Monotonic Cheap Talk*

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Abstract

This paper studies monotonic equilibria in a multi-sender version of Crawford and Sobel's (1982) cheap talk model, *i.e.* pure-strategy equilibria where senders' strategies are weakly monotonic in the state and where the receiver's strategy is strictly monotonic in the senders' messages. Monotonic equilibria have interval form, are bounded away from full revelation, and are straightforward to compute. When senders can be ranked according to bias: (i) in monotonic equilibria, senders most biased toward larger actions are informative when the receiver's desired action is smallest, and vice versa; and (ii) monotonic equilibria can be made collusion-proof by appropriately placing the receiver's off-path actions. If assumed alone, weak monotonicity of sender strategies generally has only a weak implication for the realized state-to-action function in a pure-strategy equilibrium. Strict monotonicity of the receiver's strategy is motivated by the possibility of misunderstanding.

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1 Introduction

Consider a sender-receiver model *à la* Crawford and Sobel (1982), but with multiple senders instead of one: senders observe a state $\theta \in [0, 1]$ that is unknown to the receiver and simultaneously each send a message to the receiver, who then chooses an action $a \in [0, 1]$. Senders are biased relative to the receiver in the sense that, conditional on θ, their preferred action differs from the receiver's. For example, a government may consult relatively hawkish and/or dovish experts about foreign policy, or a manager may solicit information from workers with different career concerns. Agents are unable to alter others' incentives (*e.g.* through monetary transfers), the receiver is unable to commit to a choice rule, and as θ increases, every agent desires a higher a.

It is well-known that, in such a model, there is typically a severe multiplicity of pure-strategy equilibria, which can range from full revelation to babbling (Krishna and Morgan (2001)). This paper proposes a monotonicity criterion that delivers a tractable set among pure-strategy equilibria. The requirement imposed on senders' strategies is that each equilibrium message is sent on a connected set of states: either a single point or a non-trivial interval. Therefore, for each sender, equilibrium messages can be ranked in a linear order $\geq_i$. The receiver's strategy is then required to be strictly monotonic with respect to these linear orders. That is, for any message vectors $m \neq m'$ such that $m_i \geq_i m'_i$ for all i, the receiver takes a strictly greater action after m than after m'.[1]

Section 3 shows that monotonic equilibria generalize one-sender equilibria in the following sense: each equilibrium message vector is sent on a non-trivial interval (except possibly at the endpoints of the state space), and for any pair of adjacent intervals, only one sender changes their message at the boundary θ, so that this sender must be indifferent at θ between the actions induced in the two intervals. As a result, if the differences between a sender's ideal action and the receiver's are bounded away from zero (as is assumed in this paper), the number of intervals is finite. Because

[1] This paper's main result can be obtained even if this condition is imposed *only* when at least one of m and m' occurs on path.

of the indifference condition between any two adjacent cells, every monotonic equilibrium corresponds to a version of Crawford and Sobel's (1982) forward solution extended to multiple senders, and can therefore be easily computed.

Not every equilibrium satisfying the above properties is monotonic. When senders can be ranked according to their biases, such as in the popular "uniform-quadratic" specification, an additional property of monotonic equilibria is that the boundaries where any single sender's message changes must be consecutive. That is, each sender is informative in an interval (possibly empty, if the sender babbles) of the state space, and these intervals are disjoint. Furthermore, if sender i's informative range is to the left of sender j's, then i must be more biased toward the right (or less toward the left) than j. This corresponds to the intuition that unexpected advice (*e.g.* when a right-wing expert advocates a left-wing policy) is often the most informative. However, it is noteworthy that this phenomenon occurs in *equilibrium* and is implied simply by monotonicity; it does *not* come from, for example, an expectation that right-biased experts would "lie" by reporting a higher state, so that it is surprising when they report a low state. Such an equilibrium has a simple interpretation: its on-path state-to-action function is the same as in a monotonic equilibrium where each sender reports a number from a set of consecutive integers, and the receiver's action, both on and off path, depends only on the sum of the reported integers. Thus, the receiver acts as though she were adding up the senders' messages. As shown in Section 4, in this case, the equilibrium is collusion-proof.

Monotonic equilibria yield sensible welfare properties, explored in Section 4: the receiver typically prefers senders with opposing and small biases. In fact, in the "uniform-quadratic" case, if the receiver could choose from a pool of senders, she would only need the sender with the smallest left-bias and the one with the smallest right-bias to achieve her best *ex ante* expected payoff in a monotonic equilibrium.

The assumption that sender strategies are monotonic can be motivated by senders using a language where words are ordered. It is relatively weak if made alone: Section 5 shows that it usually yields the same predictions for the on-path state-to-action function as assuming directly that this function is weakly increasing, and therefore does not rule out full revelation or many convoluted equilibria such as those constructed in Ambrus and Lu (2014) and Rubanov (2015). An alternative justification for the monotonicity of sender strategies is as follows. Consider an equilibrium where senders are "sincere" in that, at every state, each sender i prefers the action induced at that state to any other action induced by an on-path message vector where i's message is different.[2] Then there exists an equilibrium where sincerity still holds and where sender strategies are monotonic.

Therefore, the crucial part of the monotonicity assumption is the positive responsiveness of the receiver's action in response to a deviation by a sender. This responsiveness can be justified by the possibility of misunderstanding: when the receiver observes an out-of-equilibrium message vector, she is unsure whether she is misinterpreting higher messages as lower ones (in which case the state is high) or *vice versa*, and therefore picks an action somewhere between the optimal actions corresponding to these cases.

Several other papers have examined restrictions on equilibrium in one-dimensional multi-sender cheap talk, mainly in the context of robustness to noise in the senders' observation of the state. Battaglini (2002) argues that fully revealing equilibria are not robust to such noise. Ambrus and Lu (2014) and Rubanov (2015) show, however, that equilibria arbitrarily close to fully revealing are robust to the type of noise considered by Battaglini (2002) as the state space becomes large or as the number of senders grows. Lu (2017) shows that requiring robustness to a broader class of noise generically leads to the generalization of one-sender equilibria described in the third paragraph, which it calls "coordination-free equilibria." This paper thus uses a completely different (and simpler) approach to obtain a subset of the set of coordination-free equilibria. Monotonicity restrictions have been used in one-sender communication games by Kartik (2009) and Chen (2011), where payoffs were directly dependent on messages – *i.e.* these papers did not feature pure cheap talk models.

[2]Perhaps, in prior play, each sender observed only the state, their own message (not other senders') and the receiver's action, and is optimistic about what might happen after a deviation.

2 Model

A state $\theta \in \Theta = [0,1]$ is drawn according to $F(.)$, a cumulative distribution function with full support and a continuous prior density $f(.)$ such that $f(\theta) \in [d, D]$ for all θ, for some $d > 0$ and finite D. n senders $1, ..., n \in N$ observe θ and simultaneously each send a message to the receiver R, who takes an action $a \in A$ after receiving the messages. In order to simplify the statement of assumptions about preferences below, assume $A = \Theta$. Formally, sender i's pure strategies are functions $m_i : \Theta \to M_i$, where M_i, the set of messages available to i, has the same (or a greater) cardinality as Θ; the latter ensures that full revelation is possible. The receiver's pure strategies are functions $a : \times_{i=1}^{n} M_i \to \Theta$.

— The following is taken directly from Lu (2017), and will be rewritten later. —

All players' utilities depend on the state θ and the action $a \in \Theta$ taken by the receiver, but not (directly) on the message vector $m \in \times_{i=1}^{n} M_i$. Let $u_i(a, \theta)$ denote player i's utility when the action is a and state is θ, for $i = 1, ..., n, R$. The following standard assumptions are maintained throughout the paper:

1. all utility functions are Lipschitz continuous;
2. given θ, $u_R(., \theta)$ is smooth strictly concave with a maximum at $a = \theta$;
3. given θ, $u_i(., \theta)$ is single-peaked, *i.e.* is strictly increasing to the left, and strictly decreasing to the right of its unique maximum, denoted $\theta + b_i(\theta)$;
4. $\exists \eta > 0$ such that, for all $i \in N$ and $\theta \in \Theta$, either $|b_i(\theta)| > \eta$ or $\theta + b_i(\theta) \in \{0, 1\}$; and
5. for all $i \in N$, if $a < a'$, $\theta < \theta'$ and $u_i(a', \theta) \geq u_i(a, \theta)$, then $u_i(a', \theta') > u_i(a, \theta')$.

Assumption 2 implies that the receiver's best response is always unique. Assumption 4 and continuity imply that each i is either *right-biased* (for each θ, $b_i(\theta) > \eta$ or $\theta + b_i(\theta) = 1$) or *left-biased* (for each θ, $b_i(\theta) < -\eta$ or $\theta + b_i(\theta) = 0$). Assumption 5 is the commonly encountered single-crossing condition.

Let $a^\Gamma(m)$ denote the receiver's action given m in strategy profile Γ. Messages m_i and m_i' are said to be *equivalent in strategy profile* Γ if $a^\Gamma(m_i, m_{-i}) = a^\Gamma(m_i', m_{-i})$ whenever the vector m_{-i} is composed of messages that are each sent with positive probability at some θ in Γ.[3] Throughout this paper, when a given strategy profile Γ is discussed, $m_i = (\neq) m_i'$ means that m_i and m_i' are (not) equivalent in Γ, and $m = (\neq) m'$ means that each (some) component of m is (not) equivalent in Γ to the corresponding component of m'. That is, equivalent messages are treated as if they were the same message. As is standard for simultaneous multi-sender cheap talk in a continuous type space, this paper focuses on pure-strategy equilibria.[4]

The equilibrium concept is weak perfect Bayesian equilibrium (henceforth equilibrium). Pure strategy profile $\Gamma = (m_1, ..., m_n, a)$ and belief rule μ form an equilibrium if:

- for all $i \in N$ and all $\theta \in \Theta$, $m_i(\theta) \in \arg\max_{m_i' \in M_i} u_i(a(m_i', m_{-i}(\theta)), \theta)$,
- for all $m \in \times_{i=1}^n M_i$, $a(m) \in \arg\max_{a' \in \Theta} \int_{\theta \in \Theta} u_R(a', \theta) d\mu(m)$, and
- $\mu(m)$ is obtained from $f(.), m_1(.), ..., m_n(.)$ through Bayes' rule whenever $m = m^\Gamma(\theta)$ for some $\theta \in \Theta$.[5]

It will sometimes be convenient to abuse notation by using Γ to denote the equilibrium containing strategy profile Γ.

— End of content copied from Lu (2017) —

[3] This condition must hold even if m_{-i} itself is never sent in Γ. For example, suppose $a^\Gamma(1,1,1) = a^\Gamma(1,1,2) = a^\Gamma(1,2,1) = a^\Gamma(2,1,1) = a$, while $a^\Gamma(2,2,2) = a^\Gamma(2,2,1) = a^\Gamma(2,1,2) = a^\Gamma(1,2,2) = a' \neq a$. Also assume that of these eight message vectors, only $(1,1,1)$ and $(2,2,2)$ are sent in equilibrium. Then even though, on path, no sender's message affects the action, 1 and 2 are *not* equivalent for any sender: for example, if sender 1 sends 1 and sender 2 sends 2, then the actions induced by sender 3 through sending 1 and sending 2 are not the same.

[4] Here, this is in the sense that if a sender mixes between m_i and m_i', then m_i and m_i' must be equivalent. In the one-sender case, this is without loss of generality (except at the points where the sender's message changes) because any two messages leading to the same action are equivalent.

[5] Senders' strategies and μ must be such that the receiver's expected utility is well-defined. In particular, senders' strategies must be measurable, which implies that $\theta^\Gamma(m)$ is also measurable, for all m.

The following notation and terminology refer to a given pure-strategy profile Γ:

- $m_i^\Gamma(\theta)$ is sender i's message at θ, and $m^\Gamma(\theta) = (m_1^\Gamma(\theta), m_2^\Gamma(\theta), ..., m_n^\Gamma(\theta))$;
- $M_i^\Gamma = \{m_i : m_i^\Gamma(\theta) = m_i \text{ for some } \theta \in \Theta\}$;
- $\theta^\Gamma(m) = \{\theta : m^\Gamma(\theta) = m\}$ is the set of states where m is sent;
- $M^\Gamma = \{m : \theta^\Gamma(m) \neq \varnothing\}$ is the set of message vectors sent on path; and
- as in Lu (2017), a *cell in* Γ is a maximal interval of states throughout which m^Γ remains constant.[6] A *proper cell* is a cell with positive measure.

For any equilibrium Γ, there exists an equilibrium Γ' where play is the same as in Γ on path, and where the receiver's strategy is such that every out-of-equilibrium message $m_i \in M_i \backslash M_i^\Gamma$ is equivalent to a message in M_i^Γ. This paper assumes the latter for simplicity.

When two pure strategy profiles differ in the messages used by senders, they may still yield the same state-to-action function.

Definition: Strategy profiles Γ and Γ' are *outcome-equivalent* if $a^\Gamma(m^\Gamma(\theta)) = a^{\Gamma'}(m^{\Gamma'}(\theta))$ for all θ.

The following definitions relate to monotonicity.

Definition: Senders are *bias-ranked* if they are ordered $1, 2, \ldots, n$ such that for any $i, j \in \{1, \ldots, n\}$ and $a, a', \theta \in \Theta$ where $i < j$ and $a' > a$:
i) if $u_i(a, \theta) = u_i(a', \theta)$, then $u_j(a, \theta) < u_j(a', \theta)$,
ii) if $u_j(a, \theta) = u_j(a', \theta)$, then $u_i(a, \theta) > u_i(a', \theta)$.

If senders are bias-ranked, then whenever a sender i is indifferent between two actions $a < a'$, any sender $j > i$ prefers a', and any sender $j < i$ prefers a. In particular, this implies that at all states θ, senders' ideal points $\theta + b_i(\theta)$ are weakly

[6] That is, m^Γ does not remain constant in any connected strict superset of a cell. Cells can be degenerate intervals (*i.e.* they can consist of a single state).

increasing in i.[7] For example, if for every θ, $u_i(a,\theta) = u(|a - (\theta + b_i)|)$ where b_i is strictly increasing in i and u is single-peaked and has a maximum at 0, then the senders are biased-ranked - this includes the popular quadratic loss specification as long as no two senders have identical bias.

Definition: A sender strategy $m_i(.)$ is *monotonic* if, whenever $m_i(\theta) = m_i(\theta')$ for $\theta < \theta'$, $m_i(\theta'') = m_i(\theta)$ for all $\theta'' \in [\theta, \theta']$.

If a sender i uses a monotonic strategy, then the messages used by i can be unambiguously ranked from small to large. That is, when $m_i^\Gamma(.)$ is monotonic, there exists a linear order $\geq_{i,\Gamma}$ on M_i^Γ such that $m_i^\Gamma(\theta) \leq_{i,\Gamma} m_i^\Gamma(\theta')$ whenever $\theta < \theta'$. When every sender's strategy is monotonic, for message vectors m and m', let $m \leq_\Gamma m'$ denote $m_i \leq_{i,\Gamma} m_i'$ for all i and $m \neq m'$, and let $m \leqq_\Gamma m'$ denote that $m_i \leq_{i,\Gamma} m_i'$ for all i.

Definition: Given sender strategies m^Γ, a receiver strategy $a(.)$ is m^Γ*-monotonic* if for any $m \in M^\Gamma$ and any $m' \in \times_{i=1}^n M_i^\Gamma$, $a(m) < a(m')$ whenever $m \leq_\Gamma m'$ and $a(m) > a(m')$ whenever $m \geq_\Gamma m'$.

Definition: Given sender strategies m^Γ, a receiver strategy $a(.)$ is *strongly* m^Γ*-monotonic* if for any $m, m' \in \times_{i=1}^n M_i^\Gamma$, $a(m) < a(m')$ whenever $m \leq_\Gamma m'$.

A receiver plays a (strongly) m^Γ-monotonic strategy if her action is strictly increasing in the senders' messages according to the linear order derived from m^Γ; the basic definition considers only comparisons where at least one of the two message vectors appears on path in Γ, while the stronger definition considers comparisons between any two message vectors. Strictness is crucial for this paper's results: if it were left out, then the receiver could be unresponsive to unilateral deviations, which, when $n \geq 3$, are identifiable in strategy profiles where all senders reveal θ and in many others.

Definition: A strategy profile Γ is *sender-monotonic* if every sender's strategy is monotonic, and *(strongly) monotonic* if, additionally, the receiver's strategy is (strongly) m^Γ-monotonic.

[7] There is one exception to this: if every sender's ideal point is either 0 for all θ or 1 for all θ, then the above definition is vacuous because no sender is ever indifferent between two actions.

3 Basic Structure of Monotonic Equilibria

This section characterizes monotonic equilibria. First, it is shown that monotonic equilibria are a subset of the "coordination-free" equilibria identified by Lu (2017) through requiring robustness to noise in the senders' observation of θ.

Definition[8]**:** An equilibrium Γ is *coordination-free* if:

1. there are finitely many cells in Γ, and every cell (other than, if present, $\{0\}$ or $\{1\}$) is proper; and
2. the message vectors sent in any two adjacent cells in Γ differ in exactly one component.

As explained in Lu (2017), the on-path play of every coordination-free equilibrium correspond to a version of Crawford and Sobel's (1982) forward solution generalized to multiple senders: given the leftmost induced action, the receiver's optimality condition determines the right endpoint θ of the leftmost cell. Then, the indifference condition at θ of the sender whose message changes between the leftmost cell and the next cell determines the next induced action, and so on. A *generalized forward solution* is obtained when a cell endpoint is exactly 1. Like in the one-sender case, the number of solutions is finite, and the number of cells in each solution is also finite (though unlike in the one-sender case, not every generalized forward solution is part of an equilibrium). Therefore, coordination-free equilibria are tractable and bounded away from full revelation. Proposition 1 shows that monotonic equilibria are coordination-free.

Proposition 1. *If an equilibrium Γ is monotonic, then it is coordination-free.*

All omitted proofs are provided in the Appendix.

[8] By Proposition 1 of Lu (2017), this definition is equivalent to the definition in Lu (2017).

The key intuition for Proposition 1 is as follows. Consider two cells C and C' with $\sup C = \inf C' = \theta_b$, and denote the corresponding message vectors m and m' respectively. Suppose $m_1 \neq m'_1$, so that if m and m' differ in two or more components, we would have $m_{-1} \neq m'_{-1}$. It must be that, in C, sender 1 prefers $a^\Gamma(m)$ to $a^\Gamma(m'_1, m_{-1})$. By monotonicity, $a^\Gamma(m) < a^\Gamma(m'_1, m_{-1}) < a^\Gamma(m')$, so single-peakedness implies that sender 1 strictly prefers $a^\Gamma(m)$ to $a^\Gamma(m')$ in C. A symmetric argument shows that sender 1 must strictly prefer $a^\Gamma(m')$ to $a^\Gamma(m)$ in C'. The tension between these conclusions leads to a contradiction.

On the other hand, not every coordination-free equilibrium is monotonic. When senders are bias-ranked, it is especially easy to characterize the subset of coordination-free equilibria that are monotonic.

Proposition 2. *Suppose senders are bias-ranked. Then:*

a) if an equilibrium is monotonic (and therefore coordination-free), the index of the indifferent sender at cell boundaries weakly decreases from left to right; and

b) every generalized forward solution where the index of the indifferent sender at cell boundaries weakly decreases from left to right corresponds to a strongly monotonic equilibrium.

Proposition 2a implies that, if the senders can be ranked according to their biases, then in any monotonic equilibrium, the sender(s) most biased toward the right among informative senders is/are informative when θ is small, and *vice versa.* Such a phenomenon would be expected in settings where senders bias their message toward their desired action, perhaps because they believe that the receiver might be naïve or because they derive satisfaction from sending such messages. However, here, this result is obtained an implication of *equilibrium* and monotonicity in a pure cheap talk environment.

To see why this is the case, suppose $n = 2$, and that in an equilibrium Γ, at a cell boundary θ, sender 1's message changes from m_1 to m'_1, while at the next cell boundary $\theta' > \theta$, sender 2's message changesfrom m_2 to m'_2. Thus, from left to right, the message vector corresponding to the three cells at hand are (m_1, m_2), (m_1', m_2) and (m_1', m_2'). If the receiver's strategy is monotonic, we must have $a^\Gamma(m_1, m_2) < a^\Gamma(m_1, m_2') < a^\Gamma(m_1', m_2')$. At θ, sender 1 is indifferent between $a^\Gamma(m_1, m_2)$ and $a^\Gamma(m_1', m_2) > a^\Gamma(m_1, m_2)$, so that sender 2, who is more right-biased, strictly prefers $a^\Gamma(m_1', m_2)$ (and thus anything between $a^\Gamma(m_1, m_2)$ and $a^\Gamma(m_1', m_2)$) to $a^\Gamma(m_1, m_2)$. Thus, to avoid creating a profitable deviation for sender 2 in the leftmost cell, we must have $a^\Gamma(m_1, m_2') > a^\Gamma(m_1', m_2)$. But a symmetric argument for sender 1 in the rightmost cell yields the opposite conclusion. Therefore, Γ cannot be monotonic.

Moreover, Proposition 2 greatly simplifies finding all state-to-action functions that are possible in monotonic equilibria. Proposition 2a restricts the set of lists of indifferent senders that need to be tried to lists with weakly decreasing index. Proposition 2b shows that whenever such a list yields a generalized forward solution, it is possible to implement the corresponding play in a strongly monotonic equilibrium by suitably choosing the receiver's off-path actions. The construction in the proof of Proposition 2b is simple: assign consecutive integers to each sender's messages, from left to right, so that the sum of these numbers increases by 1 at every cell. For any out-of-equilibrium message vector, calculate the sum and place the corresponding action at the action following the on-path vector with the same sum. A sender i would then want to deviate to, say, a higher message only in regions where the influential sender is less right-biased. However, given the decreasing index, sender i must already be sending her highest message at those states. The receiver's strategy in this construction is particularly simple: simply add up all the senders' messages (treating a given sender's equivalent messages as equal), whether the message vector is expected on the path of play or not.

4 Properties of Monotonic Equilibria

4.1 Collusion-Proofness

This section shows that, when senders are bias-ranked, monotonic equilibria survive attempts by senders to collude.

Definition: An equilibrium Γ is *collusion-proof* if, for any $S \in 2^N \backslash \{\varnothing\}$ and any $\theta \in \Theta$, there exists no $m_S \in \times_{i \in S} M_i^\Gamma$ such that $u_i(a^\Gamma(m_S, m_{-S}^\Gamma(\theta)), \theta) > u_i(a^\Gamma(m^\Gamma(\theta)), \theta)$ for all $i \in S$. (That is, for any θ, in the complete-information simultaneous-move game induced by θ and a^Γ, senders must play a strong Nash equilibrium.)

An equilibrium fails to have this property if a group of senders can change their messages so that each of them is better off, provided that the receiver's strategy remains the same. For example, the receiver may simply be unaware that senders are able to collude.

In general, monotonicity is neither necessary (*e.g.* coordination-free equilibrium where every message vector is on path) nor sufficient (*e.g.* simple 3-cell example where bias-rankedness fails) for collusion-proofness. [EXAMPLE TO BE ADDED] However, when senders are bias-ranked, sufficiency holds in the following sense.

Proposition 3. *If senders are bias-ranked, then for every monotonic equilibrium, there is an outcome-equivalent monotonic equilibrium that is collusion-proof.*

Proof. Construct an outcome-equivalent monotonic equilibrium by placing actions following out-of-equilibrium message vectors in the same way as in proof of Proposition 2b. Then, any sender that can gain from inducing a higher (lower) action is already sending her highest (lowest) message. Therefore, it is impossible for any group of such senders, even with a joint deviation, to induce a higher (lower) action. ■

4.2 Welfare

Because monotonic equilibria are a strict subset of coordination-free equilibria, the receiver's maximum equilibrium welfare will often be lower than in Lu (2017). However, it remains generally true that, for example, given a right-biased sender, additional senders with bigger biases to the right are of limited help to the receiver:

going from left to right, these additional senders would usually cause a more rapid increase in cell sizes.

In the "uniform-quadratic" benchmark case with $\theta \sim U[0,1]$, quadratic loss functions and state-independent biases, some sharp results can be derived.

Proposition 4. *If all biases have the same sign, the best equilibrium for the receiver is simply the one-sender equilibrium with the least biased sender that has the most cells.*

Proposition 4 is the same result as in Lu (2017), whose proof applies here because the optimal coordination-free equilibrium is monotonic.

Proposition 5. *If there are senders biased in both directions, only the least biased sender in each direction is informative in the best equilibrium for the receiver.*

By Proposition 2a, we know that in equilibrium in the uniform-quadratic case, cell sizes first grow (when the influential sender(s) is/are right-biased) and then shrink from left to right. The intuition for this result is that if the influential sender at a cell boundary is replaced by a less biased one, cell sizes become more evenly distributed. For example, if the senders are right-biased, cells to the left of that boundary, which are smaller on average, become bigger at the expense of cells to the right of that boundary. This helps the receiver given her concave loss function.[9]

Another observation is that, fixing biases and taking the size of Θ to infinity, the receiver's best expected utility in a monotonic equilibrium goes to $-\infty$. This result is the same as in Crawford and Sobel (1982), and different from Lu (2017). The reason is that, going from left to right, once the size of cells has decreased, it cannot increase again in a monotonic equilibrium, unlike in coordination-free equilibria in

[9] By contrast, in Lu (2017), this result is only asymptotically true in the sense that, fixing biases and taking the size of Θ to infinity, the limit of the receiver's best equilibrium payoff depends only on the smallest bias in each direction. This is because, when all coordination-free equilibria are considered, going from left to right, cell sizes can go through multiple cycles of growing and shrinking. This paper's result is therefore stronger.

general. As a result, in monotonic equilibria, only cells near the ends of Θ can be small, and the middle cells must grow without bound as Θ grows without bound.

5 Discussion of Monotonicity

This section first discusses the monotonicity of sender strategies. It shows that, alone, this assumption does not produce strong results - which implies that its power comes when coupled with the assumption that the receiver's strategy is also monotonic - and proposes one way to rationalize sender-monotonicity as well as weak monotonicity of the receiver's strategy. A justification for the monotonicity of the receiver's strategy is then provided.

5.1 Sender-Monotonicity

Consider the following definition.

Definition: A strategy profile Γ is *action-monotonic* if $a^{\Gamma}(m^{\Gamma}(\theta))$ is weakly increasing in θ.

It is straightforward to see that sender-monotonicity implies action-monotonicity. Proposition 6 shows that the converse is true in terms of outcome predictions when there are three or more senders, or when any action is induced by at most one on-path message vector. That is, except in some special cases, the state-to-action function of any action-monotonic equilibrium can be obtained in a sender-monotonic equilibrium. This result applies to any pure-strategy fully revealing equilibrium as well as the classes of equilibria proposed by Ambrus and Lu (2014) and Rubanov (2015).

Proposition 6. *If an equilibrium Γ is action-monotonic, then there exists an outcome-equivalent sender-monotonic equilibrium except when $n = 2$ and multiple on-path message vectors induce the same action in Γ.*

When $n \geq 3$, a simple construction Γ' yields the state-to-action function of any action-monotonic equilibrium Γ: have each sender report the action that would be induced in Γ, and ignore any unilateral deviation, which is always detectable when $n \geq 3$. Γ' is sender-monotonic due to the action-monotonicity of Γ. When $n = 2$, the placement of out-of-equilibrium actions in Γ' is more delicate; this can be done by finding a corresponding message vector in Γ when no two on-path message vectors induce the same action in Γ.

The remainder of this subsection proposes a condition that leads to action-monotonicity and sender-monotonicity.

Definition: A strategy profile Γ is *sincere* if, for every $i \in N$ and $\theta \in \Theta$, $u_i(a^\Gamma(m^\Gamma(\theta)), \theta) \geq u_i(a^\Gamma(m^\Gamma(\theta')), \theta)$ for all θ' such that $m_i^\Gamma(\theta') \neq m_i^\Gamma(\theta)$.

That is, in a sincere strategy profile Γ, each sender always weakly prefers the action that results from playing according to Γ to any on-path action that might result from sending a different message. To see why sincerity is an interesting property, consider a situation where play has long occurred according to Γ, with a sender observing each time the receiver's action a, but not the other senders' messages. This sender eventually observes the set of on-path actions, and knows that each of them is inducible by some message vector. If Γ is not sincere, then there exists a state θ where the sender knows for sure that deviating would be profitable for some combination of other senders' messages. If, on the other hand, Γ is sincere, then no such state exists: a deviation can be profitable only if it results in an off-path message vector and the receiver happens to respond with an action that the sender prefers.

Definition: Given sender strategies m^Γ, a receiver strategy $a(.)$ is *weakly* m^Γ*-monotonic* if for any $m \in M^\Gamma$ and any $m' \in \times_{i=1}^n M_i^\Gamma$, $a(m) \leq a(m')$ whenever $m \leq_\Gamma m'$ and $a(m) \geq a(m')$ whenever $m \geq_\Gamma m'$. A strategy profile Γ is *weakly monotonic* if every sender's strategy is monotonic and the receiver's strategy is weakly m^Γ-monotonic.

Proposition 7. *a) If a strategy profile* Γ *is sincere and* a^Γ *is a best response to* m^Γ *on path, then it is outcome-equivalent to a sincere weakly monotonic coordination-free equilibrium.*

b) If an equilibrium Γ *is monotonic, then it is sincere.*

5.2 Receiver-Monotonicity

This subsection takes as given that senders' strategies m^Γ are monotonic, and provides one explanation why the receiver's strategy would be m^Γ-monotonic. Assume that $n = 2$, and that senders play monotonic strategies m_i^Γ. Denote sender i's messages as m_i^k, where $m_i^k <_{i,\Gamma} m_i^{k'}$ if and only if $k < k'$. Suppose the receiver observes off-path message vector (m_1^1, m_2^2), while (m_1^1, m_2^1) and (m_1^2, m_2^2) can occur on path.

The receiver may believe that she has misunderstood one of the senders' messages. Thus, if (m_1^1, m_2^1) and (m_1^2, m_2^2) are the only message vectors from which a single change yields (m_1^1, m_2^2), then the receiver would believe that the state is much more likely to lie in $\theta^\Gamma(m_1^1, m_2^1)$ or $\theta^\Gamma(m_1^2, m_2^2)$ than, say, $\theta^\Gamma(m_1^3, m_2^3)$. In that case, her best response $a^\Gamma(m_1^1, m_2^2)$ should be within $[a^\Gamma(m_1^1, m_2^1), a^\Gamma(m_1^2, m_2^2)]$. Furthermore, unless the receiver is sure that she misunderstood one of the senders' messages and not the other's, we should have $a^\Gamma(m_1^1, m_2^2) \in (a^\Gamma(m_1^1, m_2^1), a^\Gamma(m_1^2, m_2^2))$.

What if there is another message vector, say (m_1^1, m_2^0), that also differs from (m_1^1, m_2^2) in only one component? In this case, if the receiver believes that she is much more likely to confuse similar messages such as m_2^1 and m_2^2 than more distant messages such as m_2^0 and m_2^2, then we should still have $a^\Gamma(m_1^1, m_2^2) \in (a^\Gamma(m_1^1, m_2^1), a^\Gamma(m_1^2, m_2^2))$. (If not, then we could have $a^\Gamma(m_1^1, m_2^2) < a^\Gamma(m_1^1, m_2^1)$.) The definitions below and Proposition 8 formalize this intuition.

Let Π be the set of ordered set partitions of N, and for any $\pi \in \Pi$ and function h, let $\lim_\pi h(\sigma_1, \ldots, \sigma_n)$ denote

$$\lim_{\sigma_i \to \infty, \forall i \in \pi(k_\pi)} \left[\lim_{\sigma_i \to \infty, \forall i \in \pi(k_\pi - 1)} \left[\cdots \left[\lim_{\sigma_i \to \infty, \forall i \in \pi(1)} h(\sigma_1, \ldots, \sigma_n) \right] \cdots \right] \right], \qquad (1)$$

where k_π is the number of elements of π and $\pi(k)$ is the k^{th} element of π.

Definition: An equilibrium Γ is *consistent with the possibility of misunderstanding* if, for some $\{\varepsilon_{\pi,m}\}_{\pi\in\Pi, m\in\times_{i=1}^{n} M_i^\Gamma}$ and $\{g_i(.|.)\}_{i=1}^{n}$,

$$a^\Gamma(m) \in \arg\max \sum_{\pi\in\Pi} \left[\varepsilon_{\pi,m} \lim_{\pi} \frac{\int_\Theta u_R(a^\Gamma(m),\theta) f(\theta) \prod_{i=1}^{n} g_i(m_i|m_i^\Gamma(\theta))^{\sigma_i} d\theta}{\int_\Theta f(\theta) \prod_{i=1}^{n} g_i(m_i|m_i^\Gamma(\theta))^{\sigma_i} d\theta} \right] \text{ for all } m \in \times_{i=1}^{n} M_i^\Gamma, \tag{2}$$

where:

i) $\varepsilon_{\pi,m} > 0$ for all $\pi \in \Pi$ and $m \in \times_{i=1}^{n} M_i^\Gamma$;

ii) for all $i \in N$ and $m_i, m_i', m_i'' \in M_i^\Gamma$, $g_i(m_i|m_i) = 1$, $g_i(m_i|m_i')\in (0,1)$ if $m_i \neq m_i'$, and $g_i(m_i|m_i') > g_i(m_i|m_i'')$ whenever $m_i'' <_{i,\Gamma} m_i' <_{i,\Gamma} m_i$ or $m_i <_{i,\Gamma} m_i' <_{i,\Gamma} m_i''$.

The features of this definition are as follows.

1. It focuses on the limit in which the likelihood of misunderstanding vanishes ($g_i(m_i|m_i^\Gamma(\theta))^{\sigma_i} \to 0$ whenever $m_i \neq m_i^\Gamma(\theta)$ since $g_i(m_i|m_i')\in (0,1)$ if $m_i \neq m_i'$, and $\sigma_i \to \infty$).

2. Misunderstanding is more likely between closer messages than farther messages ($g_i(m_i|m_i') > g_i(m_i|m_i'')$ whenever $m_i'' <_{i,\Gamma} m_i' <_{i,\Gamma} m_i$ or $m_i <_{i,\Gamma} m_i' <_{i,\Gamma} m_i''$).

3. In the limit, some senders may be much more reliable than others (corresponding to different π; messages from senders in $\pi(1)$ are much more reliable than those in $\pi(2)$, *etc.*).

4. Which senders are more reliable is uncertain and may depend on m ($\varepsilon_{\pi,m} > 0$).

Proposition 8. *a) If an equilibrium is sender-monotonic and consistent with the possibility of misunderstanding, then it is monotonic.*

b) If an equilibrium is monotonic and no cell is $\{0\}$ *or* $\{1\}$*, then it is consistent with the possibility of misunderstanding.*

6 Conclusion

This paper shows that requiring monotonicity in sender strategies and strict monotonicity in the receiver strategy in a multi-sender cheap talk game with simultaneous messages yields a subset of coordination-free equilibria. The class of coordination-free equilibria is finite and relatively straightforward to compute. In the monotonic equilibria selected here, when the senders are bias-ranked, the most right-biased senders are informative (if anywhere) in states to the left of states where the most left-biased senders are informative (if any); these equilibria are robust to various forms of collusion. Welfare properties are intuitive (*e.g.* the receiver generally prefers senders with small and opposite biases) and, with the uniform-quadratic specification, sharp (*e.g.* given the choice between many senders, the receiver can achieve her best monotonic equilibrium payoff with just two: the senders with the smallest positive and the smallest negative biases).

Requiring sender-monotonicity is in most cases essentially equivalent, in equilibrium, to requiring that the receiver's action be (weakly) monotonic in the state. Many non-coordination-free equilibria are consistent with this assumption. The receiver's action being strictly monotonic in the senders' messages, which is key to restricting the set of equilibria, can be motivated by a possibility of misinterpreting a message as a different one that grows as the two messages get close.

7 Appendix: Proofs

Proof of Proposition 1. Since Γ is sender-monotonic, $\theta^\Gamma(m)$ is convex. Thus, for any $m, m' \in M^\Gamma$, $a^\Gamma(m) \neq a^\Gamma(m')$.

The following shows that M^Γ must be finite. Suppose not. Then $\exists m, m' \in M^\Gamma$ such that $|a^\Gamma(m) - a^\Gamma(m')| < \eta$. Let i be a sender such that $m_i \neq m_i'$, and assume without loss of generality that i is right-biased and that $a^\Gamma(m) < a^\Gamma(m')$. Then there exists $\theta \in \theta^\Gamma(m)$ such that $u_i(a^\Gamma(m'), \theta) > u_i(a^\Gamma(m), \theta)$. By monotonicity, $a^\Gamma(m) < a^\Gamma(m_i', m_{-i}) < a^\Gamma(m')$, which implies $u_i(a^\Gamma(m_i', m_{-i}), \theta) > u_i(a^\Gamma(m), \theta)$. This is a contradiction since i has a profitable deviation at θ.

Since $\theta^\Gamma(m)$ is convex, there is one cell in Γ for each $m \in M^\Gamma$. Therefore, the number of cells in Γ is also finite.

Next, it is shown that if Γ is monotonic, then for any two cells C and C' in Γ with $\sup C = \inf C' = \theta_b$, the corresponding message vectors m and m' differ in only one component. Suppose not, so that m and m' differ in two or more components. By monotonicity, $m \leq_\Gamma m'$ and $a^\Gamma(m) < a^\Gamma(m')$. Assume without loss of generality that $m_1 \neq m_1'$, and note that $m_{-1} \neq m_{-1}'$.

Consider the location of $a^\Gamma(m_1', m_{-1})$. By monotonicity, we must have $a^\Gamma(m) < a^\Gamma(m_1', m_{-1}) < a^\Gamma(m')$, and to avoid a profitable deviation by sender 1, we need $u_1(a^\Gamma(m), \theta) \geq u_1(a^\Gamma(m_1', m_{-1}), \theta)$ for all $\theta \in C$. This is possible only if $u_1(a^\Gamma(m), \theta) > u_1(a^\Gamma(m'), \theta)$ for all $\theta \in C$.

Similarly, we must have $a^\Gamma(m) < a^\Gamma(m_1, m_{-1}') < a^\Gamma(m')$ and $u_1(a^\Gamma(m'), \theta) \geq u_1(a^\Gamma(m_1, m_{-1}'), \theta)$ for all $\theta \in C'$, which is possible only if $u_1(a^\Gamma(m'), \theta) > u_1(a^\Gamma(m), \theta)$ for all $\theta \in C'$.

It follows that neither C or C' contains θ_b, and that $u_1(a^\Gamma(m), \theta_b) = u_1(a^\Gamma(m'), \theta_b)$. But by the continuity of u_1, for any $\varepsilon > 0$, if $a^\Gamma(m_1', m_{-1}) = a^\Gamma(m') - \varepsilon$, then there exists a nontrivial interval immediately to the left of θ_b where sender 1 prefers $a^\Gamma(m_1', m_{-1})$ to $a^\Gamma(m)$. This implies a profitable deviation, and contradicts that Γ is an equilibrium.

Finally, it remains to be verified that every cell (other than, if present, $\{0\}$ or $\{1\}$) is proper. If, instead, $\{\theta_b\}$ were its own cell, then by the finiteness of the number of cells, there would be two cells C and C' with $\sup C = \inf C' = \theta_b$ and corresponding message vectors m and m', respectively, such that m, m' and $m^\Gamma(\theta_b)$ differ pairwise in only one component. It follows that these three vectors differ in only one component, say i. But this implies $u_i(a^\Gamma(m), \theta_b) = u_i(\theta_b, \theta_b) = u_i(a^\Gamma(m'), \theta_b)$, which is not possible by single-peakedness since, by monotonicity, $a^\Gamma(m) < \theta_b < a^\Gamma(m')$. ■

Proof of Proposition 2. a) Suppose not, and let Γ be a monotonic equilibrium. Then there are two boundaries $\theta < \theta'$ separating adjacent cells with message vectors m, m', m'' from left to right such that $u_i(a^\Gamma(m), \theta) = u_i(a^\Gamma(m'), \theta)$, $u_j(a^\Gamma(m'), \theta') = u_j(a^\Gamma(m''), \theta')$, and $i < j$. By Proposition 1, m and m' differ only in component i, and m' and m'' differ only in component j. Now consider message vector m^*, which is equal to m except that component j is replaced by m''_j. By monotonicity, m^* does not occur on the equilibrium path, and $a^\Gamma(m) < a^\Gamma(m^*) < a^\Gamma(m'')$.

Since $i < j$, $u_j(a^\Gamma(m), \theta) < u_j(a^\Gamma(m'), \theta)$. Therefore, if $a^\Gamma(m) < a^\Gamma(m^*) \leq a^\Gamma(m')$, then by single-peakness, $u_j(a^\Gamma(m), \theta) < u_j(a^\Gamma(m^*), \theta)$. By continuity, immediately to the left of θ, sender j has a profitable deviation from m_j to m'_j. Therefore, we must have $a^\Gamma(m') < a^\Gamma(m^*) < a^\Gamma(m'')$.

Since $i < j$, $u_i(a^\Gamma(m'), \theta') > u_i(a^\Gamma(m''), \theta')$. Therefore, since $a^\Gamma(m') < a^\Gamma(m^*) < a^\Gamma(m'')$, then by single-peakness, $u_i(a^\Gamma(m''), \theta') < u_i(a^\Gamma(m^*), \theta')$. By continuity, immediately to the right of θ, sender i has a profitable deviation from m''_i to m_i. This yields the desired contradiction.

b) Let $m_i^\Gamma(0) = 0$ for all i, and let $m_i^\Gamma(\theta)$ be equal to the number of cell boundaries between 0 and θ where sender i's message changes. $a^\Gamma(m)$ is given by the generalized forward solution for on-path m; for off-path m, let $a^\Gamma(m) = a^\Gamma(m')$ where m' occurs on path and $\sum_{i=1}^n m_i = \sum_{i=1}^n m'_i$ (by construction, m' exists and is unique). Thus, in this construction, the set of the receiver's off-path actions is a subset of her on-path actions.

By construction, for any off-path m, at most two senders have a unilateral deviation that leads to m: at most one by deviating to a higher message, and at most one by deviating to a lower message. (For example, if i and j can both do the former, then (m'_i, m_j, m_{-ij}) and (m_i, m'_j, m_{-ij}) both occur on path, which is not possible in this construction since $m_i' < m_i$ and $m_j > m_j'$.) Suppose that i can do the former, and let Θ_0 be the union of cells such that i's message changes at the right endpoint. In Θ_0, sender i is influential on path, and has no desire to deviate and induce a higher action. To the left of Θ_0, the influential senders are more right-biased than i, so once again i does not want to induce a higher action. Finally, to the right of Θ_0, sender i is already sending her highest message, so she is unable to induce a higher action. Therefore, sender i never has an incentive induce m when she is able to do so; a symmetric argument rules out deviations to a lower message. ■

Proof of Proposition 5. TO BE COMPLETED as sketched in the main text ■

Proof of Proposition 6. Since Γ is action-monotonic, every on-path action is induced in an (possibly trivial) interval. Let I_a^Γ be the interval of states where action a is induced, and let A^Γ be the set of a induced on path in Γ. The construction of a sender-monotonic equilibrium Γ' that is outcome-equivalent to Γ is given below for three different cases:

- $n \geq 3$: Choose messages $\{m_i^a\}_{a \in A^\Gamma}$ such that $m_i^a \neq m_i^{a'}$ whenever $a \neq a'$, and let $m_i^{\Gamma'}(\theta) = m_i^a$ for $\theta \in I_a^\Gamma$. These sender strategies are monotonic since each message is sent on an interval. Optimality for the receiver in Γ implies that $a^{\Gamma'}(m_1^a, ..., m_n^a) = a$ is optimal when responding to $m^{\Gamma'}$, so that Γ' is outcome-equivalent to Γ. Finally, to avoid profitable deviations by senders, it suffices that any unilateral deviation - always detectable since $n \geq 3$ - does not change the receiver's action in Γ'.

- $n = 2$ and neither sender babbles: Let the senders' strategies in Γ' be as in the $n \geq 3$ case, so that the receiver's optimality still implies outcome-equivalence. By assumption, there is a single message vector inducing each $a \in A^\Gamma$; denote it by $m^{\Gamma,a}$. For each pair $a \neq a'$, let $a^{\Gamma'}(m_1^a, m_2^{a'}) = a^\Gamma(m_1^{\Gamma,a}, m_2^{\Gamma,a'})$. In Γ', the action $a^{\Gamma'}(m_1^a, m_2^{a'})$ can result from a deviation by sender 1 when $\theta \in I_{a'}^\Gamma$ or by sender 2 when $\theta \in I_a^\Gamma$. No such deviation can be profitable: otherwise, there would also be a profitable deviation in Γ either by sender 1 when $\theta \in I_{a'}^\Gamma$ or by sender 2 when $\theta \in I_a^\Gamma$.

- $n = 1$, or $n = 2$ and at least one of the senders babbles: By Crawford and Sobel (1982) and because any equivalent messages are treated as equal in this paper, Γ is itself sender-monotonic. ■

Proof of Proposition 7. a) Γ being sincere implies that if $m_i^\Gamma(\theta) \neq m_i^\Gamma(\theta')$, we must have $u_i(a^\Gamma(m^\Gamma(\theta)), \theta) \geq u_i(a^\Gamma(m^\Gamma(\theta')), \theta)$ and $u_i(a^\Gamma(m^\Gamma(\theta')), \theta') \geq u_i(a^\Gamma(m^\Gamma(\theta)), \theta')$. Thus, either $a^\Gamma(m^\Gamma(\theta)) = a^\Gamma(m^\Gamma(\theta'))$, or sender i must experience a preference reversal from $a^\Gamma(m^\Gamma(\theta))$ to $a^\Gamma(m^\Gamma(\theta'))$ as the state shifts from θ to θ'. It follows from single-crossing that Γ is action-monotonic.

Furthermore, if $a^\Gamma(m^\Gamma(\theta)) > a^\Gamma(m^\Gamma(\theta'))$, we must have $a^\Gamma(m^\Gamma(\theta)) - a^\Gamma(m^\Gamma(\theta')) > \eta$: otherwise, consider i such that $m_i^\Gamma(\theta) \neq m_i^\Gamma(\theta')$, and assume without loss of generality that $b_i(.) > 0$ and $\theta > \theta'$. Then sender i would prefer $a^\Gamma(m^\Gamma(\theta))$ to $a^\Gamma(m^\Gamma(\theta'))$ at all states equal to or above $a^\Gamma(m^\Gamma(\theta'))$, which implies by sincerity that $m^\Gamma(\theta')$ can be sent only at states below $a^\Gamma(m^\Gamma(\theta'))$. This contradicts $a^\Gamma(m^\Gamma(\theta'))$ corresponding to the receiver's optimum. As a result, only a finite number of actions are inducible on the path of Γ.

Now consider two consecutive inducible actions $a > a'$, and let θ be the state at which the induced action switches from a' to a. By sincerity, every sender whose message changes at θ must be indifferent at θ between a and a'.

Given the above conclusions, we build an outcome-monotonic strategy profile Γ' as follows:

- Call the inducible actions, from left to right, $a_1, ..., a_K$, and call the cells where these actions are induced $I_1, ..., I_K$, respectively. Γ' will feature a single on-path message vector, denoted m^k, for each cell I_k.

- Fix $m_i^1 = 1$ for all i.

- For $k = 2, ..., K$, let i_k be a sender whose message in Γ changes between I_{k-1} and I_k at the boundary. (Note that i_k is thus indifferent between a_{k-1} and a_k.) Fix $m_{i_k}^k = m_{i_k}^{k-1} + 1$ and $m_j^k = m_j^{k-1}$ for all $j \neq i_k$.

- For $k = 1, ..., K$, let $a^{\Gamma'}(m^k) = a_k$.

By construction, Γ' is monotonic, $a^{\Gamma'}$ is a best response to the senders' strategies, and Γ' is outcome-equivalent to Γ. It remains to be shown that: (i) Γ' is coordination-free; (ii) Γ' is sincere; (iii) $a^{\Gamma'}(m)$ for off-path m can be defined such that $a^{\Gamma'}$ is weakly $m^{\Gamma'}$-monotonic and no profitable sender deviation is induced.

(i) This is almost true by definition: we need only show that no cell (other than possibly I_1 and/or I_K) is a singleton. Suppose instead that $I_k = \{\theta_k\}$, and let S be the set of senders indifferent between a_{k-1} and a_k at θ_k (thus S includes at least i_k). By sincerity, in Γ, no sender outside of S sends different messages in the vectors inducing a_{k-1} and a_k, so at least one sender inside S must do so. This sender (say, without loss of generality, i_k) must, by single-peakedness, strictly prefer both a_{k-1} and a_k to a_{k+1}. The same must therefore hold slightly to the right of θ_k, *i.e.* within I_{k+1}. But this means that Γ could not be sincere: i_k strictly prefers a_{k-1} to a_{k+1} at some states in I_{k+1}, and i_k's messages in vectors inducing a_{k-1} and a_{k+1} differ.

(ii) Suppose Γ' is not sincere despite Γ being sincere. Then there exists i, k, l such that $u_i(a_k, \theta) < u_i(a_l, \theta)$ for some $\theta \in I_k$, with $m_i^\Gamma(\theta) = m_i^\Gamma(\theta')$ for all $\theta' \in I_l$ and $m_i^k \neq m_i^l$. By construction, since $m_i^k \neq m_i^l$, i's message in Γ must also change between θ and I_l (since $m_i^\Gamma(\theta) = m_i^\Gamma(\theta')$ for all $\theta' \in I_l$, it changes at least twice). Thus, for some $\theta'' \in (\theta, \inf(I_l)]$, we have $m_i^\Gamma(\theta'') \neq m_i^\Gamma(\theta)$. If $\theta'' \in I_k$, then Γ is not sincere since, by single crossing, $u_i(a_k, \theta'') < u_i(a_l, \theta'')$. If instead $\theta'' \notin I_k$, then by single-peakedness, $u_i(a_k, \theta) < u_i(a^\Gamma(m^\Gamma(\theta'')), \theta)$, so once again Γ is not sincere.

(iii) The only relevant message vectors are those that differ from some on-path message vectors in only one component (others cannot result from an unilateral deviation, and hence the action they induce can be placed anywhere, as long as monotonicity is satisfied). Due to sender-monotonicity, any such off-path m can result from a unilateral deviation by at most two different senders (only one can deviate to a higher message, and one to a lower message). Therefore, we distinguish two cases:

- Only one sender can induce m via unilateral deviation: In this case, setting $a^{\Gamma'}(m) = a^{\Gamma'}(m^*)$, where m^* is the same as m, except that the message of the deviating sender is replaced by the closest message such that m^* is on-path, satisfies weak monotonicity and does not induce any profitable deviation.

- Sender i can induce m by deviating to a higher message, while sender j can induce m by deviating to a lower message: Let I_k be the rightmost interval from which i can do so, and $I_{k'}$ be the leftmost interval from which j can do so. By construction, $k < k'$. Suppose $a^{\Gamma'}(m) = a_k$. Clearly, this does not induce any profitable deviation by i. Moreover, since Γ' is sincere, this also does not induce a profitable deviation by j.

b) Suppose Γ is not sincere. Then for some i, θ and θ', we have $u_i(a^\Gamma(m^\Gamma(\theta)), \theta) < u_i(a^\Gamma(m^\Gamma(\theta')), \theta)$, with $m_i^\Gamma(\theta') \neq m_i^\Gamma(\theta)$. Assume without loss of generality that $m_i^\Gamma(\theta') > m_i^\Gamma(\theta)$. By monotonicity, $a^\Gamma(m^\Gamma(\theta)) < a^\Gamma((m_i^\Gamma(\theta'), m_{-i}^\Gamma(\theta))) \leq a^\Gamma(m^\Gamma(\theta'))$. Thus, by single-peakedness, $u_i(a^\Gamma((m_i^\Gamma(\theta'), m_{-i}^\Gamma(\theta))), \theta) > u_i(a^\Gamma(m^\Gamma(\theta)), \theta)$. This implies that i has a profitable deviation at θ, which contradicts Γ being an equilibrium. ■

Proof of Proposition 8. a) Let $E = \{\theta : \theta^\Gamma(m)$ is the singleton $\{\theta\}$ for some $m \in M^\Gamma$, and θ is the endpoint of a cell C with positive measure$\}$. This proof proceeds in four steps:

Step 1: For all $m \in \times_{i=1}^n M_i^\Gamma$ and $m' \in M^\Gamma$ such that $\theta^\Gamma(m') \not\subseteq E$, $a^\Gamma(m) > a^\Gamma(m')$ [$a^\Gamma(m) < a^\Gamma(m')$] if $m \geq_\Gamma m'$ [$m \leq_\Gamma m'$] and for some $j \in N$ such that $m_j \neq m_j'$, there exist $\theta' < \theta_1 < \theta_2 < \theta$ [$\theta' > \theta_1 > \theta_2 > \theta$] such that $m_j' = m_j^\Gamma(\theta') \neq m_j^\Gamma(\theta_1) \neq m_j^\Gamma(\theta_2) \neq m_j^\Gamma(\theta) = m_j$.

Fix $m \in \times_{i=1}^n M_i^\Gamma$, and suppose $m' \leq_\Gamma m$, where $m' \in M^\Gamma$ and $\theta^\Gamma(m') \not\subseteq E$. Observe that for any $m'' \in M^\Gamma$ such that $m'' \leq_\Gamma m'$, $\lim_\pi \frac{\prod_{i=1}^n g_i(m_i|m_i'')^{\sigma_i}}{\prod_{i=1}^n g_i(m_i|m_i')^{\sigma_i}} \to 0$ for any $\pi \in \Pi$.

First, we show that $\arg\max_a \lim_\pi \frac{\int_\Theta u_R(a,\theta) f(\theta) \prod_{i=1}^n g_i(m_i|m_i^\Gamma(\theta))^{\sigma_i} d\theta}{\int_\Theta f(\theta) \prod_{i=1}^n g_i(m_i|m_i^\Gamma(\theta))^{\sigma_i} d\theta} \equiv a_\pi^\Gamma(m) \geq a^\Gamma(m')$ for any $\pi \in \Pi$. By sender monotonicity, there are two cases to consider: $\theta^\Gamma(m')$ is a proper interval, and $\theta^\Gamma(m') \equiv \{\theta'\}$ is a singleton. In the former case, the observation from the previous paragraph directly implies the result. In the latter case, the result follows if $\exists m'' \in M^\Gamma$ such that $m' \leq_\Gamma m'' \leqq_\Gamma m$ and $\theta^\Gamma(m'')$ is a proper interval. Otherwise, since $\theta' \notin E$, θ' is in the interior of an interval I of fully revealed states. Then, for any $\varepsilon > 0$ such that $\theta' - \varepsilon \in I$, $\lim_\pi \frac{\prod_{i=1}^n g_i(m_i|m_i^\Gamma(\theta''))^{\sigma_i}}{\prod_{i=1}^n g_i(m_i|m_i^\Gamma(\theta))^{\sigma_i}} \to 0$ whenever $\theta'' < \theta' - \varepsilon < \theta \leq \theta'$. Therefore, the result holds by the boundedness of f. This implies that $a_\pi^\Gamma(m) \geq a^\Gamma(m')$.

To see why $a^\Gamma(m) > a^\Gamma(m')$, let j be as in the statement of the step, and consider π such that $\pi(1) = \{j\}$. It is shown below that $a_\pi^\Gamma(m) > a^\Gamma(m')$. There are three cases to consider:

i) If m_j is sent on a proper interval I, then an argument analogous to the above implies that $a^\Gamma_\pi(m) \geq \inf\{a^\Gamma(m'') : m'' \in M^\Gamma \text{ and } m''_j = m_j\}$. It follows that $a^\Gamma_\pi(m) \geq \theta_2 > a^\Gamma(m')$.

ii) If m_j is sent at a single state θ that is not the right endpoint of a proper interval throughout which j sends the same message, then an argument analogous to the above implies that $a^\Gamma_\pi(m) \geq \theta > a^\Gamma(m')$.

iii) If m_j is sent at a single state θ that is the right endpoint of a proper interval throughout which j sends m^*_j, then an argument analogous to the above implies that $a^\Gamma_\pi(m) \geq \inf\{a^\Gamma(m'') : m'' \in M^\Gamma \text{ and } m''_j = m^*_j\}$. It follows that $a^\Gamma_\pi(m) \geq \theta_1 > a^\Gamma(m')$.

By the smoothness of u_R with respect to $a^\Gamma(m)$, it follows that since $a^\Gamma_\pi(m) \geq a^\Gamma(m')$ for all π and $a^\Gamma_\pi(m) > a^\Gamma(m')$ for some π, we must have $a^\Gamma(m) > a^\Gamma(m')$.

A symmetric argument shows the bracketed part of the result.

Observation 1: Note that the argument for case i) from Step 1 also shows that $a^\Gamma(m) > a^\Gamma(m')$ whenever $\theta^\Gamma(m)$ and $\theta^\Gamma(m')$ are both proper cells, and $m \geq_\Gamma m'$.

Step 2: If $\theta^\Gamma(m')$ is a singleton $\{\theta'\}$, then θ' is both the right endpoint of a cell with positive measure (unless $\theta' = 0$) and the left endpoint of a cell with positive measure (unless $\theta' = 1$).

Suppose instead $\theta' \neq 0$ is not the right endpoint of a cell with positive measure. By sender monotonicity, $\theta^\Gamma(m)$ is connected, and thus $a^\Gamma(m) \in \theta^\Gamma(m)$ for all m. It follows that there exists an infinite sequence $m^1, m^2, ...$ such that $a^\Gamma(m^k) < a^\Gamma(m^{k+1})$ for all k, and $\lim_{k\to\infty} a^\Gamma(m^k) = \theta'$. Furthermore, we may pick such a sequence with the property that $\theta^\Gamma(m^k) \nsubseteq E$ for all k (take the original sequence, replace any m such that $\theta^\Gamma(m) \subseteq E$ with the m of an adjacent cell, and remove duplicates). Finally, there must exist a subsequence $m^{j,1}, m^{j,2}, ...$ such that $m^{j,k}_j \neq m^{j,k+1}_j$ for all k, for some $j \in N$ (otherwise, all players' messages change a finite number of times in the original sequence, which contradicts the fact that it is infinite).

Take k sufficiently large such that $a^\Gamma(m^{j,k+3}) - a^\Gamma(m^{j,k}) < \eta$. By Step 1,

$$a^\Gamma(m^{j,k+3}_j, m^k_{-j}), a^\Gamma(m^{j,k}_j, m^{j,k+3}_{-j}) \in (a^\Gamma(m^{j,k}), a^\Gamma(m^{j,k+3})).$$

Therefore, sender j has a profitable deviation either from $m_j^{j,k}$ to $m_j^{j,k+3}$ at $\theta = a^\Gamma(m^{j,k})$ if right-biased, or from $m_j^{j,k+3}$ to $m_j^{j,k}$ at $\theta = a^\Gamma(m^{j,k+3})$ if left-biased.

A symmetric argument shows that $\theta' \neq 1$ is the left endpoint of a cell with positive measure.

Step 3: Every cell in Γ *(except possibly for* $\{0\}$ *or* $\{1\}$*) is proper.*

By sender monotonicity, to show that a cell in Γ has positive measure, it is sufficient to show that it is not a singleton. By Step 2, if cell $\{\theta\}$ is a singleton (and not $\{0\}$ or $\{1\}$), then it is the right endpoint of some cell $\theta^\Gamma(m)$ with positive measure and the left endpoint of some cell $\theta^\Gamma(m')$ with positive measure.

At least two senders' messages must differ in m and m': if only sender i's message differs, then by sender monotonicity, all other senders must send the same message in $\{\theta\}$ as in $\theta^\Gamma(m)$, which implies that at θ, sender i must be indifferent between $a^\Gamma(m)$, θ and $a^\Gamma(m')$. This is not possible by single-peakedness.

Observation 1 implies that, since both m and m' correspond to proper cells, any message vector m'' such that $m''_i \in \{m_i, m'_i\}$ for all i must have $a^\Gamma(m'') \in (a^\Gamma(m), a^\Gamma(m'))$ if $m'' \neq m, m'$. But then the argument in the proof of Proposition 1 applies and yields a contradiction.

Step 4: a^Γ *is* m^Γ*-monotonic.*

By Step 3, Observation 1 applies whenever $\theta^\Gamma(m') \notin \{\{0\}, \{1\}\}$.

For the case $\theta^\Gamma(m') = \{0\}$, first note that by Step 3, Observation 1, and the argument for finite M^Γ in the proof of Proposition 1, the number of cells is finite. Therefore, there exists a leftmost proper cell where the induced action, denoted a_L, satisfies $a_L > 0$. It follows that $a_\pi^\Gamma(m) \geq a_L$ for all π and all $m \geq_\Gamma m'$: *ex ante*, the state is from a proper cell with probability 1. Thus $a^\Gamma(m) \geq a_L > 0$ for all $m \geq_\Gamma m'$.

A symmetric argument takes care of the case $\theta^\Gamma(m') = \{1\}$.

b) TO BE COMPLETED. Sketch: For any out-of-eq vector m, let m^- be the highest on-path vector with $m^- \leq_\Gamma m$, and let m^+ be the lowest on-path vector with $m^+ \geq_\Gamma m$. Show that there exist π^- and π^+ such that $a_{\pi^-}^\Gamma(m) = a^\Gamma(m^-)$ and $a_{\pi^+}^\Gamma(m) = a^\Gamma(m^+)$. Then the appropriate weights $\varepsilon_{\pi,m}$ can be chosen to yield $a^\Gamma(m)$. ■

8 References

Ambrus, A. and S. Lu (2014): "Almost Fully Revealing Cheap Talk with Imperfectly Informed Senders," *Games and Economic Behavior*, 88, 174-189.

Battaglini, M. (2002): "Multiple Referrals and Multidimensional Cheap Talk," *Econometrica*, 70, 1379-1401.

Chen, Y. (2011): "Perturbed Communication Games with Honest Senders and Naïve Receivers," *Journal of Economic Theory*, 146, 401-424.

Crawford, V. and J. Sobel (1982): "Strategic Information Transmission," *Econometrica*, 50, 1431-1452.

Kartik, N. (2009): "Strategic Communication with Lying Costs," *Review of Economic Studies*, 76, 1359-1395.

Krishna, V. and J. Morgan (2001): "A Model of Expertise," *Quarterly Journal of Economics*, 116, 747-775.

Lu, S. (2017): "Coordination-Free Equilibria in Cheap Talk Games," *Journal of Economic Theory*, 168, 177-208.

Rubanov, O. (2015): "Asymptotic Full Revelation in Cheap Talk with a Large Number of Senders," *mimeo.*