Social Discounting and Intergenerational Pareto

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Abstract

The most critical issue in evaluating policies and projects that affect generations of individuals is the choice of social discount rate. This paper shows that there exist social discount rates such that the planner can simultaneously be (i) an exponential discounting expected utility maximizer; (ii) intergenerationally Pareto—i.e., if all individuals from all generations prefer one policy/project to another, the planner agrees; and (iii) strongly non-dictatorial—i.e., no individual from any generation is ignored. Moreover, to satisfy (i)–(iii), if the time horizon is long enough, it is generically sufficient and necessary for social discounting to be more patient than the most patient individual's long-run discounting, independent of the social risk attitude.

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1 Introduction

Many economic decisions are inherently dynamic and affect multiple generations, such as corporate and household long-term investment decisions, intertemporal taxation, durable public good provision, environmental policies, etc. These decisions crucially depend on one parameter, the *social discount rate*, which encapsulates the trade-off between current benefit and future benefit from the society's point of view. Unfortunately, there is no consensus on which social discount rate should be used. This disagreement has sparked debate, for example, about the cost-benefit analysis of environmental projects that affect many, if not all, future generations. Moreover, the evaluation of those projects is sensitive to the choice of social discount rate. The famous Stern review uses a near-zero social discount rate (pure rate of time preference), and suggests that we should take strong and immediate action on climate change (see Stern (2007)).[1] Nordhaus (2007) argues that Stern's conclusion does not hold if a market rate is used instead. Many economists, however, believe that using a high discount rate (such as the market rate) is ethically indefensible.

In the social discounting literature, some economists have argued that social discounting should be more patient than individual discounting (for example, see Caplin and Leahy (2004) and Farhi and Werning (2007)). The idea is that if social discounting takes into account how future generations will feel about their consumption, then because future generations will value future consumption relatively more than the current generation values future consumption, social discounting will also value future consumption more than the current generation does.[2] However, these studies usually assume that only one (representative) individual is in the society. How their insight carries over to a society with heterogeneous individuals—and which individual's discounting social discounting should be more patient than—remains unanswered.

Let us explain what will go wrong with heterogeneous individuals. Note that what is

[1]The consumption discount rate derived from the Ramsey formula used in the Stern review depends on the pure rate of time preference, the elasticity of the marginal utility of consumption, and the growth rate of per-capita consumption.

[2]Some economists have also argued that individuals' altruistic discounting for future generations should be excluded from the planner's aggregation. See Hammond (1987), Mirrlees (2007), and Boadway (2012).

common among these dynamic economic decisions is that there is a benevolent planner who needs to make choices for generations of individuals, and, as in environmental projects and many other examples, payoff uncertainty is usually involved. In such a setting, first, economists often assume that the planner's objective is an *exponential discounting expected utility function*. This assumption is widely used and normatively appealing, because it is equivalent to assuming that the planner's preference is time-consistent, time-invariant, and stationary.[3] Second, it is often assumed that a benevolent planner respects individuals' preferences. In other words, some notion of the *Pareto* property should hold: If "all" individuals agree that one project is better than another, the planner should agree that the former is better.

Despite the fact that these two assumptions are fundamental to economics, economists have established that they cannot be satisfied simultaneously (see Gollier and Zeckhauser (2005), Zuber (2011), and Jackson and Yariv (2015)). Even if every individual has an exponential discounting utility function, a planner must be dictatorial to ensure that her exponential discounting utility function satisfies some Pareto property. The negative result also raises a challenge to the conclusion that social discounting should be more patient than individual discounting. In light of the negative result, with heterogeneous individuals, perhaps we can only conclude that the planner is more patient than the only individual (dictator) she cares about.

This paper addresses these issues using a classic approach. We introduce a new Pareto property, and characterize the range of (pure-time-preference) social discount rates that are compatible with the new Pareto property. In models that generate the negative result, there is often only one generation of individuals. The Pareto property they use, which we call *current-generation Pareto*, is the key to the negative result. Current-generation Pareto requires that whenever a consumption sequence $\mathbf{p}$ is preferred to another sequence $\mathbf{q}$ by every current-generation individual, then the planner prefers $\mathbf{p}$ to $\mathbf{q}$. In many problems that

[3] A version of the definition of time consistency, time invariance, and stationarity can be found in Halevy (2015). Under the assumption that the utility function is a time-additively separable expected utility function, Halevy's version of the three properties is equivalent to assuming an exponential discounting expected utility function.

we consider, especially the environmental projects, multiple generations of individuals are involved. To determine the social discount rate, it seems natural that the planner should not only respect how the current generation discounts the future, but also care about the actual well-being of future generations—that is, how future generations will feel about their consumption and how they will discount the future. The Pareto property we introduce, *intergenerational Pareto*, captures this. It requires that whenever a consumption sequence **p** is preferred to **q** by every individual from every generation, then the planner prefers **p** to **q**.

Specifically, each generation-t individual i lives for one period, and has an arbitrary discount function $\delta_i(\tau - t)$ to discount the τ^{th} period consumption.[4] The planner is intergenerationally Pareto and has an exponential discounting utility function. To contrast with the negative result, we require that the planner be *strongly non-dictatorial* in the sense that she never ignores the preference of any individual from any generation. Under these assumptions, we show how the range of social discount rates depends on (a) individual relative discounting, average discounting, and long-run discounting, and (b) the linear dependency of individual instantaneous utility functions.

We first examine a benchmark case in which the time horizon is finite and individuals share the same instantaneous utility function. This allows us to focus on aggregating discount functions. We show that there exist two cutoffs for the social discount factor.[5] One is the lowest (across individuals) maximal (across time) relative discount factor, $\min_i \max_\tau \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$, and the other is the lowest (across individuals) asymptotic average discount factor, $\min_i \lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)}$. If the social discount factor is above the first cutoff, we show that the planner must be intergenerationally Pareto and strongly non-dictatorial. Thus, we can avoid the negative result even when individuals have arbitrary discount functions. Moreover, checking whether a planner's utility function is compatible with the Pareto property is generally difficult, but this result provides an easy way to do it. Conversely, if

[4]Each individual *altruistically* cares about future generations' consumption, as is the case when we think about environmental projects. Also note that the individual discount functions in Zuber (2011) and Jackson and Yariv (2014, 2015) are exponential. We do not make this assumption.

[5]The discount rate is equal to one minus the discount factor.

the social discount factor is below the second cutoff, we show that the planner must violate intergenerational Pareto as long as the time horizon is long enough; that is, there exist two consumption sequences such that every individual from every generation thinks that one is better than the other, but the planner disagrees. We provide examples to show that these two cutoffs are tight.

The two cutoffs merge into one cutoff when individuals exhibit *present bias*. The unique cutoff is equal to the *least patient* individual's *long-run discount factor*, in which each individual i's long-run discount factor is defined as the asymptotic relative discount factor and the asymptotic average discount factor.

Since the least patient individual's long-run discount factor could be quite low, the benchmark case does not say much about which social discount factor is reasonable. Our main result (Theorem 2) shows that if we do not assume that individuals have identical instantaneous utility functions, the result will be rather different. Generically, individual instantaneous utility functions are linearly independent in the functional space. We show that if individual instantaneous utility functions are linearly independent, the cutoff for the social discount factor jumps to the *most patient* individual's long-run discount factor, independent of the planner's choice of instantaneous utility function. This result thus supports the use of a near-zero social discount rate.

We show how the cutoff for the social discount factor changes gradually from the least patient individual's long-run discount factor to the most patient, as the number of types of individual instantaneous utility functions increases. If there is only one type, we are in the benchmark case. As the number of types increases, the cutoff moves to the most patient individual's long-run discount factor.

Lastly, we show that our main result continues to hold if the time horizon is infinite.

1.1 Related Literature

This paper is not the first to aggregate the preferences of multiple generations of individuals. Indeed, there is a long-running debate on whether future generations should be aggregated.

For example, among others, Ramsey (1928), Pigou (1920), Sen (1961), Feldstein (1964), Solow (1974), Arrow (1999), Caplin and Leahy (2004), and Farhi and Werning (2007) are in favor. On the other hand, among others, Eckstein (1957), Bain (1960), and Marglin (1963) believe that the government's or the policy maker's decision should only reflect the preferences of present individuals. Our approach is closer to Caplin and Leahy and Farhi and Werning, both of whom show that assuming there is only one individual in each generation, social discounting should be more patient than the sole individual's discounting. Our results show that having multiple heterogeneous individuals in each generation makes an important difference.

Many papers have analyzed aggregation of one generation of heterogeneous individuals. Weitzman (2001) conducts a survey on economists' discount rates to motivate a gamma discounting model. Gollier and Zeckhauser (2005) study a dynamic efficient allocation problem with heterogeneous individuals and show that even when individuals have constant discount rates, the representative agent has a decreasing discount rate. Zuber (2011) establishes that a planner cannot have an exponential discounting utility function and be (current-generation) Pareto when individuals have private consumption. Jackson and Yariv (2015) present a similar negative result in which consumption is public. Millner and Heal (2017) show that the negative result goes away if we only require that the planner's objective be time-consistent. A key difference between these papers and ours is that they aggregate only one generation of individuals, whereas we aggregate multiple generations. This distinction is important in economic decisions that have long-term impact, such as environmental policies.

Most of the studies discussed above assume that individuals have exponential discounting functions. It is well known that individuals are often time-inconsistent (see Strotz (1955), Laibson (1997), and Frederick et al. (2002), among others). Hence, it is important to understand whether the results continue to hold when we allow individuals to have more general discount functions.

There are other approaches to the study of social discounting. Our paper emphasizes the relation between social discounting and individual discounting implied by the intergenerational Pareto property. Chambers and Echenique (2017) study three models on discount rates. In the first, they characterize when a sequence of utility is always preferred to another sequence, for any discount rate between zero and one. The second model is similar to Weitzman (2001), Zuber (2011), and Jackson and Yariv (2014, 2015): The aggregate discount function is a weighted average of a set of exponential discount functions. In the last model, in order to discount a sequence of utility, the aggregate preference selects the most pessimistic from a set of discount rates. Millner (2016) shows that if heterogeneous individuals are not fully paternalistic, they will agree on parameters for the long-run social discount rate. Zuber and Asheim (2012), Asheim and Zuber (2014), Fleurbaey and Zuber (2015), and Piacquadio (2017) study models in which social discounting is due to intergenerational inequality aversion. Jonsson and Voorneveld (2017) study a welfare criterion for multiple generations. Each generation has one individual, and in the limit of the criterion, different generations are treated equally. In the first part of Drugeon and Wigniolle (2017), they characterize what exponential discounting utility functions can be written as weighted sums of the current self's and future selves' quasi-hyperbolic discounting utility functions; this is similar to a special case of our Theorem 4 or Proposition 4 and to a related result in Galperti and Strulovici (2017).

Our paper is also related to Mongin (1998), who establishes that under a standard form of the Pareto property, as long as individuals' subjective probabilities are linearly independent or their instantaneous utility functions are affinely independent, the planner must be dictatorial. Similar results can be found in Mongin (1995) and Chambers and Hayashi (2006). In our model, if we view each period as a state and discount factors as subjective probabilities, Mongin's result seems to apply. However, our planner is not dictatorial. The technical reason why our Theorem 1 can bypass Mongin's negative result is the assumption that all individuals share the same instantaneous utility function. As for Theorem 2, we first aggregate individual utility functions with identical instantaneous utility functions into a utility function whose discount factor is equal to the social discount factor. Then, we aggregate utility functions with identical discount factors (subjective probabilities). Both steps bypass

Mongin's negative result.

Lastly, related to the preference aggregation literature, our Lemma 2 extends Harsanyi's (1955) theorem and Zhou (1997) to the case with countably infinitely many individuals. Zhou shows that Pareto and utilitarianism are equivalent when the set of individuals is compact.

The paper proceeds as follows. In Sections 2 and 3, we describe individuals' and the planner's preferences. We then introduce a variant of the negative result and intergenerational Pareto. Section 4 studies the benchmark case, in which we characterize the range of social discount factors that are compatible with intergenerational Pareto. Our main results in Section 5 show how individual instantaneous utility functions interact with social discounting. Section 6 studies the infinite-horizon case, and Section 7 concludes.

2 Preferences

There are $2 < T \leq +\infty$ generations/periods. In each generation, $N < +\infty$ individuals live for one period. With an abuse of notation, we use $N := \{1, \ldots, N\}$ and $T := \{1, \ldots, T\}$ to denote the set of individuals and the set of time periods, respectively. The generation-t individual i is the parent of the generation-$(t+1)$ individual i. In each period, there is a public risky consumption good.[6] The set of consumption goods is $\Delta(X)$, in which $\Delta(X)$ is the set of probability measures on a compact set $X \subset \mathbb{R}^m$. A typical consumption sequence is denoted by $\mathbf{p} = (p_1, \ldots, p_T) \in \Delta(X)^T$.[7]

Each generation-t individual i has a preference $\succsim_{i,t}$ over the consumption sequences ($t \in T$ and $i \in N$). As is the case when we think about environmental policies, each individual *altruistically* cares about future generations' consumption. We assume throughout the paper

[6] All results we derive apply to the case in which each individual has his own consumption. We only need to view public consumption as an N-tuple of individual consumption, and let each individual care only about his own component.

[7] We discuss what may change if we allow uncertainty to resolve over time in Section S1 in the Supplemental Material.

that the generation-t individual i has the following *discounting utility* function:

$$U_{i,t}(\mathbf{p}) = \sum_{\tau=t}^{T} \delta_i(\tau - t) u_i(p_\tau), \tag{1}$$

in which $\delta_i : \{0, \ldots, T-1\} \to \mathbb{R}_{++}$ with $\delta_i(0) = 1$ is called the *discount function*, and the *instantaneous utility function* $u_i : \Delta(X) \to \mathbb{R}$ is a continuous expected utility function. When $T = +\infty$, we require $(\delta_i(\tau))_{\tau=0}^{\infty}$ to be an absolutely summable sequence (in ℓ^1). The well-known exponential, hyperbolic, and quasi-hyperbolic discounting utility functions are special cases of (1).

It is common to assume that $U_{i,t}(\mathbf{p})$ does not depend on past consumption. When a generation-t individual comes into existence, the past is sunk; that is, comparing $\mathbf{p}$ and $\mathbf{q}$ from his point of view is the same as comparing $(p_t, \ldots, p_T)$ and $(q_t, \ldots, q_T)$. This also means that there is no revealed-preference foundation for utility over past consumption.[8]

We have also assumed that the generation-$(t+1)$ individual i inherits the generation-t individual i's discount function and instantaneous utility function. This assumption does not imply that a parent and his offspring have the same preference, because the generation-$(t+1)$ individuals' discount functions are shifted one period forward. This assumption simplifies our analysis and can be relaxed (see Section S4.1 in the Supplemental Material).

In each period $t \in T$, the planner has a preference $\succsim_t$ over the consumption sequences. As in most dynamic models, we assume that the planner's objective is an exponential discounting expected utility function; that is, in each period t, the planner has a utility function of the following form:

$$U_t(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u(p_\tau), \tag{2}$$

in which $\delta > 0$ is the *social discount factor*, and u, a continuous expected utility function on $\Delta(X)$, is the planner's instantaneous utility function. When $T = +\infty$, we require $\delta < 1$.

It is well known that if the planner's objective is a discounting utility function $U_t(\mathbf{p}) =$

[8]However, see Caplin and Leahy (2004) and Ray et al. (2017) for models that allow for backward discounting for past consumption. In the Supplemental Material, we show that our results continue to hold when individuals have exponential forward and backward discounting.

$\sum_{\tau=t}^{T} \delta(\tau - t)u(p_\tau)$, the planner is time-consistent if and only if the planner's discount function is exponential.[9] More generally, (2) holds if and only if the planner's preference is time-consistent, time-invariant, and stationary (see footnote 3). Note that the above equation holds for every $t \in T$; that is, the social discount factor and the planner's instantaneous utility function never change.

Lastly, to rule out uninteresting cases and simplify the statement of our results, we assume that there are some fixed consequences $x_*, x^* \in X$ such that $u_i(x_*) = u(x_*) = 0$ and $u_i(x^*) = u(x^*) = 1$ for any $i \in N$ throughout the paper. A similar assumption, called the *minimum agreement condition*, also appears in De Meyer and Mongin (1995). Our main findings do not rely on this assumption, and we provide a more detailed discussion following Lemma 1.[10] More generally, for any continuous expected utility function v defined on $\Delta(X)$, we say that it is *normalized* if $v(x^*) = 1$ and $v(x_*) = 0$. One may think of x^* as the best consumption good and x_* as the worst, or x^* as one dollar and x_* as zero dollars.

3 Intergenerational Pareto

We want to assume that the planner's preference $(\succsim_t)_{t \in T}$ satisfies some Pareto property. In a dynamic setting, however, there are multiple ways to define the Pareto property. Different notions of Pareto lead to different results. For example, Zuber (2011) and Jackson and Yariv (2015) show that if a planner has an exponential discounting utility function and follows their Pareto property, the planner must be dictatorial. To motivate our new Pareto property, it is useful to first understand the negative result. Below, we introduce a version of the negative result.

[9] Since individuals only live for one period, time consistency may have a nonstandard interpretation for them. In contrast, the planner is a long-lived entity who tries to stick to an objective function that exhibits nice properties. The interpretation of time consistency for the planner is similar to the standard one.

[10] In Section S4.1 in the Supplemental Material, when we allow the instantaneous utility function to depend on time, the normalization assumption will play a more important role. In that case, because expected utility functions are unique up to positive affine transformations, we cannot pin down the discount function without some type of normalization assumption.

3.1 A Variant of the Negative Result

Below is a variant of the Pareto property used by Zuber (2011) and Jackson and Yariv (2015) that fits into our setting.

Definition 1 *The planner's preference* $(\succsim_t)_{t\in T}$ *is current-generation Pareto if for any consumption sequences* $\mathbf{p}, \mathbf{q} \in \Delta(X)^T$*, in each period* $t \in T$*,* $\mathbf{p} \succsim_{i,t} \mathbf{q}$ *for all* $i \in N$ *implies* $\mathbf{p} \succsim_t \mathbf{q}$*, and* $\mathbf{p} \succ_{i,t} \mathbf{q}$ *for all* $i \in N$ *implies* $\mathbf{p} \succ_t \mathbf{q}$.

This notion of the Pareto property says that in any period t, if all current-generation individuals agree that a consumption sequence $\mathbf{p}$ is preferred to another sequence $\mathbf{q}$, then the planner should agree that $\mathbf{p} \succsim_t \mathbf{q}$. The same applies when the preferences are all strict.

Consider a simple situation in which every generation-t individual i has an exponential discounting utility function. The generation-t individual i has an *exponential discounting utility* (EDU) function if $\delta_i(\tau) = \delta_i^\tau$ for some discount factor $\delta_i > 0$; that is,

$$U_{i,t}(\mathbf{p}) = \sum_{\tau=t}^{T} \delta_i^{\tau-t} u_i(p_\tau).$$

When $T = +\infty$, we require $\delta_i < 1$. Let us present below a variant of the negative result.

Proposition 1 *Suppose each generation-t individual i has an EDU function with discount factor* δ_i *and instantaneous utility function* u_i*. For a generic N-tuple of discount factors* $(\delta_i)_{i\in N}$*, the planner is current-generation Pareto if and only if for each* $t \in T$*, there exists a unique* $i \in N$ *such that* $U_t = U_{i,t}$.

The result says that if we require that the planner be current-generation Pareto and have an exponential discounting expected utility function, the planner's preference must be identical to exactly one individual's preference. Since the consumption is public, our setting is closer to Jackson and Yariv (2015). However, Jackson and Yariv's result is still different from the above proposition; they require instantaneous utility functions to be defined on a one-dimensional space and be twice continuously differentiable. We only require that individual discount factors be generic.

The intuition is as follows. First, the planner is current-generation Pareto if and only if her discounting utility function is equal to a weighted sum of the individuals' EDU functions; this is an implication of Harsanyi (1955). Next, for simplicity, suppose there are only two individuals with identical instantaneous utility functions $u_1 = u_2$. The planner attaches a weight ω to the first individual and $1 - \omega$ to the second individual. Now, for the planner to not be dictatorial, there must be some $\omega \in (0, 1)$ and $\delta > 0$ such that

$$\omega\delta_1 + (1 - \omega)\delta_2 = \delta,$$

and

$$\omega\delta_1^2 + (1 - \omega)\delta_2^2 = \delta^2.$$

However, one cannot find such a δ, unless $\omega = 0$ or 1.

3.2 Intergenerational Pareto

The key feature of environmental projects and many other economic policies is that the decisions affect multiple generations. Current-generation Pareto only takes into account the preferences of the current generation. The current generation does altruistically care about future consumption, and there are reasons why we want the planner to respect how individuals discount the future. However, how the current generation thinks about the future may well differ from how future generations will think. Since future generations will be affected by the planner's decision, the planner should take into account their actual well-being, including how they will discount their own future. The following Pareto property captures these ideas.

Definition 2 *The planner's preference* $(\succsim_t)_{t\in T}$ *is intergenerationally Pareto if for any consumption sequences* $\mathbf{p}, \mathbf{q} \in \Delta(X)^T$*, in each period* $t \in T$*,* $\mathbf{p} \succsim_{i,s} \mathbf{q}$ *for all* $i \in N$ *and all* $s \geq t$ *implies* $\mathbf{p} \succsim_t \mathbf{q}$*, and* $\mathbf{p} \succ_{i,s} \mathbf{q}$ *for all* $i \in N$ *and all* $s \geq t$ *implies* $\mathbf{p} \succ_t \mathbf{q}$.

Intergenerational Pareto says that in any period t, if all current- and future-generation individuals agree that a consumption sequence $\mathbf{p}$ is preferred to another sequence $\mathbf{q}$, then the planner should agree that $\mathbf{p} \succsim_t \mathbf{q}$. For example, suppose all current-generation individuals are extremely selfish: They are willing to sacrifice the environment in order to increase their own consumption. If the planner is current-generation Pareto, the planner must agree with them, and let them destroy the environment. However, if the planner is intergenerationally Pareto, the planner is allowed to disagree with them, because what they prefer hurts future generations.

If the planner is current-generation Pareto, she is also intergenerationally Pareto. Therefore, intergenerational Pareto is weaker than current-generation Pareto. The following lemma characterizes the consequence of intergenerational Pareto. The lemma covers a more general case than necessary for our main results. The more general case emphasizes that the following observation has nothing to do with our assumptions that the planner's discount function is exponential, that instantaneous utility functions do not change over time, etc. The more general case will also be useful in Section S4 in the Supplemental Material.

Lemma 1 *(Harsanyi (1955)) Suppose $T < +\infty$, and each generation-t individual i's utility function takes the following form:*

$$U_{i,t}(\mathbf{p}) = \sum_{\tau=t}^{T} \delta_{i,t}(\tau - t) u_i(p_\tau, \tau),$$

and the planner's utility function in period t takes the following form:

$$U_t(\mathbf{p}) = \sum_{\tau=t}^{T} \delta_t(\tau - t) u_t(p_\tau, \tau),$$

in which $\delta_{i,t}(\cdot)$ and $\delta_t(\cdot)$ are discount functions, and $u_i(\cdot, \tau)$ and $u_t(\cdot, \tau)$ are (normalized) instantaneous utility functions. The planner's preference $(\succsim_t)_{t \in T}$ is intergenerationally Pareto if and only if in each period $t \in T$, there exists a finite sequence of nonnegative numbers $(\omega_t(i,s))_{i \in N, s \geq t}$ such that

$$U_t = \sum_{i=1}^{N} \sum_{s=t}^{T} \omega_t(i,s) U_{i,s}.$$

The lemma essentially follows from Harsanyi (1955) and Fishburn (1984), and shows that intergenerational Pareto is equivalent to (intergenerational) utilitarianism in our setting; that is, the planner is intergenerationally Pareto if and only if in each period, her utility function is equal to a weighted sum of all the current- and future-generation individuals' utility functions. We omit the proof.

This lemma depends on the assumption that $U_{i,t}$'s and U_t's are expected utility functions. When $T = +\infty$, a countably infinite version of Harsanyi's theorem is required, which, to the best of our knowledge, has not been established in the literature.[11] We present this result in Section 6.

In the lemma, the instantaneous utility functions $u_i(\cdot, \tau)$ and $u_t(\cdot, \tau)$ are normalized. The normalization assumption has two implications. First, without the normalization assumption, it is possible that there do not exist two consumption sequences such that all individuals strictly prefer one to the other. In that case, if the planner is indifferent to all consumption sequences, the planner will be intergenerational Pareto trivially. When the planner is always indifferent, she has a constant instantaneous utility function and her discount function can be arbitrary. The normalization assumption rules out this uninteresting case. Second, in this lemma, since instantaneous utility functions can depend on τ, without normalizing them in some way, the discount functions will be undetermined. This will become useful in Section S4 in the Supplemental Material.

4 Social Discounting and Individual Long-Run Discounting: The Benchmark Case

We address two aspects of social discounting. First, can we bypass the negative result? If so, which social discount factors are reasonable? In particular, which social discount factors, under our assumptions, are compatible with intergenerational Pareto? Second, recall that

[11] Zhou (1997) has shown how the equivalence between Pareto and utilitarianism can be generalized to the case in which N is compact but not necessarily finite.

in the social discounting literature, economists have argued that the social discount factor should be higher (more patient) than the individual discount factor. Accordingly, with heterogeneous individuals, which individual's discount factor should the social discount factor be higher than?

To contrast with the negative results, we introduce a strong notion of the non-dictatorial property.

Definition 3 *We say that the planner is strongly non-dictatorial if for each $t \in T$,*

$$U_t(\mathbf{p}) = f_t\left(U_{1,t}(\mathbf{p}), \dots, U_{1,T}(\mathbf{p}), U_{2,t}(\mathbf{p}), \dots, U_{2,T}(\mathbf{p}), \dots, U_{N,T}(\mathbf{p})\right)$$

for some strictly increasing function f_t.

We not only want to ensure that the planner is not dictatorial, but also that every individual from every generation has a say. In light of Lemma 1, under intergenerational Pareto, this means that the planner's utility function can be written as a weighted sum of individual utility functions with strictly positive weights.

Intergenerational Pareto is weaker than current-generation Pareto. However, when combined with the strongly non-dictatorial property, the planner has more strictly positive weights to assign, and hence a more complicated task to accomplish. An analogy of this is the following: In Proposition 1, if we increase the number of individuals N, the planner has more weights to assign. However, this does not make it easier for the planner to have an exponential discounting expected utility function. If the planner is required to give strictly positive weights to the newly added individuals, this entails adding their discount factors, which renders the aggregation problem more difficult. The easiest way for the planner to have an exponential discounting expected utility function is to have only one individual and one strictly positive weight to be assigned.

Another obvious assumption that complicates our problem is that in our model, individuals have general discount functions. Under this assumption, it is not even clear what the individual discount factors are; our results show how the social discount factor depends on general individual discount functions.

4.1 The Benchmark Case

We first examine the simplest case to illustrate how social discounting is related to individual discounting. To focus on discounting, we assume that all individual instantaneous utility functions are identical; that is, there is some continuous expected utility function $u : \Delta(X) \to \mathbb{R}$ such that each generation-t individual i's utility function is

$$U_{i,t}(\mathbf{p}) = \sum_{\tau=t}^{T} \delta_i(\tau - t)u(p_\tau). \tag{3}$$

Our main result studies the case without this assumption in Section 5. An alternative interpretation of this assumption is that the planner only wants to aggregate individual discount functions. Therefore, it is without loss of generality to replace the (possibly heterogeneous) individual instantaneous utility functions with the planner's instantaneous utility function u.[12] When individuals share the same instantaneous utility function, it is straightforward to verify that the planner must also use the same instantaneous utility function in order to satisfy Pareto properties.

The benchmark case also assumes that T is finite. Although T is finite, we may vary T in part of the results below. Therefore, we assume that individual discount functions are well defined for any natural number; that is, we start with a set of individual discount functions δ_i's defined over natural numbers $\mathbb{N}$, and whenever we choose a finite T, we restrict the domain of δ_i's to T. For instance, suppose individuals have quasi-hyperbolic discounting functions. We first define $\delta_i(\tau) = \beta_i \delta_i^{\tau-1}$ for any $\tau \geq 0$. Then, we choose T and focus on $\delta_i(0), \ldots, \delta_i(T-1)$.

For each individual discount function $\delta_i(\tau)$, we call $\sqrt[\tau]{\delta_i(\tau)}$ his *average discount function*,

[12] In this interpretation, however, each individual i's preference in the definition of Pareto properties must be replaced with another preference induced by a discounting utility function with a discount function δ_i and an instantaneous utility function u chosen by the planner.

and $\frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ his *relative discount function*.[13] The average discount function measures the equivalent exponential discount factor for τ-period-ahead consumption. The relative discount function captures the additional instantaneous discounting for consumption that is $\tau+1$ periods ahead relative to consumption that is τ periods ahead.

We make two weak assumptions on the individual discount functions. The first assumption says that the average discount function has a limit; that is,

$$\lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)} \text{ exists.} \tag{4}$$

This assumption is weaker than assuming that the relative discount function has a limit. The second assumption says that the relative discount function is bounded; that is,

$$\text{there is some } \alpha > 0 \text{ such that } \frac{\delta_i(\tau+1)}{\delta_i(\tau)} < \alpha \text{ for all } \tau \geq 0. \tag{5}$$

The following theorem characterizes the set of social discount factors that are compatible with intergenerational Pareto under these assumptions.

Theorem 1 *Suppose $T < +\infty$, and each generation-t individual i's discounting utility function has an instantaneous utility function u and a discount function δ_i such that (4) and (5) hold. Then,*

1. *for each $\delta > \min_i \max_{\tau\in\{0,\ldots,T-1\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$, the planner is intergenerationally Pareto and strongly non-dictatorial;*

2. *for each $\delta < \min_i \lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)}$, there exists some $T^* > 0$ such that if $T \geq T^*$, the planner is not intergenerationally Pareto.*

The theorem shows how social discounting depends on individual discounting when there are multiple individuals with general discount functions. We can find two cutoffs for the social discount factor. If it is above the *least patient* individual's maximal relative discount factor,

[13] When $\tau = 0$, we set the average discount function's value to be 1.

the planner's preference must be intergenerationally Pareto and strongly non-dictatorial. If the social discount factor is below the *least patient* individual's asymptotic average discount factor, the planner's preference must have violated the intergenerationally Pareto property *as long as T is large enough.* The planner has a utility function in each period t, and the cutoffs apply in all periods.

In general, when we choose a social discount factor, it is not obvious whether the planner is Pareto. The first part of the theorem allows us to check whether a social discount factor is consistent with the intergenerationally Pareto property. Moreover, it shows that even if we allow individuals to have arbitrary discount functions, and require the planner to have an exponential discounting utility function, the planner can still be intergenerationally Pareto without being dictatorial. In fact, the planner can even be strongly non-dictatorial.

Conversely, the second part of the theorem says that if the social discount factor is too low, then there must be two consumption sequences such that all individuals from all generations prefer one over the other, but the planner disagrees. We do not want to use a social discount factor that allows this to happen.

Note that for any fixed T, $\max_{\tau\in\{1,\dots,T\}} \frac{\delta_i(\tau)}{\delta_i(\tau-1)} \geq \sqrt[T]{\delta_i(T)}$, because

$$\sqrt[T]{\delta_i(T)} = \sqrt[T]{\frac{\delta_i(T)}{\delta_i(T-1)} \cdot \dots \cdot \frac{\delta_i(1)}{\delta_i(0)}};$$

that is, $\sqrt[T]{\delta_i(T)}$ is the geometric mean of $\frac{\delta_i(\tau)}{\delta_i(\tau-1)}$'s. Therefore, $\max_{\tau\in\{1,\dots,T\}} \frac{\delta_i(\tau)}{\delta_i(\tau-1)}$ will be weakly higher than $\lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)}$ when T is large enough, and hence the first cutoff will eventually be higher than the second cutoff.

Although the first cutoff may be strictly higher than the second, the two cutoffs in the theorem are "tight" in the following sense. If the social discount factor is below the first cutoff, there exist some T and individual discount functions $\delta_i(\tau)$'s such that the planner is not intergenerationally Pareto. Similarly, if the social discount factor is above the second cutoff, we can find some individual discount functions $\delta_i(\tau)$'s such that for all finite T, the planner is intergenerationally Pareto and strongly non-dictatorial. To understand more concretely the two cutoffs, we examine two popular special cases in the next subsection. We also use them to illustrate why the cutoffs are tight.

The first part of the theorem can be proved in two steps. According to Lemma 1, intergenerational Pareto is equivalent to (intergenerational) utilitarianism. First, focus on one arbitrary individual i and his offspring. We show that there exist strictly positive weights such that the weighted sum of their utility functions is an EDU function with any discount factor that is strictly higher than $\max_{\tau\in\{0,\ldots,T-1\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$. Thus, without loss of generality, assume that every generation-t individual i has an EDU function with a discount factor that is sufficiently close to but strictly higher than $\max_{\tau\in\{0,\ldots,T-1\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$. Next, let all individuals' weights be equal to some small number $\varepsilon > 0$, except for the least patient individual and his offspring. We show that we can find strictly positive weights for the least patient individual and his offspring such that the weighted sum of all individuals' utility functions is an EDU function with the social discount factor δ.

For the second part, suppose we are in the first period. The planner's period-1 utility is

$$U_1 = \sum_{t=1}^{T} \sum_{i=1}^{N} \omega(i,t) U_{i,t},$$

in which $\omega(i,t) \geq 0$ is the weight the planner assigns to generation-t individual i. Consider how the planner discounts period-τ consumption. Since instantaneous utility functions are identical, the equation above implies

$$\delta^{\tau-1} = \sum_{t=1}^{\tau} \sum_{i=1}^{N} \omega(i,t) \delta_i(\tau - t).$$

By letting $\tau = 1$, $\sum_{i=1}^{N} \omega(i,1) = 1$ and hence the sum of all weights is greater than 1. Now, suppose individual 1's asymptotic average discount factor is the lowest. When τ is large enough (and hence T must be large enough), we know that $\delta_i(\tau - s) \geq \delta_1(\tau - s)$. Hence,

$$\delta^{\tau-1} = \sum_{t=1}^{\tau} \sum_{i=1}^{N} \omega(i,t) \delta_i(\tau - t) \geq \delta_1(\tau - 1) \sum_{t=1}^{\tau} \sum_{i=1}^{N} \omega(i,t) \geq \delta_1(\tau - 1).$$

Therefore, $\delta \geq \lim_{\tau\to\infty} \sqrt[\tau]{\delta_1(\tau)}$ when τ is large enough.

Although this theorem does tell us which individual the planner should be more patient than, it is not very helpful in pinning down social discount factors, because the least patient individual's discount factors can be quite low. Thus, many social discount rates can satisfy our requirements. However, as will be shown below, this is no longer the case once we relax an unrealistic assumption in the benchmark case.

4.2 Individual Quasi-Hyperbolic Discounting and Exponential Discounting

We say that the generation-t individual i has a *quasi-hyperbolic discounting utility* (QHDU) function if his discount function satisfies

$$\delta_i(\tau) = \begin{cases} 1, & \text{if } \tau = 0, \\ \beta_i \delta_i^\tau, & \text{if } \tau \in \{1, \ldots, T-1\} \end{cases}$$

for some $\beta_i \in (0, 1]$ and $\delta_i > 0$. It is immediate that if a generation-t individual i has a QHDU function, then

$$\max_{\tau \in \{0,\ldots,T-1\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)} = \lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)} = \delta_i.$$

The following result is an application of Theorem 1.

Corollary 1 *Suppose $T < +\infty$, and each generation-t individual i has a QHDU function with an instantaneous utility function u, $\beta_i \in (0, 1)$, and $\delta_i > 0$. Then,*

1. *for each $\delta > \min_i \delta_i$, the planner is intergenerationally Pareto and strongly non-dictatorial;*

2. *for each $\delta < \min_i \delta_i$, there exists some $T^* > 0$ such that if $T \geq T^*$, the planner is not intergenerationally Pareto.*

This corollary shows that the two cutoffs of Theorem 1 are identical. Moreover, the second cutoff of Theorem 1 is tight, because $\min_i \lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)} = \min_i \delta_i$, and Corollary 1 shows that for any social discount factor above $\min_i \delta_i$, the planner must be intergenerationally Pareto and strongly non-dictatorial.

In Section S2 in the Supplemental Material, we reinterpret the generation-$(t+1)$ individual i as a future self of the generation-t individual i, which also offers a reinterpretation of intergenerational Pareto and allows us to discuss how our findings are related to the time-inconsistency literature. A stronger version of Corollary 1 can also be found.

Next, we present a result that is stronger than Theorem 1 under the assumption that all individuals have EDU functions.

Proposition 2 *Suppose $T < +\infty$, and each generation-t individual i has an EDU function with discount factor δ_i and instantaneous utility function u. Then, the planner is intergenerationally Pareto and strongly non-dictatorial if and only if $\delta > \min_i \delta_i$.*

This result is different from Theorem 1, because in Theorem 1, the second cutoff works under the assumption that T is sufficiently large. Proposition 2 does not require this. Similar to Corollary 1, Proposition 2 has only one cutoff for the social discount factor.[14]

This proposition also shows that the first cutoff of Theorem 1 is tight. To see this, note that $\min\limits_i \max\limits_{\tau\in\{0,\ldots,T-1\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)} = \min_i \delta_i$. From Proposition 2, we know that any social discount factor below $\min_i \delta_i$ implies that the planner is not intergenerationally Pareto.

Because individuals have exponential discount functions and public consumption, Proposition 2 can be directly compared to Jackson and Yariv (2014, 2015). Assuming that individuals have EDU functions, Proposition 2 shows that under intergenerational Pareto, the planner can simultaneouly have an exponential discounting expected utility function and be strongly non-dictatorial.

In Jackson and Yariv (2014, 2015), adding more current-generation exponential discounting individuals to the aggregation cannot help eliminate the negative result. Compared to

[14]Imagine that we write a corollary of Theorem 1 under the additional assumption that all individuals have EDU functions, rather than the stronger proposition. The two cutoffs of this corollary will also be identical.

Jackson and Yariv, we add future-generation exponential discounting individuals to the aggregation, and this helps. To see why, first recall that when $u_i = u$, Jackson and Yariv (2014) show that utilitarian aggregation of the current generation leads to a social discount function that exhibits present bias. The fact that future generations will not care about past consumption as much as past generations did helps us remove the present bias.

In our model, past consumption does not enter future generations' utility functions; that is, $\delta_i(\tau) = 0$ for any $\tau < 0$.[15] This implies that, for example, generation-t individual i's relative discount factor applied to period-t consumption (relative to period-$(t-1)$ consumption) is equal to "$\delta_i(0)/\delta_i(-1) = +\infty$." Thus, generation-$t$ is "infinitely patient" between period $t-1$ and period t. The infinite patience can be used in the aggregation to offset the present bias generated by aggregating the current generation alone.

4.3 Individual Long-Run Discount Factors

In the two special cases above, the two cutoffs from Theorem 1 merge into one. This is not a coincidence. Let us introduce the following assumption:

$$\text{the relative discount function } \frac{\delta_i(\tau+1)}{\delta_i(\tau)} \text{ is increasing in } \tau. \tag{6}$$

In the time-inconsistency literature, when an individual has an increasing relative discount function, the individual has *present bias*.

Now, since $\frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ is increasing and bounded, we know that $\lim_{\tau\to\infty} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ exists, and is always above $\max_{\tau\in\{0,\ldots,T-1\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ for any finite T. Moreover, it can be shown that if $\lim_{\tau\to\infty} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ exists, the average discount factor has a limit, and the asymptotic relative discount factor and the asymptotic average discount factor coincide,

$$\lim_{\tau\to\infty} \frac{\delta_i(\tau+1)}{\delta_i(\tau)} = \lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)}.$$

[15]In Section S3 in the Supplemental Material, we show that when individuals use positive exponential discount factors to backward discount past consumption, our results continue to hold.

Therefore, assumptions (5) and (6) imply (4).

Definition 4 *When* $\lim_{\tau\to\infty}\frac{\delta_i(\tau+1)}{\delta_i(\tau)}$ *exists, we call* $\delta_i^* := \lim_{\tau\to\infty}\frac{\delta_i(\tau+1)}{\delta_i(\tau)} = \lim_{\tau\to\infty}\sqrt[\tau]{\delta_i(\tau)}$ *individual* i*'s long-run discount factor.*

We immediately have the following corollary.

Corollary 2 *Suppose* $T < +\infty$*, and each generation-*t *individual* i*'s discounting utility function has an instantaneous utility function* u *and a discount function* δ_i *such that (5) and (6) hold. Then,*

1. *for each* $\delta > \min_i \delta_i^*$*, the planner is intergenerationally Pareto and strongly non-dictatorial;*

2. *for each* $\delta < \min_i \delta_i^*$*, there exists some* $T^* > 0$ *such that if* $T \geq T^*$*, the planner is not intergenerationally Pareto.*

From here on, to simplify the statement of our results, we focus on the case in which each individual i's long-run discount factor δ_i^* is well defined.

5 Social Discounting and Individual Instantaneous Utility Functions

Corollary 2 shows that if all individuals share the same instantaneous utility functions, the social discount factor only has to be higher than the lowest individual long-run discount factor. The assumption that all individuals share the same instantaneous utility function is clearly unreasonable. As long as $|X| \geq N$ (i.e., the number of deterministic consumption goods is higher than the number of individuals in each generation), generically, the instantaneous utility functions should not only be different, but also linearly independent.

Definition 5 *An* N*-tuple of continuous expected utility functions* $(u_i)_{i\in N}$ *is linearly independent if there are no constants* $\alpha_1, \ldots, \alpha_N$ *that are not all zero, and* $\sum_{i\in N}\alpha_i u_i(p) = 0$ *for all* $p \in \Delta(X)$*.*

It turns out that when individual instantaneous utility functions are linearly independent, the cutoff for the social discount factor jumps from $\min_i \delta_i^*$ to $\max_i \delta_i^*$; that is, generically, social discounting must be more patient than the *most* patient individual's long-run discounting. If the social discount factor is lower than the highest individual long-run discount factor and if the time horizon is long enough, there are two consumption sequences such that all individuals from all generations prefer one to the other, but the planner disagrees.

Theorem 2 *Suppose $T < +\infty$, and each generation-t individual i's discounting utility function has an instantaneous utility function u_i and a discount function δ_i such that (5) and (6) hold and $(u_i)_{i \in N}$ is linearly independent. Let the planner's instantaneous utility function u be an arbitrary strict convex combination of $(u_i)_{i \in N}$.*[16] *Then,*

1. *for each $\delta > \max_i \delta_i^*$, the planner is intergenerationally Pareto and strongly non-dictatorial;*
2. *for each $\delta < \max_i \delta_i^*$, there exists some $T^* > 0$ such that if $T \geq T^*$, the planner is not intergenerationally Pareto.*

To understand why we assume that the planner's instantaneous utility function is a strict convex combination of individual instantaneous utility functions, note that Lemma 1 implies that the intergenerationally Pareto and strongly non-dictatorial planner's utility function is equal to a weighted sum of individual utility functions with positive weights. Thus, the planner's instantaneous utility function must also be a weighted sum of individual instantaneous utility functions. Since instantaneous utility functions are normalized, the weights must sum up to 1.

Notice that the planner's instantaneous utility function—in other words, her risk attitude—is independent of the cutoff for the social discount factor. This is somewhat surprising. Suppose there are two individuals, 1 and 2, and individual 2 is more patient. The above theorem says that even if the social discount factor is close to individual 2's discount factor,

[16] By a strict convex combination of $(u_i)_{i \in N}$, we mean that u is in the interior of the convex hull of $u_1, \ldots, u_N$.

it is not necessarily the case that the planner's risk attitude is also close to individual 2's risk attitude. We can have a planner whose risk attitude is close to individual 1's, but the social discount factor is close to individual 2's.

If there are many individuals with a wide range of long-run discount factors, this result may imply that the planner must be very patient in order to be intergenerationally Pareto. If so, perhaps the near-zero social discount rate used by Stern (2007) can be justified. If one thinks that a market rate is higher than the lowest individual discount rate, this result also rules out the use of a market rate as the social discount rate.

The theorem also shows that the cutoff for the social discount factor in Theorem 1 is not robust. When $u_i = u_j$ for all $i, j \in N$, the cutoff is $\min_i \delta_i^*$. If we introduce a small perturbation to u_i's, then generically the cutoff jumps discontinuously to $\max_i \delta_i^*$.

Theorem 2 assumes (5) and (6); that is, individual relative discount functions are increasing and bounded. If we replace (6) with (4), as in Theorem 1, the only change in the statement of Theorem 2 will be that instead of one cutoff, we will have two cutoffs similar to Theorem 1.

To prove the first part of this theorem, again, we show that there are strictly positive weights for each individual i and his offspring such that the weighted sum of their utility functions is an EDU function with any new discount factor that is higher than that individual's maximal relative discount factor. Let the new discount factor be equal to the social discount factor $\delta > \max_i \delta_i^*$. Without loss of generality, assume that every individual i has an EDU function with discount factor δ. The EDU functions only differ in u_i's, and hence can be aggregated easily. The intuition behind the second part of this theorem will be discussed following Proposition 3.

Similar to Theorem 1, the second part of Theorem 2 requires that the time horizon be long enough. In Proposition 2, we show that when individual discount functions are exponential, the second part of Theorem 1 will become independent of T. The same holds when $(u_i)_{i \in N}$ is linearly independent, as shown in the proposition below.

Proposition 3 *Suppose $T < +\infty$, and each generation-t individual i has an EDU function* *with discount factor* δ_i *and instantaneous utility function* u_i *such that* $(u_i)_{i\in N}$ *is linearly independent. Then, the planner is intergenerationally Pareto and strongly non-dictatorial if and only if* $\delta > \max_i \delta_i$.

The if part follows from the first part of Theorem 2. We explain the proof of the only-if part of Proposition 3 below, which will also explain the idea behind the second part of Theorem 2. Note that when $(u_i)_{i\in N}$ is linearly independent and u is in the interior of $\text{co}(\{u_i\}_{i\in N})$, there is a unique way to write u as a strict convex combination of $(u_i)_{i\in N}$.[17] Suppose $\sum_{i\in N}\lambda_i u_i = u$, in which $\lambda_i > 0$ (because the planner is strongly non-dictatorial) and $\sum_{i\in N}\lambda_i = 1$. Focus on the first period. The planner's utility function is

$$U_1(\mathbf{p}) = \sum_{t=1}^{T}\sum_{i=1}^{N}\omega(i,t)U_{i,t}(\mathbf{p}) = \sum_{t=1}^{T}\sum_{i=1}^{N}\omega(i,t)\sum_{\tau=t}^{T}\delta_i^{\tau-t}u_i(p_\tau),$$

in which $\omega(i,t) > 0$ is the weight the planner assigns to the generation-t individual i. Clearly, the planner's instantaneous utility function for period-1 consumption is

$$u(p_1) = \sum_{i=1}^{N}\omega(i,1)u_i(p_1) \tag{7}$$

for any p_1. Since u can be written as a unique strict convex combination of $(u_i)_{i\in N}$, it must be the case that

$$\omega(i,1) = \lambda_i \tag{8}$$

for any $i \in N$. Similarly, the planner's instantaneous utility function for period-2 consumption satisfies

$$\delta u(p_2) = \sum_{i=1}^{N}\omega(i,1)\delta_i u_i(p_2) + \sum_{i=1}^{N}\omega(i,2)u_i(p_2) \tag{9}$$

for any p_2. Then, equations (7), (8), and (9), together with the strongly non-dictatorial property, imply that

$$\lambda_i\delta = \lambda_i\delta_i + \omega(i,2) \Rightarrow \delta > \delta_i$$

[17] We use $\text{co}(\cdot)$ to denote the convex hull.

for any $i \in N$, which means $\delta > \max_i \delta_i$. We omit the proof of Proposition 3.

5.1 Gradual Transition of the Cutoff

Let us further illustrate how the cutoff changes "discountinuously" from the least patient individual's long-run discount factor to the most patient individual's. An individual's instantaneous utility function describes his risk attitude. Let Θ be some positive integer. Suppose there is a linearly independent Θ-tuple of instantaneous utility functions $(u^\theta)_{\theta=1}^{\Theta}$ that represent Θ generic types of risk attitude. Assume that individual i's instantaneous utility function $u_i \in \{u^\theta\}_{\theta=1}^{\Theta}$, and for each type u^θ, at least one individual's instantaneous utility function is equal to u^θ. If $\Theta = 1$, we are in the case of Theorem 1. When $\Theta = N$, we are in the case of Theorem 2. Define $\delta_\theta^* := \min_{k \in \{i \in N : u_i = u^\theta\}} \delta_k^*$; that is, for each θ, let δ_θ^* be the least patient individual's long-run discount factor whose type is u^θ. Define

$$\delta_{\text{maxmin}}^* := \max_\theta \delta_\theta^*.$$

Theorem 3 *Suppose $T < +\infty$, and for some linearly independent Θ-tuple of instantaneous utility functions $(u^\theta)_{\theta=1}^{\Theta}$ such that $\{u_i\}_{i \in N} = \{u^\theta\}_{\theta=1}^{\Theta}$, each generation-t individual i's discounting utility function has an instantaneous utility function $u_i \in \{u^\theta\}_{\theta=1}^{\Theta}$ and a discount function δ_i such that (5) and (6) hold. Let the planner's instantaneous utility function u be an arbitrary strict convex combination of $(u_i)_{i \in N}$. Then,*

1. *for each $\delta > \delta_{maxmin}^*$, the planner is intergenerationally Pareto and strongly non-dictatorial;*
2. *for each $\delta < \delta_{maxmin}^*$, there exists some $T^* > 0$ and some such that if $T \geq T^*$, the planner is not intergenerationally Pareto.*

Intuitively, for each type of risk attitude u^θ, we can apply Theorem 1 to show that the cutoff for the social discount factor implied by aggregating type-u^θ individuals is δ_θ^*. When aggregating across types, we apply Theorem 2 to show that the maximal δ_θ^* is the cutoff for the social discount factor.

6 Social Discounting and the Time Horizon

In many economic models (and perhaps in reality), the time horizon is infinite. In this section, we show that the finding in our main Theorem 2 with linearly independent $(u_i)_{i\in N}$ continues to hold when $T = +\infty$. In Section A.9 in the Appendix, we present a related result that does not assume that $(u_i)_{i\in N}$ is linearly independent, and show that even when u_i's are identical, the cutoff for the social discount factor will jump from $\min_i \delta_i^*$ to $\max_i \delta_i^*$ when $T = +\infty$.

One of the main challenges in extending our main result to the infinite-horizon case is to establish the equivalence between intergenerational Pareto and (intergenerational) utilitarianism. The lemma below establishes the equivalence under the setting of our main theorem (Theorem 2).

Lemma 2 *Suppose $T = +\infty$, and each generation-t individual i's discounting utility function has an instantaneous utility function u_i such that $(u_i)_{i\in N}$ is linearly independent. The planner's preference $(\succsim_t)_{t\in T}$ is intergenerationally Pareto if and only if in each period $t \in T$, there exists a sequence of nonnegative numbers $(\omega_t(i,s))_{i\in N, s\geq t}$ such that $0 < \sum_{i=1}^{N}\sum_{s=t}^{T}\omega_t(i,s) < \infty$, and*

$$U_t = \sum_{i=1}^{N}\sum_{s=t}^{T}\omega_t(i,s)U_{i,s}.$$

Note that the lemma above assumes that $(u_i)_{i\in N}$ is linearly independent.[18] This assumption holds generically, is consistent with the assumption in our main theorem (Theorem 2), and holds if we assume that $(u_i)_{i\in N}$ satisfies the "independent prospects condition," which is often imposed in the literature.[19]

If the set of individuals is compact, we can apply a result in Zhou (1997) to establish the equivalence between intergenerational Pareto and (intergenerational) utilitarianism without

[18]If discount functions and instantaneous utility functions can depend on time as in Lemma 1, we may need equicontinuity assumptions on the set of individual discount functions and instantaneous utility functions.

[19]See Fishburn (1984), Weymark (1994), and Börgers and Choo (2017). In Fishburn's proof, there are two cases to be analyzed. One is under the independent prospects condition, but Fishburn has not given the condition a name.

assuming linearly independent $(u_i)_{i\in N}$. When $T = +\infty$, we have countably infinitely many generations, and hence the set of individuals is not compact.

Theorem 4 *Suppose $T = +\infty$, and each generation-t individual i's discounting utility function has an instantaneous utility function u_i and a discount function δ_i such that (5) and (6) hold and $(u_i)_{i\in N}$ is linearly independent. Let the planner's instantaneous utility function u be an arbitrary strict convex combination of $(u_i)_{i\in N}$. Then,*

1. *for each $\max_i \delta_i^* < \delta < 1$, the planner is intergenerationally Pareto and strongly non-dictatorial;*
2. *for each $\delta < \max_i \delta_i^*$, the planner is not intergenerationally Pareto.*

An additional step in the first part of the result is to show that the weights the planner uses to aggregate individual utility functions are absolutely summable. The second step is similar to Theorem 2.

7 Conclusion

The value of a policy or a public project that affects generations of individuals often crucially depends on which social discount rate is used for the evaluation. However, there is no consensus on which social discount rate is the right one to use. This paper considers a few important and widely used assumptions in economics, and characterizes the set of social discount rates that are compatible with those assumptions. The key assumptions are (i) individuals discount future consumption in a general and heterogeneous way, (ii) the planner has an exponential discounting expected utility function, (iii) the planner takes into account every individual's preference from every generation strictly, and (iv) the planner is intergenerationally Pareto, which means that if all individuals from all generations agree that one consumption sequence is better than another, the planner must agree.

We show that for a generic set of individual instantaneous utility functions, the social discount factor should be higher than the highest individual long-run discount factor, as long as the time horizon is long enough. Therefore, using a near-zero social discount rate is justifiable.

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A Appendix

A.1 Proof of Proposition 1

Proof. If Part If there exists a unique $i \in N$ such that $U_t = U_{i,t}$ for any $t \in T$, the planner takes only individual i into account in period t. The corresponding weights in period t are $\omega_i = 1$, and $\omega_j = 0$ for all $j \neq i$. According to Lemma 1, the planner's preference $(\succsim_t)_{t \in T}$ is current-generation Pareto.

Only-If Part Suppose the planner's preference $(\succsim_t)_{t \in T}$ is current-generation Pareto. Then, according to Lemma 1, there exists an N-tuple of nonnegative weights $(\omega_i)_{i \in N}$, such that

$$\sum_{i=1}^{N} \omega_i \sum_{\tau=1}^{T} \delta_i^{\tau-1} u_i(p_\tau) = \sum_{\tau=1}^{T} \delta^{\tau-1} u(p_\tau);$$

that is, for $\tau = 1, \ldots, T-1$,

$$\sum_{i=1}^{N} \omega_i \delta_i^{\tau-1} u_i(p_\tau) = \delta^{\tau-1} u(p_\tau).$$

Let $\tau = 1$, 2, and 3. We have

$$\begin{cases} \sum_{i=1}^{N} \omega_i u_i(p) = u(p), \\ \sum_{i=1}^{N} \omega_i \delta_i u_i(p) = \delta u(p), \\ \sum_{i=1}^{N} \omega_i \delta_i^2 u_i(p) = \delta^2 u(p), \end{cases}$$

for any $p \in \Delta(X)$. Let $p = x^*$. The first equation shows that $\sum_{i \in N} \omega_i = 1$. Combining the second and the third equations above,

$$\left(\sum_{i=1}^{N} \omega_i \delta_i \right)^2 = \sum_{i=1}^{N} \omega_i \delta^2. \tag{10}$$

Since $\sum_{i \in N} \omega_i = 1$, by Jensen's inequality, equation (10) holds if and only if δ_i's are identical or there exists one $i \in N$ such that $\omega_i = 1$. With a generic N-tuple of discount factors $(\delta_i)_{i \in N}$, $\delta_i \neq \delta_j$ for any $i \neq j$. Therefore, there exists a unique $i \in N$ such that $\omega_i = 1$, and $\omega_j = 0$ for any $j \neq i$, which means that $U_t = U_{i,t}$. ■

A.2 Proof of Proposition 2

Proof. The following lemma will be useful in proving Proposition 2.

Lemma 3 *Given a positive N-tuple $(\delta_i)_{i\in N}$, if $\delta > \min_i \delta_i$, there exists a finite sequence of strictly positive numbers $(\omega_t(i,s))_{t\in T, i\in N, s\geq t}$ such that the following equation holds*

$$\sum_{i=1}^{N}\sum_{s=t}^{\tau}\omega_t(i,s)\delta_i^{\tau-s} = \delta^{\tau-t} \tag{11}$$

for any $t \in T$ and $\tau \geq t$.

Proof. Without loss of generality, we assume that $\delta_1 = \min_i \delta_i$. First, we fix all the weights other than individual 1's. Let $\omega_t(i,s) = \epsilon_t(s) > 0$ for any $i \geq 2$, $t \geq 1$, and $s \geq t$. The remaining part is to find $(\omega_t(1,s))_{t\in T, s\geq t}$ such that

1. equation (11) holds;

2. $\omega_t(1,s) > 0$, for any $t \geq 1$ and $s \geq t$.

Construct $(\omega_t(1,s))_{t\in T, s\geq t}$ by the following recursive formula:

$$\omega_t(1,s) = \begin{cases} 1 - \sum\limits_{i=2}^{N}\omega_t(i,s), & \text{if } s = t, \\ \delta^{s-t} - \sum\limits_{i=1}^{N}\omega_t(i,t)\delta_i^{s-t} - \cdots - \sum\limits_{i=1}^{N}\omega_t(i,s-1)\delta_i - \sum\limits_{i=2}^{N}\omega_t(i,s), & \text{if } s > t. \end{cases} \tag{12}$$

It can be verified that (12) ensures that equation (11) holds for any $t \in T$ and $\tau \geq t$. The remaining part is to show that $(\omega_{1,t}(s))_{t\in T, s\geq t}$ derived from (12) are strictly greater than zero, if $(\epsilon_t(s))_{t\in T, s\geq t}$ are small enough. We prove it in two steps.

Step 1 Setting $\epsilon_t(s) = 0$, the recursive formula (12) becomes

$$\omega_t(1,s) = \begin{cases} 1, & \text{if } s = t, \\ \delta^{s-t-1}(\delta - \delta_1), & \text{if } s > t, \end{cases}$$

for each $t \in T$. This can be proved by induction. Since $\delta > \delta_1$, we have $\omega_t(1,s) > 0$.

Step 2 Plugging $\epsilon_t(s)$ into formula (12), we have,

$$\begin{cases} \omega_t(1,t) = 1 - (N-1)\epsilon_t(t), \\ \omega_t(1,t+1) = \delta - \delta_1 - \left[\sum\limits_{i=2}^{N}(\delta_i - \delta_1)\right]\epsilon_t(t) - (N-1)\epsilon_t(t+1), \\ \omega_t(1,t+2) = \delta(\delta - \delta_1) - \left[\sum\limits_{i=2}^{N}\delta_i(\delta_i - \delta_1)\right]\epsilon_t(t) - \left[\sum\limits_{i=2}^{N}(\delta_i - \delta_1)\right]\epsilon_t(t+1) - (N-1)\epsilon_t(t+2), \\ \vdots \end{cases}$$

Then, we know that $\omega_t(1,s) = F_t^{(s)}(\epsilon_t(t), \ldots, \epsilon_t(s)|\delta, \delta_1, \ldots, \delta_n)$, in which $F_t^{(s)}$ is an affine (and hence continuous) function of $\epsilon_t(t), \ldots, \epsilon_t(s)$. By continuity of $F_t^{(s)}$, the weights $\omega_t(1,s)$'s are strictly greater than zero, if $\epsilon_t(s)$'s are small enough. ■

Now we are able to prove Proposition 2.

If Part Since the planner's instantaneous utility function u is identical to individual instantaneous utility function u, the if part follows from Lemma 3 immediately.

Only-If Part Suppose the planner's preference is intergenerationally Pareto and strongly non-dictatorial. For each $t \in T$, there exists a finite sequence of strictly positive numbers $(\omega_t(i,s))_{i \in N, s \geq t}$ such that

$$\begin{aligned} U_t(\mathbf{p}) &= \sum_{s=t}^{T}\sum_{i=1}^{N} \omega_t(i,s) U_{i,s}(\mathbf{p}) = \sum_{s=t}^{T}\sum_{i=1}^{N} \omega_t(i,s) \sum_{\tau=s}^{T} \delta_i^{\tau-s} u(p_\tau) \\ &= \sum_{\tau=t}^{T}\sum_{s=t}^{\tau}\sum_{i=1}^{N} \omega_t(i,s)\delta_i^{\tau-s} u(p_\tau). \end{aligned}$$

Then, for $\forall t$, $\forall \tau \geq t$, the following equality holds,

$$\sum_{s=t}^{\tau}\sum_{i=1}^{N} \omega_t(i,s)\delta_i^{\tau-s} u(p_\tau) = \delta^{\tau-t} u(p_\tau). \tag{13}$$

Let $\tau = t, t+1$ in (13). We have

$$\begin{cases} \sum_{i=1}^{N} \omega_t(i,t) u(p_t) = u(p_t), \\ \sum_{i=1}^{N} \omega_t(i,t)\delta_i u(p_{t+1}) + \sum_{i=1}^{N} \omega_t(i,t+1) u(p_{t+1}) = \delta u(p_{t+1}). \end{cases}$$

Combining the above two equations,

$$\sum_{i=1}^{N} \omega_t(i,t)\delta = \sum_{i=1}^{N} \omega_t(i,t)\delta_i + \sum_{i=1}^{N} \omega_t(i,t+1).$$

Rearranging the above equation, we have

$$\delta = \frac{\sum_{i=1}^{N} \omega_t(i,t)\delta_i + \sum_{i=1}^{N} \omega_t(i,t+1)}{\sum_{i=1}^{N} \omega_t(i,t)} > \frac{\sum_{i=1}^{N} \omega_t(i,t)\delta_i}{\sum_{i=1}^{N} \omega_t(i,t)}$$

$$> \frac{\sum_{i=1}^{N} \omega_t(i,t) \min_i \delta_i}{\sum_{i=1}^{N} \omega_t(i,t)} = \min_{i \in N} \delta_i$$

■

A.3 Proof of Theorem 1

Proof. Part I We prove Part I in two steps. First, we prove a lemma for the one-individual case. Then, we apply Proposition 2 to complete the proof.

Lemma 4 *Assume that $N = \{i\}$. Suppose $T < +\infty$, and each generation-t individual i's discounting utility function has an instantaneous utility function u and a discount function $\delta_i(\tau)$ such that (4) and (5) hold. For any $\delta > \hat{\delta}_i := \max_{\tau \in \{0,\ldots,T-1\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$, the planner is intergenerationally Pareto and strongly non-dictatorial.*

Proof. We want to show that for any $\delta > \hat{\delta}_i$ and each $t \in T$, there exists a finite sequence of strictly positive numbers $(\omega_t(i,s))_{t \in T, s \geq t}$ such that

$$U_t(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u(p_\tau) = \sum_{s=t}^{T} \omega_t(i,s) U_{i,s}(\mathbf{p}).$$

Given any $\delta > \hat{\delta}_i$, for each $t \in T$, we can construct $(\omega_t(i,s))_{s \geq t}$ according to the following formula:

$$\omega_t(i,s) = \begin{cases} 1, & \text{if } s = t, \\ \delta^{s-t-1}\left(\delta - \hat{\delta}_i\right) + \sum_{\tau=t}^{s-1} \left[\hat{\delta}_i \delta_i(s-1-\tau) - \delta_i(s-\tau)\right] \omega_t(i,\tau), & \text{if } s > t. \end{cases} \tag{14}$$

Note that by assuming $\delta > \hat{\delta}_i$, for $s > t$, the first term of $\omega_t(i,s)$ is strictly greater than 0. According to the definition of $\hat{\delta}_i$, the second term of $\omega_t(i,s)$ is greater than 0. Hence, $\omega_t(i,s) > 0$ for any $s \geq t$. Then,

$$U_t(\mathbf{p}) = \sum_{s=t}^{T} \omega_t(i,s) U_{i,s}(\mathbf{p}) = \sum_{s=t}^{T} \omega_t(i,s) \left[\sum_{\tau=s}^{T} \delta_i(\tau - s) u(p_\tau)\right] = \sum_{\tau=t}^{T} \left[\sum_{s=t}^{\tau} \delta_i(\tau - s)\omega_t(i,s)\right] u(p_\tau).$$

We want to prove that $U_t(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u(p_\tau)$ by induction. Consider $\sum_{s=t}^{\tau} \delta_i(\tau - s)\omega_t(i,s)$. When $\tau = t$, $\sum_{s=t}^{\tau} \delta_i(\tau - s)\omega_t(i,s) = \omega_t(i,t) = 1 = \delta^0$. Suppose for some $\tau \geq t$, we have proven that $\sum_{s=t}^{\tau} \delta_i(\tau - s)\omega_t(i,s) = \delta^{\tau-t}$. We want to prove that for $\tau + 1$,

$$\sum_{s=t}^{\tau+1} \delta_i(\tau + 1 - s)\omega_t(i,s) = \delta^{\tau-t+1}. \tag{15}$$

To prove (15), we only need to notice that according to (14),

$$\begin{aligned}
&\sum_{s=t}^{\tau+1} \delta_i(\tau + 1 - s)\omega_t(i,s) = \omega_t(i,\tau+1) + \sum_{s=t}^{\tau} \delta_i(\tau + 1 - s)\omega_t(i,s) \\
= \quad & \omega_t(i,\tau+1) + \hat{\delta}_i \left[\delta^{\tau-t} + \sum_{s=t}^{\tau} \frac{\delta_i(\tau+1-s)}{\hat{\delta}_i}\omega_t(i,s) - \delta^{\tau-t}\right] \\
= \quad & \omega_t(i,\tau+1) + \hat{\delta}_i \left[\delta^{\tau-t} + \sum_{s=t}^{\tau} \frac{\delta_i(\tau+1-s)}{\hat{\delta}_i}\omega_t(i,s) - \sum_{s=t}^{\tau} \delta_i(\tau - s)\omega_t(i,s)\right] \\
= \quad & \omega_t(i,\tau+1) + \hat{\delta}_i\delta^{\tau-t} + \sum_{s=t}^{\tau} \left[\delta_i(\tau+1-s) - \hat{\delta}_i\delta_i(\tau - s)\right]\omega_t(i,s) = \delta^{\tau-t+1}.
\end{aligned}$$

By induction, we know that $\sum_{s=t}^{\tau} \delta_i(\tau - s)\omega_t(i,s) = \delta^{\tau-t}$ for any $\tau \geq t$. Now, we know that $U_t(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u_i(p_\tau)$. ■

Lemma 4 states that in each period t, the planner can aggregate individual i's utility functions from the t^{th} generation to the T^{th} generation to derive an EDU function with any discount factor greater than $\hat{\delta}_i$. Then, by Proposition 2, in each period t, the planner can aggregate N exponential discounting individuals from the t^{th} generation to the T^{th} generation one more time, and obtain an EDU function with any social discount factor greater than $\min_i \hat{\delta}_i$.

Part II Define $\tilde{\delta}_i := \lim_{\tau\to\infty} \sqrt[\tau]{\delta_i(\tau)}$. Without loss of generality, we assume that $\tilde{\delta}_1$ is the unique minimum of $\tilde{\delta}_1, \ldots, \tilde{\delta}_N$. The proof can easily be extended to the case with multiple minima. We prove it by contradiction. Suppose the planner is intergenerationally Pareto. For each $t \in T$, there exists a finite sequence of nonnegative numbers $(\omega_t(i,s))_{i\in N, s\geq t}$ such that the following equality holds:

$$\sum_{s=t}^{\tau} \sum_{i=1}^{N} \omega_t(i,s)\delta_i(\tau - s)u(p_\tau) = \delta^{\tau-t}u(p_\tau) \tag{16}$$

for any $t \in T$ and $\tau \geq t$.

By letting $\tau = t$, equation (16) shows that $\sum_{i \in N} \omega_t(i,t) = 1$ for any $t \in T$. Then,

$$\delta^{\tau-t} = \frac{\sum_{s=t}^{\tau}\sum_{i=1}^{N} \omega_t(i,s)\delta_i(\tau - s)}{\sum_{i=1}^{N} \omega_t(i,t)} \geq \frac{\sum_{i=1}^{N} \omega_t(i,t)\delta_i(\tau - t)}{\sum_{i=1}^{N} \omega_t(i,t)}. \tag{17}$$

Since $\tilde{\delta}_1 = \min_i \tilde{\delta}_i$, there exists $T_1 > 0$ such that for $\forall \tau > T_1$, $\delta_1(\tau - t) = \min_i \delta_i(\tau - t)$. Hence, (17) becomes

$$\delta^{\tau-t} \geq \frac{\sum_{i=1}^{N} \omega_{i,t}(t)\delta_1(\tau - t)}{\sum_{i=1}^{N} \omega_{i,t}(t)} = \delta_1(\tau - t). \tag{18}$$

According to our assumptions, $\delta < \tilde{\delta}_1$. Then, there exists $T_2 > 0$ such that for $\forall \tau > T_2$,

$$\delta^{\tau-t} < \delta_1(\tau - t). \tag{19}$$

Let $T^* = \max\{T_1, T_2\}$. Then, (18) and (19) contradict each other. ■

A.4 Proof of Theorem 2

Proof. Part I We prove this theorem in two steps. First, again we consider the special case in which there is only one individual i to be aggregated across generations. Since the individual relative discount factor is increasing, $\delta_i^* \geq \hat{\delta}_i := \max_{\tau \in \{0,\ldots,T-1\}} \frac{\delta_i(\tau+1)}{\delta_i(\tau)}$. By Lemma 4, because the social discount factor $\delta > \max_i \delta_i^* \geq \delta_i^*$, for any $i \in N$ and $t \in T$, we can find some positive $(\omega_t(i,s))_{s \geq t}$ such that

$$\sum_{s=t}^{T} \omega_t(i,s) U_{i,s}(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u_i(p_\tau);$$

that is, we can aggregate each individual's utility functions across generations into an EDU function with discount factor δ.

Consider any N-tuple of strictly positive numbers $(\lambda_i)_{i \in N}$ such that $\sum_{i \in N} \lambda_i = 1$. Together with the weights $(\omega_t(i,s))_{i \in N, s \geq t}$ we have found above, let the planner's utility function satisfy

$$U_t(\mathbf{p}) = \sum_{i=1}^{N}\sum_{s=t}^{T}\lambda_i\omega_t(i,s)U_{i,s}(\mathbf{p}) = \sum_{i=1}^{N}\sum_{\tau=t}^{T}\delta^{\tau-t}\lambda_i u_i(p_\tau)$$
$$= \sum_{\tau=t}^{T}\delta^{\tau-t}\sum_{i=1}^{N}\lambda_i u_i(p_\tau) = \sum_{\tau=t}^{T}\delta^{\tau-t}u(p_\tau),$$

in which $u = \sum_{i\in N}\lambda_i u_i$ is an arbitrary strict convex combination of $(u_i)_{i\in N}$.

Part II We prove it by contradiction. Suppose there exists an intergenerationally Pareto planner with the social discount factor $\delta < \max_i \delta_i^*$. By intergenerational Pareto, for each $t \in T$, there exists nonnegative numbers $(\omega_t(i,s))_{i\in N, s\geq t}$ such that the following equality holds for each $t \in T$:

$$\sum_{\tau=t}^{T}\delta^{\tau-t}u(p_\tau) = \sum_{\tau=t}^{T}\sum_{i=1}^{N}\sum_{s=t}^{\tau}\omega_t(i,s)\delta_i(\tau-s)u_i(p_\tau).$$

Since p_τ's are arbitrary, the equation above implies that

$$\begin{cases} \sum_{i=1}^{N}\omega_0(i,0)u_i(p_0) = u(p_0), \\ \sum_{s=0}^{\tau}\sum_{i=1}^{N}\omega_0(i,s)\delta_i(\tau-s)u_i(p_\tau) = \delta^\tau u(p_\tau). \end{cases} \tag{20}$$

Recall that u is a strict convex combination of $(u_i)_{i\in N}$ and $(u_i)_{i\in N}$ is linearly independent. There is a unique way to write u as a convex combination of $(u_i)_{i\in N}$. Thus, the first equation of (20) implies that $\omega_0(i,0) > 0$ for each i. Combining the two equations of (20), we have

$$\sum_{i=1}^{N}\omega_0(i,0)\delta^\tau u_i = \sum_{i=1}^{N}\sum_{s=0}^{\tau}\omega_0(i,s)\delta_i(\tau-s)u_i.$$

Since $(u_i)_{i\in N}$ is linearly independent, the above equation is equivalent to

$$\omega_0(i,0)\delta^\tau u_i = \sum_{s=0}^{\tau}\omega_0(i,s)\delta_i(\tau-s)u_i.$$

for $\forall i \in N$.

Rearrange the above equation. We have

$$\delta^\tau = \frac{\sum_{s=0}^{\tau}\omega_0(i,s)\delta_i(\tau-s)}{\omega_0(i,0)} = \frac{\omega_0(i,0)\delta_i(\tau) + \sum_{s=1}^{\tau}\omega_0(i,s)\delta_i(\tau-s)}{\omega_0(i,0)} \geq \frac{\omega_0(i,0)\delta_i(\tau)}{\omega_0(i,0)} = \delta_i(\tau)$$

for $\forall i \in N$. Hence, for any $i \in N$ and $\tau \leq T-1$,

$$\delta \geq \sqrt[\tau]{\delta_i(\tau)}. \tag{21}$$

Without loss of generality, we assume δ_N^* is a maximum of $\{\delta_i^*\}_{i\in N}$. Since $\delta < \delta_N^* = \lim_{\tau\to\infty} \sqrt[\tau]{\delta_N(\tau)}$, there exists T^* such that for any $\tau \geq T^*$, $\delta < \sqrt[\tau]{\delta_N(\tau)}$, which contradicts (21). ■

A.5 Proof of Theorem 3

Proof. Part I We prove Part I in two steps. First, we aggregate individuals who share the same u^θ. For each $\theta \in \Theta$, $I^\theta := \{i \in N : u_i = u^\theta\}$ is called a "family," which is the set of i's whose instantaneous utility functions are u^θ. By Corollary 2, we know that for each θ and each $\delta > \min_{i\in I_\theta} \delta_i^*$, there exists a sequence of weights $(\omega_t(i,s))_{t\in T, i\in I_\theta, s\geq t}$ such that

$$U_t^\theta(\mathbf{p}) = \sum_{\tau=t}^{T} \delta^{\tau-t} u^\theta(p_\tau) = \sum_{s=t}^{T} \sum_{i\in I_\theta} \omega_t(i,s) U_{i,s}(\mathbf{p}).$$

for each $t \in T$. Now, we have $|\Theta|$ exponential discounting expected utility functions U_t^θ's with linearly independent instantaneous utility functions u^θ's.

Next, we apply Proposition 3 to aggregate U_t^θ's. It follows immediately that if $\delta > \max_{\theta\in\Theta} \min_{i\in I_\theta} \delta_i^*$, the planner is intergenerationally Pareto and strongly non-dictatorial.

Part II We prove its contrapositive. Suppose there exists an intergenerationally Pareto planner with the social discount factor $\delta < \delta^*_{\text{maxmin}}$. By intergenerational Pareto, for each $t \in T$, there exists a finite sequence of positive numbers $(\omega_t(i,s))_{i\in N, s\geq t}$ such that the following equality holds:

$$\delta^{\tau-t} u(p_\tau) = \sum_{\theta\in\Theta} \sum_{i\in I_\theta} \sum_{s=t}^{\tau} \omega_t(i,s) \delta_i(\tau-s) u^\theta(p_\tau). \tag{22}$$

for each $t \in T$ and $\tau \geq t$.

By letting $\tau = t$ in equation (22), we have

$$u(p_t) = \sum_{\theta\in\Theta} \sum_{i\in I_\theta} \omega_t(i,t) u^\theta(p_t). \tag{23}$$

Recall that u is a strict convex combination of $(u_i)_{i\in N}$. Equation (23) shows that $\sum_{\theta\in I_\theta} \omega_t(i,t) > 0$ for each θ. Combining equations (22) and (23), we have

$$\sum_{\theta\in\Theta} \sum_{i\in I_\theta} \delta^{\tau-t} \omega_t(i,t) u^\theta(p_\tau) = \sum_{\theta\in\Theta} \sum_{i\in I_\theta} \sum_{s=t}^{\tau} \omega_t(i,s) \delta_i(\tau-s) u^\theta(p_\tau).$$

Since $(u^\theta)_{i=1}^{\Theta}$ is linearly independent, the above equation is equivalent to

$$\sum_{i\in I_\theta}\delta^{\tau-t}\omega_t(i,t)=\sum_{i\in I_\theta}\sum_{s=t}^{\tau}\omega_t(i,s)\delta_i(\tau-s)$$

for $\forall\theta\in\Theta$. Rearranging the above equation, we obtain

$$\delta^{\tau-t}=\frac{\sum_{i\in I_\theta}\sum_{s=t}^{\tau}\omega_t(i,s)\delta_i(\tau-s)}{\sum_{i\in I_\theta}\omega_t(i,t)}>\frac{\sum_{i\in I_\theta}\omega_t(i,t)\delta_i(\tau-t)}{\sum_{i\in I_\theta}\omega_t(i,t)}. \tag{24}$$

Letting τ go to infinity, it is easy to see that (24) becomes $\delta\geq\min_{i\in I_\theta}\delta_i^*$ for $\forall\theta\in\Theta$. Hence, $\delta\geq\max_{\theta\in\Theta}\min_{i\in I_\theta}\delta_i^*=\delta^*_{\text{maxmin}}$. ■

A.6 Preliminaries of Lemma 2

The proof of Lemma 2 uses a generalization of Farkas' lemma for dual pairs due to Craven and Koliha (1977). To state the generalized Farkas' lemma, we first introduce some definitions. A dual pair is 3-tuple (A,A',ϕ) consisting of two vector spaces A and A' and a function $\phi:A\times A'\to\mathbb{R}$ such that (i) ϕ is bilinear, (ii) if $\phi(a,a')=0$ for any $a\in A$, then $a'=0$, and (iii) if $\phi(a,a')=0$ for any $a'\in A$, then $a=0$. Properties (ii) and (iii) are called the separation properties. The weak topology of A is characterized by the following: A sequence $(a_n)_{n=1}^{\infty}$ of A converges to $a\in A$ if and only if $\phi(a_n,a')$ converges to $\phi(a,a')$ for any $a'\in A'$. The weak topology of A' is similarly defined. A nonempty subset $S\subset A$ is a convex cone if $\alpha a+\beta b\in S$ for any $\alpha,\beta\geq 0$ and $a,b\in S$. We use S' to denote the anticone of the convex cone, in which $S':=\{a'\in A':\phi(a,a')\geq 0 \text{ for any } a\in S\}$.

Suppose (A,A',ϕ) and (B,B',φ) are dual pairs and $\psi:A\to B$ is a continuous linear map. Then, $\psi':B'\to A'$ is the topological adjoint of ψ if

$$\phi(a,\psi'(b'))=\varphi(\psi(a),b')$$

for any $a\in A$ and $b'\in B'$. We state Craven and Koliha's Theorem 2 below.

Theorem 5 *(Craven and Koliha (1977)) Let (A,A',ϕ) and (B,B',φ) be dual pairs, let S be a convex cone in A, and let $\psi:A\to B$ be a continuous linear map. If $\psi(S)$ is closed in weak topology and $b\in B$, the following statements are equivalent:*

1. *The equation $\psi(a)=b$ has a solution $a\in S$.*

2. *$\psi'(b')\in S'\Rightarrow\varphi(b,b')\geq 0$.*

A.7 Proof of Lemma 2

Proof. The if part is straightforward to verify.

Only-If Part Suppose the generation-t individual i's utility function is $U_{i,t}(\mathbf{p}) = \sum_{\tau=t}^{\infty} \delta_i(\tau - t) u_i(p_\tau)$, the planner's utility function is $U_t(\mathbf{p}) = \sum_{\tau=t}^{\infty} \delta_t(\tau - t) u_t(p_\tau, \tau)$, and intergenerational Pareto holds. To apply Theorem 5, let $A = \ell^1$, $A' = \ell^\infty$, $B = C_b(X^\infty)$, and $B' = ca(X^\infty)$, in which ℓ^1 is the set of absolutely summable sequences, ℓ^∞ is the set of bounded sequences, $C_b(X^\infty)$ is the set of continuous and bounded functions on X^∞, and $ca(X^\infty)$ is the set of countably additive signed measures on X^∞. Note that since X is compact, X^∞ is also compact in the product topology. The norm of A' and B is the sup norm, and the norm of B' is the total variation. By defining

$$\phi(a, a') = \sum_{n=1}^{\infty} a_n a'_n$$

and

$$\varphi(b, b') = \int_{X^\infty} b \, db'$$

for any $a \in A$, $a' \in A'$, $b \in B$, and $b' \in B'$, (A, A', ϕ) and (B, B', φ) are dual pairs (p. 211 of Aliprantis and Border (2006)).

For any sequence $\vec{\omega}_t = (\omega_t(1,t), \ldots, \omega_t(N,t), \omega_t(1,t+1), \ldots, \omega_t(N,t+1), \ldots) \in \ell^1$, define a function $\psi : A \to B$ such that

$$\psi(\vec{\omega}_t)(\mathbf{x}) = \sum_{s=t}^{\infty} \sum_{i=1}^{N} \omega_t(i,s) U_{i,s}(\mathbf{x}) \tag{25}$$

for any $\mathbf{x} = (x_1, x_2, \ldots) \in X^\infty$. In the main text, $U_{i,s}$ is defined on $\Delta(X)^\infty$ when $T = \infty$. Here, we restrict attention to degenerate lotteries of $\Delta(X)^\infty$. We claim that $(U_{i,s}(\mathbf{x}))_{i \in N, s \geq t} \in \ell^\infty$, and hence $\psi(\vec{\omega}_t)(\mathbf{x})$ is well defined. We prove this claim when we later prove that ψ is a continuous function.

We want to verify that ψ is continuous and maps from A to B. Then, let $S = \{a \in A : a \geq 0\}$. We want to verify that $\psi(S)$ is closed in weak topology. After verifying them, we apply Theorem 5.

Step 1 First, we show that $\psi(\vec{\omega}_t)$ is a continuous function on X^∞. The product topology of X^∞ is metrizable. For any $\mathbf{x}, \mathbf{y} \in X^\infty$,

$$\pi(\mathbf{x}, \mathbf{y}) := \sup_{\tau} \left\{ \frac{\min\{\|x_\tau - y_\tau\|, 1\}}{\tau} \right\}$$

induces the product topology on X^∞, in which $X \subset \mathbb{R}^m$ and $\|\cdot\|$ is the Euclidean metric of

$\mathbb{R}^m$ (p. 125 of Munkres (2000)). Intuitively, when $\mathbf{x}$ and $\mathbf{y}$ are close, x_τ and y_τ are close when τ is small, but x_τ and y_τ can be far apart when τ is large. We need to show that when γ is small enough, if $\pi(\mathbf{x},\mathbf{y}) < \gamma$, then $|\psi(\vec{\omega}_t)(\mathbf{x}) - \psi(\vec{\omega}_t)(\mathbf{y})|$ is small enough. Without loss of generality, let $\gamma < 1$. Then, $\pi(\mathbf{x},\mathbf{y}) < \gamma$ implies that $\|x_\tau - y_\tau\| < \tau\gamma$ for $\tau \leq 1/\gamma$. For $\tau > 1/\gamma$, x_τ and y_τ can be far apart.

Pick some $\varepsilon^{(1)}, \varepsilon^{(2)}, \varepsilon^{(3)} > 0$ that can be arbitrarily small. Since $\vec{\omega}_t \in \ell^1$, $(\delta_i(\tau))_{\tau=0}^\infty \in \ell^1$, and N is finite, there exists some $\varsigma^{(1)}, \varsigma^{(2)}$ such that $\sum_{s=\varsigma^{(1)}}^\infty \sum_{i=1}^N \omega_t(i,s) < \varepsilon^{(1)}$ and $\sum_{\tau=\varsigma^{(2)}}^\infty \delta_i(\tau - s) < \varepsilon^{(2)}$ for each $i \in N$. Now, let γ be chosen so that

$$\|x_\varsigma - y_\varsigma\| < \max\{\varsigma^{(1)}, \varsigma^{(2)}\} \cdot \gamma < \varepsilon^{(3)}. \tag{26}$$

Note that $\varsigma^{(1)}$ depends on $\varepsilon^{(1)}$, but not on $\varepsilon^{(2)}$ and $\varepsilon^{(3)}$. Similarly, $\varsigma^{(2)}$ depends on $\varepsilon^{(2)}$, but not on $\varepsilon^{(1)}$ and $\varepsilon^{(3)}$. Then,

$$\begin{aligned}
|\psi(\vec{\omega}_t)(\mathbf{x}) - \psi(\vec{\omega}_t)(\mathbf{y})| &= \left| \sum_{s=t}^\infty \sum_{i=1}^N \omega_t(i,s) U_{i,s}(\mathbf{x}) - \sum_{s=t}^\infty \sum_{i=1}^N \omega_t(i,s) U_{i,s}(\mathbf{y}) \right| \\
&\leq \sum_{s=t}^\infty \sum_{i=1}^N \omega_t(i,s) \cdot |U_{i,s}(\mathbf{x}) - U_{i,s}(\mathbf{y})| \\
&\leq \sum_{s=t}^\infty \sum_{i=1}^N \omega_t(i,s) \left[\sum_{\tau=s}^\infty \delta_i(\tau - s) \cdot |u_i(x_\tau) - u_i(y_\tau)| \right] \\
&\leq \sum_{s=t}^{\varsigma^{(1)}} \sum_{i=1}^N \omega_t(i,s) \left[\sum_{\tau=s}^\infty \delta_i(\tau - s) \cdot |u_i(x_\tau) - u_i(y_\tau)| \right] \\
&\quad + \sum_{s=\varsigma^{(1)}}^\infty \sum_{i=1}^N \omega_t(i,s) \left[\sum_{\tau=s}^\infty \delta_i(\tau - s) \cdot |u_i(x_\tau) - u_i(y_\tau)| \right].
\end{aligned}$$

Since u_i is continuous on a compact set, u_i is uniformly continuous and u_i is bounded by $\frac{1}{2}\upsilon > 0$ for any $i \in N$ for some $\upsilon > 0$. Thus,

$$\sum_{s=\varsigma^{(1)}}^\infty \sum_{i=1}^N \omega_t(i,s) \left[\sum_{\tau=s}^\infty \delta_i(\tau - s) \cdot |u_i(x_\tau) - u_i(y_\tau)| \right] < \upsilon \varepsilon^{(1)} \cdot \left(\sum_{\tau=0}^\infty \delta_i(\tau) \right).$$

Then,

$$\begin{aligned}
&\sum_{s=t}^{\varsigma^{(1)}} \sum_{i=1}^{N} \omega_t(i,s) \left[\sum_{\tau=s}^{\infty} \delta_i(\tau - s) \cdot |u_i(x_\tau) - u_i(y_\tau)| \right] \\
= \quad & \sum_{s=t}^{\varsigma^{(1)}} \sum_{i=1}^{N} \omega_t(i,s) \left[\begin{array}{c} \sum_{\tau=s}^{\varsigma^{(2)}} \delta_i(\tau - s) \cdot |u_i(x_\tau) - u_i(y_\tau)| \\ + \sum_{\tau=\varsigma^{(2)}}^{\infty} \delta_i(\tau - s) \cdot |u_i(x_\tau) - u_i(y_\tau)| \end{array} \right].
\end{aligned}$$

We know that

$$\sum_{\tau=\varsigma^{(2)}}^{\infty} \delta_i(\tau - s) \cdot |u_i(x_\tau) - u_i(y_\tau)| < \varepsilon^{(2)} \upsilon$$

and hence

$$\sum_{s=t}^{\varsigma^{(1)}} \sum_{i=1}^{N} \omega_t(i,s) \left[\sum_{\tau=\varsigma}^{\infty} \delta_i(\tau - s) \cdot |u_i(x_\tau) - u_i(y_\tau)| \right] < \varepsilon^{(2)} \upsilon \sum_{s=t}^{\varsigma^{(1)}} \sum_{i=1}^{N} \omega_t(i,s).$$

Since u_i is uniformly continuous on X, if $\|x_\tau - y_\tau\| < \varepsilon^{(3)}$, $|u_i(x_\tau) - u_i(y_\tau)| < \epsilon^{(3)}$, in which $\epsilon^{(3)}$ can be arbitrarily small as $\varepsilon^{(3)}$ gets arbitrarily small. Then,

$$\begin{aligned}
&\sum_{s=t}^{\varsigma^{(1)}} \sum_{i=1}^{N} \omega_t(i,s) \left[\sum_{\tau=s}^{\varsigma^{(2)}} \delta_i(\tau - s) \cdot |u_i(x_\tau) - u_i(y_\tau)| \right] \\
\leq \quad & \epsilon^{(3)} \cdot \sum_{s=t}^{\varsigma^{(1)}} \sum_{i=1}^{N} \omega_t(i,s) \cdot \sum_{\tau=s}^{\varsigma^{(2)}} \delta_i(\tau - s).
\end{aligned}$$

Thus,

$$\begin{aligned}
|\psi(\vec{\omega}_t)(\mathbf{x}) - \psi(\vec{\omega}_t)(\mathbf{y})| \quad < \quad & \upsilon \varepsilon^{(1)} \cdot \left(\sum_{\tau=0}^{\infty} \delta_i(\tau) \right) + \varepsilon^{(2)} \upsilon \sum_{s=t}^{\varsigma^{(1)}} \sum_{i=1}^{N} \omega_t(i,s) \\
& + \epsilon^{(3)} \cdot \sum_{s=t}^{\varsigma^{(1)}} \sum_{i=1}^{N} \omega_t(i,s) \cdot \sum_{\tau=s}^{\varsigma^{(2)}} \delta_i(\tau - s).
\end{aligned} \tag{27}$$

We first pick a small $\varepsilon^{(1)}$, which guarantees that the first term of the right-hand side of (27) is arbitrarily small. By picking $\varepsilon^{(1)}$, $\varsigma^{(1)}$ is determined. Then, we choose a small $\varepsilon^{(2)}$ to cause the second term of the right-hand side of (27) to be small. This step does not affect the first term of the right-hand side of (27). By choosing a small $\varepsilon^{(3)}$, we ensure that the last term of the right-hand side of (27) is also arbitrarily small. This step does not affect the first two terms of the right-hand side of (27), and γ is pinned down by $\varepsilon^{(1)}$, $\varepsilon^{(2)}$, and $\varepsilon^{(3)}$ via equation (26). Thus, we know that $\psi(\vec{\omega}_t) : X^\infty \to \mathbb{R}$ is a continuous function.

Step 2 Next, we show that $\psi : A \to B$ is continuous. Let $\psi^{\#} : B^{\#} \to A^{\#}$ be the algebraic dual of ψ such that $\phi(a, \psi^{\#}(b^{\#})) = \varphi(\psi(a), b^{\#})$ for any $a \in A$ and $b^{\#} \in B^{\#}$, in which $A^{\#}$ and $B^{\#}$ are the algebraic duals of A and B, respectively, and ϕ and φ are similarly defined for $A, A^{\#}$ and $B, B^{\#}$, respectively. It is known that ψ' is identical to the restriction of $\psi^{\#}$ to B', and that to show that $\psi : A \to B$ is continuous, it suffices to show that $\psi^{\#}(B') \subset A'$, that is, $\psi^{\#}(ca(X^\infty)) \subset \ell^\infty$. For any $\mu \in B'$ and $\vec{\omega}_t \in A$,

$$\begin{aligned}
\phi(\vec{\omega}_t, \psi^{\#}(\mu)) &= \varphi(\psi(\vec{\omega}_t), \mu) \\
&= \int_{X^\infty} \psi(\vec{\omega}_t)\ d\mu \\
&= \int_{X^\infty} \left(\sum_{s=t}^{\infty} \sum_{i=1}^{N} \omega_t(i,s) U_{i,s}(\mathbf{x}) \right)\ d\mu \\
&= \sum_{s=t}^{\infty} \sum_{i=1}^{N} \omega_t(i,s) \left[\int_{X^\infty} U_{i,s}(\mathbf{x})\ d\mu \right]
\end{aligned}$$

The last equality is by the Fubini–Tonelli theorem. By applying the theorem again,

$$\begin{aligned}
\int_{X^\infty} U_{i,s}(\mathbf{x})\ d\mu &= \int_{X^\infty} \sum_{\tau=s}^{\infty} \delta_i(\tau - s) u_i(x_\tau)\ d\mu \qquad (28) \\
&= \sum_{\tau=s}^{\infty} \left[\delta_i(\tau - s) \int_{X^\infty} u_i(x_\tau)\ d\mu \right].
\end{aligned}$$

To understand $\int_{X^\infty} u_i(x_\tau)\ d\mu$, think of $u_i(x_\tau)$ as a function defined on X^∞ that only depends on the τ^{th} component of $\mathbf{x}$, x_τ. Then, $\int_{X^\infty} u_i(x_\tau)\ d\mu = \int_X u_i(x_\tau)\ dp_\tau$, in which p_τ is μ's marginal distribution on x_τ. Since u_i is continuous on a compact set, $\int_{X^\infty} u_i(x_\tau)\ d\mu$ is bounded above by $\max_{x \in X} u_i(x)$ and below by $\min_{x \in X} u_i(x)$. Therefore, $\int_{X^\infty} U_{i,s}(\mathbf{x})\ d\mu$ is bounded above by

$$\sum_{\tau=s}^{\infty} \delta_i(\tau - s) \max_{x \in X} u_i(x) = \max_{x \in X} u_i(x) \sum_{\tau=0}^{\infty} \delta_i(\tau)$$

and below by

$$\sum_{\tau=s}^{\infty} \delta_i(\tau - s) \min_{x \in X} u_i(x) = \min_{x \in X} u_i(x) \sum_{\tau=0}^{\infty} \delta_i(\tau),$$

because $(\delta_i(\tau))_{\tau=0}^{\infty} \in \ell^1$. Both bounds only depend on i. Therefore,

$$\phi(\vec{\omega}_t, \psi^{\#}(\mu)) = \phi(\vec{\omega}_t, a'),$$

in which

$$a' = \begin{pmatrix} \int_{X^\infty} U_{1,t}(\mathbf{x})\ d\mu,\ \int_{X^\infty} U_{2,t}(\mathbf{x})\ d\mu,\ \ldots,\ \int_{X^\infty} U_{N,t}(\mathbf{x})\ d\mu, \\ \int_{X^\infty} U_{1,t+1}(\mathbf{x})\ d\mu,\ \int_{X^\infty} U_{2,t+1}(\mathbf{x})\ d\mu,\ \ldots,\ \int_{X^\infty} U_{N,t+1}(\mathbf{x})\ d\mu, \\ \cdots \end{pmatrix} \in \ell^\infty.$$

Hence, ψ is continuous. Note that the two bounds above also show that $(U_{i,s}(\mathbf{x}))_{i\in N, s\geq t} \in \ell^\infty$ for each $\mathbf{x} \in X^\infty$, which proves our claim that $\psi(\vec{\omega}_t)(\mathbf{x})$ is well defined.

Step 3 We show that $\psi(S)$ is closed in weak topology (induced by $ca(X^\infty)$). Since S is convex and ψ is linear, $\psi(S) \subset C_b(X)$ is convex. When X^∞ is compact, the topological dual of $C_b(X^\infty)$ is $ca(X^\infty)$. It is known that a convex set of a normed space ($C_b(X^\infty)$ with the sup norm) is closed in the norm topology if and only if it is closed in the weak topology induced by the topological dual. Therefore, we only need to show that $\psi(S)$ is closed in the norm topology. Take a sequence $(f^{(n)})_{n=1}^\infty$ of $\psi(S)$ such that $f^{(n)} \in \psi(S)$ converges to $f \in C_b(X^\infty)$ in sup norm. Convergence in sup norm implies pointwise convergence; that is, for any $\mathbf{x} \in X^\infty$, $f^{(n)}(\mathbf{x})$ converges to $f(\mathbf{x})$. Since $f^{(n)}$'s are functions on a compact set X^∞, by the Arzelà–Ascoli theorem, sup norm convergence implies that $(f^{(n)})_{n=1}^\infty$ is equicontinuous.

Below, we want to show that $f \in \psi(S)$; that is, there exists some $\vec{\omega}_t \in S$ such that $f = \psi(\vec{\omega}_t)$. Since $f^{(n)} \in \psi(S)$, there exists an $\vec{\omega}_t^{(n)} \in S$ such that

$$f^{(n)} = \sum_{s=t}^{\infty} \sum_{i=1}^{N} \omega_t^{(n)}(i,s) U_{i,s}.$$

Step 3-1 We first show that such an $\vec{\omega}_t^{(n)} \in S$ is unique. Suppose there exists another $\vec{\varpi}_t^{(n)} \in S$ such that

$$f^{(n)} = \sum_{s=t}^{\infty} \sum_{i=1}^{N} \varpi_t^{(n)}(i,s) U_{i,s}.$$

Suppose the smallest $s \geq t$ such that $\vec{\omega}_t^{(n)}$ and $\vec{\varpi}_t^{(n)}$ differ is $\hat{s}$. Consider the following element of X^∞,

$$\mathbf{x}^{(y,\hat{s})} = (\underbrace{x_*, \ldots, x_*}_{(\hat{s}-1)\text{ times}}, y, x_*, x_*, \ldots),$$

for any $y \in \Delta(X)$. Note that $U_{i,s}(\mathbf{x})$ depends on x_τ only if $\tau \geq s$. Because $u_i(x_*) = 0$,

$$\begin{aligned} f^{(n)}(\mathbf{x}^{(y,\hat{s})}) &= \sum_{s=t}^{\hat{s}} \sum_{i=1}^{N} \omega_t^{(n)}(i,s) U_{i,s}(\mathbf{x}^{(y,\hat{s})}) \\ &= \sum_{s=t}^{\hat{s}} \sum_{i=1}^{N} \varpi_t^{(n)}(i,s) U_{i,s}(\mathbf{x}^{(y,\hat{s})}). \end{aligned}$$

Since $\vec{\omega}_t^{(n)}(i,s)$ and $\vec{\varpi}_t^{(n)}(i,s)$ coincide for any $s < \hat{s}$, we know that

$$\begin{aligned}\sum_{i=1}^{N} \omega_t^{(n)}(i,\hat{s}) U_{i,\hat{s}}(\mathbf{x}^{(y,\hat{s})}) &= \sum_{i=1}^{N} \varpi_t^{(n)}(i,\hat{s}) U_{i,\hat{s}}(\mathbf{x}^{(y,\hat{s})}) \\ \sum_{i=1}^{N} \omega_t^{(n)}(i,\hat{s}) u_i(y) &= \sum_{i=1}^{N} \varpi_t^{(n)}(i,\hat{s}) u_i(y)\end{aligned}$$

for any $y \in \Delta(X)$. Since $(u_i)_{i\in N}$ is linearly independent, the equality above holds for any $y \in \Delta(X)$ if and only if $\omega_t^{(n)}(i,\hat{s}) = \varpi_t^{(n)}(i,\hat{s})$ for every $i \in N$, which is a contradiction.

Therefore, we have a sequence $\left(\vec{\omega}_t^{(n)}\right)_{n=1}^{\infty}$ of S such that $\psi\left(\vec{\omega}_t^{(n)}\right) = f^{(n)}$. We want to find some $\vec{\omega}_t$ in ℓ^1 such that $\psi(\vec{\omega}_t) = f$.

Step 3-2 We construct each $\vec{\omega}_t^{(n)}$ from $f^{(n)}$. First, we claim that there exists an N-tuple $(y_i)_{i=1}^N$ in X such that $\text{rank}(B) = N$, in which B is an $N \times N$ matrix

$$\begin{pmatrix} u_1(y_1) & u_2(y_1) & \dots & u_N(y_1) \\ u_1(y_2) & u_2(y_2) & \dots & u_N(y_2) \\ \vdots & \vdots & \ddots & \vdots \\ u_1(y_N) & u_2(y_N) & \dots & u_N(y_N) \end{pmatrix}.$$

Note that B does not depend on time or n.

To find y_1, let $y_1 = x^*$. Let B_1 denote a $1 \times N$ matrix that consists of the first row of B with $y_1 = x^*$. Clearly, B_1 has rank 1. Suppose we have found $y_1, \dots, y_{k-1} \in X$ such that B_{k-1} is a $(k-1) \times N$ matrix that consists of the first $(k-1)$ rows of B and has rank $1 \leq k-1 < N$. We claim that we can find $y_k \in X$ such that B_k is a $k \times N$ matrix that consists of the first k rows of B and has rank k. Suppose not; that is, for any $y_k \in X$, $\text{rank}(B_k) = \text{rank}(B_{k-1}) = k-1$. This implies that for any $y_k \in X$, there exists $\lambda_1(y_k), \dots, \lambda_{k-1}(y_k) \in \mathbb{R}$ such that

$$(u_1(y_k), \dots, u_N(y_k)) = \sum_{j=1}^{k-1} \lambda_j(y_k) \cdot (u_1(y_j), \dots, u_N(y_j));$$

that is, the k^{th} row of B_k can be written as a linear combination of the first $(k-1)$ rows of B_k.

Consider the homogeneous system of linear equations, $\sum_{i=1}^N \tilde{\lambda}_i u_i(y_j) = 0$, $j = 1, \dots, k-1$. Since there are N unknown variables ($\tilde{\lambda}_i$'s) but only $k-1$ equations and $k-1 < N$, the system always has some nontrivial solution $(\tilde{\lambda}_i)_{i=1}^N$. Therefore,

$$\begin{aligned}\sum_{i=1}^N \tilde{\lambda}_i u_i(y_k) &= \sum_{i=1}^N \left[\tilde{\lambda}_i \sum_{j=1}^{k-1} \lambda_j(y_k) u_i(y_j)\right] \\ &= \sum_{j=1}^{k-1} \lambda_j(y_k) \sum_{i=1}^N \tilde{\lambda}_i u_i(y_j) = 0\end{aligned}$$

for any $y_k \in X$, which contradicts the assumption that $(u_i)_{i \in N}$ is linearly independent. Therefore, we can find an N-tuple $(y_i)_{i=1}^N$ in X such that the $N \times N$ matrix B has full rank.

Step 3-3 We first construct each $\vec{\omega}_t^{(n)}$ from $f^{(n)}$ via matrix B constructed above. Note that

$$\begin{aligned}f^{(n)}(\mathbf{x}^{(y_k,t)}) &= \sum_{i=1}^N \sum_{\tau=t}^{\infty} \omega_t^{(n)}(i,\tau) U_{i,s}(\mathbf{x}^{(y_k,t)}) \\ &= \sum_{i=1}^N \omega_t^{(n)}(i,t) u_i(y_k),\end{aligned}$$

in which $\mathbf{x}^{(y_k,s)} = (\underbrace{x_*, \ldots, x_*}_{(s-1)\text{ times}}, y_k, x_*, x_*, \ldots)$ as defined previously, and y_k's are the elements of X that we find when constructing B. Therefore,

$$\begin{pmatrix} u_1(y_1) & \ldots & u_N(y_1) \\ \vdots & \ddots & \vdots \\ u_1(y_N) & \ldots & u_N(y_N) \end{pmatrix} \begin{pmatrix} \omega_t^{(n)}(1,t) \\ \vdots \\ \omega_t^{(n)}(N,t) \end{pmatrix} = \begin{pmatrix} f^{(n)}(\mathbf{x}^{(y_1,t)}) \\ \vdots \\ f^{(n)}(\mathbf{x}^{(y_N,t)}) \end{pmatrix},$$

in which the first matrix of the left-hand side is B. Hence,

$$\begin{pmatrix} \omega_t^{(n)}(1,t) \\ \vdots \\ \omega_t^{(n)}(N,t) \end{pmatrix} = B^{-1} \begin{pmatrix} f^{(n)}(\mathbf{x}^{(y_1,t)}) \\ \vdots \\ f^{(n)}(\mathbf{x}^{(y_N,t)}) \end{pmatrix}. \tag{29}$$

Since $f^{(n)}$ converges, by letting n go to infinity, we define $\omega_t(i,t)$ as the limit of $\omega_t^{(n)}(i,t)$. From (29), we know that $\omega_t^{(n)}(i,t)$ is a linear combination of $(f^{(n)}(\mathbf{x}^{(y_k,t)}))_{k \in N}$ that takes the following form:

$$\omega_t^{(n)}(i,t) = \sum_{k=1}^N \sum_{\tau=t}^t \zeta_t^{(i,t)}(k,\tau) \cdot f^{(n)}(\mathbf{x}^{(y_k,\tau)}).$$

It is important to note that for any $i \in N$, $\left(\zeta_t^{(i,t)}(k,\tau)\right)_{k \in N, \tau=t}$ is independent of n. The reason there is a redundant summation (τ from t to t) will become clear once we move on to $\omega_t^{(n)}(i,s)$.

Next, because

$$\begin{aligned} f^{(n)}(\mathbf{x}^{(y_k,s)}) &= \sum_{i=1}^{N}\sum_{\tau=t}^{\infty}\omega_t^{(n)}(i,\tau)U_{i,s}(\mathbf{x}^{(y_k,s)}) \\ &= \sum_{i=1}^{N}\sum_{\tau=t}^{s}\omega_t^{(n)}(i,\tau)\delta_i(s-\tau)u_i(y_k). \end{aligned}$$

Therefore,

$$\begin{aligned} \begin{pmatrix} \omega_t^{(n)}(1,s) \\ \vdots \\ \omega_t^{(n)}(N,s) \end{pmatrix} &= B^{-1}\begin{pmatrix} f^{(n)}(\mathbf{x}^{(y_1,s)}) - \sum_{i\in N}\sum_{\tau=t}^{s-1}\omega_t^{(n)}(i,\tau)\delta_i(s-\tau)u_i(y_1) \\ \vdots \\ f^{(n)}(\mathbf{x}^{(y_N,s)}) - \sum_{i\in N}\sum_{\tau=t}^{s-1}\omega_t^{(n)}(i,\tau)\delta_i(s-\tau)u_i(y_N) \end{pmatrix} \\ &= B^{-1}\begin{pmatrix} f^{(n)}(\mathbf{x}^{(y_1,s)}) \\ \vdots \\ f^{(n)}(\mathbf{x}^{(y_N,s)}) \end{pmatrix} - B^{-1}\begin{pmatrix} \sum_{i\in N}\sum_{\tau=t}^{s-1}\omega_t^{(n)}(i,\tau)\delta_i(s-\tau)u_i(y_1) \\ \vdots \\ \sum_{i\in N}\sum_{\tau=t}^{s-1}\omega_t^{(n)}(i,\tau)\delta_i(s-\tau)u_i(y_N) \end{pmatrix} \end{aligned}$$

Again, since $f^{(n)}$ converges, by letting n go to infinity, we define $\omega_t(i,s)$ as the limit of $\omega_t^{(n)}(i,s)$. Recursively, we also know that $\omega_t^{(n)}(i,s)$ is a linear combination of $(f^{(n)}(\mathbf{x}^{(y_k,\tau)}))_{k\in N, t\le\tau\le s}$ that takes the following form:

$$\omega_t^{(n)}(i,s) = \sum_{k=1}^{N}\sum_{\tau=t}^{s}\zeta_t^{(i,s)}(k,\tau)\cdot f^{(n)}(\mathbf{x}^{(y_k,\tau)}).$$

It is important to note that for any $i\in N$ and $s\ge t$, $\left(\zeta_t^{(i,s)}(k,\tau)\right)_{k\in N, t\le\tau\le s}$ is independent of n.

We have found the $\vec{\omega}_t$ such that $\omega_t(i,s) = \lim_{n\to\infty}\omega_t^{(n)}(i,s)$; that is, $\omega_t^{(n)}$ converges to ω_t "pointwisely." We can show that $\vec{\omega}_t = (\omega_t(i,s))_{i\in N, s\ge t} \in \ell^1$. Consider $x^* = (x^*, x^*, \dots) \in X^\infty$. Since $u_i(x^*) = 1$,

$$f^{(n)}(\mathbf{x}^*) = \sum_{s=t}^{\infty}\sum_{i=1}^{N}\omega_t^{(n)}(i,s)\cdot\sum_{\tau=0}^{\infty}\delta_i(\tau).$$

Since $f^{(n)}(\mathbf{x}^*)$ converges to $f(\mathbf{x}^*)$, we know that there exists some $\rho > 0$ such that $f^{(n)}(\mathbf{x}^*) \le$

ρ for any $n \in \mathbb{N}$. Because $\sum_{\tau=0}^{\infty} \delta_i(\tau) > 1$,

$$\sum_{s=t}^{\infty} \sum_{i=1}^{N} \omega_t^{(n)}(i,s) \leq \rho$$

for any $n \in \mathbb{N}$. Thus, for any fixed $n' \in \mathbb{N}$, $\sum_{s=t}^{n'} \sum_{i \in N} \omega_t^{(n)}(i,s) \leq \rho$. Let n go to infinity, we know that $\sum_{s=t}^{n'} \sum_{i \in N} \omega_t(i,s) \leq \rho$. Since $\sum_{s=t}^{n'} \sum_{i \in N} \omega_t(i,s) \leq \rho$ for any $n' \in \mathbb{N}$, we know that $\sum_{s=t}^{\infty} \sum_{i \in N} \omega_t(i,s) \leq \rho$, which means $\vec{\omega}_t \in \ell^1$.

Step 3-4 We want to show that $\vec{\omega}_t^{(n)}$ converges in ℓ^1. If we can show this, $\vec{\omega}_t^{(n)}$'s limit must be $\vec{\omega}_t$. Then, because ψ is continuous, $\psi\left(\vec{\omega}_t^{(n)}\right) = f^{(n)}$, and $f^{(n)}$ converges to f, we know that $\psi\left(\vec{\omega}_t\right) = f$, which completes the proof of Step 3. Because ℓ^1 is complete, to show that $\vec{\omega}_t^{(n)}$ converges in ℓ^1, we only need to show that $\left(\vec{\omega}_t^{(n)}\right)_{n=1}^{\infty}$ is a Cauchy sequence; that is, for any $\varepsilon > 0$, there exists some $\kappa > 0$ such that for any $n, \tilde{n} \geq \kappa$,

$$\sum_{i=1}^{N} \sum_{s=t}^{\infty} \left|\omega_t^{(n)}(i,s) - \omega_t^{(\tilde{n})}(i,s)\right| < \varepsilon. \tag{30}$$

Let $\mathbf{x}^{(*,s)} = (\underbrace{x^*, \ldots, x^*}_{s \text{ times}}, x_*, x_*, \ldots)$. Recall that for any $\mathbf{x}, \mathbf{y} \in X^\infty$, the metric of X^∞ is

$$\pi(\mathbf{x}, \mathbf{y}) := \sup_{\tau} \left\{ \frac{\min\{\|x_\tau - y_\tau\|, 1\}}{\tau} \right\}.$$

Therefore, when s is large, $\mathbf{x}^*$ and $\mathbf{x}^{(*,s)}$ are close.

Note that for any $\varepsilon' > 0$, there exists some $\kappa' > 0$ such that for any $\tilde{\kappa}' \geq \kappa'$,

$$|f^{(n')}(\mathbf{x}^*) - f^{(n')}(\mathbf{x}^{(*,\tilde{\kappa}')})| < \varepsilon'$$

for any n', because $(f^{(n)})_{n=1}^{\infty}$ is equicontinuous. Then,

$$\begin{aligned}
\varepsilon' &> |f^{(n')}(\mathbf{x}^*) - f^{(n')}(\mathbf{x}^{(*,\tilde{\kappa}')})| \\
&= \left| \sum_{i=1}^{N} \sum_{s=t}^{\infty} \omega_t^{(n')}(i,s) U(\mathbf{x}^*) - \sum_{i=1}^{N} \sum_{s=t}^{\infty} \omega_t^{(n')}(i,s) U(\mathbf{x}^{(*,\tilde{\kappa}')}) \right| \\
&= \left| \sum_{i=1}^{N} \sum_{s=t}^{\infty} \left[\omega_t^{(n')}(i,s) \sum_{\tau=0}^{\infty} \delta_i(\tau) \right] - \sum_{i=1}^{N} \sum_{s=t}^{\tilde{\kappa}'} \left[\omega_t^{(n')}(i,s) \sum_{\tau=0}^{\tilde{\kappa}'-s} \delta_i(\tau) \right] \right| \\
&\geq \sum_{i=1}^{N} \sum_{s=\tilde{\kappa}'}^{\infty} \omega_t^{(n')}(i,s).
\end{aligned}$$

This shows that for any $\varepsilon' > 0$, there exists $\kappa' > 0$ such that for any $\tilde{\kappa}' \geq \kappa'$ and any n',

$$\sum_{i=1}^{N}\sum_{s=\tilde{\kappa}'}^{\infty}\omega_t^{(n')}(i,s) < \varepsilon'. \tag{31}$$

Back to equation (30). Note that

$$\begin{aligned}\sum_{i=1}^{N}\sum_{s=t}^{\infty}\left|\omega_t^{(n)}(i,s)-\omega_t^{(\tilde{n})}(i,s)\right| \;\leq\; & \sum_{i=1}^{N}\sum_{s=t}^{\kappa'}\left|\omega_t^{(n)}(i,s)-\omega_t^{(\tilde{n})}(i,s)\right| \\ & +\sum_{i=1}^{N}\sum_{s=\kappa'}^{\infty}\omega_t^{(n)}(i,s)+\sum_{i=1}^{N}\sum_{s=\kappa'}^{\infty}\omega_t^{(\tilde{n})}(i,s).\end{aligned} \tag{32}$$

The second and third terms of the right-hand side are both less than ε' due to equation (31).

Since $f^{(n)}$ converges to f in sup norm, for any $\varepsilon'' > 0$, there exists κ'' such that if n and $\tilde{n}$ are greater than κ'', $|f^{(n)}(\mathbf{x}) - f^{(\tilde{n})}(\mathbf{x})| < \varepsilon''$ for any $\mathbf{x} \in X^\infty$. Now, recall that from Step 3-3, we know that

$$\omega_t^{(n)}(i,s) = \sum_{k=1}^{N}\sum_{\tau=t}^{s}\zeta_t^{(i,s)}(k,\tau)\cdot f^{(n)}(\mathbf{x}^{(y_k,\tau)}),$$

in which for any $i \in N$ and $s \geq t$, $\left(\zeta_t^{(i,s)}(k,\tau)\right)_{k\in N, t\leq\tau\leq s}$ is independent of n. Therefore, the first term of the right-hand side of (32) becomes

$$\begin{aligned} & \sum_{i=1}^{N}\sum_{s=t}^{\kappa'}\left|\sum_{k=1}^{N}\sum_{\tau=t}^{s}\zeta_t^{(i,s)}(k,\tau)\left[f^{(n)}(\mathbf{x}^{(y_k,\tau)})-f^{(\tilde{n})}(\mathbf{x}^{(y_k,\tau)})\right]\right| \\ \leq\; & \varepsilon''\sum_{i=1}^{N}\sum_{s=t}^{\kappa'}\sum_{k=1}^{N}\sum_{\tau=t}^{s}\left|\zeta_t^{(i,s)}(k,\tau)\right|,\end{aligned}$$

as long as n and $\tilde{n}$ are greater than κ''. The inequality above shows that the first term of the right-hand side of (32) can also be arbitrarily small.

Step 4 Finally, we want to show that the following equation

$$U_t = \sum_{s=t}^{\infty}\sum_{i=1}^{N}\omega_t(i,s)U_{i,s} = \psi(\vec{\omega}_t) \tag{33}$$

has a nonnegative solution; that is, there exists some $\vec{\omega}_t \in S$ that solves (33). If we can find such an $\vec{\omega}_t$, it must be the case that $\sum_{i=1}^{N}\sum_{s=t}^{\infty}\omega_t(i,s) > 0$, because of the normalization assumption on expected utility functions. Applying Theorem 5, we know that we can find a nonnegative solution $\vec{\omega}_t$ to (33) if and only if for any $\mu \in B'$,

$$\int_{X^\infty} U_{i,s} \; d\mu \geq 0 \tag{34}$$

for any $i \in N$ and $s \geq t$ implies $\int_{X^\infty} U_t \; d\mu \geq 0$.

To see this, first note that by the Hahn–Jordan decomposition theorem, μ can be uniquely decomposed into $\alpha\mu_+ - \beta\mu_-$ in which $\alpha, \beta \geq 0$ and μ_+ and μ_- are probability measures on X^∞. Thus, (34) becomes

$$\alpha \int_{X^\infty} U_{i,s} \; d\mu_+ \geq \beta \int_{X^\infty} U_{i,s} \; d\mu_-$$

for any $i \in N$ and $s \geq t$. Notice that $U_{i,s}$'s are time-additively separable. Again, as in equation (28), probability measures μ_+ and μ_- can be identified with $\mathbf{p} \in \Delta(X)^\infty$ and $\mathbf{q} \in \Delta(X)^\infty$, in which p_τ and q_τ are the marginal distributions of μ_+ and μ_- on x_τ, respectively. Hence, (34) becomes

$$\alpha U_{i,s}(\mathbf{p}) \geq \beta U_{i,s}(\mathbf{q})$$

for any $i \in N$ and $s \geq t$.

Suppose $\alpha \geq \beta$. The other case can be proved in a similar way. Let us use $\mathbf{x}_*$ to denote the sequence $(x_*, x_*, \dots)$. Since instantaneous utility functions are all normalized, $U_{i,s}(\mathbf{x}_*) = 0$ for any $i \in N$ and $s \geq t$. Then, (34) becomes

$$U_{i,s}(\mathbf{p}) \geq \frac{\beta}{\alpha} U_{i,s}(\mathbf{q}) + \left(1 - \frac{\beta}{\alpha}\right) U_{i,s}(\mathbf{x}_*)$$

for any $i \in N$ and $s \geq t$. Since $U_{i,s}$'s are time-additively separable, we know that for every $i \in N$ and $s \geq t$, the generation-s individual i prefers $\mathbf{p}$ to $\frac{\beta}{\alpha}\mathbf{q} + \left(1 - \frac{\beta}{\alpha}\right)\mathbf{x}_*$, in which $\frac{\beta}{\alpha}\mathbf{q} + \left(1 - \frac{\beta}{\alpha}\right)\mathbf{x}_* \in \Delta(X)^\infty$ is the period-by-period mixture between $\mathbf{q}$ and $\mathbf{x}_*$. By intergenerational Pareto, this means that

$$\begin{aligned} U_t(\mathbf{p}) &\geq \frac{\beta}{\alpha} U_t(\mathbf{q}) + \left(1 - \frac{\beta}{\alpha}\right) U_t(\mathbf{x}_*) \\ \alpha U_t(\mathbf{p}) &\geq \beta U_t(\mathbf{q}) \\ \int_{X^\infty} U_t \; d\mu &\geq 0. \end{aligned}$$

Therefore, we know that (33) has a nonnegative solution. ■

A.8 Proof of Theorem 4

Proof. Part I Since u is a strict convex combination of $(u_i)_{i\in N}$, suppose $u = \sum_i \lambda_i u_i$ for some $\lambda_1, \ldots, \lambda_N > 0$ such that $\sum_i \lambda_i = 1$. For each $i \in N$ and each $t \in T$, we want to construct a sequence of strictly positive and absolutely summable numbers $(\omega_t(i,s))_{s=t}^{\infty}$ such that

$$\sum_{s=t}^{\infty} \omega_t(i,s) U_{i,s}(\mathbf{p}) = \sum_{\tau=t}^{\infty} \delta^{\tau-t} u_i(p_\tau).$$

If this can be done, then in period t, let $\lambda_i \omega_t(i,s)$ be the planner's utilitarian weight for the generation-s individual i, in which case

$$\sum_{i=1}^{N} \sum_{s=t}^{\infty} \lambda_i \omega_t(i,s) U_{i,s}(\mathbf{p}) = \sum_{\tau=t}^{\infty} \delta^{\tau-t} u(p_\tau) = U_t(\mathbf{p}),$$

which means that the planner is intergerationally Pareto and strongly non-dictatorial.

Next, we show that the following recursive definition of $(\omega_t(i,s))_{s=t}^{\infty}$ works: For each $s \geq t$,

$$\omega_t(i,s) = \begin{cases} 1, & \text{if } s = t, \\ \sum_{\sigma=t}^{s-1} [\delta \cdot \delta_i(s-\sigma) - \delta_i(s-\sigma+1)] \omega_t(i,\sigma), & \text{if } s > t. \end{cases} \tag{35}$$

First, it can be verified that each $\omega_t(i,s)$ is strictly positive, because $\delta > \max_i \delta_i^*$ and the individual relative discount factor is increasing. Second, it can be verified inductively that for any finite τ,

$$\sum_{s=t}^{\tau} \sum_{i=1}^{N} \omega_t(i,s) \delta_i(\tau - s) u_i(p_\tau) = \delta^{\tau-t} u(p_\tau)$$

for any $p_\tau \in \Delta(X)$. These two steps are similar to the steps in the proof of Lemma 4. Thus, we only have to show that $(\omega_t(i,s))_{s=t}^{\infty}$ is summable. Clearly, $\sum_{s=t}^{n} \omega_t(i,s)$ is increasing in n. If we can show that $\sum_{s=t}^{n} \omega_t(i,s)$ is bounded above and the bound is a constant, this part of the theorem is proven.

Sum up both sides of equation (35) from $s = t$ to n. We can obtain that

$$1 = \sum_{s=t}^{n-1} \left((1-\delta) \sum_{\tau=0}^{n-1-s} \delta_i(\tau) + \delta_i(n-s) \right) \omega_t(i,s) + \omega_t(i,n).$$

Because $\sum_{\tau=0}^{n-1-s} \delta_i(\tau) > 1$ and $\delta_i(n-s) > 0$, $(1-\delta) \sum_{\tau=0}^{n-1-s} \delta_i(\tau) + \delta_i(n-s) > 1 - \delta$, which implies that

$$\begin{aligned} 1 &> \sum_{s=t}^{n-1}(1-\delta)\omega_t(i,s) + \omega_t(i,n) \\ &> (1-\delta)\sum_{s=t}^{n-1}\omega_t(i,s). \end{aligned}$$

Therefore, $\sum_{s=t}^{n-1}\omega_t(i,s)$ is bounded above by $1/(1-\delta)$ for any n.

Part II Without loss of generality, we assume that δ_N^* is the unique maximal of $\{\delta_i^*\}_{i\in N}$. The proof can easily be extended to the case with multiple maxima. We prove the contrapositive of this part. Suppose the planner is intergenerationally Pareto and strongly non-dictatorial. According to Lemma 1, for each $t \in T$, there exists a sequence of nonnegative numbers $(\omega_t(i,s))_{i\in N, s\geq t}$ such that $U_t = \sum_{i,s}\omega_t(i,s)U_{i,s}$. Hence, equality (16) holds; that is, for any t and $\tau \geq t$,

$$\sum_{s=t}^{\tau}\sum_{i=1}^{N}\omega_t(i,s)\delta_i(\tau-s)u_i(p_\tau) = \delta^{\tau-t}u(p_\tau).$$

Consider a consumption sequence that yields x^* in every period, $(x^*, x^*, \dots)$. Then, the equation above becomes

$$\sum_{s=t}^{\tau}\sum_{i=1}^{N}\omega_t(i,s)\delta_i(\tau-s) = \delta^{\tau-t}.$$

Since u_i's and u are normalized, we know that for each t, $\sum_{i\in N}\omega_t(i,t) = 1$. Due to the strongly non-dictatorial property, in particular, $\omega_t(N,t) \in (0,1)$. Then,

$$\begin{aligned} \delta^{\tau-t} &= \sum_{s=t}^{\tau}\sum_{i=1}^{N}\omega_t(i,s)\delta_i(\tau-s) \\ &> \omega_t(N,t)\delta_N(\tau-t). \end{aligned}$$

Therefore, $\delta > \sqrt[\tau-t]{\omega_t(N,t)\delta_N(\tau-t)}$ for every τ implies that $\delta \geq \delta_N^*$. ■

A.9 Utilitarianism with Infinite Time Horizon

Theorem 4 in Section 6 requires that $(u_i)_{i\in N}$ be linearly independent. Below, we state a result related to Theorem 4 without assuming that $(u_i)_{i\in N}$ is linearly independent. This result will show that even when u_i's are identical, the cutoff for the social discount factor will jump from $\min_i \delta_i^*$ to $\max_i \delta_i^*$ when $T = +\infty$. To state it, we first define utilitarianism.

Definition 6 *The planner is intergenerationally utilitarian if in each period $t \in T$, there* *exists a sequence of nonnegative numbers* $(\omega_t(i,s))_{i\in N, s\geq t}$ *such that* $0 < \sum_{i=1}^{N}\sum_{s=t}^{T}\omega_t(i,s) < \infty$*, and*

$$U_t = \sum_{i=1}^{N}\sum_{s=t}^{T}\omega_t(i,s)U_{i,s}.$$

The result below shows that if we assume intergenerational utilitarianism rather than intergenerational Pareto, we can extend Theorem 4 to the case without linear-independence assumptions on u_i's. The reason we assume intergenerational utilitarianism rather than intergenerational Pareto is that the equivalence between intergenerational utilitarianism and intergenerational Pareto is not yet established.

Proposition 4 *Suppose* $T = +\infty$*, and each generation-t individual i's discounting utility function has an instantaneous utility function* u_i *and a discount function* δ_i *such that (5) and (6) hold. Let the planner's instantaneous utility function u be an arbitrary strict convex combination of* $(u_i)_{i\in N}$*. Then,*

1. *for each* $\max_i \delta_i^* < \delta < 1$*, the planner is intergenerationally utilitarian and strongly non-dictatorial;*
2. *for each* $\delta < \max_i \delta_i^*$*, the planner is not simultaneously intergenerationally utilitarian and strongly non-dictatorial.*

The proof of this proposition turns out to be identical to the proof of Theorem 4, except that Lemma 2 is not needed here.

Proposition 4 covers the case in which u_i's are identical. Thus, Proposition 4 says that if $T = +\infty$, the cutoff for the social discount factor again jumps from $\min_i \delta_i^*$ to $\max_i \delta_i^*$, compared to Theorem 1/Corollary 2.

Note that the second part of Proposition 4 is weaker than our previous results. In Proposition 4, if the social discount factor is lower than the highest individual long-run discount factor, then either intergenerational utilitarianism is violated or the planner has ignored some individual from some generation.

However, there is still some discontinuity between Proposition 4 and Theorem 1/Corollary 2. In Theorem 1/Corollary 2, if the social discount factor is lower than the lowest individual long-run discount factor, we know that intergenerational Pareto is violated, which implies that at least one of the two conditions, intergenerational utilitarianism or the strongly non-dictatorial property, is violated as in Proposition 4.

The intuition for this discontinuity in the second part of the result is the following. For simplicity, suppose u_i's are the same. Fixing an arbitrarily large but finite T, the planner can always attach small enough utilitarian weights to individuals with high δ_i^*. In this way, the planner can keep her social discount factor low. However, if T is infinite, fixing any strictly positive weights, as τ increases to infinity, $\delta_i(\tau)$ of the individual with the highest δ_i^* dominates all the other individuals' discount factors regardless of his weight. Therefore, the social discount factor cannot be strictly less than $\max_i \delta_i^*$.