Bargaining with Rational Inattention

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Abstract

A seller makes repeated offers to a rationally inattentive buyer (Sims, 2003). The seller knows the product's quality, which is random. The buyer needs to pay attention both to the product's quality and to the seller's offers. I show that there is delay in trade that decreases in product quality, and that the buyer obtains a significant surplus, which remains significant in a frequent-offers environment with vanishing attention costs. Finally, I show that revealing the product's quality to the buyer reduces both the buyer's surplus and overall efficiency.

KEYWORDS: Complexity, bargaining, rational inattention, entropy reduction.

1 Introduction

Many high-stakes bargaining situations are complex. Mergers and acquisitions involve changing the ownership of different kinds of assets and obligations. Collective bargaining contracts set wages, benefits and work conditions for many individuals. International trade agreements determine tariffs for millions

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of products, and so on. Given the stakes involved in these transactions, it is of interest to understand the ways in which their complexity influences them. As such, a large and growing literature has studied the way that complexity influences the resulting contract (e.g., Segal, 1999; Battigalli and Maggi, 2002; Tirole, 2009; Bolton and Faure-Grimaud, 2010). The current paper studies a different distortion: complexity's influence on bargaining.

A defining feature of complex trades is that they involve costly attention. For example, one might need to carefully read long contracts, study intricate product details, or hire expensive consultants such as lawyers, bankers or engineers. Thus, I study complexity's effect on bargaining by incorporating costly attention into an otherwise standard bargaining model. In particular, I look at a rationally inattentive buyer who bargains with a fully rational seller over a product of uncertain quality. The quality, v, is determined once and for all at the beginning of the game and is observed by the seller. Each period, the seller makes the buyer an offer. The buyer then allocates her attention by choosing which among all signal structures to use to learn about v, and the seller's (past and current) offers. The signal structure, the buyer's prior, and the seller's possibly mixed strategy determine the buyer's attention cost. I follow Sims (2003) by making attention costs proportional to the expected reduction in the entropy of the buyer's beliefs. Once the buyer chooses her signal structure for the period, nature draws a signal. Given the signal, the buyer updates her prior and chooses whether to accept or reject the seller's offer. If she accepts, trade occurs and the game ends. Otherwise, the game proceeds to the next period.

I show that attention costs have two effects on bargaining. To understand these effects, consider, first, the benchmark model in which the buyer can perfectly observe both the seller's offers and product quality. It is well known that this model has a unique equilibrium characterized by two features: 1) trade happens immediately, meaning that there is no inefficiency; and 2) the seller obtains all the gains from trade. I show that these features are reversed in my model.

The paper's first major result is that there is inefficient delay in trade.

Moreover, I show that this delay persists even when the seller makes offers infinitely often (Theorem 4). To understand the intuition for delay, consider the limit as the time between offers goes to zero. Suppose that there is no delay at the limit – i.e., the buyer accepts the seller's equilibrium offers for sure in an instant. Then, the lowest-cost way for the buyer to execute this strategy is to accept *every* offer sequence in an instant. Any other strategy involves the buyer acquiring additional information, and, hence, costs more. But then the seller is best off making very high offers, implying that the buyer's strategy is suboptimal, a contradiction.

The paper's second major result is that the buyer's ex-ante payoff is positive. This result echos previous (Kessler, 1998) and concurrent (Roesler and Szentes, 2016) studies showing how imperfect information can create commitment power. My paper differs from these studies in three ways. First, the buyer in my model needs to learn about the seller's product and offers simultaneously. Second, my seller does not get to observe the buyer's signal structure before he makes his offer, meaning she would always choose full information and lose all her commitment power if attention were free. Third, my paper studies inattention in a repeated-offers model, giving the buyer an additional channel through which to generate surplus: private information. I demonstrate this channel by showing that in an environment with frequent offers, the buyer obtains a significant surplus even when attention costs are negligible. More precisely, let v be the realized quality of the product, and take v_l to be the lowest possible quality the product can attain. Theorem 5 establishes that, when offers are frequent, as attention costs vanish, trade is efficient and the buyer's expected surplus converges to $\frac{1}{2}\left(E\left[v\right]-v_l\right)$.

The intuition for Theorem 5 is based on the related literature on the Coase conjecture (see, for example, Fudenberg et al. (1985) and Gul et al. (1986)). In this literature, a seller makes repeated offers to a buyer with private information about her value. Lacking commitment, the seller attempts to go down the demand curve as fast as possible as offers become more frequent. Anticipating the decrease in prices, the buyer lowers her willingness to pay, hence flattening her demand. In the limit, the seller's offer converges to the value of the last buyer who buys the product in a calendar instant. The outcome is a high surplus to the buyer and instant trade.

A similar Coasian dynamic arises when the buyer's attention is costly. With costly attention, the buyer has private information about her signals. These signals serve a similar role to that of private values in the Coase conjecture. When attention costs are positive, the Coasian argument fails due to the above-mentioned delay. When attention costs vanish, delay disappears, thereby unleashing the Coasian effect. Therefore, in the limit, trade happens immediately at the valuation of the last buyer who purchases the product.

Therefore the key is to understand the value of the last buyer on the demand curve. This buyer's information is characterized by two features. First, she received a sequence of weakly informative but negative signals that led her to reject the seller's past offers. Second, her last signal is a strongly informative and positive signal that leads her to accept the seller's current offer. The last buyer's valuation is, therefore, a horse-race between a sequence of weak negative signals and a single positive and highly informative signal. A delicate argument shows that these two balance each other exactly, thus leading the last purchasing buyer to evaluate the good at $\frac{1}{2}(v - v_l)$.

I conclude by examining the role that quality uncertainty plays in my model. Thus, I compare my model to one in which the buyer sees the product quality before bargaining, but still needs to pay attention to the seller's offers. Proposition 4 shows that revealing the product's quality to the buyer reduces total surplus because doing so reduces attention costs but increases delay. Overall, the surplus lost due to increased delay is strictly higher than the gain from attention costs.

In addition to contributing to the literature on the effect of transaction complexity on trade, the current paper also contributes to the growing literature on rational inattention based on expected entropy reduction (e.g., Sims, 1998; Van Nieuwerburgh and Veldkamp, 2010; Dessein et al., 2016; and Yang, 2015). A key feature of my paper is that the buyer needs to pay attention to a *strategic* variable – i.e., the seller's offers. In most of the literature, the inattentive agent pays attention to variables that are controlled by nature. These include variables determined exogenously or in aggregate, such as competitive equilibrium prices.

The fact that the buyer pays attention to a strategic variable requires me to introduce an equilibrium refinement. I need a refinement because zero-probability events do not enter the expected entropy-reduction formula. By threatening to collect the right information, the buyer can deter the seller from making certain offers, hence implying that the threat costs the buyer nothing. However, I show that one can eliminate many such threats by introducing a refinement in the spirit of Selten's (1975) perfect equilibrium. In particular, for each off-equilibrium history, I require the buyer's strategy to be a limit of best responses to some sequence of belief perturbations that put positive probability on said history. These perturbations make the buyer account for the marginal cost of paying attention to off-path events, which eliminates the possibility of non-credible attention threats.

Even with my refinement, it is still true that the buyer may obtain free information in the absence of equilibrium uncertainty. After all, without uncertainty, the buyer never needs to update her beliefs. However, I view the lack of equilibrium uncertainty as a stylized, limiting case. This view is implicit in other equilibrium concepts, such as Selten's (1975) perfect equilibrium, Myerson's (1978) properness concept and Kreps and Wilson's (1982) sequential equilibrium. These equilibrium concepts see players' strategies as having some infinitesimal uncertainty that must be accounted for by their peers. In this paper, my credibility refinement expresses this infinitesimal uncertainty. This refinement leads the buyer to choose her signal structures *as if* there were uncertainty, even if there is none. In this sense, a deterministic equilibrium can be seen as an approximation for an equilibrium with infinitesimal uncertainty.

Previous papers involving agents with expected entropy-reduction costs, who pay attention to strategic uncertainty, avoided non-credible threats in other ways. In Matějka (2015), for example, a buyer needs to pay attention to a price set by a monopolistic seller. Unlike the present model, in Matějka's (2015), the seller commits to the price distribution ex-ante, and the buyer gets to observe this distribution before choosing how to allocate her attention.

Attention threats are then circumvented by standard sub-game perfection. Matějka and McKay (2012) study a static competition model in which multiple sellers compete for a buyer paying attention to quality and prices. They avoid the issue of non-credible threats by directly restricting the buyer to a subset of information structures known to arise as a solution to a full-support rational inattention problem (Matějka and McKay, 2015).

In addition to satisfying the above refinement, I require the equilibrium to satisfy two additional requirements. First, I require there to be no periods in which the buyer rejects every offer, regardless of the history. Second, when the horizon is infinite, I require the equilibrium to also be the limit of finite horizon equilibria. Both conditions have parallels in the bargaining literature (e.g., Gul and Sonnenschein, 1988). Theorem 1 proves the existence of such equilibria, while Theorem 3 characterizes them.

A main contribution of this paper is to consider a dynamic strategic interaction between a seller and a rationally inattentive buyer. Yang (2013), Matějka and McKay (2012) and Martin (2012) also consider one or more rational sellers making offers to one or more inattentive buyers with entropy-reduction costs. However, unlike my model, the sellers in the aforementioned papers' models all make a single offer. Thus, theirs are static models, while mine is dynamic. In contrast, Steiner et al. (2015) concurrently solved a dynamic rational inattention decision problem with exogenous uncertainty. In contrast, my model involves a strategic interaction with endogenous uncertainty.

Without attention costs, my model reduces to a one-sided repeated-offers bargaining model with full information. Such a model also serves as the benchmark for the literature on bargaining with one-sided incomplete information in a private-values setup (e.g., Sobel and Takahashi, 1983; Fudenberg et al., 1985; Gul et al., 1986; Ausubel and Deneckere, 1989). Unlike the model studied here, such models involve an uninformed proposer making offers to an informed receiver. As mentioned earlier, a classic result in this literature is the Coase conjecture, which implies that there is no delay whenever offers are frequent and the gains from trade are uniformly positive. The no-delay result breaks when the receiver's information is also relevant to the proposer (Deneckere and

Liang, 2006). My model differs from this literature in that I assume that it is the proposer (seller), not the receiver (buyer), that has private information. In addition, I assume that the receiver can pay attention to the proposer's information, but also needs to pay attention to the proposer's offers.

Several studies examine one-sided repeated-offers bargaining models in which both parties have private information about the gains from trade (Cramton, 1984; Cho, 1990). When offers are frequent, such models often result in no trade or a large equilibrium multiplicity with various predictions (Ausubel and Deneckere, 1992; Tsoy, 2015). The source or multiplicity was first pointed out by Rubinstein (1985), who studied an alternating-offers model in which the discount rate of one of the players was private information. He showed that one can support a large set of equilibrium outcomes by constructing belief-based threats off the equilibrium path. The multiplicity arising due to belief-based threats is similar to the one arising in my model due to attention threats. Despite the similarities, my refinement is insufficient to reduce the multiplicity in Rubinstein (1985).

Another model in which the informed party gets to make offers is Gul and Sonnenschein's (1988). Theirs is an alternating-offers bargaining model between a buyer and a seller who is uncertain about the buyer's valuation of the product. They show that taking the time between offers to zero results in immediate trade in every equilibrium satisfying their refinement. As in Gul and Sonnenschein (1988), my refinement does not identify a unique equilibrium. However, my model generates delay and a potentially negative ex-post surplus to the buyer, outcomes that cannot arise in the analysis of Gul and Sonnenschein (1988).

2 The cost of attention

A seller (S) bargains with a potential buyer (B) over a complex product. Each period, S makes an offer that B either accepts or rejects; if she accepts, the parties trade and the game ends; otherwise, the period ends and S makes a new offer in the next period. The product's complexity necessitates a complex offer, meaning that B accepts or rejects with less than perfect information about S's product and offers. B's information depends on how much attention she devotes to understanding both the product's value and S's offers. B knows that this attention is costly and, therefore, allocates her attention optimally. The model of inattention that describes B is due to Sims (2003).

Product quality Before S makes his first offer, he observes the product's quality: v. v is a random variable that takes on values according to the distribution μ_0 from a finite set $V = \{v_l, \ldots, v_h\}$, where $v_l \leq v \leq v_h$ for all $v \in V$.

Periods After the product's quality is realized, the game proceeds in periods $m = 1, 2 \ldots$. The number of periods can be either finite or infinite. The timing within each period is as follows:

1. Every period begins with S making B an **offer**, x_m.

2. B then allocates her attention – i.e., she chooses a **signal structure** for this period, P_m.

3. B observes the realized **signal**, s_m, and decides whether to accept or reject the offer.

The parties trade and the game ends if B accepts the offer. Otherwise, no trade occurs, and the game either proceeds to the next period or ends if the current period is the last.

Information Both players know their past actions and signals. S observes the product's quality (v), but B does not. B sees neither S's offers, x, nor the product's quality, v. All of B's information about v and x comes from her equilibrium knowledge and her signals, $s_1, \ldots, s_m$. S never observes B's signals or her signal structure, though he knows B's equilibrium strategy.

Offers and seller's strategy S's offer in each period is a number, $x_m \in X = [0, \bar{x}]$,[1] where $\bar{x} > v_h$. I interpret x as a reduced form of S's offers. For example, a payment plan offered by a car dealer to a potential buyer will be represented by its expected discounted value. In other words, x is the monetary value that B will give to S if a transaction takes place.[2] Letting $x^m = (x_1, \ldots, x_m)$ denote a history of offers, one can summarize a period m history for S via $\omega_m = (v, x^m)$. As such, the set $\Omega_m := V \times X^m$ denotes all of S's period m histories. Taking $\Omega_0 = V$, I often refer to $\omega_m \in \Omega_m$ as the period m state. A strategy for S, $\sigma : \cup_{m \geq 0} \Omega_m \to \Delta X$,[3] is a mapping from each period's state to the current period's (possibly random) offer.

Signal structures After S makes his period m offer, B chooses which information to collect about S's product and current and past offers. More precisely, B chooses a period m **signal structure**, which is a probability transition kernel: $P_m : \Omega_m \to \Delta\mathbb{N}$. This transition kernel specifies the conditional distribution over signals given the product's quality and S's history of offers. The set of signals that B can use is the set of all positive integers – i.e., B can use any discrete signal structure. I denote the set of all possible period m signal structures by $\mathcal{P}_m$. A period m *signal structure strategy*, ρ_m, takes a history of signal structures and signals, $\{(P_n, s_n)\}_{n=1}^{m-1}$, and maps it to a distribution over $\mathcal{P}_m$.[4]

Beliefs and attention costs The transaction's complexity makes it costly for B to pay attention to S's product and offers. As a result, each period m signal structure comes at a cost. I assume that the cost is proportional to Shannon's measure of mutual information. Formally, let μ be B's conditional

[1]The equilibrium does not depend on the upper bound, $\bar{x}$. The bound merely ensures integrability of certain functions used in solving for the buyer's strategy.

[2]One can generalize S's offers to include more dimensions. As long as preferences are quasi-linear in money, this will not alter the analysis.

[3]For $Y \subset R^n$, let $\Delta(Y)$ denote the set of all Borel probability measures on Y. I drop the brackets whenever possible without leading to confusion.

[4]Later, I show that one can restrict attention to a smaller subset of strategies. As such, I omit the specification of the appropriate σ-algebra over $\mathcal{P}_m$.

beliefs about ω_m given her past information and S's equilibrium strategy. If μ has finite support,[5] then one can define the mutual information between ω_m and the signal structure P_m, conditional on previous signals, s^{m-1}, as:

$$\mathbf{I}\left(\omega_m, P_m | s^{m-1}\right) = \mathbb{E}\left[\mathbf{H}\left(\mu\left(\cdot | s^{m-1}\right)\right) - \mathbf{H}\left(\mu\left(\cdot | s^{m-1}, s_m\right)\right) | s^{m-1}\right], \quad (1)$$

where $\mathbf{H}(\mu) = -\sum_{\omega} \mu(\omega) \ln \mu(\omega)$ is the entropy of B's beliefs. To ensure continuity, I let $0 \ln 0 = 0$, $c \ln \frac{c}{0} = \infty$ if $c > 0$, and $0 \ln \frac{0}{0} = 0$. $\mathbf{I}$ measures how many bits B acquires by using her period m signal. The above cost function comes from information theory: see Appendix A.1 for additional background.

Notice that mutual information depends on B's prior. The more informative the prior, the lower the mutual information. One can interpret this as expressing an increased ease of paying attention to familiar information. For example, B may find a non-standard transaction harder to understand than a routine one. In the model, more-routine transactions would be interpreted as less uncertain, hence resulting in lower attention costs.

Accepting or rejecting offers After observing the current signal, s_m, B chooses whether to accept or reject S's offer. A period m accept-reject strategy for B is a mapping from a sequence of signal structures and realized signals, $\{(P_n, s_n)\}_{n=1}^{m}$, to an accept-reject decision.

Outcomes and payoffs An outcome of the game is the period in which agreement is reached, m; the product's quality, v; the accepted offer, x_m; the signal structures used by the buyer each period, $(P_1, \ldots, P_m)$; the realized signals, s^m; and B's prior entering each period, $\{\mu\left(\cdot | s^{n-1}\right)\}_{n=1}^{m}$.

Both players discount time at a constant rate r. If no trade ever takes place during the game, both players obtain zero from the transaction. If the game ends with B accepting an offer x_m at period m, S's payoff is $U_S := e^{-r\Delta m} x_m$,

[5] For a general μ, one can define mutual information via:

$$\mathbf{I}\left(\omega_m, s_m | s^{m-1}\right) = \int \ln\left(\frac{P_m\left(s_m | \omega_m\right)}{\int P_m\left(s_m | \omega_m\right) \mu\left(\mathrm{d}\omega_m | s^{m-1}\right)}\right) P_m\left(\mathrm{d}s_m | \omega_m\right) \mu\left(\mathrm{d}\omega_m | s^{m-1}\right)$$

while B's transaction payoffs is $e^{-r\Delta m}(v - x_m)$. From her transaction payoff B subtracts her attention costs. Thus, B's payoff, if she accepts an offer x_m in period m, is:

$$U_B = e^{-r\Delta m}(v - x_m) - \sum_{j=1}^{m} e^{-r\Delta j} \kappa \, \mathbf{I}\left(\omega_m, s_m | s^{m-1}\right). \tag{2}$$

By assuming that the buyer's information cost is proportional to Shannon's measure of mutual information, I assume that the buyer already understands the prior joint distribution of the offers and quality but can pay further attention to these variables to understand more. Thus, she incurs attention costs *at the margin.*

Recommendation strategies As a preliminary step in the analysis, I show that an optimal strategy for B can be found within a class of recommendation strategies. A recommendation strategy is defined by three properties. First, each signal structure has only two signals; call them 0 and 1. Second, these signals are interpreted as recommendations for B – i.e., she accepts for sure if she observes 1 and rejects for sure otherwise. Third, B does not randomize among signal structures.

Formally, a **recommendation strategy** is a sequence of functions: $\beta = (\beta_m)_{m\geq 1}$, where $\beta_m(\omega_m)$ is the probability that B receives a recommendation to accept. Thus, for every m, β_m is a measurable mapping from Ω_m into $[0,1]$. The following lemma ensures that I can focus on recommendation strategies.

Lemma 1. *For every strategy for B, there is a recommendation strategy with the same distribution over trade outcomes*[6] *after every history and with weakly lower attention costs.*

To prove Lemma 1, I use the *chain rule for mutual information.* The chain rule states that the expected information gained by observing two signals consecutively is equal to the information gained from observing both signals

[6]By trade outcomes, I mean product quality, offers made, period of trade and accepted offer.

simultaneously. To use this property, I view each signal structure as composed of two different signals: an action recommendation and a residual. By the chain rule, seeing this residual with any future signal instead of with today's recommendation does not increase the total cost of attention. Therefore, one can delay paying attention to this residual until the time it is used.

By Lemma 1, for every equilibrium, there is an outcome-equivalent equilibrium in recommendation strategies. To see this, take any equilibrium in non-recommendation strategies. Switch B's strategy to the recommendation strategy from Lemma 1. By construction, the new recommendation strategy must be optimal for B given S's strategy. S's strategy also remains optimal, since he cannot distinguish between the B's old and new strategies. Therefore, switching B's strategy and S's beliefs about that strategy results in consistent beliefs and optimal play – i.e., an equilibrium – with the exact same outcomes distribution. Appealing to Lemma 1, I assume henceforth that B uses only recommendation strategies.

Recommendation strategies simplify the task of tracking B's beliefs. In particular, period m arrives if and only if B sees $m-1$ reject signals. Hence, holding β fixed, m is a sufficient statistic for B's belief. As such, I let μ_m denote B's belief over $X^m \times V$, conditional on period m arriving but before she sees the m-th signal.

Recommendation strategies also simplify B's attention costs. Abusing notation, let $\mathbf{I}\left(\beta_m, \mu_m\right)$ stand for the mutual information between ω_m and the signal generated by β_m conditional on previous signals leading to posterior μ_m. Then, one can write:

$$\mathbf{I}\left(\beta_m, \mu_m\right) = \int \left[\beta_m \ln\left(\frac{\beta_m}{\int \beta_m \mathrm{d}\mu_m}\right) + \left(1-\beta_m\right)\ln\left(\frac{1-\beta_m}{\int 1-\beta_m \mathrm{d}\mu_m}\right)\right] \mathrm{d}\mu_m.$$

3 Recommendation Perfect Equilibrium

3.1 Definition and existence

In this subsection, I provide a formal definition of recommendation perfect equilibrium and briefly discuss the issues involved. The first issue I address in my refinement is the possibility of B automatically rejecting every offer. In particular, I wish to avoid periods in which B automatically rejects S's offers, regardless of their content. Formally, I say that β is **attentive** if for every period m, there exists some price history and some quality of the product, (x^m, v), such that $\beta_m(x^m, v) > 0$. Similar assumptions are often made in bargaining models.[7]

Requiring B's strategy to be attentive assumes that B never automatically rejects S's offers. To illustrate, take any equilibrium and adjust it in the following way. In period 1, have B reject every offer, regardless of its content. At the same time, have S's first offer always be equal to some $x > v_h$. From period 2 onward, let the players play according to the original equilibrium as though period 1 never happened. Clearly, this is an equilibrium. Repeating this logic to periods $2, 3, \ldots$ then generates an equilibrium without trade after any history.[8] Assuming that B's strategy is attentive avoids equilibria such as these.

A second and more subtle issue that arises in my model is B's ability to make non-credible attention threats. Such threats involve B committing to behave in a specific way towards off-equilibrium offers. With suitably chosen off-path beliefs, one can sustain a large class of unreasonable sequential equilibria. These threats are possible because mutual information does not depend on off-path signals. As such, B can treat zero-probability offers very differently than she treats positive-probability ones. However, such an extreme differential treatment of offers is non-credible in a way which I define below.

[7] For example, Rubinstein (1985) assumes that the uninformed player never makes irrelevant offers. Similarly, Gul and Sonnenschein (1988) assume away the possibility of periods in which offers are rejected for sure and are used only for communication between the players.

[8] One can actually show that for every set of periods $M_0 \subset \{1, 2, \ldots\}$, there exists an equilibrium in which trade occurs in period m if and only if $m \in M_0$.

See Appendix A.2 for an example of such threats and how the definition below helps eliminate them.

Let $\mathbb{E}_m [U_b|\mu_m, \beta, \sigma]$ be B's expected utility conditional on: arriving at period m, B's beliefs over $X^m \times V$ being μ_m, and future play being conducted according to (β, σ). I say that the beliefs μ and strategies (β, σ) are **consistent** if μ is updated according to Bayes rule whenever possible.

Definition 1. For a consistent (μ, β, σ), β is a **credible best response** *to* σ given μ if for every (x^m, v), there is a $\mu^* \in \Delta(X^m \times V)$ with $\mu^*(x^m, v) > 0$ and a $\{\mu^n, \beta^n, \epsilon^n\}_{n=1}^{\infty}$ with $\mu^n = \epsilon^n \mu^* + (1 - \epsilon^n)\mu_m$, $\epsilon^n \downarrow 0$ and $\beta^n \to \beta$,[9] such that β^n maximizes $E_m [U_b|\mu^n, \beta^n, \sigma]$ for all n.[10]

In the infinite horizon game, I require the equilibrium to also be the limit of finite horizon equilibria. Early papers in the bargaining literature also focused on limits of finite horizon equilibria (e.g., Sobel and Takahashi, 1983; and Cramton, 1984). I do so in my analysis to exclude the players' strategies from exhibiting complicated history dependence. Other studies often avoid complicated dependencies on the past by focusing on stationary equilibria (see, for example, Gul et al. (1986), Gul and Sonnenschein (1988), Ausubel and Deneckere (1989) and Gul (2001)). B's imperfect observation of past offers makes B's behavior too rigid to allow for stationary play. Focusing on equilibria that can be approximated by finite horizon play recovers some of the simplicity lost by allowing for non-stationary strategies.

Combining all three requirements gives the following definition of a perfect recommendation equilibrium.

Definition 2. A consistent (μ, β, σ) is a **perfect recommendation equilibrium (PRE)** if: (1) β is attentive and is a credible best response to σ given μ; (2) σ is a best response to β after every history; and (3) with infinite periods, (μ, β, σ) is also a limit of finite horizon PREs with the horizon going to infinity.

[9] Convergence is defined using the topology of point-wise convergence.

[10] Note that the definition implies that β maximizes $E_m [U_b|\mu, \beta, \sigma]$ for every m due to upper hemicontinuity.

I refer to a PRE from here on simply as equilibrium unless it creates confusion. Theorem 1 below states that an equilibrium exists in both the finite and the infinite horizon versions of the game.

Theorem 1. *A PRE exists in both the finite and infinite horizon games. Moreover, in every PRE, players use simple strategies.*

The proof of Theorem 1 for the finite horizon is partially constructive and partially dependent on a fixed-point argument. The main difficulty is to ensure that β is attentive, since this property is defined via a strict inequality. Requiring β to be a credible best response to σ does not imply that β must be attentive. As such, to prove the theorem, I derive a set of necessary and sufficient conditions for (μ, β, σ) to be an equilibrium in the finite horizon. I present these conditions in Theorem 3 in the next subsection. A fixed-point argument then establishes that there is some (μ, β, σ) satisfying these conditions for every finite horizon.

For the infinite horizon, I take a generic finite horizon equilibrium sequence with a horizon going to infinity. Using the finite horizon properties stated in Theorem 3 below, one can connect the convergence of $b_m(x, v)$ and $z_{m,v}$ to the convergence of $z_{m+1,v}$ and $b_{m+1}(z_{m+1,v}, v)$. Since these are members of a countable product of compact subsets of $\mathbb{R}$, one can ensure the existence of a converging subsequence. Using the structure inherited from finite horizon equilibria, one can then prove the attentiveness of B's limit strategy. Once attentiveness is established, I prove the optimality of B's limiting strategy via sufficient conditions derived in the online appendix. Optimality of S's strategy is then attained via standard continuity at infinity arguments.

At this stage, the reader may wonder about equilibrium uniqueness. The following theorem states that in the one-shot game, the equilibrium is unique.

Theorem 2. *There exists a unique equilibrium in the one-shot game.*

When there are more than two periods, one can obtain multiple equilibria. Intuitively, the multiplicity comes from the interdependency of current and future periods. Future periods are influenced by B's posterior over the quality of the product at the end of the current period. However, behavior at the current period, and, therefore, B's posterior, depend on both players continuation values, which depend on the future. Combined, these can result in multiple equilibrium paths.

3.2 Equilibrium characterization

The equilibrium satisfies several properties that I use throughout the analysis. The first of these properties is simplicity. More precisely, S's strategy is **simple** if it prescribes a single deterministic offer, $z_{m,v}$, for every period m and every v. A v type S makes this offer in period m, regardless of S's realized offers in periods $m' < m$. B's strategy is **simple** if for every m, there is a function b_m from $X \times V$ to $[0, 1]$ such that the probability B that accepts an offer x_m made by a v type S is $b_m(x_m, v)$, regardless of S's offers in previous periods.

Theorem 3 below shows that equilibrium strategies must be simple. As such, from now on, I identify equilibrium strategies β and σ by their corresponding simple counterparts, b and z. To put it differently, I often write $b_m(x_m, v)$ instead of $\beta_m(x_1, \ldots, x_m, v)$, and say that S uses strategy z rather than σ.

The theorem presents a few additional properties, for which I need more notation. Let (μ, b, z) be an equilibrium of the game in simple strategies. Denote the marginal of μ_m over V by $\bar{\mu}_m$, and take $b_{m,v} := b_m(z_{m,v}, v)$ to be the probability that B accepts the v-seller's period m equilibrium offer, conditional on arriving at period m. Define π_m as the prior probability that the buyer accepts the m-th offer, conditional on arriving at period m – i.e., $\pi_m := \sum_v \bar{\mu}_{m,v} b_{m,v}$.

The characterization in Theorem 3 below takes the prior acceptance probabilities, $\pi_1, \pi_2 \ldots$, as given and uses them to construct simple strategies. In principle, one can construct simple strategies in this way from an arbitrary sequence of π's. Finding a sequence that generates simple strategies that average back to the same prior acceptance probability sequence is the key to proving existence in the finite horizon game. The characterization below is partial in that, while both necessary and sufficient when horizon is finite, it is only necessary when the horizon is infinite.

Theorem 3. *Equilibrium strategies are simple. Let (μ, b, z) be an equilibrium in the $M \in \mathbb{N} \cup \{\infty\}$ horizon game. Then, the following must hold:*

1. ***Delay:*** *π_m is strictly between 0 and 1 for all m.*

2. ***Logit buyers:*** *For all m, x and v:*

$$b_m(x, v) = \frac{e^{\frac{1}{\kappa}(v - x + \kappa \ln \pi_m)}}{e^{\frac{1}{\kappa}(v - x + \kappa \ln \pi_m)} + e^{\frac{1}{\kappa}(e^{-r\Delta} u_{m+1,v} + \kappa \ln(1-\pi_m))}}, \tag{3}$$

   where the continuation value, $u_{m,v}$, for $m = M+1$ is zero and for $m \leq M$ is:

$$u_{m,v} := \kappa \ln \left(e^{\frac{1}{\kappa}(v - z_{m,v} + \kappa \ln \pi_m)} + e^{\frac{1}{\kappa}\left(e^{-r\Delta} u_{m+1,v} + \kappa \ln(1-\pi_m)\right)} \right). \tag{4}$$

3. ***Seller's prices:*** *For all m and v:*

$$\left(\frac{z_{m,v} - \kappa}{\kappa}\right) - e^{-r\Delta}\left(\frac{z_{m+1,v} - \kappa}{\kappa}\right) = \left(\frac{b_{m,v}}{1 - b_{m,v}}\right), \tag{5}$$

   where $z_{M+1,v} := \kappa$.

4. ***Values:*** *B's equilibrium value is $\mathbb{E}[u_{1,v}]$, and S's equilibrium value is $\mathbb{E}[z_{1,v}] - \kappa$.*

5. ***Monotonicity:*** *$z_{m,v}$, $b_{m,v}$ and $v - z_{m,v}$ are strictly increasing in v. Moreover, $v_l - z_{m,v_l} < 0 < v_h - z_{m,v_h}$.*

The theorem's part 1 says that in equilibrium, B both accepts and rejects offers with positive probability. When offers are infrequent, B rejecting means that there is a delay in agreement. The intuition for this delay is similar to the intuition stated in the introduction. Suppose that there were no delay – i.e., B accepts S's equilibrium offers for sure. Then, the lowest-cost way of executing this strategy for B is to accept *every* offer for sure. Any other strategy will have (at least infinitesimally) positive costs. But then S is best off offering a very high price, making it best for B to reject for sure, a contradiction. A similar intuition shows that rejecting for sure on path occurs only if B's strategy is not attentive.

Equation 3 shows that B's optimal strategy takes Rust's (1987) dynamic logit form. This echos a result of Steiner et al. (2015).[11] In dynamic logit, an agent needs to take an action every period, where each action's payoffs are equal to an underlying utility plus an independent random shock with an extreme value type I distribution. The underlying payoffs in B's response are $v - x + \kappa \ln \pi_m$ from accepting and $\kappa \ln (1 - \pi_m)$ from rejecting. Given these underlying payoffs, Rust (1987) shows that $u_{m+1,v}$ is the dynamic logit agent's expected continuation utility given v and the future offer distribution. The dynamic logit continuation utilities are related to B's actual expected utility in the following way: B's expected utility conditional on arriving at period m, equals $e^{-r\Delta(m-1)}\mathbb{E}\left[u_{m,v}\right]$ plus the discounted attention costs incurred in periods $1, \ldots, m-1$. As such, B's expected utility in equilibrium is equal to $\mathbb{E}\left[u_{1,v}\right]$.

To derive B's optimal strategy, I transform B's problem to one involving a strictly concave objective functional. I then use simple calculus of variation arguments to characterize both B's optimal strategy and the continuation value. I delegate this derivation to online Appendix B.

The necessity of simple strategies comes from B's strategy and the focus on limits of finite horizon equilibria. In the last period, S knows B's equilibrium information exactly: the only way to arrive at period M is for B to receive $M-1$ reject recommendations. As such, S knows B's recommendation strategy in period M. The credibility of B's strategy and there being no future imply that B's strategy takes the above logit form. Hence, S's last-period problem, given v, is identical to the problem of a price-setting monopolist with zero

[11] The connection between Rust's (1987) model and the dynamic rational inattention solution was first pointed out by Steiner et al. (2015). Steiner et al. (2015) independently and concurrently solve for an optimal dynamic rational inattention rule, though for a different class of problems. They allow the agent to obtain free information, more general actions and more general payoffs, but require the state space to be finite. An infinite state space arises in my model due to the seller's offers. While these create some technical issues, the underlying result remains unchanged.

marginal costs facing a logit demand function. This problem is known to be strictly log-concave and has a unique solution, $z_{M,v}$ – i.e. S's last period strategy is simple. Simple future play can then be extended to simple present play via a similar argument, proving necessity by backward induction.

Characterizing S's strategy is straightforward once we know B's strategy and that strategies are simple. Since the players use simple strategies, S's continuation value when moving from m to $m+1$, conditional on v, does not depend on S's past offers. As such, the value of S's problem in period m, given v, is:

$$w_{m,v} := \max_{x} b_m(x,v)\, x + (1 - b_m(x,v))\, e^{-r\Delta} w_{m+1,v}\,. \tag{6}$$

Proving part 5 of the theorem implies that the upper bound $\bar{x}$ does not bind. One can, therefore, use equation 3 to calculate the first-order condition:

$$x - e^{-r\Delta} w_{m+1,v} = \frac{\kappa}{1 - b_m(x,v)}\,. \tag{7}$$

Rearranging this condition gives $w_{m,v} = z_{m,v} - \kappa$, where $z_{m,v}$ is the solution to S's period m problem, given v. Since the same is true for period $m+1$, one can substitute $w_{m+1,v} = z_{m+1,v} - \kappa$ and rearrange to obtain equation 5.

Part 5 presents two properties of S's offers and B's net benefit from accepting these offers. The first property is that both are strictly increasing in the product's realized quality. The second property is that B gets cheated on a low-quality product and gets good value when buying a product of high quality.[12] Thus, there are rip-offs at the bottom and bargains at the top. Given the first property, the second property follows from B both rejecting and accepting offers with positive probability. Doing so can only be optimal if B loses from some offers but benefits from others. The first property comes from B's best response function. For a rough intuition, examine S's last-period problem, given v, in a finite-horizon game. The solution to this problem is readily revealed by setting $w_{M+1,v}$ to zero in equation 7. Since $b_M(x,v)$ is increasing in $v - x$, S's offer must increase with product quality. But S's offer increases

[12]Note that these two properties imply that $z_{m,v} < v_h < \bar{x}$ for all v, thereby confirming that $\bar{x}$ is a non-binding upper bound.

only if $b_{M,v}$ increases too (equation 5), which, in turn, implies that B's net benefit from accepting, $v - z_{M,v}$, must also increase. The property for the last period follows. For previous periods, one needs to take the continuation values into account. However, these values turn out not to matter due to equal discounting and the relationship between $w_{m,v}$, $u_{m,v}$ and $z_{m,v}$.

3.3 The case of no quality uncertainty

The current section discusses the case in which B knows v. There are a few reasons to examine this case. First, this case is the main component in understanding the model in which B observes quality, but still needs to pay attention to S's offers. I return to this model in Section 6. Second, the equilibrium takes a simple form useful for understanding Proposition 2 and Theorem 4. Third, this equilibrium brings out some fundamental features of the PRE refinement worth discussing in more detail. Let $B_v(\Delta)$ be the bargaining game in which the product's quality is v with probability 1. Proposition 1 below characterizes and proves uniqueness of the equilibrium of $B_v(\Delta)$.

Proposition 1. *There exists a unique equilibrium in the game $B_v(\Delta)$. In this equilibrium:*

1. *S offers v every period with probability 1, regardless of the history.*
2. *B accepts v with probability:*

$$\pi_v^{\kappa,\Delta} = \frac{\left(1 - e^{-r\Delta}\right)(v - \kappa)}{\left(1 - e^{-r\Delta}\right)(v - \kappa) + \kappa}. \tag{8}$$

3. *B's expected utility is 0, and S's expected utility is $v - \kappa$.*

Proof. Since S's quality is known to B, one has $\bar{\mu}_{m,v} = \bar{\mu}_{m+1,v} = 1$ and $b_{m,v} = \pi_m$ for all m. Rearranging equation 3 for $x = z_{m,v}$ and using repeated substitution gives the equality:[13]

$$u_{m,v} = \kappa \sum_{j=m}^{\infty} e^{-r\Delta(j-m)} \left[\ln(1-\pi_j) - \ln(1-b_{j,v})\right], \tag{9}$$

implying that $u_{m,v} = 0$ for all m (since $b_{m,v} = \pi_m$). Equation 3 then implies that $z_{m,v} = v$ for all m. Equation 5 then establishes that $b_{m,v}$ equals to $\pi_v^{\kappa,\Delta}$ for all Δ. Proposition 3, part 4 then gives that S's expected utility is $v - \kappa$. The fact that B's expected utility equals zero follows from $u_{m,v} = 0$ $\forall m$ and Theorem 3, part 4. □

The equilibrium described by Proposition 1 has several interesting features. First, S always makes the same offer, regardless of the past. Second, the probability that B accepts an offer is independent of the period index m. Third, the total inefficiency in equilibrium caused by delay is equal to κ. Thus, the lower the attention cost parameter, the lower is the efficiency loss in equilibrium.

The fourth feature of the equilibrium in Proposition 1 is that attention is effortless. Effortless attention comes from S using a deterministic strategy. Since, in equilibrium, B knows both S's strategy and v, B's knowledge includes all there is to know in equilibrium. Note that effortless attention on the equilibrium path does not mean that B perfectly observes S's offers. This is because perfectly observing S's offers would constitute a non-credible attention threat that is ruled out by the credibility refinement. By forcing B to account for the marginal attention costs of her off-path signals, the refinement leads to B only partially adjusting her acceptance probability in reaction to off-equilibrium offers.

The reason that B obtains free information is the absence of equilibrium uncertainty. As mentioned in the introduction, I view the absence of equilib-

[13] To obtain the equality, note that: $u_{m,v} = -\kappa \ln\left((1-b_{m,v})\, e^{-\frac{1}{\kappa}\left(e^{-r\Delta} u_{m+1,v} + \kappa \ln(1-\pi_m)\right)}\right)$. One can then separate the logarithm to obtain:

$$u_{m,v} = \ln\left(\frac{1-\pi_m}{1-b_{m,v}}\right) + e^{-r\Delta} u_{m+1,v}\,.$$

Repeated substitution, gives equation 9.

rium uncertainty as a stylized, limiting case. This view is implicit in other equilibrium concepts, such as Selten's (1975) perfect equilibrium, Myerson's (1978) properness concept and Kreps and Wilson's (1982) sequential equilibrium. These equilibrium concepts see player's strategies as having some infinitesimal uncertainty that must be accounted for by their peers. In the current game, one can interpret my credibility refinement as expressing similar infinitesimal uncertainty. This refinement leads B to choose her signal structures *as if* there is uncertainty, even if there is none. In this sense, a deterministic equilibrium can be seen as an approximation for an equilibrium with infinitesimal uncertainty.

4 Delay in Trade

This section presents the result that costly attention leads to delay that is independent of the time between offers. As Gul and Sonnenschein (1988) point out, the delay that arises in an environment with infrequent offers can be misleading. In particular, restricting the time between offers conflates the calendar date of agreement with the number of offers needed to reach it. To avoid conflating the two, I show that inattention leads to delay that persists even as the time between offers converges to zero.

Consider the infinite-horizon game with some fixed time between offers $\Delta > 0$. In this game, each period m corresponds to the calendar date $\Delta(m-1)$. A sequence $\{\Delta^n, \mu^n, b^n, z^n\}$ is a **refining sequence** if (μ^n, b^n, z^n) is an equilibrium in a game with time between offers Δ^n, Δ^n converges to zero, and $\{0, \Delta^n, 2\Delta^n, \ldots\} \subset \{0, \Delta^{n+1}, \ldots\}$ for all n. A function $F : \mathbb{R} \times V \to [0,1]$ is the **agreement date distribution** for (Δ, μ, b, z) if $F_v(t)$ is the probability that trade occurred on or before calendar date t, conditional on quality being v in the equilibrium (μ, b, z). A refining sequence is **convergent** if the corresponding agreement date distribution sequence is such that F_v^n converges weakly[14] for all v to a right continuous and increasing function F_v. I refer to such an F as the sequence's **limit agreement date distribution**.

[14]Recall: H^n *converges weakly* to H if $H^n(t) \to H(t)$ for every t in which H is continuous.

Proposition 2 below shows that delay arises as Δ goes to zero and κ remains constant. I interpret a lower Δ as an increase in the rate at which new information accumulates, but not necessarily as an increase at the rate in which new information is absorbed. By fixing κ, I assume that absorbing the same amount of information at any given moment results in the same cost of attention. This assumption is in line with the chain rule of mutual information: what matters is the amount of information B absorbs, not the number of signals she uses to absorb it.

Proposition 2. *Let F be a limit agreement date distribution. Then, there exists $t > 0$ such that $F_{v_l}(t) < 1$.*

Proof. Let $\{(\Delta^n, \mu^n, b^n, z^n)\}_n$ be a convergent refining sequence, and take $\{F^n\}_n$ to be its corresponding agreement date distributions. I claim below that, for every n, the probability, given v_l, that B rejects S's equilibrium offer is at least $1-\pi_v^{\kappa,\Delta^n}$. Note that, if true, one has that for every $t \in \{\Delta^n, 2\Delta^n, \ldots\}$:

$$\begin{aligned} 1 - F_{v_l}^n(t) &\geq \mu(v_l)\left(1 - \pi_v^{\kappa,\Delta^n}\right)^{t/\Delta^n} \\ &\to \mu(v_l)\, e^{-r\left(\frac{v_h-\kappa}{\kappa}\right)t}, \end{aligned}$$

where convergence follows from L'Hopital's rule. The proposition follows.

All that remains is to show the claim. To do so, note, first, that in every period, there must be at least some v such that:

$$v - z_{m,v} \geq e^{-r\Delta}\left(v - z_{m+1,v}\right). \tag{10}$$

If equation 10 is false for all v, B is strictly better off accepting S's period $m+1$ offer over his m-th offer, meaning that B must be rejecting the m-th offer with probability 1, contradicting Theorem 3, part 1. One can now rearrange equation 10 to obtain:

$$\begin{aligned} \left(1 - e^{-r\Delta}\right) v &\geq z_{m,v} - e^{-r\Delta} z_{m,v} = \kappa\left(1 - e^{-r\Delta} + \left(\frac{b_{m,v}}{1 - b_{m,v}}\right)\right) \\ &\geq \kappa\left(1 - e^{-r\Delta} + \left(\frac{b_{m,v_l}}{1 - b_{m,v_l}}\right)\right), \end{aligned}$$

where the equality follows from equation 3, and the second inequality follows from the monotonicity of $b_{m,v}$ (Theorem 3, part 5). The claim then follows from rearranging the above inequality after noting that $v \leq v_h$. □

One can get an intuition for Proposition 2's proof by considering the game in which the product's quality is v with probability 1. For a fixed Δ, this game has a unique equilibrium that yields a unique agreement date distribution, F_Δ, satisfying:

$$F_\Delta(t) = 1 - \left(\frac{\kappa}{(1-e^{-r\Delta})(v-\kappa)+\kappa}\right)^{\frac{t}{\Delta}}.$$

Taking time between offers (Δ) to zero and using L'Hopital's rule gives the limit: $\lim_{\Delta\to 0} F_\Delta(t) = 1 - e^{-\frac{r}{\kappa}(v-\kappa)t}$. Thus, one obtains delay that persists even when offers are made infinitely often. The proposition shows that, given v_l, S's offers cannot be accepted at a higher rate than in $B_{v_h}(\Delta)$, the game in which B knows that S's is of quality v_h. Delay in the game with unknown quality, therefore, follows from the existence of delay when quality is known.

Notice that when v is known, the limit agreement date distribution is exponential with rate $\frac{r}{\kappa}(v-\kappa)$. As such, this distribution satisfies several properties. First, trade can happen at any moment. Second, while the players do trade with probability 1, trade can take an arbitrary amount of time. Third, delay is decreasing with the good's quality. Theorem 4 shows that these properties are also present in the game with unknown quality.

Theorem 4. *Let $\{\Delta^n, \mu^n, b^n, z^n\}_{n=1}^\infty$ be a convergent refining sequence*[15] *with limit agreement date distribution F. Then:*

1. *F_v is absolutely continuous for all v and satisfies $F_v(0) = 0$.*

2. *$F_v(t)$ is strictly increasing in t.*

3. *For all t: $F_v(t) < \lim_{t\to\infty} F_v(t) = 1$.*

4. *$F_v(t)$ is strictly increasing in v for all $t > 0$.*

[15] Recall that Helly's selection theorem ensures that a convergent refining sequence exists.

To prove Theorem 4, I begin by approximating each F^n by a continuous distribution, G^n, which agrees with F^n for every $t \in \{0, \Delta^n, \ldots\}$. G^n is defined by transforming $b_{m,v}$ into a date-dependent hazard rate for each v, $\lambda_{t,v}$. These hazard rates can be shown to be uniformly bounded from above by a multiple of $\frac{r}{\kappa}(v_h - \kappa)$. This bound is useful for two reasons. First, it implies that the hazard rates, as a function of calendar dates, belong to an L_2 space. Second, it allows me to evoke the Banach-Alaoglo theorem to generate a weakly convergent subsequence of the said hazard rates. The result is an absolutely continuous limit with $F_v(0) = 0$, establishing delay. The connection between $\lambda_{t,v}$ and $b_{m,v}$ then gives a continuous-time version of Theorem 3, which delivers parts 2 to 4 of Theorem 4.

5 The Value of Inattention

In this section, I show that B benefits from her costly attention. I begin with Proposition 3, which shows that B's expected utility in equilibrium is uniformly bounded away from zero. This is in stark contrast to the case without costly attenion, in which B's surplus is zero.

Proposition 3. *There exists a $\delta > 0$ such that for every equilibrium, B's expected utility is larger than δ.*

Proof. See the online appendix for the full proof. Here, I prove only that $\mathbb{E}[U_b] > 0$ for a given equilibrium (μ, b, z). Since $b_{m,v}$ is strictly increasing in v for every m (Proposition 3), $\bar{\mu}_{m,v}/\bar{\mu}_{m,v'}$ is decreasing with m for every $v > v'$. Therefore, $\bar{\mu}_m$ first-order stochastically dominates $\bar{\mu}_{m'}$ for every $m' > m$. Rearranging equation 3 for $x = z_{m,v}$ and using repeated substitution gives the equality[16]:

$$u_{m,v} = \kappa \sum_{j=m}^{\infty} e^{-r\Delta(j-m)} \left[\ln(1-\pi_j) - \ln(1-b_{j,v})\right].$$

[16] To see why this equality holds, note that: $u_{m,v} = -\kappa \ln\left((1-b_{m,v}) e^{-\frac{1}{\kappa}\left(e^{-r\Delta} u_{m+1,v} + \kappa \ln(1-\pi_m)\right)}\right)$. The equality then follows from logarithm rules and repeated substitution.

Combining equation 9 with Theorem 3, part 4 then gives:

$$\begin{aligned}\mathbb{E}\left[U_B\right] &= \kappa \sum_v \mu_{0,v} \sum_{m=1}^{\infty} e^{-r\Delta(m-1)} \left(\ln\left(1-\pi_m\right) - \ln\left(1-b_{m,v}\right)\right) \\ &> \kappa \sum_{m=1}^{\infty} e^{-r\Delta(m-1)} \left(\ln\left(1-\pi_m\right) - \sum_v \bar{\mu}_{m,v} \ln\left(1-b_{m,v}\right)\right) > 0.\end{aligned}$$

where the first inequality follows from $\bar{\mu}_m$ first-order stochastically dominating $\bar{\mu}_{m+1}$ for all m, and the second follows from Jensen's inequality. In the online appendix, I extend the above argument by taking limits to uniformly bound $\mathbb{E}\left[U_b\right]$ away from zero across all equilibria. □

Intuitively, costly attention gives B some commitment power. Consider the one-period game. The proof follows from two easily verifiable facts of mutual information, $\mathbf{I}\left(\beta_1, \mu_1\right)$. The first is that $\mathbf{I}\left(\beta_1, \mu_1\right)$ is strictly convex in β_1.[17] The second fact is that $\mathbf{I}\left(\beta_1, \mu_1\right) = 0$ whenever $\beta_1\left(\omega_1\right) = 0$ for all ω_1. The second fact implies that B can always gauarantee herself zero by rejecting S's offer for sure, while the first implies that B's equilibrium strategy is a strict best response. Since B chooses to accept S's offer with some positive probability in equilibrium, it must be that B's expected utility is strictly positive.

In a repeated-offers bargaining environment, costly attention has an additional surplus-generating effect: It generates a Coasian effect (see, for example, Fudenberg et al. (1985) and Gul et al. (1986)). Since the Coasian is at its clearest when offers are frequent, I demonstrate this effect by examining the limit in which attention costs vanish when Δ is zero. Refer to $\left(\bar{U}_S, \bar{U}_B\right)$ as *κ-frequent offer utilities* whenever there exists a refining sequence with attention cost parameter κ such that $\mathbb{E}\left[U_B^n\right]$ and $\mathbb{E}\left[U_S^n\right]$ converge to $\bar{U}_B$ and $\bar{U}_S$, respectively. Thus, $\bar{U}_S$ ($\bar{U}_B$) is S's (B's) expected utility in some frequent-offers environment. The following lemma establishes that frequent offer utilities exist.

Lemma 2. *Every refining sequence has a subsequence for which both $E[U_b^n]$ and $E\left[U_s^n\right]$ converge.*

[17]Here, β_1 is defined up to μ_1-almost sure equivalence. The fact follows from $x \ln x + (1-x)\ln(1-x)$ being strictly convex.

Theorem 5 states that B and S split the uncertain portion of the surplus when attention costs become negligible in a frequent-offers environment. S still appropriates the sure portion of the surplus, which is v_l. However, the rest of the surplus is split evenly between the two players. In addition, in this limit there is no inefficiency. Thus, no surplus is lost due either to delay or to costly attention.

Theorem 5. *For any sequence* $\left(\bar{U}_S^n, \bar{U}_B^n\right)_{n=1}^{\infty}$ *of* κ_n*-frequent offer utilities with* $\kappa_n \to 0$*,*

$$\begin{aligned} \lim_{n\to\infty} \bar{U}_S^n &= \frac{1}{2}\mathbb{E}\left[v + v_l\right] \\ \lim_{n\to\infty} \bar{U}_B^n &= \frac{1}{2}\mathbb{E}\left[v - v_l\right]. \end{aligned}$$

To understand Theorem 5, it is worth revisiting the literature on the Coase conjecture. In this literature, a seller makes repeated offers to a buyer with private information about her value. If the lowest value is above the seller's marginal cost, the buyer's equilibrium strategy is described by a maximal willingness to pay for each value. The seller then attempts to optimally price discriminate, given his impatience and the buyer's demand. As offers become infinitely frequent, the seller goes down the demand curve within an arbitrary small amount of calendar time. Anticipating the decrease in prices, the buyer lowers her willingness to pay. As a result, the seller's offers converge to the value of the last buyer on the demand curve in a calendar instant. The outcome is a high surplus to the buyer and instant trade.

A similar dynamic arises when B is rationally inattentive. With rational inattention, B has private information about her signals. These signals serve a role similar to that of values in the Coase conjecture. When κ is positive, the Coasian argument fails due to S's inability to rapidly go down the demand curve. When κ vanishes, delay disappears, thereby unleashing the Coasian effect. Therefore, in the limit, trade happens immediately at B's lowest willingness to pay across time.

The key to understanding Theorem 5 thus comes from a S's last offer, given v. One can show that this last offer is made to B, who believes that the product's quality is v_l with probability approaching 1. Intuitively, this is because the parties trade faster when S has a higher-quality product (Theorem 4). As such, the key is to characterize S's last offer as B puts a probability approaching 1 on the quality being v_l.

I now provide a heuristic derivation of S's last offer, given v. For this, consider the limit of S's offers as the initial distribution of values converges to putting a probability of 1 on v_l. From continuity, the players' strategies in the limit must still satisfy equations 3 and 5. Moreover, these equations must hold with $\pi_m = \pi_{v_l}^{\kappa,\Delta}$ for all m from Proposition 1 since, in the limit, S's quality is v_l with probability 1. Since π_m is constant, one can show that (3) and (5) can hold only if both $b_{m,v}$ and $z_{m,v}$ are fixed across periods at some $b_v^{\kappa,\Delta}$ and $z_v^{\kappa,\Delta}$. Before taking Δ to zero, define $\lambda_v^{\kappa,\Delta}$ and $\lambda_{v_l}^{\kappa,\Delta}$ as the constant hazard rates of the agreement date distribution implied by $b_v^{\kappa,\Delta}$ – i.e., $b_v^{\kappa,\Delta} = 1 - e^{-\Delta\lambda_v^{\kappa,\Delta}}$ and $\pi_{v_l}^{\kappa,\Delta} = 1 - e^{-\Delta\lambda_{v_l}^{\kappa,\Delta}}$. Substituting into equations 3 and 5, using equation 9 and rearranging gives the following two conditions:

$$\left(\frac{1 - e^{-\Delta\lambda_v^{\kappa,\Delta}}}{1 - e^{-\Delta\lambda_{v_l}^{\kappa,\Delta}}}\right) = e^{\frac{1}{\kappa}\left(v - z_v^{\kappa,\Delta} - \kappa\frac{\Delta}{1-e^{-r\Delta}}\left(\lambda_v^{\kappa,\Delta} - \lambda_{v_l}^{\kappa,\Delta}\right)\right)} \tag{11}$$

$$\left(\frac{e^{\Delta\lambda_v^{\kappa,\Delta}} - 1}{1 - e^{-r\Delta}}\right) = \frac{z_v^{\kappa,\Delta} - \kappa}{\kappa} \tag{12}$$

Since X is compact, for every sequence Δ^n that converges to zero, one can find a subsequence for which z_v^{κ,Δ^n} converges to some $\bar{z}_v^\kappa$. Consider, now, the fractions on the left-hand side of both equations, viewing both the numerator and the denominator as functions of Δ. By applying the mean value theorem to equation 12, one can obtain that $\lambda_v^{\kappa,\Delta^n}$ must converge to $\bar{\lambda}_v^\kappa = \frac{r}{\kappa}\left(\bar{z}_v^\kappa - \kappa\right)$. Similarly, applying the mean value theorem to equation 11 along with L'Hopital's rule gives that, in the limit,

$$\bar{\lambda}_v^\kappa / \bar{\lambda}_{v_l}^\kappa = e^{\frac{1}{\kappa}\left(v - \bar{z}_v^\kappa - \frac{\kappa}{r}\left(\bar{\lambda}_v^\kappa - \bar{\lambda}_{v_l}^\kappa\right)\right)}.$$

Note that by Proposition 1, we must have that $\bar{\lambda}_{v_l}^\kappa = \frac{r}{\kappa}\left(v_l - \kappa\right)$. Together with

$\bar{\lambda}_v^\kappa = \frac{r}{\kappa}\left(\bar{z}_v^\kappa - \kappa\right)$, one can rearrange the above equation to get:

$$\left(\frac{\bar{z}_v^\kappa - \kappa}{v_l - \kappa}\right) = e^{\frac{1}{\kappa}\left(v + v_l - 2\bar{z}_v^\kappa + \kappa\right)}.$$

Since the left-hand side remains finite and strictly positive as κ goes to zero, the right-hand side must do the same, which can only happen if $\bar{z}_v^\kappa$ converges to $\frac{1}{2}\left(v + v_l\right)$.

The above heuristic derivation may seem surprising to a reader familiar with the Coase conjecture, who may have conjectured that B's lowest willingness to pay is v_l. The reason is that a v type S knows that B will receive a positive ignal before her beliefs actually become an atom on v_l. When κ is small, this signal is extremely informative, partially offsetting B's very pessimistic beliefs.

6 More Information, Lower Surplus

This section explores the efficiency implications of revealing the product's quality to B. I compare the total surplus in the standard game to the payoffs when B knows quality but still needs to pay attention to S's offers. Applying Proposition 1 quality-by-quality, one obtains that revealing v to B results in a unique equilibrium in which the total surplus is given by $\mathbb{E}\left[v - \kappa\right]$. Proposition 4 below shows that this surplus is below the total surplus in the original game.

Proposition 4. *There exists a $\delta > 0$ such that for every equilibrium, the total expected surplus is larger than $E\left[v\right] - \kappa + \delta$.*

Proof. See the online appendix for the full proof. Here, I prove only that $\mathbb{E}\left[U_b + U_s\right] > v - \kappa$ for a given equilibrium (μ, b, z). Since $b_{m,v}$ is strictly increasing in v for every m (Proposition 3), $\bar{\mu}_{m,v}/\bar{\mu}_{m,v'}$ is decreasing with m for every $v > v'$. Therefore, $\bar{\mu}_m$ first-order stochastically dominates $\bar{\mu}_{m'}$ for every $m' > m$. Rearranging equation 3 for period 1, substituting equation 9 and taking expectations in equilibrium gives:

$$1 \;=\; \sum_v \mu_{0,v}\left(\frac{b_{1,v}}{\pi_1}\right) = \sum_v \mu_{0,v} \exp\frac{1}{\kappa}\left(v - z_{1,v} - \kappa\sum_{m=1}^{\infty} e^{-r\Delta(m-1)}\ln\left(\frac{1-\pi_m}{1-b_{m,v}}\right)\right).$$

By Jensen's inequality, the above implies that:

$$0 > \sum_v \mu_{0,v}\left(v - z_{1,v} - \kappa\sum_{m=1}^{\infty} e^{-r\Delta(m-1)}\ln\left(\frac{1-\pi_m}{1-b_{m,v}}\right)\right)$$

combined with $w_{1,v} = z_{1,v} - \kappa$ (equation 7) and equation 9,

$$\begin{aligned} 0 \;&>\; \sum_v \mu_{0,v} v - \kappa - \sum_v \mu_{0,v} w_{1,v} - \sum_v \mu_{0,v}\sum_{m=1}^{\infty} e^{-r\Delta(m-1)}\ln\left(\frac{1-\pi_m}{1-b_{m,v}}\right) \\ &=\; \mathbb{E}[v] - \kappa - \mathbb{E}[U_s] - \mathbb{E}[U_b], \end{aligned}$$

giving $\mathbb{E}[U_b + U_s] > \mathbb{E}[v] - \kappa$. □

Proposition 4 asserts that revealing the quality of the product to B lowers total surplus. Recall that when B knows v, S's simple strategy means that B's attention costs are zero. Hence, revealing v to B eliminates the inefficiency caused by B's attention costs. In contrast, the inefficiency due to delay is present in both models. The above proposition suggests that delay increases substantially when the product's quality is revealed to B. This is a consequence of total surplus turning out to be convex in the seller's expected profits, conditional on v. When B is uncertain about v, the distribution of S's conditional expected profits becomes more concentrated, thus reducing overall inefficiency. Therefore, keeping B in the dark with respect to v results in less delay, which more than compensates for B's positive attention costs.

7 Conclusion

Many transactions are inherently complex. As a consequence, the transactors often need to invest valuable resources to study their contents. In other words, transactors need to pay costly attention to the transaction's details. This paper studies the way in which complexity influences bargaining by looking at seller who makes repeated offers to a rationally inattentive buyer in an attempt to sell an indivisible product. It shows that a rationally inattentive buyer earns a strictly positive surplus (Proposition 3), even when attention costs are negligible (Theorem 5). When attention costs are positive, trade occurs with delay (Proposition 2), which is decreasing with the value of the product (Theorem 4). The resulting delay is accompanied by the buyer being unhappy ex-post after buying cheap, low-quality products, and pleased ex-post when buying expensive products of higher quality (Theorem 3, part 5). Finally, I show that in the presence of rational inattention, total surplus is higher when the buyer does not know the product's quality (Proposition 4).

References

Ausubel, LM and Raymond J Deneckere, "Reputation in bargaining and durable goods monopoly," *Econometrica*, 1989, *57* (3), 511–531.

_ **and RJ Deneckere**, "Durable goods monopoly with incomplete information," *The Review of Economic Studies*, 1992, *59* (4), 795–812.

Battigalli, P and Giovanni Maggi, "Rigidity, discretion, and the costs of writing contracts," *The American Economic Review*, 2002, *92* (4), 798–817.

Bolton, Patrick and Antoine Faure-Grimaud, "Satisficing Contracts," *Review of Economic Studies*, jan 2010, *77* (3), 937–971.

Cho, IK, "Uncertainty and delay in bargaining," *The Review of Economic Studies*, 1990, *57* (4), 575–595.

Cover, Thomas M and Joy A Thomas, *Elements of information theory 2nd edition*, Wiley-interscience, 2006.

Cramton, Peter C, "Bargaining with incomplete information: An infinite-horizon model with two-sided uncertainty," *The Review of Economic Studies*, 1984, *51* (4), 579–593.

Deneckere, Raymond J and MY Liang, "Bargaining with interdependent values," *Econometrica*, 2006, *74* (5), 1309–1364.

Dessein, Wouter, Andrea Galeotti, and Tano Santos, "Rational inattention and organizational focus," *The American Economic Review*, 2016, *106* (6), 1522–1536.

Fudenberg, Drew, David K Levine, and Jean Tirole, "Infinite-Horizon Models of Bargaining with One-Sided Incomplete Information," in "Game Theoretic Models of Bargaining, By Alvin E. Roth," Cambridge University Press, 1985.

Gul, Faruk, "Unobservable Investment and the Hold-Up Problem," *Econometrica*, 2001, *69* (2), 343–376.

_ and Hugo Sonnenschein, "On delay in bargaining with one-sided uncertainty," *Econometrica*, 1988, *56* (3), 601–611.

_ , _ , and Robert Wilson, "Foundations of dynamic monopoly and the coase conjecture," *Journal of Economic Theory*, jun 1986, *39* (1), 155–190.

Kessler, Anke S, "The value of ignorance," *The Rand Journal of Economics*, 1998, pp. 339–354.

Kreps, DM and Robert Wilson, "Sequential Equilibria," *Econometrica*, 1982, *50* (4), 863–894.

Lieb, Elliot and Michael Loss, *Analysis* 2010.

Martin, Daniel, "Strategic Pricing with Rational Inattention to Quality," *Working Paper*, 2012.

Matějka, Filip, "Rationally Inattentive Seller: Sales and Discrete Pricing," *Review of Economic Studies*, 2015, pp. 1–44.

— **and Alisdair McKay**, "Simple market equilibria with rationally inattentive consumers," *American Economic Review: Papers and Proceedings*, may 2012, *102* (3), 1–11.

— **and** — , "Rational Inattention to Discrete Choices: A New Foundation for the Multinomial Logit Model," *American Economic Review*, 2015, *105* (1), 272–298.

Myerson, Roger B, "Refinements of the Nash Equilibrium Concept," *International Journal of Game Theory*, 1978, *7* (2), 73–80.

Roesler, Anne-Katrin and Balázs Szentes, "Buyer-Optimal Learning and Monopoly Pricing," *Working Paper*, 2016, pp. 1–12.

Rubinstein, Ariel, "A bargaining model with incomplete information about time preferences," *Econometrica*, 1985, *53* (5), 1151–1172.

Rust, John, "Optimal Replacement of GMC Bus Engines: An Empirical Model of Harold Zurcher," *Econometrica*, 1987, *55* (5), pp.999–1033.

Segal, Ilya, "Complexity and Renegotiation: A Foundation for Incomplete Contracts," *Review of Economic Studies*, 1999, *66* (1), 57–82.

Selten, R, "Reexamination of the perfectness concept for equilibrium points in extensive games," *International journal of game theory*, 1975, *4* (1).

Shannon, C. E., "A Mathematical Theory of Communication," *Bell System Technical Journal*, jul 1948, *27* (3), 379–423.

Sims, Christopher A., "Stickiness," *Carnegie-Rochester Conference Series on Public Policy*, dec 1998, *49*, 317–356.

— , "Implications of rational inattention," *Journal of Monetary Economics*, apr 2003, *50* (3), 665–690.

Sobel, Joel and Ichiro Takahashi, "A multistage model of bargaining," *The Review of Economic Studies*, 1983, *50* (3), 411–426.

Steiner, Jakub, Colin Stewart, and Filip Matějka, "Rational Inattention Dynamics : Inertia and Delay in Decision-Making," 2015.

Tirole, Jean, "Cognition and incomplete contracts," *The American Economic Review*, 2009, *99* (1), 265–294.

Tsoy, Anton, "Alternating-Offer Bargaining with Private Correlated Values," 2015, pp. 1–82.

Van Nieuwerburgh, Stijn and Laura Veldkamp, "Information acquisition and under-diversification," *The Review of Economic Studies*, 2010, *77* (2), 779–805.

Woodford, Michael, "Inattention as a Source of Randomized Discrete Adjustment," 2008.

Yang, Ming, "Optimality of Debt under Flexible Information Acquisition," *Working Paper*, 2013, pp. 1–48.

_ , "Coordination with Flexible Information Acquisition," *Journal of Economic Theory*, 2015, *158*, 721–738.

A Main Paper Appendix

A.1 About Shannon's measure of mutual information

Shannon (1948) was the first to suggest the use of entropy to measure information. According to Shannon, to learn a random variable's outcome is to obtain information equal to its distribution's entropy. Based on this idea, Shannon (1948) suggested a measure of how much one learns about a variable by observing a signal. His answer is the expected difference between the entropy of the variable's unconditional and conditional distributions upon observing the signal. This quantity is now known as Shannon's measure of mutual information (Cover and Thomas, 2006).

Formally, let $\mu \in \Delta\Omega$ be a prior on Ω; that is, μ is a Borel probability measure on $\Omega \subset \mathbb{R}^k$. In period m of my bargaining game, Ω is $X^m \times V$. Suppose that μ has a finite support $\{\omega_1, \ldots, \omega_n\}$. Then, $\mathbf{H}$, the entropy of μ, is:

$$\mathbf{H}(\mu) = -\sum_{i=1}^{n} \mu(\omega_i) \ln \mu(\omega_i)$$

In information theory, entropy is interpreted as a measure of the information that one can learn about a random variable. For example, one can show that $\mathbf{H}(\mu)$ is proportional to the minimal expected number of yes or no questions needed for learning ω's value. In my model, I interpret entropy as the level of exertion needed to understand or process the information in question. That is, it is the level of attention that the buyer needs to fully understand the offer and the product's value.

Let $\bar{P}$ be the prior distribution of the signal: $P(s) = \int P(s|\omega)\mu(\mathrm{d}\omega)$. Then, given the signal $s \in S_{\mu P} := \{s \in S \mid \int P(s|\omega)\mu(\mathrm{d}\omega) > 0\}$, the posterior on Ω is:

$$\mu(E|s) = \frac{\int_E P(s|\omega)\mu(\mathrm{d}\omega)}{\int_\Omega P(s|\omega)\mu(\mathrm{d}\omega)}$$

for any Borel set $E \subset Y$. Then, *Shannon's measure of mutual information*, $\mathbf{I}(s, \omega)$, is the expected change in entropy between the prior μ and the posterior, given the signal structure P:

$$\mathbf{I}(\omega, s) = \sum_{s \in S_{\mu P}} [\mathbf{H}(\mu(\cdot)) - \mathbf{H}(\mu(\cdot|s))] \bar{P}(s), \tag{13}$$

which is the average change in entropy between the prior and the posterior distribution that results from seeing the signal generated by P.

For the general case – that is, if μ is not discrete – one can define Shannon's measure of mutual information as:

$$\mathbf{I}(\omega, s) = \int \ln\left(\frac{P(s|\omega)}{\bar{P}(s)}\right) P(\mathrm{d}s|\omega)\,\mu(\mathrm{d}\omega) \tag{14}$$

which becomes the same as equation 13 when μ is discrete.

Note that attention costs depend on B's prior. To illustrate, suppose that B only needs to pay attention to S's offers, and that the value of the product is 2. Let P be the signal structure that sends 0 if S's first offer is strictly above 2, and 1 otherwise. If B's prior about S's first offer is uniform over $[0, 2]$, then P will send 1 for sure. In this case, P is completely uninformative and, therefore, costs nothing. In contrast, P's cost would be positive had B's prior been a uniform distribution over $[1, 3]$. Hence, each single structure's informativeness and, therefore, the attention cost depends on B's prior information.

A.2 Understanding Non-Credible Attention Threats

Requiring B's best response to be credible rules out non-credible attention threats, which are most easily seen in the one-period model. In this model, there is an extreme equilibrium in which B obtains a large surplus. To illustrate, assume that $V = \{2, 4\}$, $\kappa = 1$ and that both qualities can occur with equal probability. Suppose, further, that S offers 2 for sure regardless of the product's quality. Let μ be B's beliefs, given S's strategy, σ. One can show that a necessary and sufficient condition for β to be optimal for B in this setting is to have $\beta(x, v) = 1$ μ-almost surely.[18] Therefore, the strategy defined by $\beta(x, v) = 1$ if $x = 2$ and 0 otherwise is optimal for B. Clearly, it is also optimal for S to offer 2 for sure, given β. Thus, (μ, β, σ) is a sequential equilibrium. It turns out that by using a similar construction one can support S offering for sure any x in $[2, 2 + \delta]$, where $\delta > 0$ depends on the probability of $v = 2$.

To understand how the above definition rules out non-credible attention threats, consider my previous example. Suppose that we perturb B's belief from the example by adding a probability of ϵ that S offers 6 whenever the quality of the product is 4. Let μ^ϵ denote B's perturbed beliefs. Recall that the probability of $v = 4$ is $\frac{1}{2}$. Calculating B's expected utility from using β,

[18] I solve for the buyer's general optimal strategy in the dynamic game in appendix B. To obtain that $\beta(x, v) = 1$ μ-almost surely is optimal here, one can also use the results of Woodford (2008), Yang (2015) and Matějka and McKay (2015).

given μ^ϵ, gives:

$$\mathbb{E}\left[U_b|\mu^\epsilon,\beta,\sigma\right] = 1-\epsilon-\frac{1}{2}\ln\left(\frac{2}{2-\epsilon}\right)-\frac{1}{2}\left((1-\epsilon)\ln\left(\frac{2-2\epsilon}{2-\epsilon}\right)+\epsilon\ln\left(\frac{2}{\epsilon}\right)\right),$$

where $1-\epsilon$ is B's expected transaction payoffs, while the remainder is B's attention costs.

Compare β to the following alternative strategy: $\beta'(x,v)=1$ for all x and v. That is, B accepts every x offered by any quality with probability 1. B's expected transaction payoff under β' is $1-2\epsilon$. Moreover, since β' is completely uninformative, its attention cost is 0. Therefore, B's expected utility from β', given μ^ϵ, is $1-2\epsilon$. Hence, β' is strictly better for the buyer than β if and only if:

$$\frac{1}{\epsilon}\ln\left(\frac{2}{2-\epsilon}\right)+\frac{1}{\epsilon}\ln\left(\frac{2-2\epsilon}{2-\epsilon}\right)-\ln\left(\frac{2-2\epsilon}{2-\epsilon}\right)+\ln\left(\frac{2}{\epsilon}\right)>2.$$

As ϵ goes to zero, an application of L'Hopital's rule reveals that the left-hand side goes to infinity. In other words, for all small enough ϵ B prefers β' over β. One can prove that this kind of logic extends to all perturbations involving S offering 4.

Definition 1 is a variation on Selten's (1975) perfect equilibrium and Myerson's (1978) proper equilibrium. Similar to these equilibrium concepts, I require B's strategy to be robust to mistakes. However, my robustness requirement differs from those of Selten (1975) and Myerson (1978) in two ways. First, in my formulation, B is aware of possible mistakes in her *beliefs*. While I do so for analytical convenience, the difference between mistakes in beliefs and mistakes in strategies is insubstantial in my setup. This is because by the time B chooses her period m signal structure, S's m-th offer has already been determined. All that matters for B are her beliefs over that offer – i.e., μ_m. Using beliefs that are consistent with perturbed strategies is, therefore, equivalent to perturbing beliefs directly.

Second, Definition 1 allows each history to have its own sequence of trembles. In contrast, Selten (1975) and Myerson (1978) use a single tremble sequence that puts positive weight on all histories. Putting positive weight on all histories in my setup is impossible due to the set of offers being a continuum. I circumvent the continuum issue by using a different tremble sequence for each history.

A.3 Proof of lemma 1

The proof is based on the following facts, which I bring without proof. See a standard information theory textbook (e.g. Cover and Thomas (2006)). The first fact is the *chain rule of mutual information.* The second set of facts come from two independent variables having zero mutual information.

Fact 1. *For any two signals, s_m and $s_m^{'}$:* $\mathbf{I}\left(\left(s_m, s_m^{'}\right), \omega_m | s^{m-1}\right) = \mathbf{I}\left(s_m, \omega_m | s^{m-1}\right) + \mathbb{E}\left[\mathbf{I}\left(s_m^{'}, \omega_m | s^{m-1}, s_m\right) | s^{m-1}\right]$.

Fact 2. *Let $s_m^{'}$ be independent of ω_m given (s_m, s^{m-1}). Then:*

1. $\mathbf{I}\left(s_m, \omega_m | s^{m-1}\right) = \mathbf{I}\left(\left(s_m, s_m^{'}\right), \omega_m | s^{m-1}\right)$.
2. $\mathbf{I}\left(s_{m+1}, \omega_{m+1} | s_m, s^{m+1}\right) = \mathbf{I}\left(s_{m+1}, \omega_{m+1} | s_m, s_m^{'}, s^{m-1}\right)$

I now prove that the buyer only ever randomizes over recommendation strategies. For that, let $a_m \in \{0, 1\}$ denote the buyer's choice to reject ($a_m = 0$) or accept ($a_m = 1$) the seller's offer. Fix any past signal realizations s^{m-1}. Consider replacing s_m and s_{m+1} with a_m and (s_m, s_{m+1}), respectively. Then:

$$
\begin{aligned}
&\mathbf{I}\left(s_m, \omega_m | s^{m-1}\right) + \mathbb{E}\left[e^{-r\Delta}\left(1 - a_m\right)\mathbf{I}\left(s_{m+1}, \omega_{m+1} | s^m\right) | s^{m-1}\right] = \\
&\mathbf{I}\left(\left(s_m, a_m\right), \omega_m | s^{m-1}\right) + \mathbb{E}\left[e^{-r\Delta}\left(1 - a_m\right)\mathbf{I}\left(s_{m+1}, \omega_{m+1} | s^m\right) | s^{m-1}\right] = \\
&\mathbf{I}\left(a_m, \omega_m | s^{m-1}\right) + \mathbb{E}\left[\mathbf{I}\left(s_m, \omega_m | a_m\right) + e^{-r\Delta}\left(1 - a_m\right)\mathbf{I}\left(s_{m+1}, \omega_{m+1} | s^m\right) | s^{m-1}\right] \geq \\
&\mathbf{I}\left(a_m, \omega_m | s^{m-1}\right) + e^{-r\Delta}\mathbb{E}\left[\left(1 - a_m\right)\left(\mathbf{I}\left(s_m, \omega_m | a_m, s^{m-1}\right) + \mathbf{I}\left(s_{m+1}, \omega_{m+1} | s^m\right)\right) | s^{m-1}\right] = \\
&\mathbf{I}\left(a_m, \omega_m | s^{m-1}\right) + e^{-r\Delta}\mathbb{E}\left[\left(1 - a_m\right)\left(\mathbf{I}\left(s_m, \omega_m | a_m, s^{m-1}\right) + \mathbf{I}\left(s_{m+1}, \omega_{m+1} | a_m, s^m\right)\right) | s^{m-1}\right] = \\
&\mathbf{I}\left(a_m, \omega_m | s^{m-1}\right) + e^{-r\Delta}\mathbb{E}\left[\left(1 - a_m\right)\left(\mathbf{I}\left(s_m, \omega_{m+1} | a_m, s^{m-1}\right) + \mathbf{I}\left(s_{m+1}, \omega_{m+1} | a_m, s^m\right)\right) | s^{m-1}\right] = \\
&\mathbf{I}\left(a_m, \omega_m | s^{m-1}\right) + e^{-r\Delta}\mathbb{E}\left[\left(1 - a_m\right)\left(\mathbf{I}\left(\left(s_m, s_{m+1}\right), \omega_{m+1} | a_m, s^{m-1}\right)\right) | s^{m-1}\right]
\end{aligned}
$$

where the second and last equalities following from the chain rule, and all other the equalities following from a_m being independent of ω_m given s^m.

The inequality follows from conditional mutual information being positive. Conduct now the following replacements: first replace s_1 with a_1, and s_2 with (s_1, s_2), then replace (s_1, s_2) in period 2 with a_2 and s_3 with (s_1, s_2, s_3) and so on. By the above inequality, each such replacement weakly lowers the buyer's expected costs. Since a's distribution is the same as the original, the replacements do not influence the buyer's transaction payoffs. Thus, the buyer is weakly better off with the resulting recommendation strategy.

I now prove that it is without loss to assume that the buyer does not mix. A buyer playing a mixed strategy is equivalent to one that condition a_m's distribution on y, random variable independent of v which the buyer observes for free at the beginning of the game. The law of iterated expectations implies that the buyer's expected transaction payoffs do not change if she does not condition a_m on y. I now show that not conditioning on y does not increase the buyer's expected costs, proving that one can restrict attention to pure strategies. To do so, I show by induction that the buyer cannot lose in expectation by waiting to condition a_m's distribution on y. Note that y is independent of ω_1 and so $\mathbf{I}(y, \omega_1) = 0$. Therefore,

$$
\begin{aligned}
\mathbb{E}\left[\mathbf{I}\left(a_1, \omega_1 | y\right)\right] &= \mathbb{E}\left[\mathbf{I}\left(a_1, \omega_1 | y\right)\right] + \mathbf{I}\left(y, \omega_1\right) \\
&= \mathbb{E}\left[\mathbf{I}\left(\left(a_1, y\right), \omega_1\right)\right] \\
&= \mathbf{I}\left(a_1, \omega_1\right) + \mathbb{E}\left[\mathbf{I}\left(y, \omega_1 | a_1\right)\right] \\
&= \mathbf{I}\left(a_1, \omega_1\right) + \mathbb{E}\left[\mathbf{I}\left(y, \omega_2 | a_1\right)\right] \\
&\geq \mathbf{I}\left(a_1, \omega_1\right) + e^{-r\Delta}\mathbb{E}\left[\left(1 - a_1\right)\mathbf{I}\left(y, \omega_2 | a_1\right)\right]
\end{aligned}
$$

Where the second equalities follow from the chain rule of mutual information, the fourth equality follows from x_2 being independent of (y, a_1) conditional on $\omega_1 = (x_1, v)$, and the last from positivity of conditional mutual information. Hence, delaying the conditioning on y by one period does not increase costs. I now show that delaying the conditioning on y from period k to period $k+1$

does not reduce expected costs. In particular,

$$
\begin{aligned}
&\mathbb{E}\left[\mathbf{I}\left(a_{k+1},\omega_{k+1}|a^{k},y\right)+\mathbf{I}\left(y,\omega_{k+1}|a^{k}\right)|a^{k}\right]= \\
&\mathbb{E}\left[\mathbf{I}\left(\left(a_{k+1},y\right),\omega_{k+1}|a^{k}\right)|a^{k}\right]= \\
&\mathbb{E}\left[\mathbf{I}\left(a_{k+1},\omega_{k+1}|a^{k}\right)+\mathbf{I}\left(y,\omega_{k+1}|a^{k+1}\right)|a^{k}\right]= \\
&\mathbb{E}\left[\mathbf{I}\left(a_{k+1},\omega_{k+1}|a^{k}\right)+\mathbf{I}\left(y,\omega_{k+2}|a^{k+1}\right)|a^{k}\right]\geq \\
&\mathbb{E}\left[\mathbf{I}\left(a_{k+1},\omega_{k+1}|a^{k}\right)+\left(1-a_{k+1}\right)e^{-r\Delta}\mathbf{I}\left(y,\omega_{k+2}|a^{k+1}\right)|a^{k}\right]
\end{aligned}
$$

where the steps follow the same logic as in one period. Therefore, indefinitely delaying the conditioning on y can only increase the buyer's expected value. As such, it is without loss to assume that the buyer does not mix.

A.4 Other proofs

See online appendix.

Online Proofs Appendix

(For Online Publication)

B The Buyer's Problem

In the current section I provide a set of necessary and sufficient conditions for solving B's problem. The key to the solution is recasting B's problem of choosing a recommendation strategy as a problem of choosing a conditional cumulative distribution function given ω_m. Recasting B's problem in this way gives a concave objective function which is amenable to elementary variational techniques.

Thus, let $M = \{1, \ldots, M\}$ be the game's periods and fix some strategy for S, σ. Combining σ with μ generates a Borel probability measure, μ, over $\Omega := V \times X^M$. As in the main text, ω_m represents ω's projection onto $V \times X^m$. In what follows, I extend any function on $V \times X^m$ to the domain Ω by taking projections. I start with stating B's original problem, namely finding the optimal recommendation strategy. Let $\mu_{\beta,m}$ be B's posterior over Ω conditional on rejecting S's first $m-1$ offers using the recommendation strategy β:

$$\mu_{\beta,m}(\mathrm{d}\omega) = \left[\frac{\prod_{j=1}^{m-1}(1-\beta_j(\omega))}{\int \prod_{j=1}^{m-1}(1-\beta_j(\omega))\,\mu(\mathrm{d}\omega)}\right]\mu(\mathrm{d}\omega)$$

Define $\pi_m(\beta) := \int \beta_m \mu_{\beta,m}(\mathrm{d}\omega)$. Then B's objective is:

$$\mathcal{U}(\beta) := \int \left\{\sum_{m=1}^{M} e^{-r\Delta j}\left[\prod_{j=1}^{m-1}(1-\beta_j)\left(\beta_m v_m - \kappa \mathbf{I}_m(\beta_m)\right)\right]\right\}\mu(\mathrm{d}\omega) \tag{15}$$

where $v_m(\omega) = v - x_m$ is B's transaction payoff, and $\mathbf{I}_m$ is the mutual information between β_m and $\mu_{\beta,m}$:

$$\mathbf{I}_m(\beta_m) = \int \left[\beta_m \ln\left(\frac{\beta_m}{\pi_m(\beta)}\right) + (1-\beta_m)\ln\left(\frac{1-\beta_m}{1-\pi_m(\beta)}\right)\right]\mu_{\beta,m}(\mathrm{d}\omega)$$

Let $\mathbf{B}$ be the set of all recommendation strategies endowed with $L^1(\mu)$ norm. The goal of what follows is to prove a characterization theorem for the optimal β. To introduce this theorem, let:

$$V_m(\beta,\omega) = \beta_m v_m - \kappa\beta_m \ln\left(\frac{\beta_m}{\pi_m(\beta)}\right) - \kappa(1-\beta_m)\ln\left(\frac{1-\beta_m}{1-\pi_m(\beta)}\right)$$

take $\mu(\cdot|\omega_m)$ to be a version of the μ's conditional probability, and define for every $n \geq m$:

$$\tilde{\mathcal{U}}_{n,m}(\beta|\omega_m) = \int \left\{\sum_{k=n}^{M} e^{-r\Delta k}\left[\prod_{j=n}^{k-1}(1-\beta_j)\,V_k(\beta,\omega)\right]\right\}\mu(\mathrm{d}\omega|\omega_m)$$

Given the above definition, we can prove the following Theorem:

Theorem 6. *Solving B's problem:*

1. *β maximizes (15) in $\mathbf{B}$ only if:*

$$\beta_m(\omega_m) = \frac{\pi_m(\beta)\,e^{\frac{1}{\kappa}v_m(\omega_m)}}{\pi_m(\beta)\,e^{\frac{1}{\kappa}v_m(\omega_m)} + (1-\pi_m(\beta))\,e^{\frac{1}{\kappa}\tilde{\mathcal{U}}_{m+1,m}(\beta|\omega_m)}} \tag{16}$$

   where it is understood that for $\beta_m(\omega_m) = \pi_m(\beta)$ whenever $\pi_m(\beta) \in \{0,1\}$.

2. *If β satisfies equation 16 with $\pi_m(\beta) \in (0,1)$ for all m as well as:*

$$\lim_{j\to\infty}\mathbb{E}\left[e^{-r\Delta(j-m)}\ln\left(\frac{1-\pi_j(\beta)}{1-\beta_j(\omega_j)}\right)|\omega_m\right] = 0 \tag{17}$$

   μ-almost surely, then β_m is optimal.

To prove the result I begin by recasting B's objective as being defined over a different domain. Denote by $\mathbf{F}$ the set of all measurable functions, $f : M \times \Omega \to [0,1]$ satisfying:

1. $f_m(\omega) = f_m(\omega')$ whenever ω and ω' satisfy $\omega_m = \omega'_m$.

2. $F_m(\omega) := \sum_{j=1}^{m} f_j(\omega) \leq 1$ for all m.

I endow $\mathbf{F}$ with the $L^1(\mu)$ norm. Note that every β generates a f via: $f_m^\beta := \beta_m \prod_{j=1}^{m-1}(1-\beta_j)$. Moreover, every f generates a β via:

$$\beta_m^f = \frac{f_m}{1-F_{m-1}}$$

unless $F_{m-1}(\omega)=1$, in which case one can define $\beta_m^f(\omega)=0$. Let:

$$u_m(f,\omega) = f_m v_m - \kappa\left(f_m \ln\left(\frac{\beta_m^f}{\pi_m(\beta^f)}\right) + (1-F_m)\ln\left(\frac{1-\beta_m^f}{1-\pi_m(\beta^f)}\right)\right)$$

then B's objective can be written as:

$$\mathcal{U}(f) := \mathcal{U}\left(\beta^f\right) = \int \sum_{m=1}^{M} e^{-r\Delta m} u_m(f,\omega)\,\mu(\mathrm{d}\omega)$$

I therefore use $\mathcal{U}(f)$ to obtain the conditions for maximizing B's utility . I begin by proving that $\mathcal{U}(f)$ is concave. Concavity of $\mathcal{U}(f)$ is useful for two reasons. First, it allows me to show integrability of certain limits. This is acheived using Lemma 6 below. Second, concavity of $\mathcal{U}(f)$ means that the having a Gateaux derivative of zero is sufficient for obtaining a maximum.

To prove that $\mathcal{U}(f)$ is concave, let $\mathbf{G}$ be the set of all measurable functions $g:\Omega\to[0,1]$. Define:

$$\begin{aligned}\varphi : \Omega\times\mathbf{G} &\to \mathbb{R}\\ (g,\omega) &\mapsto g(\omega)\ln\left(\frac{g(\omega)}{\int g(\omega)\,\mu(\mathrm{d}\omega)}\right)\end{aligned}$$

(setting φ to zero whenever $g(\omega)=0$ or $\int g(\omega)\,\mu(\mathrm{d}\omega)=0$). Noting that:

$$u_m(f,\omega) \;=\; f_m(\omega)\,v_m(\omega) - \kappa\left(\varphi_\omega(f_m) + \varphi_\omega(1-F_m) - \varphi_\omega(1-F_{m-1})\right)$$

then one can take $K_m = 1 - e^{-r\Delta}\mathbf{1}_{[m<M]}$ and define,

$$u_m^*(f,\omega) \;=\; f_m v_m - \kappa\varphi_\omega(f_m) - \kappa K_m \varphi_\omega(1-F_m)$$

which, after some algebra reveals, that:

$$\mathcal{U}(f) = \int \sum_{m=1}^{M} e^{-r\Delta m} u_m^* (f, \omega) \, \mu (\mathrm{d}\omega) \tag{18}$$

Concavity of $\mathcal{U}(f)$ follows if φ_ω is convex. I prove convexity of φ_ω in the next two lemmas.

Lemma 3 (Log-Sum inequality)**.** *Let $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ be non-negative numbers. Then:*

$$\sum_{i=1}^{n} a_i \ln \frac{a_i}{b_i} \geq \left(\sum_{i=1}^{n} a_i \right) \ln \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

with equality if and only if $\frac{a_i}{b_i}$ is constant.

Proof. The function $g(c) = c \ln c$ is strictly convex since $g''(c) = \frac{1}{c} > 0$. Set $c_i = \frac{a_i}{b_i}$ and set $\alpha_i = \frac{b_i}{\sum_{i=1}^n b_i}$. Then by Jensen's inequality:

$$\sum_{i=1}^{n} \frac{a_i}{\sum_j b_j} \ln \frac{a_i}{b_i} = \sum_i \alpha_i g(c_i) > g\left(\sum_i \alpha_i c_i \right) = \frac{\sum_{i=1}^n a_i}{\sum_{j=1}^n b_j} \ln \left(\frac{\sum_{i=1}^n a_i}{\sum_{j=1}^n b_j} \right)$$

the lemma follows. □

Lemma 4. *$\varphi_\omega(g)$ is convex.*

Proof. Fix g and g'. Then by the log-sums inequality (3):

$$\begin{aligned} &\alpha g(\omega) \ln \left(\frac{g(\omega)}{\int g \mathrm{d}\mu} \right) + (1-\alpha) g'(\omega) \ln \left(\frac{g'(\omega)}{\int g' \mathrm{d}\mu} \right) \\ \geq &\left(\alpha g(\omega) + (1-\alpha) g'(\omega) \right) \ln \left(\frac{\alpha g(\omega) + (1-\alpha) g'(\omega)}{\int \alpha g + (1-\alpha) g' \mathrm{d}\mu} \right) \end{aligned}$$

as required. □

Lemma 5. *$\mathcal{U}(f)$ is concave.*

Proof. As noted in equation 18above, $\mathcal{U}(f) = \int \sum_{m=1}^{M} e^{-r\Delta m} u_m^* (f, \omega) \mu(\mathrm{d}\omega)$. The result immediately follows from lemma 4. □

I now turn to proving Theorem 6. I begin with the necessary conditions for maximum, and then continue to sufficiency.

B.1 Proof of Theorem 6 Part 1

To prove Theorem 6 part 1, I suppose f is optimal. Given that, I show that the functions $\frac{\varphi_\omega(1-F_m)}{1-F_m}$, $\sum_{j=m}^{M} e^{-r\Delta(j-1)} \frac{\varphi_\omega(1-F_j)}{1-F_j}$, and $\frac{1}{f_m}\varphi_\omega(f_m)$ are integrable. Integrability of these functions then assures me two things. First, the Gateaux derivative of $\mathcal{U}$ at f is well defined. Second, f_m (F_m) attains a value of zero (one) only if f_m (F_m) is zero (one) μ-almost surely. This therefore assures me that $\beta_m^f(\omega)$ is zero (one) only if $\pi_m\left(\beta^f\right)$ is zero (one). In other words, $\beta_m^f(\omega) = 0$ $(= 1)$ with positive probability if and only if $\beta_m^f = 0$ $(= 1)$ almost surely. It remains then to deal with an interior f. For an interior f, I consider what happens to B's objective as a result of a particular perturbations. This perturbation increases $f_m(\omega)$ while reducing $f_{m+j}(\omega)$ for all $j \geq 1$ proportionally. f being interior means that the perturbation is feasible both in the positive and in the negative direction. As such, the Gateaux derivative at that point must be zero, which implies part 1 of Theorem 6.

I now turn to proving that $\frac{\varphi_\omega(1-F_m)}{1-F_m}$, $\sum_{j=m}^{M} e^{-r\Delta(j-1)} \frac{\varphi_\omega(1-F_j)}{1-F_j}$, and $\frac{1}{f_m}\varphi_\omega(f_m)$ are integrable. I do so in a few steps:

Lemma 6. *Let $g : [0,1] \to \mathbb{R}$ be concave. Then for all $0 \leq \gamma \leq \epsilon < 1$:*

$$\frac{g(1) - g(\epsilon)}{1-\epsilon} \leq \frac{g(1) - g(\gamma)}{1-\gamma}$$

Proof. Set $\delta = \frac{1-\epsilon}{1-\gamma}$. By concavity: $g(\epsilon) \geq \delta g(\gamma) + (1-\delta) g(1)$. Therefore: $g(\epsilon) \geq g(1) - \delta(g(1) - g(\gamma))$, or: $\delta(g(1) - g(\gamma)) \geq g(1) - g(\epsilon)$. Dividing both sides by $1-\epsilon$ gives the desired inequality. $\square$

Lemma 7. *For every $g : \Omega \to [0,1]$: (1)For all $a \neq 0$: $\varphi_\omega(ag) = a\varphi_\omega(g)$; (2) φ_ω is bounded from below by $-1/e$; and (3) $\varphi_\omega(g)$ is integrable.*

Proof. (1) By definition: $\varphi_\omega(ag) = ag(\omega)\ln\left(\frac{ag(\omega)}{\int ag\mu(\mathrm{d}\omega)}\right) = a\varphi_\omega(g)$. (2) Since $g \in [0,1]$: $\varphi(g) \geq g\ln g$. The result then follows from $\min\{\alpha\ln\alpha : \alpha \in [0,1]\} =$

$-1/e$. (3) By log-sums inequality: $\int g \ln \frac{g}{\int g\mu(\mathrm{d}\omega)} \mu(\mathrm{d}\omega) \leq 0$. Result then follows from (2). $\square$

Lemma 8. *Suppose f maximizes $\mathcal{U}(f)$. Then both $\frac{\varphi(1-F_m)}{1-F_m}$ and $\sum_{j=m}^{M} e^{-r\Delta(j-1)} \frac{\varphi(1-F_j)}{1-F_j}$ are μ-integrable for all m.*

Proof. We will begin by showing that the second expression is integrable for $m = 1$. Suppose otherwise. Since $\sum_{j=1}^{M} e^{-r\Delta(j-1)} \varphi(1-F_j) \geq 0$ is integrable, this implies that $\sum_{j=1}^{M} e^{-r\Delta(j-1)} \frac{F_j}{1-F_j} \varphi(1-F_j)$ is not integrable. Take f^0 to be such that $f_j^0 = 0$ for all j. Then by optimality of f and lemma 6 we have that for all $0 \leq \epsilon < 1$:

$$0 \leq \frac{\mathcal{U}(f) - \mathcal{U}(\epsilon f + (1-\epsilon) f^0)}{1-\epsilon} \leq \mathcal{U}(f) - \mathcal{U}(f^0) \tag{19}$$

Letting:

$$\zeta_j(\omega,\epsilon) = \frac{1}{1-\epsilon} \Big(\varphi_\omega(1-F_j) - \varphi_\omega(1-\epsilon F_j) \Big)$$

one can write $(1-\epsilon)^{-1} (\mathcal{U}(f) - \mathcal{U}(\epsilon f + (1-\epsilon) f^0))$ as:

$$\int \sum_{j=1}^{M} e^{-r\Delta(j-1)} \Big(f_j v_j - \kappa\varphi(f_j) - \kappa C_j \zeta_j(\omega,\epsilon) \Big) \mu(\mathrm{d}\omega)$$

Since v_j is bounded, and both φ and ζ are integrable, equation 19 can be rewritten as:

$$\begin{aligned}
\frac{1}{\kappa} \int \sum_{j=1}^{M} e^{-r\Delta(j-1)} (f_j v_j - \kappa\varphi(f_j)) \mu(\mathrm{d}\omega) &\geq \int \sum_{j=1}^{M} K_j e^{-r\Delta(j-1)} \zeta_j(\omega,\epsilon) \mu(\mathrm{d}\omega) \\
&\geq \frac{1}{\kappa} \int \sum_{j=1}^{M} e^{-r\Delta(j-1)} (f_j v_j - \kappa\varphi(f_j)) \mu(\mathrm{d}\omega) - \big(\mathcal{U}(f) - \mathcal{U}(f^0) \big)
\end{aligned}$$

Note that the above holds for all $\epsilon < 1$. As such, there exists a sequence ϵ_l with $\epsilon_l \to 1$ and that $\int \sum_{j=1}^{M} K_j e^{-r\Delta(j-1)} \zeta_j(\omega,\epsilon) \mu(\mathrm{d}\omega)$ converges to some finite limit $L^{\infty} < \infty$. Note that:

$$\begin{aligned}
\lim_{l\to\infty} \zeta_j\left(z_M, \epsilon_l\right) &= \left.\frac{\mathrm{d}\varphi\left(1-\epsilon F_j\right)}{\mathrm{d}\epsilon}\right|_{\epsilon=1} \\
&= -\frac{F_j}{1-F_j}\varphi\left(1-F_j\right) + \left(\frac{\int F_j \mu\left(\mathrm{d}\omega\right)}{\int\left(1-F_j\right)\mu\left(\mathrm{d}\omega\right)} - \frac{F_j}{1-F_j}\right)\left(1-F_j\right)
\end{aligned} \tag{20}$$

Applying the log sums inequality,

$$\varphi\left(1-F_j\right) + \varphi\left(\left(1-\epsilon\right)F_j\right) \geq \varphi\left(1-\epsilon F_j\right)$$

which implies:

$$\zeta_j\left(\omega, \epsilon\right) \geq -\varphi\left(F_j\right) \tag{21}$$

for every ω and all ϵ, and therefore:

$$\sum_{j=1}^{M} e^{-r\Delta(j-1)}\zeta_j\left(\omega, \epsilon\right) \geq -\sum_{j=1}^{M} e^{-r\Delta(j-1)}\varphi\left(F_j\right)$$

which is integrable. Hence by Fatou's lemma:

$$\begin{aligned}
\lim_{l\to\infty}\int \sum_{j=1}^{M} e^{-r\Delta(j-1)}\zeta_j\left(z_M, \epsilon_l\right)\mu\left(\mathrm{d}\omega\right) &= \liminf_{l\to\infty}\int \sum_{j=1}^{M} e^{-r\Delta(j-1)}\zeta_j\left(\omega, \epsilon_l\right)\mu\left(\mathrm{d}\omega\right) \\
&\geq \int \liminf_{l\to\infty}\sum_{j=1}^{M} e^{-r\Delta(j-1)}\zeta_j\left(\omega, \epsilon_l\right)\mu\left(\mathrm{d}\omega\right) \\
&= \int \sum_{j=1}^{M} e^{-r\Delta(j-1)}\lim_{l\to\infty}\zeta_j\left(z_M, \epsilon_l\right)\mu\left(\mathrm{d}\omega\right)
\end{aligned}$$

But:

$$\int\left(\frac{\int F_j \mu\left(\mathrm{d}\omega\right)}{\int\left(1-F_j\right)\mu\left(\mathrm{d}\omega\right)} - \frac{F_j}{1-F_j}\right)\left(1-F_j\right)\mu\left(\mathrm{d}\omega\right) = 0$$

for all j then suggests that $\int \sum_{j=1}^{M} e^{-r\Delta(j-1)}\frac{F_j}{1-F_j}\varphi\left(1-F_j\right)\mu\left(\mathrm{d}\omega\right)$ is bounded from above. Moreover, since equation 21 holds for all ϵ, it must also hold in the limit, i.e.

$$\sum_{j=1}^{M} e^{-r\Delta(j-1)} \lim_{l\to\infty} \zeta_j(\omega, \epsilon_l) \geq -\sum_{j=1}^{M} e^{-r\Delta(j-1)} \varphi(F_j)$$

implying that $\int \sum_{j=1}^{M} e^{-r\Delta(j-1)} \frac{F_j}{1-F_j} \varphi(1-F_j) \mu(\mathrm{d}\omega)$ is also bounded from below, a contradiction. Therefore $\sum_{j\geq 1} e^{-r\Delta(j-1)} \frac{\varphi(1-F_j)}{1-F_j}$ is integrable.

I now prove that $\frac{\varphi(1-F_m)}{1-F_m}$ and $\sum_{j=m+1}^{M} e^{-r\Delta(j-1)} \frac{\varphi(1-F_j)}{1-F_j}$ are both integrable for all m. Suppose both are for all $j \leq m-1$, but that one of them is not for $j = m$. Then both must not be integrable and in opposite directions since:

$$\begin{aligned} \sum_{j=1}^{M} e^{-r\Delta(j-1)} \frac{\varphi(1-F_j)}{1-F_j} &= \sum_{j=1}^{m-1} e^{-r\Delta(j-1)} \frac{\varphi(1-F_j)}{1-F_j} \\ &\quad + \sum_{j=m+1}^{M} e^{-r\Delta(j-1)} \frac{\varphi(1-F_j)}{1-F_j} + e^{-r\Delta m} \frac{\varphi(1-F_m)}{1-F_m} \end{aligned}$$

is integrable. Using equations 21 and 20 for every j we obtain that:

$$-\frac{F_j}{1-F_j} \varphi(1-F_j) + \left(\frac{\int F_j \mu(\mathrm{d}\omega)}{\int (1-F_j) \mu(\mathrm{d}\omega)} - \frac{F_j}{1-F_j} \right) (1-F_j) \geq -\varphi(F_j)$$

for all j. Since

$$\int \left(\frac{\int F_j \mu(\mathrm{d}\omega)}{\int (1-F_j) \mu(\mathrm{d}\omega)} - \frac{F_j}{1-F_j} \right) (1-F_j) \mu(\mathrm{d}\omega) = 0$$

it must be that $\frac{F_j}{1-F_j} \varphi(1-F_j) \leq \infty$ for all j, a contradiction. $\square$

The above lemma immediately leads to the following result:

Corollary 1. *Assume f maximizes $\mathcal{U}(f)$. Then $\mu\{F_m = 1\} > 0$ implies $\mu\{F_m = 1\} = 1$.*

Lemma 9. *Suppose f maximizes $\mathcal{U}$. Then the function $\frac{1}{f_m} \varphi(f_m)$ is integrable for all m.*

Proof. Suppose, by contradiction, that m is such that $\frac{1}{f_m}\varphi(f_m)$ is not integrable. Since $\varphi(f_m)$ is integrable, it must be that $\frac{f_m-1}{f_m}\varphi(f_m)$ is not. Let f^+ be such that $f_j^+=0$ for all $j\neq m$ and $f_m^+=1$ for every ω. Then by concavity of $\mathcal{U}$ and optimality of f, we have for all $0\leq\epsilon<1$:

$$0\leq\frac{\mathcal{U}(f)-\mathcal{U}(\epsilon f+(1-\epsilon)f^+)}{1-\epsilon}\leq\mathcal{U}(f)-\mathcal{U}\left(f^+\right)=\mathcal{U}(f)-e^{-r\Delta m}\int v_m\mu(\mathrm{d}\omega)$$

Define $\zeta_j(\omega,\epsilon)$ as in lemma 8 and $\xi_m(\omega,\epsilon)=\frac{1}{1-\epsilon}(\varphi_\omega(f_m)-\varphi_\omega(\epsilon f_m+1-\epsilon))$, we have that $\frac{1}{1-\epsilon}(\mathcal{U}(f)-\mathcal{U}(\epsilon f+(1-\epsilon)f^+))$ is equal to:

$$\int\left(-v_m+\sum_{j=1}^{M}e^{-r\Delta(j-1)}f_jv_j\right)\mu(\mathrm{d}\omega)-\kappa\sum_{j\neq m}e^{-r\Delta(j-1)}\varphi(f_j)-\kappa\sum_{j=m}^{M}e^{-r\Delta(j-1)}K_j\varphi(1-F_j)$$
$$-\kappa e^{-r\Delta m}\int\xi_m(\omega,\epsilon)\mu(\mathrm{d}\omega)-\kappa\left(1-e^{-r\Delta}\right)\int\sum_{j=1}^{m-1}e^{-r\Delta(j-1)}\zeta_j(\omega,\epsilon)\mu(\mathrm{d}\omega)$$

and therefore we obtain that there are $\underline{L}$ and $\bar{L}$ such that:

$$\underline{L}\leq e^{-r\Delta(m-1)}\int\xi_m(\omega,\epsilon)\mu(\mathrm{d}\omega)+\left(1-e^{-r\Delta}\right)\int\sum_{j=1}^{m-1}\zeta_j(\omega,\epsilon)\mu(\mathrm{d}\omega)\leq\bar{L}$$

for all ϵ. Thus, there exists a sequence $\epsilon_l\to1$ such that:

$$\int e^{-r\Delta(m-1)}\xi_m(\omega,\epsilon_l)+\left(1-e^{-r\Delta}\right)\sum_{j=1}^{m-1}e^{-r\Delta(j-1)}\zeta_j(\omega,\epsilon_l)\mu(\mathrm{d}\omega)\to\bar{L}_\infty$$

where $\bar{L}_\infty\in\left[\underline{L},\bar{L}\right]$. Using the log-sum inequality (Lemma 3) we have

$$\varphi(1-F_j)+\varphi((1-\epsilon)F_j)\geq\varphi(1-\epsilon F_j)$$

implying that for all ϵ:

$$\sum_{j=1}^{m-1}e^{-r\Delta(j-1)}\zeta_j(\omega,\epsilon)\geq-\sum_{j=1}^{m-1}e^{-r\Delta(j-1)}\varphi(F_j)$$

while using the log-sum inequality (3):

$$\varphi\left(f_m\right)+\varphi\left(\left(1-\epsilon\right)\left(1-f_m\right)\right)\geq\varphi\left(\epsilon f_m+1-\epsilon\right)$$

gives:

$$\xi_m\left(\omega,\epsilon\right)\geq\varphi\left(1-f_m\right) \tag{22}$$

we can therefore use Fatou's lemma to obtain:

$$\begin{aligned}\bar{L}_\infty &\geq \int\liminf_{l\to\infty}\left(e^{-r\Delta m}\xi_m\left(\omega,\epsilon\right)+\left(1-e^{-r\Delta}\right)\sum_{j=0}^{m-1}e^{-r\Delta j}\zeta_j\left(\omega,\epsilon\right)\right)\mu\left(\mathrm{d}\omega\right)\\ &\geq \int e^{-r\Delta m}\varphi\left(1-f_m\right)\mu\left(\mathrm{d}\omega\right)\\ &\quad+\left(1-e^{-r\Delta}\right)\sum_{j=0}^{m-1}e^{-r\Delta j}\int\varphi\left(F_j\right)\mu\left(\mathrm{d}\omega\right)=:\underline{L}_\infty>-\infty\end{aligned}$$

However, for every ω:

$$\begin{aligned}\lim_{l\to\infty}\xi_m\left(\omega,\epsilon_l\right) &= \left.\frac{\mathrm{d}\varphi\left(\epsilon f_m+1-\epsilon\right)}{\mathrm{d}\epsilon}\right|_{\epsilon=1}\\ &= \frac{f_m-1}{f_m}\varphi\left(f_m\right)+f_m\left(\frac{f_m-1}{f_m}-\frac{\int f_m\mu\left(\mathrm{d}\omega\right)-1}{\int f_m\mu\left(\mathrm{d}\omega\right)}\right)\end{aligned}$$

and:

$$\begin{aligned}\lim_{l\to\infty}\zeta_j\left(\omega,\epsilon_l\right) &= \left.\frac{\mathrm{d}\varphi\left(1-\epsilon F_j\right)}{\mathrm{d}\epsilon}\right|_{\epsilon=1}\\ &= -\frac{F_j}{1-F_j}\varphi\left(1-F_j\right)+\left(1-F_j\right)\left(\frac{\int F_j\mu\left(\mathrm{d}\omega\right)}{\int\left(1-F_j\right)\mu\left(\mathrm{d}\omega\right)}-\frac{F_j}{1-F_j}\right)\end{aligned}$$

since

$$\int f_m\left(\frac{f_m-1}{f_m}-\frac{\int f_m\mu\left(\mathrm{d}\omega\right)-1}{\int f_m\mu\left(\mathrm{d}\omega\right)}\right)\mu\left(\mathrm{d}\omega\right)=0$$

and

$$\int\left(1-F_j\right)\left(\frac{-F_j}{1-F_j}+\frac{\int F_j\mu\left(\mathrm{d}\omega\right)}{\int\left(1-F_j\right)\mu\left(\mathrm{d}\omega\right)}\right)\mu\left(\mathrm{d}\omega\right)=0$$

we have that:

$$\bar{L}_\infty \geq \int \left(e^{-r\Delta m} \left(\frac{f_m - 1}{f_m} \right) \varphi\left(f_m\right) - \left(1 - e^{-r\Delta}\right) \sum_{j=1}^{m-1} e^{-r\Delta(j-1)} \frac{F_j}{1-F_j} \varphi\left(1 - F_j\right) \right) \mu\left(\mathrm{d}\omega\right) \geq \underline{L}_\infty$$

The result then follows from $\frac{F_j}{1-F_j}\varphi\left(1-F_j\right)$ being integrable for every j (lemma 8). □

Lemma 9 immediately implies the corollary:

Corollary 2. *Suppose that f maximizes $\mathcal{U}\left(f\right)$. Then $\mu\left\{f_m = 0\right\} > 0$ implies $\mu\left\{f_m = 0\right\} = 1$.*

With the above integrability results in hand, I now turn to discussing the effect of perturbing the optimal f slightly. I begin with the following standard definitions.

Definition 3. For any $f \in \mathbf{F}$, we say that the measurable function $\eta : \Omega \rightarrow \mathbb{R}$ is an *f-feasible direction* if there exists $\bar{\epsilon} > 0$ such that for all $0 < \epsilon < \bar{\epsilon}$: $f + \epsilon\eta \in \mathbf{F}$. We denote the set of f-feasible directions by $\mathbf{F}_f$.

Definition 4. $\mathcal{U}$ is *Gateaux differentiable* at f if there exists a bounded linear functional $\mathrm{d}\mathcal{U}_f : \mathbf{F}_f \rightarrow \mathbb{R}$ such that for every $\eta \in \mathbf{F}_f$:

$$\lim_{\epsilon \rightarrow 0} \left| \frac{1}{\epsilon} \left(\mathcal{U}\left(f + \epsilon\eta\right) - \mathcal{U}\left(f\right) - \mathrm{d}\mathcal{U}_f\left(\eta\right)\right) \right| = 0 \tag{23}$$

we then say that $\mathrm{d}\mathcal{U}_f$ is $\mathcal{U}$'s Gateaux derivative.

Lemma 10. *Suppose f maximizes $\mathcal{U}$. Let:*

$$\Lambda_{f,m}\left(\omega\right) = e^{-r\Delta m}\left(v_m - \kappa \frac{1}{f_m}\varphi\left(f_m\right)\right) + \kappa \sum_{j=m}^{M} e^{-r\Delta j} K_j \frac{\varphi\left(1 - F_j\right)}{1 - F_j}$$

and define $\mathrm{d}\mathcal{U}_f\left(\eta\right) = \int \sum_{m=1}^{M} \Lambda_{f,m}\eta_m \mu\left(\mathrm{d}\omega\right)$. Then $\mathrm{d}\mathcal{U}_f\left(\eta\right)$ is: (1) bounded; (2) is the Gateaux derivative of $\mathcal{U}$.

Proof. Fix some f-feasible direction η. Assume without loss of generality that $f+\eta\in\mathbf{F}$, and let $\alpha_\epsilon = 1-\epsilon$, while defining $f^\alpha = f+(1-\alpha)\eta$. Then:

$$\begin{aligned}
0 \geq \frac{1}{\epsilon}\left(\mathcal{U}\left(f+\epsilon\eta\right)-\mathcal{U}\left(h\right)\right) &= \frac{1}{1-\alpha_\epsilon}\left(\mathcal{U}\left(f^{\alpha_\epsilon}\right)-\mathcal{U}\left(f\right)\right)\\
&= \int\sum_{m=1}^{M} e^{-r\Delta m}\left(\frac{u_m^*\left(f^{\alpha_\epsilon},\omega\right)-u_m^*\left(f,\omega\right)}{1-\alpha_\epsilon}\right)\mu\left(\mathrm{d}\omega\right)
\end{aligned}$$

since u_m^* is concave (4), we have by 6:

$$\begin{aligned}
\frac{u_m^*\left(f^\alpha,\omega\right)-u_m^*\left(f,\omega\right)}{1-\alpha} &= -\left(\frac{u_m^*\left(f,\omega\right)-u_m^*\left(f^\alpha,\omega\right)}{1-\alpha}\right)\\
&\geq -\left(u_m^*\left(f,\omega\right)-u_m^*\left(f^1,\omega\right)\right)
\end{aligned}$$

and therefore:

$$\left(\frac{u_m^*\left(f^\alpha,\omega\right)-u_m^*\left(f,\omega\right)}{1-\alpha}\right)+\left(u_m^*\left(f,\omega\right)-u_m^*\left(f^1,\omega\right)\right)\geq 0$$

for all ω. Moreover, lemma 6 implies that $\left(1-\alpha\right)^{-1}\left(u_m^*\left(f^\alpha,\omega\right)-u_m^*\left(f,\omega\right)\right)$ is increasing with α. Therefore, by the monotone convergence theorem:

$$\begin{aligned}
&\lim_{\alpha\to 1}\int\sum_{m=1}^{M} e^{-r\Delta m}\left(\frac{u_m^*\left(f^\alpha,\omega\right)-u_m^*\left(f,\omega\right)}{1-\alpha}\right)\mu\left(\mathrm{d}\omega\right)\\
&=\int\sum_{m=1}^{M} e^{-r\Delta m}\lim_{\alpha\to 1}\left(\frac{u_m^*\left(f^\alpha,\omega\right)-u_m^*\left(f,\omega\right)}{1-\alpha}\right)\mu\left(\mathrm{d}\omega\right)
\end{aligned}$$

note that: $\left(\frac{f_m^\alpha v_m - f_m v_m}{1-\alpha}\right) = v_m\eta_m$. In addition:

$$\begin{aligned}
\lim_{\alpha\to 1}\left(1-\alpha\right)^{-1}\left(\varphi\left(f_m\right)-\varphi\left(f_m^\alpha\right)\right) &= \left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\varphi\left(f_m^\alpha\right)\right|_{\alpha=1}\\
&= -\eta_m\frac{\varphi\left(f_m\right)}{f_m}+\left(\frac{f_m\int\eta_m\mu\left(\mathrm{d}\omega\right)}{\int f_m\mu\left(\mathrm{d}\omega\right)}\right)-\eta_m
\end{aligned}$$

and:

$$\begin{aligned}
\lim_{\alpha \to 1} (1-\alpha)^{-1} \left(\varphi\left(1-F_m\right) - \varphi\left(1-F_m^{\alpha}\right)\right) &= \left.\frac{\mathrm{d}}{\mathrm{d}\alpha} \varphi\left(1-F_m^{\alpha}\right)\right|_{\alpha=1} \\
&= \left(\sum_{j=1}^{m} \eta_j\right) \frac{\varphi\left(1-F_m\right)}{1-F_m} \\
&\quad + \sum_{j=1}^{m} \eta_j - \frac{\left(1-F_m\right) \int \sum_{j=1}^{m} \eta_j \mu\left(\mathrm{d}\omega\right)}{\int \left(1-F_m\right) \mu\left(\mathrm{d}\omega\right)}
\end{aligned}$$

since:

$$\int \left[\left(\frac{f_m \int \eta_m \mu\left(\mathrm{d}\omega\right)}{\int f_m \mu\left(\mathrm{d}\omega\right)}\right) - \eta_m\right] \mu\left(\mathrm{d}\omega\right) = 0$$

and:

$$\int \left[\sum_{j=1}^{m} \eta_j - \frac{\left(1-F_m\right) \int \sum_{j=1}^{m} \eta_j \mu\left(\mathrm{d}\omega\right)}{\int \left(1-F_m\right) \mu\left(\mathrm{d}\omega\right)}\right] \mu\left(\mathrm{d}\omega\right) = 0$$

we obtain that equation 23 holds. Boundedness of $\mathrm{d}\mathcal{U}_f(\eta)$ for all feasible η follows from concavity of $\mathcal{U}$ and lemma 6 which imply:

$$\begin{aligned}
0 &\geq \lim_{\alpha \to 1} \frac{1}{1-\alpha} \left(\mathcal{U}\left(f^{\alpha}\right) - \mathcal{U}(f)\right) \\
&\geq \mathcal{U}\left(f^{1}\right) - \mathcal{U}(f) \geq \underline{v} - \left(\frac{\ln 2}{1-e^{-r\Delta}}\right) - \bar{v}
\end{aligned}$$

thereby concluding the proof. □

Proof of Theorem 6 Part 1. Suppose f maximizes $\mathcal{U}$. Note that by corollaries 1 and 2 the condition holds for any m such that $\mu\{f_m = 0\} > 0$ or $\mu\{f_m = 1\} > 0$. Suppose then that $\mu\{0 < f_m < 1\} = 1$. Note that:

$$\begin{aligned}
\pi_m\left(\beta^f\right) &= \int \beta_m^f \mu_{\beta^f, m}\left(\mathrm{d}\omega\right) \\
&= \int \left(\frac{f_m}{1-F_{m-1}}\right) \left(\frac{1-F_{m-1}}{\int \left(1-F_{m-1}\right) \mu\left(\mathrm{d}\omega\right)}\right) \mu\left(\mathrm{d}\omega\right) = \frac{\int f_m \mu\left(\mathrm{d}\omega\right)}{\int \left(1-F_{m-1}\right) \mu\left(\mathrm{d}\omega\right)}
\end{aligned}$$

Note that:

$$\begin{aligned}
\frac{\varphi\left(f_m\right)}{f_m}-\frac{\varphi\left(1-F_m\right)}{1-F_m} &= \ln\left(\frac{f_m/\left(1-F_{m-1}\right)}{\int\left(\frac{f_m}{1-F_{m-1}}\right)\left(\frac{1-F_{m-1}}{\int 1-F_{m-1}\mu\left(\mathrm{d}\omega'\right)}\right)\mu\left(\mathrm{d}\omega\right)}\right)\\
&\quad -\ln\left(\frac{\left(1-F_m\right)/\left(1-F_{m-1}\right)}{\int\left(\frac{1-F_m}{1-F_{m-1}}\right)\left(\frac{1-F_{m-1}}{\int 1-F_{m-1}\mu\left(\mathrm{d}\omega'\right)}\right)\mu\left(\mathrm{d}\omega\right)}\right)\\
&= \ln\left(\frac{\beta^f_m}{\pi_m\left(\beta^f\right)}\right)-\ln\left(\frac{1-\beta^f_m}{1-\pi_m\left(\beta^f\right)}\right)
\end{aligned}$$

and:

$$\begin{aligned}
\frac{\varphi\left(1-F_j\right)}{1-F_j}-\frac{\varphi\left(1-F_{j-1}\right)}{1-F_{j-1}} &= \ln\left(\frac{\left(1-F_j\right)/\left(1-F_{j-1}\right)}{\int\left(\frac{1-F_j}{1-F_{j-1}}\right)\frac{1-F_{j-1}}{\int\left(1-F_{j-1}\right)\mu\left(\mathrm{d}\omega'\right)}\mu\left(\mathrm{d}\omega\right)}\right)\\
&= \ln\left(\frac{1-\beta^f_j}{1-\pi_j\left(\beta^f\right)}\right)
\end{aligned}$$

and therefore:

$$\begin{aligned}
\Lambda_{f,m} &= e^{-r\Delta m}\left(v_m-\kappa\frac{1}{f_m}\varphi\left(f_m\right)\right)+\kappa\sum_{j=m}^{M}e^{-r\Delta j}K_j\frac{\varphi\left(1-F_j\right)}{1-F_j}\\
&= e^{-r\Delta m}v_m-\kappa e^{-r\Delta m}\left(\frac{1}{f_m}\varphi\left(f_m\right)-\frac{\varphi\left(1-F_m\right)}{1-F_m}\right)\\
&\quad +\kappa\sum_{j=m+1}^{M}e^{-r\Delta j}\left(\frac{\varphi\left(1-F_j\right)}{1-F_j}-\frac{\varphi\left(1-F_{j-1}\right)}{1-F_{j-1}}\right)\\
&= e^{-r\Delta m}\left(v_m-\kappa\ln\left(\frac{\beta^f_m}{\pi_m\left(\beta^f\right)}\right)\right)+\kappa\sum_{j=m}^{M}e^{-r\Delta j}\ln\left(\frac{1-\beta^f_j}{1-\pi_j\left(\beta^f\right)}\right) \qquad (24)
\end{aligned}$$

Fix any ω_m in the support of μ, and define f^{ω_m} as following: (1) $f^{\omega_m}_j=f_j$ if either $\omega'_m\neq\omega_m$ or both $j<m$ and $j=m$ hold; (2) $f^{\omega_m}_m\left(\omega'\right)=1-F_{m-1}\left(\omega'\right)$ whenever $\omega'_m=\omega_m$; and (3) $f^{\omega_m}_j\left(\omega'\right)=0$ if $\omega'_m=\omega_m$ and $j>m$. Let

$f^{\alpha} = \alpha f + (1-\alpha) f^{\omega_m}$. Then obviously the following:

$$\eta^{\alpha} = f^{\alpha} - f = (1-\alpha)\left(f^{\omega_m} - f\right)$$

is a feasible direction for all $\alpha \in [0,1]$. Note that $\eta = \eta^1$ satisfies $\eta_j\left(\omega'\right) = 0$ if $j < m$ or $\omega'_m \neq \omega_m$, $\eta_m(\omega_m) = 1 - F_m(\omega_m)$, while $\eta_j\left(\omega'\right) = -f_j\left(\omega'\right)$ whenever both $j > m$ and $\omega'_m = \omega_m$. Since η is feasible and f is optimal, we must have that $\mathrm{d}\mathcal{U}_f(\eta) \leq 0$. Note that the perturbation $-\eta$ is also feasible, and therefore $0 \geq \mathrm{d}\mathcal{U}_f(-\eta) = -\mathrm{d}\mathcal{U}_f(\eta)$. Hence, we must have $\mathrm{d}\mathcal{U}_f(\eta) = 0$. Therefore:

$$\int \left(\Lambda_{f,m}(\omega_m)(1-F_m) - \sum_{j=m+1}^{M} f_j \Lambda_{f,j}\right) \mu(\mathrm{d}\omega|\omega_m) = 0$$

where $\mu(\mathrm{d}\omega|\omega_m)$ is a version of μ's conditional probability. The above can be rewritten as:

$$\begin{pmatrix} (1-F_m(\omega_m))\, e^{-r\Delta m}\left(v_m - \kappa \ln\left(\dfrac{\beta^f_m}{\pi_m(\beta^f)}\right)\right) \\ +\kappa e^{-r\Delta m}(1-F_m(\omega_m)) \ln\left(\dfrac{1-\beta^f_m}{1-\pi_m(\beta^f)}\right) \end{pmatrix} = \tag{25}$$

$$\begin{pmatrix} -\kappa(1-F_m(\omega_m)) \displaystyle\int \sum_{j=m+1}^{M} e^{-r\Delta(j-1)} \ln\left(\dfrac{1-\beta^f_j}{1-\pi_j(\beta^f)}\right) \mu(\mathrm{d}\omega|\omega_m) \\ +\kappa \displaystyle\int \sum_{j=m+1}^{M} \sum_{k=j}^{M} f_j e^{-r\Delta k} \ln\left(\dfrac{1-\beta^f_k}{1-\pi_k(\beta^f)}\right) \mu(\mathrm{d}\omega|\omega_m) \\ + \displaystyle\int \sum_{j=m+1}^{M} f_j e^{-r\Delta j}\left(v_j - \kappa \ln\left(\dfrac{\beta^f_j}{\pi_j(\beta^f)}\right)\right) \mu(\mathrm{d}\omega|\omega_m) \end{pmatrix}$$

But:

$$
\begin{aligned}
&\int \sum_{j=m+1}^{M} \sum_{k=j}^{M} f_j e^{-r\Delta k} \ln\left(\frac{1-\beta_k^f}{1-\pi_k\left(\beta^f\right)}\right) \mu\left(\mathrm{d}\omega|\omega_m\right) \\
&= \int \sum_{k=m+1}^{M} \left(\sum_{j=m+1}^{k} f_j\right) e^{-r\Delta k} \ln\left(\frac{1-\beta_k^f}{1-\pi_k\left(\beta^f\right)}\right) \mu\left(\mathrm{d}\omega|\omega_m\right) \\
&= \int \sum_{k=m+1}^{M} \left(F_k - F_m\right) e^{-r\Delta(k-1)} \ln\left(\frac{1-\beta_k^f}{1-\pi_k\left(\beta^f\right)}\right) \mu\left(\mathrm{d}\omega|\omega_m\right)
\end{aligned}
$$

and therefore we obtain the equality:

$$
\begin{aligned}
&\begin{pmatrix} \left(1-F_m\left(\omega_m\right)\right) e^{-r\Delta m}\left(v_m - \kappa \ln\left(\frac{\beta_m^f}{\pi_m\left(\beta^f\right)}\right)\right) \\ +\kappa e^{-r\Delta m}\left(1-F_m\left(\omega_m\right)\right) \ln\left(\frac{1-\beta_m^f}{1-\pi_m\left(\beta^f\right)}\right) \end{pmatrix} = \\
&\int \sum_{j=m+1}^{M} e^{-r\Delta j}\left(f_j v_j - \kappa f_j \ln\left(\frac{\beta_j^f}{\pi_j\left(\beta^f\right)}\right) - \kappa\left(1-F_j\right) \ln\left(\frac{1-\beta_j^f}{1-\pi_j\left(\beta^f\right)}\right)\right) \mu\left(\mathrm{d}\omega|\omega_m\right) = \\
&\int \sum_{j=m+1}^{M} e^{-r\Delta j} u_j\left(f,\omega\right) \mu\left(\mathrm{d}\omega|\omega_m\right)
\end{aligned}
$$

thus, dividing both sides of equation 25 by $e^{-r\Delta m}\left(1-F_m\left(\omega_m\right)\right)$ gives:

$$
\begin{aligned}
v_m - \kappa \ln\left(\frac{\beta_m^f}{\pi_m\left(\beta^f\right)}\right) + \kappa \ln\left(\frac{1-\beta_m^f}{1-\pi_m\left(\beta^f\right)}\right) &= \int \sum_{j=m+1}^{M} e^{-r\Delta j}\left(\frac{u_j\left(f,\omega\right)}{1-F_m\left(\omega_m\right)}\right) \mu\left(\mathrm{d}\omega|\omega_m\right) \\
&= \tilde{\mathcal{U}}_{m+1,m}\left(\beta|\omega_m\right)
\end{aligned}
$$

dividing both sides by κ, exponentiating and solving for β_m^f proves that β_m^f satisfies equation 16. The theorem follows.

B.2 Proof of Theorem 6 Part 2

I now turn to proving part 2 of Theorem 6. I begin by providing an expression for B's expected utility for any β satisfying the conditions of Part 2. Using this expression, I show that the Gateaux derivative of B's objective is zero at any β satisfying part 2 of the theorem. The result then follows from concavity of $\mathcal{U}(f)$.

Lemma 11. *Suppose β satisfies equation 16. If $\mu\{0<\beta_1<1\}>0$. Then:*

$$\mathcal{U}(\beta)=\kappa\int\ln\left(\pi_1(\beta)\,e^{\frac{1}{\kappa}v_1(\omega)}+(1-\pi_1(\beta))\,e^{\frac{1}{\kappa}e^{-r\Delta}\tilde{\mathcal{U}}_2(\beta,\omega_1)}\right)\mu(\mathrm{d}\omega)\tag{26}$$

moreover, for every m such that $\mu\{0<\beta_m<1\}>0$:

$$\tilde{\mathcal{U}}_m(\beta,\omega_{m-1})=\kappa\int\ln\left(\pi_m(\beta)\,e^{\frac{1}{\kappa}v_m(\omega)}+(1-\pi_m(\beta))\,e^{\frac{1}{\kappa}e^{-r\Delta}\tilde{\mathcal{U}}_{m+1}(\beta|\omega_m)}\right)\mu(\mathrm{d}\omega|\omega_{m-1})\tag{27}$$

Proof. I prove equation 26. The proof of 27 is similar and therefore omitted. If $\mu\{0<\beta_1<1\}>0$ and β satisfies 16 then:

$$\begin{aligned}
V_1(\beta,\omega) &= \beta_1 v_1-\kappa\left(\beta_1\ln\left(\frac{\beta_1}{\pi_1(\beta)}\right)+(1-\beta_1)\ln\left(\frac{1-\beta_1}{1-\pi_1(\beta)}\right)\right)\\
&= \beta_1 v_1-\kappa\left(\beta_1\frac{1}{\kappa}v_1+(1-\beta_1)\frac{1}{\kappa}e^{-r\Delta}\mathcal{U}_2(\beta|\omega_1)\right)\\
&\quad+\kappa\ln\left(\pi_1(\beta)\,e^{\frac{1}{\kappa}v_1(\omega)}+(1-\pi_1(\beta))\,e^{\frac{1}{\kappa}e^{-r\Delta}\mathcal{U}_2(\beta|\omega_1)}\right)\\
&= \kappa\ln\left(\pi_1(\beta)\,e^{\frac{1}{\kappa}v_1(\omega)}+(1-\pi_1(\beta))\,e^{\frac{1}{\kappa}e^{-r\Delta}\mathcal{U}_2(\beta|\omega_1)}\right)\\
&\quad-e^{-r\Delta}\tilde{\mathcal{U}}_2(\beta,\omega_1)
\end{aligned}$$

but:

$$\mathcal{U}(\beta)=\int\left(V_1(\beta,\omega)+(1-\beta_1)\,e^{-r\Delta}\tilde{\mathcal{U}}_2(\beta,\omega_1)\right)\mu(\mathrm{d}\omega)$$

the conclusion follows. □

I now turn to proving the sufficient condition for β to be an optimum. Note that:

$$v_m-\kappa\ln\left(\frac{\beta_m}{\pi_m(\beta)}\right) = e^{-r\Delta}\mathcal{U}_{m+1}(\beta,\omega_m)-\kappa\ln\left(\frac{1-\beta_m}{1-\pi_m(\beta)}\right)\tag{28}$$

therefore:

$$\begin{aligned}\frac{V_m(\beta,\omega)}{1-F_{m-1}} &= \beta_m v_m - \kappa\beta_m \ln\left(\frac{\beta_m}{\pi_m(\beta)}\right) - \kappa(1-\beta_m)\ln\left(\frac{1-\beta_m}{1-\pi_m(\beta)}\right) \\ &= \beta_m e^{-r\Delta}\mathcal{U}_{m+1}(\beta,\omega_m) - \kappa\ln\left(\frac{1-\beta_m}{1-\pi_m(\beta)}\right)\end{aligned}$$

which implies:

$$\mathcal{U}_m(\beta,\omega_{m-1}) = \int\left(e^{-r\Delta}\mathcal{U}_{m+1}(\beta,\omega_m) - \kappa\ln\left(\frac{1-\beta_m}{1-\pi_m(\beta)}\right)\right)\mu(\mathrm{d}\omega|\omega_m)$$

by iterative substitution,

$$\mathcal{U}_m(\beta,\omega_{m-1}) = -\kappa\int\sum_{j=m}^{M} e^{-r\Delta(j-m)}\ln\left(\frac{1-\beta_j}{1-\pi_j(\beta)}\right)\mu(\mathrm{d}\omega|\omega_m)$$

using equations 24 and 28 then gives $\int\Lambda_{f^\beta,m}\mu(\mathrm{d}\omega|\omega_m) = 0$ for all m and μ-almost every ω_m. Therefore, $\mathrm{d}\mathcal{U}_{f^\beta} = 0$. Optimality of β follows from $\mathcal{U}$ being concave.

C Infrequent Offers Environment

The current appendix includes proofs pertaining to an infrequent offers environment. These include Theorem 3, Theorem 1 and Proposition 4. The results regarding frequent offer limits, lemma 1, and as well as proof of several auxiliary results are delegated to the online proofs appendix.

A simple seller's problem Let $W:\mathbb{R}_+\to\mathbb{R}_+$ to be Lambert's W function, defined by: $W(ze^z) = z$, or, equivalently, as: $W(z)e^{W(z)}$. Also, define the function $D(x,c) = \left(1+e^{-\frac{1}{\kappa}(c-x)}\right)^{-1}$. The lemma below turns out to tightly characterize the seller's strategy in equilibrium.

Lemma 12. *Consider the function:* $H(x,c,d) = D(x,c)\,x + (1-D(x,c))\,d$*, where* $d\geq 0$*. Then:*

1. *$H-d$ is strictly log-concave over $x \in \mathbb{R}_+$, and the problem:* $\max_x H(x,c,d)$ *has a unique solution in $\mathbb{R}_+$.*

2. $x^* = \arg\max\{H(x,c,d)\,|x \in \mathbb{R}_+\}$ *if and only if one of the following holds:*

$$x^* = d + \kappa + \kappa W\left(e^{\frac{1}{\kappa}(c-d-\kappa)}\right) \tag{29}$$

$$H(x^*,c,d) = x^* - \kappa \tag{30}$$

$$\frac{D(x^*,c)}{1-D(x^*,c)} = \frac{1}{\kappa}(x^* - \kappa - d) \tag{31}$$

   moreover, x^* is strictly increasing in c.

3. *The solution to* $\max\{H(x,c,d) : x \in X\}$ *is unique, equal to* $\min\{x^*, \bar{x}\}$*, and is weakly increasing in c.*

Proof. **Part 1:** note that $\ln(H-d) = \ln(x-d) - \ln\left(1+e^{\frac{1}{\kappa}(x-c)}\right)$. Part 1 follows from concavity of $\ln(x-d)$ and convexity of $\ln\left(1+e^{\frac{1}{\kappa}(x-c)}\right)$. **Part 2:** Since $\ln(H-d)$ is strictly concave, the following FOC is both necessary and sufficient for a solution:

$$(x^*-d)^{-1} - \frac{1}{\kappa}(1-D(x^*,c)) = 0$$

which can be rearranged to obtain equation 30 and: $D(x^*,c) = (x^*-d-\kappa)/(x^*-d)$, giving 31. The FOC can be rearranged to $e^{\frac{1}{\kappa}(c-x)} = \frac{1}{\kappa}(x-d-\kappa)$which can be rearranged to be: $e^{\frac{1}{\kappa}(c-d-\kappa)} = \frac{1}{\kappa}(x-d-\kappa)\,e^{\frac{1}{\kappa}(x-d-\kappa)}$ or:

$$\frac{1}{\kappa}(x-d-\kappa) = W\left(e^{\frac{1}{\kappa}(c-d-\kappa)}\right)$$

which can be easily rearranged to give the desired equality. x^* increasing in c follows from equation 30. **Part 3:** Follows from log concavity which implies that $\partial \ln(H-d)/\partial x > 0$ for all $x < x^*$. □

C.1 Proof of Theorem 3

Finite Horizon Suppose first that the game's horizon is some finite M. I prove the lemma in steps. I begin by using a backward induction argument to prove that the player's strategies are simple, and that the buyer's strategy satisfies equation 3 with π_m being strictly between 0 and 1. I then turn to proving that the seller's strategy satisfies equation 5 and that the monotonicity condition (part 5) holds. I conclude by proving part 4.

Definition 5. Suppose (β, σ, μ) are consistent. We say that a sequence $\{\mu^n, \beta^n, \epsilon^n\}_{n=1}^{\infty}$ is a (x^m, v) *perturbation sequence* for some (x^m, v) if there exists a $\mu^* \in \Delta(X^m \times V)$ with $\mu^*(x^m, v) > 0$ such that: (1) $\mu^n = \epsilon^n \mu^* + (1 - \epsilon^n)\mu_m$; (2) $\epsilon^n > 0$, $\epsilon^n \to 0$ and $\beta^n \to \beta$; and (3) β^n maximizes $E_m[U_b|\mu^n, \beta^n, \sigma]$ for all n.

Given some (x^m, v)-perturbation sequence, $\{\mu^n, \beta^n, \epsilon^n\}_{n=1}^{\infty}$, let $\pi_m^n = \int \beta_m^n \mathrm{d}\mu^n$. Note that β is a credible best response to σ if and only if there is (x^m, v)-perturbation sequence for every (x^m, v).

Period M**:** I begin by proving the lemma for period M.

Lemma 13. *For all* (x^M, v)*:* $\beta_M(x^M, v) = D\left(x_M, v + \kappa \ln \frac{\pi_M}{1-\pi_M}\right)$.

Proof. Let $\{\mu^n, \beta^n, \epsilon^n\}_{n=1}^{\infty}$ be a (x^M, v)-perturbation sequence. By Lebesgue's dominated convergence theorem: $\int \beta_M^n \mathrm{d}\mu_M \to \int \beta_M \mathrm{d}\mu_M$, which implies $\int \beta_M^n \mathrm{d}\mu^n \to \int \beta_M \mathrm{d}\mu_M$. Using theorem 6:

$$\begin{aligned} \beta_M(x^M, v) &= \lim_{n\to\infty} \beta_M^n(x^M, v) = \lim_{n\to\infty} \frac{\pi_M^n e^{\frac{1}{\kappa}(v - x_M)}}{1 - \pi_M^n + \pi_M^n e^{\frac{1}{\kappa}(v - x_M)}} \\ &= \frac{\pi_M e^{\frac{1}{\kappa}(v - x_M)}}{1 - \pi_M + \pi_M e^{\frac{1}{\kappa}(v - x_M)}} = D\left(x_M, v + \kappa \ln \frac{\pi_M}{1 - \pi_M}\right) \end{aligned}$$

as required. □

Lemma 14. $0 < \pi_M < 1$.

Proof. By lemma 13, if $\pi_M = \int \beta_M \mathrm{d}\mu_M = 0$ then $\beta_M \left(x^M, v\right) = 0$ for all $\left(x^M, v\right)$ contradicting β_M being attentive, while $\int \beta_M \mathrm{d}\mu_M = 1$ implies $\beta_M \left(x^M, v\right) = 1$ for all $\left(x^M, v\right)$ which cannot possibly be in equilibrium since then the seller's best response in M is to offer ∞. $\square$

Note that the seller's expected value conditional on arriving to period M, the history $\left(x^{M-1}, v\right)$ and offering x_M is: $H\left(x_M, v + \kappa \ln \frac{\pi_M}{1-\pi_M}, 0\right)$. Let $x^*_{M,v}$ and $z_{M,v}$ be the unique solutions for this problem in $\mathbb{R}_+$ and X respectively (Lemma 12). Clearly, $\sigma\left(z_{M,v}, x^{M-1}, v\right) = 1$ for all $\left(x^{M-1}, v\right)$. Suppose $z_{M,v} \geq \bar{x}$ for some v. Then $z_{M,v_h} \geq \bar{x} > v_h$ by weak monotonicity of $z_{M,v}$ (Lemma 12). Moreover,

$$x^*_{M,v_h} = \kappa + \kappa W\left(\frac{\pi_M}{1-\pi_M} e^{\frac{v_h - \kappa}{\kappa}}\right) \geq \bar{x} > v_h$$

which implies:

$$W\left(\frac{\pi_M}{1-\pi_M} e^{\frac{v_h - \kappa}{\kappa}}\right) > \frac{v_h - \kappa}{\kappa} \iff \pi_M > \frac{v_h - \kappa}{v_h}$$

But this means that $\pi_M > \frac{1}{v}(v - \kappa)$ for all v, implying $x^*_{M,v} > v$ for all v. But then $z_{M,v} = \min\{x_{M,v}, \bar{x}\} > v$ for all v, meaning that the buyer's best response is to set $b_{M,v} = 0$ for all v, giving $\pi_M = 0$, a contradiction. Since $z_{M,v} = x^*_{M,v}$ for all v, Lemma 12 gives equation 5, and strict monotonicity of both $z_{M,v}$ and $b_{M,v}$ in v. Equation 3 then gives monotonicity of $v - z_{M,v}$. Note that the expected value of $1 - b_{M,v}$ is $1 - \pi_M$. Therefore, $1 - b_{M,v_l} > 1 - \pi_M > 1 - b_{M,v_h}$. Equation 3 then implies that $(1 - b_{M,v}) / (1 - \pi_M)$ is equal to $\left(1 - \pi_M + \pi_M e^{\frac{1}{\kappa}(v - z_{M,v})}\right)^{-1}$. The conclusion that $v_l - z_{M,v_l} < 0 < v_h - z_{M,v_h}$ follows.

Inductive Step: *Simple buyer strategy and part 3:* Suppose now that for periods $j \geq m+1$ Proposition 3 holds. For some (x^m, v), let $\{\mu^n, \beta^n, \epsilon^n\}_{n=1}^{\infty}$ be a (x^m, v)-perturbation sequence. Since $\beta^n_j\left(x^j, v\right) \to \beta_j\left(x^j, v\right)$ for all (x^j, v), Lebesgue's dominated convergence gives: $\pi^n_{m+j} \to \pi_{m+j}$ for all relevant $j \geq 0$. Moreover, since $\pi_{m+j} \in (0,1)$ for all $j \geq 1$: $u^n_{m+j,v} \to u_{m+j,v}$. Therefore, by

Theorem 6:

$$\begin{aligned}\beta_m\left(x^m,v\right) &= \lim_{n\to\infty}\beta_m^n\left(x^m,v\right)=\lim_{n\to\infty}\frac{\pi_m^n e^{\frac{1}{\kappa}(v-x_m)}}{\left(1-\pi_m^n\right)e^{\frac{e^{-r\Delta}}{\kappa}u_{m+1,v}^n}+\pi_m^n e^{\frac{1}{\kappa}(v-x_m)}}\\ &= \frac{\pi_m e^{\frac{1}{\kappa}(v-x_m)}}{\left(1-\pi_m\right)e^{\frac{e^{-r\Delta}}{\kappa}u_{m+1,v}}+\pi_m e^{\frac{1}{\kappa}(v-x_m)}}\\ &= D\left(x_m, v+\kappa\ln\frac{\pi_m}{1-\pi_m}-e^{-r\Delta}u_{m+1,v}\right)\end{aligned}$$

$\pi_m\in(0,1)$: Note that $\pi_m=0$ contradicts β being attentive, and $\pi_m=1$ implies seller's offer is $\bar{x}$ regardless of past, contradicting β being optimal. Hence, buyer's m period strategy is simple and satisfies equation 2.

Simple seller strategy: The above together with the induction assumption imply that a v seller's m period problem is:

$$\max_{x_m} H\left(x, v+\kappa\ln\frac{\pi_m}{1-\pi_m}-e^{-r\Delta}u_{m+1,v}, e^{-r\Delta}\left(z_{m+1,v}-\kappa\right)\right)$$

independent of x^{m-1}. Hence seller's strategy is simple.

Parts 2 and 4 of lemma: Let $x_{m,v}^*$ and $z_{m,v}$ be the seller's problem's unique solutions in $\mathbb{R}_+$ and X respectively (Lemma 12). I now prove that $b_{m,v}$ is strictly increasing. Rearranging equation 3 for any j implies that $b_{j,v}=e^{\frac{1}{\kappa}(v-z_{j,v}-\kappa\ln\pi_j-u_{j,v})}$. Hence,

$$u_{j,v}=\frac{1}{\kappa}\left(v-z_{j,v}-\kappa\ln\left(\frac{b_{j,v}}{\pi_j}\right)\right) \tag{32}$$

Substituting the above for $j=m+1$ in equation 3 for m and rearranging gives:

$$\frac{b_m\left(x,v\right)}{1-b_m\left(x,v\right)}=\left(\frac{\pi_m}{1-\pi_m}\right)\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)^{e^{-r\Delta}} e^{\frac{1}{\kappa}\left(v-x-e^{-r\Delta}\left(v-z_{m+1,v}\right)\right)} \tag{33}$$

But equation 31 of Lemma 12:

$$\frac{b_m\left(x^*_{m,v},v\right)}{1-b_m\left(x^*_{m,v},v\right)}=\frac{1}{\kappa}\left(x^*_{m,v}-\kappa-e^{-r\Delta}\left(z_{m+1,v}-\kappa\right)\right)$$

combining the above two display equations gives:

$$\left(\frac{b_m\left(x^*_{m,v},v\right)}{1-b_m\left(x^*_{m,v},v\right)}\right)e^{\left(\frac{b_m\left(x^*_{m,v},v\right)}{1-b_m\left(x^*_{m,v},v\right)}\right)}=\left(\frac{\pi_m}{1-\pi_m}\right)\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)^{e^{-r\Delta}}\left(e^{\left(\frac{v-\kappa}{\kappa}\right)}\right)^{\left(1-e^{-r\Delta}\right)} \tag{34}$$

Since $b_{m+1,v}$ is strictly increasing in v, the above implies that $b_m\left(x^*_{m,v},v\right)$ is strictly increasing in v. Equations 29 and 31 together along with the above give that $x^*_{m,v}$ is also strictly increasing. Take now any $v<v^{'}$. The above implies that $b_{m,v}<b_{m,v'}$ whenever $x^*_{m,v'}\leq\bar{x}$. If $x^*_{m,v}\leq\bar{x}<x^*_{m,v'}$ then $b_{m,v'}>b_m\left(x^*_{m,v'},v^{'}\right)>b_{m,v}$. Suppose then $x^*_{m,v}>\bar{x}$. Then equation 33 gives:

$$\frac{b_{m,v'}}{1-b_{m,v'}}-\frac{b_{m,v}}{1-b_{m,v}}=\left(\frac{\pi_m e^{-\frac{\bar{x}}{\kappa}}}{1-\pi_m}\right)\left(\begin{array}{c}\left(\frac{b_{m+1,v'}}{\pi_{m+1}}\right)^{e^{-r\Delta}}\left(e^{\left(1-e^{-r\Delta}\right)v'+e^{-r\Delta}z_{m+1,v'}}\right)^{\frac{1}{\kappa}}\\-\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)^{e^{-r\Delta}}\left(e^{\left(1-e^{-r\Delta}\right)v+z_{m+1,v}}\right)^{\frac{1}{\kappa}}\end{array}\right)>0$$

where the inequality follows $b_{m+1,v}$ and $z_{m+1,v}$ being both strictly increasing in v. Therefore $b_{m,v}$ is strictly increasing. I now show that $x^*_{m,v}=z_{m,v}$. Suppose otherwise. Monotonicity of $x^*_{m,v}$ in v and Lemma 12 part 4 mean that $z_{m,v_h}=\bar{x}$. But:

$$z_{m,v_h}=v_h-\kappa\ln\left(\frac{b_{m,v_h}}{\pi_m}\right)+\kappa\sum_{j=m}^{M}e^{-r\Delta(j-m)}\ln\left(\frac{1-b_{j,v_h}}{1-\pi_j}\right)<v_h-\kappa\ln\left(\frac{b_{m,v_h}}{\pi_m}\right)<v_h \tag{35}$$

where the equality follows equation 9 for $m+1$ and equation 3, and both inequalities follow from $b_{j,v}$ being strictly increasing in v for all $j\geq m$. A contradiction. Therefore $z_{m,v}$ is also increasing. Finally, note that equation 9 means that $u_{m,v}$ is also strictly increasing in v, giving monotonicity of $v-z_{m,v}$

by equation 32. Equation 35 establishes $v_h - z_{m,v_h} > 0$. A straightforward derivation shows that an equivalent of equation 35 holds for for v_l but with the reverse inequalities, and hence $v_l - z_{m,v_l} < 0$. Equation 5 follows from Lemma 12, which also implies $E\left[U_s|v\right] = z_{0,v} - \kappa$.

Part 4: $E\left[U_s\right] = E\left[z_{1,v}\right] - \kappa$ follows from above. Theorem 11 implies $E\left[U_b\right] = E\left[u_{1,v}\right]$.

Infinite Horizon Consider now the equilibrium in the infinite horizon. The equilibrium is simple since it is the limit of finite horizon equilibria, which are simple. Let (μ, b, z) be the infinite horizon equilibrium. A simple convergence argument establishes equation 5. Similarly, one can use a simple convergence argument to establish that

$$b_m\left(x, v\right) = D\left(x, v + \kappa \ln \frac{\pi_m}{1-\pi_m} - e^{-r\Delta} u_{m+1,v}\right)$$

and hence equation 3 as long as $\pi_m \in (0,1)$ for all m. Take $\left\{\left(\mu^M, b^M, z^M\right)\right\}_{M \in N}$ to be such that $N \subset \mathbb{N}$ is an infinite set of integers, $\left(\mu^M, b^M, z^M\right)$ is an equilibrium in the M horizon game, and $\left(\mu^M, b^M, z^M\right)$ converge to (μ, b, z). Suppose by contradiction that $\pi_m = 1$. Clearly, we must have $b_m\left(z_{m,v}, v\right) = 1$ for all v. Therefore,

$$\exp\left[-\frac{1}{\kappa}\left(v + \kappa \ln \frac{\pi_m^M}{1-\pi_m^M} - e^{-r\Delta} u_{m+1,v}^M - x\right)\right] \to 0$$

Take any $x \in \mathbb{R}_+$. Since $b_m^M\left(x, v\right)$ satisfies equation 3,

$$\begin{aligned} b_m^M\left(x, v\right) &= \left(1 + e^{-\frac{1}{\kappa}\left(v - x + \kappa \ln\left(\frac{\pi_m}{1-\pi_m}\right) - e^{-r\Delta} u_{m+1,v}\right)}\right)^{-1} \\ &= \left(1 + e^{\frac{1}{\kappa}\left(x - z_{m,v}\right)} e^{-\frac{1}{\kappa}\left(v - z_{m,v} + \kappa \ln\left(\frac{\pi_m}{1-\pi_m}\right) - e^{-r\Delta} u_{m+1,v}\right)}\right)^{-1} \end{aligned}$$

implying $b_m^M\left(x, v\right) \to 1$. Therefore $b_m\left(x, v\right) = 1$ for all x, a contradiction to $z_{m,v}$ being optimal. Suppose now by contradiction that $\pi_m = 0$. A similar argument to above implies $b_m\left(x, v\right) = 0$ for all x and v, contradicting b being attentive. Equation 5 follows from convergence of $z_{m,v}^M$. Since $b_{m+1,v}^M$ is strictly monotone in v, $b_{m+1,v}$ must be weakly increasing in v. Since equation 34 holds in limit (but with $z_{m,v}$ replacing $x^*_{m,v}$), strict monotonicity of $b_{m,v}$ follows. Analogous arguments to those made in the finite horizon establish that $z_{m,v}$ and $v-z_{m,v}$ are both strictly increasing in v, and that $v_h-z_{m,v_h}>0>v_l-z_{m,v_l}$. Part 5 follows from Theorem 11 and Lemma 12.

C.2 Proof of Theorem 1

Sufficient condition for credible best response I begin by stating the lemma bellow that helps characterize the player's optimal strategy in equilibrium. The lemma's proof is based on Theorem 6 above.

Lemma 15. *Let (μ,b,z) be a consistent simple strategy profile such that the buyer's strategy satisfies equation 3 and that $\pi_m\in(0,1)$ for all m. Suppose further that either the game's horizon is finite or that:*

$$\lim_{m\to\infty} e^{-r\Delta m}\ln\left(\frac{1-b_{m,v}}{1-\pi_m}\right)=0$$

for all v. Then the buyer's strategy is a credible best response to z given μ.

Proof. Fix any (x^m,v). Suppose wlog that $x_m<z_{m,v}$. Clearly, $b_m(x_m,v)>b_{m,v}$. Let α be such that $b_{m,v}=\alpha b_m(\bar{x},v)+(1-\alpha)b_m(x_m,v)$. Let $\tilde{x}^m=(z_{1,v},\dots,z_{m-1,v},\bar{x})$, and define μ^S via setting $\mu^S(x^m,v)$ equal to $(1-\alpha)\mu_m(z_{1,v},\dots,z_{m,v},v)$, $\mu^S(\tilde{x}^m,v)$ equal to $\alpha\mu_m(z_{1,v},\dots,z_{m,v},v)$, and for every $v'\neq v$: $\mu^S(\cdot,v')=\mu_m(\cdot,v')$. By construction the buyer's posterior over V after using b_m and reaching period $m+1$ is the same under μ_m and μ^S. The same holds for every $\mu^\epsilon=\epsilon\mu^S+(1-\epsilon)\mu_m$ for $\epsilon\in(0,1)$. It is therefore straightforward to show that $(\beta_m,\beta_{m+1},\dots)$ satisfies the conditions of Theorem 6 when the distribution over $X^m\times V$ is $\mu^\epsilon=\epsilon\mu^S+(1-\epsilon)\mu_m$ for $\epsilon\in(0,1)$ and future offers determined according to z. b being a credible best response follows. □

Proof of Theorem in Finite Horizon Define the function $\Phi:[0,1]^M\to[0,1]^M$ via the following process:

1. Start by setting $u^\pi_{M+1,v}=0$ and $z^\pi_{M+1,v}=\kappa$.

2. Given $u^{\pi}_{m+1,v}$ define the function b^{π}_{m} as in (3).

3. Given $z^{\pi}_{m+1,v}$ and b^{π}_{m}, define $z^{\pi}_{m,v}$ as the solution to (5).

4. Let $\mu^{\pi}_{1},\ldots,\mu^{\pi}_{M}$ to be (some) beliefs consistent with the players using $(b^{\pi}_{m}, z^{\pi}_{m})$.

5. Define: $\Phi_m = \sum_v \mu^{\pi}_{m,v} b^{\pi}_{m,v}$, where $b^{\pi}_{m,v} := b^{\pi}_{m}\left(z^{\pi}_{m,v}, v\right)$.

I divide the Theorem's proof into three parts. I begin by proving a lemma that bounds the ratio $\frac{b^{\pi}_{m,v}}{\pi_m}$ from above. The second part establishes that Φ has a fixed point in $(0,1)^M$. The third and final part uses this fixed point to construct an equilibrium.

Part 1

Lemma 16. *Suppose that parts (2) and (3) of Theorem 3 hold for (b^{π}, z^{π}) where $\pi > 0$. Then for all m:* $\frac{b^{\pi}_{m,v}}{\pi_m} \leq e^{\frac{v-\kappa}{\kappa}}$.

Proof. Define $\frac{b^{\pi}_{M+1,v}}{\pi_{M+1}}$ to be 1 for $M+1$. Then the claim clearly holds for period $M+1$. Suppose it holds for period $m+1$. Note that (3) and (4) imply $u^{\pi}_{m,v} = v - z^{\pi}_{m,v} - \kappa \ln \frac{b^{\pi}_{m,v}}{\pi_m}$. Therefore,

$$\begin{aligned}
\frac{b^{\pi}_{m,v}}{\pi_m} &= \left(\pi_m + (1-\pi_m)\, e^{-\frac{1}{\kappa}\left(v - z_{m,v} - e^{-r\Delta}\left(v - z_{m+1,v} - \kappa \ln \frac{b^{\pi}_{m+1,v}}{\pi_{m+1}}\right)\right)}\right)^{-1} \\
&\leq \left(\pi_m + (1-\pi_m)\, e^{-\frac{1}{\kappa}\left(\left(1-e^{-r\Delta}\right)(v-\kappa) + e^{-r\Delta}(v-\kappa)\right)}\right)^{-1} \leq e^{\frac{v-\kappa}{\kappa}}.
\end{aligned}$$

The first inequality follows from both equation (5) (which can be rearrange to give $z_{m,v} - e^{-r\Delta} z_{m+1,v} \geq \left(1 - e^{-r\Delta}\right)\kappa$) and the induction assumption. The second inequality following from $v > \kappa$ for all v. $\square$

Part 2

Lemma 17. Φ *has a fixed point in* $(0,1)^M$.

Proof. Fix any $\epsilon \in \left(0, \frac{1}{2}\right)$. Begin define the mapping $\Phi^{\epsilon} : [\epsilon, 1-\epsilon]^{M} \rightarrow [\epsilon, 1-\epsilon]^{M}$ via: $\Phi_{m}^{\epsilon}(\pi) = \min\{\max\{\epsilon, \Phi_{m}(\pi)\}, 1-\epsilon\}$. Note that Φ^{ϵ} is continuous it is a composition of continuous functions. Therefore, by Brouwer's theorem Φ^{ϵ} admits a fixed point. Let π^{ϵ} be such a fixed point. By compactness, there exists a sequence $\epsilon \rightarrow 0$ such that $\pi^{\epsilon} \rightarrow \pi^{0} \in [0,1]^{M}$. Below I claim that $\pi_{m}^{0} \in (0,1)$ for all m. Therefore, there exists an $\bar{\epsilon} > 0$ such that for all $\epsilon < \bar{\epsilon}$, $\pi_{m}^{\bar{\epsilon}}$ is a fixed point of Φ^{ϵ}. But that implies that $\pi_{m}^{\bar{\epsilon}} \in [\bar{\epsilon}, 1-\bar{\epsilon}] \subset (\epsilon, 1-\epsilon)$ for all m, implying that $\pi^{\bar{\epsilon}}$ is an interior fixed point of Φ^{0}, as required.

It remains to prove the claim that $\pi_{m}^{0} \in (0,1)$. Plan: Prove that $\pi_{m}^{0} < 1$ for all m by induction from period 1 and that $\pi_{m}^{0} > 0$ for all m by inducting backwards from period M.

Suppose first that $\pi_{1}^{0} = 1$. Then there exists a subsequence of ϵ such that $\sum_{v} \mu_{1,v}^{\pi^{\epsilon}} b_{1,v}^{\pi^{\epsilon}} \geq \pi_{1}^{\epsilon}$ for all ϵ, implying that $\sum_{v} \bar{\mu}_{1,v}^{\pi^{\epsilon}} b_{1,v}^{\pi^{\epsilon}} \rightarrow 1$. As such, for all v $b_{1,v}^{\pi^{\epsilon}} \rightarrow 1$. (5) then implies that $z_{1,v}^{\pi^{\epsilon}} \rightarrow \infty$. But this can only hold if $\pi_{1}^{0} \rightarrow 0$, a contradiction. To make the inductive argument, note that $\pi_{j}^{0} < 1$ for all $j < m$ implies $\bar{\mu}_{m,v} > 0$ for all v. Reapplying the period 1 argument proves then claim.

Suppose now that $\pi_{M}^{0} = 0$. (5) implies $z_{M,v}^{\pi^{\epsilon}} \rightarrow \kappa$, and therefore:

$$
\begin{aligned}
1 \geq \sum_{q} \bar{\mu}_{M,v}^{\pi^{\epsilon}} \left(\frac{b_{M,v}^{\pi^{\epsilon}}}{\pi_{M}^{\epsilon}}\right) &= \sum_{q} \bar{\mu}_{M,v}^{\pi^{\epsilon}} \left(\pi_{M}^{\epsilon} + (1-\pi_{M}^{\epsilon})\, e^{-\frac{1}{\kappa}\left(v - z_{M,v}^{\pi^{\epsilon}}\right)}\right)^{-1} \\
&\rightarrow \sum_{q} \bar{\mu}_{M,v}^{\pi^{0}} e^{\frac{v-\kappa}{\kappa}} > 1
\end{aligned}
$$

where the first inequality comes from $\pi_{M}^{\epsilon} \geq \sum_{q} \bar{\mu}_{M,v}^{\pi^{\epsilon}} b_{M,v}^{\pi^{\epsilon}}$ and the last from $v > \kappa$ for all v. Suppose now that $\pi_{m}^{0} = 0$ and $\pi_{j}^{0} > 0$ for all $j \geq m+1$. Then $b_{m,v}^{\pi^{\epsilon}} \rightarrow 0$ meaning that $z_{m,v}^{\pi^{\epsilon}} - e^{-r\Delta} z_{m+1,v}^{\pi^{\epsilon}} \rightarrow \left(1 - e^{-r\Delta}\right)\kappa$ by (5). Together with $u^{\pi_\epsilon}_{m+1,v} = v - z^{\pi_\epsilon}_{m+1,v} - \kappa \ln \left(\frac{b^{\pi^\epsilon}_{m+1,v}}{\pi^\epsilon_{m+1}} \right)$ this implies:

$$\begin{aligned}
\sum_q \bar{\mu}^{\pi^\epsilon}_{m,v} \left(\frac{b^{\pi^\epsilon}_{m,v}}{\pi^\epsilon_m} \right) &= \sum_q \bar{\mu}^{\pi^\epsilon}_{m,v} \left(\pi^\epsilon_m + (1-\pi^\epsilon_m) \left(\frac{b^{\pi^\epsilon}_{m+1,v}}{\pi^\epsilon_{m+1}} \right)^{-e^{-r\Delta}} e^{-\frac{1}{\kappa}\left(\left(1-e^{-r\Delta}\right)v - \left(z^{\pi^\epsilon}_{m,v} - e^{-r\Delta} z^{\pi^\epsilon}_{m+1,v}\right)\right)} \right)^{-1} \\
&\to \sum_q \bar{\mu}^0_{m,v} \left(\frac{b^0_{m+1,v}}{\pi^0_{m+1}} \right)^{e^{-r\Delta}} e^{\left(1-e^{-r\Delta}\right)\left(\frac{v-\kappa}{\kappa}\right)} \\
&= \sum_q \bar{\mu}^0_{m,v} \left(\frac{1}{e^{\frac{v-\kappa}{\kappa}}} \frac{b^0_{m+1,v}}{\pi^0_{m+1}} \right)^{e^{-r\Delta}} e^{\frac{v-\kappa}{\kappa}} \\
&= \sum_q \bar{\mu}^0_{m+1,v} \left(\frac{1}{e^{\frac{v-\kappa}{\kappa}}} \frac{b^0_{m+1,v}}{\pi^0_{m+1}} \right)^{e^{-r\Delta}} e^{\frac{v-\kappa}{\kappa}} \\
&> \sum_q \bar{\mu}^0_{m,v} \frac{1}{e^{\frac{v-\kappa}{\kappa}}} \frac{b^0_{m+1,v}}{\pi^0_{m+1}} e^{\frac{v-\kappa}{\kappa}} = 1
\end{aligned}$$

where the inequality follows from Lemma 16 and $v > \kappa$ for all v, and the last equality follows from (3). However, by choice of π^ϵ: $1 \geq \sum_q \mu^{\pi^\epsilon}_{m,v} \left(b^{\pi^\epsilon}_{m,v} / \pi^\epsilon_m \right)$ for all ϵ, a contradiction. $\square$

Part 3 Let π be the interior fixed point of Φ, and let $(b_m, z_{m,v}) = \left(b^\pi_m, z^\pi_{m,v}\right)$. I begin by proving tha t z is optimal for S. By construction, b_m satisfies (3) and is therefore b is a credible best response to S's strategy by Lemma 15. I prove that $z_{m,v}$ is optimal inductively backward from period M. S's period M problem given v is

$$\max_{x_M} H \left(x, v + \kappa \ln \frac{\pi_M}{1-\pi_M}, 0 \right).$$

Lemma (12) implies that $z_{M,v}$ solves the above problem, and that the problem's value is $z_{M,v} - \kappa$. Assuming that the value of S's problem from period $m+1$ conditional on v is $z_{m+1,v} - \kappa$ and that $z_{m+1,v}$ solves that problem, we obtain that S's period m problem given v is:

$$\max_{x_m} H\left(x, v + \kappa \ln \frac{\pi_m}{1-\pi_m} - e^{-r\Delta} u_{m+1,v}, e^{-r\Delta}\left(z_{m+1,v} - \kappa\right)\right)$$

Then (29) from Lemma 12 gives that $z_{m,v}$ is optimal.

Infinite Horizon I prove this result in steps. In the first step I establish some bounds that must hold in every equilibrium, finite or infinite horizon, and that π_m is uniformly bounded from below across all finite horizon equilibria. One therefore obtains that the same bound must hold in the infinite horizon as well via limits. In the second step I establish that convergence of the buyer's strategy in every period and the seller's best response imply that the limit is optimal for the seller. The final step simply connects all these results together to give the said theorem.

The first step itself is divided into two lemmas. The first and more complicated of the two bounds the ratio $\frac{b_{m,v}}{\pi_m}$ from below. The second step establishes additional bounds that must hold for every equilibrium. The bounds appearing in both Lemmas give an appropriate compactness condition that gives the existence of a convergent equilibrium sequence.

Lemma 18. *Let (μ, b, z) be an equilibrium for horizon $M \in \mathbb{N} \cup \{\infty\}$. Then $\frac{b_{m,v}}{\pi_m} \geq \frac{1}{2}$ for all m.*

Proof. I prove the Lemma for finite horizon. The property follows for infinite horizon via limits. Define the functions

$$\varrho(a, v, q) = \frac{a}{1-a} e^{\left(1-e^{-r\Delta}\right)\left(\frac{v-\kappa}{\kappa}\right)} q^{e^{-r\Delta}} \tag{36}$$

$$R(a, v, q) = \frac{W(\varrho(a, v, q))}{a\left(1 + W(\varrho(a, v, q))\right)} \tag{37}$$

$$R^c(a, v, q) = \left[(1-a)\left(1 + W(\varrho(a, v, q))\right)\right]^{-1} \tag{38}$$

I prove the lemma in steps. In the first step I establish that $\frac{b_{m,v}}{\pi_m} = R\left(\pi_m, v, \frac{b_{m+1,v}}{\pi_{m+1}}\right)$ for all v and m. In the second step, I prove that either $\arg\min_{a \in [0,1]} R(a, v, q) =$

0 or $R(a,v,q) \geq 1/2$ for all $a \in [0,1]$, v and q. In the third I establish that $R(0,v,q) = e^{\left(1-e^{-r\Delta}\right)\left(\frac{v-\kappa}{\kappa}\right)} q^{e^{-r\Delta}}$. In the fourth and final step I show by inducting back from M that $R\left(0, v, \frac{b_{m+1,v}}{\pi_{m+1}}\right) > \frac{1}{2}$ for all m, and therefore $\frac{b_{m,v}}{\pi_m} \geq \frac{1}{2}$, which completes the proof.

Step 1: $\frac{b_{m,v}}{\pi_m} = R\left(\pi_m, v, \frac{b_{m+1,v}}{\pi_{m+1}}\right)$ for all v and m, where $\frac{b_{M+1,v}}{\pi_{M+1}} := e^{\frac{1}{\kappa}(v-\kappa)}$.

Proof. (16) and (4) give $b_{m,v} = e^{\frac{1}{\kappa}(v - z_{m,v} - \kappa \ln \pi_m - u_{m,v})}$ and therefore

$$\ln\left(\frac{b_{m,v}}{\pi_m}\right) = \frac{1}{\kappa}\left(v - z_{m,v} - u_{m,v}\right)$$

Recall that $z_{M+1,v} := \kappa$ and $u_{M+1,v} := 0$, and so the definition $\frac{b_{M+1,v}}{\pi_{M+1}} := e^{\frac{1}{\kappa}(v-\kappa)}$ is also consistent with the above equation. Recall further that a v seller's $m \leq M$ period problem is:

$$\max_{x \geq 0} H\left(x | v + \kappa \ln\left(\frac{\pi_m}{1-\pi_m}\right) - e^{-r\Delta} u_{m+1,v}, e^{-r\Delta}\left(z_{m+1,v} - \kappa\right)\right)$$

where $z_{M+1,v} := \kappa$ and $u_{m+1,v} := 0$. Using equation 32 above and equations 29 and 31 from Lemma 12,

$$b_{m,v} = \frac{W\left(\left(\frac{\pi_m}{1-\pi_m}\right)\left(e^{\left(\frac{v-\kappa}{\kappa}\right)}\right)^{\left(1-e^{-r\Delta}\right)}\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)^{e^{-r\Delta}}\right)}{1 + W\left(\left(\frac{\pi_m}{1-\pi_m}\right)\left(e^{\left(\frac{v-\kappa}{\kappa}\right)}\right)^{\left(1-e^{-r\Delta}\right)}\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)^{e^{-r\Delta}}\right)} = \frac{W\left(\varrho\left(\pi_m, v, \frac{b_{m+1,v}}{\pi_{m+1}}\right)\right)}{1 + W\left(\varrho\left(\pi_m, v, \frac{b_{m+1,v}}{\pi_{m+1}}\right)\right)} \tag{39}$$

dividing by π_m completes the proof of the current step. □

Step 2: Either $\arg\min_{a \in [0,1]} R(a,v,q) = 0$ or $R(a,v,q) \geq 1/2$ for all $a \in [0,1]$, v and q.

Proof. Note first that $R(\cdot, v, q)$ is continuous, and therefore,

$$R(1,v,q) = \lim_{a \to 1} \frac{W\left(\frac{a}{1-a} e^{\left(1-e^{-r\Delta}\right)\left(\frac{v-\kappa}{\kappa}\right)} q^{e^{-r\Delta}}\right)}{a\left(1 + W\left(\frac{a}{1-a} e^{\left(1-e^{-r\Delta}\right)\left(\frac{v-\kappa}{\kappa}\right)} q^{e^{-r\Delta}}\right)\right)} = 1$$

I now claim that either $R(0,v,q) < \frac{1}{2}$, in which case $\arg\min_{a\in[0,1]} R(a,v,q) = 0$, or that $R(a,v,q) \geq \frac{1}{2}$ for all a. Note that if $1 \in \arg\min_{a\in[0,1]} R(a,v,q)$ then we are done. Suppose then that the minimizer of R is in the interior, i.e. $a \in (0,1)$. Let $\underline{a}$ be that minimizer. Then the first order condition, $\frac{\partial R}{\partial a} = 0$, must hold. Taking derivative of R:

$$\begin{aligned}
\frac{\partial R}{\partial a} &= -\frac{1}{a^2}\frac{W}{1+W} + \frac{1}{a}\frac{\partial \varrho}{\partial a}\left(\frac{W}{\varrho(1+W)^2} - \frac{W^2}{\varrho(1+W)^3}\right) \\
&= -\frac{R}{a} + \frac{\varrho}{a^2(1-a)}\frac{W}{\varrho(1+W)^3} \\
&= -\frac{R}{a} + \frac{R}{a}\frac{R^c}{(1+W)} = \frac{R}{a}\left(\frac{R^c}{(1+W)} - 1\right)
\end{aligned}$$

where the second equality follows from $\frac{\partial \varrho}{\partial a} = \varrho/a(1-a)$. Hence, at $\underline{a}$: $R^c = (1+W)$, or $W = (1-\underline{a})^{-1/2} - 1$. Therefore,

$$\begin{aligned}
R(\underline{a},v,q) &= \frac{(1-\underline{a})^{-1/2} - 1}{\underline{a}(1-\underline{a})^{-1/2}} \\
&= \left(1 - (1-\underline{a})^{1/2}\right)/\underline{a} \\
&= 1/\left(1+\sqrt{1-\underline{a}}\right)
\end{aligned}$$

let $f(a) = 1/\left(1+\sqrt{1-a}\right)$. The above suggests that $R(\underline{a},q,v) \geq \min f(a)$. Note that:

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}a} &= \frac{\frac{1}{2}(1-a)^{-1/2}a - 1 + (1-a)^{1/2}}{a^2} \\
&= \frac{1-(1-a)^{1/2}}{2a^2(1-a)^{1/2}}
\end{aligned}$$

which is never 0 in $(0,1)$. Therefore the minimum of f is either $f(0)$ or $f(1)$. Since $f(1) = 1$ and $f(0) = 1/2$, one has $\min f = 1/2$. $\square$

Step 3: $R(0,v,q) = e^{\left(1-e^{-r\Delta}\right)\left(\frac{v-\kappa}{\kappa}\right)} q^{e^{-r\Delta}}$

By continuity,

$$
\begin{aligned}
R(0,v,q) &= \lim_{a\to 0} \frac{W(\varrho(a,v,q))}{a(1+W(\varrho(a,v,q)))} \\
&= \lim_{a\to 0} \frac{\frac{\partial\varrho}{\partial a}(a,v,q)\left(e^{W(\varrho(a,v,q))}(1+W(\varrho(a,v,q)))\right)^{-1}}{(1+W(\varrho(a,v,q)))+\frac{\partial\varrho}{\partial a}(a,v,q)\left(e^{W(\varrho(a,v,q))}(1+W(\varrho(a,v,q)))\right)^{-1}} \\
&= \lim_{a\to 0} \frac{\partial\varrho}{\partial a}(a,v,q) = e^{\left(1-e^{-r\Delta}\right)\left(\frac{v-\kappa}{\kappa}\right)} q^{e^{-r\Delta}}
\end{aligned}
$$

where the second equality follows from L'Hopitals rule and $\mathrm{d}W(z)/\mathrm{d}z = \left(e^{W(z)}(1+W(z))\right)^{-1}$,[19] and the last equality follows from $\frac{\partial\varrho}{\partial a} = (1-a)^{-2} e^{\left(1-e^{-r\Delta}\right)\left(\frac{v-\kappa}{\kappa}\right)} q^{e^{-r\Delta}}$.

Step 4: $\frac{b_{m,v}}{\pi_m} \geq \frac{1}{2}$ for all m.

Proof. By definition, $\frac{b_{M+1,v}}{\pi_{M+1}} = e^{\frac{v-\kappa}{\kappa}}$. Hence Step 3 gives $R\left(0,v,\frac{b_{M+1,v}}{\pi_{M+1}}\right) = e^{\frac{v-\kappa}{\kappa}} > \frac{1}{2}$. which implies that $\frac{b_{M,v}}{\pi_M} = R\left(\pi_M,v,\frac{b_{M+1,v}}{\pi_{M+1}}\right) \geq \frac{1}{2}$ by Step 2. Suppose Step 4 holds for $m+1$. Then by Step 3: $R\left(0,v,\frac{b_{m+1,v}}{\pi_{m+1}}\right) = e^{\left(1-e^{-r\Delta}\right)\left(\frac{v-\kappa}{\kappa}\right)}\left(\frac{b_{m+1,v}}{\pi_{m+1}}\right)^{e^{-r\Delta}}$, which is a geometric mean of two numbers larger than $\frac{1}{2}$. Step 2 then implies: $\frac{b_{m,v}}{\pi_m} = R\left(\pi_m,v,\frac{b_{m+1,v}}{\pi_{m+1}}\right) \geq \frac{1}{2}$, thereby concluding the proof. □

□

.

Lemma 19. *Let (μ,b,z) be an equilibrium of the game. Then: (1) $b_{m,v} \leq (v_h-\kappa)/v_h$ for all v and m; (2) $v-2v_h \leq u_{m,v} \leq v_h+2\kappa$; (3) Then for every m there exists a constant $\eta_m > 0$ such that for every equilibrium with horizon $M \in \{m+1,\ldots,\infty\}$: $\pi_m \geq \eta_m$.*

Proof. **Part 1**: Suppose otherwise. Then $b_{m,v_h} > \frac{1}{v_h}(v_h-\kappa)$ since $b_{m,v}$ Add increases with v. Together with (5) we obtain $z_{m,v_h} > v_h + e^{-r\Delta}(z_{m+1,v_h}-\kappa)$. Combined with $z_{m,v_h} < v_h$ (Theorem (3) part (5)) we obtain $z_{m+1,v_h} < \kappa$. But a simple rearranging of (5) shows that $z_{m+1,v_h} \geq \kappa$, a contradiction.

[19] This follows frm the implicit function Theorem. I include some facts about Lambert's W function in appendix ... [REFERENCE].

Part 2: To prove the lower bound,

$$
\begin{aligned}
u_{m,v} &= v - z_{m,v} - \kappa \ln \frac{b_{m,v}}{\pi_m} \\
&\geq v - z_{m,v} - (v_h - \kappa) \\
&\geq v - 2v_h
\end{aligned}
$$

where first inequality follows from part 2 of the current lemma, and second inequality from $z_{m,v} \leq z_{m,v_h} < v_h$ for all v. To prove the upper bound,

$$
\begin{aligned}
u_{m,v} &= v - z_{m,v} - \kappa \ln \frac{b_{m,v}}{\pi_m} \\
&\leq v_h - \kappa \ln \frac{b_{m,v}}{\pi_m} \\
&\leq v_h + 2\kappa
\end{aligned}
$$

where the second inequality follows from Lemma 18.

Part 3: Suppose otherwise. Then there exists a sequence $\{(\mu^n, b^n, z^n, M^n)\}_{n\geq 0}$ where (μ^n, b^n, z^n) is an equilibrium with horizon $M^n \geq m+1$ such that $\pi^n_m \to 0$. Note that one can assume without loss that $\left(\bar{\mu}^n_{j,v}, \pi^n_{j,v}, b^n_{j,v}, z^n_{j,v}\right) \to \left(\bar{\mu}^\infty_{j,v}, \pi^\infty_{j,v}, b^\infty_{j,v}, z^\infty_{j,v}\right)$ converges along the sequence for all $j \leq m+1$ by compactness. Since $\pi^n_m \leq \frac{v_h - \kappa}{v_h} < 1$ (current lemma, part 1), it must be that $b^n_{m,v} \to 0$ for all v, which gives $z^n_{m,v} - e^{-r\Delta} z^n_{m+1,v} \to \left(1 - e^{-r\Delta}\right)\kappa$ by (5). Together with $u^n_{m+1,v} = v - z^n_{m+1,v} - \kappa \ln\left(\frac{b^n_{m+1,v}}{\pi^n_{m+1}}\right)$ this implies the following contradiction:

$$
\begin{aligned}
1 = \sum_q \bar{\mu}^n_{m,v}\left(\frac{b^n_{m,v}}{\pi^n_m}\right) &= \sum_v \bar{\mu}^n_{m,v}\left(\pi^n_m + (1-\pi^n_m)\left(\frac{b^n_{m+1,v}}{\pi^n_{m+1}}\right)^{-e^{-r\Delta}} e^{-\frac{1}{\kappa}\left(\left(1-e^{-r\Delta}\right)v - \left(z^n_{m,v} - e^{-r\Delta} z^n_{m+1,v}\right)\right)}\right) \\
&\to \sum_v \bar{\mu}^\infty_{m,v}\left(\frac{b^\infty_{m+1,v}}{\pi^\infty_{m+1}}\right)^{e^{-r\Delta}} e^{\left(1-e^{-r\Delta}\right)\left(\frac{v-\kappa}{\kappa}\right)} \ \left(\forall n\!: \ \frac{b^n_{m+1,v}}{\pi^n_{m+1}} \in \left[\frac{1}{2}, e^{\frac{1}{\kappa}(v-\kappa)}\right] \ \right) \\
&= \sum_v \bar{\mu}^\infty_{m,v}\left(\frac{1}{e^{\frac{v-\kappa}{\kappa}}}\frac{b^\infty_{m+1,v}}{\pi^\infty_{m+1}}\right)^{e^{-r\Delta}} e^{\frac{v-\kappa}{\kappa}} \\
&= \sum_v \bar{\mu}^\infty_{m+1,v}\left(\frac{1}{e^{\frac{v-\kappa}{\kappa}}}\frac{b^\infty_{m+1,v}}{\pi^\infty_{m+1}}\right)^{e^{-r\Delta}} e^{\frac{v-\kappa}{\kappa}} \ \left(\pi^\infty_m = 0 \text{ implies } \forall v:\ b^\infty_{m,v} = 0\right)
\end{aligned}
$$

Notice, however, that $b_{j,v}/\pi_j \leq e^{\frac{1}{\kappa}(v_h-\kappa)}$ for all j and n, and $\sum_v \bar{\mu}^n_{m,v} \frac{b^n_{m+1,v}}{\pi^n_{m+1}} = 1 < e^{\frac{v_l-\kappa}{\kappa}}$. Therefore one has $\frac{b^\infty_{m+1,v}}{\pi^\infty_{m+1}} \leq e^{\frac{v-\kappa}{\kappa}}$ with a strict inequality for at least one v. Since for all $x \in (0,1)$ one has $x^{e^{-r\Delta}} > x$, one obtains that:

$$\begin{aligned}\sum_v \bar{\mu}^\infty_{m+1,v} \left(\frac{1}{e^{\frac{v-\kappa}{\kappa}}} \frac{b^\infty_{m+1,v}}{\pi^\infty_{m+1}}\right)^{e^{-r\Delta}} e^{\frac{v-\kappa}{\kappa}} &> \sum_v \bar{\mu}^\infty_{m+1,v} \frac{1}{e^{\frac{v-\kappa}{\kappa}}} \frac{b^\infty_{m+1,v}}{\pi^\infty_{m+1}} e^{\frac{v-\kappa}{\kappa}} \\ &= 1 \text{ (For all } n\text{: } \sum_v \bar{\mu}^n_{m,v} \frac{b^n_{m+1,v}}{\pi^n_{m+1}} = 1)\end{aligned}$$

a contradiction. □

The next Lemma shows that convergence of the player's strategies means that the seller's strategy at the limit is still optimal.

Lemma 20. *Let* $(c_m)^\infty_{m=0}$ *and* $(c^n_m)^\infty_{n,m=0}$ *be such that* $c^n_m, c_m \geq 0$ *and* $c^n_m \to c_m$ *for every* m*. Define* $J : X^\infty \to \mathbb{R}_+$ *and* $J_n : X^\infty \to \mathbb{R}_+$ *by:*

$$\begin{aligned} J(x^\infty) &= \sum_{j=0}^{\infty} e^{-r\Delta j} \prod_{k=0}^{j-1} (1 - D(x_k, c_k)) D(x_j, c_j) x_j \\ J_n(x^\infty) &= \sum_{j=0}^{n} e^{-r\Delta j} \prod_{k=0}^{j-1} (1 - D(x_k, c^n_k)) D\left(x_j, c^n_j\right) x_j \end{aligned}$$

And let $J^* = \max J$ *and* $J^*_n = \max J_n$*. Note that both exist since both functions are continuous and* X^∞ *is compact. Then: (1)* $J^*_n \to J^*$*. (2) If* $x^{\infty(n)} \in \arg\max J_n$ *for every* n *is such that* $x^{\infty(n)} \to x^\infty$ *for some* $x^\infty \in X^\infty$*, then* $x^\infty \in \arg\max J$*.*

Proof. Since $c^n_m \to c_m$ for every m, for every N and $\epsilon > 0$, there is an $N_\epsilon > N$ such that for all $n > N_\epsilon$:

$$\left|\sum_{j=0}^{N} e^{-r\Delta j} \left(\prod_{k=0}^{j-1} (1 - D(x_k, c_k)) D(x_j, c_j) - \prod_{k=0}^{j-1} (1 - D(x_k, c^n_k)) D\left(x_j, c^n_j\right)\right) x_j\right| < \epsilon$$

Hence, for every $n > N$ and every x^∞: $|J(x^\infty) - J_n(x^\infty)| < \epsilon + \frac{e^{-r\Delta N}}{1-e^{-r\Delta}}\bar{x}$. This is also true for any $x^\infty \in \arg\max J$, implying that: $|J^* - J^*_n| < \epsilon + \frac{e^{-r\Delta N}}{1-e^{-r\Delta}}\bar{x}$.

Since N was arbitrary, $J_n^* \to J^*$. If $x^{\infty(n)} \in \arg\max J_n$ for every n is such that $x^{\infty(n)} \to x^{\infty}$ for some $x^{\infty} \in X^{\infty}$. Fix an N and $\epsilon > 0$. Then since $c_m^n \to c_m$ and $x_m^n \to x_m$ for all m, there exists an $N_\epsilon > N$ such that for all $n > N_\epsilon$:

$$\left|\sum_{j=0}^{N} e^{-r\Delta j}\left(\prod_{k=0}^{j-1}\left(1-D\left(x_k,c_k\right)\right)D\left(x_j,c_j\right)x_j-\prod_{k=0}^{j-1}\left(1-D\left(x_k^n,c_k^n\right)\right)D\left(x_j^n,c_j^n\right)x_j^n\right)\right|<\epsilon$$

since $x^{\infty} \in X^{\infty}$, this implies that for every $n > N$: $\left|J\left(x^{\infty}\right)-J_n\left(x^{\infty(n)}\right)\right| < \epsilon + \frac{e^{-r\Delta N}}{1-e^{-r\Delta}}\bar{x}$, thereby implying that $J_n^* = J_n\left(x^{\infty(n)}\right) \to J\left(x^{\infty}\right)$. Therefore:

$$\begin{aligned}\left|J\left(x^{\infty}\right)-J^*\right| &\leq \left|J\left(x^{\infty}\right)-J_n\left(x^{\infty(n)}\right)\right|+\left|J_n\left(x^{\infty(n)}\right)-J^*\right| \\ &= \left|J\left(x^{\infty}\right)-J_n\left(x^{\infty(n)}\right)\right|+\left|J_n^*-J^*\right| \to 0\end{aligned}$$

as required. □

Proof of Theorem 1. Take any sequence $\left\{\left(\mu^M,b^M,z^M\right)\right\}_M$ of M horizon equilibria with $M \to \infty$. By Lemma 19, $u_{m,v}^M$ belongs to a compact interval. Since the same is true for $\left(\pi_m^M,b_{m,v}^M,z_{m,v}^M\right)$, Cantor's diagonal method implies that there exists a subsequence for which π_m^M, $b_{m,v}^M$, $z_{m,v}^M$ and $u_{m,v}^M$ all converge. Let $\left\{\left(\mu^M,b^M,z^M\right)\right\}_M$ be that sequence. Lemma 19 implies that $\pi_m^M \leq \left(v_h-\kappa\right)/v_h$ for all M,m, implying the same for π_m. Convergence of $b_m\left(x,v\right)$ the follows from equation 3, while $\pi_m > 0$ follows from Lemma 19. Lemma 20 gives that following z is optimal for the seller, and Lemma 15 implies that b is a credible best response for the buyer, thereby concluding the proof.

C.3 Proof of Propositions 3 and 4

In the text I proved that for every equilibrium we have $\sum_v \mu_0\left(v\right)u_{1,v} > 0$ $\left(\sum_v \mu_0\left(v\right)\left(u_{1,v}+w_{1,v}\right) > \mathbb{E}\left[v\right]-\kappa\right)$. I now show that the same argument bounds $\sum_v \mu_0\left(v\right)u_{1,v}$ $\left(\sum_v \mu_0\left(v\right)\left(u_{1,v}+w_{1,v}\right)\right)$ away from zero ($\mathbb{E}\left[v\right]-\kappa$). Suppose otherwise. Then there is a sequence of equilibria $\left(\mu^n,b^n,z^n\right)$ with a corresponding u^n such that $\sum_v \mu_0\left(v\right)u_{1,v}^n \to 0$ $\left(\sum_v \mu_0\left(v\right)\left(u_{1,v}+w_{1,v}\right) \to \mathbb{E}\left[v\right]-\kappa\right)$. Let $p_{m,v}^n = b_{m,v}^n/\pi_m^n$. Note that for every n, m and v, $\left(\bar{\mu}_{m,v}^n,b_{m,v}^n,z_{m,v}^n,p_{m,v}^n\right)$ is in a compact interval of $\mathbb{R}_+$(for $p^n_{m,v}$ see Lemma 19). Therefore there exists a convergent subsequence. Let that subsequence be the sequence itself with limits $\left(\bar{\mu}^\infty_{m,v}, b^\infty_{m,v}, z^\infty_{m,v}, p^\infty_{m,v}\right)$. Note that $\pi^n_m \leq \frac{v_h-\kappa}{v_h}$ for all m and n by Lemma 19 and therefore $\pi_m = \lim_{n\to\infty} \pi^n_m \leq \frac{v_h-\kappa}{v_h} < 1$. Similarly, Lemma 19 implies that $\pi_m > 0$.

Below I prove Lemma 21 which shows that $p_{m,v}$ is strictly increasing, which implies that $b_{m,v}$ is strictly increasing. But then one can apply exactly the same argument as for the single equilibrium case to obtain that $\lim_{n\to\infty} \sum_v \mu_0(v)\, u^n_{1,v} > 0$ ($\lim_{n\to\infty} \sum_v \mu_0(v) \left(u^n_{1,v} + w^n_{1,v}\right) > \mathbb{E}[v] - \kappa$), thereby concluding the proof. I now turn to proving the Lemma.

Lemma 21. *Let $\{(\mu^n, b^n, z^n)\}^\infty_{n=1}$ be a sequence of equilibria with horizons $\{M_n\}^\infty_{n=1}$ all in $\{m+1, \ldots, \infty\}$. Define $p^n_{k,v} := b^n_{k,v}/\pi^n_k$. Suppose that $\pi^n_k \to \pi^\infty_k \in (0,1)$ and $p^n_{k,v} \to p^\infty_{k,v}$ for every $k \leq m+1$. Then $p^\infty_{k,v}$ is strictly increasing for all $k \leq m$.*

Proof. Note that every infinite horizon equilibrium is a limit of a sequence of finite horizon equilibria. Therefore, by Cantor's diagonal method, any limit of a sequence that includes infinite horizon equilibria is also the limit of equilibria with only finite horizons. Suppose then that M_n is finite for all n. Recall from Step 1 of Lemma 18 that: $p^n_{m,v} = \frac{b^n_{m,v}}{\pi^n_m} = R\left(\pi^n_m, v, p^n_{m+1,v}\right)$ for all v and m, where $p^n_{M_n+1,v} := e^{\frac{1}{\kappa}(v-\kappa)}$ and R is defined via:

$$\varrho(a, v, q) = \frac{a}{1-a} e^{\left(1-e^{-r\Delta}\right)\left(\frac{v-\kappa}{\kappa}\right)} q^{e^{-r\Delta}}$$
$$R(a, v, q) = \frac{W(\varrho(a, v, q))}{a\left(1 + W(\varrho(a, v, q))\right)}$$

where $W : \mathbb{R}_+ \to \mathbb{R}_+$ is Lambert's W function, defined by the equation: $W(ye^y) = y$. Clearly, $p^n_{M_n+1,v}$ is strictly increasing in v for all $n < \infty$. Assuming that $p^n_{m+1,v}$ strictly increases in v for all n, I now prove by induction that $p^n_{m,v}$ is. Notice that $\varrho(a, v, q)$ strictly increases in q and in v since $W(\cdot)$ is strictly increasing. Therefore $R(a, v, q)$ also strictly increases in q and in v. Since $p^n_{m+1,v}$ and v are both strictly increasing functions of v, we have that $p^n_{m,v} = R\left(\pi^n_m, v, p^n_{m+1,v}\right)$ strictly increases with v.

I now prove the claim. Fix any m. Since $\pi_m^\infty > 0$, $p_{m,v}^\infty = R\left(\pi_m^\infty, v, p_{m+1,v}^\infty\right)$ by continuity of ϱ and W. Since $p_{m+1,v}^n$ strictly increases with v for all n, $p_{m+1,v}^\infty$ weakly increases with v. But $R(a,v,q)$ strictly increases in v even when q is constat. Therefore $p_{m,v}^\infty = R\left(\pi_m^\infty, v, p_{m+1,v}^\infty\right)$ strictly increases in v, as required. □

C.3.1 Proof of Theorem 2

The Theorem's proof is based on various pieces of the proof of Lemma 18. The proof of the Theorem requires several steps. The first step establishes that two equilibria in the one-shot game are distinct if and only if they have different π's. The second step proves that the function:

$$\begin{aligned} f_v : [0,1] &\to \mathbb{R}_+ \\ \pi &\mapsto R\left(\pi, v, e^{\frac{1}{\kappa}(v-\kappa)}\right) \end{aligned}$$

is strictly convex. The third step then uses the convexity of the above function to show that (μ, b, z) and (μ', b', z') are both equilibria of the one shot game if and only if they have the same ex-ante choice probability, i.e. same π. Equilibrium uniqueness in the 1-shot game follows.

Step 1: Let (μ, b, z) and (μ', b', z') be equilibria of the one shot game. Then$(\mu, b, z) = (\mu', b', z')$ if and only if:

$$\pi_1 = \sum_v \mu_{0,v} b_{1,v} = \sum_v \mu_{0,v} b_{1,v}^{'} = \pi_1^{'}$$

Proof. Only if is obvious. Take now any two equilibria, (μ, b, z) and (μ', b', z') such that $\pi_1 = \pi_1^{'}$. Since there is only one period, we have $b = b^{'}$ by Theorem 3, part 2. By Lemma 12,

$$\begin{aligned} z_{1,v} &= \kappa + \kappa W\left(e^{\frac{1}{\kappa}\left(v-\kappa+\kappa \ln \frac{\pi_1}{1-\pi_1}\right)}\right) \\ &= \kappa + \kappa W\left(\frac{\pi_1}{1-\pi_1} e^{\frac{v-\kappa}{\kappa}}\right) = z_{1,v}^{'}, \end{aligned}$$

where the last equality follows from $\pi_1 = \pi_1^{'}$. $\square$

Step 2: f as defined above is strictly convex.

Proof. Rewrite $R\left(\pi, v, e^{\frac{1}{\kappa}(v-\kappa)}\right)$:

$$\begin{aligned} R(\pi, v, q) &= \frac{W\left(\frac{\pi e^{\frac{v-\kappa}{\kappa}}}{1-\pi}\right)}{\pi\left(1+W\left(\frac{\pi e^{\frac{v-\kappa}{\kappa}}}{1-\pi}\right)\right)} \\ &= \left(\frac{\pi}{W\left(\frac{\pi e^{\frac{v-\kappa}{\kappa}}}{1-\pi}\right)} + \pi\right)^{-1} \end{aligned}$$

so a sufficient condition for R to be strictly convex is for:

$$\phi(a) = a\left(1 + 1/W\left(\frac{ae^{\frac{v-\kappa}{\kappa}}}{1-a}\right)\right)$$

to be strictly concave and increasing in a. Note that the first derivative of ϕ is:

$$1 + \left[1 + W\left(\frac{ae^{\frac{v-\kappa}{\kappa}}}{1-a}\right)\right]^{-1} W\left(\frac{ae^{\frac{v-\kappa}{\kappa}}}{1-a}\right) > 0$$

while the second derivative of ϕ is:

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}a^2} = \frac{(1-a)^2\left(1+2W\left(\frac{ae^{\frac{v-\kappa}{\kappa}}}{1-a}\right)\right) - \left(1+W\left(\frac{ae^{\frac{v-\kappa}{\kappa}}}{1-a}\right)\right)^2}{a(1-a)^2 W\left(\frac{ae^{\frac{v-\kappa}{\kappa}}}{1-a}\right)\left(1+W\left(\frac{ae^{\frac{v-\kappa}{\kappa}}}{1-a}\right)\right)^3}$$

ϕ is concave if:

$$(1-a)^2\left(1+2W\left(\frac{ae^{\frac{v-\kappa}{\kappa}}}{1-a}\right)\right) - \left(1+W\left(\frac{ae^{\frac{v-\kappa}{\kappa}}}{1-a}\right)\right)^2 < 0$$

for all a in the range. However:

$$\begin{aligned}(1-a)^2\left(1+2W\left(\frac{ae^{\frac{v-\kappa}{\kappa}}}{1-a}\right)\right)&\\ -\left(1+W\left(\frac{ae^{\frac{v-\kappa}{\kappa}}}{1-a}\right)\right)^2 &= a\,(a-2)\left(1+2W\left(\frac{ae^{\frac{v-\kappa}{\kappa}}}{1-a}\right)\right)\\ &\quad-\left(W\left(\frac{ae^{\frac{v-\kappa}{\kappa}}}{1-a}\right)\right)^2\end{aligned}$$

which is strictly negative for all $a\in(0,1)$. Therefore ϕ is strictly concave, implying the desired result. □

Step 3: The one shot game has a unique equilibrium.

Proof. By step 1 of Lemma 18 's proof, π must satisfy: $\sum_v \mu_{0,v} f_v(\pi)=1$. But by step 2 of the same lemma, $f_v(1)=1$ for all v, implying $\sum_v \mu_{0,v} f_v(1)=1$. Notice that strict convexity of f_v implies strict convexity of $a\mapsto\sum_v \mu_{0,v} f_v(a)$. But then $\sum_v \mu_{0,v} f_v(1)=1=\sum_v \mu_{0,v} f_v(\pi)$ implies that $\sum_v \mu_{0,v} f_v(a)\neq 1$ for all $a\in[0,1]\setminus\{\pi,1\}$. Hence, all equilibria of the one shot game must have the same π, implying (by step 1) that they are all identical. □

D Frequent offer results from main paper

D.1 Proof of theorem 4

In what follows, let $\{(\Delta^n,\mu^n,b^n,z^n)\}_{n=1}^{\infty}$ be a refining sequence. Take $F^n:\mathbb{R}\times V\to[0,1]$ to be the agreement date distribution function of the n-th equilibrium. Let $\bar{F}^n$ be the unconditional distribution of the calendar agreement time, i.e $\bar{F}^n(t)=\sum_v \mu_0(v)\,F_v^n(t)$. By Helly's selection theorem and Cantor's diagonal argument, there exists a subsequence such that $F_v^n(t)$ and $\bar{F}^n$ weakly converge to some F_v and $\bar{F}$. Let that subsequence be the sequence itself. Define $\mathcal{T}_n=\{0,\Delta^n,2\Delta^n,\ldots\}$ as the set of possible agreement dates for the n-th element. In what follows I often use a t subscript instead of the period subscript $m = 1 + t/\Delta$ to referring to equilibrium quantities, μ_t^n, $b_{t,v}^n$, $z_{t,v}^n$, π_t^n, $u_{t,v}^n$ and $w_{t,v}^n$, whenever $t \in \mathcal{T}_n$.

I prove the theorem through a sequence of lemmas. Lemma 22 establishes an important bound on π_m. Lemma 23 proves that the limits of F_v and $\bar{F}$ are absolutely continuous. Lemma 24 shows that the expected utility-related objects, $u_{t,v}$ and $w_{t,v}$, have well-defined limits, and that these limits satisfy the appropriate limit versions of equations 9 (for u) and equation 7 (for w).[20] Lemma 25 characterizes the hazard rates of the limiting date distributions, which are then used in Lemma 26 to prove that said hazard rates are positive. Lemma 27 then shows that these hazard rates increase with v, while Lemma 28 proves that trade occurs with probability 1. The theorem follows.

Lemma 22. *For all m: $\pi_m \leq 2\pi_{v_h}^{\kappa,\Delta}$.*

Proof. By proof of Proposition 2, for all m there exists at least one v such that: $b_{m,v} \leq \pi_v^{\kappa,\Delta}$. Applying Lemma 18 implies:

$$\pi_m \leq 2b_{m,v} \leq 2\pi_v^{\kappa,\Delta} \leq 2\pi_{v_v}^{\kappa,\Delta}$$

as required. □

Lemma 23. *$\bar{F}$ and F_v are absolutely continuous and are equal to zero at $t = 0$. Moreover, the hazard rate of $\bar{F}$, $\bar{\lambda}_t$, is bounded from above by $3r\left(\frac{v_h-\kappa}{\kappa}\right)$ for all t.*

Proof. For every n, t and $v \in V$ define:

$$\begin{aligned}
\bar{\lambda}_t^n &= = -\frac{1}{\Delta_n}\ln\left(1 - \pi^n_{\lceil \frac{t}{\Delta^n}\rceil}\right) = -\frac{1}{\Delta_n}\ln\left(\frac{1-\bar{F}\left(\left\lfloor\frac{t}{\Delta^n}\right\rfloor + \Delta^n\right)}{1-\bar{F}\left(\left\lfloor\frac{t}{\Delta^n}\right\rfloor\right)}\right) \\
\lambda_{t,v}^n &= -\frac{1}{\Delta_n}\ln\left(1 - b^n_{\lceil \frac{t}{\Delta^n}\rceil, v}\right)
\end{aligned}$$

[20] For w, Lemma 24, part (2), corresponds to:

$$w_{m,v} = \sum_{j=0}^{\infty} e^{-\Delta j}\left(\frac{b_{m+j,v}}{1-b_{m+j,v}}\right)$$

which is obtained by combining equation 5 and 7 to obtain $w_{m,v}$ as a function of $w_{m+1,v}$, and using repeated substitution.

and define for every t: $\bar{G}^n(t) = 1 - e^{-\int_0^t \bar{\lambda}_s^n \mathrm{d}s}$ and $G_v^n(t) = 1 - e^{-\int_0^t \lambda_{s,v}^n \mathrm{d}s}$. Note that for every $t \in \mathcal{T}_n$:

$$\int_0^t \bar{\lambda}_s^n \mathrm{d}s = = \sum_{j=1}^{t/\Delta^n} \Delta^n \left(-\frac{1}{\Delta^n} \ln\left(1 - \pi_j^n\right) \right) = -\sum_{j=1}^{t/\Delta^n} \ln\left(1 - \pi_j^n\right)$$

and therefore:

$$\bar{G}^n(t) = 1 - \prod_{j=1}^{t/\Delta^n} \left(1 - \pi_j^n\right) = \bar{F}^n(t)$$

and similarly $G_v^n(t) = F_v^n(t)$ for all $t \in \mathcal{T}_n$. By lemma 22,

$$\bar{\lambda}_t^n \leq -\frac{1}{\Delta^n} \ln\left(1 - 2\pi_{v_v}^{\kappa,\Delta_n}\right)$$

note that:

$$\begin{aligned} 1 - 2\pi_{v_v}^{\kappa,\Delta_n} &= 1 - 2\left(\frac{\left(1 - e^{-r\Delta^n}\right)(v_h - \kappa)}{\left(1 - e^{-r\Delta^n}\right)(v_h - \kappa) + \kappa} \right) \\ &= \frac{\kappa - 2\left(1 - e^{-r\Delta^n}\right)(v_h - \kappa)}{\left(1 - e^{-r\Delta^n}\right)(v_h - \kappa) + \kappa} \end{aligned}$$

and therefore:

$$\frac{-1}{\Delta^n} \ln\left(1 - 2\pi_{v_v}^{\kappa,\Delta_n}\right) = \frac{-1}{\Delta^n} \ln\left(\frac{\kappa - 2\left(1 - e^{-r\Delta^n}\right)(v_h - \kappa)}{\left(1 - e^{-r\Delta^n}\right)(v_h - \kappa) + \kappa} \right) \to 3\frac{r}{\kappa}(v_h - \kappa)$$

Note that together lemma 22 and lemma 19 part 2 imply that $b_{m,v}^n \leq 2e^{\left(\frac{v_h - \kappa}{\kappa}\right)} \pi_{v_v}^{\kappa,\Delta_n}$ for all n. Therefore:

$$\begin{aligned} \lambda_{s,v}^n &\leq -\frac{1}{\Delta^n} \ln\left(1 - 2e^{\left(\frac{v_h - \kappa}{\kappa}\right)} \pi_{v_v}^{\kappa,\Delta_n}\right) \\ &= \frac{1}{\Delta_n} \left(\ln\left(\left(1 - e^{-r\Delta^n}\right)(v_h - \kappa) + \kappa\right) - \ln\left(\kappa - 2e^{\left(\frac{v_h - \kappa}{\kappa}\right)} \left(1 - e^{-r\Delta^n}\right)(v_h - \kappa)\right) \right) \\ &\to \left(1 + 2e^{\left(\frac{v_h - \kappa}{\kappa}\right)}\right) r \left(\frac{v_h - \kappa}{\kappa}\right) \end{aligned}$$

for all v. Thus, for every $\epsilon > 0$, there exists an N_ϵ such that for all $n > N_\epsilon$:

$0 < \bar{\lambda}_s^n \leq 3r\left(\frac{v_h-\kappa}{\kappa}\right) + \epsilon$ and $0 < \lambda_{s,v}^n \leq \left(1 + 2e^{\left(\frac{v_h-\kappa}{\kappa}\right)}\right) r\left(\frac{v_h-\kappa}{\kappa}\right) + \epsilon$ for all s and v. This implies that:

$$\left\|\bar{\lambda}_s^n\right\|_2, \left\|\lambda_{s,v}^n\right\|_2 \leq \left(\left(1 + 2e^{\left(\frac{v_h-\kappa}{\kappa}\right)}\right) r\left(\frac{v_h-\kappa}{\kappa}\right) + \epsilon\right)^2$$

and therefore, by the sequential Banach-Alaoglu theorem (theorem 10), there exists a subsequence in which $\bar{\lambda}^{n_k} \rightharpoonup \lambda$ and $\lambda_v^{n_k} \rightharpoonup \lambda_v$. Note that φ is absolutely continuous with respect to Lebesgue measure, with density $-re^{-rt}$. Letting $g_t(s) = -\mathbf{1}_{[s\leq t]}\frac{e^{rs}}{r}$, note that the linear functional defined by:

$$\begin{aligned} |L(f)| &= \left|\int_{\mathbb{R}_+} g_t(s) f(s)\,\mathrm{d}\varphi\right| \\ &\leq \left|\frac{e^{rt}}{r}\right|\left|\int_{\mathbb{R}_+} f(s)\,\mathrm{d}\varphi\right| \leq \left|\frac{e^{rt}}{r}\right|\|f\|_2 \end{aligned}$$

therefore: $\int g_t(s)\bar{\lambda}_s^{n_k}\mathrm{d}\varphi \to \int g_t(s)\bar{\lambda}_s\mathrm{d}\varphi$ for all t. However, $\int g_t(s) f_s\mathrm{d}\varphi = \int_0^t f_s\mathrm{d}t$, and therefore we've obtain that $\bar{G}^{n_k}(t) \to 1 - e^{-\int_0^t \bar{\lambda}_s\mathrm{d}t} \equiv \bar{G}(t)$ for all t. Clearly, $\bar{G}$ is continuous everywhere. Since $\bar{F}^{n_k}(t) = \bar{G}^{n_k}(t)$ for all $t \in \mathcal{T}(\Delta^{n_k})$, this implies that for all $t \in \cup_{k\geq 0}\mathcal{T}(\Delta^{n_k})$: $\bar{F}^{n_k}(t) \to \bar{G}(t)$, and therefore $\bar{G}(t) = \bar{F}(t)$ for all $t \in \cup_{k\geq 0}\mathcal{T}(\Delta^{n_k})$. For every $t \notin \cup_{k\geq 0}\mathcal{T}(\Delta^{n_k})$, there exists a sequence $(t_a^i)_{i\geq 0}$ in $\cup_{k\geq 0}\mathcal{T}(\Delta^{n_k})$ such that $t_a^i \downarrow t$. Since $\bar{F}$ is right-continuous, we have that $\bar{F}(t_a^i) \to \bar{F}(t)$. But $\bar{F}(t_a^i) = \bar{G}(t_a^i) \to \bar{G}(t)$. Therefore: $\bar{G}(t) = \bar{F}(t)$ for all t. A similar argument establishes that $F_v^{n_k}(t) \to F(t) = 1 - e^{-\int_0^t \lambda_{s,v}\mathrm{d}s}$ for all t. $F_v(t) = 0$ for all v follows from $F_v^n(0) = b_{1,v} \to 0$. $\square$

Lemma 24. *Let $\lambda_{t,v}$ be the time-dependent hazard rate of F_v. Then for every v there are two functions continuous in t, $u_v : \mathbb{R}_+ \to \mathbb{R}$ and $w_v : \mathbb{R}_+ \to X$ such that: (1) For every $t \in \cup_n \mathcal{T}_n$, $u_{t,v}^n \to u_{t,v}$ and $w_{t,v}^n \to w_{t,v}$; and (2) For* *every v and t,*

$$\begin{aligned} u_{t,v} &= \kappa \int_t^{\infty} e^{-r(s-t)} \left(\lambda_{s,v} - \bar{\lambda}_s\right) \mathrm{d}s \\ w_{t,v} &= \kappa \int_t^{\infty} e^{-r(s-t)} \lambda_{s,v} \mathrm{d}s \end{aligned}$$

Proof. Define λ_v^n and $\bar{\lambda}^n$ as in lemma 23. For every $t \in \mathcal{T}_n$:

$$\ln\left(\frac{1-\pi_t^n}{1-b_{t,v}^n}\right) = \Delta\left(\lambda_{t,v}^n - \bar{\lambda}_t^n\right)$$

and therefore:

$$u_{t,v}^n = \kappa \sum_{j=t/\Delta^n}^{\infty} e^{-r(j\Delta^n - t)} \Delta^n \left(\lambda_{j\Delta^n,v}^n - \bar{\lambda}_{j\Delta^n}^n\right)$$

let: $g_{\Delta^n}(t) = -\frac{1}{r} e^{r\left(t - \Delta^n \left\lfloor \frac{t}{\Delta^n} \right\rfloor\right)}$. Then:

$$\begin{aligned} u_{t,v}^n &= \kappa e^{rt} \int_t^{\infty} g_{\Delta^n}(s) \left(\lambda_{s,v}^n - \bar{\lambda}_s^n\right) \mathrm{d}\varphi \\ &= \kappa e^{rt} \left(\int_t^{\infty} g_{\Delta^n}(s)\, \lambda_{s,v}^n \mathrm{d}\varphi - \int_t^{\infty} g_{\Delta^n}(s)\, \bar{\lambda}_s^n \mathrm{d}\varphi\right) \end{aligned}$$

however, for every $\Delta > \Delta^n$: $\left(g_{\Delta^n}(t)\right)^2 \le \frac{1}{r^2} e^{2r\Delta^n}$, and therefore by the dominated convergence theorem: $g_{\Delta^n} \to -\frac{1}{r}$ in $L^2(\mathbb{R}_+, \mathrm{d}\varphi)$. Hence, by lemma 35:

$$\begin{aligned} u_{t,v}^n &= \kappa e^{rt} \left(\int_t^{\infty} g_{\Delta^n}(s)\, \lambda_{s,v}^n \mathrm{d}\varphi - \int_t^{\infty} g_{\Delta^n}(s)\, \bar{\lambda}_s^n \mathrm{d}\varphi\right) \\ &\to -\frac{\kappa e^{rt}}{r} \int_t^{\infty} \left(\lambda_{s,v} - \bar{\lambda}_s\right) \mathrm{d}\varphi = \kappa \int_t^{\infty} e^{-r(s-t)} \left(\lambda_{s,v} - \bar{\lambda}_s\right) \mathrm{d}s \end{aligned}$$

for all $t\in\cup_{n\geq 0}\mathcal{T}_n$. Similarly, for every $t\in\mathcal{T}_n$:

$$\begin{aligned}w_{t,v}^n &= \kappa\sum_{j=t/\Delta^n}^{\infty}e^{-r(j\Delta^n-t)}\left(\frac{b_{j\Delta^n,v}}{1-b_{j\Delta^n,v}}\right)\\ &= \kappa\sum_{j=t/\Delta^n}^{\infty}e^{-r(j\Delta^n-t)}\left(e^{\Delta^n\lambda_{j\Delta^n,v}^n}-1\right)\end{aligned}$$

using the mean value theorem, there exists a $\Delta_n^*\in(0,\Delta^n)$ such that:

$$\begin{aligned}w_{t,v}^n &= \kappa e^{rt}\sum_{j=t/\Delta^n}^{\infty}e^{-rj\Delta^n}\Delta^n\lambda_{j\Delta^n,v}^n e^{\Delta_n^*\lambda_{j\Delta^n,v}^n}\\ &= \kappa e^{rt}\int_t^{\infty}e^{\Delta_n^*\lambda_{j\Delta^n,v}^n}g_{\Delta^n}(s)\,\lambda_{j\Delta^n,v}^n\mathrm{d}\varphi\end{aligned}$$

since we have $\left(\lambda_{j\Delta^n,v}^n\right)^2<\left(\left(1+2e^{\left(\frac{v_h-\kappa}{\kappa}\right)}\right)r\left(\frac{v_h-\kappa}{\kappa}\right)+\epsilon\right)^2$, we can again use the dominated convergence theorem to obtain that $e^{\Delta^*\lambda_{j\Delta^n,v}^n}g_{\Delta^n}\to-\frac{1}{r}$ in $L^2(\mathbb{R}_+,\mathrm{d}\varphi)$, thereby implying: $w_{t,v}^n\to\kappa\int_t^{\infty}e^{-r(s-t)}\lambda_{s,v}\mathrm{d}s$. For every n and $t\notin\mathcal{T}_n$ set:

$$\begin{aligned}u_{t,v}^n &= u_{\Delta^n\lceil t/\Delta^n\rceil,v}^n\\ w_{t,v}^n &= w_{\Delta^n\lceil t/\Delta^n\rceil,v}^n\end{aligned}$$

Clearly, these converge to the obvious extensions of $u_{t,v}$ and $w_{t,v}$ to all t: $u_{t,v}=\kappa\int_t^{\infty}e^{-r(s-t)}\left(\lambda_{s,v}-\bar{\lambda}_s\right)\mathrm{d}s$ and $w_{t,v}=\kappa\int_t^{\infty}e^{-r(s-t)}\lambda_{s,v}\mathrm{d}s$ which are continuous. □

Lemma 25. *The ratio $\lambda_{t,v}/\bar{\lambda}_t$ is: (1) equal to $\exp\frac{1}{\kappa}\left(v-\kappa-(w_{t,v}+u_{t,v})\right)$; (2) is in the interval $\left[\frac{1}{2},e^{\frac{v-\kappa}{\kappa}}\right]$; and (3) The following equality holds:*

$$\sum_v\mu_0(v)\left(\frac{1-F(t,v)}{1-\bar{F}(t)}\right)\left(\frac{\lambda_{t,v}}{\bar{\lambda}_t}\right)=1$$

Proof. For every n and t (not necessarily in $\mathcal{T}_n$), set: $p_{t,v}^n=\exp\frac{1}{\kappa}\left(v-\kappa-\left(w_{t,v}^n+u_{t,v}^n\right)\right)$ (where $w_{t,v}^n$ and $u_{t,v}^n$ are defined as in lemma 24's proof). Since $u_{t,v}^n\to u_{t,v}$ and

$w^n_{t,v} \to w_{t,v}$, we have that $p^n_{t,v} \to p_{t,v} \equiv \exp \frac{1}{\kappa}(v - \kappa - w_{t,v} - u_{t,v})$. Note that $p_{t,v}$ is continuous in t since $u_{t,v}$ and $w_{t,v}$ are. By equations 3 and 5 of Proposition 3 one has that for every $t \in \mathcal{T}_n$: $p^n_{t,v} = b^n_{t,v}/\pi^n_t$. Lemma 19 for all $t \in \mathcal{T}_n$ and v then gives: $\frac{1}{2} \leq p^n_{t,v} \leq e^{\frac{v-\kappa}{\kappa}}$. Thus, $\frac{1}{2} \leq p_{t,v} \leq e^{\frac{v-\kappa}{\kappa}}$ for all $t \in \cup_{n \geq 0} \mathcal{T}_n$, implying $\frac{1}{2} \leq p_{t,v} \leq e^{\frac{v-\kappa}{\kappa}}$ for all t by continuity of $p_{t,v}$. Therefore, $\left(p^n_{t,v}\right)^2 \leq e^{2\left(\frac{v-\kappa}{\kappa}\right)}$. Hence, by the dominated convergence theorem, $p^n_{t,v} \to p_{t,v}$ in $L^2(\mathbb{R}_+, \mathrm{d}\varphi)$. But this means that for every $g \in L^2(\mathbb{R}_+, \mathrm{d}\varphi)$, $g_t p^n_{t,v} \to g_t p_{t,v}$ in $L^2(\mathbb{R}_+, \mathrm{d}\varphi)$. Thus, by Lemma 35 and Riesz representation theorem (theorem 9) we have that $p^n_{t,v}\bar{\lambda}^n_t \rightharpoonup p_{t,v}\bar{\lambda}_t$. Note, however, that every t:

$$p_{t,v} = \frac{1 - e^{-\Delta^n \lambda^n_{t,v}}}{1 - e^{-\Delta^n \bar{\lambda}^n_t}}$$

and therefore by the mean-value theorem, there exists $\Delta^n_1, \Delta^n_2 \in (0, \Delta^n)$ such that:

$$p^n_{t,v} = \frac{\lambda^n_{t,v} e^{-\Delta^n_1 \lambda^n_{t,v}}}{\bar{\lambda}^n_t e^{-\Delta^n_2 \bar{\lambda}^n_t}}$$

and therefore:

$$e^{-\Delta^n\left(\left(1+2e^{\left(\frac{v_h-\kappa}{\kappa}\right)}\right)r\left(\frac{v_h-\kappa}{\kappa}\right)+\epsilon\right)}\lambda^n_{t,v} < \bar{\lambda}^n_t p^n_{t,v} < \lambda^n_{t,v} e^{\Delta^n\left(3r\left(\frac{v_h-\kappa}{\kappa}\right)+\epsilon\right)}$$

therefore, for every bounded linear operator $L \in L^2(\mathbb{R}_+, \mathrm{d}\varphi)$:

$$\begin{aligned} L\left(\bar{\lambda}^n_t p^n_{t,v}\right) &< e^{\Delta^n\left(3r\left(\frac{v_h-\kappa}{\kappa}\right)+\epsilon\right)} L\left(\lambda^n_v\right) \to L\left(\lambda_v\right) \\ L\left(\bar{\lambda}^n_t p^n_{t,v}\right) &> e^{-\Delta^n\left(\left(1+2e^{\left(\frac{v_h-\kappa}{\kappa}\right)}\right)r\left(\frac{v_h-\kappa}{\kappa}\right)+\epsilon\right)} L\left(\lambda^n_v\right) \to L\left(\lambda_v\right) \end{aligned}$$

and therefore $p^n_v \bar{\lambda}^n \rightharpoonup \lambda_v$. Hence, by theorem 7, $p_{t,v}\bar{\lambda}_t = \lambda_{t,v}$ in $L^2(\mathbb{R}_+, \mathrm{d}\varphi)$. Therefore: $\lambda_{t,v}/\bar{\lambda}_t = \exp \frac{1}{\kappa}(v - \kappa - w_{t,v} - u_{t,v})$. Note that for every k and every $t \in \mathcal{T}_n$:

$$\begin{aligned}1 &= \sum_v \mu_0(v)\left(\frac{1-F_v^n(t-\Delta^n)}{1-\bar{F}^n(t-\Delta_{n_k})}\right)p_{t,v}^n \\ &\to \sum_v \mu_0(v)\left(\frac{1-F_v(t)}{1-\bar{F}(t)}\right)p_{t,v} = \sum_v \mu_0(v)\left(\frac{1-F_v(t)}{1-\bar{F}(t)}\right)p_{t,v}\end{aligned}$$

which extends to all t by continuity in t of $F_v(t)$, $\bar{F}(t)$ and $p_{t,v}$. The desired equality follows. □

Lemma 26. *$\bar{\lambda}_t > 0$ and $\lambda_{t,v} > 0$ for almost every t.*

Proof. I claim that $\bar{\lambda}_t > 0$ almost everywhere. $\lambda_{t,v} > 0$ almost everywhere then follows from lemma 25 part 2. Suppose by contradiction that there is an open ball, $(t, t+\epsilon)$ such that $s \in (t, t+\epsilon)$ implies $\bar{\lambda}_s = 0$. Then by lemma 25:

$$\begin{aligned}\lambda_{t,v}/\bar{\lambda}_t &= \exp\frac{1}{\kappa}\left(v-\kappa-e^{-r\epsilon}\left(w_{t+\epsilon,v}+u_{t+\epsilon,v}\right)\right) \\ &= \left(e^{\frac{v-\kappa}{\kappa}}\right)^{1-e^{-r\epsilon}}\left(\lambda_{t+\epsilon,v}/\bar{\lambda}_{t+\epsilon}\right)^{e^{-r\epsilon}} \geq \lambda_{t+\epsilon,v}/\bar{\lambda}_{t+\epsilon}\end{aligned}$$

for all v with a strict inequality for v_l since $\left(\lambda_{t,v_l}/\bar{\lambda}_t\right) \leq 1 < e^{\frac{v_l-\kappa}{\kappa}}$ for all t (since $p_{t,v_l}^n < 1$ for all $t \in \cup_n \mathcal{T}_n$). Therefore:

$$\sum_v \mu_0(v)\, e^{-\int_0^{t+\epsilon}\left(\lambda_{s,v}-\bar{\lambda}_s\right)\mathrm{d}s}\left(\frac{\lambda_{t,v}}{\bar{\lambda}_t}\right) > \sum_v \mu_0(v)\, e^{-\int_0^{t+\epsilon}\left(\lambda_{s,v}-\bar{\lambda}_s\right)\mathrm{d}s}\left(\frac{\lambda_{t+\epsilon,v}}{\bar{\lambda}_{t+\epsilon}}\right)$$

I now show that the above inequality leads to a contradiction. By lemma 25 part 2, $\lambda_{s,v} \leq e^{\frac{v-\kappa}{\kappa}}\bar{\lambda}_s = 0$ for all $s \in (t, t+\epsilon)$. Thus,

$$\begin{aligned}\sum_v \mu_0(v)\, e^{-\int_0^{t+\epsilon}\left(\lambda_{s,v}-\bar{\lambda}_s\right)\mathrm{d}s}\left(\frac{\lambda_{t,v}}{\bar{\lambda}_t}\right) &= \sum_v \mu_0(v)\, e^{-\int_0^{t}\left(\lambda_{s,v}-\bar{\lambda}_s\right)\mathrm{d}s}\left(\frac{\lambda_{t,v}}{\bar{\lambda}_t}\right) \\ &= 1 \\ &= \sum_v \mu_0(v)\, e^{-\int_0^{t+\epsilon}\left(\lambda_{s,v}-\bar{\lambda}_s\right)\mathrm{d}s}\left(\frac{\lambda_{t+\epsilon,v}}{\bar{\lambda}_{t+\epsilon}}\right)\end{aligned}$$

where the first equality follows from $\lambda_{s,v} = e^{\frac{v-\kappa}{\kappa}}\bar{\lambda}_s = 0$ for all $s \in (t, t+\epsilon)$, and second and third equalities follow from lemma 25 part 3. $\square$

Lemma 27. *$\lambda_{t,v}$ is strictly increasing in v for almost all t.*

Proof. Note that for every $t \in \cup\mathcal{T}_n$, $\lambda_{t,v}/\bar{\lambda}_t = \lim_{n\to\infty} b^n_{t,v}/\pi^n_t$ (see proof of lemma 25). Therefore $\lambda_{t,v}$ must be weakly increasing for all $t \in \cup\mathcal{T}_n$, and therefore weakly increasing in v for all t due to continuity. Suppose there is an open ball $(t, t+\epsilon)$ for $\epsilon > 0$ and $v < v^{'}$ such that $\lambda_{s,v} = \lambda_{s,v'}$ for all $s \in (t, t+\epsilon)$. Since $\lambda_{t,v}/\bar{\lambda}_t$ is continuous, we must also have $\lambda_{t,v} = \lambda_{t,v'}$. By lemma 25 parts 2 and 3 and lemma 24:

$$\begin{aligned} 1 &= \frac{\lambda_{t,v'}}{\lambda_{t,v}} \\ &= \exp\frac{1}{\kappa}\left(v^{'} - v - \left(\left(w_{t,v'} + u_{t,v'}\right) - \left(w_{t,v} + u_{t,v}\right)\right)\right) \\ &= \exp\frac{1}{\kappa}\left(v^{'} - v - 2\kappa\int_{t+\epsilon}^{\infty} e^{-r(s-t)}\left(\lambda_{s,v'} - \lambda_{s,v}\right)\mathrm{d}s\right) \end{aligned}$$

implying that $\kappa\int_{t+\epsilon}^{\infty} e^{-r(s-t)}\left(\lambda_{s,v'} - \lambda_{s,v}\right)\mathrm{d}s = 2\left(v^{'} - v\right) > 0$. But:

$$\begin{aligned} 1 &\leq \frac{\lambda_{t+\epsilon,v'}}{\lambda_{t+\epsilon,v}} \\ &= \exp\frac{1}{\kappa}\left(v^{'} - v - 2e^{r\epsilon}\kappa\int_{t+\epsilon}^{\infty} e^{-r(s-t)}\left(\lambda_{s,v'} - \lambda_{s,v}\right)\mathrm{d}s\right) \\ &< \exp\frac{1}{\kappa}\left(v^{'} - v - 2\kappa\int_{t+\epsilon}^{\infty} e^{-r(s-t)}\left(\lambda_{s,v'} - \lambda_{s,v}\right)\mathrm{d}s\right) = 1 \end{aligned}$$

since $e^{r\epsilon} > 1$, a contradiction. Therefore $\lambda_{t,v}$ is strictly increasing almost everywhere. $\square$

Lemma 28. *For every v:$\lim_{t\to\infty} F_v(t) = 1$. Therefore, $\lim_{t\to\infty}\bar{F}(t) = 1$.*

Proof. Suppose otherwise. Then $\left(\lambda_{t,v}/\bar{\lambda}_t\right) \geq 1/2$ for all t and v implies that $\bar{\lambda}_t \to 0$. But, since $\bar{\lambda}_t$ is bounded, we can use the dominated convergence theorem to obtain that: $\left(\lambda_{t,v_l}/\bar{\lambda}_t\right) = \exp\frac{1}{\kappa}\left(v_l - \kappa - \kappa\int_t^{\infty} e^{-r(s-t)}\left(2\lambda_{s,v_l} - \bar{\lambda}_s\right)\mathrm{d}s\right) \to e^{\frac{v_l-\kappa}{\kappa}} > 1$, a contradiction. $\square$

Proof of Theorem 4. Part 1 follows from lemma 23. Part 2 follows from 26. For Part 3, $\lim_{t\to\infty} F_v(t) = 1$ follows from Lemma 28. $F_v(t) < 1$ for all t and v follows from Lemma 23 which implies $\bar{\lambda}_t < 3r\left(\frac{v_h-\kappa}{\kappa}\right)$, and Lemma 25, which implies that $\lambda_{t,v} < 3re^{-r\left(\frac{v_h-\kappa}{\kappa}\right)}\bar{\lambda}_t$. Together we obtain that $\int_0^s \lambda_{t,v}\mathrm{d}t < \infty$ for all $s \in \mathbb{R}_+$, hence implying that $F_v(s) = 1 - e^{-\int_0^s \lambda_t \mathrm{d}t} < 1$ for all s. Part 4 follows form Lemma 27.

D.2 Proof of lemma 2

By Proposition 3, for every equilibrium: $E[U_b] = \sum_v \mu_0(v)\, u_{1,v}$, and $E[U_s] = \sum_v \mu_0(v)\, w_{1,v}$. The result then follows from lemma 24.

D.3 Proof of Theorem 5

Let $\left(\lambda_v, \bar{\lambda}, u_v, w_v\right)$ be a *κ-frequent offers collection* if there exists a convergent refining sequence $\{\Delta^n, \mu^n, b^n, z^n\}$ with limit F such that: (1) $\lambda_{t,v}$ is the hazard rate of F_v; (2) $\bar{\lambda}_t$ is the hazard rate of $\bar{F} = \sum_v \mu_0(v) F_v$; (3) $u_v : \mathbb{R}_+ \to \mathbb{R}_+$ and $w_v : \mathbb{R}_+ \to X$ are continuous functions such that $u^n_{t,v} \to u_{t,v}$ and $w^n_{t,v} \to w_{t,v}$ for all $t \in \cup_n \mathcal{T}_n$. Note that every convergent refining sequence generates a κ-frequent offers collection by lemmas 23 and 24. Lemmas 23, 24, 25, 26 and 27 all prove various properties of these collections. The lemma below proves another such property:

Lemma 29. *Let $\left(\lambda_v, \bar{\lambda}, u_v, w_v\right)$ be a κ-frequent offers collection. Then $w_{t,v_h} + u_{t,v_h} \le v_h - \kappa$*

Proof. Follows from lemmas 25 and 27. □

I now turn to proving the theorem. Let $\left\{\left(\bar{U}^n_s, \bar{U}^n_b\right)\right\}_{n=1}^{\infty}$ be a sequence of κ_n-frequent offer utilities with $\kappa_n \to 0$ and take $\left(\lambda^n_v, \bar{\lambda}^n, u^n_v, w^n_v\right)$ to be the corresponding κ_n-frequent offers collections. By Proposition 3 part 4, it follows that $\bar{U}^n_s = \sum_v \mu_0(v)\, w^n_{0,v}$ and $\bar{U}^n_b = \sum_v \mu_0(v)\, u^n_{0,v}$.

I now prove the theorem through a sequence of lemmas. Lemma 30 proves that the total surplus, conditional on quality being v_l, converges to v_l. Lemma

31 proves that the hazard rates must satisfy a few properties at the limit at κ goes to zero. The next few lemmas use these properties to prove the theorem for (almost) every $t > 0$: Lemma 32 establishes that the total surplus conditional on any v converges to v. Combining this result with the characterization of u and w from Lemma 24, I prove Lemma 33, which shows that, conditional on v, B and S split the difference in surplus above v_l. Lemma 34 shows that B's surplus conditional on v_l is zero, hence proving the desired result for all $t > 0$. Extending the result to $t = 0$ using continuity concludes the proof.

Lemma 30. *For all t: $w^n_{t,v_l} + u^n_{t,v_l} \to v_l$.*

Proof. By lemmas 27 and 25: $\left(\lambda^n_{t,v_l}/\bar{\lambda}^n_t\right) \in \left[\frac{1}{2}, 1\right]$ for all t. Lemma 25 then implies that $\kappa_n^{-1}\left(v_l - \kappa_n - w^n_{t,v_l} - u^n_{t,v_l}\right)$ must remain finite. $w^n_{t,v_l} + u^n_{t,v_l} \to v_l$ follows. □

Lemma 31. *For every t:*

1. $\kappa_n \int_t^{t+\epsilon} e^{-r(s-t)}\left(2\lambda^n_{s,v_l} - \bar{\lambda}^n_s\right) \mathrm{d}s \to \left(1 - e^{-r\epsilon}\right) v_l$
2. $\lim_{n\to\infty} \kappa_n \int_t^{t+\epsilon} \bar{\lambda}^n_s \mathrm{d}s \geq \left(1 - e^{-r\epsilon}\right) v_l$
3. *For almost all t:$\bar{\lambda}^n_t \to \infty$.*

Proof. Part 1: By lemma 24:

$$w^n_{t,v_l} + u^n_{t,v_l} = \kappa_n \int_t^{t+\epsilon} e^{-r(s-t)}\left(2\lambda^n_{s,v_l} - \bar{\lambda}^n_s\right) \mathrm{d}s + e^{-r\epsilon}\left(w^n_{t+\epsilon,v_l} + u^n_{t+\epsilon,v_l}\right)$$

But by lemma 30, $w^n_{t,v_l} + u^n_{t,v_l} \to v_l$ for all t. The lemma follows for all t and $\epsilon > 0$.

Parts 2 and 3: By lemmas 27 and 25, $\bar{\lambda}^n_s > \lambda^n_{s,v_l}$ for all s. Therefore:

$$\begin{aligned}
\kappa_n \int_t^{t+\epsilon} \bar{\lambda}^n_t \mathrm{d}s &= \kappa_n \int_t^{t+\epsilon} \left(2\bar{\lambda}^n_t - \bar{\lambda}^n_t\right) \mathrm{d}s \\
&> \kappa_n \int_t^{t+\epsilon} \left(2\lambda^n_{s,v_l} - \bar{\lambda}^n_s\right) \mathrm{d}s \\
&> \kappa_n \int_t^{t+\epsilon} e^{-r(s-t)}\left(2\lambda^n_{s,v_l} - \bar{\lambda}^n_s\right) \mathrm{d}s \to \left(1 - e^{-r\epsilon}\right) v_l
\end{aligned}$$

implying part 2. Part 3 then follows from $\kappa_n \to 0$ and the fact that choice of t and ϵ were arbitrary. □

Lemma 32. *For every v, for $t = 0$, and for almost all $t > 0$: $w^n_{t,v} + u^n_{t,v} \to v$.*

Proof. I begin by proving that $w^n_{t,v_h} + u^n_{t,v_h} \to v_h$ for almost all $t > 0$ by contradiction. Suppose otherwise. Then there exists a subsequence $\left(\lambda^m_v, \bar{\lambda}^m, w^m, u^m\right)$, an interval $[t_1, t_2]$, and a Borel measurable function $f : \mathbb{R}_+ \to \mathbb{R}$ satisfying $f > 0$ almost everywhere in $[t_1, t_2]$ such that $v_h - \kappa_m - w^m_{t,v_h} - u^m_{t,v_h} > f(t)$ almost everywhere in $[t_1, t_2]$ for all m larger than some M. Then for $m > M$:

$$\begin{aligned} w^m_{t_1,v_h} + u^m_{t_1,v_h} &> \int_{t_1}^{t_2} e^{-r(s-t_1)} \kappa_m \left(2\lambda^m_{t,v_h} - \bar{\lambda}^m_t\right) \mathrm{d}s \\ &\geq \int_{t_1}^{t_2} e^{-r(s-t_1)} \left(2e^{\frac{f(s)}{\kappa_m}} - 1\right) \kappa_m \bar{\lambda}^m_t \mathrm{d}s \\ &\geq \int_{t_1}^{t_2} e^{-r(s-t_1)} \liminf_{m\to\infty} \left(2e^{\frac{f(s)}{\kappa_m}} - 1\right) \kappa_m \bar{\lambda}^m_t \mathrm{d}s \end{aligned}$$

where the second inequality follows from Fatou's lemma. By lemma 31,

$$\liminf_{m\to\infty} \kappa_m \int_{t_1}^{t_2} \bar{\lambda}^m_s \mathrm{d}s \geq \left(1 - e^{-r(t_2-t_1)}\right) v_l$$

and hence, $2\kappa_m \int_{t_1}^{t_2} \bar{\lambda}^m_s e^{\frac{f(s)}{\kappa_m}} \mathrm{d}s \to \infty$. Together, these imply $w^m_{t_1,v_h} + u^m_{t_1,v_h} \to \infty$ a contradiction to $w^m_{t_1,v_h} + u^m_{t_1,v_h} \leq v_h$ from lemma 29. To prove the result for all v, note that $\lambda^n_{t,v}$ is weakly increasing in v for all t (lemma 27). As such, $v - \kappa - w^n_{t,v} - u^n_{t,v}$ is increasing in v (lemma 25 part 1). The result for almost all $t > 0$ then follows from convergence for v_h and v_l (lemma 30). To prove the lemma for $t = 0$, note that by lemma 25, $\sum_v \mu_0(v) \left(\lambda^n_{0,v}/\bar{\lambda}^n_0\right) = 1$ for all n and $\left(\lambda^n_{0,v}/\bar{\lambda}^n_0\right) \geq \frac{1}{2}$ for all v. Therefore, $\exp \frac{1}{\kappa}\left(v - \kappa - w^n_{0,v} - u^n_{0,v}\right)$ converges to a strictly positive but finite number, which can only occur if $v - \kappa_n - w^n_{0,v} - u^n_{0,v} \to 0$. □

Lemma 33. *For every v, for $t = 0$, and for almost all $t > 0$: $u^n_{t,v} - u^n_{t,v_l} = w^n_{t,v} - w^n_{t,v_l} \to \frac{1}{2}(v - v_l)$.*

Proof. Applying lemma 32:

$$w^n_{t,v} + u^n_{t,v} - \left(w^n_{t,v_l} + u^n_{t,v_l}\right) \to v - v_l$$

But by lemma 24, $u^n_{t,v} - u^n_{t,v_l} = \int_t^\infty e^{-r(s-t)}\left(\lambda^n_{s,v} - \lambda^n_{s,v_l}\right)\mathrm{d}s = w^n_{t,v} - w^n_{t,v_l}$. Therefore, $u^n_{t,v} - u^n_{t,v_l} = w^n_{t,v} - w^n_{t,v_l} \to \frac{1}{2}(v - v_l)$ for almost all t and all v. □

Lemma 34. *For almost all $t > 0$, and every v: $w^n_{t,v} \to \frac{1}{2}(v + v_l)$ and $u^n_{t,v} \to \frac{1}{2}(v - v_l)$.*

Proof. By lemma 33, for almost every $t, s > 0$: $\left(w^n_{t,v} - w^n_{t,v_l}\right) - e^{-rs}\left(w^n_{t+s,v} - w^n_{t+s,v_l}\right)$ converges to $\frac{1}{2}\left(1 - e^{-rs}\right)(v - v_l)$. Note that for every n, and almost every $t > \epsilon > 0$:

$$\begin{aligned}
\int_0^t \left(\lambda^n_{s,v} - \lambda^n_{s,v_l}\right)\mathrm{d}s &> \int_\epsilon^t \left(\lambda^n_{s,v} - \lambda^n_{s,v_l}\right)\mathrm{d}s \\
&> \int_\epsilon^t e^{-r(s-\epsilon)}\left(\lambda^n_{s,v} - \lambda^n_{s,v_l}\right)\mathrm{d}s \\
&= \frac{1}{\kappa_n}\left(\left(w^n_{\epsilon,v} - w^n_{\epsilon,v_l}\right) - e^{-r(t-\epsilon)}\left(w^n_{t,v} - w^n_{t,v_l}\right)\right) \to \infty
\end{aligned}$$

where the first inequality follows from lemma 27 and the last equality from lemma 24. Hence,

$$\frac{\mu_0(v)\left(1 - F^n_v(t)\right)}{\mu_0(v_l)\left(1 - F^n_{v_l}(t)\right)} = \frac{\mu_0(v)}{\mu_0(v_l)} e^{-\int_0^t \left(\lambda^n_{s,v} - \lambda^n_{s,v_l}\right)\mathrm{d}s} \to 0$$

for almost all t and for all $v > v_l$. Therefore:

$$\begin{aligned}
\frac{1 - \bar{F}^n(t)}{1 - F^n_{v_l}(t)} &= \sum_v \mu_0(v)\left(\frac{1 - F^n_v(t)}{1 - F^n_{v_l}(t)}\right) \\
&= \mu_0(v_l) + \sum_{v > v_l} \mu_0(v)\left(\frac{1 - F^n_v(t)}{1 - F^n_{v_l}(t)}\right) \to \mu_0(v_l)
\end{aligned}$$

And thus:

$$\begin{aligned}
0 &\leq \left(\sum_v \mu_0(v)\left(\frac{1-F_v^n(t)}{1-\bar{F}^n(t)}\right)u_{t,v}^n\right)-u_{t,v_l}^n \\
&= \left(\frac{1-F_{v_l}^n(t)}{1-\bar{F}^n(t)}\sum_v \mu_0(v)\left(\frac{1-F_v^n(t)}{1-F_{v_l}^n(t)}\right)u_{t,v}^n\right)-u_{t,v_l}^n \\
&= \mu_0(v)\left(\frac{1-F_{v_l}^n(t)}{1-\bar{F}^n(t)}\right)u_{t,v_l}^n \\
&\quad +\left(\frac{1-F_{v_l}^n(t)}{1-\bar{F}^n(t)}\right)\sum_{v>v_l}\mu_0(v)\left(\frac{1-F_v^n(t)}{1-F_{v_l}^n(t)}\right)u_{t,v}^n-u_{t,v_l}^n \to 0
\end{aligned}$$

note that by lemmas 27 and 24:

$$u_{t,v_l}=\int_t^\infty e^{-r(s-t)}\left(\lambda_{s,v_l}-\bar{\lambda}_s\right)\mathrm{d}s<0$$

and from lemma 25:

$$\begin{aligned}
\sum_v \mu_0(v)\left(\frac{1-F_v^n(t)}{1-\bar{F}^n(t)}\right)u_{t,v}^n &= \int_t^\infty e^{-r(s-t)}\left(\sum_v \mu_0(v)\left(\frac{1-F_v^n(t)}{1-\bar{F}^n(t)}\right)\lambda_{s,v}-\bar{\lambda}_s\right)\mathrm{d}s \\
&> \int_t^\infty e^{-r(s-t)}\left(\sum_v \mu_0(v)\left(\frac{1-F_v^n(s)}{1-\bar{F}^n(s)}\right)\lambda_{s,v}-\bar{\lambda}_s\right)\mathrm{d}s \\
&= 0
\end{aligned}$$

where the inequality follows from lemma 27. Therefore:

$$0\leq -u_{t,v_l}^n \leq \left(\sum_v\left(\frac{\mu_0(v)\left(1-F_v^n(t)\right)}{1-\bar{F}^n(t)}\right)u_{t,v}^n\right)-u_{t,v_l}^n$$

for all n. But this implies: $\lim_{n\to 0} u_{t,v_l}^n \to 0$ for almost all t. As such, $w_{t,v_l}^n \to v_l$ for almost all t, and therefore $w_{t,v}^n \to \frac{1}{2}(v+v_l)$ for all v. $\square$

Proof of Theorem 5. Note that for every $t > 0$:

$$\begin{aligned} 0 > u^n_{0,v_l} &= \kappa_n \int_0^t e^{-rs}\left(\lambda^n_{s,v_l} - \bar{\lambda}^n_s\right) \mathrm{d}s + e^{-rt} u^n_{t,v_l} \\ &> \kappa_n \int_0^t e^{-rs}\left(\lambda^n_{s,v_l} - \lambda^n_{s,v_h}\right) \lambda^n_s \mathrm{d}s + e^{-rt} u^n_{t,v_l} \\ &= \left(\left(w^n_{0,v_h} - w^n_{0,v_l}\right) - e^{-rt}\left(w^n_{t,v_h} - w^n_{t,v_l}\right)\right) + e^{-rt} u^n_{t,v_l} \\ &\to -\left(1 - e^{-rt}\right)\left(\frac{v_h - v_l}{2}\right) \end{aligned}$$

where the both equalities follows from lemma 24, the inequality follows from lemma 27, and convergence follows from lemmas 33 and 34. Since t is arbitrary, $u^n_{0,v_l} \to 0$. Lemma 33 then implies $w^n_{0,v_l} \to v_l$, $w^n_{0,v} \to \frac{1}{2}(v + v_l)$ and $u^n_{0,v} \to \frac{1}{2}(v - v_l)$. The theorem follows from $\bar{U}^n_s = \sum_v \mu_0(v) w^n_{0,v}$ and $\bar{U}^n_b = \sum_v \mu_0(v) u^n_{0,v}$.

D.4 Technical Results for Frequent Offers Limit

The following section states (and proves when necessary) some technical results used in the proofs of Theorem *4* and Theorem *5*.

Let φ be the Lebesgue measure; i.e. the unique σ-additive measure over $\Omega = \left(\mathbb{R}_+, \mathcal{B}_{\mathbb{R}_+}\right)$, where $\mathcal{B}_{\mathbb{R}_+}$ is the Borel σ-algebra, satisfying $\varphi([t, t+s]) = \left(e^{-rt} - e^{-r(t+s)}\right)$ for all $t, s \geq 0$. As usual, let $L^2(\Omega, \mathrm{d}\varphi)$ be the set of all equivalence classes of measurable functions satisfying: $f : \Omega \to \mathbb{R}$, f is φ-measurable and: $\int_{\mathbb{R}_+} |f|^2 \mathrm{d}\varphi < \infty$, equipped with the norm: $\|f\|_2 = \left(\int_{\mathbb{R}_+} |f|^2 \mathrm{d}\varphi\right)^{1/2}$. A map L from $L^2(\Omega, \mathrm{d}\varphi)$ to the real numbers is a *linear functional* if: $L(af_1 + bf_2) = aL(f_1) + bL(f_2)$. A linear functional is *continuous* if $L(f^n) \to L(f)$ whenever $f^n \to f$ (according to the $\|\cdot\|_2$), and it is *bounded* if $|L(f)| \leq K\|f\|_2$ for some finite number K. It is well known that a functional is continuous if and only if it is bounded. We let $L^2(\Omega, \mathrm{d}\varphi)^*$ be the set of continuous linear funtionals, also known as the *dual* of $L^2(\Omega, \mathrm{d}\varphi)$. A sequence of functions $(f^n) \in L^2(\Omega, \mathrm{d}\varphi)$ is said to *converge weakly* to $f \in L^2(\Omega, \mathrm{d}\varphi)$, denoted by $f^n \rightharpoonup f$ if: $L(f^n) \to L(f)$ for every $L \in L^2(\Omega, \mathrm{d}\varphi)^*$. Below is a statement of a few famous theorems from functional analysis, specialized to the current setting. The next theorem is often seen as a consequence of the Hahn-Banach theorem.

Theorem 7. *Suppose $f \in L^2(\Omega, \mathrm{d}\varphi)$ satisfies $L(f) = 0$ for all $L \in L^2(\Omega, \mathrm{d}\varphi)^*$. Then $f = 0$, and therefore if $f^n \rightharpoonup g$ and $f^n \rightharpoonup h$ then $g = h$*

Proof. Lieb and Loss (2010), pages 56 to 57. □

Theorem 8. *Let $(f^n)_{n\geq 0}$ be a sequence of functions in $L^2(\Omega, \mathrm{d}\varphi)$ such that for every $L \in L^2(\Omega, \mathrm{d}\varphi)^*$, the sequence $L(f^n)$ is bounded. Then there exists a finite $C > 0$ such that $\|f^n\|_2 < C$ for all n.*

Proof. Lieb and Loss (2010), pages 58 to 59. □

The theorem below is a specialization of the Riesz representation theorem specific for our purposes.

Theorem 9. *For every $L \in L^2(\Omega, \mathrm{d}\varphi)^*$ there exists a unique $g \in L^2(\Omega, \mathrm{d}\varphi)$ such that: $L(f) = \int_{\mathbb{R}_+} g(x) f(x) \varphi(\mathrm{d}x)$. Moreover, for every $g \in L^2(\Omega, \mathrm{d}\varphi)$, $L_g(f) = \int_{\mathbb{R}_+} g(x) f(x) \varphi(\mathrm{d}x)$ is a bounded linear functional.*

Proof. Lieb and Loss (2010), pages 61 to 63. □

The following is a version of the Banach-Alaoglu theorem.

Theorem 10. *Let $(f^n)_{n\geq 0}$ be a sequence of functions bounded in $L^2(\Omega, \mathrm{d}\varphi)$. Then there exists a subsequence $(f^{n_k})_{k\geq 0}$ and an $f \in L^2(\Omega, \mathrm{d}\varphi)$ such that $f^{n_k} \rightharpoonup f$.*

Proof. Lieb and Loss (2010), pages 68 to 69. □

Lemma 35. *Let $(f^n)_{n\geq 0}, (g^n)_{n\geq 0}$ be two sequences in $L^2(\Omega, \mathrm{d}\varphi)$. Suppose $f^n \rightharpoonup f$ and $g^n \to g$ for some f and g in $L^2(\Omega, \mathrm{d}\varphi)$. Then: $\int_{\mathbb{R}_+} f^n(x) g^n(x) \mathrm{d}\varphi \to \int_{\mathbb{R}_+} f(x) g(x) \mathrm{d}\varphi$.*

Proof. Note that: $f^n g^n - fg = f^n(g^n - g) + (f^n - f) g$. Then:

$$\left| \int_{\mathbb{R}_+} f^n(x) (g^n(x) - g(x)) \mathrm{d}\varphi \right| \leq \|f^n\|_2 \|g^n - g\|_2 \leq C \|g^n - g\|_2 \to 0$$

Since $g \in L^2(\Omega, \mathrm{d}\varphi)$, we have that $L(h) = \int_{\mathbb{R}_+} h(x)\, g(x)\, \mathrm{d}\varphi \in L^2(\Omega, \mathrm{d}\varphi)^*$ and therefore $\int_{\mathbb{R}_+} (f^n(x) - f(x))\, g(x)\, \mathrm{d}\varphi \to 0$. The conclusion follows. $\square$