A Neutral Mediator's Favoritism between Symmetric Potential Adversaries*

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Abstract

This paper presents a case in conflict management where a neutral, benevolent mediator should propose a biased peaceful split between two potential adversaries despite that the two are stochastically identical and that the mediator puts equal weight on their welfare. A biased proposal, when rejected, induces asymmetric posterior beliefs conducive to mitigating the detriment of the conflict, modeled as an all-pay auction. This positive effect of induced asymmetry, however, may be offset by the negative effect on the probability of peace settlements, as the paper notes two classes of biased proposals inferior to the unbiased one. We provide an explicit biased proposal that outperforms the unbiased one. This better alternative is so lopsided that the favored player always accepts it, while only the weak type of the opponent may accept it.

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1 Introduction

Would a neutral, benevolent mediator play favor between two potentially conflicting parties? If the two parties are symmetric—their types independently drawn from the same distribution—from the mediator's perspective, the answer would be No in the traditional, static framework of mechanism design, where the mediator is assumed to have full control on the probability distribution of the final outcome. There, a mechanism designer's favoritism arises only when the players, the potentially conflicting parties here, are asymmetric ex ante (c.f. Myerson and Satterthwaite [6, Section 5]). In handling potential conflicts, whereas, a mediator does not have full control of the final outcome lottery. She proposes merely a peace settlement between the two parties, and if it is not mutually accepted by both then the two fight their battle out beyond her control. In such dynamic situations, only indirectly can the mediator influence the outcome: the potential contenders' responses to her proposal affect their posterior beliefs about each other, and these posteriors affect how the two will play in the event of conflict. As conflicts often resemble an all-pay auction, where each side, win or lose, has to bear its own cost of the fight, the two parties may reduce their efforts to hurt each other if the posteriors are asymmetric between them, one believed to be strong with some higher probability than the other, so that the stochastically weak side is intimidated, and the other side complacent. Thus there is a positive effect, from the neutral mediator's perspective, to somehow induce an asymmetric posterior belief system, and to induce such asymmetry through a biased proposal that favors one against the other despite that the two parties are stochastically identical from her standpoint. However, not all biased proposals can induce socially desirable asymmetric posteriors, as the biased proposals may reduce the chance for a peace settlement that preempts the conflict. Is there a biased proposal whose positive effect of posterior asymmetry outweighs the negative effect? If yes, what is it? What are the biased proposals, where the relation between the two effects is reversed, which the mediator should not choose over the unbiased proposal?

To address these questions in a tractable model, this paper considers a two-stage interaction between two players, whose types are each independently drawn from the same binary distribution. The mediator, uninformed of their types, proposes a peaceful split of a prize between the two players. The two players, each privately informed of its own type, simultaneously announce whether to accept or reject the proposal. If it is accepted by both, they split the prize as proposed thereby ending the game; otherwise they play an all-pay auction game to determine who gets the prize. The primitives of the model are so chosen that it is impossible for the mediator to fully preempt conflict: there does not exist any peace proposal that admits a perfect Bayesian equilibrium (PBE) in which conflict occurs with zero probability. The objective for the mediator is to maximize the social surplus, or the sum of the two players' expected payoffs before realization of types, that incorporates both peace settlement and conflict as possible events on path.

Given that the two players are ex ante identical stochastically and that the mediator puts equal weight to both players, the unbiased proposal is the equal split of the prize. To answer the questions posed earlier, we examine the set of all proposal-equilibrium pairs, each consisting of a peace proposal and a PBE of the continuation game given the proposal. Among those that outperform laissez faire, the situation where the mediator, as if absent, makes a proposal that results in conflict for sure, there are only four classes of proposal-equilibrium pairs (Section 3): the lopsided, the hybrid, the mutually totally mixed (MTM) and the mutually partially mixed (MPM). We find that the unbiased proposal is the social-surplus maximum within the last class, MPM (Theorem 1), and generates larger social surplus than any member of the MTMs (Theorem 2). Thus the positive effect of posterior asymmetry induced by biased proposals need not dominate its negative effects. However, when the ex ante probability to be the weak type belongs to an intermediate range, the unbiased proposal is outperformed by a lopsided proposal, so lopsided that the favored player always accepts it, while the opponent rejects it when his type is strong, and mixes between acceptance and rejection when his type is weak (Theorem 4). Furthermore, this better, though biased, proposal is the social-surplus maximum among all lopsided ones, explicitly characterized by Theorem 3. Going through such trouble to find a better alternative than the unbiased proposal is necessary, because we also find that, when the aforementioned ex ante probability of the weak type is sufficiently high, even this optimal lopsided proposal is outperformed by the unbiased one (Theorem 5).

Thus this paper provides an affirmative answer, which might sound morally repugnant, to the question posed at the beginning. Even between ex ante identical potential contestants, a neutral, benevolent mediator sometimes should offer a biased proposal in order to maximize social surplus. Our answer, however, is not unequivocal, as our results imply that the unbiased proposal, even when it is suboptimal, is not that bad. Furthermore, our results tell the mediator which among biased proposals to choose over the unbiased one, and which to not choose.

A mediator's design of peace proposals before a potential conflict between two players has been considered in the static, traditional mechanism design framework by Bester and Wärneryd [2], Compte and Jehiel [3], Fey and Ramsay [4], Hörner, Morelli and Squintani [5] and Spier [7], who assume that in the event of conflict the outcome is determined by an exogenous lottery contingent on the potential contestants' types. Recently, two working papers, Balzer and Schneider [1] and Zheng [8], have extended the frontier to dynamic situations where the outcome in the event of conflict is determined by an all-pay auction game between the two conflicting players. Balzer and Schneider consider the mediator's problem whose objective is to minimize the probability of conflict. Zheng identifies a necessary and sufficient condition, in terms of the primitives, under which the mediator can completely preempt conflict. A question not yet considered in this literature is What the mediator in this dynamic framework should do in order to maximize the social surplus, incorporating both possibilities of peace settlement and conflict. This question is particularly relevant to cases where it is impossible to completely preempt conflict, which are the cases handled in this paper. In such cases, minimizing the probability of conflict need not maximize the social welfare, as it matters how much resource each party will spend in the continuation equilibrium conditional on conflict.

In the following, Section 2 defines the model and presents a preliminary result. Section 3 classifies all the possible proposal-equilibrium pairs. Section 4 shows that two of such classes are outperformed by the unbiased proposal. Section 5 presents the main results on the suboptimality of the unbiased proposal and the better, biased alternative. Proofs are in the appendix, in the order of appearance of the claims.

2 Preliminaries

Two players, named 1 and 2, compete for a prize. Each player's type is independently drawn from the same binary distribution, whose realization is either a, with probability θ, or z with probability $1-\theta$, such that $z > a > 0$ and

$$(1 - a/z)\,\theta > 1/2. \tag{1}$$

First, for each $i \in \{1, 2\}$, player i's type is drawn and privately known to i himself. Second, a neutral mediator proposes to the two players a peaceful split $(s_1, s_2) \in [0, 1]^2$ of the prize, with $s_1 + s_2 = 1$; Third, each player independently and publicly announces whether to accept or reject the split. If both accept the split then the game ends with player i getting a payoff equal to s_i. If at least one of them rejects the split, then the game enters its fourth stage, the *conflict*, where each player i, after observing the rejection-acceptance actions of both, submits a sealed bid $b_i \in \mathbb{R}_+$. The player who submits the higher bid wins the prize, with ties broken randomly with equal probabilities. With $\mathbf{1}_i$ denoting the indicator function of the event that player i wins the prize, i's payoff is equal to $\mathbf{1}_i - b_i/t_i$.

An *equilibrium* given a peaceful split (s_1, s_2) means a perfect Bayesian equilibrium (PBE) of the continuation game given that (s_1, s_2) has been proposed. Ineq. (1) means that the primitives modeled above violate the condition for existence of peace-ensuring mechanisms (Zheng [8]). Thus, no matter what peaceful split the mediator proposes, conflict occurs with strictly positive probability at any equilibrium. Nevertheless, the mediator's proposal determines which equilibrium gets to be played. In particular, the continuation play during conflict depends on the posterior beliefs that is derived through Bayes's rule from the players' mutually best responses to the mediator's proposal. We shall assess the mediator's proposal according to the social surplus, i.e., the sum of the two players' expected payoff before realization of types, generated by the equilibrium given the proposal.

For the rest of the paper, we use the notations

$$r := 1/(1 - a/z), \quad x_i := rs_i. \tag{2}$$

Thus, Ineq. (1) is equivalent to

$$\theta > r/2, \tag{3}$$

and the definition "$(s_1, s_2) \in [0, 1]^2$ and $s_1 + s_2 = 1$" for any peaceful split becomes

$$(x_1, x_2) \in [0, r]^2, \quad x_1 + x_2 = r. \tag{4}$$

We call such (x_1, x_2) *peace proposal*. We also use the standard notation $y^+ := \max\{y, 0\}$.

Lemma 1 *For each $i \in \{1, 2\}$, if π_i is equal to the posterior probability of "i's type is equal to a" conditional on the start of the conflict stage, then the expected payoff for each player i ($i \in \{1, 2\}$) conditional on the start of the conflict stage is equal to* $\max\{\pi_1, \pi_2\}/r$ *if i's type is z, and equal to* $(\pi_{-i} - \pi_i)^+/r$ *if i's type is a.*

From Lemma 1 we can see the positive effect of inducing asymmetry between the posteriors. Say $\pi_1 \geq \pi_2$. When we further enlarge π_1, the expected payoff for the low type of player 1 remains zero in the conflict, while that for the low type of player 2, and those for the high type of both players, get larger. Thus, conditional on the occurrence of conflict, the more disparate are the posteriors the larger is the social surplus. The total effect of such induced asymmetry is of course more complicated, as inducing such asymmetry alters the equilibrium probability with which conflict does not occur.

Consider, as a benchmark, *laissez faire*, where the posteriors are the same as the prior, as if the mediator were absent or made a proposal rejected for sure: $\pi_1 = \pi_2 = \theta$. Then Lemma 1 implies that each player's expected payoff is equal to θ/r if his type is high ($= z$), and equal to zero if his type is low ($= a$), so the social surplus is equal to

$$S_{\mathrm{LF}} := 2\theta(1-\theta)/r. \tag{5}$$

3 All Possible Cases of an Equilibrium

Given any peace proposal (x_1, x_2), for each player i, let $\sigma_i(t_i)$ denote the equilibrium probability with which i with type t_i rejects (x_1, x_2). Further denote q_i^A for his ex ante probability of accepting it, and q_i^R for that of rejecting it. Hence

$$q_i^A = \theta\left(1-\sigma_i(a)\right) + (1-\theta)\left(1-\sigma_i(z)\right), \tag{6}$$

$$q_i^R = \theta\sigma_i(a) + (1-\theta)\sigma_i(z). \tag{7}$$

Let π_i^A be the posterior probability with which $t_i = a$ conditional on i's accepting the proposal, and π_i^R the posterior probability with which $t_i = a$ conditional on i's rejecting it. Thus, if $q_i^A > 0$ then

$$\pi_i^A = \theta\left(1-\sigma_i(a)\right)/q_i^A = \frac{\theta}{\theta + (1-\theta)(1-\sigma_i(z))/(1-\sigma_i(a))}, \tag{8}$$

with the second equality true if $\sigma_i(a) < 1$; likewise, if $q_i^R > 0$ then

$$\pi_i^R = \theta\sigma_i(a)/q_i^R = \frac{\theta}{\theta + (1-\theta)\sigma_i(z)/\sigma_i(a)}, \tag{9}$$

with the second equality true if $\sigma_i(a) > 0$. Let

$$\Delta_i := \begin{bmatrix} \Delta_i(z) \\ \Delta_i(a) \end{bmatrix} := q_{-i}^A \begin{bmatrix} \max\left\{\pi_i^R, \pi_{-i}^A\right\} - x_i \\ \left(\pi_{-i}^A - \pi_i^R\right)^+ - x_i \end{bmatrix} + q_{-i}^R \begin{bmatrix} \max\left\{\pi_i^R, \pi_{-i}^R\right\} - \max\left\{\pi_i^A, \pi_{-i}^R\right\} \\ \left(\pi_{-i}^R - \pi_i^R\right)^+ - \left(\pi_{-i}^R - \pi_i^A\right)^+ \end{bmatrix}. \tag{10}$$

Note from Eqs. (8) and (9) that, for any $i \in \{1,2\}$, if $0 < \sigma_i(a) < 1$ then

$$\pi_i^R < (\text{resp.} \leq)\ \pi_i^A \Longleftrightarrow \sigma_i(z) > (\text{resp.} \geq)\ \sigma_i(a) \Longleftrightarrow \pi_i^R < (\text{resp.} \leq)\ \theta < (\text{resp.} \leq)\ \pi_i^A. \quad (11)$$

Lemma 2 *For any proposal-equilibrium pair $(x_i, \sigma_i)_{i=1}^2$, with the corresponding $\left(q_i^A, q_i^R, \pi_i^A, \pi_i^R\right)_{i=1}^2$ defined by Eqs. (6)–(9), for any $i \in \{1,2\}$ and any $t_i \in \{a, z\}$, Reject is a best response for player i of type t_i if and only if $\Delta_i(t_i) \geq 0$, Accept a best response for i of type t_i if and only if $\Delta_i(t_i) \leq 0$, and*

$$\Delta_i(z) - \Delta_i(a) = \pi_i^R - q_{-i}^R \pi_i^A. \quad (12)$$

Lemma 3 *For any proposal-equilibrium pair $(x_i, \sigma_i)_{i=1}^2$, and any $i \in \{1,2\}$, if player i's strategy σ_i is specified by a row and column in the following table, then the equilibrium has the property in the corresponding cell provided that the cell contains a property.*

	$\sigma_i(z) = 0$	$0 < \sigma_i(z) < 1$	$\sigma_i(z) = 1$
$\sigma_i(a) = 0$		$\sigma_{-i}(a) = \sigma_{-i}(z) = 0$	$\sigma_{-i}(a) = \sigma_{-i}(z) = 0$
$0 < \sigma_i(a) < 1$	*impossible*		
$\sigma_i(a) = 1$	*impossible*	*impossible*	*laissez faire*

By Lemma 3, if $(x_1, x_2; \sigma_1, \sigma_2)$ constitutes a *proposal-equilibrium pair*, with (x_1, x_2) a peace proposal and (σ_1, σ_2) constituting an equilibrium given (x_1, x_2), and if the pair yields larger social surplus than laissez faire, then for any $i \in \{1,2\}$, exactly one of the following alternatives is true:

i. (pure) $\sigma_i(a) = \sigma_i(z) = 0$;

ii. (totally mixed) $0 < \sigma_i(a) < 1$ and $0 < \sigma_i(z) < 1$;

iii. (mixed by a) $0 < \sigma_i(a) < 1$ and $\sigma_i(z) = 1$;

iv. (mixed by z) $\sigma_i(a) = 0$ and $0 < \sigma_i(z) \leq 1$.

That renders up to sixteen possible combinations between σ_1 and σ_2. The next lemma reduces the number to nine.

Lemma 4 *For any proposal-equilibrium pair $(x_i, \sigma_i)_{i=1}^2$ that yields larger social surplus than laissez faire, for any $i \in \{1,2\}$, it is impossible to have $\sigma_i(a) = \sigma_i(z) = 0 = \sigma_{-i}(a)$.*

Corollary 1 *For any proposal-equilibrium pair* $(x_i, \sigma_i)_{i=1}^2$ *that yields larger social surplus than laissez faire, if* $\sigma_i(a) = \sigma_i(z) = 0$*, then* $\sigma_{-i}(z) = 1$ *and* $0 < \sigma_{-i}(a) < 1$*.*

As listed previously, there are at most four alternatives for a player i's strategy in any proposal-equilibrium pair better than laissez faire. Among them, alternative (iv) is ruled out by Corollary 1, because the alternative means $\sigma_i(a) = 0$ and $0 < \sigma_i(z) \leq 1$, which implies, by the table in Lemma 3, that $\sigma_{-i}(a) = \sigma_{-i}(z) = 0$, which combined with $\sigma_i(a) = 0$ is impossible by the corollary applied to player $-i$. Thus, all the possible cases for any equilibrium that generates larger surplus than laissez faire are listed in the following table. Hence we can restrict attention to only four kinds of proposal-equilibrium pairs:

lopsided: for some $i \in \{1, 2\}$, $\sigma_i(a) = \sigma_i(z) = 0$, $\sigma_{-i}(z) = 1$ and $0 < \sigma_{-i}(a) < 1$;

mutually totally mixed (MTM): for any $i \in \{1, 2\}$, $0 < \sigma_i(a) < 1$ and $0 < \sigma_i(z) < 1$;

mutually partially mixed (MPM): for any $i \in \{1, 2\}$, $0 < \sigma_i(a) < 1$ and $\sigma_i(z) = 1$;

hybrid: for some $i \in \{1, 2\}$, $0 < \sigma_i(a) < 1$, $0 < \sigma_i(z) < 1$, $0 < \sigma_{-i}(a) < 1$ and $\sigma_{-i}(z) = 1$.

	$\sigma_{-i}(a) = \sigma_{-i}(z) = 0$	σ_{-i} is totally mixed	σ_{-i} is mixed by a
$\sigma_i(a) = \sigma_i(z) = 0$	impossible	impossible	lopsided, $\sigma_{-i}(z) = 1$
σ_i is totally mixed	impossible	mutually totally mixed	hybrid
σ_i is mixed by a	lopsided, $\sigma_i(z) = 1$	hybrid	mutually partially mixed

4 Proposals Inferior to the Unbiased One

The *unbiased proposal* is to split the prize equally between the players, i.e., to have $x_1 = x_2 = r/2$. This section shows that the unbiased proposal, together with an equilibrium it admits, generates larger social surplus than two classes of peace proposals: those that admit MPM equilibriums, and those that admit MTM ones.

4.1 Mutually Partially Mixed (MPM) Proposal-Equilibrium Pairs

Consider any MPM proposal-equilibrium pair $(x_i, \sigma_i)_{i=1}^2$. As characterized in Section 3, for any $i \in \{1, 2\}$, $0 < \sigma_i(a) < 1$ and $\sigma_i(z) = 1$. Thus, the expected probabilities of

Accept/Reject and the posterior beliefs are

$$\begin{array}{cc} q_i^A = \theta(1-\sigma_i(a)), & q_i^R = \theta\sigma_i(a) + (1-\theta), \\ \pi_i^A = \frac{\theta(1-\sigma_i(a))}{q_i^A} = 1, & \pi_i^R = \frac{\theta\sigma_i(a)}{\theta\sigma_i(a)+(1-\theta)}. \end{array} \tag{13}$$

Note that, for any $i \in \{1, 2\}$,

$$0 < \pi_i^R < \theta, \tag{14}$$

$$\sigma_i(a) \geq \sigma_{-i}(a) \iff \pi_i^R \geq \pi_{-i}^R \iff q_1^A \leq q_2^A. \tag{15}$$

Thus, without loss of generality, let

$$\sigma_1(a) \geq \sigma_2(a), \quad \pi_1^R \geq \pi_2^R. \tag{16}$$

Hence Eq. (10) becomes

$$\begin{aligned} \Delta_1 &= q_2^A \begin{bmatrix} 1-x_1 \\ 1-\pi_1^R - x_1 \end{bmatrix} + q_2^R \begin{bmatrix} \pi_1^R - 1 \\ 0 \end{bmatrix}, \\ \Delta_2 &= q_1^A \begin{bmatrix} 1-x_2 \\ 1-\pi_2^R - x_2 \end{bmatrix} + q_1^R \begin{bmatrix} \pi_1^R - 1 \\ \pi_1^R - \pi_2^R \end{bmatrix}. \end{aligned}$$

By Lemam 2, the necessary and sufficient condition for each σ_i to best reply to σ_{-i} is: $\Delta_i(a) = 0$ and $\Delta_i(z) \geq 0$. The condition $\Delta_i(a) = 0$ for each i is equivalent to

$$\pi_1^R = 1 - x_1, \tag{17}$$

$$\pi_2^R = \frac{\theta(x_1 - x_2) + x_2(1-x_1)}{x_1}. \tag{18}$$

The above equations pin down all MPM proposal-equilibrium pairs:

Lemma 5 *(i) An MPM proposal-equilibrium pair exists if and only if*

$$\theta \geq \frac{3}{4} + \frac{(r-1)^2}{4}. \tag{19}$$

(ii) A peace proposal (x_1, x_2) *admits an MPM equilibrium if and only if*

$$\frac{r + 1 - 2\theta + \sqrt{r^2 - 2r + 4\theta^2 - 8\theta + 5}}{2} \leq x_1 \leq \frac{r}{2}. \tag{20}$$

From this lemma we obtain the social-surplus maximum among all MPM proposal-equilibrium pairs, which is exactly the one corresponding to the unbiased proposal.

Theorem 1 *If Ineq. (19) holds, then the unbiased proposal admits an MPM equilibrium, and the pair generates a social surplus that is equal to θ and is larger than any other MPM proposal-equilibrium pair.*

This theorem also implies that the unbiased proposal also outperforms laissez faire (c.f. Eq. (5)), due to Ineqs. (3) and the fact $r > 1$ by Eq. (2).

4.2 Mutually Totally Mixed (MTM) Proposal-Equilibrium Pairs

Given any peace proposal (x_1, x_2), denote (σ_1, σ_2) for an MTM equilibrium it admits. As characterized in Section 3, for any $i \in \{1, 2\}$,

$$0 < \sigma_i(a) < 1, \quad 0 < \sigma_i(z) < 1. \tag{21}$$

This implies the expected probabilities of Accept/Reject and the posterior beliefs:

$$\begin{aligned} q_i^A &= \theta(1-\sigma_i(a)) + (1-\theta)(1-\sigma_i(z)), \quad & q_i^R &= \theta\sigma_i(a) + (1-\theta)\sigma_i(z), \\ \pi_i^A &= \theta(1-\sigma_i(a))/q_i^A, & \pi_i^R &= \theta\sigma_i(a)/q_i^R. \end{aligned} \tag{22}$$

By (21), the necessary and sufficient condition for each σ_i to best rely σ_{-i} is $\Delta_i(a) = \Delta_i(z) = 0$ (Lemma 2), which, by the identity $\Delta_i(z) - \Delta_i(a) = \pi_i^R - q_{-i}^R \pi_i^A$ (Eq. (12)), is equivalent to the condition that "$\Delta_i(a) = 0$ or $\Delta_i(z) = 0$" and

$$\pi_i^R = q_{-i}^R \pi_i^A \tag{23}$$

for each $i \in \{1, 2\}$. Since $0 < q_i^R < 1$ due to (21), Eq. (23) implies $\pi_i^R < \pi_i^A$, which means

$$\pi_i^R < \theta < \pi_i^A \tag{24}$$

for each i according to (11), Thus, by Eq. (10), the necessary and sufficient condition for (21) to constitute PBE is simultaneous satisfaction of Eq. (23) and

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = q_{-i}^A \begin{bmatrix} \pi_{-i}^A - x_i \\ \pi_{-i}^A - \pi_i^R - x_i \end{bmatrix} + q_{-i}^R \begin{bmatrix} \max\left\{\pi_i^R, \pi_{-i}^R\right\} - \pi_i^A \\ \left(\pi_{-i}^R - \pi_i^R\right)^+ \end{bmatrix}. \tag{25}$$

Relabeling the players if necessary, let

$$\pi_1^R \geq \pi_2^R. \tag{26}$$

Then Eq. (25) is equivalent to

$$x_1 = \pi_2^A - \pi_1^R, \tag{27}$$

$$q_1^A \pi_1^A + q_1^R \pi_1^R = q_1^A x_2 + q_1^R \pi_2^A. \tag{28}$$

Thus—

Lemma 6 *(i) For any (θ, r) with $\theta > r/2$, there exists a peace proposal (x_1, x_2) that admits an MTM equilibrium: pick any $i \in \{1, 2\}$ and let*

$$\frac{r + 2\theta - 1 - \sqrt{(r-1)^2 + 4(\theta - 1)^2}}{2} < x_i < \theta.$$

(ii) The unbiased proposal admits an MTM equilibrium if and only if

$$\theta < \frac{3}{4} + \frac{(r-1)^2}{4}. \tag{29}$$

Based on this lemma, we calculate the social surplus generated by any MTM proposal-equilibrium pair, thereby obtaining:

Theorem 2 *If $\left(2 - r + \sqrt{r^2 - r + 1}\right)/3 \leq \theta$, then any MTM proposal-equilibrium pair generates less social surplus than the unbiased proposal coupled with its MPM equilibrium.*

5 Suboptimality of the Unbiased Proposal

Now that the unbiased proposal outperforms the two classes proposals characterized in the previous section, the only possible alternatives that may outperform the unbiased proposal are either the lopsided or the hybrid proposal-equilibrium pairs. Since a peace proposal may admit multiple equilibriums, to fully assess the performance of the unbiased proposal we need to consider all equilibriums that it admits. Such labor is saved by the next lemma.

Lemma 7 *The unbiased proposal does not admit any hybrid equilibrium.*

Thus, to prove suboptimality of the unbiased proposal, it suffices to prove that the best among lopsided proposal-equilibrium pairs produces larger social surplus than the unbiased proposal when it is coupled with an MPM equilibrium. To that end, we start by characterizing all lopsided equilibriums.

Given any peace proposal (x_1, x_2), let (σ_1, σ_2) denote a lopsided equilibrium. Thus, by Corollary 1, for some $i \in \{1,2\}$, $\sigma_i(a) = \sigma_i(z) = 0$, $\sigma_{-i}(z) = 1$ and $0 < \sigma_{-i}(a) < 1$. By Eqs. (6)–(9),

$$q_i^A = 1, q_i^R = 0, \pi_i^A = \theta;$$
$$q_{-i}^A = \theta(1 - \sigma_{-i}(a)), q_{-i}^R = \theta\sigma_{-i}(a) + 1 - \theta;$$
$$\pi_{-i}^A = 1, \pi_{-i}^R = \theta/(\theta + (1-\theta)/\sigma_{-i}(a)) < \theta;$$

where the off-path posterior π_i^R is undetermined. By Lemma 2, (σ_1, σ_2) constitutes an equilibrium if and only if $\Delta_i \leqq [0,0]$, $\Delta_{-i}(z) \geq 0$ and $\Delta_{-i}(a) = 0$. By Eq. (12),

$$\begin{aligned} \Delta_i &= q_{-i}^A \begin{bmatrix} 1 - x_i \\ 1 - \pi_i^R - x_i \end{bmatrix} + q_{-i}^R \begin{bmatrix} \max\left\{\pi_i^R, \pi_{-i}^R\right\} - \theta \\ \left(\pi_{-i}^R - \pi_i^R\right)^+ \end{bmatrix}, \\ \Delta_{-i} &= \begin{bmatrix} \max\left\{\pi_{-i}^R \pi_i^A\right\} - x_{-i} \\ \left(\pi_i^A - \pi_{-i}^R\right)^+ - x_{-i} \end{bmatrix} = \begin{bmatrix} \theta - x_{-i} \\ \theta - \pi_{-i}^R - x_{-i} \end{bmatrix}. \end{aligned}$$

Thus, the condition "$\Delta_{-i}(z) \geq 0$ and $\Delta_{-i}(a) = 0$" is equivalent to $\Delta_{-i}(a) = 0$, i.e.,

$$\pi_{-i}^R = \theta - x_{-i}. \tag{30}$$

Eq. (30) determines the equilibrium uniquely modulo the off-path posterior π_i^R:

$$\begin{aligned} \sigma_{-i}(a) &= \frac{1 - x_{-i}/\theta}{1 + x_{-i}/(1-\theta)}, \\ q_{-i}^A &= \frac{x_{-i}}{1 - \theta + x_{-i}}, \\ q_{-i}^R &= \frac{1-\theta}{1 - \theta + x_{-i}}. \end{aligned}$$

Thus, given any peace proposal (x_1, x_2), (σ_1, σ_2) constitutes a lopsided equilibrium if and only if $\Delta_i \leqq [0,0]$, i.e., the following inequalities are satisfied:

$$\begin{aligned} q_{-i}^A (1 - x_i) + q_{-i}^R \left(\max\left\{\pi_i^R, \pi_{-i}^R\right\} - \theta\right) &\leq 0, \\ q_{-i}^A \left(1 - \pi_i^R - x_i\right) + q_{-i}^R \left(\pi_{-i}^R - \pi_i^R\right)^+ &\leq 0, \end{aligned}$$

where π_{-i}^R, $\sigma_{-i}(a)$, q_{-i}^A and q_{-i}^R take the values determined by Eq. (30). With these values plugged in and the fact that $1 - \theta + x_{-i} > 0$, the above inequalities are equivalent to

$$x_{-i}(1 - x_i) + (1-\theta)\left(\max\left\{\pi_i^R, \theta - x_{-i}\right\} - \theta\right) \leq 0, \tag{31}$$

$$x_{-i}\left(1 - \pi_i^R - x_i\right) + (1-\theta)\left(\theta - x_{-i} - \pi_i^R\right)^+ \leq 0. \tag{32}$$

Then we obtain—

Lemma 8 *A peace proposal* (x_1, x_2) *admits a lopsided equilibrium if and only if, for some* $i \in \{1, 2\}$,

$$0 < x_{-i} < \theta \tag{33}$$

and at least one of the following conditions holds:

$$x_i \geq \max\{\theta, (r + 1 - \theta)/2\}, \tag{34}$$

$$(1 - x_i)(1 - \theta + r - x_i) \leq \theta(1 - \theta) \quad \textit{and} \quad x_i \geq \theta. \tag{35}$$

While the constraint (33) is a pair of strict inequalities, they are satisfied by the social-surplus maximum subject to the other constraint listed in the lemma. Thus, we characterize the optimum among lopsided proposal-equilibrium pairs:

Theorem 3 *There exists a social-surplus maximum among all lopsided proposal-equilibrium pairs, and the peace proposal* (x_1, x_2) *of this maximum, is defined by*

$$x_{-i} = \begin{cases} \frac{1}{2}\left(r + \theta - 2 + \sqrt{r^2 - 2r\theta + 4\theta - 3\theta^2}\right) & \text{if } r \geq 3\theta - 1 \\ r - \theta & \text{if } r \leq 3\theta - 1, \end{cases} \tag{36}$$

where $-i$ *can be either 1 or 2.*

Theorem 4 *If Ineq. (19) and* $\theta \leq (r + 1)/3$ *are true, then the social-surplus maximum among lopsided mechanism-equilibrium pairs generates larger social surplus than the unbiased proposal does for any equilibrium that the latter admits.*

We need to go through the trouble of beating the unbiased proposal with the social surplus maximum among lopsided proposal-equilibrium pairs because even such maximum can be inferior to the unbiased proposal given some parameter values:

Theorem 5 *If* $\theta > 2(1+r)/5$, *then the unbiased proposal, coupled with its MPM equilibrium, generates larger social surplus than any lopsided proposal-equilibrium pair.*

6 Conclusion

The notion of "honest broker," in the news and in common sense, is usually identified with a neutral mediator proposing an unbiased deal to the conflicting parties. The results presented above shatters this notion. Theorem 4 points out a case where a neutral, benevolent mediator should propose a biased deal to two potential contestants in order to maximize their total surplus, despite the fact that the two are ex ante identical. In addition, our results go beyond this punchline. Theorem 3 provides an explicit lopsided proposal that outperforms the unbiased one. Theorem 5 furthermore cautions that such lopsided proposals are outperformed by the unbiased one when the ex ante probability of being the weak type is sufficiently high. These results motivate broader investigations. What is the exact condition under which a neutral mediator should exercise favoritism? What are other situations that also see an honest broker's biased deal? These are interesting questions for further research.

A Proofs

A.1 Proof of Lemma 1

For each $i \in \{1,2\}$, let H_i denote the c.d.f. of contestant i's bid at the Bayesian Nash equilibrium (BNE) of the contest game. By the characterization in Zheng [8],

$$H_i'(b) = \begin{cases} 1/z & \text{if } H_{-i}(b) > \pi_{-i} \\ 1/a & \text{if } H_{-i}(b) < \pi_{-i}. \end{cases}$$

Without loss of generality, let $\pi_1 \leq \pi_2$. Since the zero bid cannot be an atom for both players at equilibrium, $H_1(0) = 0 \leq H_2(0)$, and

$$\begin{aligned} H_1(b) &= \begin{cases} b/a & \text{if } H_2(b) \leq \pi_2 \\ H_2^{-1}(\pi_2)/a + \left(b - H_2^{-1}(\pi_2)\right)/z & \text{if } H_2(b) \geq \pi_2 \end{cases} \\ H_2(b) &= \begin{cases} H_2(0) + b/a & \text{if } H_1(b) \leq \pi_1 \\ H_2(0) + H_1^{-1}(\pi_1)/a + \left(b - H_1^{-1}(\pi_1)\right)/z & \text{if } H_1(b) \geq \pi_1 \end{cases} \end{aligned}$$

Inspecting the graphs of H_1 and H_2, we have $H_1(b) = b/a$ when $H_1(b) = \pi_1$; hence

$$H_1^{-1}(\pi_1) = a\pi_1.$$

The same inspection also gives the fact that, when $H_2(b) = \pi_2$,

$$\begin{aligned} H_2(b) &= H_2(0) + H_1^{-1}(\pi_1)/a + \left(b - H_1^{-1}(\pi_1)\right)/z \\ &= H_2(0) + (1 - a/z)\pi_1 + b/z, \end{aligned}$$

with the second equality due to $H_1^{-1}(\pi_1) = a\pi_1$. Thus,

$$H_2^{-1}(\pi_2) = (\pi_2 - (1 - a/z)\pi_1 - H_2(0))\, z.$$

To find out $H_2(0)$, let $\bar{b}$ be the supremum of the common support of H_1 and H_2. Since

$$1 = H_2(\bar{b}) = H_2(0) + H_1^{-1}(\pi_1)/a + \left(\bar{b} - H_1^{-1}(\pi_1)\right)/z = H_2(0) + \bar{b}/z + (1 - a/z)\pi_1,$$

we have $\bar{b}/z = 1 - H_2(0) - (1 - a/z)\pi_1$. Plugging this into the fact $1 = H_1(\bar{b})$, we have

$$\begin{aligned} 1 = H_1(\bar{b}) &= H_2^{-1}(\pi_2)/a + \left(\bar{b} - H_2^{-1}(\pi_2)\right)/z \\ &= \bar{b}/z + (z/a - 1)\left(\pi_2 - (1 - a/z)\pi_1 - H_2(0)\right) \\ &= 1 - H_2(0) - (1 - a/z)\pi_1 + (z/a - 1)\left(\pi_2 - (1 - a/z)\pi_1 - H_2(0)\right). \end{aligned}$$

Thus,

$$\begin{aligned} H_2(0) &= (1 - a/z)\left(\pi_2 - \pi_1\right), \\ \bar{b}/z &= 1 - (1 - a/z)\pi_2. \end{aligned}$$

Thus, the expected payoffs $U_i(t_i)$ at this equilibrium are, for each $i \in \{1, 2\}$,

$$\begin{aligned} U_i(z) &= 1 - \bar{b}/z = (1 - a/z)\pi_2, \\ U_1(a) &= H_2(0) = (1 - a/z)\left(\pi_2 - \pi_1\right), \\ U_2(a) &= 0. \end{aligned}$$

One readily generalizes the above to the conclusion of the lemma.

A.2 Lemmas 2, 3 and 4 and Corollary 1

Proof of Lemma 2 By Lemma 1, the expected payoff for player i of type z to play Reject is equal to

$$q_{-i}^A(1 - a/z)\max\left\{\pi_i^R, \pi_{-i}^A\right\} + q_{-i}^R(1 - a/z)\max\left\{\pi_i^R, \pi_{-i}^R\right\},$$

and that for it to play Accept is equal to

$$q_{-i}^A x_i + q_{-i}^R(1 - a/z)\max\left\{\pi_i^A, \pi_{-i}^R\right\}.$$

The difference between these two displayed expressions is equal to $\Delta_i(z)(1 - a/z)$ by the notation x_i defined in (2). Analogously, the payoff difference between Reject and Accept for player i of type a is equal to $\Delta_i(a)(1-a/z)$. To prove Eq. (12), use Eq. (10) and the fact $\max\{x,y\}-(y-x)^+ = x$ (for any $x,y\in\mathbb{R}$) to obtain

$$\begin{aligned}\Delta_i(z)-\Delta_i(a) &= q^A_{-i}\left(\max\left\{\pi^R_i,\pi^A_{-i}\right\}-\left(\pi^A_{-i}-\pi^R_i\right)^+\right)\\ &\quad +q^R_{-i}\left(\max\left\{\pi^R_i,\pi^R_{-i}\right\}-\left(\pi^R_{-i}-\pi^R_i\right)^+-\max\left\{\pi^A_i,\pi^R_{-i}\right\}+\left(\pi^R_{-i}-\pi^A_i\right)^+\right)\\ &= q^A_{-i}\pi^R_i+q^R_{-i}\left(\pi^R_i-\pi^A_i\right),\end{aligned}$$

which equals $\pi^R_i - q^R_{-i}\pi^A_i$ since $q^A_{-i}+q^R_{-i}=1$. ■

Lemma 3 First, suppose that $\sigma_i(a)=0$ and $0<\sigma_i(z)\le 1$, the case corresponding to the first row and the second and third columns in the table. Then $\Delta_i(a)\le 0$ and $\Delta_i(z)\ge 0$ by Lemma 2, and $\pi^R_i=0$ and $\pi^A_i>\theta$ by Eqs. (8) and (9). Thus,

$$0\le \Delta_i(z)-\Delta_i(a)\overset{(12)}{=}\pi^R_i-q^R_{-i}\pi^A_i=-q^R_{-i}\pi^A_i.$$

Hence $0\ge q^R_{-i}\pi^A_i$. This, with $\pi^A_i>\theta>0$, implies $q^R_{-i}=0$, i.e., $\sigma_{-i}(z)=\sigma_{-i}(a)=0$, as asserted in the cells.

Second, suppose $0<\sigma_i(a)<1$ and $\sigma_i(z)=0$. Then $\Delta_i(a)=0$ and $\Delta_i(z)\le 0$ by Lemma 2, $\pi^R_i=1$ by definition, and $\pi^A_i=\theta/((\theta+(1+\theta)/(1-\sigma_i(a)))<\theta$ by Eq. (8). Thus,

$$0\ge \Delta_i(z)-\Delta_i(a)\overset{(12)}{=}\pi^R_i-q^R_{-i}\pi^A_i=1-q^R_{-i}\pi^A_i>0,$$

with the last inequality due to $\pi^A_i<\theta<1$. The contradiction displayed above implies this case impossible, as asserted in the cell.

Third, suppose $\sigma_i(a)=1$ and $0\le\sigma_i(z)<1$, which corresponds to the cells of the third row and the first and second columns. Then $\Delta_i(a)\ge 0$ and $\Delta_i(z)\le 0$ by Lemma 2, $\pi^A_i=0$ by definition, and $\pi^R_i=\theta/(\theta+(1-\theta)\sigma_i(z))>\theta$ by Eq. (9). Thus,

$$0\ge \Delta_i(z)-\Delta_i(a)\overset{(12)}{=}\pi^R_i-q^R_{-i}\pi^A_i=\pi^R_i>\theta>0,$$

contradiction. Hence this case is impossible, as asserted in the cells.

Finally, consider the case $\sigma_i(a)=\sigma_i(z)=1$, the cell of Row Three and Column Three. Then $q^A_i=0$, $q^R_i=1$ and $\pi^R_i=\theta$ by definition. Apply Eq. (10) to the opponent $-i$ to obtain

$$\begin{bmatrix}\Delta_{-i}(z)\\ \Delta_{-i}(a)\end{bmatrix}=\begin{bmatrix}\max\left\{\pi^R_{-i},\theta\right\}-\max\left\{\pi^A_{-i},\theta\right\}\\ \left(\theta-\pi^R_{-i}\right)^+-\left(\theta-\pi^A_{-i}\right)^+\end{bmatrix}. \tag{37}$$

We claim that the posterior probability π_{-i} with which player $-i$'s type equals a is the same as the prior: $\pi_{-i}=\theta$. Suppose otherwise. We derive a contradiction for all possibilities:

1. $\sigma_{-i}(a) = 0$. Then $\sigma_{-i}(z) > 0$, otherwise the claim $\pi_{-i} = \theta$ is true. Thus, $\pi^A_{-i} > \theta$ by Eq. (8) applied to $-i$, and $\Delta_{-i}(z) \geq 0$ by Lemma 2. Then Eq. (37) implies $\pi^R_{-i} \geq \pi^A_{-i} > \theta$. But since $\sigma_{-i}(a) = 0$ and $\sigma_{-i}(z) > 0$, $\pi^R_{-i} = 0$ by Bayes's rule: contradiction.

2. $\sigma_{-i}(a) = 1$. Then $\sigma_{-i}(z) < 1$, otherwise the claim $\pi_{-i} = \theta$ is true. Thus, $\pi^R_{-i} > \theta$ by Eq. (9) applied to $-i$, and $\Delta_{-i}(a) \geq 0$ by Lemma 2. Then Eq. (37) implies $\pi^A_{-i} \geq \theta$. But since $\sigma_{-i}(a) = 1$ and $\sigma_{-i}(z) < 1$, $\pi^A_{-i} = 0$ by Bayes's rule: contradiction.

3. $0 < \sigma_{-i}(a) < 1$. Then Eq. (11) is applicable to player $-i$. Thus, either $\pi^R_{-i} < \theta < \pi^A_{-i}$ or $\pi^R_{-i} > \theta > \pi^A_{-i}$. Suppose $\pi^R_{-i} < \theta < \pi^A_{-i}$. Then Eq. (37) implies $\Delta_{-i}(z) < 0$ and $\Delta_{-i}(a) > 0$; hence $\sigma_{-i}(a) = 1$ and $\sigma_{-i}(z) = 0$ (Lemma 2). But that is impossible according to the proved assertion in the cell of Row 3 and Column 1, with $-i$ playing the role of i in the table. Thus consider the only possibility, $\pi^R_{-i} > \theta > \pi^A_{-i}$. Then Eq. (37) implies $\Delta_{-i}(z) > 0$ and $\Delta_{-i}(a) < 0$; hence $\sigma_{-i}(a) = 0$ and $\sigma_{-i}(z) = 1$ (Lemma 2). But that implies, according to the proved assertion in the cell of Row 1 and Column 3, that $\sigma_i(a) = \sigma_i(z) = 0$, contradicting the condition $0 < \sigma_{-i}(a)$ assumed throughout this subcase.

All possible cases considered, we have derived a contradiction. Thus, the claim $\pi_{-i} = \theta$ is true. It follows that, in the conflict stage, which occurs of sure because $\sigma_i(a) = \sigma_i(z) = 1$, the posteriors are $\pi_i = \pi_{-i} = \theta$. Then Lemma 1 implies that each player's expected payoff is equal to θ/r if his type is high, and equal to zero if his type is low, hence the social surplus is equal to the laissez faire level S_{LF} in Eq. (5), as asserted in the last cell of the table. ∎

Proof of Lemma 4 Since $\sigma_i(a) = \sigma_i(z) = 0 = \sigma_{-i}(a)$, we have $\sigma_{-i}(z) > 0$ because conflict occurs with strictly positive probability due to Ineq. (1). Thus $q^R_{-i} > 0$ and $q^A_{-i} > 0$, and by Bayes's rule, $\pi^R_{-i} = 0$ and $\pi^A_{-i} > \theta$. With $\sigma_i(a) = \sigma_i(z) = 0$, we have $q^R_i = 0$, $q^A_i = 1$ and, by Bayes's rule, $\pi^A_i = \theta$. By Lemma 2, for this (σ_1, σ_2) to constitute an equilibrium, the necessary and sufficient condition is that $\Delta_i(z) \leq 0$, $\Delta_i(a) \leq 0$ and $\Delta_{-i}(z) \geq 0 \geq \Delta_{-i}(a)$. By Eq. (10) applied to player $-i$,

$$\begin{bmatrix} \Delta_{-i}(z) \\ \Delta_{-i}(a) \end{bmatrix} = \begin{bmatrix} \max\left\{\pi^R_{-i}, \pi^A_i\right\} - x_{-i} \\ \left(\pi^A_i - \pi^R_{-i}\right)^+ - x_{-i} \end{bmatrix} = \begin{bmatrix} \max\left\{0, \theta\right\} - x_{-i} \\ (\theta - 0)^+ - x_{-i} \end{bmatrix} = \begin{bmatrix} \theta - x_{-i} \\ \theta - x_{-i} \end{bmatrix}.$$

Thus, $\Delta_{-i}(z) \geq 0 \geq \Delta_{-i}(a)$ implies $\theta - x_{-i} = 0$, which, by Eq. (4), implies

$$x_i = r - \theta. \tag{38}$$

By Eq. (10) applied to player i,

$$\begin{aligned}\Delta_i &= q^A_{-i}\begin{bmatrix}\max\left\{\pi^R_i,\pi^A_{-i}\right\}-x_i\\ \left(\pi^A_{-i}-\pi^R_i\right)^+-x_i\end{bmatrix}+q^R_{-i}\begin{bmatrix}\max\left\{\pi^R_i,0\right\}-\max\left\{\theta,0\right\}\\ \left(0-\pi^R_i\right)^+-\left(0-\theta\right)^+\end{bmatrix}\\ &= q^A_{-i}\begin{bmatrix}\max\left\{\pi^R_i,\pi^A_{-i}\right\}-x_i\\ \left(\pi^A_{-i}-\pi^R_i\right)^+-x_i\end{bmatrix}+q^R_{-i}\begin{bmatrix}\pi^R_i-\theta\\ 0\end{bmatrix}.\end{aligned}$$

Thus, the equilibrium conditions $\Delta_i(z)\leq 0$ and $\Delta_i(a)\leq 0$ are equivalent to, respectively,

$$x_i \geq \max\left\{\pi^R_i,\pi^A_{-i}\right\}+\frac{q^R_{-i}}{q^A_{-i}}\left(\pi^R_i-\theta\right), \tag{39}$$

$$x_i \geq \left(\pi^A_{-i}-\pi^R_i\right)^+. \tag{40}$$

Eq. (38) coupled with Ineq. (1) implies $x_i<\theta$; on the other hand, the right-hand side of (39) is greater than $\theta+\frac{q^R_{-i}}{q^A_{-i}}\left(\pi^R_i-\theta\right)$, as $\pi^A_{-i}>\theta$. Thus, Ineq. (39) implies $\pi^R_i<\theta$. Hence Ineqs. (39) and (40) become

$$\begin{aligned}x_i &\geq \pi^A_{-i}+\frac{q^R_{-i}}{q^A_{-i}}\left(\pi^R_i-\theta\right),\\ x_i &\geq \pi^A_{-i}-\pi^R_i.\end{aligned}$$

Together the two inequalities imply that

$$\pi^A_{-i}-x_i\leq\pi^R_i\leq\theta-\frac{q^A_{-i}}{q^R_{-i}}\left(\pi^A_{-i}-x_i\right),$$

which in turn implies $\pi^A_{-i}-x_i\leq\theta-\frac{q^A_{-i}}{q^R_{-i}}\left(\pi^A_{-i}-x_i\right)$, i.e.,

$$\left(1+\frac{q^A_{-i}}{q^R_{-i}}\right)\left(\pi^A_{-i}-x_i\right)\leq\theta,$$

i.e., $\frac{1}{q^R_{-i}}\left(\pi^A_{-i}-x_i\right)\leq\theta$. This, combined with the fact $x_i<\theta$, implies $\frac{1}{q^R_{-i}}\left(\pi^A_{-i}-\theta\right)<\theta$, i.e.,

$$\frac{1}{(1-\theta)\sigma_{-i}(z)}\left(\frac{\theta}{\theta+(1-\theta)(1-\sigma_{-i}(z))}-\theta\right)<\theta,$$

i.e.,

$$\frac{1}{\theta+(1-\theta)(1-\sigma_{-i}(z))}<1,$$

which is impossible because the denominator on the left-hand side is less than one. ■

Proof of Corollary 1 Since $\sigma_i(a) = \sigma_i(z) = 0$, $q_i^R = 0$, and Lemma 4 implies $\sigma_{-i}(a) > 0$. Hence $\pi_{-i}^R > 0$ by Bayes's rule, and $\Delta_{-i}(a) \geq 0$ by Lemma 2. Thus, Eq. (12) coupled with $q_i^R = 0$ implies

$$\Delta_{-i}(z) \geq \Delta_{-i}(z) - \Delta_{-i}(a) = \pi_{-i}^R - q_i^R \pi_{-i}^A = \pi_{-i}^R > 0.$$

Then Lemma 2 implies $\sigma_{-i}(z) = 1$. This in turn implies $\sigma_{-i}(a) < 1$, as the equilibrium is by hypothesis better than laissez faire. Thus, $\sigma_{-i}(z) = 1$ and $0 < \sigma_{-i}(a) < 1$, as asserted. ■

A.3 Lemma 5 and Theorem 1

Proof of Lemma 5 Combining Eqs. (13), (17) and (18), we have

$$\sigma_1(a) = \frac{1-\theta}{\theta}\frac{1-x_1}{x_1}, \tag{41}$$

$$\sigma_2(a) = \frac{1-\theta}{\theta}\frac{\theta(x_1-x_2)+x_2(1-x_1)}{x_1-\theta(x_1-x_2)-x_2(1-x_1)}, \tag{42}$$

which are equivalently to

$$q_1^A = \frac{x_1+\theta-1}{x_1}, \tag{43}$$

$$q_2^A = \frac{x_2(\theta+x_1-1)}{x_1-(\theta(x_1-x_2)+x_2(1-x_1))}, \tag{44}$$

and $q_i^R = 1 - q_i^A$ for each i. Thus, given peace proposal (x_1, x_2), (σ_1, σ_2) with $\sigma_1(a) \geq \sigma_2(a)$ constitutes an MPM equilibrium if and only if $\Delta_i(z) \geq 0$ for each $i \in \{1, 2\}$, with the values of π_i^R, q_i^A and q_i^R satisfying (14), (17), (18), (43) and (44). Combined with Eq. (17), the conditions $\Delta_1(z) \geq 0$ and $\Delta_2(z) \geq 0$ are equivalent to, respectively,

$$q_2^A \geq x_1, \tag{45}$$

$$q_1^A(1-x_2) \geq (1-q_1^A)x_1. \tag{46}$$

By Eqs. (17) and (18), the condition (14) is equivalent to

$$0 < 1 - x_1 < \theta, \tag{47}$$

$$0 < \tfrac{\theta(x_1-x_2)+x_2(1-x_1)}{x_1} < \theta. \tag{48}$$

By Eqs. (15), (17) and (18),

$$\begin{aligned}\sigma_1(a) \geq \sigma_2(a) &\iff \pi_1^R \geq \pi_2^R \\ &\iff 1 - x_1 \geq \frac{\theta(x_1 - x_2) + x_2(1 - x_1)}{x_1} \\ &\iff (x_1 - x_2)(1 - \theta - x_1) \geq 0 \\ &\iff x_1 \leq x_2,\end{aligned}$$

with the last line due to $1 - \theta - x_1 = \pi_1^R - \theta < 0$ by (47). Thus, Ineq. (16), which we assume without loss of generality, is equivalent to

$$x_1 \leq x_2. \tag{49}$$

We have thus proved that, given any peace proposal $(x_i)_{i=1}^2$, there is at most one MPM equilibrium, and such an equilibrium exists if and only if, with the players labeled so that $\sigma_1(a) \geq \sigma_2(a)$, the following conditions are simultaneously satisfied: (43), (44), (45), (46), (47), (48) and (49). Among them, (45) is redundant because it is implied by (46) and (49):

$$q_1^A(1 - x_1) \overset{(49)}{\geq} q_1^A(1 - x_2) \overset{(46)}{\geq} \left(1 - q_1^A\right) x_1 \Longrightarrow q_1^A \geq x_1 \overset{(15)}{\Longrightarrow} q_2^A \geq q_1^A \geq x_1.$$

Consequently, Eq. (44), called upon only by (45), is also redundant. Among the remaining conditions, (48) is redundant because it is implied by (47) and (49): By (47) and (49), $x_2 \geq x_1 > 1 - \theta > 0$ and $1 - x_1 > 0$, hence $\theta(x_1 - x_2) + x_2(1 - x_1) > 0$; and

$$\frac{\theta(x_1 - x_2) + x_2(1 - x_1)}{x_1} < \theta \iff x_2(1 - \theta) - x_1 x_2 < 0,$$

which is true by (47) and $x_2 > 0$. By the identity $x_2 = r - x_1$, (49) is equivalent to

$$x_1 \leq \frac{r}{2}. \tag{50}$$

Furthermore, the part "$0 < 1 - x_1$" in (47) is also redundant, as it is implied by (50) and the fact $r < 2$ (by definition of r). Thus, the necessary and sufficient condition for existence of an MPM equilibrium becomes simultaneous satisfaction of (43), (46), (50) and

$$1 - x_1 < \theta. \tag{51}$$

Combined with the identity $x_2 = r - x_1$ and Eq. (43), Ineq. (46) is equivalent to

$$(\theta + x_1 - 1)(1 - r + x_1) \geq x_1(1 - \theta),$$

which is equivalent to

$$(x_1)^2 + x_1(2\theta - r - 1) + (\theta - 1)(1 - r) \geq 0,$$

i.e., either

$$x_1 \geq \frac{r + 1 - 2\theta + \sqrt{r^2 - 2r + 4\theta^2 - 8\theta + 5}}{2} \tag{52}$$

or

$$x_1 \leq \frac{r + 1 - 2\theta - \sqrt{r^2 - 2r + 4\theta^2 - 8\theta + 5}}{2}. \tag{53}$$

We claim that Ineq. (53) is impossible. To prove the claim, note from (46) and (51) that

$$q_1^A(1 - x_2) > 0 \Longrightarrow 1 - x_2 > 0 \Longrightarrow 1 - r + x_1 > 0 \Longrightarrow x_1 > r - 1.$$

Thus, Ineq. (53), if true, would imply

$$r - 1 < \frac{r + 1 - 2\theta - \sqrt{r^2 - 2r + 4\theta^2 - 8\theta + 5}}{2},$$

i.e.,

$$-\sqrt{r^2 - 2r + 4\theta^2 - 8\theta + 5} \geq r + 2\theta - 3,$$

which is impossible: if $r + 2\theta - 3 > 0$, the contradiction is obvious; if $r + 2\theta - 3 \leq 0$, then

$$\begin{aligned} & -\sqrt{r^2 - 2r + 4\theta^2 - 8\theta + 5} \geq r + 2\theta - 3 \\ \Longleftrightarrow \quad & \sqrt{r^2 - 2r + 4\theta^2 - 8\theta + 5} \leq |r + 2\theta - 3| \\ \Longleftrightarrow \quad & (1 - \theta)(r - 1) \leq 0, \end{aligned}$$

which is false because $r - 1 > 0$ by definition of r. Thus, only (52) is the equivalence of Ineq. (46). Therefore, the necessary and sufficient condition for existence of an MPM equilibrium becomes simultaneous satisfaction of (50), (51) and (52). Among them, (51) is redundant because it is implied by (52):

$$x_1 + \theta \overset{(52)}{>} \frac{r + 1 - 2\theta}{2} + \theta = \frac{r + 1}{2} > 1,$$

with the last inequality due to the fact $r > 1$ from the definition of r. Thus, we have reduced the necessary and sufficient condition to simultaneous satisfaction of (50) and (52), i.e., Ineq. (20), as asserted by Part (ii) of the lemma. Obviously an x_1 that satisfies (20) exists if and only if

$$\frac{r + 1 - 2\theta + \sqrt{r^2 - 2r + 4\theta^2 - 8\theta + 5}}{2} \leq \frac{r}{2},$$

i.e., Ineq. (19), as asserted by Part (i) of the lemma. ■

Proof of Theorem 1 Denote $S_{\text{part}}(x_1)$ for the social surplus generated by the MPM equilibrium given peace proposal (x_1, x_2) such that, with players relabeled if necessary, $x_1 \leq x_2$. At this equilibrium, Reject is a best response for each type of each player, hence each player's surplus is equal to his expected payoff from Reject. Thus, by Lemma 1,

$$\begin{aligned} rS_{\text{part}}(x_1) &= \theta q_2^A(1-\pi_1^R) + (1-\theta)\left(q_2^A + q_2^R\pi_1^R\right) \\ &\quad +\theta\left(q_1^A(1-\pi_2^R) + q_1^R(\pi_1^R - \pi_2^R)\right) + (1-\theta)\left(q_1^A + q_1^R\pi_1^R\right) \\ &= q_2^A(1-\theta\pi_1^R) + q_1^A(1-\theta\pi_2^R) + q_2^R(1-\theta)\pi_1^R + q_1^R(\pi_1^R - \theta\pi_2^R) \\ &= q_2^A - \theta\pi_1^R + q_1^A - \theta\pi_2^R + q_2^R\pi_1^R + q_1^R\pi_1^R \\ &= \theta(1-\sigma_2(a)) - \theta(1-x_1) + \theta(1-\sigma_1(a)) - \theta\pi_2^R + q_2^R(1-x_1) + \theta\sigma_1(a) \\ &= \theta + \theta(1-\sigma_2(a)) - \theta(1-x_1) - \theta\pi_2^R + q_2^R(1-x_1), \end{aligned} \tag{54}$$

with the second last line due to Eqs. (13) and (17). By Lemma 5, the maximum of the range of x_1 for such equilibriums is equal to $r/2$, corresponding to the unbiased proposal, which by the same lemma admits a unique equilibrium, also MPM. Thus, it suffices to show that $rS_{\text{part}}(r/2) = r\theta$ and $\frac{d}{dx_1}\left(rS_{\text{part}}(x_1)\right) > 0$. To that end, we calculate:

$$\begin{aligned} \frac{d}{dx_1}\left(rS_{\text{part}}(x_1)\right) &= -\theta\tfrac{d}{dx_1}\pi_2^R + \theta - \theta\tfrac{d}{dx_1}\sigma_2(a) - q_2^R + (1-x_1)\theta\tfrac{d}{dx_1}\sigma_2(a) \\ &= -x_1\theta\tfrac{d}{dx_1}\sigma_2(a) + \theta - \theta\tfrac{d}{dx_1}\left(\tfrac{\theta\sigma_2(a)}{q_2^R}\right) - q_2^R \\ &= \theta - q_2^R - \theta\left[\frac{\theta\frac{d\sigma_2(a)}{dx_1}q_2^R - \theta^2\sigma_2(a)\frac{d\sigma_2(a)}{dx_1}}{(q_2^R)^2}\right] - x_1\theta\tfrac{d\sigma_2(a)}{dx_1} \\ &= \theta - q_2^R - \theta^2\tfrac{d\sigma_2(a)}{dx_1}\left[\tfrac{1-\pi_2^R}{q_2^R}\right] - x_1\theta\tfrac{d\sigma_2(a)}{dx_1} \\ &= \theta - q_2^R - \left[\theta x_1 + \theta^2\tfrac{1-\pi_2^R}{q_2^R}\right]\tfrac{d\sigma_2(a)}{dx_1}, \end{aligned} \tag{55}$$

where, by Eq. (42) and the identity $x_2 = r - x_1$,

$$\frac{d\sigma_2(a)}{dx_1} = \frac{(x_1)^2 + r(\theta-1)}{(x_1 - [\ \theta(x_1-x_2) + x_2(1-x_1)]\ )^2}\frac{1-\theta}{\theta}. \tag{56}$$

Denote

$$M := x_1 - [\ \theta(x_1 - x_2) + x_2(1-x_1)].$$

Note that

$$M = r(\theta + x_1 - 1) + 2x_1(1-\theta) - (x_1)^2.$$

By the restriction $\theta > \frac{r}{2} \geq x_1$ (due to Ineqs. (3) and (20)), it is straightforward to show

$$2(x_1)^2 \leq r(\theta + x_1 - 1) + 2x_1(1-\theta) \leq rx_1$$

and hence

$$(x_1)^2 \leq M \leq x_1(r - x_1). \tag{57}$$

By Eq. (44),

$$q_2^R = 1 - q_2^A = \frac{x_1(1-\theta)}{M}.$$

Plug this equation and Eq. (56) into Eq. (55) to obtain

$$\begin{aligned}
\frac{d}{dx_1}\left(rS_{\text{part}}(x_1)\right) &= \theta - \frac{x_1(1-\theta)}{M} - \left[\frac{x_1(1-\theta)}{M^2}((x_1)^2 + r(\theta-1)) + \frac{\theta}{(x_1)^2}((x_1)^2 + r(\theta-1))\right] \\
&= \frac{r\theta(1-\theta)}{(x_1)^2} - \frac{x_1(1-\theta)}{M}\left[1 + \frac{(x_1)^2 + r(\theta-1)}{M}\right] \\
&= \frac{r\theta(1-\theta)}{(x_1)^2} - \frac{x_1(1-\theta)}{M^2}\left[x_1(2-2\theta+r) + 2r(\theta-1)\right].
\end{aligned}$$

Thus, $\frac{d}{dx_1}\left(rS_{\text{part}}(x_1)\right) > 0$ is equivalent to

$$r\theta M^2 > (x_1)^3(x_1(2-2\theta+r) + 2r(\theta-1)).$$

Since

$$x_1(2-2\theta+r) + 2r(\theta-1) = r(\theta+x_1-1) + 2x_1(1-\theta) - r(1-\theta),$$

the inequality displayed above is equivalent to

$$r\theta M^2 > (x_1)^3\left[r(\theta+x_1-1) + 2x_1(1-\theta) - r(1-\theta)\right]. \tag{58}$$

By $r \geq 2x_1$ and $M \geq (x_1)^2$ (Ineq. (57)), the left-hand side of (58) is greater than or equal to $r\theta(x_1)^4$, and the right-hand side less than or equal to $(x_1)^3 r(\theta+x_1-1)$. Thus, (58) holds if

$$r\theta(x_1)^4 > (x_1)^3 r(\theta+x_1-1),$$

i.e., $\theta x_1 > \theta + x_1 - 1$, which is true because $\theta x_1 - (\theta + x_1 - 1) = (1-\theta)(1-x_1) > 0$ by Ineq. (47). Thus, we have proved $\frac{d}{dx_1}\left(rS_{\text{part}}(x_1)\right) > 0$.

Finally, given the unbiased proposal, $x_1 = x_2 = r/2$; by Eqs. (13), (18) and (41).

$$\begin{aligned}
\sigma_2(a) &= \frac{1-\theta}{\theta}\frac{2-r}{r}, \\
\pi_2^R &= 1 - r/2, \\
q_2^R &= \frac{1-\theta}{r/2}.
\end{aligned}$$

Plug them into Eq. (54) to obtain

$$rS_{\text{part}}(r/2) = \theta + \theta\left(1 - \frac{1-\theta}{\theta}\frac{2-r}{r}\right) - \theta(1-r/2) - \theta(1-r/2) + \frac{1-\theta}{r/2}(1-r/2) = r\theta,$$

as asserted. ■

A.4 Lemma 6 and Theorem 2

Proof of Lemma 6 By Eqs. (22) and (23), Eq. (28) is equivalent to $\theta(1-\sigma_1(a))+\theta\sigma_1(a) = q_1^A x_2 + \pi_2^R$, i.e., $\theta = q_1^A x_2 + \pi_2^R$. Furthermore,

$$\begin{aligned}
\theta = q_1^A x_2 + \pi_2^R \quad &\iff \quad \theta = \frac{\pi_2^A - \pi_2^R}{\pi_2^A} x_2 + \pi_2^R \\
&\iff \quad (\theta - \pi_2^R)\pi_2^A = (\pi_2^A - \pi_2^R) x_2 \\
&\iff \quad (\theta - \pi_2^R)\frac{\pi_2^R \theta(1-\sigma_2(a))}{\pi_2^R - \theta\sigma_2(a)} = \frac{\pi_2^R(\theta - \pi_2^R)}{\pi_2^R - \theta\sigma_2(a)} x_2 \\
&\iff \quad \theta(1-\sigma_2(a)) = x_2,
\end{aligned}$$

where the first line is due to $q_1^A = 1 - q_1^R = 1 - \pi_2^R/\pi_2^A$ by Eq. (23), the third line due to

$$\pi_2^A = \frac{\theta(1-\sigma_2(a))}{1-q_2^R} = \frac{\pi_2^R\theta(1-\sigma_2(a))}{\pi_2^R - \pi_2^R q_2^R} \overset{(22)}{=} \frac{\pi_2^R\theta(1-\sigma_2(a))}{\pi_2^R - \theta\sigma_2(a)},$$

and the fourth line due to Ineqs. (21) and (24). Thus, Eq. (28) is equivalent to

$$\sigma_2(a) = 1 - \frac{x_2}{\theta}. \tag{59}$$

Consequently, since $\sigma_2(a) < 1$,

$$\theta > x_2. \tag{60}$$

Plug Eq. (27) into Eq. (28) to obtain $q_1^A\pi_1^A + q_1^R\pi_1^R = q_1^A x_2 + q_1^R x_1 + q_1^R\pi_1^R$, i.e.,

$$q_1^A \pi_1^A = q_1^A x_2 + q_1^R x_1,$$

which by Eq. (22) is equivalent to

$$\theta(1-\sigma_1(a)) = x_2 - q_1^R\left(x_2 - x_1\right). \tag{61}$$

In the meantime,

$$\begin{aligned}
x_1 = \pi_2^A - \pi_1^R \quad &\iff \quad x_1 = \frac{x_2}{q_2^A} - \pi_1^R \\
&\iff \quad x_1 = \frac{x_2}{1 - \pi_1^R/\pi_1^A} - \pi_1^R \quad \iff \quad \pi_1^A x_2 = \left(\pi_1^A - \pi_1^R\right)\left(\pi_1^R + x_1\right),
\end{aligned}$$

with the first line due to

$$q_2^A \pi_2^A = \theta(1-\sigma_2(a)) \overset{(59)}{=} x_2,$$

and the second due to $q_2^A = 1 - q_2^R = 1 - \pi_1^R/\pi_1^A$ by Eq. (23). Thus, Eq. (27) is equivalent to

$$\pi_1^A x_2 = \left(\pi_1^A - \pi_1^R\right)\left(\pi_1^R + x_1\right). \tag{62}$$

Plug the fact that $\pi_1^R = \theta\sigma_1(a)/q_1^R$ and

$$\pi_1^A - \pi_1^R = \frac{\theta(1-\sigma_1(a))}{1-q_1^R} - \frac{\theta\sigma_1(a)}{q_1^R} = \frac{\theta(q_1^R - \sigma_1(a))}{q_1^A q_1^R}$$

into the right-hand side of Eq. (62) to note that Eq. (62) is equivalent to

$$q_1^A \pi_1^A x_2 = \frac{\theta(q_1^R - \sigma_1(a))}{q_1^R} \cdot \frac{\theta\sigma_1(a) + q_1^R x_1}{q_1^R}.$$

The left-hand side of this equation is equal to

$$\theta(1-\sigma_1(a))x_2 = x_2(x_2 - q_1^R(x_2 - x_1))$$

by Eq. (61), and the right-hand side equal to

$$\frac{\theta q_1^R - \theta + x_2 - q_1^R(x_2 - x_1)}{q_1^R} \cdot \frac{\theta - x_2 + q_1^R(x_2 - x_1) + q_1^R x_1}{q_1^R}$$

again by Eq. (61). Thus, Eqs. (61) and (62) together are equivalent to

$$x_2(x_2 - q_1^R(x_2 - x_1)) = \frac{x_2 - \theta - q_1^R(x_2 - x_1 - \theta)}{q_1^R} \cdot \frac{\theta - x_2 + q_1^R x_2}{q_1^R},$$

which is equivalent to

$$\left(q_1^R\right)^2 x_2 \left(q_1^R x_1 + (1-q_1^R)x_2\right) = \left(q_1^R x_1 - (1-q_1^R)(\theta - x_2)\right)\left(q_1^R x_2 + \theta - x_2\right). \tag{63}$$

In sum, (21) constitutes a PBE if and only if Eq. (63) admits a solution for $q_1^R \in (0,1)$. If $q_2^R = 0$, the left-hand side of (63) is equal to zero while the right-hand side of (63) equal to $-(\theta - x_2)$, which is negative by Ineq. (60). If $q_2^R = 1$, the left-hand side of (63) is equal to $x_2 x_1$ while the right-hand side equal to θx_1, which is bigger than the left-hand side due to Ineq. (60). Thus, Eq. (63) admits a solution for $q_1^R \in (0,1)$, hence (21) constitutes a PBE if and only if the values of (σ_1, σ_2) determined by the solution for q_1^R satisfies (21). Moreover, it can be shown that this

By Eq. (61) and $q_1^R = \theta\sigma_1(a) + (1-\theta)\sigma_1(z)$, we have

$$\begin{aligned} \sigma_1(a) &= \frac{\theta + x_1 - r + q_1^R(r - 2x_1)}{\theta}, \\ \sigma_1(z) &= \frac{q_1^R - \theta\sigma_1(a)}{1-\theta}. \end{aligned}$$

To characterize the restriction of (21) on σ_1, observe that

$$0 < \sigma_1(a) = \frac{\theta + x_1 - r + q_1^R(r - 2x_1)}{\theta} < 1 \iff 0 < x_2 + q_1^R(x_1 - x_2) < \theta;$$

and

$$0 < \sigma_1(z) = \frac{q_1^R - \theta\sigma_1(a)}{1-\theta} < 1 \iff \theta < q_1^R + x_2 + q_1^R\,(x_1 - x_2) < 1$$
$$\iff \theta < x_2 + q_1^R\,(1 + r - 2x_2) < 1$$
$$\iff \frac{\theta - x_2}{1 + r - 2x_2} < q_1^R < \frac{1 - x_2}{1 + r - 2x_2},$$

with the last line due to the fact

$$1 + r - 2x_2 = 1 + x_1 - x_2 > \theta - x_2 + x_1 \overset{(60)}{>} 0.$$

By $q_1^R = \pi_2^R/\pi_2^A$ and Eq. (59), we pin down $\sigma_2(z)$:

$$q_1^R = \frac{\theta\sigma_2(a)}{\theta(1-\sigma_2(a))} \cdot \frac{\theta(1-\sigma_2(a) + (1-\theta)(1-\sigma_2(z))}{\theta\sigma_2(a) + (1-\theta)\sigma_2(z)} = \frac{\theta - x_2}{x_2} \cdot \frac{x_2 + (1-\theta)(1-\sigma_2(z))}{\theta - x_2 + (1-\theta)\sigma_2(z)}$$

and hence

$$\sigma_2(z) = \frac{\theta - x_2}{1-\theta} \cdot \frac{1 - \theta + x_2(1 - q_1^R)}{\theta + x_2(q_1^R - 1)}.$$

To characterize the restriction of (21) on $\sigma_2(z)$, note that

$$\sigma_2(z) > 0 \iff \theta + x_2(q_1^R - 1) > 0,$$

where the second inequality is true due to Ineq. (60); also note that

$$\sigma_2(z) < 1 \iff \frac{\theta - x_2}{1-\theta} \cdot \frac{1 - \theta + x_2(1 - q_1^R)}{\theta + x_2(q_1^R - 1)} < 1$$
$$\iff \frac{1 - \theta + x_2(1 - q_1^R)}{\theta - x_2(1 - q_1^R)} < \frac{1-\theta}{\theta - x_2}$$
$$\iff \frac{1}{\theta - x_2(1 - q_1^R)} < \frac{1 - x_2}{\theta - x_2}$$
$$\iff q_1^R > \frac{\theta - x_2}{1 - x_2},$$

where the second, third and fourth lines each use Ineq. (60). To characterize the restriction of (21) on $\sigma_2(a)$, note from Eq. (59) that

$$0 < \sigma_2(a) < 1 \iff \theta > x_2 > 0.$$

Therefore, the necessary and sufficient condition for q_1^R to constitute an MTM equilibrium is simultaneous satisfaction of

$$0 < x_2 + q_1^R\,(x_1 - x_2) < \theta, \tag{64}$$
$$\tfrac{\theta - x_2}{1 + r - 2x_2} < q_1^R < \tfrac{1 - x_2}{1 + r - 2x_2}, \tag{65}$$
$$q_1^R > \tfrac{\theta - x_2}{1 - x_2}, \tag{66}$$
$$0 < x_2 < \theta. \tag{67}$$

By $1 + r - 2x_2 = 1 + x_1 - x_2 > 1 - x_2$ due to (67),

$$\frac{\theta - x_2}{1 - x_2} > \frac{\theta - x_2}{1 + r - 2x_2}.$$

Thus, the necessary and sufficient condition is equivalent to simultaneous satisfaction of (64), (67) and

$$\frac{\theta - x_2}{1 - x_2} < q_1^R < \frac{1 - x_2}{1 + r - 2x_2}. \tag{68}$$

Ineq. (68) admits a solution for q_1^R if and only if $\frac{\theta - x_2}{1 - x_2} < \frac{1 - x_2}{1+r-2x_2}$, i.e.,

$$(x_2)^2 + (1 - r - 2\theta)x_2 + (1 + r)\theta - 1 < 0,$$

i.e.,

$$\underbrace{\frac{r + 2\theta - 1 - \sqrt{(r-1)^2 + 4(\theta - 1)^2}}{2}}_{=:X} < x_2 < \underbrace{\frac{r + 2\theta - 1 + \sqrt{(r-1)^2 + 4(\theta - 1)^2}}{2}}_{=:Y}.$$

Clearly, $X < \theta < Y$. Thus, (68) coupled with (67) is equivalent to

$$\frac{r + 2\theta - 1 - \sqrt{(r-1)^2 + 4(\theta - 1)^2}}{2} < x_2 < \theta. \tag{69}$$

Hence an MTM equilibrium exists if and only if (64) and (69) are both satisfied. To characterize (64), consider the only two possible cases:

i. $x_1 < x_2$, i.e., $x_2 > r/2$. Then (64) is satisfied due to $\theta > x_2 > 0$, implied by (69). Since $r/2 < \theta$, an MTM equilibrium exists in this case if and only if

$$\max\{r/2, X\} < x_2 < \theta.$$

ii. $x_1 \geq x_2$, i.e., $x_2 \leq r/2$. Then (64) is equivalent to

$$q_1^R < \frac{\theta - x_2}{x_1 - x_2}.$$

This inequality is implied by the second inequality in (68), because

$$\frac{\theta - x_2}{x_1 - x_2} > \frac{1 - x_2}{1 + r - 2x_2} \iff \theta - r(1 - \theta) + x_2(1 - 2\theta) > 0,$$

where the second inequality follows from the fact that $x_2 \leq r/2$, $1 - 2\theta < 0$ and $r/2 < \theta$. Since (68) has been incorporated into (69), (64) is again redundant, and an MTM equilibrium exists in this case if and only if

$$X < x_2 \leq r/2.$$

Thus, for any configuration of the parameters (θ, r) there exists a peace proposal (x_1, x_2) that admits an MTM equilibrium: pick any $i \in \{1, 2\}$; let $X < x_i < \theta$.

Consequently, the unbiased proposal ($x_i = r/2$) admits an MTM equilibrium if and only if $X < r/2$, i.e.,

$$\frac{r + 2\theta - 1 - \sqrt{(r-1)^2 + 4(\theta-1)^2}}{2} < \frac{r}{2},$$

i.e.,

$$\theta < \frac{3}{4} + \frac{(r-1)^2}{4},$$

as asserted. ■

Proof of Theorem 2 Consider any MTM proposal-equilibrium pair. Denote S_{total} for the social surplus it generates. Then

$$rS_{\text{total}} = q_2^A \pi_2^A - \theta \pi_1^R + q_2^R \pi_1^R + q_1^A \pi_1^A + q_1^R \pi_1^R - \theta \pi_2^R \tag{70}$$

$$= \theta + q_2^A \pi_2^A + \left(q_2^R - \theta\right) \pi_1^R - \theta \pi_2^R. \tag{71}$$

By Eq. (59),

$$q_2^R = \theta \sigma_2(a) + (1-\theta)\sigma_2(z) = \theta - x_2 + (1-\theta)\sigma_2(z).$$

Hence

$$q_2^R \geq \theta \iff \theta - x_2 + (1-\theta)\sigma_2(z) \geq \theta \iff \sigma_2(z) \geq \frac{x_2}{1-\theta}$$

Since $\sigma_2(z) < 1$ requires that $x_2 < 1 - \theta$, if we have $x_2 \geq 1 - \theta$ then $q_2^R \geq \theta$ will violate the equilibrium condition that $\sigma_2(z) < 1$. In other words, $x_2 \geq 1 - \theta$ implies $q_2^R < \theta$.

Relabeling the two players if necessary, let $\pi_1^R \geq \pi_2^R$. It is clear that whenever $q_2^R < \theta$ then by Eq. (71) the upper bound for the social surplus is when π_1^R takes it minimal value, which is $\pi_1^R = \pi_2^R$:

$$rS_{total} = \theta + q_2^A \pi_2^A + \left(q_2^R - \theta\right) \pi_1^R - \theta \pi_2^R < 2\theta(1 - \pi_2^R).$$

For the rest of the proof, we shall show: First, $\pi_1^R = \pi_2^R$ is only admitted by the unbiased proposal. Thus, whenever $q_2^R < \theta$ the optimal split is the unbiased one which achieves the upper bound of social-surplus. Second, characterize the primitive condition based on Lemma 6 such that it also satisfy the condition $x_2 \geq 1 - \theta$.

First, we show that $\pi_1^R = \pi_2^R$ implies $\pi_1^A = \pi_2^A$. To that end we use the necessary and sufficient conditions (25) coupled with $\pi_1^R = \pi_2^R = \pi^R$ to obtain

$$\theta = q_2^A x_1 + \pi^R$$
$$\pi_2^A - \pi^R = x_2$$
$$\theta = q_1^A x_2 + \pi^R$$
$$\pi_1^A - \pi^R = x_2.$$

These set of four equations imply

$$\frac{q_2^A}{q_1^A} = \frac{x_2}{x_1}$$
$$\frac{\pi_2^A - \pi^R}{\pi_1^A - \pi^R} = \frac{x_2}{x_1}.$$

By Eq. (23) we know that for any $i \in \{1,2\}$ $\pi^R = q_i^R \pi_{-i}^A \iff q_i^A = \frac{\pi_{-i}^A - \pi_{-i}^R}{\pi_{-i}^A}$. Hence,

$$\frac{q_2^A}{q_1^A} = \frac{x_2}{x_1} \iff \frac{\pi_1^A\left(\pi_2^A - \pi^R\right)}{\pi_2^A\left(\pi_1^A - \pi^R\right)} = \frac{x_2}{x_1}$$
$$\iff \frac{\pi_1^A\left(\pi_2^A - \pi^R\right)}{\pi_2^A\left(\pi_1^A - \pi^R\right)} = \frac{\pi_2^A - \pi^R}{\pi_1^A - \pi^R} \iff \pi_2^A = \pi_1^A = \pi^A.$$

Thus, if $\pi_1^R = \pi_2^R = \pi^R$ then $\pi_1^A = \pi_2^A = \pi^A$. This case corresponds to the symmetric mutually mixed PBE which we know is admitted by the unbiased split.

Thus, if $x_2 \geq 1-\theta$ then $q_2^R < \theta$; by Eq. (71), the upper bound for the social surplus is when π_1^R takes it minimal value which is $\pi_1^R = \pi_2^R$. We have shown that this posterior beliefs is associated with the MTM equilibrium that is admitted by the unbiased split.

Second, we characterize the primitive condition corresponding to $x_2 \geq 1-\theta$. By Eq. (69), one of the necessary and sufficient conditions for this class of equilibrium is.

$$\frac{r + 2\theta - 1 - \sqrt{(r-1)^2 + 4(\theta-1)^2}}{2} < x_2 < \theta. \tag{72}$$

Note that

$$\frac{r + 2\theta - 1 - \sqrt{(r-1)^2 + 4(\theta-1)^2}}{2} < 1-\theta \iff r + 4\theta - 3 < \sqrt{(r-1)^2 + 4(\theta-1)^2}.$$

By $1 < r < 2$ and $\frac{r}{2} < \theta$ we can verify that $r + 4\theta - 3 = r - 1 + 4(\theta - 2) > 0$. Hence

$$r + 4\theta - 3 < \sqrt{(r-1)^2 + 4(\theta-1)^2} \iff (r + 4\theta - 3)^2 < (r-1)^2 + 4(\theta-1)^2$$
$$\iff 4(3\theta^2 + (2r-8)\theta + 1 - r) < 0.$$

Which is equivalent to

$$\frac{2-r-\sqrt{r^2-r+1}}{3} < \theta < \frac{2-r+\sqrt{r^2-r+1}}{3}$$

It is straightforward to note that the lower bound is not binding in the presence of $\frac{r}{2} < \theta$.

Thus, if $\frac{r}{2} < \theta < \frac{2-r+\sqrt{r^2-r+1}}{3}$ there exists a split that satisfies the necessary and sufficient condition

$$\frac{r+2\theta-1-\sqrt{(r-1)^2+4(\theta-1)^2}}{2} < x_2 < 1-\theta < r/2 < \theta.$$

If $\frac{2-r+\sqrt{r^2-r+1}}{3} \leq \theta < 1$ then there exists a split that satisfies the necessary and sufficient condition

$$1-\theta \leq \frac{r+2\theta-1-\sqrt{(r-1)^2+4(\theta-1)^2}}{2} < x_2 < \theta.$$

In sum, for any $1 < r < 2$ and $\frac{2-r+\sqrt{r^2-r+1}}{3} \leq \theta < 1$ there exist x_2 that satisfies both of the necessary and sufficient condition for existence of totally mutually mixed equilibriums, i.e, (64) and (69) and the sufficient condition $1-\theta \leq x_2$. Moreover, for this primitive condition the unbiased split maximize social-surplus among all MTM proposal-equilibrium pairs. ■

A.5 Lemma 7

Given any peace proposal (x_1, x_2), denote (σ_1, σ_2) for a hybrid equilibrium it admits. As characterized in Section 3, for some $i \in \{1, 2\}$,

$$0 < \sigma_i(a) < 1, \quad 0 < \sigma_i(z) < 1, \quad 0 < \sigma_{-i}(a) < 1, \quad 0 < \sigma_{-i}(z) = 1. \tag{73}$$

Without loss of generality let $i = 1$. This implies the expected probabilities of Accept/Reject and the posterior beliefs:

$$\begin{aligned} q_1^A &= \theta(1-\sigma_1(a)) + (1-\theta)(1-\sigma_1(z)), & q_1^R &= \theta\sigma_1(a) + (1-\theta)\sigma_1(z), \\ \pi_1^A &= \theta(1-\sigma_1(a))/q_1^A, & \pi_1^R &= \theta\sigma_1(a)/q_1^R, \\ q_2^A &= \theta(1-\sigma_2(a)), & q_2^R &= \theta\sigma_2(a) + (1-\theta), \\ \pi_2^A &= 1, & \pi_2^R &= \theta\sigma_2(a)/q_2^R. \end{aligned} \tag{74}$$

By (73), the necessary and sufficient condition for each σ_1 to best rely σ_2 is $\Delta_1(a) = \Delta_1(z) = 0$ (Lemma 2), which, by the identity $\Delta_1(z) - \Delta_1(a) = \pi_1^R - q_2^R \pi_1^A$ (Eq. (12)), is equivalent to the condition that "$\Delta_1(a) = 0$ or $\Delta_1(z) = 0$" and

$$\pi_1^R = q_2^R \pi_1^A \tag{75}$$

By (73), the necessary and sufficient condition for each σ_2 to best rely σ_1 is $\Delta_2(a) = 0$ and $\Delta_2(z) \geq 0$ (Lemma 2), which, by the identity $\Delta_1(z) - \Delta_1(a) = \pi_1^R - q_2^R \pi_1^A$ (Eq. (12)), is equivalent to the condition "$\Delta_2(a) = 0$ or $\Delta_2(z) \geq 0$" and

$$\pi_2^R \geq q_1^R \pi_2^A = q_1^R. \tag{76}$$

Since $0 < q_i^R < 1$ due to (73), Eq. (75) and Ineq. (76) implies $\pi_i^R < \pi_i^A$, which means

$$\pi_i^R < \theta < \pi_i^A \tag{77}$$

for each i according to (11). Hence Eq. (10) becomes

$$\Delta_1 = q_2^A \begin{bmatrix} \pi_2^A - x_1 \\ \pi_2^A - \pi_1^R - x_1 \end{bmatrix} + q_2^R \begin{bmatrix} \max\{\pi_1^R, \pi_2^R\} - \pi_1^A \\ \left(\pi_2^R - \pi_1^R\right)^+ \end{bmatrix}, \tag{78}$$

$$\Delta_2 = q_1^A \begin{bmatrix} \pi_1^A - x_2 \\ \pi_1^A - \pi_2^R - x_2 \end{bmatrix} + q_1^R \begin{bmatrix} \max\{\pi_1^R, \pi_2^R\} - \pi_2^A \\ \left(\pi_1^R - \pi_2^R\right)^+ \end{bmatrix}. \tag{79}$$

The necessary and sufficient condition for each σ_1 to best rely σ_2 is $\Delta_1(a) = \Delta_1(z) = 0$. Also, the necessary and sufficient condition for each σ_2 to best rely σ_1 is $\Delta_2(a) = 0$ and $\Delta_2(z) \geq 0$. There are only two possible cases: (i) $\pi_2^A > \pi_1^A > \pi_2^R \geq \pi_1^R$ or (ii) $\pi_2^A > \pi_1^A > \pi_1^R > \pi_2^R$. By the definition of posterior probabilities, it is easy to show:

i. $\pi_2^A > \pi_1^A > \pi_2^R \geq \pi_1^R \iff \sigma_1(z) > \sigma_1(a) \quad \text{and} \quad \sigma_2(a) \geq \frac{\sigma_1(a)}{\sigma_1(z)}$;

ii. $\pi_2^A > \pi_1^A > \pi_1^R > \pi_2^R \iff \sigma_1(z) > \sigma_1(a) \quad \text{and} \quad \sigma_2(a) < \frac{\sigma_1(a)}{\sigma_1(z)}$.

Case (i): $\pi_2^A > \pi_1^A > \pi_2^R \geq \pi_1^R$. By Eq. (75) and Ineq. (76), $\Delta_1(a) = \Delta_1(z) = 0$, $\Delta_2(a) = 0$, and $\Delta_2(z) \geq 0$, given any peace proposal (x_1, x_2), the necessary and sufficient condition for (73) to constitute PBE is simultaneous satisfaction of

$$\theta = q_2^A x_1 + q_2^R \pi_1^A \tag{80}$$

$$q_2^R = \frac{\pi_1^R}{\pi_1^A} \tag{81}$$

$$\pi_2^R \geq q_1^R \tag{82}$$

$$\pi_1^A - \pi_2^R = x_2. \tag{83}$$

Lemma 9 *At any hybrid proposal-equilibrium pair $(x_1, x_2; \sigma_1, \sigma_2)$ such that $\sigma_2(a) \geq \frac{\sigma_1(a)}{\sigma_1(z)}$,*

$$r - \theta < x_2 < \frac{r}{2}. \tag{84}$$

Proof By Eqs. (80), (83), and $q_2^A = \theta(1-\sigma_2(a))$,

$$\begin{aligned}
\theta = q_2^A x_1 + q_2^R \pi_1^A \quad &\iff \quad \theta = q_2^A x_1 + q_2^R(\pi_2^R + x_2) \\
&\iff \quad q_2^A = \frac{x_2}{1 + x_2 - x_1} \\
&\iff \quad \sigma_2(a) = \frac{\theta(1-r+2x_2) - x_2}{\theta(1-r+2x_2)}. \qquad (85)
\end{aligned}$$

Hence,

$$\pi_2^R = \frac{\theta \sigma_2(a)}{q_2^R} = \frac{\theta(1-r+2x_2) - x_2}{1-r+x_2}. \qquad (86)$$

Moreover, using Eq. (83)

$$\pi_1^A = \pi_2^R + x_2 = \frac{\theta(1-r+2x_2) - (r-x_2)x_2}{1-r+x_2}. \qquad (87)$$

Next, plugging $q_2^R = 1 - q_2^A$ and π_1^A in Eq. (81) yields

$$\pi_1^R = q_2^R \pi_1^A = \frac{\theta(1-r+2x_2) - x_2(r-x_2)}{1-r+2x_2}. \qquad (88)$$

Hence, using the definition of π_1^R via Eq. (74) we have

$$\sigma_1(a) = \frac{(1-\theta)}{\theta} \frac{[\theta(1-r+2x_2) - (r-x_2)x_2]}{[(1-\theta)(1-r+2x_2) + x_2(r-x_2)]} \sigma_1(z)$$

Using this relationship between $\sigma_1(a)$ and $\sigma_1(z)$ along with equation (81):

$$\begin{aligned}
\frac{\pi_1^R}{\pi_1^A} = q_2^R \quad &\iff \quad \frac{\sigma_1(a)}{1-\sigma_1(a)} \frac{1-q_1^R}{q_1^R} = \frac{1-r+x_2}{1-r+2x_2} \\
&\iff \quad \sigma_1(z) = \frac{(\theta + x_2 - r)\left[(1-\theta)(1-r+2x_2) + x_2(r-x_2)\right]}{(1-\theta)\left[-x_2(r-x_2) + \theta(1-r+2x_2)\right]} \qquad (89) \\
&\iff \quad \sigma_1(a) = \frac{\theta + x_2 - r}{\theta}. \qquad (90)
\end{aligned}$$

By the fact (i) displayed above, $\pi_2^R \geq \pi_1^R \iff \sigma_2(a) \geq \frac{\sigma_1(a)}{\sigma_1(z)}$, which also implies that $\sigma_2(a) > \sigma_1(a)$. Hence using Eqs. (85) and (90),

$$\sigma_2(a) > \sigma_1(a) \iff \frac{\theta(1-x_1+x_2) - x_2}{\theta(1-x_1+x_2)} > \frac{\theta - x_1}{\theta} \iff x_2 < x_1 \iff x_2 < \frac{r}{2} < x_1.$$

Moreover, characterizing the restriction of (73) on $\sigma_1(a)$,

$$0 < \sigma_1(a) < 1 \iff 0 < \frac{\theta - x_1}{\theta} < 1 \iff 0 < x_1 < \theta.$$

Thus, we obtain (84). ■

Case (ii): $\pi_2^A > \pi_1^A > \pi_1^R > \pi_2^R$ Using Eq. (75), Ineq. (76), $\Delta_1(a) = \Delta_1(z) = 0$, $\Delta_2(a) = 0$, and $\Delta_2(z) \geq 0$, given any peace proposal (x_1, x_2), the necessary and sufficient condition for (73) to constitute PBE is simultaneous satisfaction of

$$q_2^R = \frac{\pi_1^R}{\pi_1^A} \tag{91}$$

$$1 - \pi_1^R = x_1 \tag{92}$$

$$\theta = q_1^A x_2 + \pi_2^R \tag{93}$$

$$\pi_2^R \geq q_1^R \tag{94}$$

Lemma 10 *At any hybrid proposal-equilibrium pair* $(x_1, x_2; \sigma_1, \sigma_2)$ *such that* $\sigma_2(a) < \frac{\sigma_1(a)}{\sigma_1(z)}$, *Ineq. (84) is satisfied.*

Proof By Eq. (92):

$$\pi_1^R = 1 - x_1 \iff \frac{\theta\sigma_1(a)}{\theta\sigma_1(a) + (1-\theta)\sigma_1(z)} = 1 - x_1$$
$$\iff \sigma_1(a) = \frac{(1-\theta)(1-x_1)}{\theta x_1}\sigma_1(z) \tag{95}$$

Note that,

$$\sigma_1(a) < \sigma_1(z) \iff \theta + x_1 - 1 > 0 \tag{96}$$

$$\sigma_1(a) > 0 \iff \theta < 1 \quad \text{and} \quad x_1 < 1. \tag{97}$$

Using equation (95),

$$q_1^A = \theta(1 - \sigma_1(a)) + (1-\theta)(1 - \sigma_1(z)) = \frac{1 - x_1 - \theta\sigma_1(a)}{1 - x_1}. \tag{98}$$

Note that,

$$0 < q_1^A < 1 \iff 0 < \sigma_1(a) < \frac{1 - x_1}{\theta}. \tag{99}$$

Plugging Eqs. (92) and (98) in Eq. (91),

$$q_2^R = \frac{1 - x_1}{\pi_1^A} = \frac{1 - x_1}{\theta(1 - \sigma_1(a))} q_1^A \iff \theta\sigma_2(a) + 1 - \theta = \frac{1 - x_1 - \theta\sigma_1(a)}{\theta(1 - \sigma_1(a))}$$
$$\iff \sigma_2(a) = \frac{1 - x_1 - \theta + \theta^2(1 - \sigma_1(a))}{\theta^2(1 - \sigma_1(a))}. \tag{100}$$

By equation (93) and plugging in for q_1^A and $\sigma_2(a)$ by equations (98) and (100)

$$\theta = q_1^A x_2 + \pi_2^R \iff \theta = \frac{1 - x_1 - \theta\sigma_1(a)}{1 - x_1} x_2 + \frac{1 - x_1 + \theta^2(1 - \sigma_1(a)) - \theta}{1 - x_1 - \theta\sigma_1(a)}.$$

Using the properties of Ineqs. (99) and (97), and multiplying the both side of the latter equation by $(1 - x_1 - \theta\sigma_1(a))(1 - x_1)$ will yield to the following quadratic equation:

$$x_2\theta^2\sigma_1^2(a) - 2x_2(1 - x_1)\theta\sigma_1(a) + (1 - x_1)\left((1 - \theta)(1 - x_1 - \theta) + x_2(1 - x_1)\right) = 0,$$

which is equivalent to:

$$\sigma_1(a) = \frac{(1 - x_1)(r - x_1) \pm \sqrt{(1 - \theta)(1 - x_1)(\theta + x_1 - 1)(r - x_1)}}{\theta(r - x_1)}.$$

Moreover, using equation (99) the acceptable root is,

$$\sigma_1(a) = \frac{(1 - x_1)(r - x_1) - \sqrt{(1 - \theta)(1 - x_1)(\theta + x_1 - 1)(r - x_1)}}{\theta(r - x_1)}. \tag{101}$$

Denote $Z := \sqrt{(1 - \theta)(1 - x_1)(\theta + x_1 - 1)(r - x_1)}$. Note that Ineqs. (96) and (97) imply that $Z > 0$. By Eq. (95),

$$\sigma_1(z) = \frac{x_1(1 - x_1)(r - x_1) - x_1 Z}{(r - x_1)(1 - \theta)(1 - x_1)}. \tag{102}$$

By Eq. (100),

$$\sigma_2(a) = \frac{\theta Z - (1 - \theta)(\theta + x_1 - 1)(r - x_1)}{\theta\left(Z + (\theta + x_1 - 1)(r - x_1)\right)}. \tag{103}$$

By Eqs. (74), (101) and (102), player 1's ex-ante probability of Reject and associated posterior beliefs can be summarized as

$$q_1^R = \frac{(1 - x_1)(r - x_1) - Z}{(r - x_1)(1 - x_1)}, \tag{104}$$

$$\pi_1^R = 1 - x_1, \tag{105}$$

$$\pi_1^A = \frac{(\theta + x_1 - 1)(r - x_1)(1 - x_1) + (1 - x_1)Z}{Z}., \tag{106}$$

and likewise for player 2:

$$q_2^R = \frac{Z}{Z + (\theta + x_1 - 1)(r - x_1)}, \tag{107}$$

$$\pi_2^R = \frac{\theta Z - (1 - \theta)(\theta + x_1 - 1)(r - x_1)}{Z}, \tag{108}$$

$$\pi_2^A = 1. \tag{109}$$

Next we will use Eqs. (104) and (108) and impose equilibrium condition (94), i.e., $q_1^R \leq \pi_2^R$ to get

$$q_1^R \leq \pi_2^R \iff (1-\theta)(1-x_1)(r-x_1) \leq (1-x_2)Z. \tag{110}$$

Moreover, by Eqs. (105) and (108)

$$\pi_1^R > \pi_2^R \iff Z < (1-\theta)(r-x_1). \tag{111}$$

Eq. (110) coupled with Eq. (111) implies that:

$$(1-\theta)(1-x_1)(r-x_1) \leq (1-x_2)Z < (1-\theta)(r-x_1)(1-x_2) \Rightarrow x_2 < \frac{r}{2} < x_1. \tag{112}$$

Furthermore, using equations (106), (108), and the restriction $\pi_1^A > \pi_2^R$ leads to the following necessary condition:

$$\pi_1^A > \pi_2^R \iff (r-x_1)(\theta-x_1) > Z > 0 \Rightarrow x_1 < \theta \tag{113}$$

Thus, given (x_1, x_2) the necessary condition for (σ_1, σ_2) to constitutes hybrid PBEs is $r-\theta < x_2 < \frac{r}{2} < x_1 < \theta$, as asserted. ■

A.6 Lemma 8

Proof of Lemma 8 Note that Eq. (30) and Ineqs. (31) and (32) constitute the necessary and sufficient condition for a lopsided equilibrium. By the definition of lopsided equilibriums and Bayes's rule, $0 < \pi_{-i}^R < \theta$ at any lopsided equilibrium. Thus, existence of a π_{-i}^R that satisfies Eq. (30) is equivalent to $0 < \theta - x_{-i} < \theta$, which by Eq. (2) is equivalent to Ineq. (33). Therefore, the necessary and sufficient condition for a lopsided equilibrium becomes: Ineq. (33) holds and there exists a (off-path posterior) $\pi_i^R \in [0,1]$ that satisfies both (31) and (32). For such π_i^R, there are only two possible cases:

Case (i): $\pi_i^R \leq \theta - x_{-i}$. In this case, Ineqs. (31) and (32) are equivalent to

$$\begin{aligned} x_{-i}(1-x_i) - (1-\theta)x_{-i} &\leq 0, \\ x_{-i}\left(1-\pi_i^R - x_i\right) + (1-\theta)\left(\theta - x_{-i} - \pi_i^R\right) &\leq 0. \end{aligned}$$

These inequalities, by Ineq. (33), are equivalent to

$$\begin{aligned} \theta - x_i &\leq 0, \\ x_{-i}(\theta - x_i) + \theta(1-\theta) &\leq \pi_i^R(1 + x_{-i} - \theta). \end{aligned}$$

The second inequality displayed above is equivalent to

$$\pi_i^R \geq \frac{x_{-i}\left(\theta - x_i\right) + \theta(1-\theta)}{1 + x_{-i} - \theta}.$$

This, coupled with the defining condition $\pi_i^R \leq \theta - x_{-i}$ for this case, means that Ineqs. (31) and (32) in this case are equivalent to

$$\frac{x_{-i}\left(\theta - x_i\right) + \theta(1-\theta)}{1 + x_{-i} - \theta} \leq \pi_i^R \leq \theta - x_{-i}. \tag{114}$$

That is, a desired π_i^R in this case exists if and only if $\theta - x_i \leq 0$ and

$$\frac{x_{-i}\left(\theta - x_i\right) + \theta(1-\theta)}{1 + x_{-i} - \theta} \leq \theta - x_{-i}.$$

i.e.,

$$x_{-i}\left(\theta - x_i\right) + \theta(1-\theta) \leq \left(1 + x_{-i} - \theta\right)\left(\theta - x_{-i}\right),$$

which is reduced to

$$x_i - x_{-i} \geq 1 - \theta.$$

Thus, the necessary and sufficient condition in this case becomes $x_i \geq \theta$ and $x_i - x_{-i} \geq 1 - \theta$. By Eq. (2) and $x_1 + x_2 = r$, the condition is equivalent to $x_i \geq \theta$ and $x_i - (r - x_1) \geq 1 - \theta$. The two inequalities together become Ineq. (34).

Case (ii): $\pi_i^R \geq \theta - x_{-i}$. In this case, Ineqs. (31) and (32) are equivalent to

$$\begin{aligned} x_{-i}\left(1 - x_i\right) + (1-\theta)\left(\pi_i^R - \theta\right) &\leq 0, \\ x_{-i}\left(1 - \pi_i^R - x_i\right) &\leq 0. \end{aligned}$$

Since $x_{-i} > 0$ due to neq. (33), the second inequality displayed above is equivalent to

$$1 - \pi_i^R - x_i \leq 0.$$

Thus, Ineqs. (31) and (32) are equivalent to

$$1 - x_i \leq \pi_i^R \leq \theta - \frac{x_{-i}(1-x_i)}{1-\theta}.$$

This, coupled with the defining condition $\pi_i^R \geq \theta - x_{-i}$ for this case, means that a desired π_i^R exists in this case if and only if

$$\max\left\{\theta - x_{-i}, 1 - x_i\right\} \leq \theta - \frac{x_{-i}(1-x_i)}{1-\theta},$$

which is equivalent to $\theta \leq x_i$ and $1 - x_i \leq \theta - \frac{x_{-i}(1-x_i)}{1-\theta}$. Combined with Eq. (2) and $x_1 + x_2 = r$, these two inequalities together become (35).

A.7 Theorem 3

Lemma 11 *A social-surplus maximum among lopsided proposal-equilibrium pairs is the one with maximum x_2 among those (x_1, x_2) that, with $i = 1$ and $-i = 2$, satisfy Ineq. (33) and at least one of Conditions (34) and (35).*

Proof Denote S_{lop} for the social surplus rendered by (x_1, x_2) coupled with its lopsided equilibrium (σ_1, σ_2). Since $\sigma_i(a) = \sigma_i(z) = 0$, $\sigma_{-i}(z) = 1$ and $0 < \sigma_{-i}(a) < 1$ at this equilibrium, the surplus for player i is equal to his expected payoff from Accept, and that for player $-i$ equal to her expected payoff from Reject. Thus, by Lemma 1,

$$\begin{aligned}
rS_{\text{lop}} &= \underbrace{\theta q^A_{-i}x_i + (1-\theta)\left(q^A_{-i}x_i + q^R_{-i}\theta\right)}_{\text{player } i} + \underbrace{\theta\left(\theta - \pi^R_{-i}\right) + (1-\theta)\theta}_{\text{player } -i} \\
&= q^A_{-i}x_i + \theta(1-\theta)q^R_{-i} + \theta\left(1 - \pi^R_{-i}\right) \\
&\overset{(30)}{=} \frac{x_{-i}}{1-\theta+x_{-i}}x_i + \theta(1-\theta)\frac{1-\theta}{1-\theta+x_{-i}} + \theta\left(1-\theta+x_{-i}\right) \\
&= \frac{x_{-i}}{1-\theta+x_{-i}}\left(r - x_{-i}\right) + \theta(1-\theta)\frac{1-\theta}{1-\theta+x_{-i}} + \theta\left(1-\theta+x_{-i}\right).
\end{aligned}$$

Denote

$$y := 1 - \theta + x_{-i},$$

so $x_{-i} = y - 1 + \theta$. Then

$$\begin{aligned}
rS_{\text{lop}} &= \frac{y-1+\theta}{y}\left(r - y + 1 - \theta\right) + \frac{\theta(1-\theta)^2}{y} + \theta y \\
&= \frac{1}{y}\left(-y^2 + \left(r + 2(1-\theta)\right)y - (1-\theta)\left(r + (1-\theta)^2\right)\right) + \theta y \\
&= -(1-\theta)y - \frac{(1-\theta)\left(r + (1-\theta)^2\right)}{y} + r + 2(1-\theta) \\
&= -(1-\theta)\left(y + \frac{r + (1-\theta)^2}{y}\right) + r + 2(1-\theta). \qquad (115)
\end{aligned}$$

We claim that rS_{lop} is strictly increasing in y, because

$$-\frac{1}{1-\theta}\frac{d}{dy}\left(rS_{\text{lop}}\right) = 1 - \frac{r + (1-\theta)^2}{y^2} = 1 - \frac{1/(1-a/z) + (1-\theta)^2}{\left(1 - (\theta - x_{-i})\right)^2},$$

which is negative because Ineq. (33) implies

$$\left(1 - (\theta - x_{-i})\right)^2 < 1 < 1/(1-a/z) + (1-\theta)^2.$$

Thus, rS_{lop} is strictly increasing in y, i.e., the social surplus is strictly increasing in x_{-i}, and hence strictly increasing in x_{-i}. ■

Proof of Theorem 3 By Lemma 8, the necessary and sufficient condition for a lopsided equilibrium is equivalent to simultaneous satisfaction of the following conditions:

$$x_{-i} > 0,$$
$$x_i \geq \theta,$$
$$\pi_i^R \leq \theta - x_{-i} \Longrightarrow x_i \geq \tfrac{1}{2}(r + 1 - \theta),$$
$$\pi_i^R \geq \theta - x_{-i} \Longrightarrow (1 - x_i)(1 - \theta + r - x_i) \leq \theta(1 - \theta).$$

The above is the same as the condition obtained by the proof of Lemma 8 except that the condition $x_{-i} < \theta$ in the lemma is removed here. The condition is removed without loss of generality because it is implied by the other condition $x_i \geq \theta$ here:

$$x_{-i} = r - x_i \leq r - \theta < \theta,$$

with the last strict inequality due to Ineq. (3).

Second, dissect the class of lopsided equilibriums into two cases according to the off-path posterior π_i^R: (i) $\pi_i^R \leq \theta - x_{-i}$; (ii) $\pi_i^R \geq \theta - x_{-i}$. For each subset we find the social-surplus maximum:

Case (i): $\pi_i^R \leq \theta - x_{-i}$. Then the necessary and sufficient condition becomes $x_{-i} > 0$ and $x_i \geq \max\{\theta, (r + 1 - \theta)/2\}$. With $x_{-i} = r - x_i$, the latter inequality is equivalent to $x_{-i} \leq \min\{r - \theta, (r + \theta - 1)/2\}$. Thus, Lemma 11 implies that the optimal split within Case (i) is $x_{-i} = \min\{r - \theta, (r + \theta - 1)/2\}$. Note that the condition $x_{-i} > 0$ is satisfied because $r > 1 > \theta$ and $r + \theta > 1$.

Case (ii): $\pi_i^R \geq \theta - x_{-i}$. Then the necessary and sufficient condition becomes $x_{-i} > 0$, $x_i \geq \theta$ and $(1 - x_i)(1 - \theta + x_{-i}) \leq \theta(1 - \theta)$. The second and third inequalities, in terms of x_{-i}, are equivalent to $x_{-i} \leq r - \theta$ and $(1 - r + x_{-i})(1 - \theta + x_{-i}) \leq \theta(1 - \theta)$. The last inequality is equivalent to

$$(x_{-i})^2 + (2 - r - \theta)x_{-i} + (\theta - 1)(r - 1) \leq \theta(1 - \theta),$$

i.e.,

$$\frac{r + \theta - 2 - \sqrt{r^2 - 2r\theta + 4\theta - 3\theta^2}}{2} \leq x_{-i} \leq \frac{r + \theta - 2 + \sqrt{r^2 - 2r\theta + 4\theta - 3\theta^2}}{2},$$

where the square root is real because $r^2 - 2r\theta + 4\theta - 3\theta^2 = (r - \theta)^2 + 4\theta(1 - \theta) > 0$. Thus, by Lemma 11, the optimal split within Case (ii) is that x_{-i} is equal to either $r - \theta$ or

$\left(r + \theta - 2 + \sqrt{r^2 - 2r\theta + 4\theta - 3\theta^2}\right)/2$, whichever is smaller. Note that the condition $x_{-i} > 0$ is satisfied: $r - \theta > 0$ as in Case (i); and

$$r + \theta - 2 + \sqrt{r^2 - 2r\theta + 4\theta - 3\theta^2} > r + \theta - 2 + \sqrt{(r-\theta)^2} > 2(r-1) > 0.$$

Third, it follows from the second step, as well as Lemma 11, that the social-surplus maximum among lopsided proposal-equilibrium pairs exists and is either the maximum within Case (i) or that within Case (ii), whichever has a larger x_{-i}. Note

$$(r + \theta - 1)/2 \le r - \theta \iff r \ge 3\theta - 1,$$

and

$$\begin{aligned} & \frac{r + \theta - 2 + \sqrt{r^2 - 2r\theta + 4\theta - 3\theta^2}}{2} \le r - \theta \\ \iff & \sqrt{r^2 - 2r\theta + 4\theta - 3\theta^2} \le 2 + r - 3\theta \\ \iff & 4(\theta - 1)(r + 1 - 3\theta) \le 0 \\ \iff & r \ge 3\theta - 1. \end{aligned}$$

Thus, when $r \ge 3\theta - 1$, the optimal split within Case (i) is $x_{-i} = (r + \theta - 1)/2$, and that within Case (ii) is $x_{-i} = \frac{r+\theta-2+\sqrt{r^2-2r\theta+4\theta-3\theta^2}}{2}$. Furthermore note that $\frac{r+\theta-2+\sqrt{r^2-2r\theta+4\theta-3\theta^2}}{2} \ge (r + \theta - 1)/2$. This is true because:

$$\begin{aligned} & \frac{r + \theta - 2 + \sqrt{r^2 - 2r\theta + 4\theta - 3\theta^2}}{2} \ge \frac{r + \theta - 1}{2} \iff \sqrt{r^2 - 2r\theta + 4\theta - 3\theta^2} \ge 1 \\ & \iff r^2 - 2r\theta + 4\theta - 3\theta^2 - 1 \ge 0 \iff (r + \theta - 1)(r - 3\theta + 1) \ge 0. \end{aligned}$$

Where the last inequality is satisfied by assumption that $r \ge 3\theta - 1$. Thus, when $r \ge 3\theta - 1$, the optimum among the entire lopsided equilibriums is that x_{-i} is equal to the maximum between Case (i)'s optimal split, i.e. $x_{-i} = (r + \theta - 1)/2$, and Case (ii)'s optimal split, i.e. $x_{-i} = \frac{r+\theta-2+\sqrt{r^2-2r\theta+4\theta-3\theta^2}}{2}$, which is the upper branch of Eq. (36) asserted by the theorem.

When $r \le 3\theta - 1$, the optimal split within either case is that $x_{-i} = r - \theta$, which is the lower branch of Eq. (36). ■

A.8 Theorems 4 and 5

Proof of Theorem 4 By Ineq. (19), Theorem 1 implies that the social surplus generated by the unbiased proposal, coupled with its MPM equilibrium, is equal to θ. This, combined with Theorem 2 and Lemma 7, implies that it suffices to prove that the maximum social surplus S^*_{lop} among lopsided proposal-equilibrium pairs is greater than θ.

By Theorem 3 and the hypothesis $\theta \le (r+1)/3$, S^*_{lop} is equal to the social surplus induced by a peace proposal (x^*_1, x^*_2) such that, for some $-i \in \{1,2\}$,

$$x^*_{-i} = \frac{1}{2}\left(r + \theta - 2 + \sqrt{r^2 - 2r\theta + 4\theta - 3\theta^2}\right),$$

which generates the social surplus S^*_{lop} such that

$$rS^*_{\text{lop}} =: \tilde{S}_{\text{lop}}(x^*_{-i}) = \frac{x^*_{-i}(r - x^*_{-i})}{1 - \theta + x^*_{-i}} + \frac{\theta(1-\theta)^2}{1 - \theta + x^*_{-i}} + \theta\left(1 - \theta + x^*_{-i}\right).$$

Note from the above equations that

$$\begin{aligned} \frac{\partial}{\partial r}\tilde{S}_{\text{lop}}(x^*_{-i}) &= \frac{x^*_{-i}}{1 - \theta + x^*_{-i}} > 0, \\ \frac{d}{dr}x^*_{-i} &= \frac{1}{2} + \frac{r - \theta}{2\sqrt{r^2 - 2r\theta + 4\theta - 3\theta^2}} > 0, \end{aligned}$$

with the second inequality due to the fact $r > 1 > \theta$. Recall from the proof of Lemma 11 that $\tilde{S}_{\text{lop}}(x_{-i})$ is strictly increasing in x_{-i}. Thus, by the chain rule, one can show that $\tilde{S}_{\text{lop}}(x^*_{-i})$ is strictly increasing in r:

$$\frac{d}{dr}\left(rS^*_{\text{lop}}\right) = \frac{d}{dr}\tilde{S}_{\text{lop}}(x^*_{-i}) = \frac{\partial}{\partial r}\tilde{S}_{\text{lop}}(x^*_{-i}) + \frac{\partial}{\partial x^*_{-i}}\tilde{S}_{\text{lop}}(x^*_{-i})\frac{d}{dr}x^*_{-i} > 0.$$

Thus, for any parameter value configuration (r,θ) that satisfy the hypothesis $\theta \le (r+1)/3$, $\tilde{S}_{\text{lop}}(x^*_{-i})$ is larger than

$$\begin{aligned} \tilde{S}_{\text{lop}}(x^*_{-i})\Big|_{r=3\theta-1} &= \tilde{S}_{\text{lop}}(2\theta - 1) \\ &= \frac{(2\theta-1)(3\theta - 1 - (2\theta - 1)}{1 - \theta + 2\theta - 1} + \frac{(\theta(1-\theta)^2}{1 - \theta + 2\theta - 1} + \theta(1 - \theta + 2\theta - 1) \\ &= 2\theta^2. \end{aligned}$$

Thus, if $\theta \le (r+1)/3$ then $\tilde{S}_{\text{lop}}(x^*_{-i})$ implies

$$S^*_{\text{lop}} \ge \frac{1}{r}\,\tilde{S}_{\text{lop}}(x^*_{-i})\Big|_{r=3\theta-1} = \frac{2\theta^2}{r}, \tag{116}$$

which by Ineq. (3) is greater than θ, as desired. ■

Proof of Theorem 5 When $\theta > 2(1+r)/5$, $\theta > (1+r)/3$, thus Theorem 3 implies that S^*_{lop} is equal to the social surplus induced by an unequal split such that, for some $-i \in \{1,2\}$,

$$x_{-i} = r - \theta$$

Thus, by Eq. (115),

$$rS^*_{\text{lop}} = -(1-\theta)\left(x + \frac{r+(1-\theta)^2}{x}\right) + r + 2(1-\theta),$$

where

$$x = 1 - \theta + x_{-i} = 1 - \theta + +r - \theta = 1 + r - 2\theta.$$

Thus,

$$\begin{aligned} rS^*_{\text{lop}} &= -(1-\theta)\left(r+1-2\theta + \frac{r+(1-\theta)^2}{r+1-2\theta}\right) + r + 2(1-\theta) \\ &= -(1-\theta)\left(r+1-2\theta + \frac{r+1-2\theta+\theta^2}{r+1-2\theta}\right) + r + 2(1-\theta) \\ &= -(1-\theta)\left(r+2(1-\theta) + \frac{\theta^2}{r+1-2\theta}\right) + r + 2(1-\theta) \\ &= \theta(r+2(1-\theta)) - (1-\theta)\left(\frac{\theta^2}{r+1-2\theta}\right). \end{aligned}$$

The asserted conclusion, $S^*_{\text{lop}} < \theta$, follows from the following chain of equivalent statements:

$$\begin{aligned} & \theta\left(r+2(1-\theta)\right) - (1-\theta)\left(\frac{\theta^2}{r+1-2\theta}\right) < r\theta \\ \iff & (1-\theta)\left(2\theta - \frac{\theta^2}{r+1-2\theta}\right) < 0 \\ \iff & (1-\theta)\left(\frac{2\theta r + 2\theta - 4\theta^2 - \theta^2}{r+1-2\theta}\right) < 0 \\ \iff & \theta(1-\theta)\left(\frac{2+2r-5\theta}{r+1-2\theta}\right) < 0, \end{aligned}$$

with the last " $\iff$ " due to the fact that $r > \theta$ and the hypothesis $\theta \leq \frac{2+2r}{5}$. ∎

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