in \hat{P}(p) \right\}, \tag{8}$$

where $\hat{P}(p)$ is the set of beliefs that would be $\gamma^0$-attainable if $p^0 = p$. Intuitively, $\mathrm{env}_{\mathrm{q}}\, v$ operates by "flooding the valleys" while $\mathrm{att}_{\gamma^0}\, v$ is obtained by "widening the hills." See the left and center panels of Figure 2 for an example.

Using $\gamma^0$-attainable beliefs allows me to establish $v^* \geq \mathrm{att}_{\gamma^0}\, v^0(p^0)$. The results from Lipnowski and Ravid (2018) imply that $v^* \geq \mathrm{env}_{\mathrm{q}}\, v^0(p^0)$. Combining both ideas yields the following theorem.

**Theorem 5** *Under assumption 3, $v^* = \mathrm{att}_{\gamma^0}\, \mathrm{env}_{\mathrm{q}}\, v^0(p^0)$.*

Lemma 1 implies that $\mathrm{att}_1\, v^0 = v^0$. Hence, Theorem 5 reduces to Theorem 2 in Lipnowski and Ravid (2018) when $\gamma^0 = 1$. However, in view of Proposition 4, the two results differ whenever there is an excess audience and Assumption 1 holds. The difference between the results corresponds to the value of excess audience defined in the following subsection.

### 4.2. The value of excess audience and private communication

What happens to the value of the sender when she has to persuade a larger or smaller fraction of her audience? The maximum sender equilibrium can depend on the size of her audience and the number of people she wishes to persuade. However, Assumption 3 guarantees that it is always measured in the same units. Define the *value of excess audience* to be the difference between $v^*$ and the maximum equilibrium payoff to the sender in an alternative environment with $n = n^0$. By

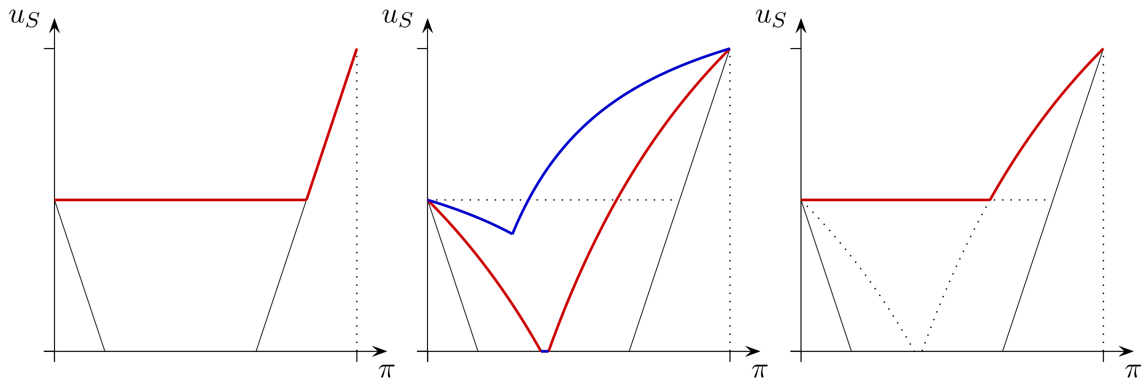


**Figure 2** – Value functions for the bookie example with $w = 0.5$, $\eta^2 = 2$, $b_1 = 1/2$, and $b_0 = 1/5$. Left panel: $\mathrm{env_q}\, v^0$. Center panel: $\mathrm{att}_{1/3}\, v^0$ (red) and $\mathrm{att}_{1/10}\, v^0$ (blue). Right panel: $v^*$ with $\gamma^0 = 1/3$.

Theorem 5 and Lemma 1, the value of excess audience is given by the difference $v^* - \mathrm{env_q}\, v^0(p^0)$.

**Proposition 6** *Under assumptions 1–3, there exist a function $\bar{\gamma}(p) > 0$ and an open set $P' \subseteq \Delta^\circ(\Theta)$ such that the value of excess audience is*

*(i) non-negative and non-increasing in $\gamma^0$ for every $p^0 \in \Delta^\circ(\Theta)$,*

*(ii) strictly positive whenever $p^0 \in P'$ and $\gamma^0 < 1$,*

*(iii) and equal to $\tilde{u}_S(a^*) - v^0(p^0)$ whenever $\gamma^0 \leq \bar{\gamma}(p)$.*

Consider an alternative model in which the sender can use only public messages. She still cares only about the actions of part of her audience, but she has to persuade all of the receivers. The *value of private communication* is the gap between $v^*$ and the maximum sender equilibrium value in this alternative model. Without private messages, the only $\gamma^0$-attainable belief is the prior. Hence, the value of private communication coincides with the value of excess audience.

*Example 3 [Bookie]* The state $\theta^0 \in \{0, 1\}$ indicates the winner of a rigged boxing match. The sender is a bookie who knows the state, and would like to persuade the receivers to place large bets. Each sender places a bet $a_r \in [-w, w]$, where a positive number denotes a bet on $\theta^0 = 1$ and a negative number denotes a bet on $\theta^0 = 0$. Bets on different states have different exogenous returns $b_0, b_1 > 0$ with $b_0 b_1 < 1$. Receivers have logarithmic Bernoulli utility functions, and initial wealth $w$. For example, a receiver that places a bet on $\theta^0 = 1$ maximizes the expectation

$\pi \log(w + a_r b_1) + (1 - \pi) \log(w - a_r)$ subject to $a_r \in [0, w]$, where $\pi = p(1)$. The bookie can accept at most $n_0 \leq n$ bets, and his total utility equals $V_0 + \eta^0 V_1$, where $V_\theta$ is the total volume of bets on $\theta$, and $\eta^0 > 1$ is a fixed parameter.

This example deviates slightly from our environment because $A$ is an interval and the sender cares about the tail of the distribution of the *absolute value* of the actions, but Theorem 5 still applies. Best responses are given by

$$\mathrm{BR}(\pi) = \begin{cases} \frac{w}{b_1}[(1+b_1)\pi - 1] & \text{if} \quad \pi \geq \frac{1}{1+b_1} \\ -\frac{w}{b_2}[(1+b_2)(1-\pi) - 1] & \text{if} \quad \pi \leq \frac{b_2}{1+b_2} \\ 0 & \text{otherwise} \end{cases}. \tag{9}$$

Just as in the election example, the set of $\gamma^0$-attainable posteriors corresponds to the interval

$$\left[\frac{\pi^0}{\pi^0 + (1-\pi^0)/\gamma^0}, \frac{\pi^0}{\pi^0 + (1-\pi^0)\gamma^0}\right]. \tag{10}$$

The $\gamma^0$-attainable value is obtained by substituting these bounds into $u_S(\mathrm{BR}(p))$. Figure 2 illustrates $\mathrm{env}_\mathrm{q}\, v^0$ (left), $\mathrm{att}_{\gamma^0}\, v^0$ for $\gamma^0 \in \{1/3, 1/10\}$ (middle), and $v^*$ for $\gamma^0 = 1/3$ (right). The gap between the envelopes in the left and right panels corresponds to both the value of private communication, and the value of having more receivers than capacity to take bets.

# 5. Extensions

## 5.1. Information design

Suppose now that the sender chooses and commits to a communication strategy *before* learning the state of nature. This timing corresponds to the the information design paradigm used by Kamenica and Gentzkow (2011). See also Bergemann and Morris (2016). Define a *commitment protocol* to be a tuple $(x, \mathbf{y}, \mathbf{q})$ satisfying conditions $(i)$ and $(ii)$ in Definition 1. The sender's *maximum commitment value* $v^{**}(p)$ is the maximum utility that the sender can obtain in any commitment

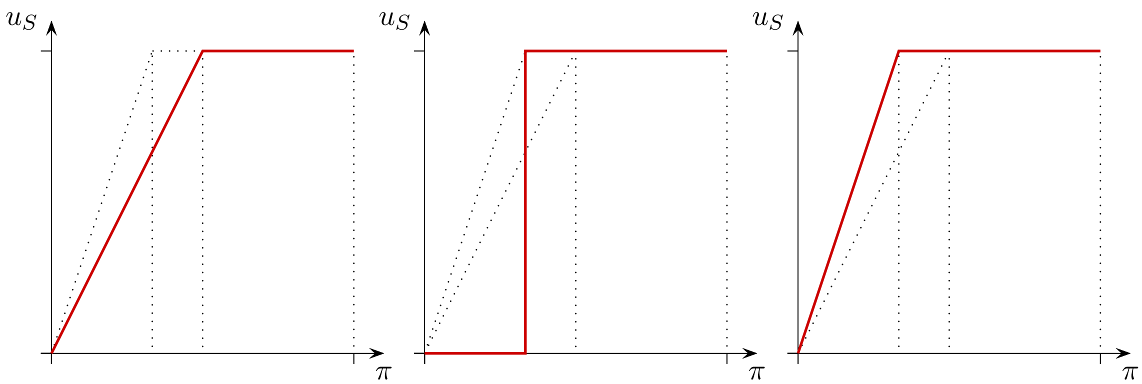


**Figure 3** – Commitment value for election example with $\gamma^0 = \eta^0 = 1/2$.

protocol.

Since every equilibrium is a commitment protocol, Propositions 3 and 4 continue to hold in the game with commitment. Also, it is possible to obtain a geometric characterzation of $v^{**}$ Let, $\operatorname{env} v^0$ denote the concave envelope of $v^0$, that is, the pointwise-minimum, concave function that majorizes $v^0$.

**Theorem 7** *Under assumption 3,* $v^{**} \geq \operatorname{att}_{\gamma^0} \operatorname{env} v^0(p^0)$.

Figure 3 illustrates this result for the election example with $\gamma^0 = \eta^0 = 1/2$. The well known figure on the left panel shows $\operatorname{env} v^0$. This would be the maximum commitment value for the sender if she was restricted to use public messages. The middle panel shows $v^* = \operatorname{att}_{\gamma^0} v^0$. The right panel shows $v^{**}$. The value of commitment is thus given by the gap between the right and the middle panel. The value of excess audience under commitment is given by the gap between the right and the left panel.

## 5.2. Lack of transparent motives

Assuming that the sender does not care about the state simplifies the analysis and plays a crucial role in Theorem 5. However, an excess audience can play an important role even when the sender's motives are not transparent. Consider the classic quadratic-loss game from Crawford and Sobel (1982), with the twist that the sender faces excess audience.

The state is distributed $\theta^0 \sim \operatorname{unif}[0, 1]$. The receivers take actions in $A = [0, 1]$, and their utility is $u_R(a_r, \theta^0) = -(\theta^0 - a_r)^2$. The sender's utility is given by

$$u_S(\mathbf{a}, \theta^0) = -\frac{1}{n_0} \sum_{i=1}^{n_0} \left(\theta^0 + b - a^{(n+1-i)}\right)^2, \tag{11}$$

where $b > 1/4$ is a fixed parameter measuring the *bias* of the sender relative to the receivers. Suppose that the sender is only allowed to use private messages.

When $n_0$ is equal to the total number of receivers, this is a particular instance of the environment studied by Goltsman and Pavlov (2011). Since all the receivers are biased in the same direction, there are only babbling equilibira. In contrast, when $n$ is much larger than $n^0$, there can be effective information transmission.

**Proposition 8** *For all $\epsilon > 0$, there exists $\underline{n} < \infty$ such that whenever $n \geq \underline{n}$ the maximum sender equilibrium value in the quadratic-loss game is greater than $-\epsilon$ if $\theta^0 + b \leq 1$, and greater than $-b^2 - \epsilon$ otherwise.*

The proof is constructive. The sender can create a finite partition of $[0, 1]$. She randomly splits the receivers into two groups of sizes $n - n_0$ and $n_0$. She then reveals truthfully which block of the partition contains $\theta^0$ to the members of the first group. She misleads the members of the second group so that they choose her preferred action. When $n$ is very large, it is possible to construct incentive compatible equilibria of this sort with very fine partitions.

## 6. Closing remarks

When talk is cheap, information transmission requires the sender to be indifferent between all messages he uses. I found a novel mechanism that can create indifference. When the sender only cares about persuading a strict subset of her audience, she is indifferent between the messages she sends to the rest of the receivers. It is possible for her to gain credibility by being truthful with some receivers while lying to others. This mechanism can greatly facilitate information transmission and increase the sender's power to persuade.

The present work provides a full characterization of the sender-optimal equilibrium assuming that the receivers do not care about each other actions, the sender has transparent motives, and that her preferences are monotone and satisfy a separability condition. These restrictions greatly simplify the analysis. They make it possible to characterize the set of equilibria combining $\gamma^0$-attainability with the techniques from Lipnowski and Ravid (2018). However, they appear to be inessential for many of the results. The value of having a large audience in general settings is left as an open problem.

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# A. Proofs

## A.1. Attainable posteriors

*Proof of Lemma 1.* (⇒) Suppose that $p'$ is $\gamma$-attained by some $x$ and $\mathbf{q}$, and fix any two states $\theta'$ and $\theta''$. Let $M_r$ be the set of compound messages $m$ such that $q_r(m) = p'$, let $\chi_r$ be the indicator that $m_r \in M_r$, and let $n' = \sum_{r\in R} \chi_r$ be the number of receivers whose posterior equals $p'$. The prior beliefs $p^0$, communication strategy $x$ and updating rule $\mathbf{q}$ induce a joint probability measure over $\theta^0$, $\chi_r$ and $n'$. Note that

$$\mathbb{E}\left[\, n' | \theta^0 = \theta' \,\right] = \sum_{r \in R} \mathbb{E}[\, \chi_r | \theta' \,] = \sum_{r \in R} \Pr(m_r \in M_r | \theta'). \tag{12}$$

where I am using short notation for the conditioning event $\theta^0 = \theta'$. Since $p'$ is $\gamma^0$-attainable, it follows that

$$\Pr(n' \geq n^0 | \theta') = 1 \qquad \Rightarrow \qquad \mathbb{E}[\, n' | \theta' \,] \geq n^0. \tag{13}$$

Combining (12) and (13) it follows that

$$\frac{1}{n} \sum_{r \in R} \Pr(m_r \in M_r | \theta') \geq \gamma^0. \tag{14}$$

Therefore there exists at least one receiver $r'$ such that $\Pr(m_{r'} \in M_{r'} | \theta') \geq \gamma^0$.

It follows from Bayes' rule that

$$p'(\theta) = \frac{p^0(\theta) \Pr(m_{r'} \in M_{r'} | \theta)}{\Pr(m_{r'} \in M_{r'})}, \quad \text{and} \quad p'(\theta') = \frac{p^0(\theta') \Pr(m_{r'} \in M_{r'} | \theta')}{\Pr(m_{r'} \in M_{r'})}. \tag{15}$$

Taking the ratio of these equations yields:

$$\Lambda(\theta, \theta'; p') = \frac{\Pr(m_{r'} \in M_{r'} | \theta)}{\Pr(m_{r'} \in M_{r'} | \theta')} \cdot \Lambda(\theta, \theta'; p^0) \geq \gamma^0 \Lambda(\theta, \theta'; p^0). \tag{16}$$

($\Leftarrow$) Suppose $p'$ satisfies (2), and fix an arbitrary state $\theta^*$. For each $\theta$, let $\rho(\theta)$ denote the ratio of likelihood ratios:

$$\rho(\theta) := \frac{\Lambda(\theta, \theta^*; p')}{\Lambda(\theta, \theta^*; p^0)} = \frac{p^0(\theta^*) p'(\theta)}{p^0(\theta) p'(\theta^*)}. \tag{17}$$

Consider the following communication strategy. A target set $T \subseteq R$ consisting of exactly $n^0$ receivers is chosen uniformly from $\{R' \subseteq R | \|R'\| = n^0\}$. All receivers in $T$ receive the (compound) message $m' = (m^\emptyset, m^1)$ with probability 1. Receivers $r$ *not* in $T$ receive message $m'$ with conditional probability

$$x_r(m' | \theta) = \phi(\theta) := \frac{1}{1 - \gamma} \left( \frac{\Lambda(\theta)}{\sum_{\theta' \in \Theta} \Lambda(\theta')} - \gamma \right), \tag{18}$$

and with the remaining probability they receive a different fixed compound message $m'' = (m^\emptyset, m^2)$.[2] Since $p'$ satisfies (2), it follows that $\rho(\theta) \in [\gamma, 1/\gamma]$ and, consequently, $\phi(\theta) \in [0,1]$ for every state $\theta$.

A receiver $r$ who receives message $m'$ does not know whether he belongs to $T$ or not. Hence, Bayes' rule dictates that

$$q_r(\theta|m') = \frac{p^0(\theta)\Big[(1-\gamma)\phi(\theta) + \gamma\Big]}{\Pr(m_r = m')} = \frac{p^0(\theta)\rho(\theta)}{\Pr(m_r = m')\sum_{\theta' \in \Theta} \rho(\theta')}, \tag{19}$$

where the second inequality follows from substituting with (18). Therefore, for every pair of states $\theta$ and $\theta'$ we have that

$$\frac{q_r(\theta|m')}{q_r(\theta'|m')} = \frac{p^0(\theta)\rho(\theta)}{p^0(\theta')\rho(\theta')}. \tag{20}$$

Substituting with (17) yields

$$\Lambda\Big(\theta, \theta'; q_r(\,\cdot\,|m')\Big) = \frac{p^0(\theta)\dfrac{p^0(\theta^*)p'(\theta)}{p^0(\theta)p'(\theta^*)}}{p^0(\theta')\dfrac{p^0(\theta^*)p'(\theta')}{p^0(\theta')p'(\theta^*)}} = \Lambda(\theta, \theta'; p') \tag{21}$$

Since $\theta$ and $\theta'$ were arbitrary, it follows that $q_r(\,\cdot\,|m') = p'$ for every receiver $r$. This implies that $p'$ is $\gamma^0$-attained by $x$ and $\mathbf{q}$, because there are always at least $n^0$ receivers with $m_r = m'$. ■

*Proof or Lemma 2.* (⇒) Let $p^v$ be a vertex of $\hat{P}$. If $\gamma^0 = 1$, then we must have $p^v = p^0$ and we can set $\Theta^+ = \Theta$. For the remainder of the proof, suppose that $\gamma^0 < 1$. In this case, $\hat{P} \subseteq \Delta^\circ(\Theta)$.

Let $g : \Delta^\circ(\Theta) \times \Delta(\Theta) \to \mathbb{R}$ be given by

$$g(p, p') := \min_{\theta, \theta' \in \Theta} \frac{\Lambda(\theta, \theta'; p)}{\Lambda(\theta, \theta'; p')}. \tag{22}$$

Note that full-support beliefs $p$ are $\gamma^0$-attainable if and only if $g(p, p^0) \geq \gamma^0$. Berge's theorem implies that $g$ is continuous on its first argument. Hence, every interior belief $p$ with $g(p, p^0) > \gamma^0$ is in the interior of $\hat{P}$. Since $p^v$ is in the boundary of $\hat{P}$, we must have $g(p^v, p^0) = \gamma^0$. Therefore, there exist states $\theta^-$ and

[2] Note that the sender only uses two private messages, and that public messages are not informative. These facts play no role in this proof, but they are used in the proof of other propositions which rely on on this communication strategy.

$\theta^+$ such that

$$\Lambda(\theta^-, \theta^+; p^v) = \gamma^0 \Lambda(\theta^-, \theta^+; p^0). \tag{23}$$

Define the sets $\Theta^-$ and $\Theta^+$ by

$$\Theta^- = \left\{ \theta \in \Theta \middle| \quad \Lambda(\theta, \theta^+; p^v) = \gamma^0 \Lambda(\theta, \theta^+; p^0). \right\}, \tag{24}$$

and

$$\Theta^+ = \left\{ \theta \in \Theta \middle| \quad \Lambda(\theta^-, \theta; p^v) = \gamma^0 \Lambda(\theta^-, \theta; p^0). \right\}. \tag{25}$$

These sets are nonempty because $\theta^- \in \Theta^-$ and $\theta^+ \in \Theta^+$. I will show that they conform a partition of $\Theta$.

For any state $\theta \in \Theta^+$,

$$\frac{p^0(\theta)}{p^v(\theta)} = \gamma^0 \frac{p^0(\theta^-)}{p^v(\theta^-)} = \frac{p^0(\theta^+)}{p^v(\theta^+)} \neq \gamma^0 \frac{p^0(\theta^+)}{p^v(\theta^+)}, \tag{26}$$

where the first equality follows from (25), the second one from (25), and the inequality from $\gamma^0 < 1$. Rearranging terms, equation (26) implies that $\theta \notin \Theta^-$. Hence, $\Theta^+ \cap \Theta^- = \emptyset$.

It remains to show that $\Theta^+ \cup \Theta^- = \Theta$. Let $\kappa$ be the number given by

$$\kappa = \frac{\|\Theta\| - \|\Theta^+ \cup \Theta^-\|}{p^*(\Theta^+ \cup \Theta^-)}. \tag{27}$$

Since $p^*$ has full support and $0 < \|\Theta^+ \cup \Theta^-\| \leq \|\Theta\|$, we know that $\kappa \in (0, \infty)$. For $\epsilon > 0$, construct $p^+_\epsilon$ and $p^-_\epsilon$ as follows.

$$p^+_\epsilon(\theta) = \begin{cases} (1 + \kappa\epsilon) p^v(\theta) & \text{if} \quad \theta \in \Theta^+ \cup \Theta^- \\ p^v(\theta) - \epsilon & \text{if} \quad \theta \notin \Theta^+ \cup \Theta^- \end{cases}, \tag{28}$$

and

$$p^-_\epsilon(\theta) = \begin{cases} (1 - \kappa\epsilon) p^v(\theta) & \text{if} \quad \theta \in \Theta^+ \cup \Theta^- \\ p^v(\theta) + \epsilon & \text{if} \quad \theta \notin \Theta^+ \cup \Theta^- \end{cases}. \tag{29}$$

We can express $p^v$ as the convex combination $0.5 p^-_\epsilon + 0.5 p^+_\epsilon$. I will show that there exists some $\bar{\epsilon} > 0$ such that $p^-_\epsilon, p^+_\epsilon \in \hat{P}$ f or all $\epsilon \in (0, \bar{\epsilon})$. Since $p^v$ is an extreme point of $\hat{P}$, this implies that $p^v = p_\epsilon^-$ and $p^* = p_\epsilon^+$. And therefore $\Theta^+ \cup \Theta^- = \Theta$.

Note that,

$$\sum_{\theta \in \Theta} p_\epsilon^+(\theta) = -(\|\Theta\| - \|\Theta^+ \cup \Theta^-\|)\epsilon + \kappa \sum_{\theta \in \Theta^+ \cup \Theta^-} \epsilon p^*(\theta)$$
$$= \epsilon \left[ -(\|\Theta\| - \|\Theta^+ \cup \Theta^-\|) + \kappa p^*(\Theta^+ \cup \Theta^-) \right] = 0. \tag{30}$$

Moreover, let $\bar{\epsilon}_1$ be the number

$$\bar{\epsilon}_1 = \min_{\theta \in \Theta} \min \left\{ p^v(\theta) \, , \, \frac{1 - p^v(\theta)}{\kappa p^v(\theta)} \right\}. \tag{31}$$

From (28), it is easy to show that if $\epsilon < \bar{\epsilon}_1$, then $p_\epsilon^+(\theta) \in (0, 1)$ for every state $\theta$ and, consequently, $p_\epsilon^+ \in \Delta^\circ(\Omega)$. Similarly, there exists some $\bar{\epsilon}_2$ such that if $\epsilon < \bar{\epsilon}_2$, then $p_\epsilon^- \in \Delta^\circ(\Omega)$.

It remains to establish $\gamma^0$-attainability. If $\Theta^+ \cup \Theta^- = \Theta$, then $p_\epsilon^+ = p_\epsilon^- = p^v$ and we are done. Otherwise, there would exist some $\theta \in \Theta^+ \cup \Theta^-$. Since $p^v$ satisfies (2) and $\theta \notin \Theta^-$,

$$\Lambda(\theta, \theta^+, p^v) > \gamma^0 \Lambda(\theta, \theta^+, p^0) \quad \Rightarrow \quad \frac{p^v(\theta)}{p^0(\theta)} > \gamma^0 \frac{p^v(\theta^+)}{p^0(\theta^+)} = \frac{p^v(\theta^-)}{p^0(\theta^-)}, \tag{32}$$

where the last equality follows from (23). By a similar argument, since $\theta \notin \Theta^+$, it would follow that

$$\frac{p^v(\theta^-)}{p^0(\theta^-)} < \frac{p^v(\theta)}{p^0(\theta)} < \frac{p^v(\theta^+)}{p^0(\theta^+)}. \tag{33}$$

For $\epsilon$ sufficiently small these inequalities continue to hold when we replace $p^v$ with $p_\epsilon^+$. Reversing the previous steps, it follows that $\Lambda(\theta, \theta^+, p_\epsilon^+) > \gamma^0$ and $\Lambda(\theta^-, \theta, p_\epsilon^+) > \gamma^0$, which in turn implies that $p_\epsilon^+ \in \Delta(\gamma^0, p^0)$. An analogous argument implies that for $\epsilon$ sufficiently small we also have that $p_\epsilon^- \in \Delta(\gamma^0, p^0)$, thus completing this direction of the proof.

($\Rightarrow$) Now take any two-block partition $\{\Theta^-, \Theta^+\}$ of $\Theta$. Let $p^v$ be given by

$$p^v(\theta) = \begin{cases} \dfrac{p^0(\theta)}{p^0(\Theta^+) + \gamma^0 p^0(\Theta^-)} & \text{if} \quad \theta \in \Theta^+ \\[2ex] \dfrac{\gamma^0 p^0(\theta)}{p^0(\Theta^+) + \gamma^0 p^0(\Theta^-)} & \text{if} \quad \theta \in \Theta^- \end{cases}. \tag{34}$$

It is straightforward to verify that $p^v(\theta) \in (0,1)$ for every state $\theta$ and that $p^v$ satisfies (4) for every pair of states. Also, note that

$$\sum_{\theta \in \Theta} p^v(\theta) = \sum_{\theta \in \Theta^+} \frac{p^0(\theta)}{p^0(\Theta^+) + \gamma^0 p^0(\Theta^-)} + \sum_{\theta \in \Theta^-} \frac{\gamma^0 p^0(\theta)}{p^0(\Theta^+) + \gamma^0 p^0(\Theta^-)} \tag{35}$$

$$= \frac{1}{p^0(\Theta^+) + \gamma^0 p^0(\Theta^-)} \left[ \sum_{\theta \in \Theta^+} p^0(\theta) + \gamma^0 \sum_{\theta \in \Theta^-} p^0(\theta) \right] = 1. \tag{36}$$

Therefore, $p^v \in \Delta^\circ(\Theta)$. Since all likelihood ratios have been scaled by a factor of $\gamma^0$, $1/\gamma^0$, or remained constant, condition (2) holds. Hence, Lemma 1 implies that $p^v$ is $\gamma^0$-achievable.

It remains to show that $p^v$ is an extreme point of $\hat{P}$. Since, $\Delta\Theta$ has finite dimension, the Krein-Milman theorem implies that there exist a finite set $P = \{p_1, \ldots, p_k\}$ consisting of *extreme points* of $\hat{P}$ and a vector $\mu \in \Delta^k$ such that $p^v = \sum_{i=i}^k \mu_i p_i$. Without loss of generality suppose that $\mu_i > 0$ for all $i = 1, \ldots, k$. I will show that $p_i = p^v$ for all $i = 1, \ldots, k$.

For that purpose, I will use the only-if part of Lemma 2. For every $i = 1, \ldots, k$, there exists a partition $\{\Theta_i^-, \Theta_i^+\}$ such that $p_i$ satisfies (4) .

Fix any two states $\theta^+ \in \Theta^+$ and $\theta^- \in \Theta^-$ and let $H$ be the hyperplane

$$H = \left\{ p \in \Delta(\Theta) \mid p(\theta^-) - \lambda^0 \Lambda(\theta^-, \theta^+; p^0) p(\theta^+) = 0 \right\}. \tag{37}$$

Since $\Lambda(\theta^-, \theta^+; p^v) = \lambda^0 \Lambda(\theta^-, \theta^+; p^0)$, it follows that $p^v \in H$. Also, note that

$$\begin{aligned} \Lambda(\theta^-, \theta^+; p^v) &= \frac{\sum_{i=1}^k \mu_i p_i(\theta^-)}{\sum_{i=1}^k \mu_i p_i(\theta^+)} = \frac{\sum_{i=1}^k \mu_i p_i(\theta^+) \Lambda(\theta^-, \theta^+; p_i)}{\sum_{i=1}^k \mu_i p_i(\theta^+)} \\ &\geq \frac{\sum_{i=1}^k \mu_i p_i(\theta^+)}{\sum_{i=1}^k \mu_i p_i(\theta^+)} \cdot \min_{i=1,\ldots,k} \Lambda(\theta^-, \theta^+; p_i) = \min_{i=1,\ldots,k} \Lambda(\theta^-, \theta^+; p_i). \end{aligned} \tag{38}$$

Since all the $p_i$s are $\gamma^0$-attainable, Lemma 1 implies that

$$\lambda^0 \Lambda(\theta^-, \theta^+; p^0) = \Lambda(\theta^-, \theta^+; p^v) \geq \min_{i=1,\ldots,k} \Lambda(\theta^-, \theta^+; p_i) \geq \lambda^0 \Lambda(\theta^-, \theta^+; p^0), \tag{39}$$

and thus $\Lambda(\theta^-, \theta^+; p^v) = \min_{i=1,\ldots,k} \Lambda(\theta^-, \theta^+; p_i)$. Hence, for all $i = 1, \ldots, k$ we have $\Lambda(\theta^-, \theta^+; p^v) \leq \Lambda(\theta^-, \theta^+; p_i)$ and, consequently,

$$\Lambda(\theta^-, \theta^+; p_i) \geq \lambda^0 \Lambda(\theta^-, \theta^+; p^0) \quad \Rightarrow \quad p_i(\theta^-) - \lambda^0 \Lambda(\theta^-, \theta^+; p^0) p_i(\theta^+) \geq 0. \tag{40}$$

That is, all the $p_s$s lie on the same side of $H$. Since $p^v \in H$ is their convex combination, all of them must lie exactly on $H$. This implies that $\theta^- \in \Theta_i^-$ and $\theta^+ \in \Theta_i^+$ for all $i = 1, \ldots, k$. Since $\theta^-$ and $\theta^+$ were arbitrary, this implies that $\Theta_i^- = \Theta^-$ and $\Theta_i^+ = \Theta^+$ for all $i = 1, \ldots, k$, thus completing the proof. ■

### A.2. Effective communication and persuasion

*Proof of Proposition 3.* The threshold is given by $\bar{\gamma} = g(p^*, p^0)$, where $g$ is the function defined in (22). This threshold is well defined and strictly positive because $\Theta$ is finite and $p^0$ and $p^*$ have full support. If $\gamma^0 < \bar{\gamma}$, then

$$\frac{\Lambda(\theta, \theta'; p^*)}{\Lambda(\theta, \theta'; p^0)} \geq g(p^*, p^0) > \gamma^0 \quad \Rightarrow \quad \Lambda(\theta, \theta'; p^*) > \gamma^0 \Lambda(\theta, \theta'; p^0), \tag{41}$$

for every pair of states $\theta$ and $\theta$. From Lemma 1, it follows that $p^*$ is $\gamma^0$-attainable by some $x$ and $\mathbf{q}$. Consider any strategy profile $\mathbf{y}$ such that all receivers choose best responses and, in particular, $y_r(m_r, \omega) = a^*$ whenever $q_r(\cdot|m_r) = p^*$. The tupple $(x, \mathbf{q}, \mathbf{y})$ constitutes an equilibrium.

Let $\mathbf{a}$ be any action profile that results with positive probability in this equilibrium. By construction, at least $\gamma^0$ of the receivers satisfy $a_r = a^*$. Hence, it maximizes the sender's utility. ■

*Proof of Proposition 4.* Let $P^*$ be the set of beliefs in $\Delta(\Theta)$ for which $a^*$ is a best response. This set is closed because expected utility is linear in probabilities and best responses are defined by weak inequalities. Since $\mathrm{BR}(\theta) \cap \mathrm{BR}(\theta') = \emptyset$ for some $\theta, \theta' \in \Theta$, there exists some $\theta \in \Theta$ such that $a^* \notin \mathrm{BR}(\theta)$. Therefore, $\Delta^\circ(\Theta) \setminus P^*$ is open and nonempty.

Since both $P^* \cap \Delta^\circ(\Omega)$ and $\Delta^\circ(\Theta) \setminus P^*$ are nonempty, there exists some full-support belief $p_1 \in \Delta^\circ(\Omega)$ in the boundary of $P^*$. Lemma 1 thus implies that there exists some $p_2 \in \Delta^\circ(\Theta) \setminus P^*$ that would be $\gamma^0$-achievable if $p^0 = p_1$. Fix any closed set $\bar{P} \subseteq \Delta^\circ(\Theta)$ containing both $p_1$ and an open neighborhood of $p_2$. Define the function $d : \bar{P} \to \mathbb{R}$ by

$$d(p) = \max\left\{ g(p, p^{**}) \,\middle|\, p^{**} \in P^* \cap \bar{P} \right\}, \tag{42}$$

where $g$ is the function defined in (22). Since $g$ is continuous on $\bar{P} \times \bar{P} \subseteq \Delta^\circ(\Theta) \times \Delta^\circ(\Theta)$, and $P^* \cap \bar{P}$ is compact, it follows that $d$ is well defined and continuous.

Moreover, for all $p \in \bar{P}$ we have $d(p) = 1$ if $p \in P^*$, and $d(p) < 1$ if $p \in P'$. Let $P'$ be the set of full-support beliefs $p' \in \bar{P} \setminus P^*$ such that $d(p') \in (\gamma^0, 1)$. Since $p_2 \in P'$ and $d$ is continuous, $P'$ it is a nonempty open set.

For every $p' \in P'$, $d(p') > \gamma^0$ and thus there exists some $p^{**} \in P^*$ such that

$$\frac{\Lambda(\theta, \theta'; p')}{\Lambda(\theta, \theta'; p^{**})} > \gamma^0 \qquad \Rightarrow \qquad \Lambda(\theta', \theta; p^{**}) > \gamma^0 \Lambda(\theta', \theta; p') \tag{43}$$

for every pair of states $\theta, \theta' \in \Theta$. Lemma 1 thus implies that, if $p^0 = p'$, then $p^{**}$ would be $\gamma^0$-attainable. Hence, we can employ the equilibrium constructed in the proof of Proposition 3. In this equilibrium, message $m'$ is sent with positive probability, and $y_r(m', \omega) = a^* \notin \text{BR}(p')$ for every sunspot realization $\omega$. ■

### A.3. Sender-optimal equilibrium

**Lemma 9** *Every equilibrium value can be attained by a symmetric equilibrium* $(x, \mathbf{y}, \mathbf{q})$ *with in that* $x_r = x_{r'}$, $y_r = y_{r'}$ *and* $q_r = q_{r'}$ *for all receivers* $r$ *and* $r'$.

*Proof.* Suppose that an equilibrium value $u_S^*$ is generated by some equilibrium $(x, \mathbf{y}, \mathbf{q})$. Consider the alternative strategies $\tilde{x}$ and $\tilde{\mathbf{y}}$ obtained by shuffling identities as follows. First, the sender draws messages from the original distribution. Then, she shuffles the identity of the receivers uniformly. She tells each receiver which function $y : [0, 1] \to A$ they would have used in the original equilibrium with their swapped identity and message.

Note that $y : [0, 1] \to A$ would also be a best response had $r$ been told the shuffling and the message. The sure thing principle then implies that it is also a best response when this information is garbled. This strategy profile is thus also an equilibrium that generates $u_S^*$. But every receiver has the same marginal distribution of messages and uses the same strategy and update rule. Hence, we can assume without loss of generality that $(x, \mathbf{y}, \mathbf{q})$ is symmetric. ■

**Lemma 10** (Lipnowski and Ravid (2018)) *Under assumption 3,* $v^*(p) \geq \text{env}_\text{q}\, v^0$.

*Proof.* Let $u_S^* = \text{env}_\text{q}\, v^0(p^0)$. Consider the alternative environment with $\tilde{n} = 1$ and $\tilde{u}_S(a) = u_s(a, \ldots, a)$, and $\Theta$, $A$, $p^0$, and $u_R$ unchanged. Assumption 3 implies that the ex-ante value of the alternative environment coincides with the ex-ante value of the original environment. Since there is only one receiver, and the sender's utility does not depend on the state, this alternative environment satisfies the assumptions in Lipnowski and Ravid (2018). Hence by their Theorem 2, there exists an equilibrium $(\tilde{x}, \tilde{y}_1, \tilde{q}_1)$ which achieves $u_S^*$.

Consider the replica of this equilibrium given by $q_r = \tilde{q}_1$, $y_r(m_0, m_r^p, \omega) = \tilde{y}_1(m_0, \omega)$, and $x(m, m^\phi, \dots, m^\phi|\theta) = \tilde{x}(m|\theta)$, where $m^\phi$ is a fixed non-informative message. Note that this replica uses correlated strategies which guarantee that all agents receive the same message and take the same action with probability 1. It is straightforward to verify that $(x, \mathbf{y}, \mathbf{q})$ is an equilibrium of the original environment and achieves $u_S^*$. ■

*Proof of Theorem 5.* (*Step 1* — $\operatorname{att}_{\gamma^0} \operatorname{env}_{\mathrm{q}} v^0$ is well defined) Since the environment is finite, $\operatorname{env}_{\mathrm{q}} v^0$ is well defined and upper-semicontinuous. From secition A.4.1 in Lipnowski and Ravid (2018), $\operatorname{env}_{\mathrm{q}} v^0$ is also well defined and upper-semicontinuous. From Lemma 1, $\hat{P}(p)$ is compact. Hence $\operatorname{att}_{\gamma^0} \operatorname{env}_{\mathrm{q}} v^0$ is well defined by Weierstrass' Extreme-Value Theorem, and upper-semicontinuous by the generalization Berge's Theorem in Ausubel and Deneckere (1993).

(*Step 2* — $v^* \geq \operatorname{att}_{\gamma^0} \operatorname{env}_{\mathrm{q}} v^0(p^0)$) Let $u_S^* = \operatorname{att}_{\gamma^0} \operatorname{env}_{\mathrm{q}} v^0(p^0)$. There exists some $\hat{p} \in \hat{P}$ such that $u_S^* = \operatorname{env}_{\mathrm{q}} v^0(\hat{p})$. Consider the alternative environment with $\tilde{n} = 1$, $\tilde{u}_S(a) = u_s(a, \dots, a)$, $\tilde{p}^0 = \hat{p}$, and $\Theta$, $A$, and $u_R$ unchanged. Assumption 3 implies that the value function of the alternative environment coincides with $v^0$. Since there is only one receiver, and the sender's utility does not depend on the state, this alternative environment satisfies the assumptions in Lipnowski and Ravid (2018). Hence by their Theorem 2, there exists an equilibrium $(\tilde{x}, \tilde{y}_1, \tilde{q}_1)$ which achieves $\operatorname{env}_{\mathrm{q}} v^0(\hat{p}) = u_S^*$. Moreover, we must have

$$\int_0^1 u_S\left(\tilde{y}_1(m)\right)\, d\omega = u_S^*, \tag{44}$$

for every message $m$ such that $\tilde{x}(m) > 0$.

Consider the replica $(\hat{x}, \hat{\mathbf{y}}, \hat{\mathbf{q}})$ of this equilibrium given by $\hat{y}_r(m_0, m_r^p, \omega) = \tilde{y}_1(m_0, \omega)$, $\hat{x}(m, m^\phi, \dots, m^\phi|\theta) = \tilde{x}(m|\theta)$, and $\hat{q}_r(m, m^\phi) = \tilde{q}_1(m)$, where $m^\phi$ is a fixed non-informative private message. It is straightforward to verify that $(\hat{x}, \hat{\mathbf{y}}, \hat{\mathbf{q}})$ is an equilibrium of the original environment. Let $\bar{x}$ and $\bar{\mathbf{q}}$ be the communication strategy and updating rule that $\gamma^0$-attain $\hat{p}$ from the proof of Lemma 1.

Consider the tuple $(x, \mathbf{y}, \mathbf{q})$ described as follows. The sender first draws (but does not deliver) profiles of messages $\hat{m}$ using $\hat{x}$, and $\bar{\mathbf{m}} \in \{m', m''\}^n$ using $\bar{x}$. If $\bar{m}_r = m'$, then $r$ receives the *private* message $m_r^p = (m', \hat{m}_0)$. Otherwise, he receives the private message $m_r^p = (m'')$. $\mathbf{q}$ is derived from $x$ using Bayes rule. Actions are given by $y_r((m', \hat{m}_0), \omega) = \tilde{y}_1(\hat{m}_0, \omega)$, and $y_r(m'', \omega) = a''$ with $a'' = \min \mathrm{BR}(q_r(m''))$.

By construction, we have $q(m', \hat{m}_0) = \tilde{q}_1(\hat{m}_0)$, which implies that $y_r(m', \hat{m}_0, \omega)$ is a best response. Since $(\hat{x}, \hat{\mathbf{y}}, \hat{\mathbf{q}})$ is an equilibrium, the sender cannot benefit from manipulating $\hat{m}_0$. Hence, if $u_S^* \geq \tilde{u}_S(a'')$, then $(x, \mathbf{y}, \mathbf{q})$ is an equilibrium. Since there are always $n^0$ players who receive message $m'$ and their actions lead to $u_S^*$ (because of (44), this would imply $v^* \geq \mathrm{att}_{\gamma^0} \mathrm{env_q}\, v^0(p^0)$.

Suppose instead that $u_S^* < \tilde{u}_S(a'')$. For this case, note that $q^0$ is a convex combination of $\hat{p}$ and $q_r(m'')$. And, in turn $\hat{p}$ is a convex combination of $\{q(m', \hat{m}_0) | \tilde{x}(\hat{m}_0) > 0\}$. Note that $v^0(q_r(m'')) \geq \tilde{u}_S(a'') > u_S^*$, and $v^0(q(m', \hat{m}_0)) u_S^*$ (because of (44). Hence, $\mathrm{env_q}\, v^0(p^0) \geq u_S^*$ From Lemma 10.it follows that $v^* \geq \mathrm{att}_{\gamma^0} \mathrm{env_q}\, v^0(p^0)$.

(*Step 3* — $v^* \leq \mathrm{att}_{\gamma^0} \mathrm{env_q}\, v^0(p^0)$) From Lemma 9, there exists a symmetric equilibrium $(x, \mathbf{y}, \mathbf{q})$ that generates $v^*(p)$. Let $\mathbf{m}$ be a message profile such that $x(\mathbf{m}) > 0$. Under assumption 3, there must exist a set $R(\mathbf{m})$ with $\|R(\mathbf{m})\| \geq n^0$ and such that $\max\{y_r(m_r, \omega) : | \, \omega \in [0,1]\} \geq v^*$ for every $r \in R(\mathbf{m})$. Let $P^*$ be the set corresponding set of posterior beliefs,

$$P^* = \sup \left\{ q_r(m_r) \middle| \quad x(\mathbf{m}) > 0 \text{ and } r \in R(\mathbf{m}) \right\}. \tag{45}$$

It follows that

$$v^* \leq \min_{p \in P^*} v(p). \tag{46}$$

Consider the alternative communication strategy $x'$ with only two messages $m'$ and $m''$ described as follows. The sender first chooses a profile $\mathbf{m}$ according to $x$ (but does *not* deliver it). Receiver $r$ receives message $m'$ if and only if $r \in R(\mathbf{m})$. Since it is a symmetric equilibrium, $\bar{p} := q_r(m')$ does not depend on $r$

Bayes rule implies that $\bar{p} \in \mathrm{co}(P^*)$.[3] Since there are always at least $n^0$ receivers in $R(\mathbf{m})$, it follows that $\bar{p}$ is $\gamma^0$-attainable. Therefore

$$v^* \leq \min_{p \in P^*} v(p) \leq \min_{p \in P^*} \mathrm{env_q}\, v(p) \leq \min_{p \in \mathrm{co}(P^*)} \mathrm{env_q}\, v(p)$$

[3] Let $(\Omega, 2^\Omega, \Pr)$ be a finite probability space, and let $\{A_1, \ldots, A_k\}$ be disjoint non-null events. For any event $B$

$$\Pr\left(B | \cup_{i=1}^k A_i\right) = \frac{\Pr(B) \sum_{i=1}^k \Pr(A_i | B)}{\sum_{i=1}^k \Pr(A_i)} = \sum_{i=1}^k \left( \frac{\Pr(A_i)}{\sum_{i=1}^k \Pr(A_i)} \right) \Pr(B | A_i)$$

$$\leq \operatorname{env}_{\mathrm{q}} v(\bar{p}) \leq \operatorname{att}_{\gamma^0} \operatorname{env}_{\mathrm{q}} v(p^0). \tag{47}$$

The first inequality is just (46). The second inequality follows because $\operatorname{env}_{\mathrm{q}} v$ majorizes $v$. The third one because $\operatorname{env}_{\mathrm{q}} v$ is quasiconcave. The fourth one from the definition of minimum. The last one from the fact that $\bar{p}$ is $\gamma^0$-attainable. ■

*Proof of Proposition 6.* Since $\hat{P}$ is $\subseteq$-decreasing in $\gamma^0$, $\operatorname{att}_{\gamma^0} \operatorname{env}_{\mathrm{q}} v^0$ is weakly decreasing. Hence $(i)$ follows from Theorem 5. $(iii)$ is a corollary of Proposition 3, and $(ii)$ is a corollary of Proposition 4 and Assumption 3. ■

## A.4. Extensions

*Proof of Theorem 7.* Let $u_S^* = \operatorname{att}_{\gamma^0} \operatorname{env} v^0(p^0)$. There exists some $\hat{p} \in \hat{P}$ such that $u_S^* = \operatorname{env} v^0(\hat{p})$. By Caratheodory's theorem there exist beliefs $p_1, \ldots, p_K$ and weights $\mu \in \Delta^K$ with $K \leq n_\Theta + 1$ such that $\hat{p} = \sum_{k=1}^K \mu_k p_k$, and $u_S^* = \sum_{k=1}^K \mu_k v^0(p_k)$. Under Assumption 3, there exist actions $a_1, \ldots, a_K$ such that $v^0(p_k) = \tilde{u}_S(a_k)$ and $a_k \in \operatorname{BR}(p_k)$. The result then follows from Proposition 1 in Kamenica and Gentzkow (2011). ■

*Proof of Proposition 8.* Let $\mathscr{I}_K$ be the partition of $[0, 1)$ into $K$ intervals of the form $\Theta_k = [(k-1)/K, k/K)$, $k = 1, \ldots, K$. Let $\bar{\theta}_k = (2k-1)/2K$ be the midpoint of the $k$th interval. Let $k(\theta)$ denote the block of $\mathscr{I}_K$ containing $\theta$. Let $k^*(\theta) = \arg\min_k \{(\theta + b - \bar{\theta}_k)^2\}$.

Consider the tuple $(x, \mathbf{y}, \mathbf{q})$ described as follows. The audience is randomly divided into a target set $T \subseteq R$ consisting of exactly $n^0$ receivers, and $R \setminus T$. When the state is $\theta$, receivers $r \notin T$ receive message $k(\theta)$, while receivers $r \in T$ receive message $k^*(\theta)$. The update rule $\mathbf{q}$ is derived using Bayes' rule. Receivers choose $y_r(m_r) = \mathbb{E}[\,\theta^0 | m_r\,]$.

Fix $K$. Since $n^0$ is also fixed, there exists $\bar{n}$ such that whenever $n \geq \bar{n}$ $\mathbb{E}[\,\theta^0|k\,] \in \Theta_k$, and $k^*(\theta) = \arg\min_k \{(\theta + b - \mathbb{E}[\,\theta^0|k\,])^2\}$. For such values of $n$, the proposed tuple is an equilibrium and it yields the sender's payoff

$$u_S^* = -\left(\theta^0 + b - \mathbb{E}\left[\,\theta^0 | k^*(\theta_0)\,\right]\right)^2. \tag{48}$$

If $\theta^0 + b \leq 1$, then $\theta^0 + b \in \Theta_{k^*(\theta)}$ and therefore $u_S^* \geq -1/K$. If not, then $k^*(\theta) = K$ and $u_S^* \geq -(b + 1/K)^2$. ■