Common-Value Auctions With an Uncertain Number of Bidders*

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Abstract

This paper studies a common-value first-price auction in which bidders are uncertain about the number of competitors they have. This uncertainty affects the nature of the inference from winning ("winner's curse"). In particular, the expected value conditional on winning is usually not monotone and features a stronger winner's curse at intermediate bids. Consequently, bidders have incentives to pool on common bids. At these pooling bids ("atoms"), payoffs change discontinuously. Due to this discontinuity, no equilibrium exists unless the expected number of bidders is sufficiently small. To the ensure the existence for any number of bidders, we extend the auction mechanism by a compound cheap talk message that enables bidders to indicate their eagerness to win. This extended auction mechanism can be used to easily derive properties for auctions on a discretized bidding space, where an equilibrium always exists.

Keywords: common-value auctions, random player games, numbers uncertainty, Poisson games, endogenous tiebreaking, non-existence

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1 Introduction

Bidders in most auctions are uncertain about the number of competitors they have. This is true not only for many well-known examples such as eBay[1], Christie's[2], or the Dutch flower auctions, but also for many auction-like trading mechanisms such as the call market that initiates trading at the NYSE.[3]

We study the effect of this "numbers uncertainty" in a first-price common-value auction. In common-value auctions, winning enables for inferences about the value of the good. As a benchmark, consider a setting in which the number of competitors is known and the bidding strategy is symmetric and strictly increasing in the bidder's estimate of the value of the good. Then, the winning bidder knows that all of her competitors have a lower estimate of the good's value than herself. This is bad news about the value of the good, which is known as the "winner's curse". The winner's curse is more severe if there are a large number of competitors, or if the winning bid is low, thereby implying that the expected value conditional on winning is increasing with the size of the winning bid.

When bidders are uncertain about the number of competitors they have, winning is also informative about this number. In particular, winning with a low bid is more likely when there are fewer competitors, and this reduces the winner's curse. Therefore, winning with a low bid is not necessarily bad news about the value of the good, and the expected value conditional on winning does not need to be monotone. The random number of competitors adds a second dimension of uncertainty that breaks the affiliation between the winning bid and the value of the good.

Following Myerson (1998), we model auctions with numbers uncertainty as a standard common-value first-price auction where the number of bidders is Poisson-distributed. The Poisson distribution is tractable and also arises endogenously as the result of some entry process.[4] All bidders in the auction compete for a single, indivisible good of common-value. The value is either high or low, and every bidder receives a conditionally independent signal;

[1]eBay provides information about the number of bidders who actually placed a bid, but does not disclose how many prospective bidders follow the object via their watch list etc. In particular, eBay does not disclose how many bidders are online, waiting to place their bid in the last seconds of the auction ("snipe" - cf. Roth & Ockenfels (2002)).

[2]The Wall Street Journal reports that personal attendance in auction rooms is in decline, as bidders prefer to phone in or place their bids online. Therefore, "[...] they know even less about who they're bidding against, which in some cases can leave them wondering how high they should go" (https://www.wsj.com/articles/with-absentee-bidding-on-the-rise-auction-rooms-seem-empty-these-days-1402683887 - cf. Akbarpour & Li (2018)).

[3]Data from the stock market informs market participants about the stream of (un-)filled buy and sell orders, but reveals neither the number nor the identity of buyers and sellers in the market. In fact, market participants often try to hide large transactions by splitting orders into smaller ones or trading in dark pools.

[4]Among others, compare Välimäki & Murto (2017), and Lauermann et al. (2018). Most of our results extend to arbitrary distributions, and we discuss the significance of the Poisson assumption in the last section of the paper.

high signals indicating a high value. Each bidder simultaneously submits a bid, the highest bidder wins and pays her bid. Ties are broken at random.

In this setup, we find that the expected value is U-shaped in the first-order statistic of signals (Lemma 2). When the expected number of bidders is high, this non-monotonicity implies that no strictly increasing equilibrium exists (Prop. 1). The problem does not arise, when the expected number of bidders is sufficiently small (Prop. 2). We conclude that if there is an equilibrium in a large auction, it has to contain pooling bids, that is, atoms in the bid distribution.

Bidders actually have an incentive to tie on low bids, because it reduces the winner's curse. Under a uniform tiebreaking rule, winning the auction with a bid that ties with positive probability is more likely if there are fewer competitors, which is good news for the value of the good. However, when the equilibrium bid distribution contains atoms, the bidder's utility is not continuous in the bid. This discontinuity implies that no equilibrium exists when the expected number of bidders is sufficiently large (Prop. 3).

To solve the existence problem and create a useful approximation tool for equilibria on the grid, we extend the auction mechanism by cheap talk communication, as in Jackson et al. (2002). In this Communication Extension, bidders report two signals in addition to their bid, which indicates their eagerness to win. The extension ensures that equilibria exist, which we characterize in Proposition 5. Thereafter, we consider auctions on the grid and find that equilibria on a sufficiently fine grid are structurally equivalent to equilibria in the communication extension (Prop. 6). The characterization of equilibria on the grid is helpful to understand why the standard continuous auction is not the limit of auctions on the grid.

Then, we investigate the robustness of our results and argue that the findings do not hinge on the precise assumptions on the distribution of signals, distribution of bidders, or the auction format.

When the good is of a private rather than common-value and there is numbers uncertainty, McAfee & McMillan (1987) and Harstad et al. (1990) show that the optimal bidding strategy is a weighted average of what would have been chosen if the number of bidders was known. Our analysis shows that this is no longer true when bidders have interdependent valuations for the good. Consequently, a simple extension of the results for auctions with a known number of bidders to auctions with numbers uncertainty is not possible. We discuss implications and the related literature in Section 8 of the paper.

In addition to this substantive contribution, our analysis provides a robust example where equilibrium existence fails in a simple game, but is regained in a mechanism with cheap talk, following the concept by Jackson et al. (2002).[5] Further, we show that this

[5]Contrary to Jackson et al. (2002), we do not need to make the auctioneer a player of the game, but can provide a mechanism that guarantees existence.

mechanism with cheap talk is not only of technical interest but can be used to approximate equilibria on the sufficiently fine grid. Thus, the Communication Extensions is the "correct" mechanism to derive equilibrium properties for equilibria on the grid. This is not only true for common-value auction with numbers uncertainty, but whenever the bidding strategy is monotone but can contain atoms. Therefore, the Communication Extensions particularly lends itself to the analysis of other non-affiliated common value auctions.

2 The Model

A single indivisible good is sold in a first-price sealed-bid auction. The good's value is either high v_h, or low v_ℓ, with $v_h > v_\ell \geq 0$, depending on the unknown state of the world $\omega \in \{h, \ell\}$. The world is in state $\omega = h$ with probability ρ and in state $\omega = \ell$ with probability $1-\rho$, where $\rho \in (0,1)$. The number of bidders is a Poisson-distributed random variable with mean η, such that there are n bidders in the auction with probability $\mathbb{P}(n) = e^{-\eta}\frac{\eta^n}{n!}$. The realization of the variable is unknown to the bidders.

Every bidder receives a signal s from the compact set $[\underline{s}, \bar{s}]$. Conditional on the state of the world, the signals are independent and identically distributed according to the cumulative density functions F_h and F_ℓ, respectively. Both distributions have continuous densities f_ω, and the likelihood ratio of these densities, $\frac{f_h(s)}{f_\ell(s)}$, satisfies the (weak) monotone likelihood ratio property, that is, for all $s < s'$ it holds that $\frac{f_h(s)}{f_\ell(s)} \leq \frac{f_h(s')}{f_\ell(s')}$. Furthermore, $0 < \frac{f_h(\underline{s})}{f_\ell(\underline{s})} < \frac{f_h(\bar{s})}{f_\ell(\bar{s})} < \infty$, such that signals do contain information but never reveal the state of the world perfectly. For convenience, we assume that there is only one unique s^*, such that $\frac{f_h(s^*)}{f_\ell(s^*)} = 1$.

Having received their signals, every bidder submits a bid b. We assume that there is a reserve price at v_ℓ and exclude (without loss) bids above v_h, such that $b \in [v_\ell, v_h]$. The bidder with the highest bid wins the auction, receives the object, and pays her bid. Ties are broken uniformly. If there is no bidder, the good is not allocated. Bidders are risk neutral.[6]

It is useful to mention two special properties of the Poisson distribution (see Myerson (1998) for a detailed derivation and discussion). First, when participating in the auction, a bidder does not change her belief regarding the number of other bidders in the auction. Therefore, her belief about the number of her competitors is again a Poisson distribution with mean η. This property is analogous to a stationary Poisson process, where an event does not allow for inferences about the number of other events.[7]

[6] We discuss the significance of the assumptions on the distribution of bidders, signals, the reserve price, and auction format in Section 8. In the appendix, we allow the number of bidders to be state-dependent by considering a Poisson random variable with state-dependent means η_ω.

[7] In fact, the Poisson distribution is the only distribution with this "environmental equivalence" (Myerson (1998)).

Second, the Poisson distribution implies that we have to restrict attention to symmetric equilibria.[8] Since the Poisson distribution has an unbounded support, it draws bidders from a hypothetical infinite urn. Any individual bidder and, thus, any individual bidding strategy is thereby drawn with zero probability, and no bidder expects to face such an individual. One could imagine certain proportions of the bidders in the urn following divergent strategies, such that those are encountered with positive probability. However, this would be equivalent to drawing the bidders first and having them mix strategies afterward.

Accordingly, we consider symmetric strategies, which are functions mapping from the signals into the set of probability distributions over bids[9] $\beta : [\underline{s}, \bar{s}] \to \Delta[v_\ell, v_h]$. Let $\pi_\omega(b; \beta)$ denote the probability to win the auction with a bid b in state ω, given that the other bidders use strategy β. Using Bayes' rule, the interim expected utility for a bidder with signal s choosing bid b is

$$U(b|s;\beta) = \frac{\rho f_h(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} \pi_h(b;\beta)(v_h - b) + \frac{(1-\rho) f_\ell(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} \pi_\ell(b;\beta)(v_\ell - b). \quad (1)$$

Strategy β^* is a *best response* to a strategy β, if, for almost all s, $b \in \operatorname{supp} \beta^*(s)$ implies that $b \in \arg\max_{\hat{b} \in [v_\ell, v_h]} U(\hat{b}|s;\beta)$. Two strategies are *equivalent,* if they correspond to the same distributional strategy after merging all signals that share the same likelihood ratio $\frac{f_h}{f_\ell}$.

Lemma 1 (Monotonicity)**.** *Let β be some strategy and β^* a best response to it. If the likelihood ratio of signals $\frac{f_h}{f_\ell}$ is strictly increasing, then β^* is essentially pure and non-decreasing. If the likelihood ratio is only weakly increasing, then there exists an equivalent best response $\hat{\beta}^*$, which is pure and non-decreasing.*

We look for Bayes-Nash equilibria, that is, strategies β^* which are best-responses to themselves. Lemma 1 implies that it is without loss to restrict attention to pure and non-decreasing equilibria.[10] Henceforth, we denote pure strategies as functions mapping the signals into bids – $\beta : [\underline{s}, \bar{s}] \to [v_\ell, v_h]$ – and only consider non-decreasing ones.

[8]This necessity fits our aim of analyzing how uncertainty about the number of bidders (as opposed to identity) affects the equilibrium behavior.

[9]We consider functions that are measurable and probability distributions which that Borel probability measures.

[10]Suppose that β^* is a symmetric Bayes-Nash equilibrium strategy. Using Lemma 1, there is pure and non-decreasing best response $\hat{\beta}^*$ to β^*, such that $\hat{\beta}^*$ is equivalent to β^* . This means that the implied distribution of bids is the same under either strategy. Since this is the only manner in which the strategy enters the bidders' utilities (1), $\hat{\beta}^*$ is a best response to $\hat{\beta}^*$ and, hence, an equilibrium as well.

3 Equilibrium of the Standard Auction

3.1 Non-pooling bids

To analyze the model, we first consider bids that, given a non-decreasing strategy β, never tie. We derive the winning probabilities and the expected value conditional on winning with such a bid. Last, we use our findings to analyze equilibria where β is strictly increasing.

Fix some non-decreasing bidding strategy β. A bid b is a *non-pooling bid* if it is selected with zero probability by any bidder. Given strategy β, this is the case if b is either not in the support of β, or when there is only a single signal s, such that $\beta(s) = b$. In any case, a bidder who chooses b wins whenever all of her competitors select bids smaller than b. Given that β is non-decreasing, this implies that they all received lower signals than $\hat{s} := \sup\{s\colon\ \beta(s) \le b\}$. Thus, the bidder wins whenever $s_{(1)} \le \hat{s}$, where

$$s_{(1)} := \sup\{s_{-i}\}$$

is the highest of the opponents' signals. We employ the convention that $\sup\{\emptyset\} = -\infty$, which means that $s_{(1)} = -\infty < \underline{s}$ denotes the situation when there is no competitor. In state ω, the generalized first-order statistic $s_{(1)}$, therefore, has a cumulative density function $F^{\omega}_{s_{(1)}}(s) = e^{-\eta(1-F_\omega(s))}$ for $s \in [\underline{s}, \bar{s}]$.[11] Since bid b wins whenever $s_{(1)} \le \hat{s}$, bid b wins with probability $\pi_\omega(b;\beta) = F^{\omega}_{s_{(1)}}(\hat{s}) = e^{-\eta(1-F_\omega(\hat{s}))}$ for $\omega \in \{h, \ell\}$.

A defining feature of common-value auctions is that winning the auction is informative about the value of the good. We aim to analyze how different non-pooling bids affect this inference. Since any non-pooling bid induces some cutoff $\hat{s}$, we can work directly with this cutoff and analyze how the expected value

$$\begin{aligned} \mathbb{E}[v|s_{(1)} \le \hat{s}] &= \frac{\rho e^{-\eta(1-F_h(\hat{s}))} v_h + (1-\rho) e^{-\eta(1-F_\ell(\hat{s}))} v_\ell}{\rho e^{-\eta(1-F_h(\hat{s}))} + (1-\rho) e^{-\eta(1-F_\ell(\hat{s}))}} \\ &= \frac{\rho e^{\eta(F_h(\hat{s})-F_\ell(\hat{s}))} v_h + (1-\rho) v_\ell}{\rho e^{\eta(F_h(\hat{s})-F_\ell(\hat{s}))} + (1-\rho)} \end{aligned} \tag{2}$$

changes in $\hat{s}$. If β is strictly increasing (all bids are non-pooling bids) and continuous, this is the same as considering $\mathbb{E}[v|\text{win with } b;\beta]$ for different $b \in [\beta(\underline{s}), \beta(\bar{s})]$.

[11] Conditional on state ω, any competitor (independently) receives a signal larger than s with probability $1 - F_\omega(s)$. By the decomposition and environmental equivalence property of the Poisson distribution (Myerson (1998)), any bidder believes that the number of competitors with signals larger than s is Poisson distributed with mean $\eta(1 - F_\omega(s))$. The probability that $s_{(1)} \le s$ is the probability that there is non competitor with a signal above s – $\mathbb{P}(n = 0) = e^{-\eta(1-F_\omega(s))} \frac{[\eta(1-F_\omega(s))]^0}{0!} = e^{-\eta(1-F_\omega(s))}$.

Lemma 2. *The expected value* $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ *is strictly decreasing when* $\hat{s} < s^*$*, has unique global minimum at* $\hat{s} = s^*$ *and is strictly increasing when* $\hat{s} > s^*$*.*

Proof. Note that $\frac{av_h+v_\ell}{a+1} > \frac{bv_h+v_\ell}{b+1}$ if and only if $a > b$. Thus, $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ is strictly increasing if and only if $e^{\eta(F_h(\hat{s})-F_\ell(\hat{s}))}$ is strictly increasing. The derivative $e^{\eta(F_h(\hat{s})-F_\ell(\hat{s}))}\eta[f_h(\hat{s}) - f_\ell(\hat{s})]$ is positive if and only if $f_h(\hat{s}) > f_\ell(\hat{s})$. The monotone likelihood ratio property and the assumption that $f_h(s^*) = f_\ell(s^*)$ is unique imply that $f_h(\hat{s}) < f_\ell(\hat{s})$ for $\hat{s} < s^*$, and $f_h(\hat{s}) > f_\ell(\hat{s})$ for $\hat{s} > s^*$. □

Lemma 2 implies that $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ is U-shaped in $\hat{s}$ with it's minimum at s^*. The intuition behind the shape may be explained best with the help of Figure 1, which depicts $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ against $\hat{s} \in [\underline{s}, \bar{s}]$.

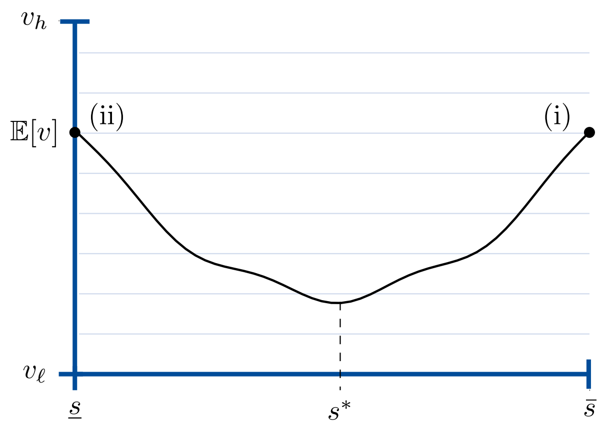

Figure 1: The expected value $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$

First, consider point (i) on the top right, which marks $\mathbb{E}[v|s_{(1)} \leq \bar{s}]$. By the way we defined $s_{(1)}$ it takes values on $\{-\infty\} \cup [\underline{s}, \bar{s}]$, such that it is always true that $s_{(1)} \leq \bar{s}$, independent of the state. Hence, the condition does not allow for any inferences about the value of the good, and the expected value conditional on winning is the unconditional one, $\mathbb{E}[v|s_{(1)} \leq \bar{s}] = \mathbb{E}[v]$. This reasoning applies for any distribution on the number of bidders; in particular, it is also true when the number of bidders is fixed and known, as in the standard Milgrom & Weber (1982) model.

Second, consider point (ii) on the top left, denoting $\mathbb{E}[v|s_{(1)} \leq \underline{s}]$. The event that $s_{(1)} = \underline{s}$ occurs with zero probability (the signal distribution has no atoms), while there are no competitors and $s_{(1)} = -\infty$ with positive probability. Consequently, $\mathbb{E}[v|s_{(1)} \leq \underline{s}] = \mathbb{E}[v|s_{(1)} = -\infty]$. However, the event that there is no competitor does not contain information about the state, because the distribution of bidders is independent of that state. As a result, no inference is possible and $\mathbb{E}[v|s_{(1)} \leq \underline{s}] = \mathbb{E}[v]$. Thus, there is no winner's curse at the bottom (ii) or at the top (i).

In the middle when $\hat{s} \in (\underline{s}, \bar{s})$, the winner's curse comes into play. With positive probability, there are competitors, all of which received signals below $\hat{s}$. This is bad news about the state of the world because it excludes high signals. Consequently, for $\hat{s} \in (\underline{s}, \bar{s})$, the expected value is smaller than the unconditional one, $\mathbb{E}[v|s_{(1)} \leq \hat{s}] < \mathbb{E}[v]$, with the global minimum at s^*, where $f_h(s^*) = f_\ell(s^*)$.

Observe, that as η (the expected number of competitors) increases, the winner's curse grows more severe on $\hat{s} \in (\underline{s}, \bar{s})$. Since the bidder expects to face more competitors, the negative inference from winning grows in η. For $\hat{s} \in (\underline{s}, \bar{s})$, it follows that $\mathbb{E}[v|s_{(1)} \leq \hat{s}] \xrightarrow{\eta} v_\ell$.[12] For points (i) and (ii) the arguments remain unaltered, however, such that $\mathbb{E}[v|s_{(1)} \leq s]$ converges in η to a $\sqcup$-shape.

While the precise form of $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ follows from the Poisson distribution, similar effects play a role for any distribution of bidders[13]. When the number of bidders is random, the winning bidder simultaneously updates her belief over two random variables: the number of bidders and their signal realization. Since these two can push the expected value in opposite directions, her inference will generally not be monotone in $\hat{s}$. The numbers uncertainty breaks the affiliation between the value of the good and the first order statistic of (other bidders') signals. Accordingly, bidding a higher, non-tieing bid does not necessarily increase the expected value conditional on winning, as in Milgrom & Weber (1982), and equilibrium behavior can substantially diverge from the one in auctions with affiliation.

3.1.1 Strictly increasing equilibria – Non-existence

Proposition 1. *Holding all other parameters fixed, for a sufficiently large η, no strictly increasing equilibrium exists.*

To see why this is true, suppose to the contrary that there was a strictly increasing equilibrium β^*, such that all bids are non-pooling bids. In this case, a bidder with signal s, following the bidding strategy β^* and considering both, the inference from winning as well as her own signal, expects the good to be of value

$$\begin{aligned}\mathbb{E}[v|\text{win with } \beta^*(s), s; \beta^*] &= \mathbb{E}[v|s_{(1)} \leq s, s] \\ &= \frac{\rho f_h(s) e^{-\eta(1-F_h(s))} v_h + (1-\rho) f_\ell(s) e^{-\eta(1-F_\ell(s))} v_\ell}{\rho f_h(s) e^{-\eta(1-F_h(s))} + (1-\rho) f_\ell(s) e^{-\eta(1-F_\ell(s))}}.\end{aligned}$$

[12] The monotone likelihood ratio property implies that $F_h(s) < F_\ell(s)$ for all $s \in (\underline{s}, \bar{s})$. Thus, $\eta(F_h(s) - F_\ell(s)) \to -\infty$ for all $s \in (\underline{s}, \bar{s})$ when $\eta \to \infty$. The convergence then follows by equation (2).

[13] Consider, for example, a truncated Poisson distribution in which $n \geq 2$ always. This distribution of the number of bidders would lead to a similar shape of the expected value for a sufficiently large η. In particular, the non-monotonicity of the inference does not hinge on the possibility of being alone in the auction.

The general idea of the proof can be described in the following manner: When η is large (when there are many competitors), the inference from winning is more relevant for the expected value than the bidder's own signal. Consequently, for a η sufficiently large, $\mathbb{E}[v|s_{(1)} \le s, s]$ is U-shaped in s. Further, when competition is fierce, equilibrium bids must be close to the expected value conditional on winning, $\beta^*(s) \approx \mathbb{E}[v|s_{(1)} \le s, s]$ for $s \in (\underline{s}, \bar{s}]$. However, given that $\mathbb{E}[v|s_{(1)} \le s, s]$ is U-shaped this would imply that β^* is U-shaped, which is a contradiction. The crucial step of the proof is the check that $\beta^*(s)$ converges to $\mathbb{E}[v|s_{(1)} \le s, s]$ sufficiently quick, such that the U-shape can be exploited. Otherwise, the argument might fail because $\mathbb{E}[v|s_{(1)} \le s, s]$ converges to v_ℓ for all $s \in (\underline{s}, \bar{s})$.

Consider any three signals $s_- < s < s_+$ with $s_+ < s^*$. The necessary condition $U(\beta^*(s_+)|s_+;\beta^*) \ge U(v_\ell|s_+;\beta^*)$[14] implies that $\beta^*(s_+) \le \mathbb{E}[v|s_{(1)} \le s_+, s_+]$, which rearranges to

$$\frac{\beta^*(s_+) - v_\ell}{v_h - \beta^*(s_+)} \le \frac{\rho}{1-\rho}\frac{f_h(s_+)}{f_\ell(s_+)}\frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}}. \tag{3}$$

Further, we show in the appendix that there exists a function $A(\eta) > 1$, such that for any $s_- < s$ and η, it must hold that

$$\frac{\beta^*(s) - v_\ell}{v_h - \beta^*(s)} \ge \frac{\rho}{1-\rho}\frac{f_h(s_-)}{f_\ell(s_-)}\frac{e^{-\eta(1-F_h(s))}}{e^{-\eta(1-F_\ell(s))}}A(\eta). \tag{4}$$

Otherwise, a bidder with signal s_- would have a strict incentive to deviate and bid $\beta^*(s)$ instead of $\beta^*(s_-)$. As η increases and competition grows more fierce, $A(\eta)$ decreases. In the limit when $A(\eta) = 1$, inequality (4) rearranges to $\beta^*(s) \ge \mathbb{E}[v|s_{(1)} \le s, s_-]$ which implies that for η large $\beta^*(s) \approx \mathbb{E}[v|s_{(1)} \le s, s]$.

Combining equations (3) and (4) and using that $\frac{\beta^* - v_\ell}{v_h - \beta^*}$ is increasing in β^* yields

$$\frac{\rho}{1-\rho}\frac{f_h(s_+)}{f_\ell(s_+)}\frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}} \ge \frac{\beta^*(s_+) - v_\ell}{v_h - \beta^*(s_+)} > \frac{\beta^*(s) - v_\ell}{v_h - \beta^*(s)} \ge \frac{\rho}{1-\rho}\frac{f_h(s_-)}{f_\ell(s_-)}\frac{e^{-\eta(1-F_h(s))}}{e^{-\eta(1-F_\ell(s))}}A(\eta)$$
$$\iff \frac{f_h(s_+)}{f_\ell(s_+)}\Big(\frac{f_h(s_-)}{f_\ell(s_-)}\Big)^{-1} > \Big(\frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}}\Big)^{-1}\frac{e^{-\eta(1-F_h(s))}}{e^{-\eta(1-F_\ell(s))}}A(\eta).$$

As η increases, $A(\eta) \to 1$; more importantly, however, the monotone likelihood ratio property implies that[15]

$$\Big(\frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}}\Big)^{-1}\frac{e^{-\eta(1-F_h(s))}}{e^{-\eta(1-F_\ell(s))}} = e^{\eta([F_h(s_+)-F_h(s)]-[F_\ell(s_+)-F_\ell(s)])} \to \infty.$$

[14] Given some strategy β, the utility from bidding b can be rewritten as $U(b|s;\beta) = \mathbb{P}(\text{win with } b|s;\beta)\Big(\mathbb{E}[v|\text{win with } b, s;\beta] - b\Big)$. Thus, a bid larger than the expected value results in a negative utility and is dominated by bidding v_ℓ, at which the utility is strictly positive.

[15] $[F_h(s_+) - F_h(s)] - [F_\ell(s_+) - F_\ell(s)] = \int_s^{s_+} f_h(z) - f_\ell(z)dz \le \int_s^{s_+} f_\ell(z)(\frac{f_h(s_+)}{f_\ell(s_+)} - 1)dz < 0$.

The negative inference from $s_{(1)} \leq s_+$ grows unboundedly stronger than from $s_{(1)} \leq s$, such that for a sufficiently large η it dominates the difference in signals $\frac{f_h(s_+)}{f_\ell(s_+)}\left(\frac{f_h(s_-)}{f_\ell(s_-)}\right)^{-1} > 1$, thereby implying that $\mathbb{E}[v|s_{(1)} \leq s, s]$ becomes U-shaped. The inequality cannot hold and β^* cannot be a strictly increasing equilibrium.

Note the two roles a large η plays in this argument. First, the increased competition ensures that bids are close to the expected value conditional on winning. Second, it implies that the inference from winning is more decisive for the winning bidder's belief than her own signal, thereby making the expected value conditional on winning non-monotone. Both effects and, hence, the non-existence are not tied to the Poisson distribution, but are more general. Whenever the inference is non-monotone in the winning bid[16] and competition is fierce, such that bids have to be close to this expected value, a strictly increasing equilibrium will not exist. Therefore, Proposition 1 extends to other distributions of the number of bidders and even has its counterpart for other auction formats.[17]

To conclude the non-existence argument, we want to provide an example that illustrates the argument once more and highlights how large "a sufficiently large η" is.

Example 1: Assume that $v_h = 1$, $v_\ell = 0$, and that both states are equally likely. Let the signal space be $[0, 1]$ and that the likelihood ratio $\frac{f_h}{f_\ell}$ is constant on $[0, \frac{1}{2}]$. Therefore, bidders with signals $s \in [0, \frac{1}{2}]$ are essentially equal and, in equilibrium, have to be indifferent over each other's bids. We want to find the critical η, such that no strictly increasing equilibrium exists. To this end, suppose that β^* is a strictly increasing equilibrium. Then, it follows from the indifference that

$$U(\beta^*(\underline{s})|\underline{s}; \beta^*) = U(\beta^*(s)|s; \beta^*) \quad \forall s \in [0, \frac{1}{2}],$$

and by standard arguments $\beta^*(\underline{s}) = v_\ell = 0$. Thus, we can solve for $\beta^*(s)$ (steps in the appendix), and take the derivative with respect to s. We find that the slope is positive if and only if

$$\Big(\frac{f_h(s)}{f_\ell(s)}\Big)^2 > e^{\eta F_\ell(s)}(1 - \frac{f_h(s)}{f_\ell(s)} - e^{-\eta F_h(s)}).$$

Now, assume that $f_h(s) = \frac{3}{4}$ and $f_\ell(s) = \frac{5}{4}$ for $s \in [0, \frac{1}{2}]$. Setting $s = \frac{1}{2}$ and solving for η yields a critical value of $\eta \approx 2.9$. For any larger η, a strictly increasing equilibrium does not exist. The problem in this example is particularity pronounced, since all signals below $\frac{1}{2}$

[16] That is, whenever the order statistics and the value of the good are not affiliated. Consider Atakan & Ekmekci (2014) and Pesendorfer & Swinkels (2000) for other examples of non-affiliated auctions.

[17] A second-price auction will, for example, not have a strictly increasing equilibrium, either (compare footnote 20). Harstad et al. (2008) provide an example for the SPA, where the distribution of bidders is binary.

imply the same belief, which means that $\mathbb{E}[v|s_{(1)} \leq s, s]$ is decreasing on $[0, \frac{1}{2}]$ (independent of η). When the monotone likelihood ratio property holds strictly, the critical η is generally slightly higher. Nevertheless, the non-existence is more the rule than the exception.

3.1.2 Strictly increasing equilibria – Existence

After considering (not so) large η, we analyze what happens when η is small. For $s, \hat{s} \in [\underline{s}, \bar{s}]$, let $F_{s_{(1)}}(s|\hat{s})$ denote the expected cumulative density function of $s_{(1)}$ conditional on observing $\hat{s}$

$$\begin{aligned} F_{s_{(1)}}(s|\hat{s}) &:= \frac{\rho f_h(\hat{s}) F^h_{s_{(1)}}(s) + (1-\rho) f_\ell(\hat{s}) F^\ell_{s_{(1)}}(s)}{\rho f_h(\hat{s}) + (1-\rho) f_\ell(\hat{s})} \\ &= \frac{\rho f_h(\hat{s}) e^{-\eta(1-F_h(s))} + (1-\rho) f_\ell(\hat{s}) e^{-\eta(1-F_\ell(s))}}{\rho f_h(\hat{s}) + (1-\rho) f_\ell(\hat{s})}, \end{aligned}$$

and let $f_{s_{(1)}}(s|\hat{s})$ be the associated density.

Proposition 2 (Strictly Increasing Equilibria)**.** *The ordinary differential equation*

$$\hat{\beta}'(s) = \Big(\mathbb{E}[v|s_{(1)} = s, s] - \hat{\beta}(s)\Big) \frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)} \qquad \text{with } \hat{\beta}(\underline{s}) = v_\ell \tag{5}$$

has a unique solution, $\hat{\beta}$.

(i) If $\hat{\beta}$ is strictly increasing, then it is a unique equilibrium in the class of strictly increasing equilibria.

(ii) If $\hat{\beta}$ is not strictly increasing, no strictly increasing equilibrium exists.

(iii) If

$$2\Big(\frac{\partial}{\partial s} \frac{f_h(s)}{f_\ell(s)}\Big) \frac{f_\ell(s)}{f_h(s)} + \eta f_h(s) - \eta f_\ell(s) > 0 \text{ for a.e. } s \in [\underline{s}, \bar{s}], \tag{6}$$

but in any case when η is sufficiently small, a strictly increasing equilibrium exists.

The proof is provided in the appendix.[18] When arguing why a strictly increasing equilibrium does not exist for η sufficiently large, we used two implications of a large η: that $\mathbb{E}[v|s_{(1)} \leq s, s]$ is non-monotone and that competition is sufficiently fierce. Both effects reoccur in the conditions sufficient for the existence of a strictly increasing equilibrium (*iii*).

If η is sufficiently small, such that the expected value conditional on winning is monotone, the existence problem described above does not arise. Even when bids are close to the expected value conditional on winning, the bidding function can be strictly increasing. In fact, we can provide a slightly tighter[19] sufficient condition: $\hat{\beta}(s)$ is strictly increasing if

[18] Apart from the slightly different definition of $s_{(1)}$, this is the standard ODE in the literature.

[19] $\mathbb{E}[v|s_{(1)} \leq s, s]$ is strictly increasing in s when $\left(\frac{\partial}{\partial s} \frac{f_h(s)}{f_\ell(s)}\right) \frac{f_\ell(s)}{f_h(s)} + \eta f_h(s) - \eta f_\ell(s) > 0 \;\; \forall s \in [\underline{s}, \bar{s}]$

$\mathbb{E}[v|s_{(1)} = s, s]$ is strictly increasing in s,[20] which is the case if and only if condition (6) holds. Note that $\frac{f_h(s)}{f_\ell(s)}$ is differentiable almost everywhere because it is monotonic.

Even if this first condition fails and $\mathbb{E}[v|s_{(1)} = s, s]$ is decreasing over some interval (as in Example 1), a strictly increasing equilibrium exists for η small. In this situation, we can utilize the second effect of η – the degree of competition. If η is small, such that competition is very weak, bids are far away from the expected value conditional on winning. Therefore, the problem described above does not arise, and a strictly increasing equilibrium always exists.

While a strictly increasing equilibrium will only exist when η is small, the bidding function might be partially flat and contain jumps. Next, we take a closer look at these flat parts to understand why it might be beneficial for bidders with different signals to pool on the same bid.

3.2 Pooling Bids

In this subsection, we consider bids that are selected by bidders with different signals and tie with positive probability. We derive the winning probabilities and revisit Example 1 to construct an equilibrium when no strictly increasing equilibrium exists. Thereafter, we analyze the effects of these pooling bids more formally.

Fix some non-decreasing strategy β, and assume $\beta(s) = b_p$ for some b_p and all s from an interval I, but $\neq b_p$ otherwise. We generally refer to these intervals as *pools*, to b_p as a *pooling bid* and, without loss, always think about the closure of interval I, which we denote by $[s_-, s_+]$. In the appendix (proof of Lemma 3), we show by simple computation that the probability to win with b_p is

$$\pi_\omega(b_p; \beta) = \frac{\mathbb{P}(s_{(1)} \in [s_-, s_+] \,|\omega)}{\mathbb{E}[\#s \in [s_-, s_+] \,|\omega]} = \frac{e^{-\eta(1-F_\omega(s_+))} - e^{-\eta(1-F_\omega(s_-))}}{\eta(F_\omega(s_+) - F_\omega(s_-))}. \tag{7}$$

In the section above, we considered Example 1 and found that no strictly increasing equilibrium exists for $\eta > 2.9$. Now, we want to revisit the example and show that an equilibrium with a pooling bid can exist when $\eta > 2.9$.

Example 1 continued: Extend the densities from Example 1 to

[20] For a second-price auction, standard arguments imply that the equilibrium bid in a symmetric and strictly increasing equilibrium is the expected value conditional on being tied at the top $\mathbb{E}[v|s_{(1)} = s, s]$. Thus, condition (6) is necessary and sufficient for the existence of a strictly increasing equilibrium in a second-price auction.

$$f_h(s) = \begin{cases} \frac{3}{4} & s \in [0, \frac{1}{2}] \\ 2s - \frac{1}{4} & s \in (\frac{1}{2}, 1] \end{cases} \qquad f_\ell(s) = \begin{cases} \frac{5}{4} & s \in [0, \frac{1}{2}] \\ -2s + \frac{9}{4} & s \in (\frac{1}{2}, 1] \end{cases}$$

and consider the following strategy: All bidders with signals at or below 0.5 select the same bid $b_p = 0.12$, while all bidders with a signal above 0.5 follow a strictly increasing bidding strategy (5) with an initial value $b_p = 0.12$. We show in the appendix that there is an $\eta^* \approx 4.98$, such that this constitutes an equilibrium. For intuition, assume that $\eta = \eta^*$, and consider the relevant incentives:

• At $s = 0.5$, the expected value $\mathbb{E}[v|s_{(1)} \leq 0.5, 0.5] \approx 0.147 > 0.12$, and for $s \geq 0.5$ the sufficient condition *(iii)* of Proposition 2 holds. Thus, bidders with $s \geq 0.5$ do not want to deviate to $v_\ell = 0$, and the differential equation (5) with initial value b_p is strictly increasing.

• Bidders with signal $s = 0.5$ are indifferent between selecting b_p or marginally overbidding it

$$\lim_{\epsilon \to 0} U(b_p + \epsilon|0.5; \beta) = \mathbb{P}(s_{(1)} \leq 0.5|0.5)\Big(\mathbb{E}[v|s_{(1)} \leq 0.5, 0.5] - b_p\Big) \approx 0.0031,$$
$$U(b_p|0.5; \beta) = \mathbb{P}(\text{win with } b_p|0.5;\ \beta)\Big(\mathbb{E}[v|\text{win with } b_p, 0.5;\ \beta] - b_p\Big) \approx 0.0031.$$

• Last, for bidders with $s = 0$, a deviation to $v_\ell = 0$ is unprofitable because

$$U(0|0; \beta) \approx 0.0026 < U(b_p|0; \beta) \approx 0.0031.$$

The example shows that pooling bids can ensure the existence of an equilibrium when no strictly increasing equilibrium exists. The central feature that makes this possible is that the expected value conditional on winning with b_p is larger than the expected value conditional on winning with a bid marginally above b_p. Formally, a strategy β can only be an equilibrium with some $b_p = \beta(s)$ for exactly $s \in [s_-, s_+]$, if

$$\mathbb{E}[v|\text{win with } b_p; \beta] > \lim_{\epsilon \to 0} \mathbb{E}[v|\text{win with } b_p + \epsilon; \beta] = \mathbb{E}[v|s_{(1)} \leq s_+].$$

Suppose this was not the case, that is $\mathbb{E}[v|\text{win with } b_p, s_+; \beta] \leq \mathbb{E}[v|s_{(1)} \leq s_+, s_+]$. Then, a deviation to a bid marginally above b_p would discretely raise the winning probability (no random tiebreak) and weakly increase the expected value. Since $\beta(s_+) = b_p < \mathbb{E}[v|\text{win with } b_p, s_+; \beta]$ (cf. footnote 14), such a deviation would always be profitable and β could be no equilibrium.

To gain intuition with regard to how winning with b_p can be a blessing compared to winning with a marginally larger bid, consider the following reasoning: With positive probability, multiple bidders tie on the pooling bid b_p, such that the winner is decided by the uniform tiebreaking rule. Consequently, a bidder is more likely to win when there are fewer competitors who also chose b_p, that is when there are fewer other bidders with signals from $[s_-, s_+]$. If those signals are low, such that they are more likely to be realized in the low state of the world, there is more competition in the low state and the bidder wins less often in the low state than in the high state. This is good news about the value of the good, a blessing the bidder would lose when marginally overbidding the pooling bid.

For this effect to work, the number of competitors must be random. Otherwise, winning more often when there are fewer competitors form $[s_-, s_+]$ implies winning more often when there are more bidders with signals below s_-. This worsens the winner's curse. When the number of bidders is Poisson-distributed, then the number of bidders with signals below s_- is independent of the number of bidders with signals from $[s_-, s_+]$. Therefore, the blessing occurs whenever the expected number of bidders with signals from $[s_-, s_+]$, that is $\eta[F_\omega(s_+) - F_\omega(s_-)]$ is larger in the low state than in the high state. We group all remaining results on pooling bids in the following lemma and discuss them thereafter.

Lemma 3. *Assume β is such that there exists an interval $I := [s_-, s_+]$ and a bid b_p, such that $b_p = \beta(s)$ for all $s \in I$ and $\beta(s) < b_p < \beta(s')$ for all $s < s_- < s_+ < s'$.*

Then,

$$\mathbb{E}[v|\textit{win with } b_p; \beta] \in \Big[\mathbb{E}[v|s_{(1)} \leq s_-],\ \mathbb{E}[v|s_{(1)} \leq s_+]\Big]. \tag{8}$$

If β is an equilibrium bidding strategy, then

$$\eta[F_h(s_+) - F_h(s_-)] < \eta[F_\ell(s_+) - F_\ell(s_-)],$$

and, consequently,

$$\mathbb{E}[v|s_{(1)} \leq s_-] > \mathbb{E}[v|\textit{win with } b_p; \beta] > \mathbb{E}[v|s_{(1)} \leq s_+]. \tag{9}$$

The expected value and bounds (8) follows from straight-up computation, which is found in the appendix. It states that the expected value conditional on winning with b_p always takes a value between the expected value conditional on marginally underbidding or overbidding b_p. Combining equation (8) and Figure 1, it follows directly that in equilibrium, it must hold that $\mathbb{E}[v|s_{(1)} \leq s_-] > \mathbb{E}[v|s_{(1)} \leq s_+]$. Otherwise, $\mathbb{E}[v|s_{(1)} \leq s_+] \geq \mathbb{E}[v|\text{win with } b_p; \beta]$ and by the reasoning above, bidders would have a strict incentive to marginally overbid b_p. The condition $\mathbb{E}[v|s_{(1)} \leq s_-] > \mathbb{E}[v|s_{(1)} \leq s_+]$ holds, if and only if[21]

$$\begin{aligned} &\frac{e^{-\eta(1-F_h(s_-))}}{e^{-\eta(1-F_\ell(s_-))}} > \frac{e^{-\eta(1-F_h(s_+))}}{e^{-\eta(1-F_\ell(s_+))}} \\ \Leftrightarrow\ & e^{-\eta(F_h(s_-)-F_h(s_-))} > e^{-\eta(F_\ell(s_-)-F_\ell(s_-))} \\ \Leftrightarrow\ & \eta[F_h(s_+) - F_h(s_-)] < \eta[F_\ell(s_+) - F_\ell(s_-)], \end{aligned}$$

which proves the rest of the Lemma. Note that it follows that there can be no pool in equilibrium where $s_- \geq s^*$.

[21]Recall equation (2) and that $\frac{av_h + v_\ell}{a+1} > \frac{bv_h + v_\ell}{b+1}$ if and only if $a > b$.

When β is partially flat, the utility is not continuous in the bid. The probability of winning the auction with a bid just below or above a pooling bid b_p is discretely different from the probability at b_p, so is the expected value conditional on winning. Further, equilibria of the game will, generally, not be unique. The equilibrium bidding strategy does not follow a unique differential equation but can contain a mixture of strictly increasing and flat parts, as well as jumps. Last, as equation (9) reveals, the expected value conditional on winning with a bid just below the pooling bid is discretely larger than winning with the pooling bids. Thus, there is an open set of bids below b_p with a discretely lower winner's curse attached to them. As will become evident in the next section, this open set is detrimental to equilibrium existence when η is sufficiently large.

4 Non-existence

Proposition 3. *Holding all other parameters fixed, for a sufficiently large η, no equilibrium exists.*

The formal proof follows as a corollary to Proposition 5. For now, we only want to provide an intuition for the result.

So far, we already know from Proposition 1 that for a sufficiently large η there is no strictly increasing equilibrium. A quick review of the proof will reveal that we can conclude even more. For η sufficiently large, there can never be a (substantial) interval below s^* where the bidders follow a strictly increasing bidding strategy. In particular, the equilibrium we constructed for Example 1 does not exist when η is large, because $\mathbb{E}[v|s_{(1)} \leq s]$ is decreasing on $[\frac{1}{2}, \frac{5}{8}]$.

One idea to potentially circumvent this problem is to construct an equilibrium β^* in which all signals below s^* pool on one bid b_p, while all higher signals follow a strictly increasing bidding strategy. This candidate equilibrium is depicted in the left frame of Figure 2 and we want to eliminate it for large η . The two arrows indicate two possible deviations, which would have to be unprofitable in equilibrium.

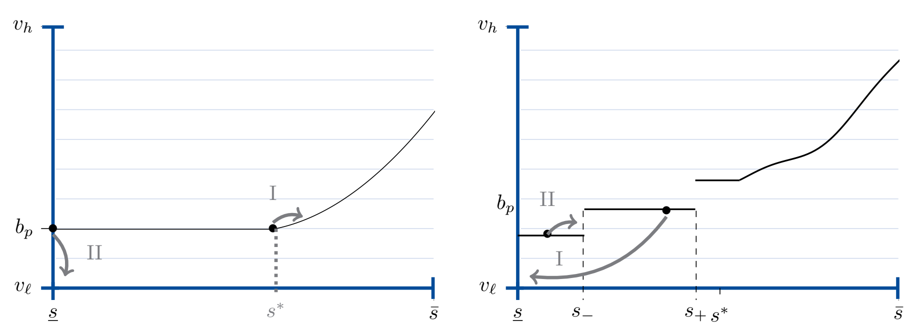

Figure 2: Candidate equilibria

As a simplification, we abbreviate the winning probabilities when selecting b_p and marginally overbidding it by

$$\pi_\omega := \pi_\omega(b_p;\beta^*) = \frac{e^{-\eta(1-F_\omega(s^*))} - e^{-\eta}}{\eta F_\omega(s^*)} \qquad \pi_\omega^+ := \lim_{\epsilon\searrow 0} \pi_\omega(b_p+\epsilon;\beta^*) = e^{-\eta(1-F_\omega(s^*))}.$$

When a bidder with signal s^* deviates to a bid marginally above b_p (deviation I), she wins the tiebreak for sure. This deviation is unprofitable if $U(b_p|s^*;\beta^*) \geq \lim_{\epsilon\searrow 0} U(b_p+\epsilon|s^*;\beta^*)$, which can be expressed as (more steps in appendix (D))

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho}\frac{f_h(s^*)}{f_\ell(s^*)}\frac{\pi_h^+ - \pi_h}{\pi_\ell^+ - \pi_\ell}. \tag{10}$$

As η increases, the bidder wins infinitely more often when marginally overbidding the pooling bid instead of bidding b_p, that is, $\frac{\pi_\omega^+}{\pi_\omega} \to \infty$. Consequently, there exists a function $B(\eta) < 1$ with $\lim B(\eta) = 1$, such that $\frac{\pi_h^+ - \pi_h}{\pi_\ell^+ - \pi_\ell} = B(\eta)\frac{\pi_h^+}{\pi_\ell^+}$, which implies that b_p is at least $\approx \mathbb{E}[v|s_{(1)} \leq s^*, s^*]$ for η large.

To ensure that any signal s does not deviate from $\beta^*(s)$ to v_ℓ, it has to hold that $\beta^*(s) \leq \mathbb{E}[v|\text{win with } \beta^*(s), s; \beta^*]$ (cf. footnote 14). For signal $\underline{s}$ with $\beta^*(\underline{s}) = b_p$ this rearranges to

$$\frac{b_p - v_\ell}{v_h - b_p} \leq \frac{\rho}{1-\rho}\frac{f_h(\underline{s})}{f_\ell(\underline{s})}\frac{\pi_h}{\pi_\ell}. \tag{11}$$

Inspecting π_ω and π_ω^+, we observe that there exists a function $D(\eta) > 1$ with $\lim D(\eta) = 1$, such that $\frac{\pi_h}{\pi_\ell} = \frac{\pi_h^+}{\pi_\ell^+}\frac{F_\ell(s^*)}{F_h(s^*)}D(\eta)$ – the blessing from winning with b_p as opposed to marginally higher bid is bounded and of the order $\frac{F_h(s^*)}{F_\ell(s^*)}$. The problem is that for large η, this blessing does not suffice to reconcile the two conditions (10) and (11). Either s^* wants to marginally outbid b_p, or $\underline{s}$ makes a strict loss. To see this formally, combine inequalities (10) and (11) and use that $\frac{f_h(s^*)}{f_\ell(s^*)} = 1$. This yields the following necessary condition

$$\begin{aligned} \frac{\rho}{1-\rho}\frac{f_h(\underline{s})}{f_\ell(\underline{s})}\frac{\pi_h^+}{\pi_h^+}\frac{F_\ell(s^*)}{F_h(s^*)}D(\eta) &\geq \frac{\rho}{1-\rho}\frac{\pi_h^+}{\pi_\ell^+}B(\eta) \\ \iff \frac{f_h(\underline{s})}{f_\ell(\underline{s})}\frac{F_\ell(s^*)}{F_h(s^*)} &\geq \frac{B(\eta)}{D(\eta)}. \end{aligned}$$

Since we assume that $\frac{f_h(\underline{s})}{f_\ell(\underline{s})} < \frac{f_h(\bar{s})}{f_\ell(\bar{s})}$, the monotone likelihood ratio property implies that the expression on the left side is strictly smaller than 1, while that on the right side converges to 1. Thus, either condition (10) or (11) is violated for large η and, as a result, β^* cannot take the presumed form. The problem is the same if all signals up to $s_+ > s^*$ select bid b_p.

At this point, we have eliminated the possibilities that β^* may be strictly increasing over any (significant) interval below s^*, or is constant below s^*. This implies that if there is an equilibrium, there has to be an interval $[s_-, s_+]$, with $s_- \in (\underline{s}, s^*)$ and $\beta^*(s) = b_p$ for exactly $s \in [s_-, s_+]$. Suppose that this was the case and, as a simplification, assume that $s_+ \leq s^*$. This candidate equilibrium is depicted in the right frame of Figure 2. Denote the winning probabilities for bidding b_p and marginally overbidding and underbidding b_p by

$$\pi_\omega := \pi_\omega(b_p; \beta^*) = \frac{e^{-\eta(1-F_\omega(s_+))} - e^{-\eta(1-F_\omega(s_-))}}{\eta(F_\omega(s_+) - F_\omega(s_-))}$$
$$\pi_\omega^- := \lim_{\epsilon \searrow 0} \pi_\omega(b_p - \epsilon; \beta^*) = e^{-\eta(1-F_\omega(s_-))} \qquad \pi_\omega^+ := \lim_{\epsilon \searrow 0} \pi_\omega(b_p + \epsilon; \beta^*) = e^{-\eta(1-F_\omega(s_+))}.$$

Sine bidding v_ℓ dominates any bid that is above the expected value conditional on winning, the pooling bid $b_p \leq \mathbb{E}[v|\text{win with } b_p, s_-; \beta^*]$ (deviation I; cf. footnote 14), which rearranges to

$$\begin{aligned}\frac{b_p - v_\ell}{v_h - b_p} &\leq \frac{\rho}{1-\rho}\frac{f_h(s_-)}{f_\ell(s_-)}\frac{\pi_h}{\pi_\ell} \\ &= \frac{\rho}{1-\rho}\frac{f_h(s_-)}{f_\ell(s_-)}\frac{\pi_h^+}{\pi_\ell^+}\frac{F_\ell(s_+) - F_\ell(s_-)}{F_h(s_+) - F_h(s_-)}\hat{B}(\eta). \end{aligned} \tag{12}$$

The second equality with $\hat{B}(\eta) \searrow 1$ follows in the same manner as $B(\eta)$ above. Again, steps are found in appendix (D). In order to ensure that a bidder with signal $s \in [\underline{s}, s_-)$ does not want to deviate from $\beta^*(s)$ to a bid marginally below b_p (deviation II), the pooling bid b_p must not be too low. In the appendix, we use this necessary condition to derive a function $E_s(\eta) < 1$, with $\lim E_s(\eta) = 1$ and a lower bound on b_p

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho}\frac{f_h(s)}{f_\ell(s)}\frac{\pi_h^-}{\pi_h^-}E_s(\eta). \tag{13}$$

Putting equations (12) and (13) together yield

$$\frac{\rho}{1-\rho}\frac{f_h(s_-)}{f_\ell(s_-)}\frac{\pi_h^+}{\pi_\ell^+}\frac{F_\ell(s_+) - F_\ell(s_-)}{F_h(s_+) - F_h(s_-)}\hat{B}(\eta) \geq \frac{\rho}{1-\rho}\frac{f_h(s)}{f_\ell(s)}\frac{\pi_h^-}{\pi_\ell^-}E_s(\eta).$$

The crucial observation here is that because $s_+ < s^*$, it follows that (c.f. footnote 15)

$$\frac{\pi_h^+}{\pi_\ell^+}\left(\frac{\pi_h^-}{\pi_\ell^-}\right)^{-1} = e^{-\eta[(F_h(s_+) - F_h(s_-)) - (F_\ell(s_+) - F_\ell(s_-))]} \to 0,$$

which implies that for η sufficiently large, either equation (12) or (13) is violated.

Walking through the argument once more, since equilibrium bids can at most be the expected value conditional on winning, $\mathbb{E}[v|\text{win with } b_p, s_-; \beta]$ puts an upper bound on b_p (12). For large η and any $s < s_-$, this upper bound is smaller than the expected value conditional on marginally underbidding the pooling bid $\mathbb{E}[v|s_{(1)} \leq s_-, s]$ (9). Hence, the expected profits when selecting a bid marginally below b_p are strictly positive. When η is large, competition by bidders with signals below s_- is fierce and a Bertrand competition emerges. Bidders compete for the highest bid below b_p which maximizes their chances to win the auction but is subject to a strictly smaller winner's curse than b_p. Such a bid does not exist because the set of bids below b_p is open which yields the contradiction.

The arguments presented here are no complete proof, but highlight the effects that prevent the existence of an equilibrium. First, there can neither be an equilibrium strategy that is strictly increasing over an interval below s^*, nor one in which all bidders with signals below s^* pool. Thus, there has to be a pool that starts strictly to the right of s_-, at which the utility is discontinuous, thereby creasing an openness problem in the bidding space. This openness is characteristic of the continuous bidding space. When we consider auctions on a grid, there is a maximal bid below the pooling bid b_p, such that the problem does not arise and an equilibrium exists. Another way to solve the problem is to introduce an extended auction mechanism, which allows bidders to send a cheap talk message alongside their bid.

5 Communication Extension

To ensure equilibrium existence for any η and develop a tool to analyze auctions on the sufficiently fine grid, we extend the auction mechanism and allow bidders to send two cheap talk messages alongside their bid. This mechanism is an implementation of the endogenous tiebreaking rule by Jackson et al. (2002). We call it the *Communication Extension* of the auction and denote it by Γ^c. In the following account, we will describe the new mechanism before characterizing the set of equilibria. Last, we use our findings to prove Proposition 3.

In the Communication Extension, every bidder simultaneously selects three actions. To begin with, she reports a set of number $C \subseteq [\underline{s}, \bar{s}]$ that partitions the signal space into (potentially trivial) intervals. Given partition C, two signals s and s' belong to different intervals if and only if there is a number $c \in C$, such that $s < c \leq s'$. To ensure measurability, we require bidders to play pure strategies over C, which, as we will see later, is not a binding constraint. As a second cheap talk message, each bidder reports a signal $s^c \in [\underline{s}, \bar{s}]$ which selects an interval from partition C. Multiple reports s^c may be from the same interval, which creates an equivalence relation over reports: $s^c \sim \hat{s}^c$ if $\nexists c \in C$ such that $s^c < c \leq \hat{s}^c$. Last, every bidder selects a bid $b \in [v_\ell, v_h]$.

Thus, the (symmetric) strategy of a bidder is a function[22] $\sigma : [\underline{s}, \bar{s}] \to \mathcal{P}[\underline{s}, \bar{s}] \times \Delta\Big([\underline{s}, \bar{s}] \times$

[22]We consider functions that are measurable and probability distributions which that Borel probability measures.

$[v_\ell, v_h]\Big)$, which maps the signals into a partition and distribution over reports and bids. The auction mechanism selects the winner according to the following rule: First, it checks whether all bidders reported the same partition C. If not, the good is not allocated. In case all bidders reported the same partition, the good is allocated to the highest bidder. If there are multiple highest bidders, the good is allocated randomly among those who reported a signal from the highest interval of the partition, that is, the highest equivalence class of signal reports s^c. The winner receives the object and pays her bid.

Denoting the probability to win with action (C, s^c, b), if all other bidders follow strategy σ by $\pi^c_\omega(C, s^c, b; \sigma)$, the interim expected utility for a bidder with signal s who selects (C, s^c, b) is

$$
\begin{aligned}
U^c(C, s^c, b|s; \sigma) = & \frac{\rho f_h(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} \pi^c_h(C, s^c, b; \sigma)(v_h - b) \\
& + \frac{(1-\rho) f_\ell(s)}{\rho f_h(s) + (1-\rho) f_\ell(s)} \pi^c_\ell(C, s^c, b; \sigma)(v_\ell - b).
\end{aligned}
\tag{14}
$$

Given the utility, a strategy σ^* is a *best response* to a strategy σ, if, for almost every s, $(C, s^c, b) \in \operatorname{supp} \sigma^*(s)$, implies that $(C, s^c, b) \in \arg\max_{(\hat{P}, \hat{s}^c, \hat{b})} U(\hat{C}, \hat{s}^c, \hat{b}|s; \sigma)$. We make one assumption that restricts the set of best responses (and thereby equilibria) we take into consideration.

Assumption 1. *In any best response σ^*, all signals report the same partition C.*

Generally, the two cheap talk messages allow for various forms of coordination that are not feasible under the rules of a standard first-price auction.[23] Since we want to use the Communication Extension as a tool to approximate equilibria of auctions without cheap talk on a grid, we eliminate these forms of coordination. Observe that under Assumption 1, our restriction to strategies in which bidders do not mix over partitions becomes innocuous. In equilibrium, every bidder plays a best response and, therefore, selects the same partition. A deviation to another partition cannot be profitable, because it is detected unless the bidder is alone in the auction and would have won anyhow. If a deviation to another partition is not profitable, neither is a deviation to any sort of mixture over partitions.

Lemma 4 (Monotonicity in the Communication Extension)**.** *Consider a Communication Extension Γ^c, any strategy σ, and any best response σ^* to it. Then, there exists another best response $\hat{\sigma}^*$, which has the following properties:*

(i) If $(C, s^c, b), (C, s^{c\prime}, b) \in \operatorname{supp} \hat{\sigma}^*$ and $\pi^c_h(C, s^c, b; \sigma) = \pi^c_h(C, s^{c\prime}, b; \sigma)$, then $s^c = s^{c\prime}$;

(ii) It is pure in all three actions;

[23] Assume, for example, that the signals space is $[0, 1]$ and bids are strictly increasing in the signal, but signals $[0, \frac{1}{2}]$ select C, while signals $(\frac{1}{2}, 1]$ select $C' \neq C$. Then, signal $\frac{3}{4}$ only wins whenever there is no bidder with a signal from $[0, \frac{1}{2}]$ and no signal above $\frac{3}{4}$. Signal $\frac{1}{4}$ only wins when there is no higher signal. Such an outcome cannot be achieved in a standard first-price auction.

(iii) *Bids are non-decreasing in the signal* s*;*

(iv) *For a given bid* b*, the report* s^c *is non-decreasing in the signal* s*;*

(v) $U(\hat{\sigma}^*(s)|s;\sigma) = U(\sigma^*(s)|s;\sigma)$ *for almost every* s*;*

(vi) $\pi^c_\omega(C, s^c, b; \hat{\sigma}^*) = \pi^c_\omega(C, s^c, b; \sigma^*)$ *for all* $(C, s^c, b) \in \mathcal{P}[\underline{s}, \bar{s}] \times [\underline{s}, \bar{s}] \times [v_\ell, v_h]$ *and* $\omega \in \{h, \ell\}$.

Given a partition C and bid b, we can identify every equivalence class of the partition (see above) by a unique cheap talk signal *(i),* which simplifies notation. Properties *(ii)–(iv)* are analogous to the result in Lemma 1. Bidders with higher signals are more optimistic, select higher bids/reports, and win more often. If multiple signals induce the same belief, the actions can be reordered such that they are monotone, but without altering the utilities (v), or attainable outcomes (vi). We prove the results in the appendix.

Again, we look for Bayes-Nash equilibria, that are, strategies which are best responses to themselves. By Lemma 4, we can restrict attention to equilibria which fulfill properties *(i) - (iv)*. Henceforth, we only consider equilibria that are pure and where the bidders with higher signals win weakly more often. We denote pure strategies that fulfill *(i)*, *(iii),* and *(iv)* by $\sigma : [\underline{s}, \bar{s}] \to \mathcal{P}[\underline{s}, \bar{s}] \times [\underline{s}, \bar{s}] \times [v_\ell, v_h]$.

We can now explicitly state the winning probabilities π^c_ω. To do so, fix some strategy σ under which all signals report partition C. Let $s^c(s)$ and $b(s)$ be functions such that $\sigma(s) = (C, s^c(s), b(s))$ for all s. If action (C, s^c, b) is selected with zero probability by another bidder, then it wins whenever $s_{(1)} \leq \hat{s}$ with $\hat{s} := \sup\Big(\{s\colon\ b(s) < b\} \cup \{s\colon\ b(s) = b \text{ and } s^c(s) < s^c\}\Big)$. This happens in state $\omega \in \{h, \ell\}$ with probability

$$\pi^c_\omega(C, s^c, b; \sigma) = e^{-\eta(1-F_\omega(\hat{s}))}.$$

In case the bidder deviates to some other partition $C' \neq C$, she wins only when she is alone and, hence, not detected which happens with probability

$$\pi^c_\omega(C', s^c, b; \sigma) = e^{-\eta}.$$

If $\sigma(s) = (C, s^c, b)$ for $s \in [s_-, s_+]$, and $\neq (C, s^c, b)$ for all other signals, then the action wins in state $\omega \in \{h, \ell\}$ with probability

$$\pi^c_\omega(C, s^c, b; \sigma) = \frac{e^{-\eta(1-F_\omega(s_+))} - e^{-\eta(1-F_\omega(s_-))}}{\eta(F_\omega(s_+) - F_\omega(s_-))}.$$

All three expressions are analogous to the ones in the standard auction and are derived the same manner. Contrary to the standard auction, however, the Communication Extension always has an equilibrium.

Proposition 4 (Existence in the Communication Extension)**.** *Any Communication Extension Γ^c has an equilibrium. The equilibrium is pure and bids b as well as reports s^c are non-decreasing in the signal s.*

In the appendix, we construct this equilibrium as the limit of a sequence of equilibria on an ever finer grid. Even though there are, generally, multiple equilibria, we can characterize their form up to some ϵ environment around $\underline{s}$ and s^*.

Proposition 5 (Form of the Equilibria in the Communication Extension)**.** *Fix any $\epsilon \in (0, \frac{s^*-\underline{s}}{2})$. For η sufficiently large (given ϵ), any equilibrium σ^* of the Communication Extension Γ^c takes the following form: There are two disjoint, adjacent intervals of signals I, J such that*

(i) $[\underline{s}+\epsilon, s^*-\epsilon] \subset I \cup J$;

(ii) $\sigma^*(s_I) = (C, s^c_I, b)$ *for all* $s_I \in I$ *and* $\sigma^*(s_J) = (C, s^c_J, b)$ *for all* $s_J \in J$, *with* $s^c_I < s^c_J$;

(iii) $\nexists (C, s^c, b)$ *s.th.* $\pi^c_\omega(\sigma^*(s_I); \sigma^*) < \pi^c_\omega(C, s^c, b; \sigma^*) < \pi^c_\omega(\sigma^*(s_J); \sigma^*)$ *for* $\omega \in \{h, \ell\}$;

(iv) $\int_I \eta f_\omega(z)dz > \frac{1}{\epsilon}$, *and* $\int_J \eta f_\omega(z)dz > \frac{1}{\epsilon}$ *for* $\omega \in \{h, \ell\}$;

(v) On $s \in (s^*+\epsilon, \bar{s}]$, *the bids are strictly increasing and the report* s^c *is irrelevant.*

The proof is provided in the appendix. The following figure summarizes the results:

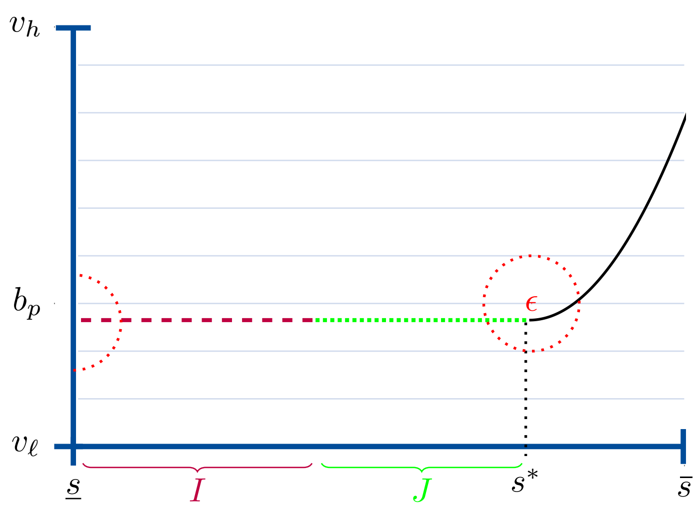

Figure 3: Form of equilibria σ^* of the Communication Extension

There are two adjacent intervals I and J (purple and green), which span the signals between $\underline{s}+\epsilon$ and $s^*-\epsilon$ *(i)*. Bidders with signals from both intervals select the same bid b_p *(ii)*, but separate by sending two different messages. Thus, signals from I receive the good whenever there are no signals from J or above and they win the tiebreak against other signals from I. Signals from J win when there is no signal above J and they win the tiebreak against other signals from J. Further, there is no action that wins whenever there is no signal from J or higher (*iii*). The intervals I and J can vary in length as η increases, but the expected number of bidders in both intervals grows without bound *(iv)*. Above $s^* + \epsilon$, bids are strictly increasing and follow the ordinary differential equation from Proposition 2 with the appropriate initial value *(v)*. Observe that Figure 2 only depicts one of multiple equilibria. Thus, while interval J is drawn to start to the left of $s^* - \epsilon$, this is not guaranteed. Rather, the proposition states that J does not end to the left of $s^* - \epsilon$. Hence, J may be contained in the ϵ-environment around s^*. Furthermore, equilibria can assume different forms within the ϵ-environment to the right of J, or around $\underline{s}$.

To understand why an equilibrium exists in the Communication Extension and why it has to assume such a form, it is helpful to recall the arguments in Section 4. The reasons why the bids cannot be strictly increasing over an interval below s^* and why $\underline{s}$ and s^* have to select different actions remain unchanged. Thus, any equilibrium must be similar to the second candidate equilibrium. For this, we argued that whenever an interval (J) pools on some bid b_p and η is sufficiently large (competition is fierce), all bidders with lower signals (I) compete for the highest bid below b_p. In the standard auction, no such bid exists, which results in a contradiction. This problem can be solved with cheap talk. By sending two different messages, bidders from I and J can differentiate themselves, while leaving no room for signals in I to marginally deviate (property *(iii)*). Since signals from I do not want to mimic or outbid signals from J due to the stronger winner's curse, there is no profitable deviation for them.

One immediate implication of Proposition 5 is that there can be no equilibrium in the standard auction (Proposition 3). All equilibria of the standard auction are also equilibria of the Communication Extension, where $C = \emptyset$, which makes the reports s^c irrelevant. Thus, the equilibria in the auction without cheap talk are a subset of the equilibria in the Communication Extension. Since Proposition 5 describes every equilibrium of the Communication Extension and I and J cannot be separated when $C = \emptyset$, and the standard auction does not have an equilibrium.

In the next section, we consider auctions on a grid, where equilibria exist without cheap talk. We show that these equilibria are approximated by the equilibria of the Communication Extension. In particular, we show that every equilibrium on the sufficiently fine grid basically inherits the properties derived in Proposition 5.

6 Equilibria on a Grid

Definition 1 (Auction on a Grid)**.** *Consider a variation of the auction without cheap talk in which the bids are constrained to a set with $k \geq 2$ equidistant bids from $[v_\ell, v_h]$, including v_ℓ and v_h. Denoting the distance between two bids by $\Delta := \frac{v_h - v_\ell}{k-1}$, we summarize such an auction by $\Gamma(k)$. Accordingly, the auction on the continuous bidding space is $\Gamma(\infty)$.*

The assumption of equidistance is for expositional purposes, only. The following results hold for any discretization, as long as the grid becomes dense on $[v_\ell, v_h]$ as $k \to \infty$. Like finite games with a fixed number of players, Poisson games with finitely many actions always have an equilibrium. Since the proof of Lemma 1 did not rely on the form of the bidding space, the result applies to auctions on the grid as well. Therefore, without loss, we can restrict attention to pure and non-decreasing equilibria.

Lemma 5 (Existence on the Grid)**.** *Any auction on the grid $\Gamma(k < \infty)$ has an equilibrium in pure, non-decreasing strategies.*

The proof, an adaptation of Myerson (2000), is provided in the appendix. We now relate equilibria on an arbitrary fine grid with equilibria in the Communication Extension.

Lemma 6 (Limit Equilibrium)**.** *Consider any sequence of auctions on the ever-finer grid $(\Gamma(k))_{k\in\mathbb{N}}$ and the corresponding sequence of equilibria $(\beta_k^*)_{k\in\mathbb{N}}$. There exists a subsequence of auctions $(\Gamma(n))_{n\in\mathbb{N}}$ with equilibria $(\beta_n^*)_{n\in\mathbb{N}}$ and an equilibrium σ^* in the Communication Extension, such that*

- *β_n^* converges pointwise to some non-decreasing β^**
- *$\beta^*(s) = b$ if and only if $\sigma^*(s) = (\bullet, \bullet, b)$;*
- *$\lim \pi_\omega(\beta_n^*(s); \beta_n) = \pi_\omega^c(\sigma^*(s); \sigma)$ for $\omega \in \{h, \ell\}$*
- *$\lim U(\beta_n^*(s)|s; \beta_n^*) = U^c(\sigma^*(s)|s; \sigma^*)$*

The proof is provided in the appendix. Combining Lemma 6 and Proposition 5, we can characterize the equilibria on the fine grid.

Proposition 6 (Form of the Equilibria on the Grid)**.** *Fix any $\epsilon \in (0, \frac{s^* - \underline{s}}{2})$. For η sufficiently large (given ϵ) and Δ sufficiently small (given ϵ and η), any equilibrium β^* of the discretized auction $\Gamma(k < \infty)$ takes the following form: There are two disjoint, adjacent intervals of signals I, J such that:*

(i) $[\underline{s} + \epsilon, s^* - \epsilon] \subset I \cup J$;

(ii) $\beta^*(s_I) = b$ *for all* $s_I \in I$ *and* $\beta^*(s_J) = b + \Delta$ *for all* $s_J \in J$;

(iii) $\int_I \eta f_\omega(z)dz > \frac{1}{\epsilon}$, *and* $\int_J \eta f_\omega(z)dz > \frac{1}{\epsilon}$ *for* $\omega \in \{h, \ell\}$;

(iv) On $s \in (s^* + \epsilon, \bar{s}]$, the bids tie with probability smaller than $\frac{1}{\epsilon}$.

Proposition 6 describes the discrete analog of the equilibria in the Communication Extension. Again, the result is summarized best in the following figure:

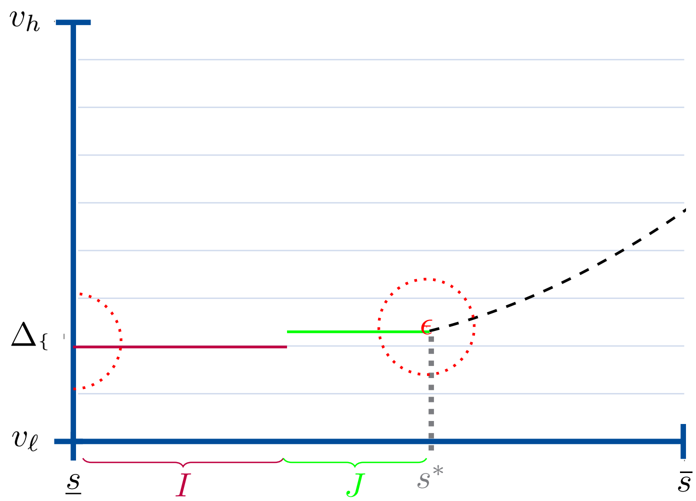

Figure 4: Form of equilibria on the grid

There are two adjacent intervals I and J (purple and green) and any signal between $\underline{s}+\epsilon$ and $s^* - \epsilon$ is part of one of the two intervals *(i)*. Bidders with signals from interval I pool on a lower bid b_p, while bidders on the interval J select the next bid on the grid $b_p + \Delta$ *(ii)*. The intervals can vary in length as η increases, but the expected number of bidders in both intervals grows without bound *(iii)*. As the grid becomes finer, the bidding function above $s^* + \epsilon$ becomes smooth and strictly increasing *(iv)*.

This characterization highlights why the auction on the continuous bidding space is not the limit of the auction on an arbitrary fine grid. As $\Delta \to 0$, the difference between the two pooling bids b_p and $b_p + \Delta$ vanishes. In the limit, when the discretized space becomes continuous, I and J can no longer be separated, and the utility changes discontinuously. Therefore, the limit of the equilibrium strategies is generally no equilibrium of the limit (i.e continuous) auction[24] and existence proofs that rely on this continuity do not work. While the standard auction cannot represent the limit of equilibria on the grid, by Lemma 6, the Communication Extension can. Equilibria in the Communication Extension inherit the characteristics of equilibria on the sufficiently fine grid, which is why we can use the

[24]In particular, in the limit, the bidding strategy becomes the first candidate equilibrium of Section 4 (a single large pool followed by a strictly increasing interval), which, as we argued, cannot be an equilibrium.

extension to characterize the equilibria on the sufficiently fine grid.

To prove Proposition 6, fix η sufficiently large, such that Proposition 5 applies for the ϵ given. Contrary to Proposition 6, suppose that for every k at least one of the properties (i)-(iv) is violated. Then, there exists a sequence of equilibria on the ever finer grid $(\beta_k^*)_{k\in\mathbb{N}}$, along which one of the properties *(i)-(iv)* never holds. By Lemma 6, there exists a subsequence of games $(\Gamma(n))_{n\in\mathbb{N}}$ with equilibria $(\beta_n^*)_{n\in\mathbb{N}}$ and an equilibrium strategy σ^* of the Communication Extension, such that $\lim \pi_\omega(\beta_n^*(s);\beta_n^*) = \pi_\omega^c(\sigma^*(s);\sigma^*)$ and $\lim U(\beta_n^*(s)|s,\beta_n^*) = U^c(\sigma^*(s)|s^*;\sigma)$ for almost every s. Strategy σ^* has the properties described in Proposition 5. Using this result, we show that none of the properties *(i)-(iv)* of Proposition 6 are violated for infinitely many n, which is a contradiction.

First, consider property *(iv)*. By Lemma 6, if the bids under σ^* are strictly increasing over some interval, so is $\beta^* = \lim \beta_n^*$. Since β_n^* converges to β^*, for n sufficiently large (Δ sufficiently small), the bids tie with probability smaller than $\frac{1}{\epsilon}$ on $s \in (s^* + \epsilon, \bar{s}]$.

Next, turn to properties *(i)–(iii)*: We make two preliminary observations.

Claim 1: If $s_- < s_+$ pool in Γ^c i.e. $\sigma^*(s_-) = \sigma^*(s_+)$, then $\beta_n^*(s_-) = \beta_n^*(s_+)$ for any n sufficiently large. Fix any such $s_- < s_+$ and suppose the claim was not true. Since β_n^* is non-decreasing, this implies that there exists a subsequence of equilibria along which $\beta_n^*(s_-) < \beta_n^*(s_+)$. Thus, $\{s\colon \beta_n(s) \in [\beta_n(s_-), \beta_n(s_+)]\} \not\to \emptyset$ which, in turn, implies that $|\pi_\omega(\beta_n^*(s_+);\beta_n^*) - \pi_\omega(\beta_n^*(s_-);\beta_n^*)| \not\to 0$.[25] It follows that $|\pi_\omega(\beta_n^*(s_+);\beta_n^*) - \pi_\omega(\sigma^*(s_+);\sigma^*)| + |\pi_\omega(\beta_n^*(s_-);\beta_n^*) - \pi_\omega^c(\sigma(s_-);\sigma)| \geq |\pi_\omega(\beta_n^*(s_+);\beta_n^*) - \pi_\omega(\beta_n^*(s_-);\beta_n^*)| \not\to 0$, which contradicts that $\pi_\omega(\beta_n^*(s);\beta_n^*)$ converges to $\pi_\omega^c(\sigma^*(s);\sigma^*)$.

Claim 2: If $s_- < s_+$ separate in Γ^c i.e. $\sigma^*(s_-) \neq \sigma^*(s_+)$, then $\beta_n^*(s_-) < \beta_n^*(s_+)$ for any n sufficiently large. Fix any such $s_- < s_+$ and suppose the claim was not true. Then, there exists a subsequence of equilibria along which $\beta_n^*(s_-) = \beta_n^*(s_+)$. Since $\pi_\omega(\beta_n^*(s);\beta_n^*)$ converges, this implies that $\lim \pi_\omega(\beta_n^*(s_-);\beta_n^*) = \lim \pi_\omega(\beta_n^*(s_+);\beta_n^*)$, which is a contradiction since $\lim \pi_\omega(\beta_n^*(s);\beta_n^*) = \pi_\omega^c(\sigma^*(s);\sigma^*)$ for all s.

Next, consider I and J as defined in Proposition 6, and choose from the interior any $s_I \in I^\circ$ and $s_J \in J^\circ$. Further, define $I^n = \{s\colon \beta_n^*(s) = \beta_n^*(s_I)\}$ as well as $J^n = \{s\colon \beta_n^*(s) = \beta_n^*(s_J)\}$. By Claims 1 and 2, $I^n \to I$ and $J^n \to J$. Thus, property *(iii)* cannot be violated for n sufficiently large.

What remains to be shown is that for n sufficiently large $\beta_n^*(s_I) + \Delta = \beta_n^*(s_J)$ *(ii)*. In

[25] If $\{s\colon \beta_n^*(s) \in (\beta_n^*(s_-), \beta_n^*(s_+))\} \not\to \emptyset$ the implication follows directly. Otherwise, either $\{s\colon \beta_n^*(s) = \beta_n^*(s_-)\} \not\to \emptyset$, in which case $\pi_\omega(\beta_n^*(s_+);\beta_n^*)$ stays bounded above $\pi_\omega(\beta_n^*(s_-);\beta_n^*)$, because it wins the random tiebreak on $\beta_n^*(s_-)$ with certainty; and/or $\{s\colon \beta_n^*(s) = \beta_n^*(s_+)\} \not\to \emptyset$, in which case $\pi_\omega(\beta_n^*(s_-);\beta_n^*)$ stays bounded below $\pi_\omega(\beta_n^*(s_+);\beta_n^*)$ because $\beta_n^*(s_-)$ only wins when no bid at or above $\beta_n^*(s_+)$ is made.

this case, it follows that for n sufficiently large $(\underline{s}+\epsilon, s^*-\epsilon) \subset I^n \cup J^n$ *(i)* which completes the proof of Proposition 5. Suppose to the contrary that there exists a subsequence along which $\beta_n^*(s_I) + \Delta < \beta_n^*(s_J)$. Without loss, let this be the original sequence. Since $I^n \to I$ and $J^n \to J^n$, it follows that $\{s \colon \beta_n^*(s_I) < \beta_n^*(s) < \beta_n^*(s_J)\} \to \emptyset$. Denote $\hat{s} := \sup I = \inf J$. Then, $\lim \pi_\omega(\beta_n^*(s_I) + \Delta; \beta_n^*) = \mathbb{P}(s_{(1)} \leq \hat{s}|\omega) = e^{-\eta(1-F_\omega(\hat{s}))}$.[26] Because strategy β_n^* is an equilibrium, it follows for all $s_n \in I^n \cup J^n$ that $U(\beta_n^*(s_n)|s_n; \beta_n^*) \geq U(\beta_n^*(s_I) + \Delta|s_n; \beta_n^*)$. Hence, continuity of the utility in s_n implies that in the limit

$$\lim U(\beta_n^*(s)|\hat{s}; \beta_n^*) = U^c(\sigma^*(s)|\hat{s}; \sigma^*) \geq \lim U(\beta_n^*(s_I) + \Delta|\hat{s}; \beta_n^*) \quad \text{for } s \in \{s_I, s_J\},$$

thereby implying that in the Communication Extension, bidders prefer action $\sigma^*(s_I)$ or $\sigma^*(s_J)$ over some hypothetical action that wins whenever $s_{(1)} \leq \hat{s}$. Thus, there could be an equilibrium with a signal/bid combination that wins whenever $s_{(1)} \leq \hat{s}$, since bidders would not deviate. However, this is a contradiction to property (*iii*) of Proposition 5, which completes the proof.[27]

The proof of Proposition 6 illustrates how the Communication Extension can be employed to characterize equilibria on the sufficiently fine grid. In contrast to the standard auction that cannot handle non-vanishing atoms in the equilibrium bid distribution, it is, thereby, the "correct" mechanism to analyze auctions on the grid. This is not only true for the Poisson distribution, or even common-value auctions under numbers uncertainty, but whenever one establishes that the equilibrium strategy is symmetric and non-decreasing. Thus, the Communication Extension particularly lends itself to the analysis of other non-affiliated common-value auctions where, generally, there are atoms in the equilibrium bid distribution that severely complicate the establishment and characterization of equilibria.

In the next section, we revisit our model assumptions, before discussing the substantive and technical implications of our results in the final chapter.

7 Robustness

7.1 State-dependent Competition

One natural modification of the model is the introduction of state-dependant participation, expressed by a state-dependent mean η_ω. This extension combines our numbers uncertainty with the fixed but state dependent participation in Lauermann & Wolinsky (2017). When

[26] Since $L^n \to \emptyset$, it follows that ever fewer signals pool on $\beta_n(s_I) + \Delta$. In the limit, $\beta_n(s_I) + \Delta$ wins when all present bidders received a signal from I or lower, i.e. whenever $s_{(1)} \leq \hat{s}$.

[27] The proof follows equivalently if η is state dependent.

the number of bidders depends on the state, being solicited to the auction is revealing about the state. Conditional on participation, the bidder updates her belief to

$$\mathbb{P}(\omega = h|\text{participation}) = \frac{\rho\eta_h}{\rho\eta_h + (1-\rho)\eta_\ell}.$$

Knowledge of the number of competitors now has two effects. Apart from determining the intensity of the winner's curse, it is also directly informative about the state of the world. By virtue of the Poisson distribution, we can pin down how these two effects jointly determine the inference of the winning bidder. The expected value $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ is increasing in $\hat{s}$, if and only if $\eta_h f_h(\hat{s}) > \eta_\ell f_\ell(\hat{s})$.

For the most part, the introduction of state dependent participation leaves our results unaltered. The reader merely needs to replace $f_\omega(s)$ with $\eta_\omega f_\omega(s)$ for $\omega \in \{h, \ell\}$. In the appendix, we prove every result for this more general case. Only when $\frac{\eta_h}{\eta_\ell}$ is such that s^* does not exist, Propositions 3, 5, and 6 are no longer valid. If $\frac{\eta_h f_h(\underline{s})}{\eta_\ell f_\ell(\underline{s})} \geq 1$, claim *(iii)* of Proposition 2 ensures the existence of a strictly increasing bidding strategy; by Lemma 3, this is the only symmetric equilibrium.[28] If, on the other hand, $\frac{\eta_h f_h(\bar{s})}{\eta_\ell f_\ell(\bar{s})} < \frac{f_h(\underline{s})}{f_h(\underline{s})}$ and η_h, η_ℓ are sufficiently large, then there exists an equilibrium in which every bidder selects the same bid.

7.2 Distribution of the Number of Bidders

Independent of the distribution, uncertainty about the number of competitors breaks the affiliation between the winning bid and the value of the good. This creates room for the presence of atoms in the equilibrium bid distribution. While atoms can always be problematic for equilibrium existence, it is unclear whether existence can fail in a broader class of distributions other than Poisson. At the very least, the Poisson distribution is not a "knife-edge"-case, in the sense that we can truncate the distribution to always have at least n bidders, or marginally change the probabilities of the realizations. Our results stay valid for any distribution in which there are n bidders with probability $\mathbb{P}(n)$ and where $\sum_n |\mathbb{P}(n) - e^{-\eta}\frac{\eta^n}{n!}|$ is sufficiently small.

What extends more easily than the general non-existence is the non-existence of strictly increasing equilibria. When the expected number of bidders is sufficiently large, the winner's curse plays an important role, and bids are close to the expected value conditional on winning. If the lower end of the support of the bidder distribution is sufficiently small, then the expected value conditional on winning is non-monotone, such that no strictly increasing equilibrium exists. In this case, any equilibrium bid distribution has to contain atoms, thereby making the Communication Extension the correct auction mechanism to approximate equilibria on the sufficiently fine grid.

[28] Such an equilibrium arises in Lauermann et al. (2018).

7.3 Signal Structure

The assumption that s^* is unique is only for convenience. If there is an interval of signals along which $f_h(s) = f_\ell(s)$, the propositions just become more lengthy. For example, n Proposition 5 the bids are constant between $\underline{s}+\epsilon$ and $\inf\{s : f_h(s) = f_\ell(s)\} - \epsilon$, and strictly increasing at or above $\sup\{s : f_h(s) = f_\ell(s)\}+\epsilon$. Moreover, unboundedly informative signals leave our results unaltered, but complicate some proofs.

As a more substantial change, we can allow for finitely many jumps in f_h and f_ℓ. This also captures problems with finitely many discrete signals, which can be modeled as intervals of signals sharing the same likelihood ratio. In case the densities are discontinuous, all results up to Propositions 3, 5, and 6 still apply. However, the strictly increasing bidding strategy from Proposition 2 will have kinks and be no longer differentiable at points where the densities jump. A more profound change is that the discontinuities can solve the existence problem. The characterization of the equilibria and non-existence relies on the continuity of the $\frac{f_h}{f_\ell}$ around s^*. To be precise, our results remain valid as long as there exists an open interval of signals S, such that for all $s \in S$ it holds that $\frac{f_h(s)}{f_\ell(s)} \leq 1$, but $\frac{f_h(\underline{s})}{f_\ell(\underline{s})} \frac{F_\ell(s)}{F_h(s)} < \frac{f_h(s)}{f_\ell(s)}$. If there is no such interval S and η is sufficiently large, an equilibrium exists which takes the form depicted in the left frame of Figure 2: all signals below s^* pool on the same bid and all higher signals follow a strictly increasing bidding strategy. Note that this is always true when signals are binary, thereby making this signal structure a special case.[29]

7.4 Reserve Price

The result of Lemma 1 that any best response can be reordered to be non-decreasing relies on the fact that $b \geq v_\ell$, such that the winning bidder incurs a loss in state $\omega = \ell$. If, to the contrary, the reserve price was $0 < v_\ell$ and participation was state-dependent with $\eta_h << \eta_\ell$ small, then equilibrium strategies can be strictly decreasing. In this case, bidders with high signals expect less competition and are, therefore, inclined to bid less. The bidder with the highest signal bids the lowest bid, gambling to be alone in the auction.

However, if $\eta = \eta_h = \eta_\ell$ are sufficiently large the assumption on the reserve price can be omitted. As η increases, the probability of being alone in the auction vanishes, and by Bertrand logic all signals above some $\underline{s} + \epsilon$ bid something at or above v_ℓ and follow a non-decreasing strategy. Alternatively, one can assume that the good is only allocated when there are at least two bidders present, which ensures that any equilibrium bid is weakly larger than v_ℓ and leaves our results qualitatively unaltered.

[29] Välimäki & Murto (2017) make use of this fact.

7.5 Auction Format

As indicated in footnote 20, whenever η is sufficiently large, there is no strictly increasing equilibrium in the second-price auction, either. By standard arguments, in any such equilibrium, bidders bid their expected value conditional on being tied at the top $\beta(s) = \mathbb{E}[v|s_{(1)} = s, s]$, which is increasing if and only if condition (6) holds. Since this condition is violated for η large, any equilibrium bid distribution necessarily contains atoms, which are problematic for auctions without cheap talk. In fact, one can check that for η large, there is no equilibrium in which all signals below s^* pool, while the others follow a strictly increasing bidding strategy. Thus, we conjecture that for η sufficiently large, then there is no non-decreasing equilibrium in the second-price auction either. However, one can construct an analogous Communication Extension for the second-price auction, which captures the bidding behavior on the sufficiently fine grid.

8 Discussion

A common rationale for analyzing auctions on the continuous bidding space is to make the problem easier to solve, while providing a good approximation for equilibria on a sufficiently fine grid. The non-existence highlights that this is no longer the case when the number of competitors is unknown and the good is of interdependent value. Contrary to the pure private value case (c.f. McAfee & McMillan (1987), and Harstad et al. (1990)) a direct extension of the results and techniques for auctions with a known number of bidders to settings with numbers uncertainty is not possible. In particular, equilibrium bidding strategies are not just a weighted average of what would have been selected if the number of bidders was known. Consequently, any combination of signal- and bidder-distribution requires a separate analysis on whether an equilibrium exists and which form it can assume. Generally, equilibrium behavior under numbers uncertainty involves pooling at lower bids, which is very different from the strictly increasing behavior predicted for a fixed numbers of bidders. The pooling behavior not only results in the possibility of non-existent or non-unique equilibria but has some interesting economic implications.

First, even though the model is purely competitive, bidders with low signals engage in a cooperative behavior to reduce the winner's curse. Contrary to a common-value auction with affiliation, they have an incentive to coordinate on certain bids. Consequently, equilibria resemble collusive behavior, even though they are the outcome of independent utility-maximizing behavior of the bidders.

Second, the presence of atoms in the bid distribution implies that the bidding function cannot be inverted to back out the distribution of signals. When signals are unobservable, many empirical studies utilize the (presumed) strict monotonicity of the equilibrium strategy to estimate the bidders' signals; we show that this can result in a misspecification. In the

Poisson case, our model predicts that lower bids are more concentrated around $\beta^*(s^*)$, as bidders (attempt to) employ pooling strategies. Thus, an inversion of the bidding function would overestimate the density signals around s^*. As a side note, the pooling behavior of pessimistic bidders may make small changes in their beliefs undetectable, as multiple signal distributions can have the same pooling equilibrium.

Last, if signals are observable, the equilibrium distribution of bids may look like the bidders do not (fully) internalize the winner's curse. Since pooling bids increase the expected value conditional on winning, bidders are willing to place higher bids compared to the ones in a strictly increasing equilibrium.

As a technical contribution, we present a very simple and robust model in which the existence of an equilibrium fails due to an openness problem which arises endogenously. The problem stems from the two-dimensional uncertainty the bidders face, which breaks the affiliation between the winning bid and the value of the good. The observation that equilibrium existence can fail in a non-affiliated setup has been noted before. Among others, Jackson (2009) provides an example in a setting where the value of the good has a discrete private and a common-value component.[30] In our setup, we can explicitly identify how the existence fails and why the standard auction is unsuited to approximate equilibria on the grid when the bid distribution can contain atoms. To solve this problem, we implement the Communication Extension by Jackson et al. (2002) as a mechanism that extends the auction by cheap talk. With a simple trick, we can do so without making the auctioneer an implicit player of the game. We illustrate that this extended mechanism is not only of theoretical interest but can help to characterize equilibria on the sufficiently fine grid. Thereby, it is the "correct" mechanism to consider when analyzing auctions in which the equilibrium bid distribution can contain atoms, in particular, in other non-affiliated auctions.

Our model is not the first to consider a non-affiliated common-value auction. In Lauermann & Wolinsky (2017) and Lauermann & Wolinsky (2018) the number of bidders is deterministic but state-dependent. When more bidders participate in the low state, this state dependence implies that the winning bid and value of the good are non-affiliated. In Lauermann & Wolinsky (2017), the state-dependent participation is the outcome of a strategic solicitation decision by an informed seller. The authors construct an equilibrium for binary signals in which bidders with high signals pool. Lauermann & Wolinsky (2018) also consider exogenously given state-dependence and focus on large auctions. When the number of bidders is large and exogenous, they show that any equilibrium is either of this "pooling type" or of a "separating type", in which the price partially reveals the correct

[30] In a working paper, Lauermann & Speit (2018) show that the existence problem in Jackson's setup can be circumvented by assuming that the private types are continuously distributed.

state.

Atakan & Ekmekci (2014) analyze a model in which winning bidders have an additional valuation for correct knowledge of the state. This additional valuation raises the value from winning with a low, as opposed to an intermediate bid, such that expected value conditional on winning is non-monotone. The authors construct one equilibrium in which low signals pool while high signals follow a strictly increasing strategy.

Pesendorfer & Swinkels (2000) consider k-th price multiunit auctions in which the valuation has a common and private value component. The authors assume that an atomless equilibrium bid distribution exists and investigate the efficiency properties of such an auction when the number of goods and bidders becomes large.

In a setting of pure common values, Harstad et al. (2008) and Atakan & Ekmekci (2016) consider the effect of numbers uncertainty on information aggregation properties of auctions with many goods and bidders. In Harstad et al. (2008), the distribution of bidders is exogenously given. The authors find that even if the equilibrium strategy is strictly increasing (which aids aggregation), information aggregation fails unless the numbers uncertainty is negligible. In contrast, Atakan & Ekmekci (2016) assume that bidders have a type-dependent outside option such that the numbers uncertainty arises endogenously and is correlated with the state. In particular, this includes multiple, competing auctions. They find that even when there are many goods and bidders, the self-selection by bidders can be detrimental to information aggregation.

Closest to our paper, Välimäki & Murto (2017) consider a common-value auction where bidders have to pay a participation cost. If the pool of potential bidders is large, this results in a Poisson-distributed number of bidders. The authors concentrate on the case where bidders make their entry decision after observing their signal. Signals in their setup are binary, which circumvents the existence problem described in this paper (cf. Section 7.3), and enables the authors to compare revenues across auction formats.

Appendix A Overview

The appendix is divided into five parts. After this overview and some general comments (A) follow the proofs skipped in the body of the text (B), before proving Example (C) and the results to Section 4 (D). The appendix concludes with the references (E).

Maintained Assumptions We give all proofs for the more general case where the mean of the Poisson distribution is state dependent η_ω. To that end, we redefine $s^* : \frac{\eta_h f_h(s^*)}{\eta_\ell f_\ell(s^*)} = 1$ and sometimes have to restate the claims for this more general case. For convenience, we distinguish between claims that hold everywhere and almost everywhere only when it is central to the argument. Unless specified otherwise, results hold for almost all s. Apart from the proofs of Lemma 1 and 4 we assume that strategies are pure, bids b are non-decreasing. Furthermore, in the Communication Extension reports s^c are unique in their indifference class of reports and non-decreasing given a fixed bid.

As a reminder for the reader, we restate the most important symbols:

$\omega \in \{h, \ell\}$	states of the world	ρ	prior probability $\omega = h$
η_ω	mean of the number of bidders	v_ω	value of the good
β	standard bidding strategy	$b \in [v_\ell, v_h]$	bid
$s \in [\underline{s}, \bar{s}]$	signals	$s_{(1)}$	highest (other) signal
f_ω	signal density	F_ω	signal cdf
$C \subset [\underline{s}, \bar{s}]$	interval partition	s^c	signals report
σ	Comm. Extension strategy	s^*	$s^* : \frac{\eta_h f_h(s^*)}{\eta_\ell f_\ell(s^*)} = 1$

Interim Expected Utility (Standard Auction):

$$U(b|s;\beta) = \frac{\rho\eta_h f_h(s)}{\rho\eta_h f_h(s) + (1-\rho)\eta_\ell f_\ell(s)}\pi_h(b;\beta)(v_h - b) + \frac{(1-\rho)\eta_\ell f_\ell(s)}{\rho\eta_h f_h(s) + (1-\rho)\eta_\ell f_\ell(s)}\pi_\ell(b;\beta)(v_\ell - b).$$

Interim Expected Utility (Communication Extension):

$$U^c(C, s^c, b|s;\sigma) = \frac{\rho\eta_h f_h(s)}{\rho\eta_h f_h(s) + (1-\rho)\eta_\ell f_\ell(s)}\pi_h^c(C, s^c, b;\sigma)(v_h - b) + \frac{(1-\rho)\eta_\ell f_\ell(s)}{\rho\eta_h f_h(s) + (1-\rho)\eta_\ell f_\ell(s)}\pi_\ell^c(C, s^c, b;\sigma)(v_\ell - b).$$

Because many of our results rely on the comparison of expected values, recall that for any two events ϕ and ϕ' it holds that $\mathbb{E}[v|\phi] > \mathbb{E}[v|\phi']$ if and only if $\frac{\mathbb{P}(\phi|h)}{\mathbb{P}(\phi|\ell)} > \frac{\mathbb{P}(\phi'|h)}{\mathbb{P}(\phi'|\ell)}$.

Appendix B Proofs Skipped

Proof of Lemma 1

Proof. **Claim 1:** *If $b' > b \geq v_\ell$ and $U(b'|s;\beta) \geq U(b|s;\beta)$, then $U(b'|s';\beta) \geq U(b|s';\beta)$ for $s' > s$. The second inequality is strict if $\frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$.*

Because $b' > b \geq v_\ell$ it follows that $(v_\ell - b') < (v_\ell - b) \leq 0$ and since the winning probability π_ω is weakly increasing in the bid and never zero (the bidder is alone with positive probability), $\pi_\omega(b';\beta) \geq \pi_\omega(b;\beta) \geq \pi_\omega(v_\ell;\beta) > 0$. Together this yields $\pi_\ell(b';\beta)(v_\ell - b') < \pi_\ell(b;\beta)(v_\ell - b) \leq 0$. Hence, $U(b'|s;\beta) \geq U(b|s;\beta)$ requires that $\pi_h(b';\beta)(v_h - b') > \pi_h(b;\beta)(v_h - b)$. Rearranging $U(b'|s;\beta) \geq U(b|s;\beta)$ yields

$$\frac{\rho\eta_h f_h(s)}{(1-\rho)\eta_\ell f_\ell(s)}[\pi_h(b';\beta)(v_h - b') - \pi_h(b;\beta)(v_h - b)] \geq \pi_\ell(b;\beta)(v_\ell - b) - \pi_\ell(b';\beta)(v_\ell - b').$$

If $s' > s$ is such that $\frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$, the left side is strictly larger for s' and thus $U(b'|s',\beta) > U(b|s',\beta)$. ■

Claim 2: *The set of interim beliefs which imply indifference between two bids $L := \{\frac{f_h(s)}{f_\ell(s)} : \exists b, b'$ with $b \neq b'$ and $U(b|s;\beta) = U(b'|s;\beta)\}$ is countable.*

By construction, $\forall l \in L$ there exist two bids $b^l_- < b^l_+$ such that a bidder $s : \frac{f_h(s)}{f_\ell(s)} = l$ is indifferent between these two bids, $U(b^l_-|s;\beta) = U(b^l_+|s;\beta)$. Furthermore, there exists a $q^l \in \mathbb{Q}$ s.t. $b^l_- < q^l < b^l_+$. By Claim 1, $b^l_+ \leq b^{l'}_-$ for all $l < l'$, which implies that $q^l < q^{l'}$. Because $\mathbb{Q}$ is countable, so is L. ■

Claim 3: *For any strategy, if the likelihood ratio $\frac{f_h}{f_\ell}$ is constant on some interval I, the bids can be reordered in such a way that they are pure, non-decreasing and the distribution of bids remains the same.*

C.f Pesendorfer & Swinkels (2000) footnote 8. ■

Up to the set of beliefs at which bidders are indifferent between multiple bids, the best response is pure and non-decreasing (Claim 1). There are at most countably many such beliefs at which bidders are indifferent (Claim 2). Thus, we can consider the countable set of intervals of signals $\{I^l\}$ which induce a belief at which bidders are indifferent. If an interval from the set is trivial, i.e. only contains a single signal $\hat{s}$, we can, without loss, assume that $\hat{s}$ chooses the lowest bid in the support of its distribution over bids, only. This reassignment does not affect the implied distribution of bids and thereby not the utility of other bidders. Along the remaining non-trivial intervals I^l the likelihood ratio $\frac{f_h}{f_\ell}$ is constant. To those intervals, we can sequentially apply Claim 3 obtaining a best response which is pure and non-decreasing. Furthermore, this reordering leaves the distribution of bids and thereby outcomes unaltered. □

Proof of Lemma 2*

Lemma 2*. *The expected value $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ is strictly decreasing when $\hat{s} < s^*$, has unique global minimum at $\hat{s} = s^*$ and is strictly increasing when $\hat{s} > s^*$.*

Proof. Note that $\frac{av_h+v_\ell}{a+1} > \frac{bv_h+v_\ell}{b+1}$ if and only if $a > b$. Thus, $\mathbb{E}[v|s_{(1)} \leq \hat{s}]$ is strictly increasing if and only if $e^{-\eta_h(1-F_h(\hat{s}))+\eta_\ell(1-F_\ell(\hat{s}))}$ is strictly increasing. The derivative $e^{-\eta_h(1-F_h(\hat{s}))+\eta_\ell(1-F_\ell(\hat{s}))}[\eta_h f_h(\hat{s}) - \eta_\ell f_\ell(\hat{s})]$ is positive if and only if $\eta_h f_h(\hat{s}) > \eta_\ell f_\ell(\hat{s})$. The monotone likelihood ratio property and the assumption that $\eta_h f_h(s^*) = \eta_\ell f_\ell(s^*)$ is unique imply that $\eta_h f_h(\hat{s}) < \eta_\ell f_\ell(\hat{s})$ for $\hat{s} < s^*$, and $\eta_h f_h(\hat{s}) > \eta_\ell f_\ell(\hat{s})$ for $\hat{s} > s^*$. Thus the Lemma follows. □

Proof of Proposition 1*

Proposition 1*. *Fix $\frac{\eta_h}{\eta_\ell} = l < \frac{f_\ell(\underline{s})}{f_h(\underline{s})}$. Holding all other parameters fixed, for a sufficiently large η_h, no strictly increasing equilibrium exists.*

Proof. Fix $\frac{\eta_h}{\eta_\ell} = l < \frac{f_\ell(\underline{s})}{f_h(\underline{s})}$ and suppose to the contrary that a strictly increasing equilibrium β^* exists for η_h arbitrary large. Fix $s_-, s \in [\underline{s}, s^*)$ with $s_- < s$ and, for ease of notation, abbreviate the winning probabilities $\pi_\omega := \pi_\omega(\beta^*(s); \beta^*) = e^{-\eta_\omega(1-F_\omega(s))}$ as well as $\pi_\omega^- := \pi_\omega^-(\beta^*(s_-); \beta^*) = e^{-\eta_\omega(1-F_\omega(s_-))}$. Since β^* is an equilibrium, for all $s \in [\underline{s}, \bar{s}]$ it has to hold that

$$
\begin{aligned}
&U(\beta^*(s_-)|s_-; \beta^*) \geq U(\beta^*(s)|s_-; \beta^*) \\
\iff &\frac{\rho\eta_h f_h(s_-)\pi_h^-(v_h - \beta^*(s_-)) + (1-\rho)\eta_\ell f_\ell(s_-)\pi_\ell^-(v_\ell - \beta^*(s_-))}{\rho\eta_h f_h(s_-) + (1-\rho)\eta_\ell f_\ell(s_-)} \\
&\qquad \geq \frac{\rho\eta_h f_h(s_-)\pi_h(v_h - \beta^*(s)) + (1-\rho)\eta_\ell f_\ell(s_-)\pi_\ell(v_\ell - \beta^*(s))}{\rho\eta_h f_h(s_-) + (1-\rho)\eta_\ell f_\ell(s_-)} \\
\Rightarrow &\rho\eta_h f_h(s_-)\pi_h^-(v_h - v_\ell) \geq \rho\eta_h f_h(s_-)\pi_h(v_h - \beta^*(s)) + (1-\rho)\eta_\ell f_\ell(s_-)\pi_\ell(v_\ell - \beta^*(s)),
\end{aligned}
$$

where we use in the last step that $\beta(s_-) \geq v_\ell$. This equation rearranges to

$$
\frac{\beta^*(s) - v_\ell}{v_h - \beta^*(s)} \geq \frac{\rho}{1-\rho} \frac{\eta_h f_h(s_-)}{\eta_\ell f_\ell(s_-)} \frac{\pi_h}{\pi_\ell} \Big(1 - \frac{\pi_h^-}{\pi_h} \frac{v_h - v_\ell}{v_h - \beta^*(s)}\Big)
$$

Since $\frac{\pi_h^-}{\pi_h} = e^{-\eta_h(F_h(s)-F_h(s_-))} \to 0$ it follows that $1-\frac{\pi_h^-}{\pi_h}\frac{v_h-v_\ell}{v_h-\beta^*(s)} \to 1$ unless $\beta^*(s) \to v_h$. If $\beta^*(s) \to v_h$, this implies that $\beta^*(\bar{s}) > \beta^*(s) \to v_h$. In that case, signal $\bar{s}$, however, would have an incentive to deviate to v_ℓ because

$$\beta^*(\bar{s}) \leq \mathbb{E}[v|\text{win with } \beta^*(\bar{s}), \bar{s}; \beta^*] = \mathbb{E}[v|s_{(1)} \leq \bar{s}, \bar{s}] = \mathbb{E}[v|\bar{s}] < v_h,$$

where the last inequality follows from our assumption that the likelihood ratio of signals is bounded. This is a contradiction. Hence, it is without loss to restrict attention to the case where there is a function $A(\eta_h) < 1$ with $\lim A(\eta_h) = 1$ such that

$$\frac{\beta^*(s) - v_\ell}{v_h - \beta^*(s)} \geq \frac{\rho}{1-\rho}\frac{\eta_h f_h(s_-)}{\eta_\ell f_\ell(s_-)}\frac{\pi_h}{\pi_\ell}A(\eta_h). \tag{15}$$

Next, consider any bidder with signal $s_+ \in (s, s^*)$. A deviation to v_ℓ would be profitable for s_+ unless

$$\begin{aligned}\mathbb{E}[v|\text{win with } \beta^*(s_+), s_+; \beta^*] = \mathbb{E}[v|s_{(1)} \leq s_+, s_+] \geq \beta^*(s_+)\\ \iff \frac{\beta^*(s_+) - v_\ell}{v_h - \beta^*(s_+)} \leq \frac{\rho}{1-\rho}\frac{\eta_h f_h(s_+)}{\eta_\ell f_\ell(s_+)}\frac{\pi_h^+}{\pi_\ell^+},\end{aligned} \tag{16}$$

where $\pi_\omega^+ := \pi_\omega(\beta^*(s_+); \beta^*) = e^{-\eta_\omega(1-F_\omega(s_+))}$. Combining equations (15) and (16) and using that $\frac{\beta^*-v_\ell}{v_h-\beta^*}$ is increasing in β^* gives

$$\frac{\rho}{1-\rho}\frac{\eta_h f_h(s_+)}{\eta_\ell f_\ell(s_+)}\frac{\pi_h^+}{\pi_\ell^+} \geq \frac{\rho}{1-\rho}\frac{\eta_h f_h(s_-)}{\eta_\ell f_\ell(s_-)}\frac{\pi_h}{\pi_\ell}A(\eta_h). \tag{17}$$

The crucial observation now is that $\frac{\pi_h^+}{\pi_\ell^+}\frac{\pi_\ell}{\pi_h} = e^{\eta_h[F_h(s_+)-F_h(s)]-\eta_\ell[F_\ell(s_+)-F_\ell(s)]} \to 0$, because

$$\begin{aligned}\eta_h[F_h(s_+) - F_h(s)] - \eta_\ell[F_\ell(s_+) - F_\ell(s)] &= \int_s^{s_+}[1 - \frac{\eta_\ell f_\ell(z)}{\eta_h f_h(z)}]\eta_h f_h(z)dz\\ &< \underbrace{\eta_h}_{\to\infty}\int_s^{s_+}\underbrace{[1 - \frac{\eta_\ell f_\ell(s_+)}{\eta_h f_h(s_+)}]}_{(1-\frac{1}{l}\frac{f_\ell(s)}{f_h(s)})<0,\text{ constant}} f_h(z)dz \to -\infty.\end{aligned}$$

Since $A(\eta_h) \to 1$, and $\frac{f_h(s_+)}{f_\ell(s_+)}\frac{f_\ell(s)}{f_h(s)}$ is bounded, this implies that equation (17) cannot hold for η_h large. Thus, we have found a contradiction. $\square$

Proof of Proposition 2*

Proposition 2*. *The ordinary differential equation*

$$\hat{\beta}'(s) = \Big(\mathbb{E}[v|s_{(1)} = s, s] - \hat{\beta}(s)\Big)\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)} \qquad \text{with } \hat{\beta}(\underline{s}) = v_\ell.$$

has a unique solution, $\hat{\beta}$.

(i) If $\hat{\beta}$ is strictly increasing, then it is a unique equilibrium in the class of strictly increasing equilibria.

(ii) If $\hat{\beta}$ is not strictly increasing, no strictly increasing equilibrium exists.

(iii) If

$$2\Big(\frac{\partial}{\partial s}\frac{f_h(s)}{f_\ell(s)}\Big)\frac{f_\ell(s)}{f_h(s)} + \eta_h f_h(s) - \eta_\ell f_\ell(s) > 0 \text{ for a.e. } s \in [\underline{s}, \bar{s}],$$

but in any case when η is sufficiently small, a strictly increasing equilibrium exists.

Proof. For $s, s' \in [\underline{s}, \bar{s}]$, let $F_{s_{(1)}}(s|s')$ denote the expected cumulative density function of $s_{(1)}$ conditional on observing s', and let $f_{s_{(1)}}$ be the associated density

$$F_{s_{(1)}}(s|s') := \frac{\rho\eta_h f_h(s')e^{-\eta_h(1-F_h(s))} + (1-\rho)\eta_\ell f_\ell(s')e^{-\eta_\ell(1-F_\ell(s))}}{\rho\eta_h f_h(s') + (1-\rho)\eta_\ell f_\ell(s')},$$

$$f_{s_{(1)}}(s|s') := \frac{\rho\eta_h^2 f_h(s')f_h(s)e^{-\eta_h(1-F_h(s))} + (1-\rho)\eta_\ell^2 f_\ell(s')f_\ell(s)e^{-\eta_\ell(1-F_\ell(s))}}{\rho\eta_h f_h(s') + (1-\rho)\eta_\ell f_\ell(s')}. \tag{18}$$

Since $s_{(1)} = -\infty$ if the bidder is alone, the cdf of $s_{(1)}$ on $[\underline{s}, \bar{s}]$ is $F_{s_{(1)}}(s|s') = \int_{\underline{s}}^{s} f_{s_{(1)}}(z|s')dz + F_{s_{(1)}}(\underline{s}|s')$. Define further $v(s,s') := \mathbb{E}[v|s_{(1)} = s, s']$ i.e.

$$v(s,s') := \frac{\rho\eta_h^2 f_h(s')f_h(s)e^{-\eta_h(1-F_h(s))}v_h + (1-\rho)\eta_\ell^2 f_\ell(s')f_\ell(s)e^{-\eta_\ell(1-F_\ell(s))}v_\ell}{\rho\eta_h^2 f_h(s')f_h(s)e^{-\eta_H(1-F_h(s))} + (1-\rho)\eta_\ell^2 f_\ell(s')f_\ell(s)e^{-\eta_\ell(1-F_\ell(s))}}. \tag{19}$$

If β is strictly increasing and continuous, $\pi_\omega(b;\beta) = \mathbb{P}(s_{(1)} \leq \beta^{-1}(b)|\omega;\beta)$ for all b in β's support. As a result, for all b in the support, the utility (1) can be rewritten as

$$U(b|s;\beta) = \int_{\underline{s}}^{\beta^{-1}(b)=s} \Big[v(z,s) - b\Big] f_{s_{(1)}}(z|s)dz + \Big[v(-\infty,s) - b\Big] F_{s_{(1)}}(\underline{s}|s). \tag{20}$$

Claim 1: *If β is a strictly increasing equilibrium, then β is differentiable. Furthermore, it solves the ODE $\frac{\partial\beta(s)}{\partial s} = \Big(\mathbb{E}[v|s_{(1)} = s, s] - \beta(s)\Big)\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}$ and $\beta(\underline{s}) = v_\ell$.*

Suppose β is a strictly increasing equilibrium (we forgo on the $*$) and, hence, continuous. If β would jump upwards, any bid just above a jump would be dominated by a bid just below the jump, which wins with the same probability but at a lower price. By the same reason, $\beta(\underline{s}) = v_\ell$.

We take any point $s \in (\underline{s}, \bar{s})$ and show that β is differentiable at this point. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence converging to s from below. Then, the sequence $b_n := \beta(s_n)$ converges to $b = \beta(s)$ from below, too. Because $b_n < b$ is a best response for $s_n < s$, it follows that $U(b_n|s_n;\beta) \geq U(b|s_n;\beta)$. Using (20), we receive

$$\int_{\underline{s}}^{\beta^{-1}(b_n)=s_n} \Big[v(z,s_n) - b_n\Big] f_{s_{(1)}}(z|s_n)dz + \Big[v(-\infty,s_n) - b_n\Big] F_{s_{(1)}}(\underline{s}|s_n)$$
$$\geq \int_{\underline{s}}^{\beta^{-1}(b)=s} \Big[v(z,s_n) - b\Big] f_{s_{(1)}}(z|s_n)dz + \Big[v(-\infty,s_n) - b\Big] F_{s_{(1)}}(\underline{s}|s_n),$$

which can be rearranged to

$$\int_{\underline{s}}^{s_n} \Big[b - b_n\Big] f_{s_{(1)}}(z|s_n)dz + \Big[b - b_n\Big] F_{s_{(1)}}(\underline{s}|s_n) \geq \int_{s_n}^{s} \Big[v(z,s_n) - b\Big] f_{s_{(1)}}(z|s_n)dz.$$

Dividing by $s - s_n > 0$, as well as $F_{s_{(1)}}(s|s_n) = \int_{\underline{s}}^{s} f_{s_{(1)}}(z|s_n)dz + F_{s_{(1)}}(\underline{s}|s_n) > 0$ and taking the lim inf yields

$$\liminf_{n\to\infty} \frac{b - b_n}{s - s_n} \geq \liminf_{n\to\infty} \frac{1}{s - s_n} \int_{s_n}^{s} \Big[v(z,s_n) - b\Big] \frac{f_{s_{(1)}}(z|s_n)}{F_{s_{(1)}}(s|s_n)} dz.$$

By inspection of equations (18) and (19), the continuity of f_h and f_ℓ ensures that $v(z,s_n)$, $f_{s_{(1)}}(z|s_n)$ and thereby $F_{s_{(1)}}(s|s_n)$ are continuous in both arguments and thereby

$$\liminf_{n\to\infty} \frac{b - b_n}{s - s_n} \geq [v(s,s) - b] \frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}. \tag{21}$$

Bid b is a best response for signal s, implying that $U(b_n|s;\beta) \leq U(b|s;\beta)$, which rearranges to

$$\int_{\underline{s}}^{\beta^{-1}(b_n)=s_n} \Big[v(z,s) - b_n\Big] f_{s_{(1)}}(z|s)dz + \Big[v(-\infty,s) - b_n\Big] F_{s_{(1)}}(\underline{s}|s)$$
$$\leq \int_{\underline{s}}^{\beta^{-1}(b)=s} \Big[v(z,s) - b\Big] f_{s_{(1)}}(z|s)dz + \Big[v(-\infty,s) - b\Big] F_{s_{(1)}}(\underline{s}|s).$$

Repeating the steps as before, but taking the lim sup instead, yields

$$\limsup_{n\to\infty} \frac{b - b_n}{s - s_n} \leq [v(s,s) - b] \frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}. \tag{22}$$

And because lim inf $\leq$ lim sup, it follows from equations (21) and (22) that

$$\lim_{n\to\infty}\frac{b-b_n}{s-s_n}=\lim_{n\to\infty}\frac{\beta(s)-\beta(s_n)}{s-s_n}=[v(s,s)-\beta(s)]\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}.$$

We can repeat the construction for any sequence of signals and bids which converges from above instead of below and obtain the same result. Therefore, β is differentiable and we can write (replacing v)

$$\frac{\partial\beta(s)}{\partial s}=\Big(\mathbb{E}[v|s_{(1)}=s,s]-\beta(s)\Big)\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}, \tag{23}$$

or, fully spelled out for future reference,

$$\frac{\partial\beta(s)}{\partial s}=\frac{\rho\eta_h^2 f_h(s)^2e^{-\eta_h(1-F_h(s))}(v_h-\beta(s))+(1-\rho)\eta_\ell^2 f_\ell(s)^2e^{-\eta_\ell(1-F_\ell(s))}(v_\ell-\beta(s))}{\rho\eta_h f_h(s)e^{-\eta_H(1-F_h(s))}+(1-\rho)\eta_\ell f_\ell(s)e^{-\eta_\ell(1-F_\ell(s))}}. \tag{24}$$

■

Claim 2: *If β is strictly increasing and solves the ODE $\frac{\partial\beta(s)}{\partial s}=\Big(\mathbb{E}[v|s_{(1)}=s,s]-\beta(s)\Big)\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}$ with initial value $\beta(\underline{s})=v_\ell$, then β is an equilibrium.*

Suppose that β is strictly increasing and solves the ODE. We want to show that $U(\beta(s)|s;\beta)\geq U(\beta(s')|s;\beta)$ for all $s'\in[\underline{s},\bar{s}]$. This suffices, because $\beta(\underline{s})=v_\ell$ denotes the lower bound of bids and any bid $b>\beta(\bar{s})$ is dominated by bidding $\beta(\bar{s})$, which also always wins but at lower cost. We show that $U(\beta(s)|s;\beta)\geq U(\beta(s')|s;\beta)$ by proving that $\frac{\partial U(\beta(s')|s;\beta)}{\partial s'}\geq 0$ for all $s'<s$ and $\frac{\partial U(\beta(s')|s;\beta)}{\partial s'}\leq 0$ for all $s'>s$ such that U is hump-shaped with a global maximum for a bidder with signal s at $\beta(s)$.

Replacing b by $\beta(s')$ in the utility function (20) and taking the derivative wrt. s' yields (note that β is differentiable by assumption of the Claim)

$$\frac{\partial}{\partial s'}U(\beta(s')|s;\beta)=\Big([v(s',s)-\beta(s')]\frac{f_{s_{(1)}}(s'|s)}{F_{s_{(1)}}(s'|s)}-\beta'(s')\Big)F_{s_{(1)}}(s'|s),$$

which is positive if and only if

$$[v(s',s)-\beta(s')]\frac{f_{s_{(1)}}(s'|s)}{F_{s_{(1)}}(s'|s)}>\beta'(s').$$

Because β solves the ODE $\beta'(s')=[v(s',s')-\beta(s')]\frac{f_{s_{(1)}}(s'|s')}{F_{s_{(1)}}(s'|s')}$, this means that $\frac{\partial}{\partial s'}U(\beta(s')|s,\beta)$ is positive if and only if

$$[v(s',s)-\beta(s')]\frac{f_{s_{(1)}}(s'|s)}{F_{s_{(1)}}(s'|s)}>[v(s',s')-\beta(s')]\frac{f_{s_{(1)}}(s'|s')}{F_{s_{(1)}}(s'|s')}.$$

Fully expanded, the left side of the equation becomes (c.f. equation (19))

$$\frac{\rho\eta_h f_h(s)e^{-\eta_h(1-F_h(s'))}}{\rho\eta_h f_h(s)e^{-\eta_h(1-F_h(s'))}+(1-\rho)\eta_\ell f_\ell(s)e^{-\eta_\ell(1-F_\ell(s'))}}\underbrace{\eta_h f_h(s')(v_h-\beta(s'))}_{>0}$$
$$+\frac{(1-\rho)\eta_\ell f_\ell(s)e^{-\eta_\ell(1-F_\ell(s'))}}{\rho\eta_h f_h(s)e^{-\eta_h(1-F_h(s'))}+(1-\rho)\eta_\ell f_\ell(s)e^{-\eta_\ell(1-F_\ell(s'))}}\underbrace{\eta_\ell f_\ell(s')(v_\ell-\beta(s'))}_{<0}.$$

As a result, the expression is nondecreasing in s, and strictly increasing in s if $\frac{f_h(s)}{f_\ell(s)}$ is increasing. This means that

$$[v(s',s)-\beta(s')]\frac{f_{s_{(1)}}(s'|s)}{F_{s_{(1)}}(s'|s)}>[v(s',s')-\beta(s')]\frac{f_{s_{(1)}}(s'|s')}{F_{s_{(1)}}(s'|s')}$$

if and only if $\frac{f_h(s')}{f_\ell(s')}<\frac{f_h(s)}{f_\ell(s)}$. It follows that

- $\frac{\partial}{\partial s'}U(\beta(s')|s,\beta)>0$ for all $s'<s:\frac{f_h(s')}{f_\ell(s')}<\frac{f_h(s)}{f_\ell(s)}$,
- $\frac{\partial}{\partial s'}U(\beta(s')|s,\beta)<0$ for all $s'>s:\frac{f_h(s')}{f_\ell(s')}>\frac{f_h(s)}{f_\ell(s)}$,
- $\frac{\partial}{\partial s'}U(\beta(s')|s,\beta)=0$ for all $s':\frac{f_h(s')}{f_\ell(s')}=\frac{f_h(s)}{f_\ell(s)}$,

and thus $\beta(s)$ is a global maximizer for s. ■

Claim 3: *β is a strictly increasing equilibrium if an only if it is strictly increasing, solves the ODE $\frac{\partial\beta(s)}{\partial s}=\Big(\mathbb{E}[v|s_{(1)}=s,s]-\beta(s)\Big)\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}$ with initial value $\beta(\underline{s})=v_\ell$. If β is an equilibrium, it is unique in the class of strictly increasing equilibria. Thus, if β is not strictly increasing, no strictly increasing equilibrium exists.*

Because the signal densities are continuous and the likelihood-ratio $\frac{f_h}{f_\ell}$, bids, and values v_ω are bounded and since $F_{s_{(1)}}(s|s)>0$, the ODE $\frac{\partial\beta(s)}{\partial s}=[\mathbb{E}[v|s_{(1)}=s,s]-\beta(s)]\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}$ is Lipschitz continuous (c.f. (18) and (19)). Thus, by the Picard Lindölf Theorem there exists a unique solution to the initial value problem $\beta(\underline{s})=v_\ell$. Combining this with Claim 1 (necessary condition) and 2 (sufficient condition), the result follows. ■

Claim 4: *If $2\Big(\frac{\partial}{\partial s}\frac{f_h(s)}{f_\ell(s)}\Big)\frac{f_\ell(s)}{f_h(s)}+\eta_h f_h(s)-\eta_\ell f_\ell(s)>0$ for almost all s, then $\hat{\beta}$ is strictly increasing.*

Since $\frac{\eta_h f_h(\underline{s})}{\eta_\ell f_\ell(\underline{s})}>0$, it follows that $v(\underline{s},\underline{s})>v_\ell$. In combination with the initial value $\hat{\beta}(\underline{s})=v_\ell$, this means that $\hat{\beta}'(\underline{s})>0$. Because the densities f_h and f_ℓ are continuous, so is $\hat{\beta}$ and $\hat{\beta}'$. Thus, $\hat{\beta}'$ can only be negative if it intersects the 0 from above. If there exists some $\hat{s}$ such that $\hat{\beta}'(\hat{s})=0$, this means that $v(\hat{s},\hat{s})-\hat{\beta}(\hat{s})=0$ (c.f. (23)). Since $\hat{\beta}'(\hat{s})=0$, marginally increasing $\hat{s}$ will not change $\hat{\beta}$. Hence, the marginal change of $v(\hat{s},\hat{s})$ decides whether $\hat{\beta}'$ is just tangent, or intersects the 0 at $\hat{s}$. The expected value $v(s,s)$ is increasing at (almost) every s if and only if $\frac{f_h(s)^2 e^{-\eta_h(1-F_h(s))}}{f_\ell(s)^2 e^{-\eta_\ell(1-F_\ell(s))}}$, is increasing in s (cf. (19)). Differentiating with respect to s yields

$$2\Big(\frac{\partial}{\partial s}\frac{f_h(s)}{f_\ell(s)}\Big)\frac{f_h(s)}{f_\ell(s)}\frac{e^{-\eta(1-F_h(s))}}{e^{-\eta(1-F_\ell(s))}}+\frac{f_h(s)^2}{f_\ell(s)^2}\frac{e^{-\eta(1-F_h(s))}e^{-\eta(1-F_\ell(s))}}{(e^{-\eta(1-F_\ell(s))})^2}(\eta_h f_h(s)-\eta_\ell f_\ell(s))>0$$

Dividing by $\frac{e^{-\eta_h(1-F_h(s))}}{e^{-\eta_\ell(1-F_\ell(s))}}>0$ and $\frac{f_h(s)^2}{f_\ell(s)^2}>0$ yields the result. Note that since $\frac{f_h}{f_\ell}$ is monotone, it is differentiable almost everywhere. ■

Claim 5: *For η_h, η_ℓ sufficiently small, $\hat{\beta}$ is strictly increasing.*

First, if $\eta_h,\eta_\ell\to 0$ and $\lim\frac{\eta_h}{\eta_\ell}>\frac{f_\ell(\underline{s})}{f_h(\underline{s})}$ then, for η_h,η_ℓ sufficiently small, $\eta_h f_h(s)\geq \eta_\ell f_\ell(s)$ for all s. Thus, by Claim 4, a strictly increasing equilibrium exists. Next, consider a sequence of auctions along which $\eta_h,\eta_\ell\to 0$ and $\lim\frac{\eta_h}{\eta_\ell}=l\leq\frac{f_\ell(\underline{s})}{f_h(\underline{s})}$. Then

$$\begin{aligned}v(s,s)&=\frac{\rho\eta_h^2 f_h(s)^2 e^{-\eta_h(1-F_h(s))}v_h+(1-\rho)\eta_\ell^2 f_\ell(s)^2 e^{-\eta_\ell(1-F_\ell(s))}v_\ell}{\rho\eta_h^2 f_h(s)^2 e^{-\eta_h(1-F_h(s))}+(1-\rho)\eta_\ell^2 f_\ell(s)^2 e^{-\eta_\ell(1-F_\ell(s))}}\\&\overset{\eta_\omega\to 0}{\to}\frac{\rho\, l^2 f_h(s)^2 v_h+(1-\rho)f_\ell(s)^2 v_\ell}{\rho\, l^2 f_h(s)^2+(1-\rho)f_\ell(s)^2}=:\phi(s)\geq\phi(\underline{s})>v_\ell.\end{aligned}$$

Using that $\hat{\beta}(s)\geq v_\ell$ and equation (24), $\hat{\beta}'(s)$ can be bounded above by $\eta_h f_h(s)(v_h-v_\ell)$. Therefore, $\hat{\beta}(s)=\int_{\underline{s}}^{s}\hat{\beta}'(z)dz+v_\ell<\phi(\underline{s})$ for η_h sufficiently small. It follows that for η_h,η_ℓ sufficiently small, $\hat{\beta}'(s)=[v(s,s)-\hat{\beta}(s)]\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}\geq[\phi(\underline{s})-\hat{\beta}(s)]\frac{f_{s_{(1)}}(s|s)}{F_{s_{(1)}}(s|s)}>0$ for all s. ■ □

Proof of Lemma 3*

Lemma 3*. *Assume β is such that there exists an interval $I:=[s_-,s_+]$ and a bid b_p, such that $b_p=\beta(s)$ for all $s\in I$ and $\beta(s)<b_p<\beta(s')$ for all $s<s_-<s_+<s'$. Then b_p wins with probability*

$$\pi_\omega(b_p;\beta)=\frac{\mathbb{P}(s_{(1)}\in[s_-,s_+]\,|\omega)}{\mathbb{E}[\#s\in[s_-,s_+]\,|\omega]}=\frac{e^{-\eta_\omega(1-F_\omega(s_+))}-e^{-\eta_\omega(1-F_\omega(s_-))}}{\eta_\omega(F_\omega(s_+)-F_\omega(s_-))}\qquad \text{for } \omega\in\{h,\ell\}.$$

Furthermore,

$$\mathbb{E}[v|\text{win with } b_p;\beta]\in\Big[\mathbb{E}[v|s_{(1)}\leq s_-],\ \mathbb{E}[v|s_{(1)}\leq s_+]\Big].$$

If β is an equilibrium bidding strategy, then

$$\eta_h[F_h(s_+) - F_h(s_-)] < \eta_\ell[F_\ell(s_+) - F_\ell(s_-)],$$

and, as a result,

$$\mathbb{E}[v|s_{(1)} \leq s_-] > \mathbb{E}[v|\textit{win with } b_p;\beta] > \mathbb{E}[v|s_{(1)} \leq s_+].$$

Proof. **Claim 1:** $\pi_\omega(b_p;\beta) = \frac{\mathbb{P}(s_{(1)}\in[s_-,s_+]\,|\omega)}{\mathbb{E}[\#s\in[s_-,s_+]\,|\omega]} = \frac{e^{-\eta_\omega(1-F_\omega(s_+))}-e^{-\eta_\omega(1-F_\omega(s_-))}}{\eta_\omega(F_\omega(s_+)-F_\omega(s_-))}$ *for* $\omega \in \{h,\ell\}$.

$$\begin{aligned}
\pi_\omega(b_p;\beta) &= \mathbb{P}(\text{no bid} > b_p|\omega)\sum_{n=0}^{\infty}\frac{1}{n+1}\mathbb{P}(\text{n other bidders bid } b_p|\omega)\\
&= e^{-\eta_\omega(1-F_\omega(s_+))}\Big(\sum_{n=0}^{\infty}\frac{1}{n+1}e^{-\eta_\omega(F_\omega(s_+)-F_\omega(s_-))}\frac{[\eta_\omega(F_\omega(s_+)-F_\omega(s_-))]^n}{n!}\Big)\\
&= e^{-\eta_\omega(1-F_\omega(s_+))}\Big(\sum_{n=1}^{\infty}e^{-\eta_\omega(F_\omega(s_+)-F_\omega(s_-))}\frac{[\eta_\omega(F_\omega(s_+)-F_\omega(s_-))]^n}{n!}\Big)\frac{1}{\eta_\omega(F_\omega(s_+)-F_\omega(s_-))}\\
&= e^{-\eta_\omega(1-F_\omega(s_+))}\Big(\sum_{n=1}^{\infty}\mathbb{P}(\text{n other bidders bid } b_p|\omega)\Big)\frac{1}{\eta_\omega(F_\omega(s_+)-F_\omega(s_-))}\\
&= e^{-\eta_\omega(1-F_\omega(s_+))}\Big(1-e^{-\eta_\omega(F_\omega(s_+)-F_\omega(s_-))}\Big)\frac{1}{\eta_\omega(F_\omega(s_+)-F_\omega(s_-))}\\
&= \frac{e^{-\eta_\omega(1-F_\omega(s_+))}-e^{-\eta_\omega(1-F_\omega(s_-))}}{\eta_\omega(F_\omega(s_+)-F_\omega(s_-))}.
\end{aligned}$$

The numerator is $\mathbb{P}(s_{(1)} \in [s_-,s_+]|\omega)$ and the denominator is the expected number of signals in $[s_-,s_+]$ in state ω i.e. $\mathbb{E}[\#s \in [s_-,s_+]\,|\omega]$. ■

Claim 2: *If* $\eta_h[F_h(s_+) - F_h(s_-)] < \eta_\ell[F_\ell(s_+) - F_\ell(s_-)]$*, then* $\mathbb{E}[v|s_{(1)} \leq s] > \mathbb{E}[v|\textit{win with } b_p;\beta] > \mathbb{E}[v|s_{(1)} \leq s_+]$*. If* $\eta_h[F_h(s_+) - F_h(s_-)] > \eta_\ell[F_\ell(s_+) - F_\ell(s_-)]$*, the inequalities reverse.*

Recall that for any two events ϕ and ϕ' it holds that $\mathbb{E}[v|\phi] > \mathbb{E}[v|\phi']$ if and only if $\frac{\mathbb{P}(\phi|h)}{\mathbb{P}(\phi|\ell)} > \frac{\mathbb{P}(\phi'|h)}{\mathbb{P}(\phi'|\ell)}$. Therefore, we have to show that when $\eta_h[F_h(s_+) - F_h(s_-)] < \eta_\ell[F_\ell(s_+) - F_\ell(s_-)]$ it holds that

$$\frac{e^{-\eta_h(1-F_h(s_-))}}{e^{-\eta_\ell(1-F_\ell(s_-))}} > \frac{\frac{e^{-\eta_h(1-F_h(s_+))}-e^{-\eta_h(1-F_h(s_-))}}{\eta_h[F_h(s_+)-F_h(s_-)]}}{\frac{e^{-\eta_\ell(1-F_\ell(s_+))}-e^{-\eta_\ell(1-F_\ell(s_-))}}{\eta_\ell[F_\ell(s_+)-F_\ell(s_-)]}} > \frac{e^{-\eta_h(1-F_h(s_+))}}{e^{-\eta_\ell(1-F_\ell(s_+))}}. \tag{25}$$

Denote $x_\omega := \eta_\omega[F_\omega(s_+) - F_\omega(s_-)]$ for $\omega \in \{h,\ell\}$. Dividing the left inequality of equations (25) by $\frac{e^{-\eta_h(1-F_h(s_-))}}{e^{-\eta_\ell(1-F_\ell(s_-))}}$, it becomes

$$1 > \frac{\frac{e^{x_h}-1}{x_h}}{\frac{e^{x_\ell}-1}{x_\ell}},$$

which holds because $\frac{e^z-1}{z}$ is strictly increasing in z. If, on the other hand, the right inequality of equation (25) is divided by $\frac{e^{-\eta_h(1-F_h(s_+))}}{e^{-\eta_\ell(1-F_\ell(s_+))}}$, it becomes

$$\frac{\frac{1-e^{x_h}}{x_h}}{\frac{1-e^{x_\ell}}{x_\ell}} > 1,$$

which is true because $\frac{1-e^z}{z}$ is strictly decreasing in z. ■

Claim 3: *β can only be an equilibrium bidding strategy if $\eta_h[F_h(s_+) - F_h(s_-)] < \eta_\ell[F_\ell(s_+) - F_\ell(s_-)]$.*

Suppose to the contrary that β is an equilibrium (we forgo on the $*$), but $\eta_h[F_h(s_+) - F_h(s_-)] \geq \eta_\ell[F_\ell(s_+) - F_\ell(s_-)]$. Observe that since $s^* : \frac{\eta_h f_h(s^*)}{\eta_\ell f_\ell(s^*)} = 1$, it follows from the monotone likelihood ratio property that $s_+ > s^*$. Consider a potential deviation to $b + \epsilon$ for any bidder $s \in [s_-, s_+]$. There are two possibilities:

First, $b_p + \epsilon$ can be a pooling bid meaning that there exists an interval of signals $[s'_-, s'_+]$ such that on exactly this interval $\beta(s) = b_p + \epsilon$. Notice that $s^* < s_+ \leq s'_-$ which means that $\eta_h[F_h(s'_+) - F_h(s'_-)] \geq \eta_\ell[F_\ell(s'_+) - F_\ell(s'_-)]$, and thus

$$\mathbb{E}[v|\text{win with } b_p + \epsilon; \beta] \overset{\text{Claim 2}}{\geq} \mathbb{E}[v|s_{(1)} \leq s'_-] \overset{\text{Lemma 2*}}{\geq} \mathbb{E}[v|s_{(1)} \leq s_+] \overset{\text{Claim 2}}{\geq} \mathbb{E}[v|\text{win with } b_p; \beta].$$

If $b_p + \epsilon$ is not played with positive probability, then it wins when the highest other signal is smaller than some cutoff $y \geq s^+$, i.e. $\mathbb{E}[v|\text{win with } b_p + \epsilon, s; \beta] = \mathbb{E}[v|s_{(1)} \leq y, s]$. This means that

$$\mathbb{E}[v|\text{win with } b_p + \epsilon; \beta] = \mathbb{E}[v|s_{(1)} \leq y] \overset{\text{Lemma 2*}}{\geq} \mathbb{E}[v|s_{(1)} \leq s_+] \overset{\text{Claim 2}}{\geq} \mathbb{E}[v|\text{win with } b_p; \beta].$$

For any $s \in [s_-, s_+]$ this implies that $\mathbb{E}[v|\text{win with } b_p + \epsilon, s; \beta] \geq \mathbb{E}[v|\text{win with } b_p, s; \beta] > b_p$.[31] Since a deviation to $b_p + \epsilon$ discretely increases the winning probability by avoiding the random tiebreak when the second highest bid is b_p, is is always profitable for ϵ sufficiently small. Thus, β cannot be an equilibrium when $\eta_h[F_h(s_+) - F_h(s_-)] \geq \eta_\ell[F_\ell(s_+) - F_\ell(s_-)]$ which proves Claim 3 and the second assertion of this lemma. Together with Claim 2 the last assertion follows as well. ■ □

Proof of Lemma 5

Denote the bidding space with $n = \frac{v_h - v_\ell}{\Delta} + 1$ equidistant bids by B_n. Existence is shown by a fixed point argument on the distribution of bids. Since those are Poisson distributed and thereby fully described by the mean, we look at the compact set of vectors

[31] A bid above the expected value is strictly dominated by bidding v_ℓ

$$\Lambda = \left\{ \begin{pmatrix} \lambda(b_1|h) & ... & \lambda(b_n|h) & \lambda(b_1|\ell) & ... & \lambda(b_n|\ell) \end{pmatrix} : \sum_{b \in B_n} \lambda(b|\omega) = \eta_\omega \right\} \subset R^{n \times 2}$$

where $\lambda(b|\omega)$ denotes the expected number of bids b in state ω.

Let $F : \Lambda \rightrightarrows \mathcal{P}(\Lambda)$ be the correspondence which maps any λ into the set of vectors $\{\tilde{\lambda}\}$ that are are induced by a pure and nondecreasing best response $\beta : [\underline{s}, \bar{s}] \to B_n$ meaning that $\tilde{\lambda}(b|\omega) = \int_{\beta^{-1}(b)} \eta_\omega f_\omega(s) ds$ for all $b \in B_n$, and $\beta(s) = \arg\max_b U(b|s, \lambda)$ for almost all s. Here, $U(b|s, \lambda)$ is the interim expected utility from bidding b, given the bidders signal s and a distribution of (other) bids described by the Poisson parameter λ.

Because Λ is compact, to apply Kakutani's fixed-point theorem we need to show that $F(\lambda)$ is nonempty, convex valued and that F has a closed graph.

$F(\lambda)$ is non-empty because on the finite set there exists a best response for any signal s. By Lemma 1 these best responses can be reordered, such that the resulting β is pure and nondecreasing.

To show that $F(\lambda)$ is convex valued, consider $\tilde{\lambda}$ and $\tilde{\lambda}'$ from its image. We have to show that $\forall \alpha \in [0, 1]$, $\alpha\tilde{\lambda} + (1 - \alpha)\tilde{\lambda}' = \tilde{\lambda}^* \in F(\lambda)$. $\tilde{\lambda}$ and $\tilde{\lambda}'$ are induced by two best responses $\tilde{\beta}$ and $\tilde{\beta}'$. Consider a mixed strategy, which follows $\tilde{\beta}$ with probability α and $\tilde{\beta}'$ with probability $1 - \alpha$. Such a strategy would be optimal for the bidders and result in a distribution of bids $\tilde{\lambda}^*$. By Lemma 1 we can find a pure, nondecreasing strategy inducing the same distribution and utilities. Thus $\tilde{\lambda}^* \in F(\lambda)$.

What remains to be shown is that F has a closed graph. Take any two sequences $\lambda_n \to \lambda$ and $\tilde{\lambda}_n \to \tilde{\lambda}$ where $\tilde{\lambda}_n \in F(\lambda_n)$. We have to show that $\tilde{\lambda} \in F(\lambda)$. For every λ_n there is a nondecreasing best response β_n inducing $\tilde{\lambda}_n$. By Helly's Selection Theorem there is a point-wise converging subsequence of those β_n with a nondecreasing limit β. Obviously, β induces $\tilde{\lambda}$. Furthermore, because $U(b|s, \lambda_n)$ is continuous in both λ_n and b, β is a best response to λ. Thus, F has a closed graph.

Kakutani's fixed-point theorem guarantees an equilibrium vector $\lambda \in \Lambda$ and by construction there exists a pure, nondecreasing bidding strategy β which is a best response and induces this λ. Thus, β is a pure, nondecreasing and symmetric equilibrium. □

Proof of Lemma 4

Proof. **Claim 1:** *If* $(C, s^{c\prime}, b')$ *and* (C, s^c, b) *are s.t.* $\pi_h^c(C, s^{c\prime}, b'; \sigma) > \pi_h^c(C, s^c, b; \sigma)$ *and* $U(C, s^{c\prime}, b'|s; \sigma) \geq U(C, s^c, b|s; \sigma)$*, then* $U(C, s^{c\prime}, b'|s'; \sigma) \geq U(C, s^c, b|s'; \sigma)$ *for* $s' > s$*. The* *second inequality is strict if and only if* $\frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$.

As a preliminary observation, note that $\pi_h^c(C, s^{c\prime}, b'; \sigma) > \pi_h^c(C, s^c, b; \sigma)$ implies that $\pi_\ell^c(C, s^{c\prime}, b'; \sigma) > \pi_\ell^c(C, s^c, b; \sigma)$ since the winning probabilities are isomorph. Now, from $\pi_h^c(C, s^{c\prime}, b'; \sigma) > \pi_h^c(C, s^c, b; \sigma)$, it follows that $b' \geq b \geq v_\ell$ which implies that $(v_\ell - b') \leq (v_\ell - b) \leq 0$. If $b' = b$, then $\pi_h^c(C, s^{c\prime}, b'; \sigma)(v_h - b') > \pi_h^c(C, s^{c\prime}, b; \sigma)(v_h - b)$. If $b' > b$, on the other hand, $\pi_\ell^c(C, s^{c\prime}, b'; \sigma) > \pi_\ell^c(C, s^c, b; \sigma)$ implies that $\pi_\ell^c(C, s^{c\prime}, b'; \sigma)(v_\ell - b') < \pi_\ell^c(C, s^c, b; \sigma)(v_\ell - b)$. Hence, $U(C, s^{c\prime}, b'|s; \sigma) \geq U(C, s^c, b|s; \sigma)$ requires that $\pi_h^c(C, s^{c\prime}, b'; \sigma)(v_h - b') > \pi_h^c(C, s^{c\prime}, b; \sigma)(v_h - b)$.

Rearranging $U(C, s^{c\prime}, b'|s; \sigma) \geq U(C, s^c, b|s; \sigma)$ yields

$$\frac{\rho\eta_h f_h(s)}{(1-\rho)\eta_\ell f_\ell(s)}[\pi_h^c(C, s^{c\prime}, b'; \sigma)(v_h - b') - \pi_h^c(C, s^c, b; \sigma)(v_h - b)]$$
$$\geq \pi_\ell^c(C, s^c, b; \sigma)(v_\ell - b) - \pi_\ell^c(C, s^{c\prime}, b'; \sigma)(v_\ell - b').$$

Since $\pi_h^c(C, s^{c\prime}, b'; \sigma)(v_h - b') > \pi_h^c(C, s^{c\prime}, b; \sigma)(v_h - b)$, if $s' > s$ is such that $\frac{f_h(s')}{f_\ell(s')} > \frac{f_h(s)}{f_\ell(s)}$, the left side is strictly larger for s' and thus $U(C, s^{c\prime}, b'|s'; \sigma) > U(C, s^c, b|s'; \sigma)$ ■

Claim 2: *Take any strategy σ and any best response σ^* to it. If (C, s^c, b) and $(C, s^{c\prime}, b')$ are in the support of σ^* with $\pi_h^c(C, s^c, b; \sigma) = \pi_h^c(C, s^{c\prime}, b'; \sigma)$, then $b = b'$. Furthermore, there exists another best response $\hat{\sigma}^*$ which has the property that*

- *if (C, s^c, b) and $(C, s^{c\prime}, b)$ are in the support of $\hat{\sigma}^*$ and $\pi_h^c(C, s^c, b; \sigma) = \pi_h^c(C, s^{c\prime}, b; \sigma)$, then $s^{c\prime} = s^c$;*
- *the winning probabilities and utilities under $\hat{\sigma}^*$ are unchanged, i.e. $\pi_\omega^c(\sigma^*(s); \sigma) = \pi_\omega^c(\hat{\sigma}^*(s); \sigma)$ as well as $U(\hat{\sigma}^*(s)|s; \sigma) = U(\sigma^*(s)|s; \sigma)$ for all s and $\omega \in \{h, \ell\}$.*

If (C, s^c, b) and $(C, s^{c\prime}, b')$ are in the support of σ^*, and $\pi_h^c(C, s^c, b; \sigma) = \pi_h^c(C, s^{c\prime}, b'; \sigma)$ (and thereby $\pi_\ell^c(C, s^c, b; \sigma) = \pi_\ell^c(C, s^{c\prime}, b'; \sigma)$), then $b = b'$. Otherwise, the action tuple with the higher bid would be dominated and could, hence, not be part of of a best response.

If (C, s^c, b) and $(C, s^{c\prime}, b')$ are in the support of σ^*, and $\pi_h^c(C, s^c, b; \sigma) = \pi_h^c(C, s^{c\prime}, b; \sigma)$, but $s^{c\prime} \neq s^c$ there are two possibilities. Either the report is irrelevant for the winning-probability (when b is chosen with zero probability), or $s^{c\prime} \sim s^c$ i.e. both reports are from the same equivalence class as defined by C. In both cases we can simply create a new best response $\hat{\sigma}^*$ where every equivalence class hence a unique identifier. Since only the equivalence classes are relevant for the auction mechanism with Communication Extension this does not alter the winning probabilities or utilities. ■

Claim 3: *For any best response σ^*, there exists another pure best response $\hat{\sigma}^* : [\underline{s}, \bar{s}] \to \mathcal{P}[\underline{s}, \bar{s}] \times [\underline{s}, \bar{s}] \times [v_\ell, v_h]$, s.t. b is nondecreasing in s and given b, s^c*

is nondecreasing in s. Furthermore, the implied winning probabilities are equal, i.e. $\pi^c_\omega(C, s^c, b; \sigma^*) = \pi^c_\omega(C, s^c, b; \hat{\sigma}^*)$ *for all* (C, s^c, b) *and* $\omega \in \{h, \ell\}$.

By Claim 2 and rules of the Communication Extension, winning with a higher probability means choosing a higher bid b and (given a fixed bid) higher report s^c.

By Claim 1, bidders with higher signals prefer to win more often. If the likelihood ratio $\frac{f_h}{f_\ell}$ is strictly increasing, this preference is strict and the Lemma follows directly.

If there are intervals of signals along which $\frac{f_h}{f_\ell}$ is constant, however, it may happen that a lower signal from the interval wins more often. In that case, we can proceed as in Lemma 1 and reorder the report/bid combinations. Since we just reorder the report/bid pairs among signals which imply the same belief, this does not change the implied joint distribution of beliefs and reports/bids, and, as a result, the winning probabilities are unchanged. ■ □

Proof of Proposition 4

Take any sequence of games on an ever finer grid $\Gamma(k))_{k\in\mathbb{N}}$. By Lemma 5, for any grid size k, a pure, non-decreasing equilibrium exists. By Lemma 6 there, thus, is an equilibrium of the Communication Extension. The properties follow by construction.

Lemma 7 (Lower Bound on Equilibrium Bids)**.** *Fix some equilibrium strategy* σ^* *and some action* (C, s^c, b) *which wins with probability* $\pi^c_\omega := \pi^c_\omega(C, s^c, b; \sigma)$ *in state* $\omega \in \{h, \ell\}$*. Assume that* $\hat{s}$ *chooses* $\sigma^*(\hat{s})$ *and wins with probability* $\pi^{c-}_\omega := \pi^c_\omega(\sigma^*(\hat{s}); \sigma^*) < \pi^c_\omega$ *in state* $\omega \in \{h, \ell\}$*. Then*

$$b \geq \frac{\rho f_h(\hat{s})\eta_h(\pi^c_h - \pi^{c-}_h)v_h + (1-\rho)f_\ell(\hat{s})\eta_\ell(\pi^c_\ell + \frac{\rho\eta_h f_h(\hat{s})}{(1-\rho)\eta_\ell f_\ell(\hat{s})\eta_\ell}\pi^{c-}_h)v_\ell}{\rho\eta_h f_h(\hat{s})(\pi^c_h - \pi^{c-}_h) + (1-\rho)\eta_\ell f_\ell(\hat{s})(\pi^c_\ell + \frac{\rho\eta_h f_h(\hat{s})}{(1-\rho)\eta_\ell f_\ell(\hat{s})}\pi^{c-}_h)}.$$

The lower bound is decreasing in π^{c-}_h.

Proof. In order for $\hat{s}$ not to deviate from $\sigma^*(\hat{s}) = (\hat{C}, \hat{s}^c, \hat{b})$ to (C, s^c, b) it has to hold that $U^c(C, s^c, b|\hat{s}; \sigma^*) \leq U^c(\sigma^*(\hat{s})|\hat{s}; \sigma^*)$. Notice that

$$\begin{aligned} U^c(\sigma^*(\hat{s})|\hat{s}; \sigma^*) &= \mathbb{P}(\text{win with } \sigma^*(\hat{s})|\hat{s}; \sigma^*)(\mathbb{E}[v|\text{win with } \sigma^*(\hat{s}), \hat{s}; \sigma^*] - b) \\ &\leq \mathbb{P}(\text{win with } \sigma^*(\hat{s})|\hat{s}; \sigma^*)(\mathbb{E}[v|\text{win with } \sigma^*(\hat{s}), \hat{s}; \sigma^*] - v_\ell). \end{aligned}$$

Thus, a necessary condition for $U^c(C, s^c, b|\hat{s};\sigma^*) \leq U^c(\sigma(\hat{s})|\hat{s};\sigma^*)$ is that

$$U^c(C, s^c, b|\hat{s};\sigma^*) \leq \mathbb{P}(\text{win with } \sigma^*(\hat{s})|\hat{s};\sigma^*)(\mathbb{E}[v|\text{win with } \sigma^*(\hat{s}), \hat{s};\sigma^*] - v_\ell)$$

$$\frac{\rho\eta_h f_h(\hat{s})\pi_h^c(v_h - b) + (1-\rho)\eta_\ell f_\ell(\hat{s})\pi_\ell^c(v_\ell - b)}{\rho\eta_h f_h(\hat{s}) + (1-\rho)\eta_\ell f_\ell(\hat{s})} \leq \frac{\rho\eta_h f_h(\hat{s})\pi_h^{c-}(v_h - v_\ell) + (1-\rho)\eta_\ell f_\ell(\hat{s})\pi_\ell^{c-}(v_\ell - v_\ell)}{\rho\eta_h f_h(\hat{s}) + (1-\rho)\eta_\ell f_\ell(\hat{s})}$$

$$\rho\eta_h f_h(\hat{s})\pi_h^c(v_h - b) + (1-\rho)\eta_\ell f_\ell(\hat{s})\pi_\ell^c(v_\ell - b) \leq \rho\eta_h f_h(\hat{s})\pi_h^{c-}(v_h - v_\ell)$$

$$\frac{\rho\eta_h f_h(\hat{s})}{(1-\rho)\eta_\ell f_\ell(\hat{s})}\pi_h^c(v_h - b) + \pi_\ell^c(v_\ell - b) \leq \frac{\rho\eta_h f_h(\hat{s})}{(1-\rho)\eta_\ell f_\ell(\hat{s})}\pi_h^{c-}(v_h - v_\ell).$$

Rearranging yields

$$b \geq \frac{\frac{\rho\eta_h f_h(\hat{s})}{(1-\rho)\eta_\ell f_\ell(\hat{s})}(\pi_h^c - \pi_h^{c-})v_h + (\pi_\ell^c + \frac{\rho\eta_h f_h(\hat{s})}{(1-\rho)\eta_\ell f_\ell(\hat{s})}\pi_h^{c-})v_\ell}{\frac{\rho\eta_h f_h(\hat{s})}{(1-\rho)\eta_\ell f_\ell(\hat{s})}\pi_h^c + \pi_\ell^c}.$$

By simple computation

$$\frac{\rho\eta_h f_h(\hat{s})}{(1-\rho)\eta_\ell f_\ell(\hat{s})}(\pi_h^c - \pi_h^{c-}) + (\pi_\ell^c + \frac{\rho\eta_h f_h(\hat{s})}{(1-\rho)\eta_\ell f_\ell(\hat{s})}\pi_h^{c-}) = \frac{\rho\eta_h f_h(\hat{s})}{(1-\rho)\eta_\ell f_\ell(\hat{s})}\pi_h^c + \pi_\ell^c.$$

Thus, we can rewrite the denominator and establish the lower bound

$$b \geq \frac{\rho\eta_h f_h(\hat{s})(\pi_h^c - \pi_h^{c-})v_h + (1-\rho)\eta_\ell f_\ell(\hat{s})(\pi_\ell^c + \frac{\rho\eta_h f_h(\hat{s})}{(1-\rho)\eta_\ell f_\ell(\hat{s})}\pi_h^{c-})v_\ell}{\rho\eta_h f_h(\hat{s})(\pi_h^c - \pi_h^{c-}) + (1-\rho)\eta_\ell f_\ell(\hat{s})(\pi_\ell^c + \frac{\rho\eta_h f_h(\hat{s})}{(1-\rho)\eta_\ell f_\ell(\hat{s})}\pi_h^{c-})}.$$

To establish that the lower bound is decreasing in π_h^{c-}, divide the numerator and denominator by $(1-\rho)\eta_\ell f_\ell(\hat{s})\pi_h^{c-}$ to receive

$$b \geq \frac{\frac{\rho\eta_h f_h(\hat{s})(\pi_h^c - \pi_h^{c-})}{(1-\rho)\eta_\ell f_\ell(\hat{s})(\pi_\ell^c + \frac{\rho f_h(\hat{s})\eta_h}{(1-\rho)f_\ell(\hat{s})\eta_\ell}\pi_h^{c-})}v_h + v_\ell}{\frac{\rho\eta_h f_h(\hat{s})(\pi_h^c - \pi_h^{c-})}{(1-\rho)\eta_\ell f_\ell(\hat{s})(\pi_\ell^c + \frac{\rho f_h(\hat{s})\eta_h}{(1-\rho)f_\ell(\hat{s})\eta_\ell}\pi_h^{c-})} + 1}.$$

Since $\frac{\rho\eta_h f_h(\hat{s})(\pi_h^c - \pi_h^{c-})}{(1-\rho)\eta_\ell f_\ell(\hat{s})(\pi_\ell^c + \frac{\rho f_h(\hat{s})\eta_h}{(1-\rho)f_\ell(\hat{s})\eta_\ell}\pi_h^{c-})}$ is decreasing in π_h^{c-}, so is the lower bound. □

Proof of Proposition 5*

Proposition 5*. *Assume that* $\frac{\eta_h}{\eta_\ell} = l \in (\frac{f_\ell(\bar{s})}{f_h(\bar{s})}, \frac{f_\ell(\underline{s})}{f_h(\underline{s})})$. *Fix any* $\epsilon \in (0, \frac{s^* - \underline{s}}{2})$. *For* η_h *sufficiently large (given* ϵ*), any equilibrium* σ^* *of the Communication Extension* Γ^c *takes the following form: There are two disjoint, adjacent intervals of signals* I, J *such that*

(i) $[\underline{s} + \epsilon, s^* - \epsilon] \subset I \cup J$*;*

(ii) $\sigma^*(s_I) = (C, s_I^c, b)$ *for all* $s_I \in I$ *and* $\sigma^*(s_J) = (C, s_J^c, b)$ *for all* $s_J \in J$*, with* $s_I^c < s_J^c$*;*

(iii) $\nexists(C, s^c, b)$ *s.th.* $\pi^c_\omega(\sigma^*(s_I);\sigma^*) < \pi^c_\omega(C, s^c, b;\sigma^*) < \pi^c_\omega(\sigma^*(s_J);\sigma^*)$ *for* $\omega \in \{h, \ell\}$*;*

(iv) $\int_I \eta f_\omega(z)dz > \frac{1}{\epsilon}$*, and* $\int_J \eta f_\omega(z)dz > \frac{1}{\epsilon}$ *for* $\omega \in \{h, \ell\}$*;*

(v) On $s \in (s^* + \epsilon, \bar{s}]$*, the bids are strictly increasing and the report* s^c *is irrelevant.*

Proof. We consider a sequence of auctions with the Communication Extension $(\Gamma^c_n)_{n\in\mathbb{N}}$, where $\frac{\eta^n_h}{\eta^n_\ell} = l \in (\frac{f_\ell(\bar{s})}{f_h(\bar{s})}, \frac{f_\ell(\underline{s})}{f_h(\underline{s})})$ and $\eta^n_h, \eta^n_\ell \to \infty$. By Proposition 4 there exists an equilibrium for each n which we call (economizing on the $*$) σ_n.

Claim 1: *For any* $\epsilon > 0$*, if* n *is sufficiently large, on* $(s^* + \epsilon, \bar{s}]$ *the bids are strictly increasing and the report* s^c *is irrelevant.*

Suppose to the contrary that this was not true. Then there exits an $\epsilon > 0$ and a subsequence of auctions with equilibria $(\sigma_n)_{n\in\mathbb{N}}$ where there is an interval of signals $[s^n_-, s^n_+]$ with $s^n_+ > s^* + \epsilon$ which choose the same bid b_n. For all $s \in [s^n_-, s^n_+]$, define a function $s^c_n(s)$ such that $\sigma_n(s) = (C_n, s^c_n(s), b_n)$.

When there aren't two distinct signals $s, s' \in [s^n_-, s^n_+]$ such that $s^c_n(s) = s^c_n(s')$, then signal $s \in [s^n_-, s^n_+]$ wins whenever $s_{(1)} \leq s$. In Proposition 2, we show that in that case, bids by signals above s^* have to follow a strictly increasing differential equation. Otherwise bidders with low reports would have an incentive to deviate. They could send a higher report, win more often and have a higher expected value for the good. Thus, this cannot be the case. By continuity of the arguments, the same is true if there was a (sub) interval of signals $[\hat{s}^n_-, \hat{s}^n_+]$ along which σ_n is constant (and different otherwise), but $\eta^n_\omega[F_\omega(\hat{s}^n_+) - F_\omega(\hat{s}^n_-)] \to 0$.

Hence, we can restrict attention to intervals $[s^n_-, s^n_+]$ with $\sigma_n(s) = (C_n, s^c_n, b_n)$ for all $s \in [s^n_-, s^n_+]$ and $\eta^n_\omega[F_\omega(s^n_+) - F_\omega(s^n_-)] \not\to 0$. Suppose that $s^n_- > s^*$ for all n sufficiently large and consider a derivation to $(C_n, s^c_n, b_n + \epsilon)$. If $\epsilon > 0$ is sufficiently small, this deviation wins whenever $s_{(1)} \leq s_\epsilon$ with $s_\epsilon \geq s^n_+$. Because $s^n_- > s^*$, it follows that $\eta^n_h[F_h(s^n_+) - F_h(s^n_-)] > \eta^n_\ell[F_\ell(s^n_+) - F_\ell(s^n_-)] >> 0$. Hence, Lemma 3 implies that $\mathbb{E}[v|\text{win with } (C_n, s^c_n, b_n);\sigma_n] < \mathbb{E}[v|s_{(1)} \leq s^n_+] \leq \mathbb{E}[v|s_{(1)} \leq s_\epsilon]$. Since the deviation also discretely increases the probability to win, it is profitable for ϵ sufficiently small. Hence, we found a contradiction.

We conclude that if there is a non-vanishing interval $[s^n_-, s^n_+]$ along which b_n is constant, then s^c_n is constant as well and $s^n_- < s^* < s^* + \epsilon \leq s^n_+$. Furthermore, we know that all higher signals follow a strictly increasing bidding strategy. To abbreviate notation, define the implied winning probabilities from bidding (C_n, s^c_n, b_n) and bidding marginally more by

$$\pi^n_\omega := \pi^c_\omega(C_n, s^c_n, b_n;\sigma_n) = \frac{e^{-\eta^n_\omega(1-F_\omega(s^n_+))} - e^{-\eta^n_\omega(1-F_\omega(s^n_-))}}{\eta^n_\omega(F_\omega(s^n_+) - F_\omega(s^n_-))},$$

$$\hat{\pi}^n_\omega := \lim_{\epsilon\to 0} \pi^c_\omega(C_n, s^c_n, b_n + \epsilon;\sigma_n) = e^{-\eta^n_\omega(1-F_\omega(s^n_+))}.$$

To ensure that s_+^n does not want to marginally overbid b_n, it has to hold that

$$U^c(C_n, s_n^c, b_n|s_+^n; \sigma_n) \geq \lim_{\epsilon \to 0} U(C_n, s_n^c, b_n + \epsilon|s_+^n; \sigma_n)$$

$$\frac{\rho\eta_h^n f_h(s_+^n)\pi_h^n(v_h - b_n) + (1-\rho)\eta_\ell^n f_\ell(s_+^n)\pi_\ell^n(v_\ell - b_n)}{\rho\eta_h^n f_h(s_+^n) + (1-\rho)\eta_\ell^n f_\ell(s_+^n)} \geq \frac{\rho\eta_h^n f_h(s_+^n)\hat{\pi}_h^n(v_h - b_n) + (1-\rho)\eta_\ell^n f_\ell(s_+^n)\hat{\pi}_\ell^n(v_\ell - b_n)}{\rho\eta_h^n f_h(s_+^n) + (1-\rho)\eta_\ell^n f_\ell(s_+^n)},$$

which rearranges to

$$b_n \geq \frac{\rho\eta_h^n f_h(s_+^n)(\hat{\pi}_h^n - \pi_h^n)v_h + (1-\rho)\eta_\ell^n f_\ell(s_+^n)(\hat{\pi}_\ell^n - \pi_\ell^n)v_\ell}{\rho\eta_h^n f_h(s_+^n)(\hat{\pi}_h^n - \pi_h^n) + (1-\rho)\eta_\ell^n f_\ell(s_+^n)(\hat{\pi}_\ell^n - \pi_\ell^n)}.$$

Observe that

$$\frac{\eta_h^n f_h(s_+^n)(\hat{\pi}_h^n - \pi_h^n)}{\eta_\ell^n f_\ell(s_+^n)(\hat{\pi}_\ell^n - \pi_\ell^n)} = \frac{\eta_h^n f_h(s_+^n)[e^{-\eta_h^n(1-F_h(s_+^n))} - \frac{e^{-\eta_h^n(1-F_h(s_+^n))} - e^{-\eta_h^n(1-F_h(s_-^n))}}{\eta_h^n(F_h(s_+^n)-F_h(s_-^n))}]}{\eta_\ell^n f_\ell(s_+^n)[e^{-\eta_\ell^n(1-F_\ell(s_+^n))} - \frac{e^{-\eta_\ell^n(1-F_\ell(s_+^n))} - e^{-\eta_\ell^n(1-F_\ell(s_-^n))}}{\eta_\ell^n(F_\ell(s_+^n)-F_\ell(s_-^n))}]}$$

$$\overset{\text{large n}}{>} \frac{\eta_h^n f_h(s^*)}{\eta_\ell^n f_\ell(s^*)} \frac{e^{-\eta_h^n(1-F_h(s_+^n))}}{e^{-\eta_\ell^n(1-F_\ell(s_+^n))}} = \frac{\eta_h^n f_h(s^*)}{\eta_\ell^n f_\ell(s^*)} \frac{\hat{\pi}_h^n}{\hat{\pi}_\ell^n},$$

where we use that because $\eta_\omega^n[F_\omega(s_+^n) - F_\omega(s_-^n)] \to \infty$, $\frac{e^{-\eta_\omega^n(1-F_\omega(s_+^n))} - e^{-\eta_\omega^n(1-F_\omega(s_-^n))}}{\eta_\omega^n(F_\omega(s_+^n)-F_\omega(s_-^n))}$ becomes negligible compared to $e^{-\eta_\omega^n(1-F_\omega(s_+^n))}$. Since $s^* < s^* + \epsilon \leq s_+^n$ the monotone likelihood ratio property and the assumption that $s^* : \frac{\eta_h^n f_h(s^*)}{\eta_h^\ell f_\ell(s^*)} = 1$ is unique establishes that $1 = \frac{\eta_h^n f_h(s^*)}{\eta_\ell^n f_\ell(s^*)} < \frac{\eta_h^n f_h(s^*+\epsilon)}{\eta_\ell^n f_\ell(s^*+\epsilon)} \leq \frac{\eta_h^n f_h(s_+^n)}{\eta_\ell^n f_\ell(s_+^n)}$ which yields the strict inequality for n sufficiently large. It thereby follows that for n sufficiently large $b_n > \mathbb{E}[v|s_{(1)} \leq s_+^n, s^*]$.

No bidder, particularly s_-^n, will bid more than her expected value conditional on winning, which is why $b_n < \mathbb{E}[v|\text{win with } (C_n, s_n^c, b_n), s_-^n; \sigma_n]$. The likelihood ratio of winning is

$$\frac{\eta_h^n f_h(s_-^n)\pi_h^n}{\eta_\ell^n f_\ell(s_-^n)\pi_\ell^n} = \frac{\eta_h^n f_h(s_-^n)\frac{e^{-\eta_h^n(1-F_h(s_+^n))} - e^{-\eta_h^n(1-F_h(s_-^n))}}{\eta_h^n(F_h(s_+^n)-F_h(s_-^n))}}{\eta_\ell^n f_\ell(s_-^n)\frac{e^{-\eta_\ell^n(1-F_\ell(s_+^n))} - e^{-\eta_\ell^n(1-F_\ell(s_-^n))}}{\eta_\ell^n(F_\ell(s_+^n)-F_\ell(s_-^n))}}$$

$$= \frac{f_h(s_-^n)}{f_\ell(s_-^n)} \frac{F_\ell(s_+^n) - F_\ell(s_-^n)}{F_h(s_+^n) - F_h(s_-^n)} \frac{e^{-\eta_h^n(1-F_h(s_+^n))}(1 - e^{-\eta_h^n(F_h(s_+^n)-F_h(s_-^n))})}{e^{-\eta_\ell^n(1-F_\ell(s_+^n))}(1 - e^{-\eta_\ell^n(F_\ell(s_+^n)-F_\ell(s_-^n))})}$$

$$\overset{\text{large n}}{<} \frac{\eta_h^n f_h(s^*)}{\eta_\ell^n f_\ell(s^*)} \frac{e^{-\eta_h^n(1-F_h(s_+^n))}}{e^{-\eta_\ell^n(1-F_\ell(s_+^n))}} = \frac{\eta_h^n f_h(s^*)}{\eta_\ell^n f_\ell(s^*)} \frac{\hat{\pi}_h^n}{\hat{\pi}_\ell^n}.$$

To receive bound, we use that $\eta_\omega^n[F_\omega(s_+^n) - F_\omega(s_-^n)] \to \infty$ and thereby $(1 - e^{-\eta_\omega^n(1-F_\omega(s_+^n))}) \to 1$. Furthermore, we employ that by the monotone likelihood ratio property it holds that $1 = \frac{\eta_h^n f_h(s^*)}{\eta_\ell^n f_\ell(s^*)} > \frac{f_h(s_-^n)}{f_\ell(s_-^n)} \frac{F_\ell(s_+^n)-F_\ell(s_-^n)}{F_h(s_+^n)-F_h(s_-^n)}$. It follows that for n sufficiently large, $b_n < \mathbb{E}[v|s_{(1)} \leq s_+^n, s^*]$, which is a contradiction to the earlier result that $b_n > \mathbb{E}[v|s_{(1)} \leq s_+^n, s^*]$. ■

Claim 2: *Fix any* $\epsilon \in (0, \frac{s^* - \underline{s}}{2})$. *For every* n *sufficiently large,* $\nexists (C, s^c, b)$ *s.t.* $\pi_\omega^c(\sigma_n(\underline{s} + \epsilon); \sigma_n) < \pi_\omega^c(C, s^c, b; \sigma_n) < \pi_\omega^c(\sigma_n(s^* - \epsilon); \sigma_n)$ *for* $\omega \in \{h, \ell\}$. *As a result, all* *bidders with signals from the interval* $[\underline{s}+\epsilon, s^*-\epsilon]$ *choose the same bid.*

Suppose to the contrary, that there exists an $\epsilon \in (0, \frac{s^*-\underline{s}}{2})$ and a subsequence of equilibria for which $\exists(C_n, s_n^c, b_n)$ s.t. $\pi_h^c(\sigma_n(\underline{s}+\epsilon);\sigma_n) < \pi_h^c(C_n, s_n^c, b_n;\sigma_n) < \pi_h^c(\sigma_n(s^*-\epsilon);\sigma_n)$. Since π_ℓ^c is isomorph to π_h^c the same is true for π_ℓ^c. We immediately not that C_n has to be the one chosen by σ_n (cf Assumption 1). Otherwise, (C_n, s_n^c, b_n) would only win when there is no other bidder in the auction and the resulting winning probability would be below $\pi_h^c(\sigma_n(\underline{s}+\epsilon);\sigma_n)$.

Either $(C_n, s_n^c, b_n) = \sigma_n(s)$ for all s from some non-empty interval $[s_-^n, s_+^n]$ and (by construction) $s_-^n, s_+^n \in (\underline{s}+\epsilon, s^*-\epsilon)$, or (C_n, s_n^c, b_n) is not in the support of σ_n. We focus on the former case and consider a subsequence where s_-^n, s_+^n converge. If (C_n, s_n^c, b_n) is not in the support of σ_n, it wins whenever $s_{(1)} \leq s_n$ for some $s_n \in (\underline{s}+\epsilon, s^*-\epsilon)$. The proof follows with the appropriate winning probability $\pi_\omega^c(C_n, s_n^c, b_n;\sigma_n) = e^{-\eta_\omega^n(1-F_\omega(s_n))}$.

The rest of this proof revolves around b_n. In a first step, we derive a lower bound on b_n by utilizing that (C_n, s_n^c, b_n) is not chosen by bidders with signals at or below $s^*-\epsilon$. In Step 2 we derive an upper bound on b_n by bounding the bid made by $s^*-\epsilon$, which will result in a contradiction.

Step 1: Action (C_n, s_n^c, b_n) wins with probability

$$\pi_\omega^c(C, s_n^c, b_n;\sigma_n) = \frac{e^{-\eta_\omega^n(1-F_\omega(s_+^n))} - e^{-\eta_\omega^n(1-F_\omega(s_-^n))}}{\eta_\omega^n(F_\omega(s_+^n) - F_\omega(s_-^n))}.$$

The highest probability with which a bidder with signal $\underline{s}$ can win is $\pi_\omega^c(\sigma_n(\underline{s}+\epsilon);\sigma_n)$. This is the case, whenever signals $\underline{s}$ to $\underline{s}+\epsilon$ pool on the same bid and same report. The probability $\pi_\omega^c(\sigma_n(\underline{s}+\epsilon);\sigma_n)$ attains the highest value in case all signals up to s_-^n pool on the same bid/partition as well, that is if $\sigma_n(\underline{s}+\epsilon) = \sigma_n(s)$ for $s < s_-^n$. As a result,

$$\pi_\omega^c(\sigma_n(\underline{s});\sigma_n) \leq \frac{e^{-\eta_\omega^n(1-F_\omega(s_-^n))} - e^{-\eta_\omega}}{\eta_\omega^n F_\omega(s_-^n)}.$$

Lemma 7 then gives the most conservative lower bound on the bid b_n, which ensures that $\underline{s}$ does not want to deviate to (C_n, s_n^c, b_n). The lower bound is

$$b_n \geq \frac{\rho \mathscr{L}_n v_h + (1-\rho)v_\ell}{\rho \mathscr{L}_n + (1-\rho)},$$

with

$$\mathscr{L}_n = \frac{\eta_h^n f_h(\underline{s})\left(\frac{e^{-\eta_h^n(1-F_h(s_+^n))} - e^{-\eta_h^n(1-F_h(s_-^n))}}{\eta_h^n(F_h(s_+^n)-F_h(s_-^n))} - \frac{e^{-\eta_h^n(1-F_h(s_-^n))} - e^{-\eta_h^n}}{\eta_h^n F_h(s_-^n)}\right)}{\eta_\ell^n f_\ell(\underline{s})\left(\frac{e^{-\eta_\ell^n(1-F_\ell(s_+^n))} - e^{-\eta_\ell^n(1-F_\ell(s_-^n))}}{\eta_\ell^n(F_\ell(s_+^n)-F_\ell(s_-^n))} + \frac{\rho\eta_h^n f_h(\underline{s})}{(1-\rho)\eta_\ell^n f_\ell(\underline{s})}\frac{e^{-\eta_h^n(1-F_h(s_-^n))} - e^{-\eta_h^n}}{\eta_h^n F_h(s_-^n)}\right)}, \tag{26}$$

which we want to investigate.

Since $\frac{e^{-\eta_h^n(1-F_h(s_+^n))}-e^{-\eta_h^n(1-F_h(s_-^n))}}{\eta_h^n(F_h(s_+^n)-F_h(s_-^n))} \geq e^{-\eta_h^n(1-F_h(s_-^n))}$ and $\eta_h^n F_h(s_-^n) \to \infty$, for n large, the numerator of (26) is of order $\eta_h^n f_h(\underline{s})\frac{e^{-\eta_h^n(1-F_h(s_+^n))}-e^{-\eta_h^n(1-F_h(s_-^n))}}{\eta_h^n(F_h(s_+^n)-F_h(s_-^n))}$. Further, $\frac{e^{-\eta_\ell^n(1-F_\ell(s_+^n))}-e^{-\eta_\ell^n(1-F_\ell(s_-^n))}}{\eta_\ell^n(F_\ell(s_+^n)-F_\ell(s_-^n))} = e^{-\eta_\ell^n(1-F_\ell(y^n))}$ for some signal $y^n \in [s_-^n, s_+^n]$.[32] Last, for large n, $e^{-\eta_h^n}$ is negligible compared to $e^{-\eta_h^n(1-F_h(s_-^n))}$. Thus we can bound equation (26) from below. For any $\lambda \in (0,1)$ and n is sufficiently large

$$\mathscr{L} \overset{\text{n large}}{>} \frac{\eta_h^n f_h(\underline{s})\frac{e^{-\eta_h^n(1-F_h(s_+^n))}-e^{-\eta_h^n(1-F_h(s_-^n))}}{\eta_h^n(F_h(s_+^n)-F_h(s_-^n))}}{\eta_\ell^n f_\ell(\underline{s})(e^{-\eta_\ell^n(1-F_\ell(y^n))}+\phi\frac{e^{-\eta_h^n(1-F_h(s_-^n))}}{\eta_h^n F_h(s_-^n)})}\lambda,$$

where $\phi = l\frac{\rho f_h(\underline{s})}{(1-\rho)f_\ell(\underline{s})}$ is a constant. We now distinguish two cases:

First, consider the case in which $-\eta_h^n(1-F_h(s_-^n))+\eta_\ell^n(1-F_\ell(y^n)) \to \infty$. By Lemma 3 and because $s_+^n < s^*$, it follows that $\frac{e^{-\eta_h^n(1-F_h(s_+^n))}-e^{-\eta_h^n(1-F_h(s_-^n))}}{\eta_h^n(F_h(s_+^n)-F_h(s_-^n))} \geq e^{-\eta_h^n(1-F_h(s_-^n))}$. Dividing the numerator and denominator by $e^{-\eta_h^n(1-F_h(s_-^n))}$ yields the lower bound

$$\mathscr{L} \overset{\text{large n}}{>} \frac{\eta_h^n f_h(\underline{s})}{\eta_\ell^n f_\ell(\underline{s})(e^{-\eta_\ell^n(1-F_\ell(y^n))+\eta_h^n(1-F_h(s_-^n))}+\phi\frac{1}{\eta_h^n F_h(s_-^n)})}\lambda \to \frac{\eta_h^n f_h(\underline{s})}{\phi\frac{\eta_\ell^n f_\ell(\underline{s})}{\eta_h^n F_h(s_-^n)}}\lambda \to \infty.$$

This implies that for any $\lambda \in (0,1)$, it holds that $b_n \overset{\text{large n}}{>} v_h\lambda$. Note, however, that because the signals are bounded and $\bar{s}$ does not pool for n large (cf. Claim 1) $\mathbb{E}[v|\text{win with } \sigma_n(\bar{s}), \bar{s}; \sigma_n] = \mathbb{E}[v|\bar{s}] < v_h$. Since $\bar{s}$ chooses a higher bid than b_n she would make strict loss, which would be dominated by choosing v_ℓ (and some arbitrary report) and making a weak profit. Thus, we found a contradiction.

If $-\eta_h^n(1-F_h(s_-^n))+\eta_\ell^n(1-F_\ell(y^n)) \not\to \infty$, then (recall that $\eta_h^n F_h(s_-^n) \to \infty$)

$$\frac{e^{-\eta_\ell^n(1-F_\ell(y^n))}+\phi\frac{e^{-\eta_h^n(1-F_h(s_-^n))}}{\eta_h^n F_h(s_-^n)}}{e^{-\eta_\ell^n(1-F_\ell(y^n))}} = 1+\phi\frac{e^{-\eta_h^n(1-F_h(s_-^n))+\eta_\ell^n(1-F_\ell(y^n))}}{\eta_h^n F_h(s_-^n)} \to 1.$$

Thus, for n large, the denominator of equation (26) is of order $e^{-\eta_\ell^n(1-F_\ell(y^n))}$. Reverting the y-substitution, for n large, equation (26) hence can be bounded below by

[32] This equivalence follows because $e^{-\eta_\ell^n(1-F_\ell(s_-^n))} \leq \frac{e^{-\eta_\ell^n(1-F_\ell(s_+^n))}-e^{-\eta_\ell^n(1-F_\ell(s_-^n))}}{\eta_\ell^n(F_\ell(s_+^n)-F_\ell(s_-^n))} \leq e^{-\eta_\ell^n(1-F_\ell(s_+^n))}$ and $e^{-\eta_\ell^n(1-F_\ell(s))}$ is monotonically increasing in s.

$$\mathscr{L} \overset{\text{large n}}{>} \frac{\eta_h^n f_h(\underline{s}) \frac{e^{-\eta_h^n(1-F_h(s_+^n))} - e^{-\eta_h^n(1-F_h(s_-^n))}}{\eta_h^n(F_h(s_+^n)-F_h(s_-^n))}}{\eta_\ell^n f_\ell(\underline{s}) \frac{e^{-\eta_\ell^n(1-F_\ell(s_+^n))} - e^{-\eta_\ell^n(1-F_\ell(s_-^n))}}{\eta_\ell^n(F_\ell(s_+^n)-F_\ell(s_-^n))}} \lambda^2$$
$$\geq \frac{\eta_h^n f_h(\underline{s}) e^{-\eta_h^n(1-F_h(s_+^n))}}{\eta_\ell^n f_\ell(\underline{s}) e^{-\eta_\ell^n(1-F_\ell(s_+^n))}} \lambda^2 \geq \frac{\eta_h^n f_h(\underline{s}) e^{-\eta_h^n(1-F_h(s^*-\epsilon))}}{\eta_\ell^n f_\ell(\underline{s}) e^{-\eta_\ell^n(1-F_\ell(s^*-\epsilon))}} \lambda^2,$$

where the latter two inequalities follow from $s_+^n < s^*$ and Lemma 3, as well as 2*. Wrapping up, this means that $b_n \overset{\text{large n}}{>} \mathbb{E}[v|\underline{s}, s_{(1)} \leq s^* - \epsilon, \lambda^2]$.

Step 2: Since s^* s.t. $\eta_h f_h(s^*) = \eta_\ell f_\ell(s^*)$ is unique, the monotone likelihood ratio property implies that $\frac{\eta_h f_h(s)}{\eta_\ell f_\ell(s)} < 1$ for all $s < s^*$ and as a result

$$\begin{aligned}
&- \eta_h^n(1 - F_h(s^* - \epsilon)) + \eta_h^n(1 - F_h(s^*)) + \eta_\ell^n(1 - F_\ell(s^* - \epsilon)) - \eta_\ell^n(1 - F_\ell(s^*)) \\
&= \eta_h^n(F_h(s^* - \epsilon) - F_h(s^*)) - \eta_\ell^n(F_\ell(s^* - \epsilon) - F_\ell(s^*)) \\
&= \int_{s^*-\epsilon}^{s^*} \eta_\ell^n f_\ell(s) \underbrace{(1 - l\frac{f_h(s)}{f_\ell(s)})}_{<0 \text{ constant}} ds \to -\infty.
\end{aligned}$$

By continuity of the arguments, the same is true for $s^* + \epsilon'$ with $\epsilon' > 0$ sufficiently small. It follows from this that for n sufficiently large,

$$\lambda^2 \frac{\eta_h^n f_h(\underline{s}) e^{-\eta_h^n(1-F_h(s^*-\epsilon))}}{\eta_\ell^n f_\ell(\underline{s}) e^{-\eta_\ell^n(1-F_\ell(s^*-\epsilon))}} > \frac{\eta_h^n f_h(s^* + \epsilon') e^{-\eta_h^n(1-F_h(s^*+\epsilon'))}}{\eta_\ell^n f_\ell(s^* + \epsilon') e^{-\eta_\ell^n(1-F_\ell(s^*+\epsilon'))}},$$

which implies that for n large $\mathbb{E}[v|\underline{s}, s_{(1)} \leq s^* - \epsilon, \lambda^2] > \mathbb{E}[v|s_{(1)} \leq s^* + \epsilon', s^* + \epsilon']$.

The probability that $s^* + \epsilon'$ ties is zero for n sufficiently large (c.f. Claim 1). Thus, $\mathbb{E}[v|\text{win with } \sigma_n(s^* + \epsilon'), s^* + \epsilon'; \sigma_n] = \mathbb{E}[v|s_{(1)} \leq s^* + \epsilon', s^* + \epsilon']$. But this is smaller than $\mathbb{E}[v|\underline{s}, s_{(1)} \leq s^* - \epsilon, \lambda^2]$ i.e. the minimum bid for b_n and therefore minimum bid in chosen by $s^* + \epsilon'$. Signal $s^* + \epsilon'$ would make a loss, which is a contradiction to the equilibrium, because she could always deviate and bid v_ℓ, making positive profits. ■

Claim 3: *Fix any $\epsilon \in (0, \frac{s^* - \underline{s}}{2})$. For every n sufficiently large, there are two disjoint, adjacent intervals I_n and J_n with $[\underline{s} + \epsilon, s^* - \epsilon] \subset I_n \cup J_n$. Bidders with signals $s_I \in I_n$ choose $\sigma_n(s_I) = (C_n, s_I^{c,n}, b_n)$ and bidders with signals $s_J \in J_n$ choose $\sigma_n(s_J) = (C_n, s_J^{c,n}, b_n)$ and $\exists! c_n \in C_n$ s.t. $s_I^{c,n} < c_n < s_J^{c,n}$. This means that , $\nexists(C, s^c, b)$ s.t. $\pi_\omega^c(\sigma_n(s_I); \sigma_n) < \pi_\omega^c(C, s^c, b; \sigma_n) < \pi_\omega^c(\sigma_n(s_J); \sigma_n)$ for $\omega \in \{h, \ell\}$. Last, the expected number of bidders in both intervals is larger than $\frac{1}{\epsilon}$.*

Fix any $\epsilon > 0$ sufficiently small, such that $\frac{f_h(\underline{s}+\epsilon)}{f_\ell(\underline{s}+\epsilon)}\frac{F_\ell(s^*-\epsilon)}{F_h(s^*-\epsilon)} < \frac{\eta_h f_h(s^*-\epsilon)}{\eta_\ell f_\ell(s^*-\epsilon)}$. Notice that such an ϵ exists, because $\frac{f_h(\underline{s})}{f_\ell(\underline{s})}\frac{F_\ell(s^*)}{F_h(s^*)} < 1 = \frac{\eta_h f_h(s^*)}{\eta_\ell f_\ell(s^*)}$ and the expressions are continuous in its arguments.

For n sufficiently large, we know that all bidders from the interval $[\underline{s}+\epsilon, s^*-\epsilon]$ bid the same bid and but send at most two different reports (Claim 2). We define I_n as the interval of signals choosing $(C_n, s_I^{c,n}, b_n) = \sigma(\underline{s}+\epsilon)$ and J_n as the largest interval of signals choosing the same bid b_n, but a report from the next interval of the partition C_n (potentially empty)[33]. For future reference in this proof, we denote the action chosen by bidders with signals from J_n by $\zeta_n := (C_n, s_J^{c,n}, b_n)$.

By construction, I_n and J_n fulfill all the properties stated above except for, potentially, the last. Thus, we have to show that as n grows large, the expected number of bidders in both intervals grows without bounds $\int_{I_n} \eta_\omega^n f_\omega(s)ds, \int_{J_n} \eta_\omega^n f_\omega(s)ds \to \infty$ for $\omega \in \{h, \ell\}$. Suppose to the contrary that this was not the case.

First, consider interval I_n with bounds $s_-^{I,n}$ and $s_+^{I,n}$ and suppose that $s_+^{I,n} - s_-^{I,n} \to 0$. . By definition of I_n, if it converges to a length of zero this means that the upper and lower bound converge to $\underline{s}+\epsilon$. If this is the case, consider the interval $(\underline{s}+\frac{\epsilon}{2}, \bar{s}-\frac{\epsilon}{2})$. By Claim 2, $\nexists (C, s^c, b)$ s.t. $\pi_\omega^c(\sigma_n(\underline{s}+\frac{\epsilon}{2}); \sigma_n) < \pi_\omega^c(C, s^c, b; \sigma_n) < \pi_\omega^c(\sigma_n(s^*-\frac{\epsilon}{2}); \sigma_n)$ for $\omega \in \{h, \ell\}$ and n sufficiently large. However, if I_n converges to a length of zero i.e. to the point $\underline{s}+\epsilon$, this means that for n sufficiently large, $\pi_\omega^c(\sigma_n(\underline{s}+\frac{\epsilon}{2}); \sigma_n) < \pi_\omega^c(\sigma_n(\underline{s}+\epsilon); \sigma_n) < \pi_\omega^c(\sigma_n(s^*-\frac{\epsilon}{2}); \sigma_n)$ for $\omega \in \{h, \ell\}$ – a contradiction. Thus, I_n cannot converge to a length of zero and the expected number of bidders in I_n has to grow without bound.

Next, turn to interval J_n with bounds $s_-^{J,n}$ and $s_+^{J,n}$ (obviously, $s_+^{I,n} = s_-^{J,n}$). Suppose to the contrary of the claim that $\eta_\omega(F_\omega(s_+^{J,n}) - F_\omega(s_+^{J,n})) \not\to \infty$ for $\omega \in \{h, \ell\}$. In this case, $s_-^{J,n}, s_+^{J,n}$ converge to some common limit s^J [34]. Notice that it cannot be that $s^J < s^* - \epsilon$. By the way we constructed J_n, if this was the case, then for n sufficiently large, $\pi_\omega^c(\sigma_n(\underline{s}+\epsilon); \sigma_n) < \pi_\omega^c(\zeta_n; \sigma_n) < \pi_\omega^c(\sigma_n(s^*-\epsilon); \sigma_n)$ for $\omega \in \{h, \ell\}$, which is a contradiction to Claim 2. Since the same is true for any $\epsilon' < \epsilon$ and $s^J < s^* - \epsilon'$, it follows that $s^J \geq s^*$. In the following, we only concentrate on this remaining case.

The idea of the remainder of the proof is the one presented in Section 4 of the paper. If J_n is arbitrary small and thereby I_n very long, I_n is approximately a single large pool and thereby such an equilibrium cannot exist.

We first show that a bidder with signal s winning with action ζ_n expects the good to

[33] Such a report might not exits. However, one can replicate the same winning probability by bidding a bid marginally above b_n, such that we can act as if such a report exists.

[34] If J_n is empty, set $s_-^{J,n} = s_+^{J,n} = s_+^{I,n}$. The proof follows with $\pi_\omega^{J,n} := \pi_\omega^c(\zeta_n; \sigma_n) = e^{-\eta_\omega(1-F_\omega(s_+^{I,n}))}$

be approximately of value $\mathbb{E}[v|s_{(1)} \leq s_-^{J,n}, s]$ (Step 1). Using this, we exploit the preference of the bidder with signal $s_+^{I,n}$ over the actions $\sigma_n(\underline{s}+\epsilon)$ and ζ_n to derive a lower bound on b_n (Step 2). Then, we use a bidder with signal $s_-^{I,n}$ and her expected value conditional on winning to find an upper bound on b_n (Step 3). In Step 4 we show that the lower bound exceeds the upper bound.

Step 1: First, if $\eta_\omega^n[F_\omega(s_+^{J,n}) - F_\omega(s_-^{J,n})] \to 0$ for $\omega \in \{h, \ell\}$ this implies that (using l'Hospital)

$$\lim \frac{\pi_h^{J,n}}{\pi_\ell^{J,n}} = \lim \frac{\frac{e^{-\eta_h^n(1-F_h(s_+^{J,n}))} - e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}{\eta_h^n(F_h(s_+^{J,n}) - F_h(s_-^{J,n}))}}{\frac{e^{-\eta_\ell^n(1-F_\ell(s_+^{J,n}))} - e^{-\eta_\ell^n(1-F_\ell(s_-^{J,n}))}}{\eta_\ell^n(F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n}))}} = \frac{e^{-\eta_h^n(1-F_h(s^J))}}{e^{-\eta_\ell^n(1-F_\ell(s^J))}}.$$

By Claim 2, the probability that $s^J > s^*$ ties is zero for n sufficiently which implies that $\eta_\omega^n[F_\omega(s_+^{J,n}) - F_\omega(s_-^{J,n})] \to 0$ for $\omega \in \{h, \ell\}$. If, on the other hand, $s^J = s^*$, but $\eta_\omega^n[F_\omega(s_+^{J,n}) - F_\omega(s_-^{J,n})] \not\to 0$ then $\eta_h^n[F_h(s_+^{J,n}) - F_h(s_-^{J,n})] - \eta_\ell^n[F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n})] \to 0$.[35] For the likelihood-ratio of winning, $\frac{\pi_h^{J,n}}{\pi_\ell^{J,n}} := \frac{\pi_h^c(\zeta_n;\sigma_n)}{\pi_\ell^c(\zeta_n;\sigma_n)}$, this means that

$$\frac{\pi_h^{J,n}}{\pi_\ell^{J,n}}\Big(\frac{e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}{e^{-\eta_\ell^n(1-F_\ell(s_-^{J,n}))}}\Big)^{-1} = \frac{\frac{e^{-\eta_h^n(1-F_h(s_+^{J,n}))} - e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}{\eta_h^n(F_h(s_+^{J,n}) - F_h(s_-^{J,n}))}}{\frac{e^{-\eta_\ell^n(1-F_\ell(s_+^{J,n}))} - e^{-\eta_\ell^n(1-F_\ell(s_-^{J,n}))}}{\eta_\ell^n(F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n}))}}\Big(\frac{e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}{e^{-\eta_\ell^n(1-F_\ell(s_-^{J,n}))}}\Big)^{-1} \to 1.$$

Summing up, if $s^J \geq s^*$

$$\frac{\pi_h^{J,n}}{\pi_\ell^{J,n}}\Big(\frac{e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}{e^{-\eta_\ell^n(1-F_\ell(s_-^{J,n}))}}\Big)^{-1} = \frac{\frac{e^{-\eta_h^n(1-F_h(s_+^{J,n}))} - e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}{\eta_h^n(F_h(s_+^{J,n}) - F_h(s_-^{J,n}))}}{\frac{e^{-\eta_\ell^n(1-F_\ell(s_+^{J,n}))} - e^{-\eta_\ell^n(1-F_\ell(s_-^{J,n}))}}{\eta_\ell^n(F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n}))}}\Big(\frac{e^{-\eta_h^n(1-F_h(s_-^{J,n}))}}{e^{-\eta_\ell^n(1-F_\ell(s_-^{J,n}))}}\Big)^{-1} \to 1. \quad (27)$$

Step 2: Consider a bidder with signal $s_+^{I,n} = s_-^{J,n}$ who is indifferent (if J_n is non-empty) or prefers (if J_n is empty) $\sigma_n(\underline{s}+\epsilon)$ over ζ_n. Using this preference, we will find a lower bound on b_n. Define $\pi_\omega^{I,n} := \pi_\omega^c(\sigma_n(\underline{s}+\epsilon);\sigma_n)$ for $\omega \in \{h, \ell\}$. Then $U(\sigma_n(\underline{s}+\epsilon)|s_+^{I,n};\sigma_n) \geq U(\zeta_n|s_+^{I,n};\sigma_n)$ implies that

[35]Toward the contradiction, we supposed that $\eta_\omega^n(F_\omega(s_+^{J,n}) - F_\omega(s_-^{J,n})) \not\to \infty$. Thus, $\eta_h^n[F_h(s_+^{J,n}) - F_h(s_-^{J,n})] - \eta_\ell^n[F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n})] = \int_{[s_-^{J,n}, s_+^{J,n}]} \eta_h^n f_h(s) - \eta_\ell^n f_\ell(s)ds \leq \int_{[s_-^{J,n}, s_+^{J,n}]}(\frac{\eta_h^n f_h(s_+^{J,n})}{\eta_\ell^n f_\ell(s_+^{J,n})} - 1)\eta_\ell^n f_\ell(s)ds = (\frac{\eta_h^n f_h(s_+^{J,n})}{\eta_\ell^n f_\ell(s_+^{J,n})} - 1)\eta_\ell^n[F_\ell(s_+^{J,n}) - F_\ell(s_-^{J,n})] \to 0$, since $\frac{\eta_h^n f_h(s_+^{J,n})}{\eta_\ell^n f_\ell(s_+^{J,n})} \to \frac{\eta_h^n f_h(s^J)}{\eta_\ell^n f_\ell(s^J)} = \frac{\eta_h^n f_h(s^*)}{\eta_\ell^n f_\ell(s^*)} = 1$ which bounds the limit from above. The bound from below follows by using $\int_{[s_-^{J,n}, s_+^{J,n}]} \eta_h^n f_h(s) - \eta_\ell^n f_\ell(s)ds \geq \int_{[s_-^{J,n}, s_+^{J,n}]}(\frac{\eta_h^n f_h(s_-^{J,n})}{\eta_\ell^n f_\ell(s_-^{J,n})} - 1)\eta_\ell^n f_\ell(s)ds$.

$$\frac{\rho\eta_h^n f_h(s_+^{I,n})\pi_h^{I,n}(v_h-b_n)+(1-\rho)\eta_\ell^n f_\ell(s_+^{I,n})\pi_\ell^{I,n}(v_\ell-b_n)}{\rho\eta_h^n f_h(s_+^{I,n})\pi_h^{I,n}+(1-\rho)\eta_\ell^n f_\ell(s_+^{I,n})\pi_\ell^{I,n}}$$
$$\geq \frac{\rho\eta_h^n f_h(s_+^{I,n})\pi_h^{J,n}(v_h-b_n)+(1-\rho)\eta_\ell^n f_\ell(s_+^{I,n})\pi_\ell^{J,n}(v_\ell-b_n)}{\rho\eta_h^n f_h(s_+^{I,n})\pi_h^{J,n}+(1-\rho)\eta_\ell^n f_\ell(s_+^{I,n})\pi_\ell^{J,n}}$$
$$\iff b_n \geq \frac{\rho\eta_h^n f_h(s_+^{I,n})(\pi_h^{J,n}-\pi_h^{I,n})v_h+(1-\rho)\eta_\ell^n f_\ell(s_+^{I,n})(\pi_\ell^{J,n}-\pi_\ell^{I,n})v_\ell}{\rho\eta_h^n f_h(s_+^{I,n})(\pi_h^{J,n}-\pi_h^{I,n})+(1-\rho)\eta_\ell^n f_\ell(s_+^{I,n})(\pi_\ell^{J,n}-\pi_\ell^{I,n}).}=\mathbb{E}[v|\psi_n,s_+^{I,n};\sigma_n]$$

where the likelihood ratio of event ψ_n is $\frac{\pi_h^{J,n}-\pi_h^{I,n}}{\pi_\ell^{J,n}-\pi_\ell^{I,n}}$.

Step 3: Next, we derive an upper bound for b_n. No bidder with a signal from I_n will bid more than his expected value conditional on winning. For a bidder with signal $\underline{s}+\epsilon \in I_n$ this means that $b_n \leq \mathbb{E}[v|\text{win with } \sigma_n(\underline{s}+\epsilon),\underline{s}+\epsilon;\sigma_n]$. Inspecting the likelihood-ratio $\frac{\rho\eta_h^n f_h(\underline{s}+\epsilon)\pi_h^{I,n}}{(1-\rho)\eta_\ell^n f_\ell(\underline{s}+\epsilon)\pi_\ell^{I,n}}$, and using that $\eta_\omega^n(F_\omega(s_+^{I,n})-F_\omega(s_-^{I,n}))\to\infty$[36] for $\omega\in\{h,\ell\}$ yields

$$\begin{aligned}
&\frac{\rho\eta_h^n f_h(\underline{s}+\epsilon)\pi_h^{I,n}}{(1-\rho)\eta_\ell^n f_\ell(\underline{s}+\epsilon)\pi_\ell^{I,n}}\Big(\frac{\eta_h^n f_h(s_+^{I,n})e^{-\eta_h^n(1-F_h(s_+^{I,n}))}}{\eta_\ell^n f_\ell(s_+^{I,n})e^{-\eta_\ell^n(1-F_\ell(s_+^{I,n}))}}\Big)^{-1}\\
&=\frac{\eta_h^n f_h(\underline{s}+\epsilon)\frac{e^{-\eta_h^n(1-F_h(s_+^{I,n}))}-e^{-\eta_h^n(1-F_h(s_-^{I,n}))}}{\eta_h^n[F_h(s_+^{I,n})-F_h(s_-^{I,n})]}}{\eta_\ell^n f_\ell(\underline{s}+\epsilon)\frac{e^{-\eta_\ell^n(1-F_\ell(s_+^{I,n}))}-e^{-\eta_\ell^n(1-F_\ell(s_-^{I,n}))}}{\eta_\ell^n[F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n})]}}\Big(\frac{\eta_h^n f_h(s_+^{I,n})e^{-\eta_h^n(1-F_h(s_+^{I,n}))}}{\eta_\ell^n f_\ell(s_+^{I,n})e^{-\eta_\ell^n(1-F_\ell(s_+^{I,n}))}}\Big)^{-1}\\
&=\frac{1-e^{-\eta_h(F_h(s_+^{I,n})-F_h(s_-^{I,n}))}}{1-e^{-\eta_\ell(F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n}))}}\frac{\eta_\ell^n[F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n})]}{\eta_h^n[F_h(s_+^{I,n})-F_h(s_-^{I,n})]}\frac{f_h(\underline{s}+\epsilon)f_\ell(s_+^{I,n})}{f_\ell(\underline{s}+\epsilon)f_h(s_+^{I,n})}\\
&\to\frac{f_h(\underline{s}+\epsilon)f_\ell(s_+^{I,n})}{f_\ell(\underline{s}+\epsilon)f_h(s_+^{I,n})}\frac{\eta_\ell^n[F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n})]}{\eta_h^n[F_h(s_+^{I,n})-F_h(s_-^{I,n})]}.
\end{aligned}$$

Since $s_+^{I,n}=s_-^{J,n}\to s^J\geq s^*$, for n sufficiently large, the monotone likelihood ratio property implies that $\frac{\eta_\ell^n[F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n})]}{\eta_h^n[F_h(s_+^{I,n})-F_h(s_-^{I,n})]}\leq\frac{\eta_\ell^n F_\ell(s_-^{I,n})}{\eta_h^n F_h(s_-^{I,n})}<\frac{\eta_\ell^n F_\ell(s^*-\epsilon)}{\eta_h^n F_h(s^*-\epsilon)}$. Furthermore, we did set $\epsilon>0$ s.t. $\frac{\eta_h^n f_h(\underline{s}+\epsilon)}{\eta_\ell^n f_\ell(\underline{s}+\epsilon)}\frac{\eta_\ell^n F_\ell(s^*-\epsilon)}{\eta_h^n F_h(s^*-\epsilon)}<\frac{\eta_h^n f_h(s^*-\epsilon)}{\eta_\ell^n f_\ell(s^*-\epsilon)}<\frac{\eta_h^n f_h(s_+^{I,n})}{\eta_\ell^n f_\ell(s_+^{I,n})}$ such that $\frac{f_h(\underline{s}+\epsilon)f_\ell(s_+^{I,n})}{f_\ell(\underline{s}+\epsilon)f_h(s_+^{I,n})}\frac{\eta_\ell^n[F_\ell(s_+^{I,n})-F_\ell(s_-^{I,n})]}{\eta_h^n[F_h(s_+^{I,n})-F_h(s_-^{I,n})]}$ stays bounded below 1. Hence, there exists a $\mu<1$ such that

$$\frac{\rho\eta_h^n f_h(\underline{s}+\epsilon)\pi_h^{I,n}}{(1-\rho)\eta_\ell^n f_\ell(\underline{s}+\epsilon)\pi_\ell^{I,n}}\overset{\text{large n}}{<}\frac{\eta_h^n f_h(s_+^{I,n})e^{-\eta_h^n(1-F_h(s_+^{I,n}))}}{\eta_\ell^n f_\ell(s_+^{I,n})e^{-\eta_\ell^n(1-F_\ell(s_+^{I,n}))}}\mu. \tag{28}$$

[36] Recall that we ruled out the possibility that the length of I^k converges to zero.

Thus, we conclude that for n sufficiently large, $b_n \geq \mathbb{E}[v|\chi_n, s_+^{I,n}; \sigma_n]$, where the likelihood ratio of event χ_n is $\frac{e^{-\eta_h^n(1-F_h(s_+^{I,n}))}}{e^{-\eta_\ell^n(1-F_\ell(s_+^{I,n}))}}\mu$.

Step 4: We now show that for n sufficiently large, the lower bound from Step 2 is larger than the upper bound from Step 3 and thus no such b_n can exist. For this, it is sufficient to consider the likelihood ratios of ψ_n and χ_n. Suppose to the contrary that $\psi_n < \chi_n$ for all n:

$$\left(\frac{\pi_h^{J,n} - \pi_h^{I,n}}{\pi_\ell^{J,n} - \pi_\ell^{I,n}}\right) < \frac{e^{-\eta_h^n(1-F_h(s_+^{I,n}))}}{e^{-\eta_\ell^n(1-F_\ell(s_+^{I,n}))}}\mu$$

$$\underbrace{\frac{\pi_h^{J,n} - \pi_h^{I,n}}{\pi_\ell^{J,n} - \pi_\ell^{I,n}}\left(\frac{\pi_h^{J,n}}{\pi_\ell^{J,n}}\right)^{-1}}_{\to 1 \text{ by inspection}} < \underbrace{\frac{e^{-\eta_h^n(1-F_h(s_+^{I,n}))}}{e^{-\eta_\ell^n(1-F_\ell(s_+^{I,n}))}}\left(\frac{\pi_h^{J,n}}{\pi_\ell^{J,n}}\right)^{-1}}_{\to 1 \text{ by equation (27)}}\mu.$$

However, because $\mu < 1$ this is violated for n sufficiently large. This means that for n large the lower bound on b_n (Step 2) is larger than the upper bound on b_n (Step 3). Therefore b_n cannot exist. Since we know that $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of equilibria, it, therefore, cannot be that the expected number of bidders who choose J_n stays bounded and for n sufficiently large, expected number of bidders in both intervals is larger than $\frac{1}{\epsilon}$. ■ □

Proof of Lemma 6

Proof. Given the sequence of games on the ever finer grid $(\Gamma(k))_{k\in\mathbb{N}}$, let B_k the respective bidding space. Consider the sequence of respective equilibria $(\beta_k)_{k\in\mathbb{N}}$, and for any k the implied winning probabilities $\hat{\pi}_\omega^k(s) := \pi_\omega(\beta_k(s); \beta_k)$ for $\omega \in \{h, \ell\}$. Furthermore, define an auxiliary function $\gamma_k : [v_\ell, v_h] \to B_k$ such that $\gamma_k(b) := \sup\{b' \in B_k : \ b' \leq b\}$. Since all of those functions are nondecreasing, we can find a subsequence on which they converge to some nondecreasing limit β, γ and $\hat{\pi}_\omega$ for $\omega \in \{h, \ell\}$. We denote this subsequence by n. Construct C as a partition of $[\underline{s}, \bar{s}]$ into (potential trivial) intervals, such that s and s' are in the same interval if and only if $\hat{\pi}_h(s) = \hat{\pi}_h(s')$. Note that because the winning probabilities are isomorph across states, this implies that $\hat{\pi}_\ell(s) = \hat{\pi}_\ell(s')$.

We claim that the following is an equilibrium: All bidders report C, reveal their type s truthfully, $s^c = s$, and bid $\beta(s)$. We call this strategy σ^*.

We first want to show that, given the rules of the Communication Extension, for any s and $\omega \in \{h, \ell\}$ it holds that $\pi_\omega^c(C, s, \beta(s)) = \hat{\pi}_\omega(s)$. We will focus on state h, the result follows for ℓ because, again, the winning probabilities are isomorph across states. To show this, fix any $\hat{s} \in [\underline{s}, \bar{s}]$ and define the sets $W_n := \{s\colon\ \hat{\pi}_h^n(s) < \hat{\pi}_h^n(\hat{s})\}$, $T_n := \{s\colon\ \hat{\pi}_h^n(s) = \hat{\pi}_h^n(\hat{s})\}$ and $L_n := \{s\colon\ \hat{\pi}_h^n(s) > \hat{\pi}_h^n(\hat{s})\}$. Furthermore, define $W := \{s\colon\ \hat{\pi}_h(s) > \hat{\pi}_h(\hat{s})\}$, and T, L respectively. Because $\hat{\pi}_h^n$ is non-decreasing and converges, $W_n \to W, T_n \to T$ and $L_n \to L$. We have to show that under the rules of the extended auction mechanism and σ^*, $\hat{s}$ loses against signals from L, wins against signals from W and ties with the signals from T.

Fix any $s \in L$. For n sufficiently large, $s \in L_n$. Further, it follows from $\hat{\pi}_h^n(\hat{s}) < \hat{\pi}_h^n(s)$ that $\beta_n(\hat{s}) < \beta_n(s)$. This, and the convergence of β_n implies that $\beta(\hat{s}) \leq \beta(s)$. Further, by definition there exists $c \in C$ such that $\hat{s} < c < s$. Following the rules of the Communication Extension, a bidder $\hat{s}$ choosing $\sigma^*(\hat{s})$ thereby never wins against $s \in L$. Either s chooses a higher bid, or $\beta(\hat{s}) = \beta(s)$ but s reports a higher interval of the partition. The symmetric argument can be made for bidders with signals from set W such that signal $\hat{s}$ following $\sigma^*(\hat{s})$ always wins against any $s \in W$.

Last, fix any $s \in T$. Again, n sufficiently large, $s \in T_n$ and $\hat{\pi}_h^n(s) = \hat{\pi}_h^n(\hat{s})$ implies that $\beta_n(s) = \beta_n(\hat{s})$, which means that $\beta(s) = \beta(\hat{s})$. By the way we defined C, signals $\hat{s}$ and s choose the same interval of the partition. By the rules of the Communication Extension, signal $\hat{s}$ thereby wins against signals from T if the random tiebreak decides in his / her favor.

Wrapping up, $\hat{s}$ choosing $\sigma^*(\hat{s})$ wins whenever there is no signal from L and the tiebreak among other signals from T decides in his / her favor. The same is true for any finite n strategy $\beta_n(s)$ and sets L_n and T_n. Since the sets converge, it follows that $\pi_\omega^c(C, s, \beta(\hat{s})) = \hat{\pi}_\omega(\hat{s})$ for $\omega \in \{h, \ell\}$ and any $\hat{s}$ and, thereby,

$$U(\sigma^*(s)|s;\sigma^*) = U(C, s, \beta(s)|s;\sigma^*) = \lim_{n\to\infty} U(\beta_n(s)|s;\beta_n). \tag{29}$$

To ensure that σ^* is an equilibrium, we now check all possible deviations:

0: Reporting C is an equilibrium because a deviating bidder will only receive the good if the deviation is not detected i.e. when she is alone. She could, however, always achieve at least the same utility by bidding v_ℓ and reporting truthfully. Thus, such a deviation is (weakly) dominated.

In the following, we, therefore, keep C fixed and consider only deviations with respect to the bid and signal report. We suppose that signal s deviates from $(s, \beta(s))$ to some (s', b'), and that the deviation affects the payoff, which implies that we only consider changes in the signal report when the bid ties with positive probability.

1: If b' does not tie with positive probability, then the report does not matter and the resulting winning probability is $\pi_\omega^c(C, s', b'; \sigma^*) = \pi_\omega^c(C, s, b'; \sigma^*)$ for $\omega \in \{h, \ell\}$. Furthermore, $\pi^c_\omega(C, s', b'; \sigma^*)$ is continuous at b'[37]. As a result, $\lim \pi_\omega(\gamma_n(b'); \beta_n) = \pi^c_\omega(C, s', b'; \sigma^*)$ for $\omega \in \{h, \ell\}$. Because the utility (1) is continuous bids and probabilities and all sequences converge, $U^c(C, s', b'|s; \sigma^*) = \lim_{n\to\infty} U(\gamma_n(b')|s; \beta_n)$. But this, and equation (29) imply that a deviation to b' cannot be strictly profitable. Otherwise, a deviation to $\gamma_n(b')$ would have been profitable for n sufficiently large.

2: If b' ties with positive probability and s' is such that $b' = \beta(s')$, then signal s mimics s'. By (29), $U^c(C, s', \beta(s')|s; \sigma^*) = \lim_{n\to\infty} U(\beta_n(s')|s; \beta_n)$. Hence, such a deviation cannot be strictly profitable. Otherwise, the bidder s would have had a strict incentive to mimic s' for n sufficiently large.

3: Last, consider the case in which b' ties with positive probability (which implies that $b' \neq v_\ell, v_h$)[38] , but $b' \neq \beta(s')$. By construction, reports and bids are non-decreasing in the signal. Thus, there are two possibilities: First, if $s' > \sup\{s\colon\ \beta(s) = b'\}$ then the deviating player wins the tiebreak for sure, but never when there is a higher bid. Because probability mass can at most be on countably many bids, and $b' < v_h$, there are bids larger, but arbitrary close to b' which tie with zero probability. Thus, for every $\epsilon > 0$, there exists a $b'' \in \{b \in (b', b' + \epsilon)\colon\ b \text{ does not tie given } \sigma^*\}$, such that $\pi^c_\omega(C, s', b'') \in \Big(\pi^c_\omega(C, s', b'; \sigma^*), \pi^c_\omega(C, s', b'; \sigma^*) + \epsilon\Big)$ for $\omega \in \{h, \ell\}$. Because b'' does not tie, the type report does not matter and by step 1, it cannot be profitable. Because this is true for any b'' for any $\epsilon > 0$, it follows that (b', s') cannot be a profitable deviation either. Second, if $s' < \inf\{s\colon\ \beta(s) = b'\}$ then deviating player always loses the tiebreak. We can redo the argument for a $b'' \in (b' - \epsilon, b')$.

Thus, no deviation is strictly profitable and σ^* is an equilibrium. □

[37]The set of bids which tie is the union of points and thereby closed. Thus, the set of those which do not tie is open. Because there is no positive mass on non-tieing bids, marginally changing the bid in the open set only marginally changes the set of signals the bidder wins against and, hence, and there exists a neighborhood where the winning probability is continuous.

[38]If k is sufficiently large, and b' is played with positive probability, then b' can be neither v_ℓ, nor v_h. If it was v_ℓ, then the winning bidders would not make a loss when the state is low. Since no signal is fully revealing of the state, she would have an incentive to marginally overbid b', discretely raising her winning probability (by circumventing the tiebreak) in exchange for an arbitrary small loss in the low state. Along the sequence of ever finer grids, such a deviation becomes available for n sufficiently large, and thereby the limiting equilibrium cannot contain a pool at v_ℓ. If b' was v_h, then every bidder choosing b' would make a loss. Because no signal is fully revealing of the state, for any signal and given any strategy, the bidder's expected value conditional on winning is strictly below v_h. Since a bidder is alone i.e. wins with positive probability, this means that she would make a strict loss. A deviation to v_ℓ would therefore be dominant.

Appendix C Numerical Example

C.1 Non Existence

In a strictly increasing equilibrium, the lowest bid equals the reserve price $v_\ell = 0$. Otherwise, $\underline{s}$ could lower her bid, win in the same situations (when she is alone) but pay less. Since $\frac{f_h(s)}{f_\ell(s)}$ is constant on $s \in [0, \frac{1}{2}]$, the bidders with these signals are essentially equal thus:

$$\begin{aligned}
& U(\beta(\underline{s})|\underline{s};\beta) = U(0|\underline{s};\beta) = U(\beta(s)|s;\beta) \quad \forall s \in [0, \frac{1}{2}] \\
\iff & \frac{\rho f_h(\underline{s})\pi_h(0;\beta)}{\rho f_h(\underline{s}) + (1-\rho) f_\ell(\underline{s})} = \frac{\rho f_h(s)\pi_h(\beta(s);\beta)(1-\beta(s)) + (1-\rho) f_\ell(s)\pi_\ell(\beta(s);\beta)(-\beta(s))}{\rho f_h(s) + (1-\rho) f_\ell(s)}.
\end{aligned}$$

Note that $f_\omega(s) = f_\omega(\underline{s})$ for all $s \in [0, \frac{1}{2}]$, $\omega \in \{h, \ell\}$ and $\rho = \frac{1}{2}$, such that we can rearrange the argument to

$$\begin{aligned}
\iff f_h(s)\pi_h(0;\beta) &= f_h(s)\pi_h(\beta(s);\beta)(1-\beta(s)) + f_\ell(s)\pi_\ell(\beta(s);\beta)(-\beta(s)) \\
\iff \beta(s) &= \frac{f_h(s)}{f_\ell(s)} \frac{\pi_h(\beta(s);\beta) - \pi_h(\beta(0);\beta)}{\pi_\ell(\beta(s);\beta) + \frac{f_h(s)}{f_\ell(s)}\pi_h(\beta(s);\beta)} \\
&= \frac{f_h(s)}{f_\ell(s)} \frac{e^{-\eta(1-F_h(s))} - e^{-\eta}}{e^{-\eta(1-F_\ell(s))} + \frac{f_h(s)}{f_\ell(s)} e^{-\eta(1-F_h(s))}} \\
&= \frac{f_h(s)}{f_\ell(s)} \frac{1 - e^{-\eta F_h(s)}}{e^{\eta(F_\ell(s)-F_h(s))} + \frac{f_h(s)}{f_\ell(s)}}.
\end{aligned}$$

To check if β is indeed strictly increasing, take the derivative with respect to s. The slope $\beta' \geq 0$ if

$$\begin{aligned}
& 0 \leq \frac{f_h(s)}{f_\ell(s)} \frac{\eta f_h(s) e^{-\eta F_h(s)}(e^{\eta(F_\ell(s)-F_h(s))} + \frac{f_h(s)}{f_\ell(s)}) - \eta(f_\ell(s) - f_h(s)) e^{\eta(F_\ell(s)-F_h(s))}(1 - e^{-\eta F_h(s)})}{(e^{\eta(F_\ell(s)-F_h(s))} + \frac{f_h(s)}{f_\ell(s)})^2} \\
\iff & 0 \leq f_h(s) e^{-\eta F_h(s)}(e^{\eta(F_\ell(s)-F_h(s))} + \frac{f_h(s)}{f_\ell(s)}) - (f_\ell(s) - f_h(s)) e^{\eta(F_\ell(s)-F_h(s))}(1 - e^{-\eta F_h(s)}) \\
\iff & 0 \leq f_h(s) e^{-\eta F_h(s)} \frac{f_h(s)}{f_\ell(s)} - f_\ell(s) e^{\eta(F_\ell(s)-F_h(s))}(1 - e^{-\eta F_h(s)}) + f_h(s) e^{\eta(F_\ell(s)-F_h(s))} \\
\iff & 0 \leq f_h(s) \frac{f_h(s)}{f_\ell(s)} - f_\ell(s) e^{\eta F_\ell(s)}(1 - e^{-\eta F_h(s)}) + f_h(s) e^{\eta F_\ell(s)}
\end{aligned}$$

which rearranges to

$$\Big(\frac{f_h(s)}{f_\ell(s)}\Big)^2 \geq e^{\eta F_\ell(s)}(1 - \frac{f_h(s)}{f_\ell(s)} - e^{-\eta F_h(s)}).$$

Plugging in the values at $s = \frac{1}{2}$ gives $\frac{9}{25} \geq e^{\eta \frac{5}{8}}(1 - \frac{3}{5} - e^{-\eta \frac{3}{8}})$ which has a unique critical value at $\eta \approx 2.9$.

C.2 Existence With a Pool

Suppose that $\beta(s) = b_p = 0.12$ for all $s \in [0, \frac{1}{2}]$ and strictly increasing otherwise. We want to show that there is an $\eta \in [4.9, 5]$ where this is an equilibrium.

First, we check that $\lim_{s\to 0.5} \mathbb{E}[v|\text{win with } \beta(s); s] = \mathbb{E}[v|s_{(1)} \leq \frac{1}{2}, \frac{1}{2}] > b_p$.

$$\mathbb{E}[v|s_{(1)} \leq \frac{1}{2}, \frac{1}{2}] = \frac{f_h(\frac{1}{2})e^{-\eta[1-\frac{3}{8}]}}{f_h(\frac{1}{2})e^{-\eta[1-\frac{3}{8}]} + f_\ell(\frac{1}{2})e^{-\eta[1-\frac{5}{8}]}} = \frac{1}{1+\frac{5}{3}e^{\eta\frac{2}{8}}} \overset{\eta\leq 5}{\geq} \frac{1}{1+\frac{5}{3}e^{\frac{10}{8}}}$$
$$= 0.146687 > 0.12 = b_p.$$

Next, we verify that condition *(iii)* of Proposition 2 is satisfied for all $s \in [\frac{1}{2}, \frac{5}{8}]$

$$2\Big(\frac{\partial}{\partial s}\frac{f_h(s)}{f_\ell(s)}\Big)\frac{f_\ell(s)}{f_h(s)} + \eta f_h(s) - \eta f_\ell(s) = \frac{2}{(\frac{9}{8}-s)(s-\frac{1}{8})} + \eta(4s - \frac{10}{4})$$
$$\geq 8 - \frac{\eta}{2}.$$

where we used that the first fraction has a local minimum at $s = \frac{5}{8}$ and the second term is minimized at $s = \frac{1}{2}$. Obviously the expression is positive for $\eta \leq 5$. We conclude that $\beta(s)$ with $\beta(\frac{1}{2}) = b_p$ is strictly increasing if $\eta \in [4.9, 5]$.

A bidder with signals $s \in [0, \frac{1}{2}]$ prefers to bid b_p and pool with other bidders with signals from $[0, \frac{1}{2}]$ over deviating to 0 if $U(0|\underline{s};\beta) \geq U(b_p|\underline{s};\beta)$:

$$U(0|\underline{s};\beta) = \frac{\rho f_h(s)\pi_h(0;\beta)}{\rho f_h(s) + (1-\rho)f_\ell(s)} = \frac{3\ \pi_h(0;\beta)}{8} = \frac{3}{8}e^{-\eta} \overset{4.9\leq\eta\leq 5}{\in} [0.0025, 0.0028],$$

$$\begin{aligned} U(b_p|\underline{s};\beta) &= \frac{\rho f_h(s)\pi_h(b_p;\beta)(1-b_p) + (1-\rho)f_\ell(s)\pi_\ell(b_p;\beta)(-b_p)}{\rho f_h(s) + (1-\rho)f_\ell(s)} \\ &= \frac{3\pi_h(b_p;\beta)(1-b_p) + 5\pi_\ell(b_p;\beta)(-b_p)}{8} \\ &= \frac{(e^{-\eta[1-\frac{3}{8}]} - e^{-\eta})(1-b_p) + (e^{-\eta[1-\frac{5}{8}]} - e^{-\eta})(-b_p)}{8\eta} \overset{\eta\leq 5, b_p=0.12}{\geq} 0.0030. \end{aligned}$$

Last, we have to check that a bidder with $s = \frac{1}{2}$ is indifferent between pooling on b_p bidding $b_p + \epsilon$ for ϵ arbitrary small (which wins whenever $s_{(1)} \leq \frac{1}{2}$). Denote the respective winning probabilities by:

$$\pi_h^p := \pi_h(b_p;\beta) = \frac{e^{-\eta[1-\frac{3}{8}]} - e^{-\eta}}{\eta\frac{3}{8}} \qquad \pi_\ell^p := \pi_\ell(b_p;\beta) = \frac{e^{-\eta[1-\frac{5}{8}]} - e^{-\eta}}{\eta\frac{5}{8}}$$
$$\pi_h := \lim_{\epsilon\searrow 0}\pi_h(b_p+\epsilon;\beta) = e^{-\eta[1-\frac{3}{8}]} \qquad \pi_\ell := \lim_{\epsilon\searrow 0}\pi_\ell(b_p+\epsilon;\beta) = e^{-\eta[1-\frac{5}{8}]}.$$

The bidder with signal $\frac{1}{2}$ is indifferent between the pooling bid and bidding marginally more if

$$
\begin{aligned}
U(b_p|\frac{1}{2};\ \beta) &= \lim_{\epsilon \searrow 0} U(b_p+\epsilon|\frac{1}{2};\beta) \\
\iff \frac{\frac{3}{4}\pi_h^p(1-b_p)+\frac{5}{4}\pi_\ell^p(0-b_p)}{\frac{3}{4}+\frac{5}{4}} &= \frac{\frac{3}{4}\pi_h(1-b_p)+\frac{5}{4}\pi_\ell(0-b_p)}{\frac{3}{4}+\frac{5}{4}} \\
\iff b &= \frac{\frac{3}{4}(\pi_h-\pi_h^p)}{\frac{3}{4}(\pi_h-\pi_h^p)+\frac{5}{4}(\pi_\ell-\pi_\ell^p)} \\
&= \frac{3(e^{\eta\frac{3}{8}}-\frac{e^{\eta\frac{3}{8}}-1}{\eta\frac{3}{8}})}{3(e^{\eta\frac{3}{8}}-\frac{e^{\eta\frac{3}{8}}-1}{\eta\frac{3}{8}})+5(e^{\eta\frac{5}{8}}-\frac{e^{\eta\frac{5}{8}}-1}{\eta\frac{5}{8}})}.
\end{aligned}
$$

Setting $b_p = 0.12$ and solving for η gives $\eta \approx 4.98225$. By the observations above, none of the other bidders wants to deviate at this η, either.

Appendix D Derivations for section 4

D.1 Candidate Equilibrium 1:

Suppose to the contrary that β^* is an equilibrium and that $\beta^*(s) = b_p$ for exactly $s \in [\underline{s}, s^*]$. To abbreviate notation, denote winning probability and the probability to win with a bid marginally above b_p by:

$$\pi_\omega := \pi_\omega(b_p;\beta^*) = \frac{e^{-\eta_\omega(1-F_\omega(s^*))}-e^{-\eta_\omega}}{\eta_\omega F_\omega(s^*)} \qquad \pi_\omega^+ := \lim_{\epsilon\searrow 0}\pi_\omega(b_p+\epsilon;\beta^*) = e^{-\eta_\omega(1-F_\omega(s^*))}.$$

Signal s^* does not want to deviate to a bid marginally above b_p if

$$
\begin{aligned}
& U(b_p|s^*;\beta^*) \geq \lim_{\epsilon\searrow 0} U(b_p+\epsilon|s^*;\beta^*) \\
\iff & \frac{\rho\eta_h f_h(s^*)\pi_h(v_h-b_p)+(1-\rho)\eta_\ell f_\ell(s^*)\pi_\ell(v_\ell-b_p)}{\rho\eta_h f_h(s^*)+(1-\rho)\eta_\ell f_\ell(s^*)} \\
& \qquad \geq \frac{\rho\eta_h f_h(s^*)\pi_h^+(v_h-b_p)+(1-\rho)\eta_\ell f_\ell(s^*)\pi_\ell^+(v_\ell-b_p)}{\rho\eta_h f_h(s^*)+(1-\rho)\eta_\ell f_\ell(s^*)} \\
\iff & (b_p-v_\ell)(1-\rho)\eta_\ell f_\ell(s^*)(\pi_\ell^+-\pi_\ell) \geq (v_h-b_p)\rho\eta_h f_h(s^*)(\pi_h^+-\pi_h) \\
\iff & \frac{b_p-v_\ell}{v_h-b_p} \geq \frac{\rho}{1-\rho}\frac{\eta_h f_h(s^*)}{\eta_\ell f_\ell(s^*)}\frac{\pi_h^+-\pi_h}{\pi_\ell^+-\pi_\ell}. \qquad (30)
\end{aligned}
$$

Furthermore, we can approximate $\frac{\pi_h^+-\pi_h}{\pi_\ell^+-\pi_\ell}$ by

$$\frac{\pi_h^+-\pi_h}{\pi_\ell^+-\pi_\ell}\left(\frac{\pi_h^+}{\pi_\ell^+}\right)^{-1}\left(\frac{\pi_h^+}{\pi_\ell^+}\right) = \frac{1-\frac{1-e^{-\eta_h F_h(s^*)}}{\eta_h F_h(s^*)}}{1-\frac{1-e^{-\eta_\ell F_\ell(s^*)}}{\eta_\ell F_\ell(s^*)}}\left(\frac{\pi_h^+}{\pi_\ell^+}\right) = B(\eta_\omega)\frac{\pi_h^+}{\pi_\ell^+}. \qquad (31)$$

where $B(\eta_\omega) = \frac{1-\frac{1-e^{-\eta_h F_h(s^*)}}{\eta_h F_h(s^*)}}{1-\frac{1-e^{-\eta_\ell F_\ell(s^*)}}{\eta_\ell F_\ell(s^*)}} \to 1$, as $\eta_h, \eta_\ell \to \infty$.

Next, individual rationality of signal $\underline{s}$ requires that

$$\mathbb{E}[v|\text{win with } b_p, \underline{s}; \beta] = \frac{\rho\eta_h f_h(\underline{s})\pi_h v_h + (1-\rho)\eta_\ell f_\ell(\underline{s})v_\ell}{\rho\eta_h f_h(\underline{s})\pi_h + (1-\rho)\eta_\ell f_\ell(\underline{s})} \geq b_p$$
$$\iff \frac{b_p - v_\ell}{v_h - b_p} \leq \frac{\rho}{1-\rho}\frac{\eta_h f_h(\underline{s})}{\eta_\ell f_\ell(\underline{s})}\frac{\pi_h}{\pi_\ell}. \tag{32}$$

Inspecting $\frac{\pi_h}{\pi_\ell}$ and $\frac{\pi_h^+}{\pi_\ell^+}$ we note that

$$\frac{\pi_h}{\pi_\ell} = \frac{\pi_h}{\pi_\ell}\Big(\frac{\pi_h^+}{\pi_\ell^+}\Big)^{-1}\frac{\pi_h^+}{\pi_\ell^+} = \frac{1-e^{-\eta_h F_h(s^*)}}{1-e^{-\eta_\ell F_\ell(s^*)}}\frac{\eta_\ell F_\ell(s^*)}{\eta_h F_h(s^*)}\frac{\pi_h^+}{\pi_\ell^+} = \frac{\pi_h^+}{\pi_\ell^+}D(\eta_\omega)\frac{\eta_\ell F_\ell(s^*)}{\eta_h F_h(s^*)} \tag{33}$$

where $D(\eta_\omega) = \frac{1-e^{-\eta_h F_h(s^*)}}{1-e^{-\eta_\ell F_\ell(s^*)}} \to 1$ as $\eta_h, \eta_\ell \to \infty$.

Combining equations (30) - (33) we receive that

$$\frac{\rho}{1-\rho}\frac{\eta_h f_h(s^*)}{\eta_\ell f_\ell(s^*)}\frac{\pi_h^+ - \pi_h}{\pi_\ell^+ - \pi_\ell} \leq \frac{\rho}{1-\rho}\frac{\eta_h f_h(\underline{s})}{\eta_\ell f_\ell(\underline{s})}\frac{\pi_h}{\pi_\ell} \iff \frac{f_h(\underline{s})}{f_\ell(\underline{s})}\frac{F_\ell(s^*)}{F_h(s^*)} \geq \frac{B(\eta_\omega)}{D(\eta_\omega)}.$$

The MLRP implies that $\frac{f_h(\underline{s})}{f_\ell(\underline{s})} < \frac{F_\ell(s^*)}{F_h(s^*)}$ which means that the left side is strictly smaller than 1. The right side, on the other hand, converges to 1, such that the condition cannot hold for η_h, η_ℓ sufficiently large.

D.2 Candidate Equilibrium 2:

Fix the ratio $\frac{\eta_h}{\eta_\ell} = l < \frac{f_\ell(\underline{s})}{f_h(\underline{s})}$. Suppose to the contrary that β^* is an equilibrium in which $\beta^*(s) = b_p$ for exactly $s \in [s_-, s_+]$ and where $s_- \in (\underline{s}, s^*)$. Assume further that $s_+ \leq s^*$. To simplify the following expressions, denote the winning probabilities for bidding b_p and marginally overbidding b_p and underbidding b_p by

$$\pi_\omega := \pi_\omega(b_p; \beta^*) = \frac{e^{-\eta_\omega(1-F_\omega(s_+))} - e^{-\eta_\omega(1-F_\omega(s_-))}}{\eta_\omega(F_\omega(s_+) - F_\omega(s_-))}$$
$$\pi_\omega^- := \lim_{\epsilon\searrow 0}\pi_\omega(b_p - \epsilon; \beta^*) = e^{-\eta_\omega(1-F_\omega(s_-))} \qquad \pi_\omega^+ := \lim_{\epsilon\searrow 0}\pi_\omega(b_p + \epsilon; \beta^*) = e^{-\eta_\omega(1-F_\omega(s_+))}.$$

Individual rationality requires that

$$\mathbb{E}[v|\text{win with } b_p, s_-; \beta^*] = \frac{\rho\eta_h f_h(s_-)\pi_h v_h + (1-\rho)\eta_\ell f_\ell(s_-)\pi_\ell v_\ell}{\rho\eta_h f_h(s_-)\pi_h + (1-\rho)\eta_\ell f_\ell(s_-)\pi_\ell} \geq b_p$$
$$\iff \frac{b_p - v_\ell}{v_h - b_p} \leq \frac{\rho}{1-\rho}\frac{\eta_h f_h(s_-)}{\eta_\ell f_\ell(s_-)}\frac{\pi_h}{\pi_\ell}$$
$$= \frac{\rho}{1-\rho}\frac{\eta_h f_h(s_-)}{\eta_\ell f_\ell(s_-)}\frac{\pi_h^+}{\pi_\ell^+}\frac{\eta_\ell(F_\ell(s_+) - F_\ell(s_-))}{\eta_h(F_h(s_+) - F_h(s_-))}\hat{B}(\eta_\omega) \tag{34}$$

where we, similar to B in the section before define $\hat{B}$ as

$$\hat{B}(\eta_\omega) = \frac{1 - e^{-\eta_h(F_h(s_+)-F_h(s_-))}}{1 - e^{-\eta_\ell(F_\ell(s_+)-F_\ell(s_-))}} \to 1$$

when $\eta_h, \eta_\ell \to \infty$.

Fix any $s < s_-$ now. To make sure that s does not want to deviate from $\beta^*(s)$ to a bid marginally below b_p, Lemma 7 provides a lower bound for b_p which is

$$b_p \geq \frac{\rho f_h(s)\eta_h(\pi_h^- - \pi_h(\beta^*(s);\beta^*))v_h + (1-\rho)f_\ell(s)\eta_\ell(\pi_\ell^- + \frac{\rho\eta_h f_h(s)}{(1-\rho)\eta_\ell f_\ell(s)\eta_\ell}\pi_h(\beta^*(s);\beta^*))v_\ell}{\rho\eta_h f_h(s)(\pi_h^- - \pi_h(\beta^*(s);\beta^*)) + (1-\rho)\eta_\ell f_\ell(s)(\pi_\ell^- + \frac{\rho\eta_h f_h(s)}{(1-\rho)\eta_\ell f_\ell(s)}\pi_h(\beta^*(s);\beta^*))}$$

$$\iff \frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho}\frac{\eta_h f_h(s)}{\eta_\ell f_\ell(s)}\frac{\pi_h^- - \pi_h(\beta^*(s);\beta^*)}{\pi_\ell^- + \frac{\rho\eta_h f_h(s)}{(1-\rho)\eta_\ell f_\ell(s)}\pi_h(\beta^*(s);\beta^*)}. \tag{35}$$

There are two possibilities now. Either $\beta^*(s)$ is a pooling bid or not. In both cases, we can find a $y \in (\underline{s}, s_-)$ such that $\pi_h(\beta^*(s);\beta^*) = e^{-\eta_h(1-F_h(y))}$. One can check that $e^{-\eta_h(1-F_h(y))} \leq \frac{e^{-\eta_h(1-F_h(s_-))} - e^{-\eta_h(1-F_h(s))}}{\eta_h(F_h(s_-)-F_h(s))}$ such that $\frac{\pi_h(\beta^*(s);\beta^*)}{e^{-\eta_h(1-F_h(s_-))}} \to 0$.

There are two possibilities. Either $e^{-\eta_h(1-F_h(y))+\eta_\ell(1-F_\ell(s_-))} \to 0$ (always if $\eta_h = \eta_\ell$), or not.

•If not, i.e. if $e^{-\eta_h(1-F_h(y))+\eta_\ell(1-F_\ell(s_-))} \to \phi > 0$, then

$$\frac{\pi_h^- - \pi_h(\beta^*(s);\beta^*)}{\pi_\ell^- + \frac{\rho\eta_h f_h(s)}{(1-\rho)\eta_\ell f_\ell(s)}\pi_h(\beta^*(s);\beta^*)} = \frac{\frac{\pi_h^-}{e^{-\eta_h(1-F_h(y))}} - 1}{e^{\eta_h(1-F_h(y))-\eta_\ell(1-F_\ell(s_-))} + \frac{\rho\eta_h f_h(s)}{(1-\rho)\eta_\ell f_\ell(s)}} \to \frac{\infty}{\frac{1}{\phi} + l\frac{\rho f_h(s)}{(1-\rho)f_\ell(s)}},$$

which means that $b_n \to v_h$. Given the form of the equilibrium, signal $\bar{s}$ always wins the auction, which means her expected value $\mathbb{E}[v|\beta^*(\bar{s}), \bar{s};\ \beta^*] = \mathbb{E}[v|\bar{s}]$. Because the signals are bounded, this is bounded away from v_h. Since $\beta^*(\bar{s}) > b_p$ signal $\bar{s}$ would make strict loss, which is a contradiction, because she could always deviate to v_ℓ. Thus, we can ignore the case in which $e^{-\eta_h(1-F_h(y))+\eta_\ell(1-F_\ell(s_-))} \to \phi > 0$.

•If not, i.e. if $e^{-\eta_h(1-F_h(y))+\eta_\ell(1-F_\ell(s_-))} \to 0$, we observe that

$$\begin{aligned}\frac{\pi_h^- - \pi_h(\beta^*(s);\beta^*)}{\pi_\ell^- + \frac{\rho\eta_h f_h(s)}{(1-\rho)\eta_\ell f_\ell(s)}\pi_h(\beta^*(s);\beta^*)} &= \frac{\pi_h^- - \pi_h(\beta^*(s);\beta^*)}{\pi_\ell^- + \frac{\rho\eta_h f_h(s)}{(1-\rho)\eta_\ell f_\ell(s)}\pi_h(\beta^*(s);\beta^*)}\left(\frac{\pi_h^-}{\pi_\ell^-}\right)^{-1}\frac{\pi_h^-}{\pi_\ell^-} \\ &= \frac{1 - e^{-\eta_h(F_h(s_-)-F_h(y))}}{1 + \frac{\rho\eta_h f_h(s)}{(1-\rho)\eta_\ell f_\ell(s)}e^{-\eta_h(1-F_h(y))+\eta_\ell(1-F_\ell(s_-))}}\frac{\pi_h^-}{\pi_\ell^-} \\ &= E_s(\eta_\omega)\frac{\pi_h^-}{\pi_\ell^-},\end{aligned}$$

such that we found a function $E_s(\eta_\omega) < 1$ with

$$E_s(\eta_\omega) = \frac{1 - e^{-\eta_h(F_h(s_-)-F_h(y))}}{1 + \frac{\rho\eta_h f_h(s)}{(1-\rho)\eta_\ell f_\ell(s)}e^{-\eta_h(1-F_h(y))+\eta_\ell(1-F_\ell(s_-))}} \to 1.$$

Hence, we can rewrite equation (35) as

$$\frac{b_p - v_\ell}{v_h - b_p} \geq \frac{\rho}{1-\rho} \frac{\eta_h f_h(s)}{\eta_\ell f_\ell(s)} E_s(\eta_\omega) \frac{\pi_h^-}{\pi_\ell^-}.$$

Combining this with equation (34) yields

$$\begin{aligned} \frac{\rho}{1-\rho} \frac{\eta_h f_h(s_-)}{\eta_\ell f_\ell(s_-)} \frac{\pi_h^+}{\pi_\ell^+} \frac{\eta_\ell(F_\ell(s_+) - F_\ell(s_-))}{\eta_h(F_h(s_+) - F_h(s_-))} \hat{B}(\eta_\omega) &\geq \frac{\rho}{1-\rho} \frac{\eta_h f_h(s)}{\eta_\ell f_\ell(s)} E_s(\eta_\omega) \frac{\pi_h^-}{\pi_\ell^-} \\ \iff \frac{f_h(s_-)}{f_\ell(s_-)} \frac{f_\ell(s)}{f_h(s)} \frac{F_\ell(s_+) - F_\ell(s_-)}{F_h(s_+) - F_h(s_-)} &\geq \frac{E_s(\eta_\omega)}{\hat{B}(\eta_\omega)} \frac{\pi_h^-}{\pi_\ell^-} \Big(\frac{\pi_h^+}{\pi_\ell^+} \Big)^{-1}. \end{aligned}$$

The left side is bounded, the first fraction of the right side converges to 1. The second

$$\frac{\pi_h^-}{\pi_\ell^-} \Big(\frac{\pi_h^+}{\pi_\ell^+} \Big)^{-1} = e^{\eta_\ell (F_\ell(s_+) - F_\ell(s_-)) - (\eta_h (F_h(s_+) - F_h(s_-))} \to \infty,$$

because

$$\begin{aligned} \eta_\ell[F_\ell(s_+) - F_\ell(s_-)] - \eta_h[F_h(s_+) - F_h(s_-)] &= \int_{s_-}^{s_+} [1 - \frac{\eta_h f_h(z)}{\eta_\ell f_\ell(z)}] \eta_\ell f_\ell(z) dz \\ &> \underbrace{\eta_\ell}_{\to\infty} \int_{s_-}^{s_+} \underbrace{[1 - \frac{\eta_h f_h(s_+)}{\eta_\ell f_\ell(s_+)}]}_{(1 - l \frac{f_h(s_+)}{f_\ell(s_+)}) > 0, \text{ constant}} f_\ell(z) dz \to \infty. \end{aligned}$$

Hence, equations (34) and (35) cannot hold simultaneously for η_h, η_ℓ large and we found a contradiction.

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