Extended Luce Rules

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Abstract.

A growing literature generalizes the classical model of random choice from Theorem 3 of Luce's monograph (1959) to accommodate zero probabilities. I establish the close connection between this literature and the very next result (Theorem 4) from Luce's monograph. In the process, I answer some long-standing questions about Luce's approach to zero probabilities; and generalize almost all of the results from the recent literature on extended Luce rules.

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From consumer purchases to regional homicide rates and international trade flows, empirical data almost always includes options that are never chosen (Gandhi et al., 2017). As big data has made granular information more accessible, zeroes have only become more prevalent in choice data. At the same time, the *multinomial logit*, which is widely-used for estimations of discrete choice, does not permit alternatives to be chosen with zero probability. In fact, the same is true for Luce's classical model of random choice (Theorem 3 in his 1959 monograph), which is the theoretical foundation for the multinomial logit.

In view of this tension, a recent literature seeks to extend *classical Luce rules* to accommodate probability-zero choices (see McCausland, 2009; Lindberg, 2012; Dogan and Yildiz, 2016; Ahumada and Ulku, 2017; and Echenique and Saito, 2017; Cerreia-Vioglio et al., 2018). The common feature of the *extended Luce rules* proposed in these papers is to add a preliminary stage which deterministically selects a subset of the feasible alternatives *before* applying a classical Luce rule to choose randomly among these alternatives.

Sadly, much of this literature—with the exception of the early papers by McCausland and Lindberg—overlooks that Luce had already proposed a solution to the zero-probability problem in his very next result (Theorem 4 in his monograph). In that result, Luce relies on three simple axioms to obtain a "partial" representation for menus where binary choice probabilities are non-zero. In the current paper, I complete Luce's result and establish the connection to the recent literature on extended Luce rules. After laying the groundwork in Sections 1 and 2, I state my main results (Theorems 1 and 2) in Section 3, before turning to some extensions and implications of my results in Sections 4 and 5.

My first result (Theorem 1) establishes that Luce's three axioms are necessary and sufficient for random choice to be represented by an extended Luce rule whose first-stage is rationalized by a semi-ordering that is "based on" the utility function used by the decision-maker in the second-stage. This type of representation has two natural interpretations. While both take the *ex ante* utility of each alternative to an independently distributed random variable, they explain the incidence of zeroes in different ways.

According to one interpretation, the first stage reflects *rational inattention* by the decision-maker. Conceptually, the notion of inattention differs from that proposed by Matejka and McKay (2015). To avoid

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considering the entire menu, the (otherwise classical Luce) decision-maker eliminates alternatives *ex ante* that are unlikely to have the highest realized utility *ex post*. Clearly, it makes sense to ignore an alternative that is "perfectly" discriminated as inferior to another feasible alternative (like, as Luce suggests on pp. 18-19, a gamble that is less likely to yield a given prize or a bundle that provides less of each available good). Even when the decision-maker discriminates "imperfectly" among the feasible alternatives however, she may still want to ignore an alternative that is sufficiently unlikely to yield more utility than another.

According to the other interpretation, the first stage reflects *measurement error* by the observer. If the sample of choice data is sufficiently small, then it is unlikely to include realizations drawn from the tails of the utility distributions.[2] In this case, the choice data of a classical Luce decision-maker will be consistent with the representation in Theorem 1. Conceptually, this interpretation shares something with the econometric approach to zero-probability choices that was recently proposed by Gandhi et al. (2017).

In turn, my second result (Theorem 2) establishes that Luce's main substantive axiom (the *Choice Axiom*) is necessary and nearly sufficient for random choice behavior to be represented by an extended Luce rule. The small "gap" is a variation on the *Product Rule* studied by Luce and Suppes (1965). What is more, this variation guarantees a representation whose first-stage is rationalized by a pre-ordering. Put differently, the rationality of the first stage comes "for free" when random choice behavior satisfies the Choice Axiom.

Together, Theorems 1 and 2 resolve some long-standing questions about Luce's approach to probability-zero choice; and generalize the important results from the recent literature on extended Luce rules.

1. Definitions

Whenever possible, I follow Luce's notation. Let U denote a countable non-empty universe of *alternatives*; and $\mathcal{F} = \{S \subseteq U : |S| \in \mathbb{N}\}$ the finite non-empty subsets (or *menus*) of alternatives in U. A *random choice function* $p : U \times \mathcal{F} \to [0,1]$ is a mapping such that, for every menu $S \in \mathcal{F}$, $p(\cdot, S)$ is a *probability measure* on S (i.e., $\sum_{x \in S} p(x,S) = 1$). In words, $p(x,S)$ reflects the probability of choosing x from the menu S.

In principle, the definition of a random choice function p allows this probability to be zero (i.e., $p(x,S) = 0$) when S contains multiple alternatives. However, much of the focus has been on models where, for every menu $S \in \mathcal{F}$, the distribution $p(\cdot, S)$ is *positive* (i.e., $p(x,S) > 0$ for all $x \in S$). As discussed in the Introduction, the most widely-used of these models is due to Luce (1959, Theorem 3).

Definition. *A* ***classical Luce rule*** $v : U \to \mathbb{R}_{++}$ *is a strictly positive utility function. It represents the random choice function* p *if, for each menu* $S \in \mathcal{F}$ *and every alternative* $x \in S$*:*

$$p(x,S) := \frac{v(x)}{\sum_{y \in S} v(y)}.$$

Each of the papers cited in the Introduction extend this model to accommodate zero probabilities. These extensions share a common structure: they add a preliminary stage where the decision maker (deterministically) focuses on a subset of the feasible alternatives *before* using the classical Luce model to choose (randomly) among the considered alternatives. Formally:

[2]Interestingly, Luce himself (p.18) points to sample size as a potential reason for observing zero-frequency choice.

Definition. *An **extended Luce rule** (Γ, v) is a pair consisting of: a* choice correspondence $\Gamma : \mathcal{F} \to \mathcal{F}$ *(i.e., for all $S \in \mathcal{F}$, $\emptyset \subset \Gamma(S) \subseteq S$); and a strictly positive utility function $v : U \to \mathbb{R}_{++}$. The pair (Γ, v) represents the random choice function p if, for each menu $S \in \mathcal{F}$ and every alternative $x \in S$:*

$$p(x, S) := \begin{cases} \frac{v(x)}{\sum_{y \in \Gamma(S)} v(y)} & \text{if} \quad x \in \Gamma(S) \\ 0 & \text{otherwise.} \end{cases}$$

While the model imposes no structure on Γ, most of the cited papers require it to be *rationalized* by a binary relation $\succ$ on U (i.e., $\Gamma(S) := \max_{\succ} S = \{x \in S : y \not\succ x \text{ for all } y \in S\}$ for all $S \in \mathcal{F}$). The next definition follows Fishburn (1975). To simplify, I write $x \sim y$ if $x \not\succ y$ and $y \not\succ x$ (and $x \succsim y$ if $y \not\succ x$).

Definition. *The following properties of a binary relation $\succ$ on U apply to all alternatives $x, y, z, w \in U$:*

- **P1 (*irreflexivity*).** *$x \sim x$.*
- **P2 (*transitivity*).** *$[x \succ y$ and $y \succ z] \implies x \succ z$.*
- **P3 (*Ferrers property*).** *$[x \succ w$ and $y \succ z] \implies [x \succ z$ or $y \succ w]$.*
- **P4 (*semi-transitivity*).** *$[x \succ y$ and $y \succ z] \implies [x \succ w$ or $w \succ z]$.*
- **P5 (*negative transitivity*).** *$x \succ z \implies [x \succ y$ or $y \succ z]$.*
- **P6 (*sorites property*).** *$x \succ y \implies [x = z_1 \sim ... \sim z_n = y$ for some $z_1, ..., z_n \in U]$.*

*A binary relation $\succ$ that satisfies P1 is: a **pre-ordering** if it satisfies P2; an **interval ordering** if it satisfies P3; a **semi-ordering** if it satisfies P3 and P4; and a **weak ordering** if it satisfies P5.*

The standard properties P1-P5 are related as follows: each of P3-P5 implies P2; and P5 implies both P3 and P4. This pins down the relationship among the three special classes of orderings defined above (i.e., a weak ordering is a semi-ordering and, in turn, a semi-ordering is an interval ordering). Notice that the indifference relation $\sim$ defined above is transitive if and only if $\succ$ is a weak ordering.

While property P6 is non-standard, it captures the essence of the classical paradox: every pair of differentiated alternatives (e.g., a "heap" of sand and a "non-heap" of sand) may be linked by a sequence of negligible differences. Since this undermines the transitivity of $\sim$, property P6 precludes P5.

Luce (1956) showed that a semi-ordering can be represented by utility intervals and later work extended this to interval orderings (see e.g., Fishburn, 1985). To state the relevant results, let $\mathcal{I}(\mathbb{R}) := \{[a, b] : a, b \in \mathbb{R} \text{ and } a \leq b\}$ denote the collection of closed real intervals. Then, define binary relations $\gg$ and $>$ on $\mathcal{I}(\mathbb{R})$ such that: $[a, b] \gg [c, d]$ if $a > d$; and $[a, b] > [c, d]$ if $a \geq c$, $b \geq d$ and $[a, b] \neq [c, d]$.

Definition. *A set of **utility intervals** (I, v) is a pair consisting of: a mapping $I : U \to \mathcal{I}(\mathbb{R})$ that assigns a real interval to each alternative $x \in U$; and a utility function $v : U \to \mathbb{R}$ such that $v(x) \in I(x)$ for all $x \in U$. The pair (I, v) represents the binary relation $\succ$ on U if for all $x, y \in U$:*

$$x \succ y \iff I(x) \gg I(y)$$

In a representation by utility intervals, one can interpret $v(x)$ as the "mean utility" of $x \in U$ and $I(x)$ as a "prediction interval" (or quantile range) around $v(x)$. This provides $\succ$ with a natural statistical meaning: x is preferred to y if and only if the interval around $v(x)$ lies entirely above the interval around $v(y)$.

The following conditions impose some additional structure on the intervals used in the representation:

Definition. *The following regularity conditions for a pair (I, v) apply to all alternatives $x, y, z \in U$:*

R1 (*co-monotonicity*). $I(x) > I(y) \implies v(x) > v(y)$.
R2 (*non-nestedness*). $I(x) \setminus I(y) \neq \emptyset \implies I(y) \setminus I(x) \neq \emptyset$.
R3 (*clustering*). $[I(x) \cap I(z) \neq \emptyset$ *and* $I(y) \cap I(z) \neq \emptyset] \implies I(x) \cap I(y) \neq \emptyset$.

In words, condition R1 requires higher intervals to correspond to higher (mean) utilities; R2 precludes nested intervals; and R3 requires two intervals that overlap with a third to have a common overlap.

The next remark follows directly from Theorems 6-8 of Fishburn (1985, Chapter 2). For orderings in each of the three classes defined above, it establishes a representation by utility intervals.

Remark. *Suppose $\succ$ is an irreflexive relation on U. Then:*

(a) *$\succ$ is an interval ordering if and only if it is represented by a pair (I, v).*
(b) *$\succ$ is a semi-ordering if and only if it is represented by a pair (I, v) that satisfies R2.*
(c) *$\succ$ is a weak ordering if and only if it is represented by a pair (I, v) that satisfies R2 and R3.*

In each of the statements (a)-(c), the pair (I, v) can be strengthened to a pair (I, v) that satisfies R1.

A representation of (an interval ordering) $\succ$ by a pair (I, v) only requires the interval $I(x)$ associated with a given alternative $x \in U$ to contain $v(x)$. When (I, v) satisfies R1, there is a much stronger connection between the intervals I and the utility function v. This suggests the following definition, which extends Beja and Gilboa's (1992) notion of a "generalized utility representation" to include interval orderings.[3]

Definition. *An (interval) ordering $\succ$ on U is **weakly based** on a utility function $v : U \to \mathbb{R}$ if it is represented by a set of utility intervals (I, v); and it is **based** on v if (I, v) also satisfies R1.*

To close, I mention the special case where $v : U \to \mathbb{R}_{++}$ and the interval around $v(x)$ takes the form $I(x) := [v(x), (1 + \alpha)v(x)]$ for some constant $\alpha \in \mathbb{R}_+$. Since the resulting pair (I, v) satisfies R1 and R2, a binary relation $\succ$ with such a representation is known as a *multiplicative* semi-ordering based on v.

2. Luce's Approach

To extend his classical model to zero probabilities, Luce relies on three axioms. To simplify the statement of his first axiom, let $p(R, S) := \sum_{x \in R} p(x, S)$ denote the probability of choosing within $R \subseteq S$.

Choice Axiom (for $T \in \mathcal{F}$). *For all $R, S \in \mathcal{F}$ such that $R \subset S \subseteq T$:*

$$p(R, T) = \begin{cases} p(R, S) \times p(S, T) & \text{if } p(x, \{x, y\}) \neq 0, 1 \text{ for all } \{x, y\} \subseteq T \\ p(R \setminus \{x\}, T \setminus \{x\}) & \text{if } p(x, \{x, y\}) = 0 \text{ for some } \{x, y\} \subseteq T. \end{cases}$$

[3]Stated in terms of (I, v), a generalized utility representation would require $I(x) > I(y) \iff v(x) > v(y)$. This requirement (which strengthens R1) ensures that the intervals are (weakly) non-nested; and thus restricts the orderings to be semi-orderings.

A random choice function p satisfies the Choice Axiom for a collection $\mathcal{G} \subseteq \mathcal{F}$ if it satisfies the axiom for each menu $T \in \mathcal{G}$. In the special case where $\mathcal{G} = \mathcal{F}$, p is simply said to satisfy the Choice Axiom (CA).

If binary choice probabilities discriminate "imperfectly" among the alternatives in T, then the Choice Axiom requires the probability of choosing within $R \subset T$ to coincide with the probability of conditioning on any intermediate menu S. Otherwise, some $x \in T$ is "perfectly" discriminated as inferior; and the Choice Axiom states that it can be eliminated without affecting the probability of choosing within $R \subset T$.

Luce's second axiom is a "stochastic" transitivity requirement:

Luce Transitivity (LT).[4] *For all $x, y, z \in U$ such that* $\min\{p(x,\{x,y\}), p(y,\{y,z\})\} \geq {}^1\!/\!_2$*:*

$$\max\{p(x,\{x,y\}), p(y,\{y,z\})\} = 1 \implies p(x,\{x,z\}) = 1.$$

This axiom is related to the three transitivity axioms from the literature (which are re-stated in Appendix A). On its own, Luce Transitivity is weaker than the most demanding, *Strong Transitivity*, which concludes $p(x,\{x,z\}) \geq \max\{p(x,\{x,y\}), p(y,\{y,z\})\}$ even when $\max\{p(x,\{x,y\}), p(y,\{y,z\})\} < 1$. When p satisfies the Choice Axiom however, the two axioms are equivalent (see Lemma 5 of Appendix B). It follows that Luce Transitivity is strictly more demanding than the two other transitivity axioms from the literature (*Moderate Transitivity* and *Weak Transitivity*) when p satisfies the Choice Axiom.

Luce describes his third axiom as a "richness" condition. Some definitions will help to simplify the statement of this axiom. For each menu $S \in \mathcal{F}$, first define the following equivalence relation:

$$x \overset{S}{\frown} y \quad \text{if} \quad \min\{p(x,S), p(y,S)\} > 0.$$

If $x \overset{S}{\frown} y$, then x and y are *imperfectly discriminated* in S. In the case where $S = \{x,y\}$, I simply write $x \frown y$. A *sequence of imperfect discriminations* from x to y is a pair $(z_i, S_i)_{i \in I}$ consisting of:

- a (potentially infinite) sequence of alternatives $z_i \in U$ starting from $z_1 := x$ and ending with $z_I := y$;[5]
- a related sequence of menus $S_i \in \mathcal{F}$ such that $z_{i-1} \overset{S_{i-1}}{\frown} z_i \overset{S_i}{\frown} z_{i+1}$ for all $z_i \in (z_i)_{i \in I}$ except z_1 and z_I.

If x is linked to y by a (finite) sequence of imperfect discriminations, then x and y are said to be *(finitely) linked.* Since linking defines an equivalence relation on U, it partitions U into *linked components.*

Finitely Connected Domain (FCD). *For every pair of alternatives $x, y \in U$ such that $p(y,\{x,y\}) = 0$, there exists a finite sequence of imperfect discriminations $(z_i, \{z_i, z_{i+1}\})_{i=1}^n$ from y to x such that:*

$$\max\{p(z_i, \{z_i, z_{i+1}\})\}_{i=1}^n \leq {}^1\!/\!_2.$$

It is worth emphasizing that this axiom is not merely a richness condition. In terms of substantive content, it restricts the odds along the particular sequence that link the two alternatives in question.

Luce relies on these three axioms to establish the following result (1959, Theorem 4):

[4] This property subsequently appeared as Axiom 8 (*Strong Dominance Transitivity*) in Echenique and Saito.

[5] When the index set $|I|$ is infinite, the resulting "sequence" $z_1, ..., z_I$ has a definite *beginning* and *end* but no definite *middle.* While this kind of seqence is a little unusual, it does not present any formal difficulties. To the contrary, it formalizes what it means for two definite alternatives to be linked by an infinite chain of imperfect discriminations.

Theorem 0 (Luce). *Suppose p satisfies the Choice Axiom for $\{T \in \mathcal{F} : |T| \leq 3\}$, Luce Transitivity and Finitely Connected Domain. Then, there exists a strictly positive utility function $v : U \to \mathbb{R}_{++}$ such that for every menu $S \in \mathcal{F}$ where (i) p satisfies the Choice Axiom and (ii) $p(y, \{y, z\}) \neq 0, 1$ for all $y, z \in S$:*

$$p(x, S) = \frac{v(x)}{\sum_{y \in S} v(y)} \text{ for all } x \in S.$$

In the representation, v is identified up to multiplication by a strictly positive scalar by the requirement that

$$\frac{v(x)}{v(y)} = \prod_{i=1}^{n-1} \frac{p(z_i, \{z_i, z_{i+1}\})}{p(z_{i+1}, \{z_i, z_{i+1}\})}$$

for every pair $x, y \in U$ and finite sequence of imperfect discriminations $(z_i, \{z_i, z_{i+1}\})_{i=1}^n$ linking x to y.

Luce's result provides sufficient conditions for a "partial" representation: if p satisfies his three axioms, then it can be represented as a classical Luce rule on those menus where all binary choice probabilities are non-zero. The problem is that menus with this feature are not likely to arise in the data. Indeed, it would seem more relevant to provide a representation for menus where some binary choice probabilities *are* zero.

3. Main Results

(a) Theorem 1

My first result characterizes the class of extended Luce rules that are consistent with Luce's axioms.

Theorem 1. *A random choice function p satisfies the Choice Axiom, Luce Transitivity and Finitely Connected Domain if and only if it is represented by an extended Luce rule (Γ, v) where: Γ is rationalized by a semi-ordering $\succ$ that is based on v; and $\succ$ satisfies the sorites property. In the representation, the preference between every pair of alternatives $x, y \in U$ is uniquely identified by the requirement that $x \succ y$ if $p(y, \{x, y\}) = 0$; and v is identified up to (multiplication by) a strictly positive scalar.*

This result undermines the view (expressed by some critics at the time) that Luce dealt with zero probabilities in a "distinctly artificial manner" (Luce and Suppes, p. 336). To the contrary, Theorem 1 shows that his three axioms ensure that the two stages of the representation are based on a single utility scale v.

The next lemma provides a number of different ways to understand the relationship between $\succ$ and v in Theorem 1. Some definitions are required. First, consider the following conditions on a binary relation $\succ$:

Definition. *For a given utility function $v : U \to \mathbb{R}$, the following conditions on a binary relation $\succ$ on U apply to all alternatives $x, y, z \in U$:*

C1 (v-*coarsening*). $x \succ y \implies v(x) > v(y)$.
C2 (v-*transitive indifference*). $[v(y) \notin (\min\{v(x), v(z)\}, \max\{v(x), v(z)\})$ *and* $x \sim y \sim z] \implies x \sim z$.
C3 (v-*transitivity*).[6] $\{[x \succ y$ *and* $v(y) \geq v(z)]$ *or* $[v(x) \geq v(y)$ *and* $y \succ z]\} \implies x \succ z$.

[6]See Fishburn (1970, p. 34) and (1985, p. 130) for similar conditions in preference settings that are somewhat different.

Condition C1 (which is standard) states that $\succsim$ cannot contradict the weak ordering induced by v. In turn, conditions C2 and C3 (which are new) stipulate that the indifference $x \sim y \sim z$ is transitive provided that $v(y)$ is *not* "between" $v(x)$ and $v(z)$; and transitivity holds for "mixed" comparisons involving $\succ$ and v.

Next, consider the following natural refinements of a pre-ordering $\succ$ discussed by Fishburn (1985):

Definition. *Given a binary relation $\succ$ on U, let $\succ^*$ and $\succ^{**}$ denote the binary relations on U defined by:*

$$\begin{aligned} x \succ^* y \quad &\text{if} \quad x \succ z \sim y \text{ or } x \sim z \succ y \text{ for some } z \in U; \text{ and} \\ x \succ^{**} y \quad &\text{if} \quad x \succ^* y \text{ and } y \not\succ^* x. \end{aligned}$$

The relation $\succ^*$ refines the original relation $\succ$ by adding "indirect" preferences that arise via intermediate alternatives. In turn, $\succ^{**}$ coarsens $\succ^*$ by removing "ambiguous" comparisons from the derived relation $\succ^*$. If $\succ$ is a pre-ordering, then $\succ^{**}$ also refines $\succ$; and, if $\succ$ is a semi-ordering, then $\succ^{**}$ is equivalent to $\succ^*$.

Lemma 1. *For a utility function $v : U \to \mathbb{R}$ and an irreflexive relation $\succ$ on U, the following are equivalent:*

- **(i)** $\succ$ *is a semi-ordering based on v;*
- **(ii)** $\succ$ *satisfies transitivity, v-coarsening and v-transitive indifference;*
- **(iii)** $\succ$ *satisfies v-transitivity; and*
- **(iv)** $\succ^*$ *satisfies v-coarsening.*

The equivalence (i) $\Leftrightarrow$ (ii) shows that $\succ$ is a semi-ordering based on v if and only if it is a pre-ordering that satisfies two natural requirements linking it to v. While the first requirement may prevent $\sim$ from inheriting all of the "indifference transitivity" of v, the second ensures that $\sim$ does display some transitivity. The equivalences (i) $\Leftrightarrow$ (iii) and (i) $\Leftrightarrow$ (iv) describe the relationship between $\succ$ and v somewhat differently. Their significance will become apparent when I discuss extensions of Theorem 1 in Section 4 below.

(b) Theorem 2

My second result characterizes the formal relationship between the Choice Axiom and extended Luce rules. It establishes that the "small" gap between the two is spanned by the following condition:

Quadruple Product Rule (4-PR). *For all distinct $x, y, z, w \in U$ such that $x \frown y \frown z \frown w \frown x$:*

$$\frac{p(x,\{x,y\})}{p(y,\{x,y\})} \times \frac{p(y,\{y,z\})}{p(z,\{y,z\})} \times \frac{p(z,\{z,w\})}{p(w,\{z,w\})} \times \frac{p(w,\{w,x\})}{p(x,\{w,x\})} = 1.$$

This axiom is closely related to Luce and Suppes' (1965) *Product Rule*, which requires

$$\frac{p(x,\{x,y\})}{p(y,\{x,y\})} \times \frac{p(y,\{y,z\})}{p(z,\{y,z\})} \times \frac{p(z,\{x,z\})}{p(x,\{x,z\})} = 1$$

when $x \frown y \frown z \frown x$. The difference is that the latter applies to cycles of *three* rather than *four* alternatives.

The result also relies on the following weakening of Finitely Connected Domain:

Compact Domain (CD). *Every pair of linked alternatives $x, y \in U$ is finitely linked.*

This axiom is what might be called is a "technical" condition. When U is finite, it is vacuous. When U is countably infinite, it ensures that the utility function v used in the representation is well-defined.

Theorem 2. *If a random choice function p satisfies Compact Domain, then the following are equivalent:*

- **(i)** *p satisfies the Choice Axiom and the Quadruple Product Rule;*
- **(ii)** *p is represented by an extended Luce rule (Γ, v) where Γ is rationalized by a pre-ordering; and*
- **(iii)** *p satisfies the Choice Axiom and it is represented by an extended Luce rule (Γ, v).*

In the representation, Γ is uniquely identified by $\Gamma(S) := \{x \in S : p(x, S) > 0\}$ for all $S \in \mathcal{F}$; and, within each linked component of U, v is unique up to (multiplication by) a strictly positive scalar.[7]

The equivalence $(i) \Leftrightarrow (iii)$ shows that, when p satisfies the Choice Axiom, the Quadruple Product Rule is necessary and sufficient for an extended Luce representation. In turn, the implication $(i) \Rightarrow (ii)$ shows that the added structure of the choice correspondence comes for free: the Quadruple Product Rule ensures that Γ is rationalized by a pre-ordering when p satisfies the Choice Axiom. Finally, the implication $(ii) \Rightarrow (iii)$ shows that the Choice Axiom is necessary for Γ to be rationalized by a pre-ordering.

4. Generalizations

In this section, I present some natural generalizations of my two main results.[8] Not only are they interesting in their own right, but these generalizations also serve to clarify the connection to the recent literature.

(a) Theorem 1

The weakest aspect of Luce's approach is Finitely Connected Domain. While it ensures the uniqueness of the utility function v, it requires the ordering $\succ$ to satisfy the sorites property. Since this property contradicts negative transitivity, Theorem 1 does not cover the "limit case" where $\succ$ is a weak ordering.

To rectify this, one would like to generalize Theorem 1 in a way that treats the properties of the first-stage relation $\succ$ separately. The key to achieve this separation is an observation that Fishburn (1978) makes in a related setting: by using the identification of the relevant parameters, one can simply "translate" the required deterministic properties into the random choice setting. By applying the definitions of $\succ$ and v (from Theorem 1) to v-transitivity, for instance, one obtains Luce Transitivity. By applying the same kind of translation to v-coarsening and the sorites property, one obtains the following random choice axioms:

Weak Coarsening (WC). *For every pair of alternatives $x, y \in U$ such that y is linked to x by a finite sequence of imperfect discriminations $(z_i, \{z_i, z_{i+1}\})_{i=1}^{n}$:*

$$p(y, \{x, y\}) = 0 \implies \prod_{i=1}^{n-1} \frac{p(z_i, \{z_i, z_{i+1}\})}{p(z_{i+1}, \{z_i, z_{i+1}\})} < 1.$$

[7] If (Γ, v) and (Γ, v') represent p, then, within each linked component $U_i \subseteq U$, there is some $k_i \in \mathbb{R}_{++}$ such that $v'|_{U_i} = k_i \cdot v|_{U_i}$.

[8] To simplify the statement of these results, I omit the uniqueness properties which are the same as in Theorems 1 and 2.

Linked Domain (LD).[9] *Every pair of alternatives $x, y \in U$ such that $p(y, \{x, y\}) = 0$ is finitely linked.*

The first of these axioms weakens the substantive requirement of Finitely Connected Domain: on a finite sequence linking perfectly discriminated alternatives, the product of odds cannot favor the alternative which is discriminated as inferior (even if some odds along the sequence are unfavorable).

In turn, the second axiom weakens the richness requirement of Finitely Connected Domain: every pair of perfectly discriminated alternatives must be finitely linked (even if some menus along the sequence are non-binary). As Luce argues (on p. 25), this kind of property seems very reasonable when the domain U contains alternatives which are distinguished by sufficiently minor "physical" differences.

For a random choice function that satisfies the Choice Axiom, Luce Transitivity is stronger than Weak Coarsening (by Lemma 4 of Appendix B). Since Luce Transitivity also strengthens the Quadruple Product Rule (by Lemma 2 of Appendix B), Lemma 1 and Theorem 2 can then be used to establish the following:

Theorem 1*. *Suppose p is a random choice function that satisfies Compact Domain. Then, p satisfies the Choice Axiom and Luce Transitivity if and only if it is represented by an extended Luce rule (Γ, v) where Γ is rationalized by a semi-ordering that is based on v. In the representation, v is unique up to (multiplication by) a non-negative scalar if and only if p satisfies Linked Domain.*

This result sharpens Theorem 1, first, by showing that the substantive content of Finitely Connected Domain is redundant; and, second, by weakening the richness requirement of Finitely Connected Domain.

Theorem 1* also generalizes the main result of Echenique and Saito (which is re-stated as Theorem E in Appendix A). For the setting where U is finite, they characterize extended Luce rules where Γ is rationalized by a multiplicative semi-ordering based on v. Their result relies on a strong version of Linked Domain as well as a "calibration" condition (called *Path Monotonicity*) that strengthens Weak Coarsening.

(b) Theorem 2

As the analysis in Section 4(a) suggests, Theorem 2 provides a simple way to characterize *any* specialized model where Γ is rationalized by a pre-ordering. By using the definition of $\succ$ (from Theorem 1) to translate properties P2-P5, one obtains the following *stochastic rationality conditions*, which apply to all $x, y, z, w \in U$:

S2.[10] $\min\{p(x, \{x, y\}), p(y, \{y, z\})\} = 1 \implies p(x, \{x, z\}) = 1.$
S3. $\max\{p(w, \{x, w\}), p(z, \{y, z\})\} = 0 \implies \min\{p(z, \{x, z\}), p(w, \{y, w\})\} = 0.$
S4. $\max\{p(y, \{x, y\}), p(z, \{y, z\})\} = 0 \implies \min\{p(w, \{x, w\}), p(z, \{z, w\})\} = 0.$
S5.[11] $\min\{p(x, \{x, y\}), p(y, \{y, z\})\} > 0 \implies p(x, \{x, z\}) > 0.$

Each of these axioms imposes a "stochastic" transitivity requirement. At one extreme, S2 requires transitivity among perfectly discriminated alternatives. At the other, S5 strengthens Strong Transitivity.

[9] In this axiom, linking may occur on arbitrary menus. So, it is strictly weaker than a related condition (see Condition 4 on p. 417 of Krantz et al., 1989; or Axiom 6 (*Richness*) of Echenique and Saito) which requires linking on binary menus.
[10] This property also appears as Axiom 7 (*Dominance Transitivity*) in Echenique and Saito.
[11] This property also appears as Axiom P2 in Fishburn (1978).

When p satisfies the Choice Axiom, it turns out that axiom S2 is automatically satisfied. What is more, axioms S3 and S5 each imply the Quadruple Product Rule. The first point is a key step in the proof of Theorem 2. To verify the second, suppose $x \frown y \frown w \frown z \frown x$. Notice that S3 requires *one* of "diagonal pairs" in this cycle to be imperfectly discriminated (i.e., $x \frown w$ or $y \frown z$). Otherwise, $min\{p(z, \{x, z\}), p(w, \{y, w\})\} = 0$, which contradicts the assumption that $z \frown x$ and $y \frown w$. Similarly, S5 requires *both* diagonal pairs to be imperfectly discriminated (i.e., $x \frown w$ and $y \frown z$). Without loss of generality, suppose $x \frown w$. As shown in the proof of Theorem 2, the Choice Axiom implies the Product Rule. By applying this rule to the cycle $x \frown y \frown w \frown x$ before applying it to cycle $x \frown w \frown z \frown x$, it then follows that

$$\left[\frac{p(x,\{x,y\})}{p(y,\{x,y\})} \times \frac{p(y,\{y,w\})}{p(w,\{y,w\})}\right] \times \frac{p(w,\{z,w\})}{p(z,\{z,w\})} \times \frac{p(z,\{x,z\})}{p(x,\{x,z\})} = \frac{p(x,\{x,w\})}{p(w,\{x,w\})} \times \frac{p(w,\{z,w\})}{p(z,\{z,w\})} \times \frac{p(z,\{x,z\})}{p(x,\{x,z\})} = 1.$$

Since axioms S3-S5 translate P3-P5, it is not difficult to derive the following result from Theorem 2:

Theorem 2*. *Suppose p is a random choice function that satisfies Compact Domain. Then:*

(a) *p satisfies the Choice Axiom and the Quadruple Product Rule if and only if it is represented by an extended Luce rule (Γ, v) where Γ is rationalized by a pre-ordering.*

(b) *p satisfies the Choice Axiom and Axiom S3 if and only if it is represented by an extended Luce rule (Γ, v) where Γ is rationalized by an interval ordering.*

(c) *p satisfies the Choice Axiom and Axioms S3-S4 if and only if it is represented by an extended Luce rule (Γ, v) where Γ is rationalized by a semi-ordering.*

(d) *p satisfies the Choice Axiom and Axiom S5 if and only if it is represented by an extended Luce rule (Γ, v) where Γ is rationalized by a weak ordering.*

Theorems 1^* and 2^* generalize almost every result from the recent literature (see e.g., Theorems B-E in Appendix A). The only notable exception is the axiomatization of extended Luce rules for finite U (which is re-stated as Theorem A in Appendix A). Two recent papers (Ahumada and Ulku; Echenique and Saito) show that this general model is characterized by an axiom (called the *Strong Product Rule* in Appendix A) which extends the Product Rule to arbitrary cycles of imperfect discriminations. Theorem 2^* shows that, in most natural applications, one can replace this requirement with the more intuitive Choice Axiom.

(c) The remaining "gap"

The fundamental difference between Theorems 1^* and 2^*(c) is the link that the first result establishes between the two stages of the representation. While it seems quite natural to conjecture that Weak Coarsening is sufficient to establish this link, a simple example shows that this is not the case.

Example. *Suppose the random choice function p on $U := \{x, y, z\}$ is represented by the pair (v, Γ) where: $\langle v(x), v(y), v(z)\rangle = \langle 3, 2, 1\rangle$; and Γ is rationalized by a semi-ordering $\succ$ such that $x \sim y \succ z \sim x$. Clearly, p satisfies S3 and S4. In fact, it also satisfies Weak Coarsening since $p(z, \{y, z\}) = 0$ and*

$$\frac{p(z,\{x,z\})}{p(x,\{x,z\})} \times \frac{p(x,\{x,y\})}{p(y,\{x,y\})} = \frac{v(z)}{v(x)} \times \frac{v(x)}{v(y)} = 1/2 < 1.$$

However, p violates Luce Transitivity since $p(x, \{x, y\}) = 3/5$ and $p(y, \{y, z\}) = 1$ but $p(x, \{x, z\}) = 3/4 < 1$.

The problem in this example is that the semi-ordering $\succ$ puts y "indirectly" above x (i.e., $y \succ^* x$) while the utility function v does precisely the opposite. In other words, $\succ^*$ violates v-coarsening. By Lemma 1, it follows that $\succ$ cannot be based on the utility function v. Since it translates the requirement that $\succ$ satisfies v-coarsening however, Weak Coarsening nonetheless ensures that $\succ$ is *weakly* based on v.

The preceding discussion suggests several natural extensions of Theorem 1*:

(1) By adding Weak Coarsening to Theorem 2*(b)/(c), one characterizes extended Luce rules where $\succ$ is an interval/semi-ordering that is *weakly* based on the utility function v.[12] Interestingly, nothing is gained by adding Weak Coarsening to Theorem 2*(d). Since the Choice Axiom and S5 together imply Luce Transitivity (by Lemma 5 of Appendix B), the weak ordering $\succ$ from that result is *already* based on v.

(2) By translating the requirement that v coarsens $\succ^{**}$, one obtains a *coarsening condition* which is stronger than Weak Coarsening. By adding this *Moderate Coarsening* axiom (stated in Appendix A) to Theorem 2*(b)/(c), one characterizes extended Luce rules where $\succ$ is an interval/semi-ordering based on v. This extension of Theorem 2*(c) re-formulates Theorem 1* in terms of Moderate Coarsening.

(3) Finally, by translating the requirement that v coarsens $\succ^*$, one obtains an even more demanding coarsening condition. As Lemma 1 suggests, this *Strong Coarsening* axiom (stated in Appendix A) is equivalent to Luce Transitivity. As such, Theorem 1* can also be re-formulated in terms of Strong Coarsening.

It is worth emphasizing that all three of these extensions follow almost directly from the proofs of Lemma 1 and Theorem 1* in Appendix B. For the sake of parsimony, I relegate the details to Appendix C.

5. Conclusion

To conclude, I briefly discuss several important implications of my results.

(a) Luce's Conjecture

In his 1959 monograph, Luce conjectured that Luce Transitivity was necessary for a random choice function p which satisfies the Choice Axiom to be *represented by a one-dimensional utility scale* (see pp. 27 and 112). Unfortunately, Luce never formalized what he meant by the phrase in italics.

The results of the current paper suggest a number of sensible (albeit "prochronistic") interpretations. One natural interpretation would require p to be represented by an extended Luce rule where the first stage is *weakly* based on the second. The discussion in Section 4(c) implies that Luce's conjecture is false when given this interpretation. A more restrictive interpretation would require the first stage to be based on the the second. The truth of this version depends on the structure of the first-stage ordering: Luce Transitivity is necessary for a semi-ordering (by Theorem 2*) but not for an interval ordering.

Remarkably, this discussion establishes that Luce's conjecture is true when interpreted in terms of extended Luce representations—representations which were not even proposed until more than fifty years later.

[12] By adding it to Theorem 2*(a), one obtains a characterization of extended Luce rules where $\succ$ satisfies v-coarsening.

More specifically, the conjecture holds for extended Luce representations whose first stage are rationalized by semi-orderings—orderings which Luce (1956) himself proposed a few years before his conjecture.

(b) "Stochastic" Transitivity

Besides the Choice Axiom, "stochastic" transitivity is the main testable implication of extended Luce rules.[13] Indeed, every rule discussed in this paper displays some degree of transitivity—from S2, which is equivalent to the usual "deterministic transitivity" to S5, which is even more demanding than Strong Transitivity.

Figure 1 summarizes the logical relationship among the various transitivity axioms satisfied by extended Luce rules (under the assumption that the Choice Axiom is satisfied). In the center (in bold) are the three main transitivity axioms from the literature. They are flanked, on either side, by the coarsening conditions discussed in Section 4(c) and the stochastic rationality conditions defined in Section 4(b).

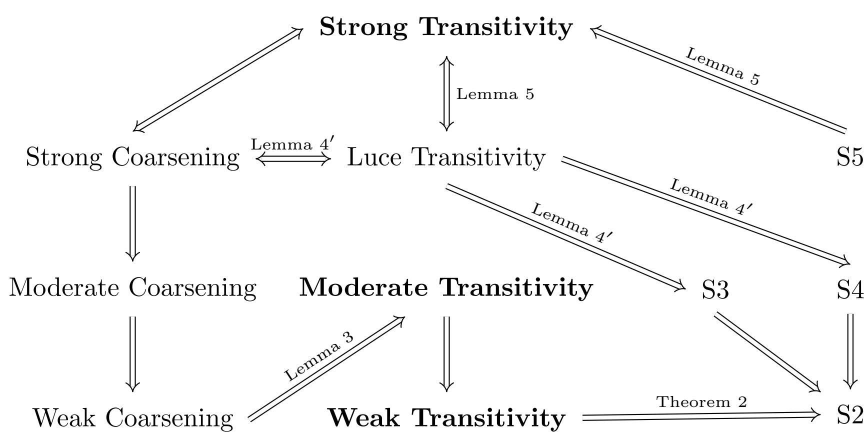

Figure 1: Logical relationship among "stochastic" transitivity axioms

For extended Luce rules, the degree of stochastic transitivity increases as the first-stage ordering $\succ$ gets "closer to" the second-stage utility function v (i.e., as p satisfies stronger coarsening conditions); or as $\succ$ becomes "more rational" (i.e., as p satisfies stronger stochastic rationality conditions). Table 1 summarizes the transitivity requirements for each of the models discussed in the paper.

	Coarsening			**Stochastic**
$\succ$ is ...	*unrelated to v*	*weakly based on v*	*based on v*	**Rationality**
a pre-ordering	–	N/A	N/A	S2
an interval ordering	–	Weak	Moderate	S3
a semi-ordering	–	Weak	Strong	S3 & S4
a weak ordering	Strong	Strong	Strong	S5

Table 1: Stochastic transitivity requirements of extended Luce models

[13]Rieskamp et al. (2006) survey the recent empirical evidence for the Choice Axiom. For earlier studies, see Luce (1977).

By and large, the empirical literature shows that violations of Strong Transitivity are commonplace while violations of Weak Transitivity are rather unusual (pp. 636 and 648 of Rieskamp et al., 2006). Interestingly, a recent study suggests that decision-makers who tend to violate Moderate Transitivity may be qualitatively different than those who violate the strong or weak variants of the axiom (McCausland and Marley, 2014). While this evidence undermines extended Luce rules that require Strong Transitivity (like those where the first-stage ordering $\succ$ is a weak ordering or a semi-ordering based on v), it says little about extended Luce rules that impose weaker transitivity requirements. To evaluate these models, it would be necessary to test the weak and moderate variants of Coarsening as well as the stochastic rationality requirements S2-S4.

(c) Related Literature

As discussed, there are two natural interpretations for extended Luce rules whose first stage is (weakly) based on the second: one which attributes the zeroes to the decision-maker; and another which attributes them to the observer. Each of these interpretations is related to a different paper from the recent literature—the former to Matejka and McKay (2015) and the latter to Gandhi et al. (2017).

Matejka and McKay propose a model where the decision-maker has a prior over the utility of feasible alternatives; and can update at a cost proportional to the reduction in entropy between her prior and posterior beliefs. Under these assumptions, they show that choice probabilities follow a generalized multinomial logit. A very different notion of attention costs provides foundations for extended Luce rules where the first stage is (weakly) based on the second. Rather than incurring an *information cost* of refining her beliefs about the feasible alternatives, the decision-maker incurs a *carrying cost* for each alternative that she actually considers. Accordingly, she ignores alternatives that are sufficiently unlikely to have the highest realized utility (according to her prior). In the result, these alternatives are chosen with zero-probability.

In turn, Gandhi et al. (2017) consider the econometric issues raised by zero-frequency choice. As they explain, it is impossible to use the standard estimation procedure proposed by Berry et al. (1995) when the data contains zeroes. To circumvent this issue, the literature has relied on one of two tricks: either "ignore" the problem by lumping zero-frequency choices with the outside option; or "correct" it by replacing zeroes with very small choice probabilities. Unfortunately, both tricks introduce bias into the estimation. To resolve the problem of bias, Gandhi et al. suggest an another approach. It is based on using the data to estimate upper and lower bounds for the "true" choice probabilities. Conceptually, this is similar to identification in an extended Luce model where the first stage is (weakly) based on the second—which relies on the choice data to determine intervals of "true" utilities for each alternative.

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Appendix A – Related Literature

(a) Additional Axioms

For the reader's convenience, I state some axioms (from the related literature) that are mentioned in the text.

(i) Stochastic Transitivity Conditions

Strong Transitivity (ST). *For all $x, y, z \in U$:*

$$\min\{p(x, \{x, y\}), p(y, \{y, z\})\} \geq 1/2 \implies p(x, \{x, z\}) \geq \max\{p(x, \{x, y\}), p(y, \{y, z\})\}.$$

Moderate Transitivity (MT). *For all $x, y, z \in U$:*

$$\min\{p(x, \{x, y\}), p(y, \{y, z\})\} \geq 1/2 \implies p(x, \{x, z\}) \geq \min\{p(x, \{x, y\}), p(y, \{y, z\})\}.$$

Weak Transitivity (WT). *For all $x, y, z \in U$:*

$$\min\{p(x, \{x, y\}), p(y, \{y, z\})\} \geq 1/2 \implies p(x, \{x, z\}) \geq 1/2.$$

(ii) Coarsening Conditions

By defining the binary relation $\succ$ as in the statement of Theorem 1 (by $x \succ y$ if $p(y, \{x, y\}) = 0$), the coarsening conditions for the derived relations $\succ^*$ and $\succ^{**}$ discussed in Section 4(b) may be stated as follows:

Strong Coarsening (SC). *For every pair of alternatives $x, y \in U$ such that y is linked to x by a finite sequence of imperfect discriminations $(z_i, \{z_i, z_{i+1}\})_{i=1}^n$:*

$$x \succ^* y \implies \prod_{i=1}^{n-1} \frac{p(z_i, \{z_i, z_{i+1}\})}{p(z_{i+1}, \{z_i, z_{i+1}\})} < 1.$$

Moderate Coarsening (MC). *For every pair of alternatives $x, y \in U$ such that y is linked to x by a finite sequence of imperfect discriminations $(z_i, \{z_i, z_{i+1}\})_{i=1}^n$:*

$$x \succ^{**} y \implies \prod_{i=1}^{n-1} \frac{p(z_i, \{z_i, z_{i+1}\})}{p(z_{i+1}, \{z_i, z_{i+1}\})} < 1.$$

Path Monotonicity (PM). *For all pairs of alternatives $x, y \in U$ and $x', y' \in U$ that are respectively linked by finite sequences of imperfect discriminations $(z_i, \{z_i, z_{i+1}\})_{i=1}^n$ and $(z'_i, \{z'_i, z'_{i+1}\})_{i=1}^m$:*

$$[p(y, \{x, y\}) = 0 \text{ and } p(x', \{x', y'\}) \neq 0, 1] \implies \prod_{i=1}^{n-1} \frac{p(z_i, \{z_i, z_{i+1}\})}{p(z_{i+1}, \{z_i, z_{i+1}\})} > \prod_{i=1}^{m-1} \frac{p(z'_i, \{z'_i, z'_{i+1}\})}{p(z'_{i+1}, \{z'_i, z'_{i+1}\})}.$$

(iii) Product Rules

Product Rule (PR).[14] *For all distinct $x, y, z \in U$ such that $x \frown y \frown z \frown x$:*

$$\frac{p(x, \{x, y\})}{p(y, \{x, y\})} \times \frac{p(y, \{y, z\})}{p(z, \{y, z\})} \times \frac{p(z, \{x, z\})}{p(x, \{x, z\})} = 1.$$

[14] See Luce and Suppes (1965, p. 341), who credit Luce (1959, Theorem 2) as the original source of this rule.

Strong Product Rule (SPR).[15] *For every pair of alternatives $x, y \in U$ such that x is linked to y by a finite sequence of imperfect discriminations $(z_i, S_i)_{i=1}^n$ and $x \overset{S}{\frown} y$ for some menu $S \in \mathcal{F}$:*

$$\frac{p(x,S)}{p(y,S)} = \prod_{i=1}^{n-1} \frac{p(z_i, S_i)}{p(z_{i+1}, S_i)}. \qquad (*)$$

(iv) Additional Conditions

Consistency (Con). *For all $R, S, T \in \mathcal{F}$ such that $S \subset T$: $p(R \cap S, T) = p(R, S) \times p(S, T)$.*

Maximization (Max).[16] *For all $x \in U$: $p(x, S) > 0 \iff p(x, \{x, y\}) > 0$ for all $y \in S$.*

McCausland considers the case where $U \subseteq \mathbb{R}^n_+$. In that setting, the *(vector) dominance* ordering $\gneq$ is defined by $x \gneq y$ if $x_i \geq y_i$ for all $i \in \{1, ..., n\}$ and $x \neq y$. Vector dominance suggests a natural concept of monotonicity:

Dominance Monotonicity (DM). *For all $x, y \in U$: $x \gneq y \iff p(y, \{x, y\}) = 0$.*

(b) Related Results

For the reader's convenience, I also state the main results from the related literature.[17]

Theorem A (Ahumada and Ulku; Echenique and Saito). *Suppose U is finite. Then, p satisfies SPR if and only if it is represented by an extended Luce rule (Γ, v).*

Theorem B (Ahumada and Ulku; Echenique and Saito).[18] *Suppose U is finite. Then, p satisfies SPR, Max and S2 if and only if is represented by an extended Luce rule (Γ, v) where Γ is rationalized by a pre-ordering.*

Theorem C (Lindberg; Cerreia-Vioglio et al). *Suppose U is finite. Then, p satisfies Con if and only if is represented by an extended Luce rule (Γ, v) where Γ is rationalized by a weak ordering.*

Theorem D (Echenique and Saito).[19] *Suppose U is finite. Then, p satisfies SPR, Max, S2 and PM if and only if is represented by an extended Luce rule (Γ, v) where Γ is rationalized by a multiplicative semi-ordering that is based on v. In the representation, v is unique up to a non-negative scalar if and only if p satisfies Linked Domain.*

Theorem E (McCausland).[20] *Suppose $U \subseteq \mathbb{R}^n_+$. Then, p satisfies CA, MT and DM if and only if is represented by an extended Luce rule (Γ, v) where: Γ is rationalized by the dominance ordering $\gneq$ on $\mathbb{R}^n_+$; and $\gneq$ satisfies v-coarsening.*

(c) Proof of Remark

(**a**) Fishburn's Theorems 6 and 8 establish that $\succ$ is an interval ordering if and only if it is represented by an interval mapping I (i.e., $x \succ y \iff I(x) \gg I(y)$ for all $x, y \in U$). To complete the result, define $v(x) := [\inf I(x) + \sup I(x)]/2$. Then, it is clear that $v(x) \in I(x)$ for all $x \in U$. What is more, (I, v) satisfies R1. ∎

(**b**) Similarly, Theorems 7 and 8 of Fishburn show that $\succ$ is a semi-ordering if and only if it is represented by an interval mapping I that satisfies R2. To complete the result, simply define v as in (a). ∎

[15] This axiom is called *Extended Cyclical Independence* by Ahumada and Ulku; and *Cyclical Independence* by Echenique and Saito. The name used here (purposely avoids both names and) instead evokes the connection to the Product Rule.

[16] This axiom (P1 in Fishburn, 1978) combines Axioms 2 (*Weak Regularity*) and 4 (*Probabilistic β*) from Echenique and Saito.

[17] To simplify the statement of these results, I omit the (obvious) uniqueness properties of the representation.

[18] Ahumada and Ulku replace Maximization with their Conditions 1-2 while Echenique and Saito replace it with their Axioms 2 and 4 (see footnote 16). In turn, both papers replace S2 with the stochastic analog of Fishburn's (1975) Axiom 2.

[19] Echenique and Saito replace Linked Domain with their Axiom 6 (*Richness*) (see footnote 9).

[20] McCausland decomposes Dominance Monotonicity into two separate conditions (his Assumptions 2 and 4); and imposes an additional condition (his Assumption 3) which ensures that v is log-concave.

(**c**) Finally, part (b) establishes that $\succ$ is a weak ordering only if it is represented by a pair (I, v) that satisfies R1 and R2. Clearly, R3 is also necessary for such a representation. Conversely, if the representation (I, v) satisfies R2 and R3, then it follows directly that $\succ$ is a weak ordering. ■

Appendix B – Proofs

(a) Proof of Theorem 2

(**i**) $\Rightarrow$ (**ii**) Suppose p satisfies CD, CA and 4-PR.

Define the binary relation $\succ$ and the mapping Γ as in the statement of Theorems 1 and 2, respectively

Step 1.[21] *$\succ$ is a pre-ordering. (Equivalently, p satisfies S2.)*

To show that $\succ$ is transitive, suppose $x \succ y \succ z$. By definition, $p(x, \{x, y\}), p(x, \{y, z\}) = 1$. So, $p(x, \{x, y, z\}) = p(x, \{x, z\}), p(x, \{x, y\})$ by CA(ii). Thus, $1 = p(x, \{x, y\}) = p(x, \{x, y, z\}) = p(x, \{x, z\})$. So, $x \succ z$. Since $\succ$ is irreflexive by definition, $\succ$ is a pre-ordering. ❙

Step 2.[22] *For all $S \in \mathcal{F}$, $\Gamma(S) = \max_{\succ} S$. (Equivalently, p satisfies Max – as defined in Appendix A above.)*

If $|S| = 1$, then there is nothing to prove. So, suppose $|S| > 1$. Starting from $S_0 := S$, recursively define $S_i := S_{i-1} \setminus \{x\}$ where $x \in S_{i-1}$ is such that $y \succ x$ for some $y \in S_{i-1}$.

By Step 1, this iterative elimination process must stop at $\widehat{S} := \max_{\succ} S$ regardless of the order of elimination. If it stops at $T \supset \widehat{S}$, then there is some $x \in T$ and $y_1 \in S \setminus T$ such that $y_1 \succ x$ and $z \not\succ x$ for all $z \in T$. So, y_1 must have been eliminated at a prior stage by some $y_2 \in S$ such that $y_2 \succ y_1$. Since $y_1 \succ x$, the transitivity of $\succ$ implies $y_2 \succ x$. Since the process stops at T, $y_2 \in S \setminus T$; and one can then repeat the same argument with y_2 in place of y_1. Since S is finite, this line of reasoning implies $T = \{x\}$ and $y \succ x$ for all $y \in S \setminus \{x\}$. This is a contradiction: since $|S| > 1$ and $T = \{x\}$, there must be some stage where x eliminates an alternative $y \in S \setminus \{x\}$.

Since the elimination process stops at $\widehat{S}$, (the second line of) CA implies $p(x, \widehat{S}) = p(x, S)$ for all $x \in \widehat{S}$. To complete the proof, suppose $p(x, \widehat{S}) = 0$ for some $x \in \widehat{S}$. Since $p(\widehat{S}, \widehat{S}) = 1$, $\widehat{S} \neq \{x\}$. By (the first line of) CA, it then follows that $p(x, \widehat{S}) = p(x, \{x, y\}) \times p(\{x, y\}, \widehat{S})$ for all $y \in \widehat{S} \setminus \{x\}$. Since $p(x, \widehat{S}) = 0$ and $p(x, \{x, y\}) > 0$, it follows that $p(\{x, y\}, \widehat{S}) = 0$. So $1 = p(\widehat{S}, \widehat{S}) \leq \sum_{y \in \widehat{S} \setminus \{x\}} p(\{x, y\}, \widehat{S}) = 0$, which is a contradiction. ❙

Step 3.[23] *If $x, y \in \Gamma(S)$, then $p(x, \{x, y\}) \times p(y, S) = p(y, \{x, y\}) \times p(x, S)$.*

From Step 2, $p(z, S) = p(z, \Gamma(S)) > 0$ for $z \in \{x, y\}$. By (the first line of) CA, $p(z, S) = p(z, \{x, y\}) \times p(\{x, y\}, \Gamma(S))$ for $z \in \{x, y\}$. Since $p(\{x, y\}, \Gamma(S)) > 0$, cross-multiplying the identities for $p(x, S)$ and $p(y, S)$ gives the result. ❙

Step 4.[24] *p satisfies PR (as defined in Appendix A above).*

Let $LHS := \frac{p(x,\{x,y\})}{p(y,\{x,y\})} \times \frac{p(y,\{y,z\})}{p(z,\{y,z\})} \times \frac{p(z,\{x,z\})}{p(x,\{x,z\})}$ and $\Pi := p(x, \{x, y, z\}) \times p(y, \{x, y, z\}) \times p(z, \{x, y, z\})$. Then:

$$LHS \times \Pi = \frac{p(x, \{x, y\}) \times p(y, \{x, y, z\})}{p(y, \{x, y\})} \times \frac{p(y, \{y, z\}) \times p(z, \{x, y, z\})}{p(z, \{y, z\})} \times \frac{p(z, \{x, z\}) \times p(x, \{x, y, z\})}{p(x, \{x, z\})} = \Pi.$$

by repeated application of the identity in Step 3. Since $\Pi \neq 0$ by Step 2, it then follows that $LHS = 1$. ❙

Note: The proofs of Steps 1 to 4 do not rely on 4-PR.

Given a sequence of imperfect discriminations $(z_i, S_i)_{i=1}^n$, let $\mathbb{P}(z_i, S_i)_{i=1}^n := \prod_{i=1}^{n-1} \frac{p(z_i, S_i)}{p(z_{i+1}, S_i)}$.

Step 5.[25] *p satisfies SPR (as defined in Appendix A above).*

[21] The argument establishing transitivity is equivalent to the argument in Luce's Lemma 4 (p. 10).

[22] The last paragraph of the proof follows the argument in Luce's Lemma 1 (p. 6).

[23] This result generalizes Luce's Lemma 3 (p. 7); but the proof relies on the same type of argument.

[24] This result is Luce's Theorem 2 (p. 16).

[25] This generalizes the argument for the same result in the classical Luce model (see fn. 9 of Luce and Suppes, 1965).

Suppose there exists a sequence of imperfect discriminations $(z_i, S_i)_{i=1}^n$ linking x to y and some $S \in \mathcal{F}$ such that $x \overset{S}{\frown} y$. For simplicity, let $(z_i, S_i)_{i=1}^{n+1}$ with $S_n := S$ and $z_{n+1} := x$ denote the related cycle of imperfect discriminations linking x to itself. To establish $(*)$, I show that

$$\mathbb{P}(z_i, S_i)_{i=1}^{n+1} = 1. \tag{$**$}$$

The proof of identity $(**)$ is by strong induction on n. The base cases $n = 2, 3, 4$ are straightforward:

- $n = 2$: $\mathbb{P}(z_i, S_i)_{i=1}^3 = \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^3 = 1$ where the first equality holds by Step 3; and the second equality by the fact that the factors of $\mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^3$ are reciprocals.
- $n = 3$: $\mathbb{P}(z_i, S_i)_{i=1}^4 = \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^4 = 1$ where the equalities hold by Steps 3 and 4, respectively.
- $n = 4$: $\mathbb{P}(z_i, S_i)_{i=1}^5 = \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^5 = 1$ where the equalities hold by Step 3 and 4-PR, respectively.

For the induction step, suppose $(**)$ holds for $n \leq m$ and let $n = m + 1$.

By Step 3, $(z_i, \{z_i, z_{i+1}\})_{i=1}^{m+2}$ defines a cycle of imperfect discriminations from x to itself. If $p(z_j, \{z_j, z_k\}) \in (0, 1)$ for some $j \in \{1, ..., , m-1\}$ and $k \in \{j+1, ..., m+1\}$, then $z_j \frown z_k$. One can use this link to divide $(z_i, \{z_i, z_{i+1}\})_{i=1}^{m+2}$ into two new cycles of imperfect discriminations, namely:

$$z_j \frown ... \frown z_k \frown z_j \quad \text{and} \quad z_k \frown ... \frown z_{m+1} \frown z_1 \frown ... \frown z_j \frown z_k$$

From the induction hypothesis, it follows that

$$\mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=j}^{k-1} \times \frac{p(z_k, \{z_j, z_k\})}{p(z_j, \{z_j, z_k\})} = 1 \quad \text{and} \quad \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=k}^{j-1} \times \frac{p(z_j, \{z_j, z_k\})}{p(z_k, \{z_j, z_k\})} = 1$$

Using Step 3 and these two identities, it follows that

$$\mathbb{P}(z_i, S_i)_{i=1}^{m+2} = \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^{m+2} = \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=j}^{k-1} \times \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=k}^{j-1} = 1.$$

To complete the proof, I show that $p(z_j, \{z_j, z_k\}) \in (0, 1)$ for some $j \in \{1, ..., , m-1\}$ and $k \in \{j+1, ..., m+1\}$. By way of contradiction, suppose $p(z_j, \{z_j, z_k\}) \in \{0, 1\}$ for all $j \in \{1, ..., , m-1\}$ and $k \in \{j+1, ..., m+1\}$. (Since $m + 1 \geq 5$, the alternatives z_1, z_2, z_3, z_4 and z_{m+1} are all distinct.)

Suppose $p(z_1, \{z_1, z_3\}) = 1$. (The case where $p(z_1, \{z_1, z_3\}) = 0$ is similar.) Then, $z_1 \succ z_3$. If $p(z_1, \{z_1, z_4\}) = 0$, then $z_4 \succ z_1$ so that $z_4 \succ z_3$ by transitivity, which contradicts the assumption that $z_3 \sim z_4$ (i.e., $z_3 \frown z_4$).[26] So, $p(z_1, \{z_1, z_4\}) = 1$ (and hence $z_1 \succ z_4$). By the same reasoning, $z_1 \succ z_4$ implies $p(z_2, \{z_2, z_4\}) = 1$ (and hence $z_2 \succ z_4$). By pursuing this line of reasoning, one obtains the following:

$$p(z_2, \{z_2, z_4\}) = 1 \Rightarrow p(z_2, \{z_2, z_{m+1}\}) = 1 \Rightarrow p(z_3, \{z_3, z_{m+1}\}) = 1 \Rightarrow p(z_3, \{z_1, z_3\}) = 1.$$

Since $p(z_1, \{z_1, z_3\}) = 1$ by assumption, the last implication above gives the desired contradiction. ∎

Let $\overset{*}{\frown}$ denote the (symmetric) relation on U where $x \overset{*}{\frown} y$ if x and y are finitely linked. Since p satisfies CD, $\overset{*}{\frown}$ defines an equivalence relation on U. Thus, $(U/\overset{*}{\frown}) := \{U_j\}_{j \in J}$ partitions the domain U into finitely linked components that are indexed by some (potentially infinite) set $J = \{1, 2, ...\} \subseteq \mathbb{N}$.

First, define a strictly increasing function $f : J \to [1, +\infty)$ such that $f(1) = 1$. For each component $U_j \in (U/\overset{*}{\frown})$, fix some $\hat{x}^j \in U_j$ and set $v(\hat{x}^j) := f(j)$. For each $y^j \in U_j \setminus \{\hat{x}^j\}$, fix a finite sequence of imperfect discriminations $(z_i, S_i)_{i=1}^n$ linking y^j to $\hat{x}^j$ and set $v(y^j) := \mathbb{P}(z_i, S_i)_{i=1}^n \times f(j)$. Since p satisfies CD, the sequence $(z_i, S_i)_{i=1}^n$ is finite and, thus, $v(y_j) \in \mathbb{R}_{++}$. So, v is a mapping such that $v : U \to \mathbb{R}_{++}$.

Step 6. p is represented by (Γ, v).

Fix some $S \in \mathcal{F}$ and $x \in S$. By Step 2, $\Gamma(S) = \max_{\succ} S$. If $x \notin \Gamma(S)$, then $p(x, S) = 0$ as required. Otherwise, $x \in \Gamma(S)$. By definition, $\Gamma(S) \subseteq U_j$ for some component $U_j \in (U/\overset{*}{\frown})$. By SPR, it then follows that

$$p(x, S) \times v(y) = \left[p(x, S) \times \frac{v(y)}{f(j)}\right] \times f(j) = \left[p(y, S) \times \frac{v(x)}{f(j)}\right] \times f(j) = p(y, S) \times v(x)$$

[26] Given the definition of $\succ$, it turns out that $\sim$ coincides with $\frown$. Since the conceptual basis for each relation is somewhat different, I continue to use both symbols (selecting the one that is more appropriate in the specific context).

for all $y \in \Gamma(S)$. By adding up these equations for all $y \in \Gamma(S)$, one obtains:

$$p(x,S) \times \left[\sum_{y\in\Gamma(S)} v(y)\right] = \sum_{y\in\Gamma(S)} [p(x,S) \times v(y)] = \sum_{y\in\Gamma(S)} [p(y,S) \times v(x)] = \left[\sum_{y\in\Gamma(S)} p(y,S)\right] \times v(x) = 1 \times v(x).$$

By re-arranging the preceding expression, one obtains $p(x,S) = v(x)/\left[\sum_{y\in\Gamma(S)} v(y)\right]$ as required. ❙

To conclude the proof, note that the uniqueness of $\succ$ follows from Steps 1-2 above. In turn, CD and SPR imply that v is unique up to scalar multiplication within each finitely linked component. ❙

(**ii**) $\Rightarrow$ (**iii**) Suppose p is represented by an extended Luce rule (Γ, v) such that Γ rationalized by a pre-ordering $\succ$. From the uniqueness of the representation (shown above), $p(y, \{x,y\}) = 0$ if $x \succ y$; and $\Gamma(S) = \max_{\succ} S$ for all $S \in \mathcal{F}$. Applying these definitions to the representation, it is straightforward to check that p satisfies CA. ❙

(**iii**) $\Rightarrow$ (**i**) Suppose p is represented by an extended Luce rule (Γ, v) (whether or not p happens to satisfy CA). Using the representation, it is straightforward to check that p satisfies 4-PR. ■

(b) Proof of Theorem 2*

When $\succ$ is defined by the identification in Theorem 1, it is easy to check that S3-S5 translate P3-P5. By the argument in Section 4(b) of the text, S3 and S5 strengthen 4-PR when p satisfies CA. So, (ii)-(iv) follow from Theorem 2. ■

(c) Proof of Lemmas 0 to 5

Lemma 0. *If $\succ$ is irreflexive and $\succ^*$ satisfies v-coarsening, then $\succ$ is a semi-ordering based on v.*

Proof. Suppose $\succ$ is irreflexive and $\succ^*$ satisfies v-coarsening.

Step 1. $\succ$ is a semi-ordering on U.

First, note that $\succ$ is transitive. To see this, let $x \succ y \succ z$. By v-coarsening, $v(x) > v(y) > v(z)$ and, consequently, $x \succsim z$. If $x \sim z$, then $y \succ z \sim x \succ y$. So, $y \succ^* z \succ^* y$. Since $\succ^*$ satisfies v-coarsening, $v(y) > v(z) > v(y)$.

To see that $\succ$ satisfies P3, let $x \succ y$ and $z \succ w$. By way of contradiction, suppose $w \succsim x$ and $z \succsim y$. Since $\succ$ is transitive, $x \succ y \sim z \succ w \sim x$. So, $x \succ^* z \succ^* x$. Since $\succ^*$ satisfies v-coarsening, $v(x) > v(z) > v(x)$.

To see that $\succ$ satisfies P4, let $x \succ y \succ z$. By way of contradiction, suppose $z \succsim w \succsim x$. Since $\succ$ is transitive, $y \succ z \sim w \sim x \succ y$. So, $y \succ^* w \succ^* y$. Since $\succ^*$ satisfies v-coarsening, $v(y) > v(w) > v(y)$. ❙

Following Fishburn (1985, pp. 21-23), first define the relations $\succ^-$ and $\succ^+$ on U by

$$\begin{aligned} x \succ^- y \quad &\text{if: } x \succ z \sim y \text{ for some } z \in U; \text{ and} \\ x \succ^+ y \quad &\text{if: } x \sim z \succ y \text{ for some } z \in U. \end{aligned}$$

Step 2. $\succ^-$ and $\succ^+$ are weak orderings on U.

Since $\succ$ is an interval ordering on U (by Step 1), this follows from Fishburn's Theorem 2 (p. 22). ❙

Let $U^- := \{x^- : x \in U\}$ and $U^+ := \{x^+ : x \in U\}$ and consider the relation $\succ_\pm$ on $U^\pm := U^- \cup U^+$ defined by

$$\begin{aligned} x^- \succ_\pm y^- \quad &\text{if:} \quad x \succ^- y \text{ or } [x \sim^- y \text{ and } x \succ^+ y]; \\ x^+ \succ_\pm y^+ \quad &\text{if:} \quad x \succ^+ y \text{ or } [x \sim^+ y \text{ and } x \succ^- y]; \\ x^- \succ_\pm y^+ \quad &\text{if:} \quad x \succ y; \text{ and} \\ x^+ \succ_\pm y^- \quad &\text{if:} \quad x \succsim y. \end{aligned}$$

Step 3.[27] *$\succ_\pm$ is a weak ordering on $U^\pm$. What is more, $x^\pm \sim_\pm y^\pm \Leftrightarrow [x \sim^- y,\ x \sim^+ y$ and $\{x^\pm, y^\pm\} \subseteq U^-, U^+]$.*

[27] The proof follows the same kind of reasoning as the proof of Fishburn's Theorem 3 (pp. 23-24).

To see that $\succ_\pm$ is irreflexive, suppose otherwise. In particular, let $x^- \succ_\pm x^-$. (The case where $x^+ \succ_\pm x^+$ is similar.) Since $x^- \succ_\pm x^-$, it follows that $x \succ^- x$ or $x \succ^+ x$. In each case, there is a contradiction with Step 2.

To see that $\succ_\pm$ is negatively transitive, suppose $x^\pm \succ_\pm z^\pm$ and consider some alternative $z^\pm \in U$. By way of contradiction, suppose $y^\pm \succsim_\pm x^\pm$ and $z^\pm \succsim_\pm y^\pm$. There are eight different cases to consider:

1. $(x^\pm, y^\pm, z^\pm) = (x^+, y^+, z^+)$. By definition, $x^+ \succ_\pm z^+$, $y^+ \succsim_\pm x^+$ and $z^+ \succsim_\pm y^+$. It follows that $x \succsim^+ z \succsim^+ y \succsim^+ x$. Since $\succ^+$ is a weak ordering by Step 2, $x \sim^+ z \sim^+ y \sim^+ x$. By definition, $x^+ \succ_\pm z^+$ and $x \sim^+ z$ imply $x \succ^- z$. So, $x \succ w \sim z$ for some $w \in U$. Regardless of the relationship between y and w, this leads to a contradiction. If $w \succsim y$, then $x \succ w \succsim y$. Since $\succ$ is transitive, it follows that $x \succ^- y$. Since $y \sim^+ x$, this contradicts $y^+ \succsim_\pm x^+$. Otherwise, $y \succ w$. Then, $y \succ w \sim z$. So, $y \succ^- z$ by definition. Since $z \sim^+ y$, this contradicts $z^+ \succsim_\pm y^+$.

2. $(x^\pm, y^\pm, z^\pm) = (x^-, y^-, z^-)$. By definition, $x^- \succ_\pm z^-$, $y^- \succsim_\pm x^-$ and $z^- \succsim_\pm y^-$. This case is similar to case 1.

3. $(x^\pm, y^\pm, z^\pm) = (x^-, y^-, z^+)$. By definition, $x \succ z$, $y^- \succsim_\pm x^-$ and $z \succsim y$. Since $\succ$ is transitive, $x \succ z \succsim y$ implies $x \succ^- y$. In turn, $x \succ^- y$ implies $x^- \succ_\pm y^-$. But, this contradicts $y^- \succsim_\pm x^-$.

4. $(x^\pm, y^\pm, z^\pm) = (x^-, y^+, z^-)$. By definition, $x^- \succ_\pm z^-$, $y \succsim x$ and $z \succ y$. Since $\succ$ is transitive, $z \succ y \succsim x$ implies $z \succ^- x$. In turn, $x^- \succ_\pm z^-$ implies $x \succsim^- z$. But, this contradicts $z \succ^- x$ by Step 2.

5. $(x^\pm, y^\pm, z^\pm) = (x^+, y^-, z^-)$. By definition, $x \succsim z$, $y \succ x$ and $z^- \succsim_\pm y^-$. This case is similar to case 3.

6. $(x^\pm, y^\pm, z^\pm) = (x^-, y^+, z^+)$. By definition, $x \succ z$, $y \succsim x$ and $z^+ \succsim_\pm y^+$. Since $\succ$ is transitive, $y \succsim x \succ z$ implies $y \succ^+ z$. In turn, $y \succ^+ z$ implies $y^+ \succ_\pm z^+$. But, this contradicts $z^+ \succsim_\pm y^+$.

7. $(x^\pm, y^\pm, z^\pm) = (x^+, y^-, z^+)$. By definition, $x^+ \succ_\pm z^+$, $y \succ x$ and $z \succsim y$. Since $\succ$ is transitive, $z \succsim y \succ x$ implies $z \succ^+ x$. In turn, $x^+ \succ_\pm z^+$ implies $x \succsim^+ z$. But, this contradicts $z \succ^+ x$ by Step 2.

8. $(x^\pm, y^\pm, z^\pm) = (x^+, y^+, z^-)$. By definition, $x \succsim z$, $y^+ \succsim_\pm x^+$, and $z \succ y$. This case is similar to case 6.

By definition of $\succ_\pm$, it follows directly that $x^\pm \sim_\pm y^\pm$ if and only if $[x \sim^- y,\ x \sim^+ y$ and $\{x^\pm, y^\pm\} \subseteq U^-, U^+]$. ∎

Next, let $\succ^v$ denote the binary relation on $U^* := U^\pm \cup U$ induced by v:

$$\begin{aligned} x^* \succ^v y^* \quad &\text{if:} \quad v(x) > v(y) \text{ and } [\{x^*, y^*\} \subseteq U^+, U \text{ or } U^-]; \\ x^\pm \succ^v y \quad &\text{if:} \quad x^\pm = y^+; \text{ and} \\ x \succ^v y^\pm \quad &\text{if:} \quad y^\pm = x^-. \end{aligned}$$

In turn, extend $\succ_\pm$ from $U^\pm$ to $U^* := U^\pm \cup U$ by letting $x^* \sim_\pm y^*$ if $\{x^*, y^*\} \cap U \neq \emptyset$.

Finally, let $\rhd^v_\pm$ denote the binary relation on U^* defined by the lexicographic composition of $\succ_\pm$ with $\succ^v$:

$$x^\pm \rhd^v_\pm y^\pm \quad \text{if:} \quad x^* \succ_\pm y^* \text{ or } [x^* \sim_\pm y^* \text{ and } x^* \succ^v y^*].$$

Step 4. $\rhd^v_\pm$ is acyclic.

By way of contradiction, suppose $\rhd^v_\pm$ contains a cycle C on $X \subseteq U$. Since $\succ_\pm$ is a weak ordering on $U^\pm$ and $\succ^v$ is a weak ordering on U, C contains a sub-cycle C' on $X' \subseteq X$ that "alternates" two-by-two between $U^\pm$ and U as follows:

$$...x^+ \rhd^v_\pm x \succ^v y \rhd^v_\pm y^- \succ_\pm z^+ \rhd^v_\pm z...$$

Since $\rhd^v_\pm$ is a weak ordering on $U^\pm$, $r^+ \succ^v_\pm s^-$ cannot hold for all r^+ and s^- in C' such that $r^+ \rhd^v_\pm r \succ^v s \rhd^v_\pm s^-$. Without loss of generality, suppose $y^- \succsim^v_\pm x^+$. Since $y^- \not\sim_\pm x^+$ by Step 3, $y^- \succ_\pm x^+$. So, $y \succ x$ by definition. Since $\succ$ satisfies v-coarsening, $v(y) > v(x)$. Then, $y \succ^v x$ by definition. But, this contradicts $x \succ^v y$. ∎

By Step 4, the transitive closure $tc(\rhd^v_\pm)$ of $\rhd^v_\pm$ is a pre-ordering. So, $tc(\rhd^v_\pm)$ admits a weak ordering extension $\trianglerighteq^v_\pm$ by the Szpilrajn Theorem. Since U is countable, $\trianglerighteq^v_\pm$ also admits a utility representation $u : U^\pm \cup U \to \mathbb{R}$ such that $u(x) = v(x)$ for all $x \in U$. Finally, define $I : U \to \mathcal{I}(\mathbb{R})$ such that $I(x) := [u(x^-), u(x^+)]$ for all $x \in U$.

Step 5. The pair (I, v) represents $\succ$.

For all $x \in U$, $x \in I(x)$. Moreover, for all $x, y \in U$: $x \succ y \iff x^- \succ^v_\pm y^+ \iff u(x^-) > u(y^+) \iff I(x) \gg I(y)$. ∎

Step 6. The pair (I, v) satisfies R1.

Let $I(x) > I(y)$. By way of contradiction, suppose $v(y) \geq v(x)$. By definition of $I(x) > I(y)$, $u(x^+) \geq u(y^+)$ and $u(x^-) \geq u(y^-)$ (where at least one of the equalities is strict). Since $v(y) \geq v(x)$, it follows that $x^+ \succsim_\pm y^+$ and $x^- \succsim_\pm y^-$ (where at least one of the preferences is strict). So, $x^+ \succ_\pm y^+$ and $x^- \succ_\pm y^-$ by Step 3. Since the asymmetric part of $\succ^*$ satisfies v-coarsening, it follows that $v(x) > v(y)$. But, this contradicts $v(y) \geq v(x)$. ❙

Step 7. The pair (I, v) satisfies R2.

Suppose $u(x^+) > u(y^+)$. (The reasoning for $u(x^-) > u(y^-) \Rightarrow u(x^+) > u(y^+)$ is similar.) Then, $x^+ \succ_\pm^v y^+$.

1. If $x^+ \succ_\pm y^+$, then $x \succ^+ y$; or $x \sim^+ y$ and $x \succ^- y$. In the second sub-case, $x^- \succ_\pm^v y^-$ follows by definition. In the first sub-case, suppose $y \succ^- x$. Then, $x \sim z \succ y \succ w \sim x$ for some $z, w \in U$. But, this contradicts the fact that $\succ$ is a semi-ordering. So, $x \succsim^- y$. Since $x \succ^+ y$, it then follows that $x^- \succ_\pm^v y^-$.

2. If $x^+ \sim_\pm y^+$ and $v(x) > v(y)$, then $x^- \sim_\pm y^-$ by Step 3. Since $v(x) > v(y)$, it then follows that $x^- \succ_\pm^v y^-$.

So, $x^- \succ_\pm^v y^-$ in both cases. Then, $u(x^-) > u(y^-)$ as required. ■

Proof of Lemma 1. Suppose $\succ$ is irreflexive.

(**i**) $\Rightarrow$ (**iv**) Suppose $\succ$ is a semi-ordering with an interval representation (I, v) that satisfies R1 and R2. Let $x \succ z \sim y$ for some $z \in U$. (The case where $x \sim z \succ y$ for some $z \in U$ is similar.) By way of contradiction, suppose $v(y) \geq v(x)$. By R1 and R2, $I(y) > I(x)$ or $I(y) = I(x)$. What is more, $x \succ z$ implies $I(x) \gg I(z)$. It follows that $I(y) \gg I(z)$. So, $y \succ z$ which, in turn, contradicts $y \sim z$. ❙

(**iv**) $\Rightarrow$ (**i**) See Lemma 0. ❙

(**iv**) $\Rightarrow$ (**iii**) Suppose $\succ^*$ satisfies v-coarsening. Let $x \succ y$ and $v(y) \geq v(z)$. (The case where $v(x) \geq v(y)$ and $y \succ z$ is similar.) By way of contradiction, suppose $z \succsim x$. Then, $z \succ^* y$ or $z \succ^* x \succ^* y$. By v-coarsening, $v(z) > v(y)$ or $v(z) > v(x) > v(y)$. In either case, this contradicts $v(y) \geq v(z)$. ❙

(**iii**) $\Rightarrow$ (**ii**) Suppose $\succ$ satisfies v-transitivity. To see that $\succ$ satisfies v-coarsening, let $x \succ y$. By way of contradiction, suppose $v(y) \geq v(x)$. Then, $x \succ x$ by v-transitivity, which is a contradiction. So, $v(x) > v(y)$.

To see that $\succ$ satisfies v-transitive indifference, let $x \sim y \sim z$ and $v(y) \geq \max\{v(x), v(z)\}$. (The case where $v(y) \leq \min\{v(x), v(z)\}$ is similar.) Towards a contradiction, suppose $x \not\sim z$. If $z \succ x$, then $y \succ x$ by v-transitivity, which contradicts $x \sim y$. Otherwise, $x \succ z$. By the same kind of reasoning, this leads to the contradiction $y \succ z$.

Finally, to see that $\succ$ is transitive, let $x \succ y \succ z$. Then, $v(y) > v(z)$ by v-coarsening. So, $x \succ z$ by v-transitivity. ❙

(**ii**) $\Rightarrow$ (**iv**) Suppose $\succ$ (is a pre-ordering that) satisfies v-coarsening and v-transitive indifference. To see that $\succ^*$ satisfies v-coarsening, let $x \succ z \sim y$ for some $z \in U$. (The case where $x \sim z \succ y$ for some $z \in U$ is similar.) By way of contradiction, suppose $v(y) \geq v(x)$. By v-coarsening, $x \succ z$ implies $v(x) > v(z)$ so that $v(y) \geq v(x) > v(z)$. Since $x \sim y \sim z$, v-transitive indifference then implies $x \sim z$. But, this contradicts $x \succ z$. ■

Lemma 2. *For a random choice function p that satisfies the Choice Axiom:*

(a) *p satisfies the Quadruple Product Rule if it satisfies Luce Transitivity; but*

(b) *p need not satisfy Luce Transitivity if it satisfies the Quadruple Product Rule.*

Proof. Fix a random choice function p that satisfies CA.

(a) Suppose $x \frown y \frown z \frown w \frown x$. By Step 4 in the proof of [(**i**) $\Rightarrow$ (**ii**) from] Theorem 2, CA implies the Product Rule. Provided $p(x, \{x, z\})$ or $p(y, \{y, w\}) \in (0, 1)$, the argument in Section 4(b) then implies the required identity.

To complete the proof, I show $p(x, \{x, z\})$ or $p(y, \{y, w\}) \in (0, 1)$. By way of contradiction, suppose $p(x, \{x, z\}) = p(y, \{y, w\}) = 1$. (The case where $p(x, \{x, z\}) = p(y, \{y, w\}) = 0$ is similar.) If $p(z, \{z, w\}) \in [1/2, 1]$, then $p(x, \{x, z\}) = 1$ implies $p(x, \{x, w\}) = 1$ by LT. But, this contradicts $w \frown x$. So, $p(z, \{z, w\}) \in [0, 1/2)$. Similarly, $p(y, \{y, w\}) = 1$ and $y \frown z$ imply $p(w, \{z, w\}) \in [0, 1/2)$. So, $p(z, \{z, w\}) + p(w, \{z, w\}) < 1/2 + 1/2 = 1$, which is a contradiction. ❙

(b) Suppose p on $U := \{x, y, z, w\}$ is represented by the pair (v, Γ) such that: $v(a) = 1$ for all $a \in U$; and Γ is rationalized by $\succ$ where $x \succ y$ and $z \succ w$ (but $a \sim b$ for all other $a, b \in U$). To see that p violates LT, notice that $p(x, \{x, y\}) = 1$ but $p(x, \{x, z\}), p(y, \{y, z\}) = 1/2$. ■

Lemma 3. *For a random choice function p that satisfies the Choice Axiom:*

(a) *p satisfies Moderate Transitivity if it satisfies Weak Coarsening; but*

(b) *p need not satisfy Weak Coarsening if it satisfies Moderate Transitivity.*

Proof. Fix a random choice function p that satisfies CA.

(a) First observe that CA implies that p satisfies S2 and PR (by Steps 1 and 4 in the proof of Theorem 2). Next, suppose $\min\{p(x,\{x,y\}), p(y,\{y,z\})\} \geq 1/2$ for distinct $x, y, z \in U$. There are several cases:

1. Let $p(x,\{x,y\}) = p(y,\{y,z\}) = 1$. By S2, it follows that $p(x,\{x,z\}) = 1 = \min\{p(x,\{x,y\}), p(y,\{y,z\})\}$.

2. Let $p(x,\{x,y\}) = 1$ and $p(y,\{y,z\}) < 1$. If $p(x,\{x,z\}) = 0$, then $p(z,\{y,z\}) = 1$ by S2, which contradicts $p(y,\{y,z\}) \geq 1/2$. If $p(x,\{x,z\}) = 1$, then $p(x,\{x,z\}) > p(y,\{y,z\}) = \min\{p(x,\{x,y\}), p(y,\{y,z\})\}$ as required. Otherwise, $p(x,\{x,z\}) \in (0,1)$. In that case,

$$\frac{p(y,\{y,z\})}{p(z,\{y,z\})} \times \frac{p(z,\{x,z\})}{p(x,\{x,z\})} < 1$$

by WC. Since $p(y,\{y,z\}) \geq 1/2$, it again follows that $p(x,\{x,z\}) > p(y,\{y,z\}) = \min\{p(x,\{x,y\}), p(y,\{y,z\})\}$.

3. Let $p(x,\{x,y\}) < 1$ and $p(y,\{y,z\})\} = 1$. This case is similar to case 2.

4. Let $\max\{p(x,\{x,y\}), p(y,\{y,z\})\} < 1$. If $p(x,\{x,z\}) = 0$, then $p(z,\{y,z\}) > p(x,\{x,y\}) \geq 1/2$ by (the reasoning in) case 2. But, this contradicts $p(y,\{y,z\}) \geq 1/2$. If $p(x,\{x,z\}) = 1$, then $p(x,\{x,z\}) \geq \min\{p(x,\{x,y\}), p(y,\{y,z\})\}$ as required. Otherwise, $p(x,\{x,z\}) \in (0,1)$. In that case,

$$\frac{p(x,\{x,z\})}{p(z,\{x,z\})} = \frac{p(x,\{x,y\})}{p(y,\{x,y\})} \times \frac{p(y,\{y,z\})}{p(z,\{y,z\})}$$

by PR. Since $\min\{p(x,\{x,y\}), p(y,\{y,z\})\} \geq 1/2$, it follows that $p(x,\{x,z\}) \geq \min\{p(x,\{x,y\}), p(y,\{y,z\})\}$. ▮

(b) The random choice function p from the proof of Lemma 2(b) satisfies MT and CA. However, it violates WC. To see this, notice that $p(y,\{x,y\}) = 0$ and $y \frown z \frown x$ but $p(y,\{y,z\}) \times p(z,\{x,z\}) = p(z,\{y,z\}) \times p(x,\{x,z\})$. ■

Lemma 4. *For a random choice function p that satisfies the Choice Axiom:*

(a) *p satisfies Weak Coarsening if it satisfies Luce Transitivity; but*

(b) *p need not satisfy Luce Transitivity if it satisfies Weak Coarsening.*

Proof. Fix a random choice function p that satisfies CA.

(a) First observe the following. Since p satisfies CA and LT, p satisfies 4-PR by Lemma 2(a). So, p satisfies SPR by Step 5 in the proof of Theorem 2. Next, suppose y is linked to x by a finite sequence of imperfect discriminations $(z_i, \{z_i, z_{i+1}\})_{i=1}^n$ and $p(y,\{x,y\}) = 0$. The proof that $\mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^n < 1$ holds is by strong induction on n.

For the base case $n = 3$, $y = z_1 \frown z_2 \frown z_3 = x$. If $p(y,\{y,z_2\}) \in [1/2, 1)$, then $p(x,\{x,z_2\}) = 1$ by LT. So, $p(y,\{y,z_2\}) < 1/2$. Similarly, $p(z_2,\{x,z_2\}) < 1/2$. Combining these two inequalities gives $\mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^3 < 1$.

For the induction step $n = m+1$, suppose that the desired identity holds for every finite sequence of imperfect discriminations $(z_i, \{z_i, z_{i+1}\})_{i=1}^n$ such that $n \leq m$. If $p(z_j, \{z_j, z_k\}) \in (0,1)$ for any $1 \leq j, k \leq m+1$ such that $k > j+1$, then $y = z_1 \frown ... \frown z_j \frown z_k \frown ... \frown z_{m+1} = x$ so that

$$\mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^j \times \frac{p(z_j, \{z_j, z_k\})}{p(z_k, \{z_j, z_k\})} \times \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=k}^{m+1} < 1$$

by the induction hypothesis. Since p satisfies SPR, $p(z_j, \{z_j, z_k\}) = \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=j}^k \times p(z_k, \{z_j, z_k\})$. By combining this with the previous inequality, it follows that

$$\mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^{m+1} = \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^j \times \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=j}^k \times \mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=k}^{m+1} < 1.$$

So, suppose $p(z_j, \{z_j, z_k\}) \in \{0, 1\}$ for all $1 \leq j, k \leq m+1$ such that $k > j+1$.

If $p(z_1, \{z_1, z_3\}) = 0$, then $p(z_1, \{z_1, z_2\}), p(z_2, \{z_2, z_3\}) < 1/2$ by the base case $n = 3$. In turn, $p(z_1, \{z_1, z_3\}) = 0$ implies $p(z_2, \{z_2, z_4\}) = 0$. Otherwise, $p(z_2, \{z_2, z_4\}) = 1$ which implies $p(z_2, \{z_2, z_3\}) > 1/2$ by the base case $n = 3$. By extending this type of reasoning, it follows that $p(z_1, \{z_1, z_3\}) = 0$ implies $p(z_i, \{z_i, z_{i+1}\}) < 1/2$ for all $1 \leq i \leq m$. Similarly, $p(z_1, \{z_1, z_3\}) = 1$ implies $p(z_i, \{z_i, z_{i+1}\}) > 1/2$ for all $1 \leq i \leq m$.

Now, consider $p(z_1, \{z_1, z_m\}) \in \{0, 1\}$. If $p(y, \{y, z_m\}) = 1$, then $p(x, \{x, y\}) = 1$ and LT imply $p(x, \{x, z_m\}) = 1$, which contradicts $p(x, \{x, z_m\}) \in (0, 1)$. So, $p(y, \{y, z_m\}) = 0$. Similarly, $p(z_2, \{z_2, x\}) = 0$. Since $y = z_1 \frown ... \frown z_m$ and $z_2 \frown ... \frown z_{m+1} = x$, the induction hypothesis then implies $\mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^{m} < 1$ and $\mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=2}^{m+1} < 1$.

Since $p(z_1, \{z_1, z_3\}) \in \{0, 1\}$, the observations from the two previous paragraphs together imply $p(z_i, \{z_i, z_{i+1}\}) < 1/2$ for all $1 \leq i \leq m$. Combining these m inequalities gives $\mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^{m+1} < 1$. ❙

(b) The random choice function p from the example in Section 4(c) satisfies WC and CA but violates LT. ■

Lemma 5. *For a random choice function p that satisfies the Choice Axiom:*

(a) *p satisfies Strong Transitivity if it satisfies Luce Transitivity; and*

(b) *p satisfies Luce Transitivity if it satisfies S5; but*

(c) *p need not satisfy S5 if it satisfies Strong Transitivity.*

It follows that Luce Transitivity is equivalent to Strong Transitivity when p satisfies the Choice Axiom.

Proof. Fix a random choice function p that satisfies CA.

(a) First observe the following. Since p satisfies CA and LT, p satisfies 4-PR by Lemma 2(a). So, p satisfies SPR by Step 5 in the proof of Theorem 2. Next, suppose $\min\{p(x, \{x, y\}), p(y, \{y, z\})\} \in [1/2, 1]$ and $p(x, \{x, z\}) < 1$. (If $p(x, \{x, z\}) = 1$, then there is nothing to prove.) If $\max\{p(x, \{x, y\}), p(y, \{y, z\})\} = 1$, then $p(x, \{x, z\})\} = 1$ by LT, which is a contradiction. So, suppose $\max\{p(x, \{x, y\}), p(y, \{y, z\})\} < 1$. If $p(x, \{x, z\}) = 0$, then $p(z, \{y, z\})\} = 1$ by LT, which is a contradiction. So, $p(x, \{x, z\}) \in (0, 1)$. By SPR, it then follows that

$$\frac{p(x, \{x, z\})}{p(z, \{x, z\})} = \frac{p(x, \{x, y\})}{p(y, \{x, y\})} \times \frac{p(y, \{y, z\})}{p(z, \{y, z\})}.$$

Without loss of generality, suppose $p(x, \{x, y\}) \geq p(y, \{y, z\}) \geq 1/2$. From the identity above, it then follows that

$$\frac{p(x, \{x, z\})}{p(z, \{x, z\})} = \frac{p(x, \{x, y\})}{p(y, \{x, y\})} \times \frac{p(y, \{y, z\})}{p(z, \{y, z\})} \geq \frac{p(x, \{x, y\})}{p(y, \{x, y\})}.$$

In turn, this implies $p(x, \{x, z\}) \geq \max\{p(x, \{x, y\}), p(y, \{y, z\})\}$. ❙

(b) First observe the following. Since p satisfies CA and S5, p satisfies 4-PR by the argument in Section 4(b) of the text. So, p satisfies SPR by Step 5 in the proof of Theorem 2. Next, suppose $p(x, \{x, y\}) = 1$ and $p(y, \{y, z\}) \in [1/2, 1]$. (The case where $p(x, \{x, y\}) \in [1/2, 1]$ and $p(y, \{y, z\}) = 1$ is similar.) By S5, it follows that $p(x, \{x, z\}) > 0$. By way of contradiction, suppose $p(x, \{x, z\}) \in (0, 1)$. Then, $p(y, \{x, y\}) > 0$ by S5, which is a contradiction. ❙

(c) Suppose p on $U := \{x, y, z\}$ is represented by the pair (v, Γ) such that: $\langle v(x), v(y), v(z)\rangle = \langle 1, 2, 3\rangle$; and Γ is rationalized by $\succ$ where $y \sim z \succ x \sim y$. It is straightforward to check that p satisfies ST. To see that it violates S5, notice that $p(x, \{x, y\}) = 1/3$ and $p(y, \{y, z\}) = 2/5$ but $p(x, \{x, z\}) = 0$. ■

(d) Proof of Theorem 1*

($\Rightarrow$) Suppose p satisfies CD, CA and LT.

Since it satisfies CA and LT, p satisfies Weak Coarsening (WC) and Weak Transitivity (WT) by Lemmas 3-4. Let $\succ_p$ denote the binary relation on U defined by $x \succ_p y$ if $p(y, \{x, y\}) < 1/2$. Clearly, $\succ_p$ is a weak ordering. By definition, it is irreflexive; and, since p satisfies WT, it is negatively transitive. (To see the latter, let $x \succ_p y$. By way of contradiction, suppose $y \succsim_p z \succsim_p x$. Since p satisfies WT, it follows that $y \succsim_p x$, which contradicts $x \succ_p y$.)

First, consider the linking relation $\overset{*}{\frown}$ from the proof of Theorem 2. Since p satisfies CD, $\overset{*}{\frown}$ defines an equivalence relation on U. By definition, the partition $(U/\overset{*}{\frown})$ coarsens the partition $(U/\sim_p)$. Recursively define $U^1 := U$, $U^{j+1} := U \setminus (\cup_{k=1}^{j} U_k)$ and $U_j \in (U/\overset{*}{\frown})$ so that $U_j \supseteq \arg\min_{\succ_p} U^j$ contains the $\succ_p$-minimal alternatives in U^j. (This amounts to numbering the components of $(U/\overset{*}{\frown})$ in increasing order of preference $\succ_p$.)

Next, define the strictly increasing function $f : J \to [1, +\infty)$ from the proof of Theorem 2 as follows:

$$f(j+1) := f(j) \times \mathbb{P}(z_i^j, \{z_i^j, z_{i+1}^j\})_{i=1}^k + 1$$

where $\mathbb{P}(z_i^j, \{z_i^j, z_{i+1}^j\})_{i=1}^k$ is a finite path of imperfect discriminations linking $\overline{x}_j \in \arg\max_{\succ_p} U_j$ to $\underline{x}_j \in \arg\min_{\succ_p} U_j$. Define $v : U \to \mathbb{R}_{++}$ as in the proof of Theorem 2 by picking $\hat{x}^j \in \arg\min_{\succ_p} U_j$ for each $U_j \in (U/\overset{*}{\frown})$.

Finally, define the binary relation $\succ$ on U and the choice correspondence $\Gamma : \mathcal{F} \to \mathcal{F}$ as in the proof of Theorem 2.

Step 1. (Γ, v) *represents* p*; and* Γ *is rationalized by* $\succ$.

By Lemma 2(a), CA and LT imply 4-PR. Since p satisfies CD, the result then follows from the proof of Theorem 2. ❙

Step 2. $\succ$ *is a semi-ordering based on* v.

Since p satisfies CA and 4-PR, p satisfies SPR by Step 5 in the proof of Theorem 2.

By Lemma 1, it suffices to show that $\succ^*$ satisfies v-coarsening. To see this, let $x \succ z \sim y$ for some $z \in U$. (The case where $x \sim z \succ y$ for some $z \in U$ is similar.) There are two possibilities.

If $x \in U_j$ and $y, z \in U_k$ for $j \neq k$, then $k < j$ since $x \succ z$. By definition of v, it then follows that

$$v(x) \geq f(j) \geq v(\overline{z}) + 1 > v(y)$$

where $\overline{z} \in \arg\max_{\succ_p} U_k$. So, $v(x) > v(y)$ as required.

Otherwise, $x, y, z \in U_j$. First observe $p(y, \{x, y\}) \in [0, 1/2)$. Otherwise, $p(y, \{x, y\}) \in [1/2, 1]$. Then, $p(x, \{x, z\}) = 1$ implies $p(y, \{y, z\}) = 1$ by LT. But, this contradicts $y \sim z$.

1. Suppose $p(y, \{x, y\}) = 0$. By definition, $u(x) := \mathbb{P}(w_i, S_i)_{i=1}^n \times f(j)$ for some sequence of imperfect discriminations $(w_i, S_i)_{i=1}^n$ linking x to $\hat{x}^j$; and $u(y) := \mathbb{P}(w_i', S_i')_{i=1}^m \times f(j)$ for some sequence of imperfect discriminations $(w_i', S_i')_{i=1}^m$ linking y to $\hat{x}^j$. Then, since <u>p satisfies WC</u> and SPR, it follows that

$$1 > \frac{\mathbb{P}(w_i', \{w_i', w_{i+1}'\})_{i=1}^m \times f(j)}{\mathbb{P}(w_i, \{w_i, w_{i+1}\})_{i=1}^n \times f(j)} = \frac{\mathbb{P}(w_i', S_i')_{i=1}^m \times f(j)}{\mathbb{P}(w_i, S_i)_{i=1}^n \times f(j)} = \frac{u(y)}{u(x)}.$$

2. Suppose $p(y, \{x, y\}) \in (0, 1/2)$. Then, SPR implies

$$1 > \frac{p(y, \{x, y\})}{p(x, \{x, y\})} = \frac{\mathbb{P}(w_i', S_i')_{i=1}^m \times f(j)}{\mathbb{P}(w_i, S_i)_{i=1}^n \times f(j)} = \frac{u(y)}{u(x)}.$$

In either case, $v(x) > v(y)$ as required. ❙

To conclude the proof, note that LD is necessary and sufficient for the uniqueness of v. ❙

($\Leftarrow$) Suppose p is represented by an extended Luce rule (Γ, v); and Γ is rationalized by a semi-ordering $\succ$ that is based on v. In addition, suppose p satisfies CD. Given Theorem 2, I show that p satisfies LT.

Fix $x, y, z \in U$ such that $p(x, \{x, y\}), p(y, \{y, z\}) \in [1/2, 1]$. Suppose $p(x, \{x, y\}) = 1$. (The case where $p(y, \{y, z\}) = 1$ is similar.) If $p(y, \{y, z\}) \in [1/2, 1)$, then $v(y) > v(z)$. Since $\succ$ satisfies v-transitivity (by Lemma 1), it follows that $p(x, \{x, z\}) = 1$. Otherwise, $p(y, \{y, z\}) = 1$. In that case, $p(x, \{x, z\}) = 1$ follows from the transitivity of $\succ$. ❙ ■

(e) Proof of Theorem 1

($\Rightarrow$) Suppose p satisfies CA, LT and FCD. As discussed in the text, FCD implies CD and LD. So, Theorem 1* implies that p is represented by (Γ, v) where Γ is rationalized by a semi-ordering $\succ$ based on v; and v is unique up to a strictly positive scalar. Finally, since p satisfies FCD, $\succ$ also satisfies the sorites property. ❙

($\Leftarrow$) Suppose p is represented by an extended Luce rule (Γ, v); and Γ is rationalized by a semi-ordering based on v and satisfies the sorites property. Since p satisfies CA and LT by Theorem 1*, I show that p satisfies FCD.

Fix $x, y \in U$ such that $p(y, \{x, y\}) = 0$. Then, $x \succ y$. Since $\succ$ satisfies the sorites property, $x = z_1 \sim ... \sim z_n = y$ for some finite sequence $z_1, ..., z_n \in U$. If $p(z_i, \{z_i, z_{i+1}\}) \geq 1/2 \geq p(z_{i+1}, \{z_{i+1}, z_{i+2}\})$ for some $1 \leq i \leq n-2$, then $v(z_i), v(z_{i+2}) \geq v(z_{i+1})$. Since $\succ$ satisfies v-transitive indifference (by Lemma 1), $z_i \sim z_{i+2}$. By deleting z_{i+1}, one obtains a shorter sequence linking x to y. (The same reasoning applies if $p(z_i, \{z_i, z_{i+1}\}) \leq 1/2 \leq p(z_{i+1}, \{z_{i+1}, z_{i+2}\})$ for some $1 \leq i \leq n-2$.) Ultimately, this leads to a sequence $z'_1, ..., z'_m \in U$ such that: $p(z'_i, \{z'_i, z'_{i+1}\}) \leq 1/2$ for $i = 1, ..., m-1$; or $p(z'_i, \{z'_i, z'_{i+1}\}) \geq 1/2$ for $i = 1, ..., m-1$. In the first case, $v(x) = v(z'_1) \leq ... \leq v(z'_m) = v(y)$. Since $\succ$ satisfies v-coarsening (by Lemma 1), $y \succsim x$. But, this contradicts $x \succ y$. So, the second case obtains. ■

Appendix C – Further Generalizations

Lemma 1′. *For a utility function $v : U \to \mathbb{R}$ and an irreflexive relation $\succ$ on U:*

(b.i) *$\succ$ is an interval ordering based on v if and only if $\succ$ satisfies P3 and $\succ^{**}$ satisfies v-coarsening.*

(b.ii) *$\succ$ is an interval ordering that is weakly based on v if and only if $\succ$ satisfies P3 and v-coarsening.*

(c.i) *$\succ$ is a semi-ordering that is weakly based on v if and only if $\succ$ satisfies P3, P4 and v-coarsening.*

(d) *$\succ$ is a weak ordering based on v if and only if $\succ$ satisfies P5 and v-coarsening.*

Proof. In each case, necessity is obvious. For sufficiency, define I as in the proof of Lemma 0. Then:

(**b.i**) Since it satisfies P3, $\succ$ is an interval ordering. By Steps 2-5 (in the proof of Lemma 0), it follows that (I, v) represents $\succ$. Since $\succ^{**}$ satisfies v-coarsening, (I, v) satisfies R1 by Step 6 (in the proof of Lemma 0). ❙

(**b.ii**) By the argument in part (b.i), $\succ$ is an interval ordering represented by (I, v). ❙

(**c.i**) Since it satisfies P3 and P4, $\succ$ is a semi-ordering. By the argument in part (b.i), (I, v) represents $\succ$. Since $\succ$ is a semi-ordering, (I, v) satisfies R2 by Step 7 (in the proof of Lemma 0). ❙

(**d**) Since it satisfies P5, $\succ$ is a weak ordering. By the argument in part (c), (I, v) represents $\succ$ and satisfies R2. Since $\succ$ is a semi-ordering, it follows that (I, v) also satisfies R3 (which is tantamount to the transitivity of $\sim$). Finally, since $\succ$ is a weak ordering, $\succ$ coincides with $\succ^{**}$. Since $\succ$ satisfies v-coarsening, $\succ^{**}$ satisfies the same property. So, satisfies R1 by Step 6 (in the proof of Lemma 0). ■

Lemma 4′. *For a random choice function p that satisfies the Choice Axiom, the following are equivalent:*

(i) *p satisfies S3-S4 and Moderate Coarsening; and*

(ii) *p satisfies Luce Transitivity; and*

(iii) *p satisfies Strong Coarsening.*

Proof. Fix a random choice function p that satisfies CA.

(**i**) $\Rightarrow$ (**ii**) First observe the following. Since p satisfies CA and S3, p satisfies 4-PR by the argument in Section 4(b) of the text. So, p satisfies SPR by Step 5 in the proof of Theorem 2. Next, suppose $p(x, \{x, y\}) = 1$ and $p(y, \{y, z\}) \in [1/2, 1]$. (The case where $p(x, \{x, y\}) \in [1/2, 1]$ and $p(y, \{y, z\}) = 1$ is similar.) By S2, $p(x, \{x, z\}) > 0$. By way of contradiction, suppose $p(x, \{x, z\}) \in (0, 1)$. Then, $z \sim x \succ y$ so that $z \succ^* y$.

In fact, $z \succ^{**} y$. If $y \succ z$, then $x \succ z$ follows by S2, which is a contradiction. If $y \sim w \succ z$ for some $w \in U$, then $x \succ z$ or $w \succ y$ by S3, which is a contradiction. If $y \succ w \sim z$ for some $w \in U$, then $x \succ z$ or $z \succ y$ by S4, which is a contradiction. Since $z \succ^{**} y$, it then follows that $p(z, \{y, z\}) > p(y, \{y, z\})$ by MC, which is a contradiction. ❙

(**ii**) $\Rightarrow$ (**iii**) As in the proof of Lemma 4, p satisfies 4-PR and SPR. Suppose y is linked to x by a finite sequence of imperfect discriminations $(z_i, \{z_i, z_{i+1}\})_{i=1}^n$ and $x \succ^* y$. If $x \succ y$, then $\mathbb{P}(z_i, \{z_i, z_{i+1}\})_{i=1}^n < 1$ by Lemma 4(a). Otherwise, $p(y, \{x, y\}) \in (0, 1)$. Since $x \succ^* y$, suppose there exists some $w \in U$ such that $y \sim x \succ w \sim y$. (The case where $x \sim w \succ y \sim x$ is similar.) Then, $p(x, \{x, y\}) \in (1/2, 1)$. Otherwise, $p(y, \{x, y\}) \in [1/2, 1)$ and $p(x, \{x, w\}) = 1$

imply $p(y,\{y,w\})=1$ by LT, which contradicts $p(y,\{y,w\})\in(0,1)$. Since $p(x,\{x,y\})\in({}^1\!/_2,1)$, the desired inequality $\mathbb{P}(z_i,\{z_i,z_{i+1}\})_{i=1}^n<1$ then follows by SPR. ❙

(iii) $\Rightarrow$ **(i)** To show S3, let $p(y,\{x,y\}),p(w,\{z,w\})=0$. By way of contradiction, suppose $p(x,\{x,w\}),p(y,\{y,z\})\in[0,1)$. Since p satisfies S2 (by the proof of Theorem 2), $p(x,\{x,w\}),p(y,\{y,z\})\in(0,1)$ and $p(x,\{x,z\})\in(0,1)$. So, $x\succ^* z\succ^* x$ by definition. Then, $p(x,\{x,z\})>p(z,\{x,z\})>p(x,\{x,z\})$ by SC, which is a contradiction.

To show S4, let $p(y,\{x,y\}),p(z,\{y,z\})=0$. By way of contradiction, suppose $p(w,\{z,w\}),p(x,\{x,w\})\in[0,1)$. Since p satisfies S2, $p(w,\{z,w\}),p(x,\{x,w\})\in(0,1)$. So, $y\succ^* w\succ^* y$ by definition. Then, $p(y,\{y,w\})>p(w,\{y,w\})>p(y,\{y,w\})$ by SC, which is a contradiction. ■

Theorem 1′.[28] *Suppose p is a random choice function that satisfies Compact Domain. Then:*

(a) *p satisfies the Choice Axiom, the Quadruple Product Rule and Weak Coarsening if and only if it is represented by an extended Luce rule (Γ,v) where Γ is rationalized by a pre-ordering that satisfies v-coarsening.*

(b.i) *p satisfies the Choice Axiom, Axiom S3 and Moderate Coarsening if and only if it is represented by an extended Luce rule (Γ,v) where Γ is rationalized by an interval ordering that is based on v.*

(b.ii) *p satisfies the Choice Axiom, Axiom S3 and Weak Coarsening if and only if it is represented by an extended Luce rule (Γ,v) where Γ is rationalized by an interval ordering that is weakly based on v.*

(c.i) *p satisfies the Choice Axiom, Axioms S3-S4 and Weak Coarsening if and only if it is represented by an extended Luce rule (Γ,v) where Γ is rationalized by a semi-ordering that is weakly based on v.*

(c.ii) *p satisfies the Choice Axiom and Strong Coarsening if and only if it satisfies the Choice Axiom, Axioms S3-S4 and Moderate Coarsening if and only if it is represented by an extended Luce rule (Γ,v) where Γ is rationalized by a semi-ordering that is based on v.*

(d) *p satisfies the Choice Axiom and Axiom S5 if and only if it is represented by an extended Luce rule (Γ,v) where Γ is rationalized by a weak ordering that is based on v.*

Proof. Fix a random choice function p that satisfies CD. In each case, necessity is obvious. For sufficiency, define (I,v) as in the proof of Theorem 1*. Then:

(a) Since p satisfies WC, it satisfies WT by Lemma 3(a). Then, by Step 1 (in the proof of Theorem 1*), (Γ,v) represents p; and Γ is rationalized by $\succ$. To see that $\succ$ satisfies v-coarsening, suppose $x\succ y$. Then, by the same type of reasoning as Step 2 (in the proof of Theorem 1*), it follows that $v(x)>v(y)$. ❙

(b.i) Since p satisfies CA and S3, it satisfies 4-PR by the argument in Section 4(b); and $\succ$ satisfies P3. Since p satisfies WC, it satisfies WT by Lemma 3(a). By the argument in part (a), (Γ,v) represents p; and Γ is rationalized by $\succ$. If $\succ^{**}$ satisfies v-coarsening, then Lemma 1′(b.i) implies that $\succ$ is an interval ordering based on v .

To see that $\succ^{**}$ satisfies v-coarsening, suppose that x covers y and $x\succ z\sim y$ for some $z\in U$. (The case where $x\sim z\succ y$ for some $z\in U$ is similar.) Then, $v(x)>v(y)$ by the same type of reasoning as Step 2 (in the proof of Theorem 1*). (The key difference is that $p(y,\{x,y\})\in[0,{}^1\!/_2)$ follows from MC rather than LT. If $p(y,\{x,y\})=1$, then $p(y,\{y,z\})=1$ by S2 (which holds by Step 1 of Theorem 2). But, this contradicts $z\sim y$. Since x strictly covers y, $p(y,\{x,y\})\in[{}^1\!/_2,1)$ contradicts MC. So, $p(y,\{x,y\})\in[0,{}^1\!/_2)$ as required.) ❙

(b.ii) By the argument in part (b.i), (Γ,v) represents p; Γ is rationalized by $\succ$; and $\succ$ satisfies P3. By Lemma 1′(b.ii), it suffices to show that $\succ$ satisfies v-coarsening. To see this, suppose $x\succ y$. There are two possibilities:

- If $x\in U_j$ and $y\in U_k$ for $j\neq k$, then $v(x)\geq f(j)\geq v(\overline{z})+1>v(y)$ as in Step 2 (from Theorem 1*).

- Otherwise, $x,y\in U_j$. By definition, $u(x):=\mathbb{P}(w_i,S_i)_{i=1}^n\times f(j)$ for some sequence $(w_i,S_i)_{i=1}^n$ linking x to $\hat{x}^j$; and $u(y):=\mathbb{P}(w_i',S_i')_{i=1}^m\times f(j)$ for some sequence $(w_i',S_i')_{i=1}^m$ linking y to $\hat{x}^j$. Since $p(y,\{x,y\})=0$, WC implies $u(x)=\mathbb{P}(w_i,S_i)_{i=1}^n\times f(j)>\mathbb{P}(w_i',S_i')_{i=1}^m\times f(j)=u(y)$ as in Step 2 (from Theorem 1*). ❙

(c.i) By the argument in part (b.ii), (Γ,v) represents p; Γ is rationalized by $\succ$; and $\succ$ satisfies P3 and v-coarsening. Since p satisfies S4, $\succ$ satisfies P4. By Lemma 1′(c), it follows that $\succ$ is a semi-ordering weakly based on v. ❙

[28]To simplify, I omit the uniqueness properties of the representation (which are the same as Theorem 2).

(c.ii) This follows directly from Lemma $4'$ and Theorem 1^*. ❙

(d) Since p satisfies CA and S5, it satisfies WC by Lemmas 4-5. Then, by the argument in part (b.i), (Γ, v) represents p; Γ is rationalized by $\succ$; and $\succ$ satisfies v-coarsening. Since p satisfies S5, it follows that $\succ$ satisfies P5. By Lemma $1'$(d), it follows that $\succ$ is a weak ordering based on v. ■