

Missing Bids

Sylvain Chassang Kei Kawai Jun Nakabayashi
New York University U.C. Berkeley Kindai University

Juan Ortner*[†]
Boston University

January 23, 2018

Preliminary - do not quote, do not distribute.

Abstract

We document a novel bidding pattern observed in procurement auctions from Japan: winning bids tend to be isolated. We prove that in a general class of models, missing bids robustly indicate non-competitive behavior. In addition, we provide evidence that missing bids coincide tightly with known cartel activity. Finally, we show that missing bids are consistent with efficient collusion in environments where it is difficult for bidders to coordinate on precise bids.

KEYWORDS: missing bids, collusion, isolated winner strategies, cartel enforcement, procurement.

*Chassang: chassang@nyu.edu, Kawai: kei@berkeley.edu, Nakabayashi: nakabayashi.1@eco.kindai.ac.jp, Ortner: jortner@bu.edu.

[†]We are especially indebted to Steve Tadelis for encouragement and detailed feedback. The paper benefited from discussions with Pierpaolo Battigali, Eric Budish, Yeon-Koo Che, Francesco Decarolis, Emir Kamenica, Roger Myerson, Ariel Pakes and Paulo Somaini, as well as comments from seminar participants at Bocconi, the 2017 Berkeley-Sorbonne workshop on Organizational Economics, the University of Chicago, and the 2017 NYU CRATE conference on theory and econometrics.

1 Introduction

This paper documents a novel bidding pattern found in multiple datasets describing public procurement auctions held in Japan: the density of bids just above the winning bid is very low. Put differently, winning bids tend to be isolated. We show that these missing bids indicate non-competitive behavior under a general class of asymmetric information models. Indeed, this missing mass of bids makes it a profitable stage-game deviation for bidders to increase their bids. Motivated by these findings, we develop structural tools that allow us to quantify the extent of non-competitive behavior in the data. Finally, we propose an explanation for why this bidding pattern arises, and discuss what it suggests about the challenges of sustaining collusion.

Our data comes from two separate datasets of public procurement auctions taking place in Japan. Our first data-set, already analyzed by Kawai and Nakabayashi (2014), assembles roughly 90,000 national-level auctions for public work projects taking place between 2001 and 2006. Our second dataset, previously studied by Chassang and Ortner (2016), assembles approximately 1,500 city-level auctions for public works projects taking place between 2007 and 2014. In both cases, we are interested in the distribution bidders' margins of victory/defeat. In other terms, for every (bidder, auction) pair, we are interested in the difference Δ between the bidder's own bid and the most competitive bid among this bidder's opponents, normalized by the reserve price. When $\Delta < 0$, the bidder won the auction. When $\Delta > 0$ the bidder lost. The finding motivating this paper is summarized by Figure 1, which plots the distribution of margins of victory Δ in the sample of national-level auctions. The distribution follows a truncated bell curve, except that there is a visible gap in the distribution at $\Delta = 0$.

Our primary goal for this paper is to clarify the sense in which this gap is suspicious. For this purpose, we consider a fairly general model of repeated play in first-price procurement auctions. A group of firms repeatedly participates in first-price procurement auction. Firms'

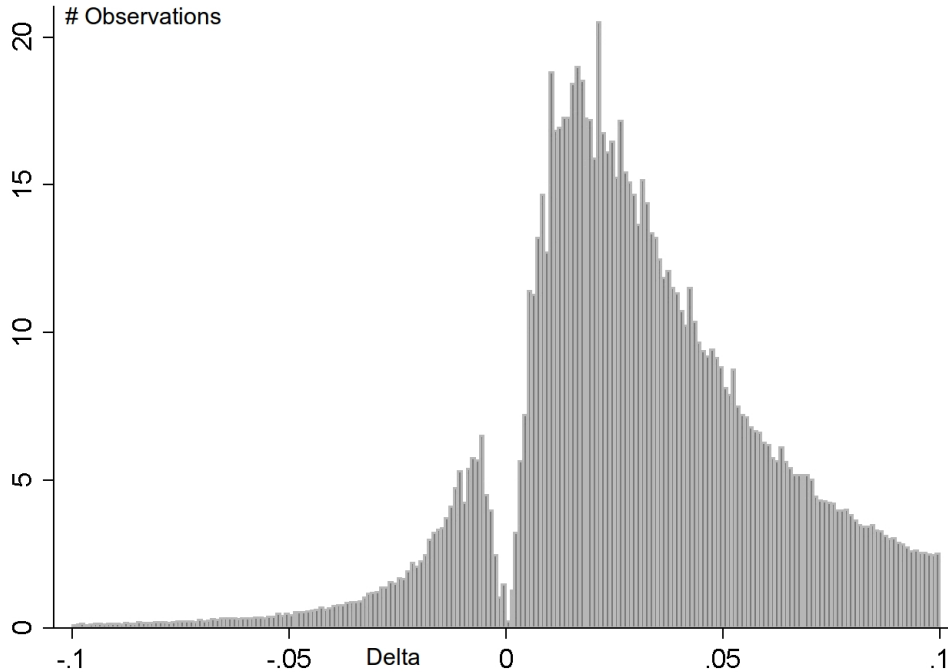


Figure 1: Distribution of margins of victory Δ

costs can be serially correlated over time, and we allow for general asymmetric information. We are interested in characterizing the extent to which players' behavior can be rationalized as competitive, in the sense of being stage-game optimal at the player level.

Our first set of results identifies conditions that any dataset arising from a competitive equilibrium must satisfy. In any competitive equilibrium, firms must not find it profitable to increase their bids. We show that this incentive constraint implies that the elasticity of firms' *counterfactual demand* (i.e., the probability of winning an auction at any given bid) must be bounded above by -1. This condition is not satisfied in our data: since winning bids are isolated, the elasticity of counterfactual demand is approximately zero for some industrial sectors in our data.

Our second set of results builds on these observations to quantify the extent of non-competitive behavior in the data. We propose a new measure of collusion corresponding to the smallest share of the data that must be excluded, in order to rationalize the remaining

data as competitive. We show that this program is computationally tractable and delineate how different patterns of demand map into restrictions on the set of possibly competitive histories.

Finally, we propose a tentative explanation for missing bids, and why they could plausibly arise as an implication of collusive behavior. This is not entirely obvious because missing bids are not rationalized by standard models of tacit collusion (i.e., Rotemberg and Saloner (1986), Athey and Bagwell (2001, 2008)). In these models, the cartel's main concern is to incentivize losers not to undercut the winning bid. The behavior of designated winners is stage game optimal. We show that missing bids arise as an optimal response to noise. Keeping the designated winner's bid isolated ensures that small trembles in play do not cause severe misallocations.

Our paper relates primarily to the literature on cartel detection.¹ Porter and Zona (1993, 1999) show that suspected cartel members use different bidding strategies than non-cartel members. Bajari and Ye (2003) design a test of collusion based on excess correlation across bids. Porter (1983), Ellison (1994) and Chassang and Ortner (2016) build on classic theories of repeated games (i.e., Green and Porter (1984), Rotemberg and Saloner (1986)) to detect collusion. Conley and Decarolis (2016) propose a test to detect collusive bidders competing in average-price auctions. Kawai and Nakabayashi (2014) analyze auctions with re-bidding, and exploit correlation patterns in bids across stages to detect collusion. We provide a new test of collusion that is robust to arbitrary information structures, and that allows us to quantify the extent of collusion in the data.

Our paper also relates to a set of papers studying the internal organization of cartels. Asker (2010) studies stamp auctions, and analyses the effect of a particular collusive scheme on non-cartel bidders and sellers. Pesendorfer (2000) studies the bidding patterns for school milk contracts and compares the collusive schemes used by strong cartels and weak cartels (i.e., cartels that used transfers and cartels that didn't). Clark and Houde (2013) document

¹See Harrington (2008) for a recent survey of this literature.

the collusive strategies used by the retail gasoline cartel in Quebec. We add to this literature by documenting a novel bidding pattern, and argue that this bidding behavior reflects some of the frictions that cartels face.

The paper is structured as follows. Section 2 describes our data and documents the bidding patterns that motivate our paper. Section 3 introduces our framework. Section 4 presents our main theoretical findings: we show that missing bids are inconsistent with competition, and derive bounds on the maximum share of competitive histories consistent with the data. Section 5 illustrates our approach with data. Section 6 proposes an interpretation of missing bids as a feature of optimal collusive behavior in noisy environments. Proofs are collected in Appendix A unless mentioned otherwise.

2 Motivating Facts

We draw on two sets of data. The first dataset, analyzed in Kawai and Nakabayashi (2014), consists of roughly 90,000 auctions held between 2001 and 2006 by the Ministry of Land, Infrastructure, Transport and Tourism in Japan (the Ministry). The auctions are first-price auctions with secret reserve price, and re-bidding in case there is no successful winner. The auctions involve construction projects, the median winning bid is USD 600K, and the median participation is 10. Our second dataset, analyzed in Chassang and Ortner (2016), consists of roughly 1,500 auctions held between 2007 and 2014 by the city of Tsuchiura in the Ibaraki prefecture. Projects are allocated using a standard first-price auction with public reserve price. The median winning bid is USD 130K, and the median participation is 4. In both cases, the bids of all participants are publicly revealed after the auctions, and reported in our data.

For any given firm, we investigate the distribution of

$$\Delta = \frac{\text{own bid} - \text{most competitive bid}}{\text{reserve price}}.$$

The value Δ represents the margin by which a bidder wins or lose an auction. If $\Delta < 0$ the bidder won, if $\Delta > 0$ he won. At $\Delta = 0$, the bidder barely won.

The left panel of Figure 2 plots the distribution of bid differences Δ for a large firm in the sample of auctions held by the Ministry. The right panel aggregates bid differences over the sample firms in the data. The mass of missing bids around a difference of 0 is starkly

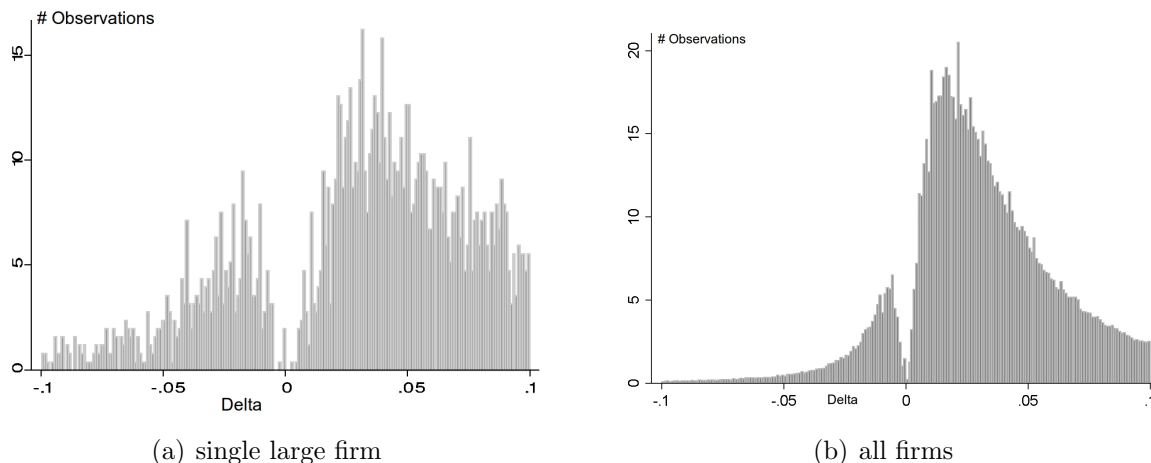


Figure 2: Distribution of bid-difference Δ – national data.

visible. This pattern is not limited to a particular firm and remains clearly noticeable when aggregating over all auctions in our sample.²

Figure 3 presents plots the distribution of Δ for auctions held in Tsuchiura. The left panel uses all the bids in the sample. Again, we see a significant mass of missing bids around zero. The right panel shows that the pattern all but disappears when we exclude winning bids from the analysis.

Our objective in this paper is to: 1) formalize why this pattern is suspicious; 2) delineate what it implies about bidding behavior and the competitiveness of auctions in our sample; 3) propose a possible explanation for why this behavior arises as a feature of optimal bidding. To do so we use a model of repeated auctions.

²Note that the distribution of normalized bid-differences is skewed to the right since the most competitive alternative bid is a minimum over other bidders' bids.

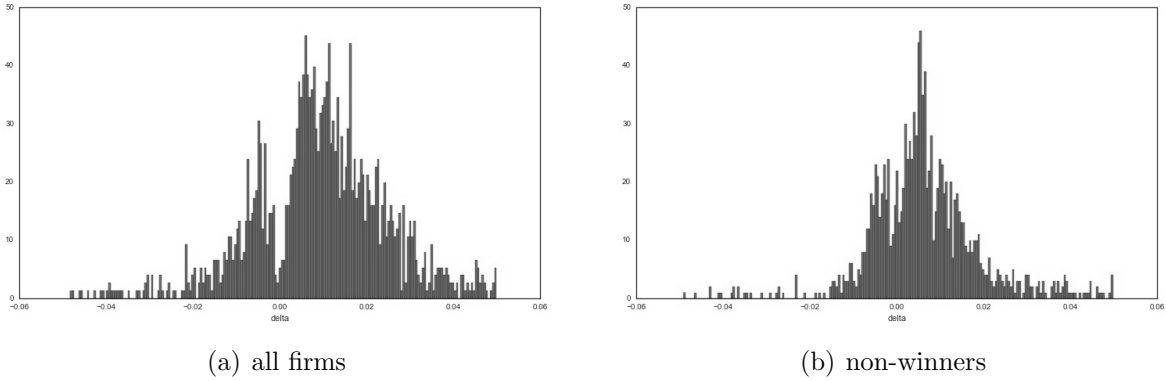


Figure 3: Distribution of bid-difference Δ – city data.

3 Framework

We consider a dynamic setting in which, at each period $t \in \mathbb{N}$, a buyer needs to procure a single project. The auction format is a first-price auction with reserve price r , which we normalize to $r = 1$.

In each period $t \in \mathbb{N}$, a set $\widehat{N}_t \subset N$ of bidders is able to participate in the auction, where N is the overall set of bidders. We think of this set of participating firms as those eligible to produce in the current period.³ The sets of eligible bidders can vary over time.

Realized costs of production for eligible bidders $i \in \widehat{N}_t$ are denoted by $\mathbf{c}_t = (c_{i,t})_{i \in \widehat{N}_t}$. Each bidder $i \in \widehat{N}_t$ submits a bid $b_{i,t}$. Profiles of bids are denoted by $\mathbf{b}_t = (b_{i,t})_{i \in \widehat{N}_t}$. We let $\mathbf{b}_{-i,t} \equiv (b_{j,t})_{j \neq i}$ denote bids from firms other than firm i , and define $\wedge \mathbf{b}_{-i,t} \equiv \min_{j \neq i} b_{j,t}$ to be the lowest bid among i 's opponents at time t . The procurement contract is allocated to the bidder submitting the lowest bid at a price equal to her bid.

In the case of ties, we follow Athey and Bagwell (2001) and let the bidders jointly determine the allocation. This simplifies the analysis but requires some formalism (which can be skipped at moderate cost to understanding). We allow bidders to simultaneously pick numbers $\gamma_t = (\gamma_{i,t})_{i \in \widehat{N}_t}$ with $\gamma_{i,t} \in [0, 1]$ for all i, t . When lowest bids are tied, the allocation

³See Chassang and Ortner (2016) for a treatment of endogenous participation by cartel members.

to a lowest bidder i is

$$x_{i,t} = \frac{\gamma_{i,t}}{\sum_{\{j \in \hat{N}_t \text{ s.t. } b_{j,t} = \min_k b_{k,t}\}} \gamma_{j,t}}.$$

Participants discount future payoffs using common discount factor $\delta < 1$. Bids are publicly revealed at the end of each period.

Costs. We allow for costs that are serially correlated over time, and that may be correlated across firms within each auction. Denoting by $\langle \cdot, \cdot \rangle$ the usual dot-product we assume that costs take the form

$$c_{i,t} = \langle \alpha_i, \theta_t \rangle + \varepsilon_{i,t} > 0 \tag{1}$$

where

- parameters $\alpha_i \in \mathbb{R}^k$ are fixed over time;
- $\theta_t \in \mathbb{R}^k$ may be unknown to the bidders at the time of bidding, but is revealed to bidders at the end of period t ; we assume that θ_t follows a Markov chain;
- $\varepsilon_{i,t}$ is i.i.d. with mean zero conditional on θ_t .

In period t , bidder i obtains profits

$$\pi_{i,t} = x_{i,t} \times (b_{i,t} - c_{i,t}).$$

Note that costs include both the direct costs of production and the opportunity cost of backlog.

The sets \hat{N}_t of bidders are independent across time conditional on θ_t , i.e.

$$\hat{N}_t | \theta_{t-1}, \hat{N}_{t-1}, \hat{N}_{t-2} \dots \sim \hat{N}_t | \theta_{t-1}.$$

Information. In each period t , bidder i gets a signal $z_{i,t}$ that is conditionally i.i.d. given $(\theta_t, (c_{j,t})_{j \in \hat{N}_t})$. This allows our model to nest many informational environments, including

asymmetric information private value auctions, common value auctions, as well as complete information. Bids \mathbf{b}_t are observable at the end of the auction.

Transfers. Bidders are able to make positive transfers from one to the other at the end of each period. A transfer from i to j is denoted by $T_{i \rightarrow j, t} \geq 0$. Transfers are costly, and we denote by $K \left(\sum_{j \neq i} T_{i \rightarrow j, t} \right)$ the cost to player i of the transfers she makes. We assume that K is positive, increasing and convex. Altogether, flow realized payoffs to player i in period t take the form

$$u_{i,t} = \pi_{i,t} + \sum_{j \neq i} T_{j \rightarrow i, t} - K \left(\sum_{j \neq i} T_{i \rightarrow j, t} \right).$$

Solution Concepts. The public history h_t at period t takes the form

$$h_t = (\theta_{s-1}, \mathbf{b}_{s-1}, \mathbf{T}_{s-1})_{s \leq t},$$

where \mathbf{T}_s are the transfers made in period s . Our solution concept is perfect public Bayesian equilibrium (σ, μ) (Athey and Bagwell (2008)), with strategies

$$\sigma_i : h_t \mapsto (b_{i,t}(z_{i,t}), (T_{i \rightarrow j, t}(z_{i,t}, \mathbf{b}_t))_{j \neq i}),$$

where bids $b_{i,t}(z_{i,t}) \in \Delta([0, r])$ and transfers $(T_{i \rightarrow j, t}(z_{i,t}, \mathbf{b}_t))_{j \neq i} \in \Delta(\mathbb{R}^{n-1})$ depend on the public history and on the information available at the time of decision making. We let \mathcal{H} denote the set of all public histories.

We emphasize the class of competitive equilibria, or in this case Markov perfect equilibria (Maskin and Tirole, 2001). In a competitive equilibrium, players condition their play only on payoff relevant parameters.

Definition 1 (competitive strategy). *We say that (σ, μ) is competitive (or Markov perfect) if and only if $\forall i \in N$ and $\forall h_t \in \mathcal{H}$, $\sigma_i(h_t, z_{i,t})$ depends only on $(\theta_{t-1}, z_{i,t})$.*

We say that a strategy profile (σ, μ) is a competitive equilibrium if it is a perfect public Bayesian equilibrium in competitive strategies.

We note that in a competitive equilibrium, firms must be playing a stage-game Nash equilibrium at every period; that is, firms must play a static best-reply to the actions of their opponents. Generally, an equilibrium may include periods in which firms collude and periods in which they compete. This leads us to define competitive histories.

Competitive histories. Fix a perfect public Bayesian equilibrium (σ, μ) . Given a public history $h_t \in \mathcal{H}$ and firm i 's private signal $z_{i,t}$, let $h_{i,t} = (h_t, z_{i,t})$. Note that, under perfect public Bayesian equilibrium (σ, μ) , firm i 's strategy at time t depends on $h_{i,t}$.

Definition 2 (competitive histories). *Fix an equilibrium (σ, μ) and a history $h_{i,t} = (h_t, z_{i,t})$. We say that (σ, μ) is competitive at $h_{i,t}$ if play at $h_{i,t}$ is stage-game optimal for firm i .*

4 Inference

In this section, we show how to exploit equilibrium conditions at different histories to obtain bounds on the share of competitive histories. The first step is to obtain aggregates of counterfactual demand that can be estimated from data, even though the players' residual demands can vary with the history.

4.1 Counterfactual demand

Fix a perfect public Bayesian equilibrium (σ, μ) . For all public histories $h_{i,t} = (h_t, z_{i,t})$ and all bids $b' \in [0, r]$, player i 's counterfactual demand at $h_{i,t}$ is

$$D_i(b'|h_{i,t}) \equiv \text{prob}_{\sigma, \mu}(\wedge \mathbf{b}_{-i,t} > b'|h_{i,t}).$$

For any finite set of histories $H = \{(h_t, z_{i,t})\} = \{h_{i,t}\}$, and any scalar $\rho \in (-1, \infty)$, define

$$\overline{D}(\rho|H) \equiv \sum_{h_{i,t} \in H} \frac{1}{|H|} D_i((1 + \rho)b_{i,t}|h_{i,t})$$

to be the average counterfactual demand for histories in H .

Fix a set of histories $H = \{(h_t, z_{i,t})\} = \{h_{i,t}\}$, and define

$$\widehat{D}(\rho|H) \equiv \sum_{h_{i,t} \in H} \frac{1}{|H|} \mathbf{1}_{\wedge \mathbf{b}_{-i,t} > (1+\rho)b_{i,t}}.$$

We say that set H is adapted to the players' information if and only if the event $h_{i,t} \in H$ is measurable with respect to player i 's information at time t prior to bidding. For instance, the set of auctions for a specific industry with reserve prices above a certain threshold is adapted. In contrast, the set of auctions in which the margin of victory is below a certain level is not.

Theorem 1. *Consider a sequence of adapted sets H such that $|H| \rightarrow \infty$. Under any perfect public Bayesian equilibrium (σ, μ) , with probability 1, $\widehat{D}(\rho|H) - \overline{D}(\rho|H) \rightarrow 0$.*

In other words, in equilibrium, the sample residual demand conditional on an adapted set of histories converges to the true subjective aggregate conditional demand. This result can be viewed as a weakening of the equilibrium requirement that beliefs be correct. It may fail in settings with sufficiently strong non-common priors.

The ability to legitimately vary the conditioning set H lets us explore the competitiveness of auctions in particular subsettings of interest.

4.2 A Test of Non-Competitive Behavior

The pattern of bids illustrated in Figures 1, 2 and 3 is striking. Our first main result shows that its more extreme forms are inconsistent with competitive behavior.

Proposition 1. *Let (σ, μ) be a competitive equilibrium. Then,*

$$\forall h_i, \quad \frac{\partial \log D_i(b'|h_i)}{\partial \log b'} \Big|_{b'=b_i^+(h_i)} \leq -1, \quad (2)$$

$$\forall H, \quad \frac{\partial \log \bar{D}(\rho|H)}{\partial \rho} \Big|_{\rho=0^+} \leq -1. \quad (3)$$

In other terms, under any non-collusive equilibrium, the elasticity of counterfactual demand must be less than -1 at every history. The data presented in the left panel of Figure 2 contradicts the results in Proposition 1. Note that for every $i \in N$ and every h_i ,

$$\begin{aligned} D_i(b'|h_i) &= \text{prob}_\sigma(b' - \wedge \mathbf{b}_{-i} < 0 | h_i) \\ &= \text{prob}_\sigma(b' - b_i + \Delta_i < 0 | h_i), \end{aligned}$$

where we used $\Delta_i = \frac{b_i - \wedge \mathbf{b}_{-i}}{r} = b_i - \wedge \mathbf{b}_{-i}$ (since we normalized $r = 1$). Since the density of Δ_i at 0 is essentially 0 for some sets of histories in our data, the elasticity of demand is approximately zero as well in these histories.

Proof. Consider a competitive equilibrium (σ, μ) . Let u_i denote the flow payoff of player i , and let $V(h_{i,t}) \equiv \mathbb{E}_{\sigma, \mu} \left(\sum_{s \geq t} \delta^{s-t} u_{i,s} \mid h_{i,t} \right)$ denote her discounted expected payoff at history $h_{i,t} = (h_t, z_{i,t})$.

Let $b_{i,t} = b$ be the bid that bidder i places at history $h_{i,t}$. Since $b_{i,t} = b$ is an equilibrium bid, it must be that for all bids $b' > b$,

$$\begin{aligned} \mathbb{E}_{\sigma, \mu} \left[(b - c_{i,t}) \mathbf{1}_{\wedge \mathbf{b}_{-i,t} > b} + \delta V(h_{i,t+1}) \mid h_{i,t}, b_{i,t} = b \right] \\ \geq \mathbb{E}_{\sigma, \mu} \left[(b' - c_{i,t}) \mathbf{1}_{\wedge \mathbf{b}_{-i,t} > b'} + \delta V(h_{i,t+1}) \mid h_{i,t}, b_{i,t} = b' \right] \end{aligned}$$

Since (σ, μ) is competitive, $\mathbb{E}_{\sigma, \mu} [V(h_{i,t+1}) \mid h_{i,t}, b_{i,t} = b] = \mathbb{E}_{\sigma, \mu} [V(h_{i,t+1}) \mid h_{i,t}, b_{i,t} = b']$. Hence,

we must have

$$\begin{aligned} bD_i(b|h_{i,t}) - b'D_i(b'|h_{i,t}) &= \mathbb{E}_{\sigma,\mu} [b\mathbf{1}_{\mathbf{b}_{-i,t} > b} - b'\mathbf{1}_{\mathbf{b}_{-i,t} > b'} | h_{i,t}] \\ &\geq \mathbb{E}_{\sigma,\mu} [c_{i,t}(\mathbf{1}_{\mathbf{b}_{-i,t} > b} - \mathbf{1}_{\mathbf{b}_{-i,t} > b'}) | h_{i,t}] \geq 0, \end{aligned} \quad (4)$$

where the last inequality follows since $c_{i,t} \geq 0$. Inequality (4) implies that, for all $b' > b$,

$$\frac{\log D_i(b'|h_i) - \log D_i(b|h_i)}{\log b' - \log b} \leq -1.$$

Inequality (2) follows from taking the limit as $b' \rightarrow b$. Inequality (3) follows from summing (4) over histories in H , and performing the same computations. ■

As the proof highlights, this result exploits the fact that in procurement auctions, zero is a natural lower bound for costs (see inequality (4)). In contrast, for auctions where bidders have a positive value for the good, there is no obvious upper bound to valuations to play that role. One would need to impose an ad hoc upper bound on values to establish similar results.

An implication of Proposition 1 is that, in our data, bidders have a short-term incentive to increase their bids. To keep participants from bidding higher, for every $\epsilon > 0$ small, there exists $\nu > 0$ and a positive mass of histories $h_{i,t} = (h_t, z_{i,t})$ such that,

$$\delta \mathbb{E}_{\sigma,\mu} [V(h_{i,t+1}) | h_{i,t}, b_i(h_{i,t})] - \delta \mathbb{E}_{\sigma,\mu} [V(h_{i,t+1}) | h_{i,t}, b_i(h_{i,t})(1 + \epsilon)] > \nu. \quad (5)$$

In other terms, equilibrium (σ, μ) must give bidders a dynamic incentive not to overcut the winning bid.

Proposition 1 proposes a simple test of whether a dataset H can be generated by a competitive equilibrium or not. We now refine this test to obtain bounds on the minimum share of non-competitive histories needed to rationalize the data. We begin with a simple

loose bound and then propose a more sophisticated program resulting in tighter bounds.

4.3 A simple bound on the share of competitive histories

Fix a perfect public Bayesian equilibrium (σ, μ) and a finite set of histories H . Let $H^{\text{comp}} \subset H$ be the set of competitive histories in H , and let $H^{\text{coll}} = H \setminus H^{\text{comp}}$. Define $s_{\text{comp}} \equiv \frac{|H^{\text{comp}}|}{|H|}$ to be the fraction of competitive histories in H .

For all histories $h_{i,t} = (h_t, z_{i,t})$ and all bids $b' \geq 0$, player i 's *counterfactual revenue* at $h_{i,t}$ is

$$R_i(b'|h_{i,t}) \equiv b' D_i(b'|h_{i,t}).$$

For any finite set of histories H and scalar $\rho \in (-1, \infty)$, define

$$\bar{R}(\rho|H) \equiv \sum_{h_{i,t} \in H} \frac{1}{|H|} (1 + \rho) b_{i,t} D_i((1 + \rho) b_{i,t} | h_{i,t})$$

to be the average counterfactual revenue for histories in H . Our next result builds on Proposition 1 to derive a bound on s_{comp} .

Proposition 2. *The share s_{comp} of competitive auctions is such that*

$$s_{\text{comp}} \leq 1 - \sup_{\rho > 0} \frac{\bar{R}(\rho|H) - \bar{R}(0|H)}{\rho}.$$

Proof. For any $\rho > 0$,

$$\begin{aligned} \frac{1}{\rho} [\bar{R}(\rho|H) - \bar{R}(0|H)] &= s_{\text{comp}} \frac{1}{\rho} [\bar{R}(\rho|H^{\text{comp}}) - \bar{R}(0|H^{\text{comp}})] \\ &\quad + (1 - s_{\text{comp}}) \frac{1}{\rho} [\bar{R}(\rho|H^{\text{coll}}) - \bar{R}(0|H^{\text{coll}})] \\ &\leq 1 - s_{\text{comp}}. \end{aligned}$$

The last inequality follows from two observations. First, since the elasticity of counterfactual

demand is bounded above by -1 for all competitive histories (Proposition 1), it follows that $\overline{R}(\rho|H^{\text{comp}}) - \overline{R}(0|H^{\text{comp}}) \leq 0$. Second,

$$\frac{1}{\rho}[\overline{R}(\rho|H^{\text{coll}}) - \overline{R}(0|H^{\text{coll}})] \leq \frac{1}{\rho}((1 + \rho)\overline{R}(0|H^{\text{coll}}) - R(0|H^{\text{coll}})) = \overline{R}(0|H^{\text{coll}}) \leq r = 1.$$

■

In words, if total revenue in histories H increases by more than $\kappa \times \rho$ when bids are uniformly increased by $(1 + \rho)$, the share of competitive auctions in H is bounded above by $1 - \kappa$.

For each each $\rho \in (-1, \infty)$, define

$$\widehat{R}(\rho|H) \equiv \sum_{h_{i,t} \in H} \frac{1}{|H|} (1 + \rho)b_{i,t} \mathbf{1}_{\wedge b_{-i,t} > (1+\rho)b_{i,t}}.$$

Note that $\widehat{R}(\rho|H)$ is the sample analog of counterfactual revenue. A result identical to Theorem 1 establishes that $\widehat{R}(\rho|H)$ is an unbiased estimate of $\overline{R}(\rho|H)$, whenever set H is adapted. We have the following corollary to Proposition 2.

Corollary 1. *Suppose there exists $\rho > 0$ and $\kappa > 0$ such that*

$$\frac{\widehat{R}(\rho|H) - \widehat{R}(0|H)}{\rho} \geq 2\kappa.$$

Then, with probability at least $1 - 4 \exp(-\frac{(\rho\kappa^2)|H|}{8})$, $s_{\text{comp}} \leq 1 - \kappa$.

Corollary 1 allows us to obtain an estimate on the share of competitive histories.

4.4 Tight inference

We now seek to establish tight bounds on the set of competitive histories. We expand on Section 4.3 by fully exploiting the empirical content of upwards deviations. In addition we

consider the empirical content of downward deviations.

Indeed, although the residual demand in our data is inelastic immediately around winning bids, it is very elastic for large downward deviations. Moderate drops in price (a few percentage points) lead to large increases in the likelihood of winning the contract. This suggests that jointly considering upward and downward deviations will provide a tighter bound on the share of competitive histories than the bound in Proposition 2 and Corollary 1.

For simplicity, we assume that players interact in a private value environment. Take as given an adapted set of histories H and scalars $(\rho_n)_{\{n=-\underline{n}, \dots, 0, \dots, \bar{n}\}}$, with $\rho_n \in (-1, \infty)$, $\rho_0 = 0$ and $\rho_n < \rho_{n'}$ for all $n' > n$. For each history $h_{i,t} \in H$, let $d_{h_{i,t},n} = D_i((1 + \rho_n)b_{h_{i,t}} | h_{i,t})$. That is, $(d_{h_{i,t},n})_{\{n=-\underline{n}, \dots, 0, \dots, \bar{n}\}}$ is firm i 's subjective counterfactual demand at history $h_{i,t}$ if it had set a bid equal to $(1 + \rho_n)b_{i,t}$. For each n , define

$$D_n \equiv \frac{1}{|H|} \sum_{h_{i,t} \in H} d_{h_{i,t},n} \quad \text{and} \quad \widehat{D}_n \equiv \frac{1}{|H|} \sum_{h_{i,t} \in H} \mathbf{1}_{(1+\rho_n)b_{h_{i,t}} < \wedge b_{-i,h_{i,t}}}.$$

Under perfect Bayesian equilibrium, subjective counterfactual demand at competitive histories must satisfy four types of constraints: feasibility constraints, individual optimality constraints, aggregate consistency constraints, and ad hoc economic plausibility constraints. Formally, for every history $h \in H$ there must exist costs c_h and subjective demands $(d_{h,n})_{n=-\underline{n}, \dots, \bar{n}}$ satisfying the following conditions

Feasibility. Costs and beliefs must be feasible, satisfying

$$\forall h \in H, \quad c_h \in [0, b_h]; \quad \forall n, \quad d_{h,n} \in [0, 1]; \quad \forall n, n' > n, \quad d_{h,n} \geq d_{h,n'}. \quad (6)$$

Individual optimality. Bidding b_h must be optimal, given cost and subjective beliefs:

$$\forall n, \quad [(1 + \rho_n)b_h - c_h] d_{h,n} \leq ((1 + \rho_0)b_h - c_h) d_{h,0} \quad (7)$$

Aggregate consistency. Bidders' subjective demand must be consistent with aggregate data. Given a tolerance level $T > 0$, aggregate subjective demand at histories $h \in H$ is consistent with the data if and only if

$$\forall n, \quad D_n = \frac{1}{|H|} \sum_{h \in H} d_{h,n} \in \left[\widehat{D}_n - T, \widehat{D}_n + T \right] \quad (8)$$

The aggregate consistency conditions must hold since, by Theorem 1, for all n \widehat{D}_n is an unbiased estimator of aggregate counterfactual demand $D_n = \frac{1}{|H|} \sum_{h \in H} d_{h,n}$.

Economic plausibility. In addition to incentive compatibility and aggregate consistency, one may be able to impose plausible ad hoc constraints on the bidder's economic environment at each history h . We focus on two intuitive constraints on the bidder's costs c_h and interim beliefs ($d_{h,n}$):

$$\frac{b_h}{c_h} \leq 1 + m \quad (9)$$

and

$$\forall n, \quad \left| \log \frac{d_{h,n}}{1 - d_{h,n}} - \log \frac{D_n}{1 - D_n} \right| \leq k \quad (10)$$

where $m \in [0, +\infty]$ is a maximum markup, and $k \in [0, +\infty)$ provides an upper bound to the information contained in any signal.⁴

From now on, we assume that, for each $h \in H$, there exists ($d_{h,n}$) and a cost $c_h \in [b_h/(1 + m), b_h]$ satisfying (6), (7) and (10).⁵

⁴To see why, that that $\log \frac{d_{h,n}}{1 - d_{h,n}} = \log \frac{\text{prob}(Z|h)}{\text{prob}(\neg Z|h)}$ for Z the event that $\wedge \mathbf{b}_{-i} > (1 + \rho_n)b_h$. Hence, k is a bound on the log-likelihood ratio of signals that bidders get. One focal case in which $k = 0$ is that of i.i.d. types.

⁵If this was not true for some history $h \in H$, then history h could not be rationalized by a competitive model that satisfies conditions (9) and (10).

The following Proposition shows that, if the histories in H are all competitive, then with high probability the conditions above all hold simultaneously.

Proposition 3. *Consider an economic environment in which conditions (9) and (10) hold. There exists $\alpha > 0$ and $\beta > 0$ such that, for all PBEs (σ, μ) and all adapted sets H , whenever (σ, μ) is competitive at histories $h \in H$, then with probability at least $1 - \beta \exp(-\alpha|H|)$, conditions (6), (7) and (8) hold simultaneously.*

We define the share of non-competitive histories as the minimum share of histories that must be excluded from the data so that the remaining histories are consistent with competitive play. Formally:

Definition 3 (share of competitive histories). *For any set of histories H , we define the maximum share of competitive histories in H as*

$$\widehat{s}_{comp} \equiv \frac{1}{|H|} \max_{\mathbf{p}^C = (p_h^C)_{h \in H} \in [0,1]^{|H|}} \sum_{h \in H} p_h^C \quad (11)$$

such that there exists $((d_{h,n}), c_h)_{h \in H}$ satisfying history-level constraints (6), (7), (9), and modified aggregate constraints and information constraints

$$\forall n, \quad \frac{1}{\sum_{h \in H} p_h^C} \sum_{h \in H} p_h^C d_{h,n} \in [\widehat{D}_n(\mathbf{p}^C) - T, \widehat{D}_n(\mathbf{p}^C) + T] \quad (8')$$

$$\forall n, \quad \left| \log \frac{d_{h,n}}{1 - d_{h,n}} - \log \frac{\widehat{D}_n(\mathbf{p}^C)}{1 - \widehat{D}_n(\mathbf{p}^C)} \right| \leq k \quad (10')$$

where

$$\forall n, \quad \widehat{D}_n(\mathbf{p}^C) \equiv \frac{1}{\sum_{h \in H} p_h^C} \sum_{h \in H} p_h^C \mathbf{1}_{(1+\rho_n)b_h < \wedge b_{-i,h}}$$

Note that Program (11) allows us to discard fractions $p_h \in [0, 1]$ of each history $h \in H$. As the following result shows, this convexification of the problem implies that \widehat{s}_{comp} is an

upper bound on the true share s_{comp} of competitive histories in H .

Corollary 2. *Consider a public perfect Bayesian equilibrium (σ, μ) and an economic environment in which conditions (9) and (10) hold. Let H be an adapted set of histories such that a share $s_{\text{comp}} \in (0, 1]$ is competitive. Then, there exists $\alpha > 0$ and $\beta > 0$ such that, with probability at least $1 - \beta \exp(-\alpha s_{\text{comp}} |H|)$, $\widehat{s}_{\text{comp}} \geq s_{\text{comp}}$.*

The proof of Corollary 2 shows that constants α and β are equal to $T^2/2$ and $2(1 + \underline{n} + \bar{n})$, respectively. Corollary 2 can be used to derive the following statistical test. Let $H_0 = s_{\text{comp}} \geq s$ for some $s \in (0, 1]$, and let $H_1 = s_{\text{comp}} < s$. Pick the tolerance level T such that $1 - \beta \exp(-\alpha s |H|) = 0.05$, where $\alpha = T^2/2$ and $\beta > 0$ are the constants in the statement of Corollary 2. We then reject the null hypothesis if $\widehat{s}_{\text{comp}} < s$.

A relaxed program. A difficulty with Problem (11) is that the optimization variable \mathbf{p}^C belongs to $[0, 1]^{|H|}$ and the set of constraints is non-convex, making it computationally intractable. We now propose a convex relaxation that is more amenable to computation.

For each history $h \in H$, let

$$(y_{h,n})_{n=-\underline{n}, \dots, \bar{n}} \equiv (\mathbf{1}_{(1+\rho_n)b_h < \wedge \mathbf{b}_{-i,h}})_{n=-\underline{n}, \dots, \bar{n}}.$$

Vector y_h records the bidding outcomes of each history h , and can take values in $Y \equiv \{(0, 0, \dots, 0), (1, 0, \dots, 0), \dots, (1, 1, \dots, 1)\}$. It turns out that $((y_{h,n}))_{h \in H}$ is a sufficient statistic of data for Problem (11). As we show now, this allows us to consider solutions $\mathbf{p}^C = (p_h^C)_{h \in H}$ such that $p_h^C = q^C(y)$ for all $h \in H$ with $y_h = y$. Indeed, note that for any $\mathbf{p}^C = (p_h^C)_{h \in H}$,

$$\begin{aligned} (\widehat{D}_n(\mathbf{p}^C))_{n=-\underline{n}, \dots, \bar{n}} &= \frac{1}{\sum_h p_h^C} \sum_{y \in Y} \left(y \times \sum_{h|y_h=y} p_h^C \right) \\ &= \frac{1}{|H|} \sum_{y \in Y} y \times |\{h|y_h = y\}| q^C(y) \end{aligned}$$

for $q^C(y) = \frac{\sum_{h|y_h=y} p_h^C}{|\{h|y_h=y\}|}$. Looking for solutions of the form $q : Y \rightarrow [0, 1]$ makes Problem (11) significantly easier in terms of computation.

Let Z be the set of beliefs $z_h = (d_{h,n})_{n=\underline{n}, \dots, \bar{n}}$ such that there exists a cost $c_h \in [\frac{1}{1+m}b_h, b_h]$ satisfying (6) and (7). Let

$$A(\mathbf{p}^C) = \Pi_n[\widehat{D}_n(\mathbf{p}^C) - T, \widehat{D}_n(\mathbf{p}^C) + T]$$

denote the set of aggregate constraints. Finally, for any D_n , define

$$\underline{B}(D_n, k) \equiv \frac{\frac{D_n}{1-D_n} \exp(-k)}{1 + \frac{D_n}{1-D_n} \exp(-k)} \quad \text{and} \quad \overline{B}(D_n, k) \equiv \frac{\frac{D_n}{1-D_n} \exp(k)}{1 + \frac{D_n}{1-D_n} \exp(k)},$$

and let

$$I(\mathbf{p}^C) = \Pi_n[\underline{B}(\widehat{D}_n(\mathbf{p}^C), k), \overline{B}(\widehat{D}_n(\mathbf{p}^C), k)]$$

denote the information constraints.

For any function $q : Y \rightarrow [0, 1]$ define $\tilde{s}_{\text{comp}}(q) \equiv \frac{1}{|H|} \sum_{y \in Y} q(y) |\{h|y_h = y\}|$, and let $\mathbf{p}(q) \in [0, 1]^{|H|}$ be such that, for all $h \in H$, $p(q)_h = q(y_h)$. For any set C , let $\text{Conv}(C)$ denote the convex hull of C .

Proposition 4. *We have that*

$$\widehat{s}_{\text{comp}} \leq \max_{q: Y \rightarrow [0,1]} \{\tilde{s}_{\text{comp}}(q) \mid \text{Conv}[Z \cap I(\mathbf{p}(q))] \cap A(\mathbf{p}(q)) \neq \emptyset\}.$$

5 Case Studies

This Section takes the results of Section 4 to data. We start by using our results to analyze two collusion cases of firms participating in auctions in our national data that were implicated by the Japanese Fair Trade Commission (JFTC). The two cases are: (i) prestressed concrete

providers, and (ii) firms installing electric traffic signs.⁶ It is worth highlighting that firms in case (ii) admitted that they were violating anti-trust laws soon after the JFTC started investigating them. In contrast, firms in case (i) denied the cases against them and the case went to trial. As it turns out, firms in case (i) continued colluding for some time after the JFTC launched its investigation.

Figure 4 shows the bidding behavior of implicated firms. The left panels plot the distribution of Δ for prestressed concrete providers, before and after the JFTC started its investigation. Consistent with the fact that firms in this market continued to collude for a while after the investigation, the after-period features missing bids around bid difference $\Delta = 0$, but to a lesser extent than the before-period.

The panels on the right plot the same distributions for firms installing electric traffic signs. Consistent with Proposition 1, the distribution of Δ has missing mass around zero only during the non-competitive period.

Next, we apply our results of Section 4.4 to these two markets. We proceed as follows. First, we fix two downward deviations: $\rho_{-2} = \rho^- \in (-1, 0)$ and $\rho_{-1} = \lim_{\epsilon \uparrow 0} \epsilon = 0^-$, and one upward deviation $\rho_1 = \rho^+ \in (0, \infty)$. Second, we pick mark-up and information constraint parameters $m \geq 0$ and $k \geq 0$. Third, for each possible null hypothesis $H_0 = s_{\text{comp}} \geq s$ with $s \in (0, 1]$, we pick tolerance level $T(s)$ such that $1 - \beta \exp(-\alpha s |H|) = 0.05$, where $\alpha > 0$ and $\beta > 0$ are the constants in the statement of Corollary 2.⁷ Then, our estimate of the share of competitive histories in the data is the largest $s \in (0, 1]$ such that $\hat{s}_{\text{comp}} \geq s$ given tolerance level $T(s)$.

Figure 5 plots our estimate of the share of competitive histories in each of these markets, before and after prosecution. Auctions in both markets became more competitive in the period post-investigation. In the case of traffic signs, our estimates suggest that collusion

⁶See JFTC Recommendation and Ruling #5-8 (2005) for case (ii), and JFTC Recommendation #27-28 (2004) and Ruling #26-27 (2010) for case (i).

⁷Recall that $\alpha = T^2/2$ and $\beta = (1 + \underline{n} + \bar{n})$.

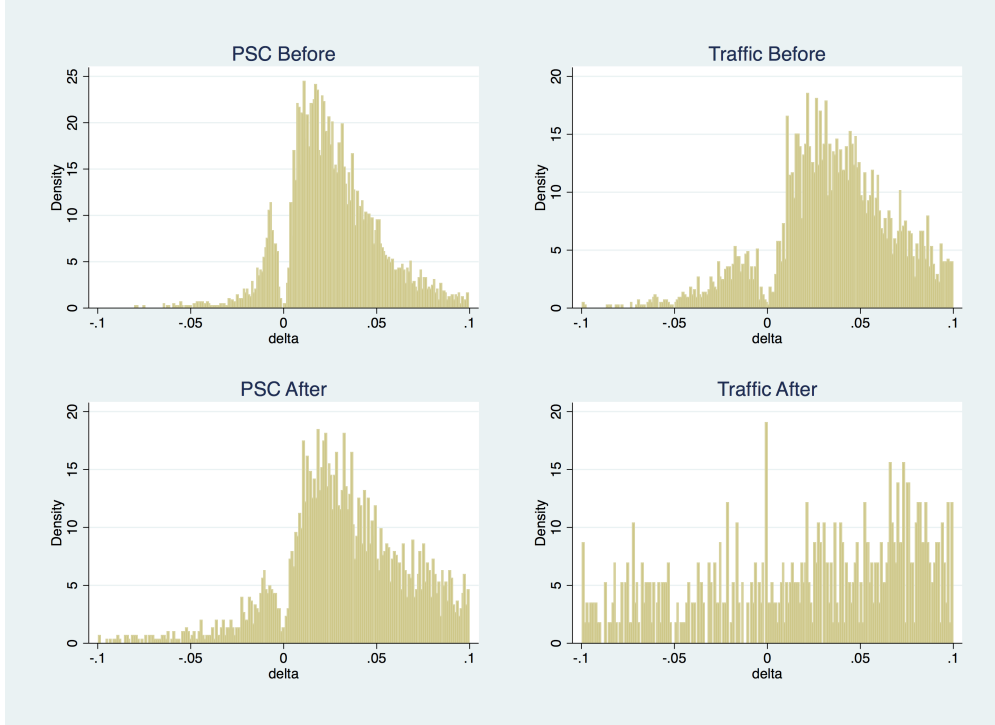


Figure 4: Distribution of bid-difference Δ . Left-panel: prestressed concrete. Right-panel: traffic signs.

stopped completely after the investigation. In contrast, our estimates suggest that there was still some collusion after the investigation in the market for prestressed concrete.

Lastly, we look at the auctions run by the city of Tsuchiura. Chassang and Ortner (2016) find evidence consistent with collusion in these auctions. Moreover, they show that the extent of collusion fell after October 2009, when the city changed its procurement auction format and introduced price floors.⁸

Figure 6 plots the distribution of bid differences Δ for these auctions. The left-panel plots the distribution for auctions taking place before October 2009, and the right-panel plots the same distribution for auctions taking place after October 2009. Consistent with Proposition 1 and with the evidence in Chassang and Ortner (2016), auctions before the policy change

⁸In October 2009, the city of Tsuchiura switched from a standard first-price auction format to a first-price auction with a minimum price; i.e., an auction in which bids below the minimum price are discarded.

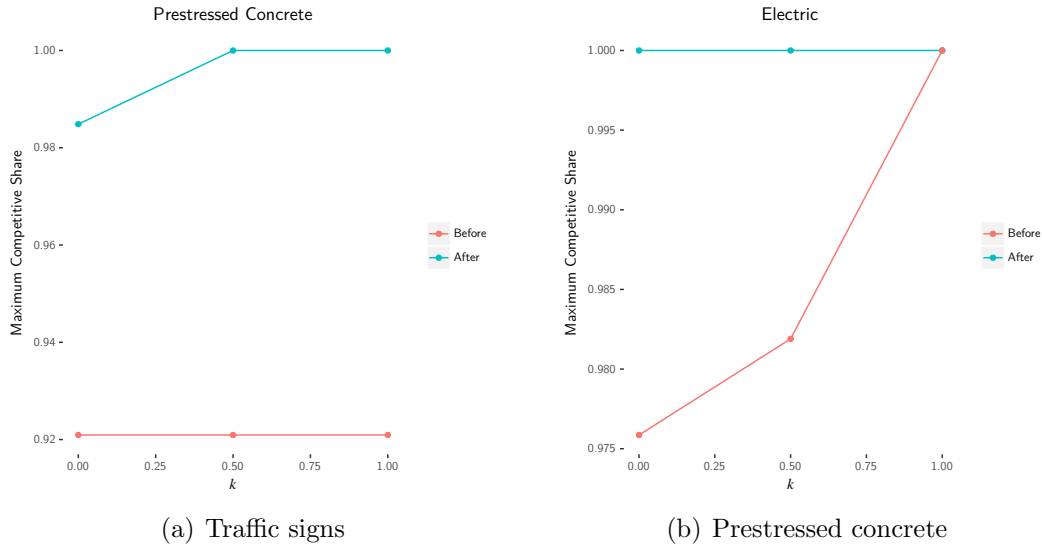


Figure 5: Share of competitive histories. Parameters: $\rho^- = -0.01$, $\rho^+ = 0.001$ and $m = 1.5$.

feature a more pronounced “gap” in the distribution of bid differences around $\Delta = 0$.

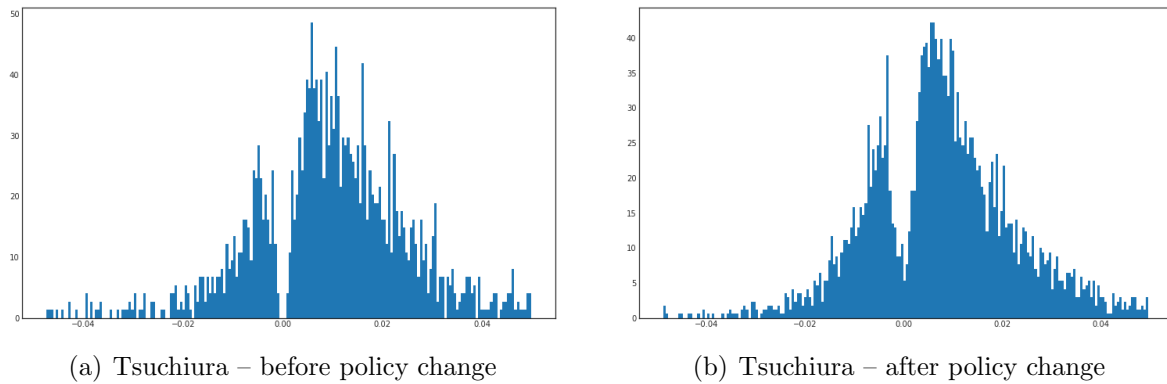


Figure 6: Distribution of bid-difference Δ in Tsuchiura.

Figure 7 plots our estimates of the share of competitive histories for auctions run in Tsuchiura, before and after the change in the auction format. Our estimates are broadly consistent with the idea that the extent of collusion fell after the city introduced price floors into the auctions.

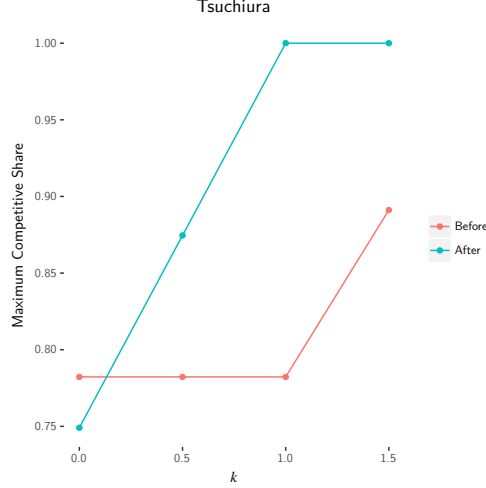


Figure 7: Share of competitive histories. Parameters: $\rho^- = -0.01$, $\rho^+ = 0.001$ and $m = 1.5$.

6 Interpreting Missing Bids

This section has two objectives. First, we want to highlight that the bidding behavior we observe in our data is not easily explained by standard models of collusion. Second, we put forward an explanation for the bidding patterns we observe in these two datasets.

Workhorse model. We specialize the model in Section 3 as follows. We assume: (i) costs are i.i.d. across firms and across periods, (ii) cost realizations are publicly observed by all firms, and (iii) utility is perfectly transferable.

We denote by Σ the set of Subgame Perfect Equilibria of this game. For any $\sigma \in \Sigma$ and any history h_t , let

$$V(\sigma, h_t) = \mathbb{E}_\sigma \left[\sum_{i \in N} \sum_{s \geq 0} \delta^s x_{i,t+s} (b_{i,t+s} - c_i) \mid h_t \right]$$

denote the total surplus generated by σ at history h_t . Define

$$\bar{V} \equiv \max_{\sigma \in \Sigma} V(\sigma, h_0)$$

to be the highest surplus sustainable in equilibrium.

For any cost realization $\mathbf{c} = (c_i)_{i \in N}$, we denote by $\mathbf{x}^*(\mathbf{c}) = (x_i^*(\mathbf{c}))_{i \in N}$ the efficient allocation (i.e., the allocation that assigns the contract to the lowest cost bidder and breaks ties randomly). We denote by $b_{(1)}$ and $b_{(2)}$ the lowest and second lowest bids. The following result, which is proved in Chassang and Ortner (2016), characterizes bidding behavior in any equilibrium that attains \bar{V} .

Proposition 5. *Let σ be an equilibrium that attains \bar{V} . Then:*

(i) *equilibrium σ is stationary on-path, and generates surplus \bar{V} at every history.*

(ii) *for any cost realization $\mathbf{c} = (c_i)_{i \in N}$, the lowest cost bidder wins at bid $b^*(\mathbf{c})$ defined by*

$$b^*(\mathbf{c}) \equiv \sup \left\{ b \leq r : \sum_{i \in N} (1 - x_i^*(\mathbf{c})) [b - c_i]^+ \leq \delta \bar{V} \right\}.$$

(iii) *there is no money left on the table under equilibrium σ : $b_{(2)} - b_{(1)} \approx 0$ at all periods.*

By Proposition 5, the bidding patterns in our data cannot be rationalized by optimal collusion. In an optimal equilibrium, firms never use strategies under which the winning bidder has a short-run incentive to undercut the winning bid. Indeed, this would mean that firms have to spend continuation surplus to provide incentives to the winner not to bid higher. This creates efficiency losses relative to equilibria in which the winner is given incentives not to undercut by having the second lowest bid right on top of the winning bid.

As a result, bids will be clustered in an optimal collusive equilibria, and the “money left on the table” (i.e., the difference between the winning bid and the second lowest bid) will be negligible. As Figures 1 and 8 show, this is in sharp contrast with the bidding patterns we observe in our data, under which winning bids are isolated and the money left on the table is significant.

Missing bids as coordination challenges. The fact that winning bids are isolated implies that the allocation that this bidding behavior induces is robust to trembles or impre-

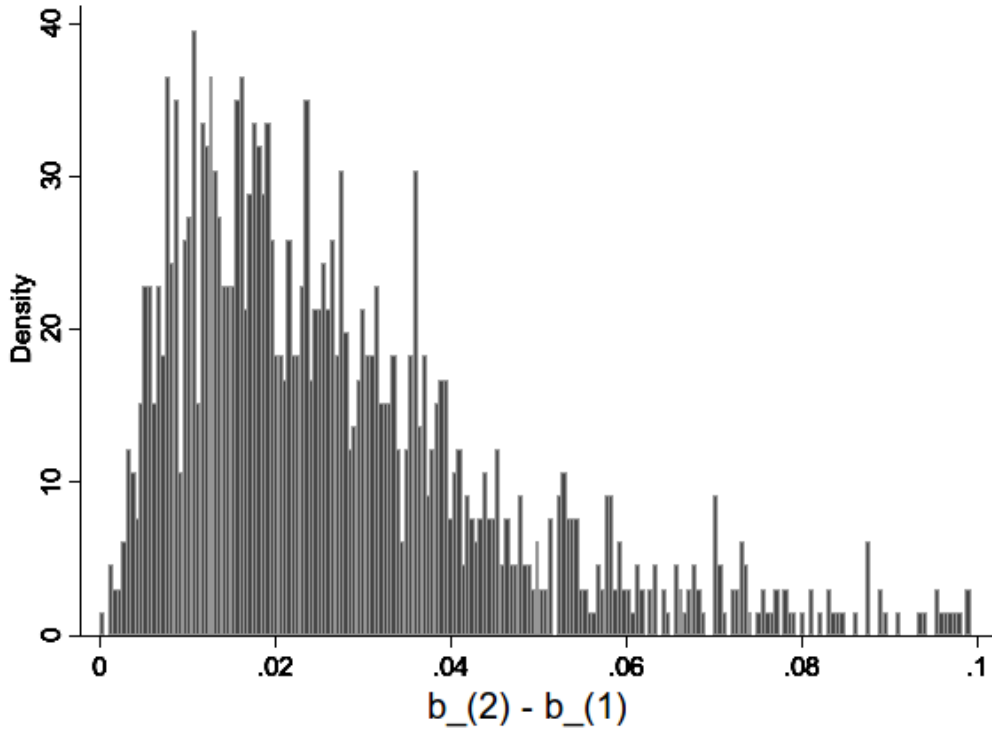


Figure 8: Distribution of normalized bid difference $b_{(2)} - b_{(1)}$ – national data.

cisions in the communication among cartel members. We now lay out a simple model to illustrate how isolated winning bids may emerge as a response to such imperfections.

Suppose $N = \{1, 2\}$. We continue to assume that cost realizations are publicly observed by all firms, that utility is perfectly transferable, and that costs are drawn i.i.d. across firms and across periods from distribution F . Let $\epsilon \sim G$, where $\text{supp } G = [-1, 1]$ and $g(\epsilon) \equiv G'(\epsilon) \in (\underline{g}, \bar{g})$ for all $\epsilon \in [-1, 1]$ and some $0 < \underline{g} < \bar{g}$. Firms choose intended bids (\hat{b}_1, \hat{b}_2) . With probability $1/2$ firm 1 trembles and realized bids are $(\hat{b}_1 + \sigma\epsilon, \hat{b}_2)$, and with probability $1/2$ firm 2 trembles and realized bids are $(\hat{b}_1, \hat{b}_2 + \sigma\epsilon)$. For simplicity, we assume that distribution F has finite support and that $\text{prob}_F(c_i < 1 - \sigma) = 1$.⁹

For each cost realization $\mathbf{c} = (c_1, c_2)$, let $c_{(1)}$ be the lowest cost. Let $\hat{b}_{(1)}$ denote the intended bid of the bidder with cost $c_{(1)}$, and let $\hat{b}_{(2)}$ be the intended bid of her opponent.

⁹These assumptions simplify the exposition, but are not essential.

As above, we let \bar{V} denote the highest surplus sustainable in equilibrium.

Proposition 6. *There exists $\bar{\sigma} > 0$ and $\bar{\delta} < 1$ such that, if $\sigma < \bar{\sigma}$ and $\delta > \bar{\delta}$, then in any equilibrium that attains \bar{V} and for any cost realization \mathbf{c} with $c_1 \neq c_2$:*

(i) $\hat{b}_{(1)} = \hat{b}_{(2)} - \sigma$ and the lowest cost bidder wins with probability 1.

(ii) The lowest cost bidder does not have a stage-game incentive to increase $\hat{b}_{(1)}$.

Proposition 6(i) shows that the bidding patterns we observe in the data can be rationalized by a model in which bids are subject to trembles: in an optimal equilibrium, winning bids will be isolated to guarantee that the lowest cost bidder always wins. Proposition 6(ii) shows that, in an optimal equilibrium, there will be sufficient density of bids above intended bid $\hat{b}_{(1)}$, so that the winner does not have an incentive to increase her intended bid. We stress that this feature of the model is consistent with our data, as Figure 8 illustrates.

7 Conclusion

This paper documents a novel bidding pattern from Japanese procurement auctions: winning bids tend to be isolated. We show that this bidding behavior is a strong marker for collusion, and propose structural methods to estimate the extent of collusion in a given dataset.

Lastly, we show that isolated winning bids can be rationalized by a model with trembles. Indeed, isolated winning bids make the allocation robust, and guarantee that contracts are allocated to the designated winner.

Appendix

A Proofs

A.1 Proofs of Section 3

Proof of Theorem 1. Let H be a set of histories, and fix $\rho \in (-1, \infty)$. For each history $h_{i,t} = (h_t, z_{i,t}) \in H$, define

$$\begin{aligned} \varepsilon_{i,t} &\equiv \mathbb{E}_{\sigma,\mu}[\mathbf{1}_{\wedge b_{-i,t} > b_{i,t}(1+\rho)} | h_{i,t}] - \mathbf{1}_{\wedge b_{-i,t} > b_{i,t}(1+\rho)} \\ &= \text{prob}_{\sigma,\mu}(\wedge \mathbf{b}_{-i,t} > b_{i,t}(1+\rho) | h_{i,t}) - \mathbf{1}_{\wedge b_{-i,t} > b_{i,t}(1+\rho)}. \end{aligned}$$

Note that $\widehat{D}(\rho|H) - \overline{D}(\rho|H) = \frac{1}{|H|} \sum_{h_{i,t} \in H} \varepsilon_{i,t}$.

Note further that, by the law of iterated expectations, for all histories $h_{j,t-s} \in H$ with $s \geq 0$, $\mathbb{E}_{\sigma,\mu}[\varepsilon_{i,t} | h_{j,t-s}] = \mathbb{E}_{\sigma,\mu}[\mathbb{E}_{\sigma,\mu}[\mathbf{1}_{\wedge b_{-i,t} > b_{i,t}(1+\rho)} | h_t, z_{i,t}] - \mathbf{1}_{\wedge b_{-i,t} > b_{i,t}(1+\rho)} | h_{t-s}, z_{j,t-s}] = 0$.¹⁰

Number the histories in H as $1, \dots, |H|$ such that, for any pair of histories $k = (h_s, z_{i,s}) \in H$ and $k' = (h_{s'}, z_{j,s'}) \in H$ with $k' > k$, $s' \geq s$. For each history $k = (h_t, z_{i,t})$, let $\varepsilon_k = \varepsilon_{i,t}$, so that

$$\widehat{D}(\rho|H) - \overline{D}(\rho|H) = \frac{1}{|H|} \sum_{k=1}^{|H|} \varepsilon_k.$$

Note that, for all $\hat{k} \leq |H|$, $S_{\hat{k}} \equiv \sum_{k=1}^{\hat{k}} \varepsilon_k$ is a Martingale, with increments $\varepsilon_{\hat{k}}$ whose absolute value is bounded above by 1. By the Azuma-Hoeffding Inequality, for every $\alpha > 0$, $\text{prob}(|S_{|H|}| \geq |H|\alpha) \leq 2 \exp\{-\alpha^2|H|/2\}$. Therefore, with probability 1, $\frac{1}{|H|} S_{|H|} = \widehat{D}(\rho|H) - \overline{D}(\rho|H)$ converges to zero as $|H| \rightarrow \infty$. ■

¹⁰This holds since, in a perfect public Bayesian equilibrium, bidders' strategies at any time t depend solely on the public history and on their private information at time t .

A.2 Proofs of Section 4

Proof of Corollary 1. Fix scalars $\rho > 0$ and $\kappa > 0$ satisfying the statements of the Corollary. Then, Note that

$$\begin{aligned}
2\kappa &\leq \frac{1}{\rho}[\widehat{R}(\rho|H) - \widehat{R}(0|H)] \\
&= \frac{1}{\rho}[\overline{R}(\rho|H) - \overline{R}(0|H) + \widehat{R}(\rho|H) - \overline{R}(\rho|H) + \widehat{R}(0|H) - \overline{R}(0|H)] \\
&\leq 1 - s_{\text{comp}} + \frac{1}{\rho}[\widehat{R}(\rho|H) - \overline{R}(\rho|H) - \widehat{R}(0|H) + \overline{R}(0|H)], \tag{12}
\end{aligned}$$

where the second inequality follows since, by the arguments in the proof of Proposition 2, $\frac{1}{\rho}[\overline{R}(\rho|H) - \overline{R}(0|H)] \leq 1 - s_{\text{comp}}$.

Next, note that for any scalar $\rho' \in (-1, \infty)$,

$$\overline{R}(\rho'|H) - \widehat{R}(\rho'|H) = \sum_{h_{i,t} \in H} \varepsilon_{i,t},$$

where

$$\varepsilon_{i,t} = \mathbb{E}_{\sigma,\mu}[(1 + \rho')b_{i,t}\mathbf{1}_{\wedge b_{-i,t} > b_{i,t}(1+\rho')} | h_{i,t}] - (1 + \rho)b_{i,t}\mathbf{1}_{\wedge b_{-i,t} > b_{i,t}(1+\rho)}.$$

By the law of iterated expectations, for all $h_{j,t-s} \in H$ with $s \geq 0$,

$$\mathbb{E}_{\sigma,\mu}[\varepsilon_{i,t} | h_{j,t-s}] = \mathbb{E}_{\sigma,\mu}[\mathbb{E}_{\sigma,\mu}[(1+\rho)b_{i,t}\mathbf{1}_{\wedge b_{-i,t} > b_{i,t}(1+\rho')} | h_t, z_{i,t}] - (1+\rho')b_{i,t}\mathbf{1}_{\wedge b_{-i,t} > b_{i,t}(1+\rho)} | h_{t-s}, z_{j,t-s}] = 0.$$

As in the proof of Theorem 1, number the histories in H as $1, \dots, |H|$ such that, for any pair of histories $k = (h_s, z_{i,s}) \in H$ and $k' = (h_{s'}, z_{j,s'}) \in H$ with $k' > k$, $s' \geq s$. For each history $k = (h_t, z_{i,t})$, let $\varepsilon_k = \varepsilon_{i,t}$, so that

$$\overline{R}(\rho'|H) - \widehat{R}(\rho'|H) = \frac{1}{|H|} \sum_{k=1}^{|H|} \varepsilon_k.$$

Note that, for all $\hat{k} \leq |H|$, $S_{\hat{k}} \equiv \sum_{k=1}^{\hat{k}} \varepsilon_k$ is a Martingale, with increments $\varepsilon_{\hat{k}}$ whose absolute value is bounded above by 1.¹¹ By the Azuma-Hoeffding Inequality, for all $\alpha > 0$, $\text{prob}(|S_{|H|}| \geq |H|\alpha) = \text{prob}(|\bar{R}(\rho'|H) - \hat{R}(\rho'|H)| \geq \alpha) \leq 2 \exp(-\alpha^2|H|/2)$.

Since this bound holds for all $\rho' \in (-1, \infty)$, it follows that

$$\text{prob}(|\bar{R}(\rho|H) - \hat{R}(\rho|H)| \geq \frac{\rho\kappa}{2} \text{ and } |\bar{R}(0|H) - \hat{R}(0|H)| \geq \frac{\rho\kappa}{2}) \leq 4 \exp(-(\rho\kappa)^2|H|/8).$$

Combining this with equation (12), it follows that with probability at least $1 - 4 \exp(-(\rho\kappa)^2|H|/8)$, $s_{\text{comp}} \leq 1 - \kappa$. ■

Proof of Proposition 3. Note first that conditions (6) and (7) must automatically hold at every competitive history $h \in H$.

Note next that, by the arguments in Theorem 1, for all n , $\text{prob}(|\hat{D}_n - D_n| \geq T) \leq 2 \exp(-T^2|H|/2)$. It follows that

$$\text{prob}(\forall n, |\hat{D}_n - D_n| \geq T) \leq 2(\underline{n} + \bar{n} + 1) \exp(-T^2|H|/2).$$

Therefore, with probability at least $1 - 2(\underline{n} + \bar{n} + 1) \exp(-\frac{T^2}{2}|H|)$, conditions (6), (7) and (8) hold simultaneously. ■

Proof of Corollary 2. Let $H^{\text{comp}} \subset H$ be the set of competitive histories in H , so that $s_{\text{comp}} = \frac{|H^{\text{comp}}|}{|H|}$. Consider the vector $\mathbf{p}^{\text{comp}} = (p_h^{\text{comp}})_{h \in H}$ with $p_h^{\text{comp}} = 1$ for all $h \in H^{\text{comp}}$ and $p_h^{\text{comp}} = 0$ otherwise.

Note first that, for all histories $h \in H^{\text{comp}}$, the firms' true beliefs and costs $((d_{h,n}), c_h)$ must satisfy conditions (6), (7), (9) and (10). Hence, for all $h \in H^{\text{comp}}$, set beliefs and costs equal to the firms' true beliefs and costs. For all $h \notin H^{\text{comp}}$, pick any beliefs and costs

¹¹This follows since we normalized reserve price to 1.

$((d_{h,n}), c_h)$ that satisfy conditions (6), (7), (9) and (10).

For every $\mathbf{p}^C \in [0, 1]^{|H|}$ and every n , define

$$D_n(\mathbf{p}^C) \equiv \frac{1}{\sum_{h \in H} p_h} \sum_{h \in H} p_h^C d_{h,n}.$$

Since \mathbf{p}^{comp} is such that $p_h^{\text{comp}} = 1$ for all $h \in H^{\text{comp}}$ and $p_h^{\text{comp}} = 0$ for all $h \notin H^{\text{comp}}$, it follows that, for all n ,

$$D_n(\mathbf{p}^{\text{comp}}) = \frac{1}{|H^{\text{comp}}|} \sum_{h \in H^{\text{comp}}} d_{h,n}.$$

Similarly, note that for all n ,

$$\hat{D}_n(\mathbf{p}^{\text{comp}}) = \frac{1}{|H^{\text{comp}}|} \sum_{h \in H^{\text{comp}}} \mathbf{1}_{(1+\rho_n)b_h < \wedge b_{-i,h}}.$$

Using the arguments as in the proof of Theorem 1,

$$\forall n, \quad \text{prob}(|\hat{D}_n(\mathbf{p}^{\text{comp}}) - D_n(\mathbf{p}^{\text{comp}})| \geq T) \leq 2 \exp(-T^2 |H^{\text{comp}}|/2)$$

These inequalities imply that, for $\mathbf{p}^C = \mathbf{p}^{\text{comp}}$, conditions (8) hold simultaneously with probability at least $1 - 2(\underline{n} + \bar{n} + 1) \exp(-T^2 |H^{\text{comp}}|/2)$. The result follows by noting that $|H^{\text{comp}}| = s_{\text{comp}}(H)|H|$. ■

Proof of Proposition 4. Let $\mathbf{p}^C \in [0, 1]^C$ be a solution to Problem (11), and let $((d_{h,n}), c_h)_{h \in H}$ be the corresponding beliefs satisfying all the constraints of the problem. For every $y \in Y$, define $H(y) \equiv \{h \in H : y_h = y\}$.

Consider any permutation $\alpha : H \rightarrow H$ such that, for all $y \in Y$ and all $h \in H(y)$, $\alpha(h) \in H(y)$. Let $\tilde{\mathbf{p}}^C = (\tilde{p}_h^C)_{h \in H}$ be such that, for all $h \in H$, $\tilde{p}_h^C = p_{\alpha(h)}^C$. Note that $\tilde{\mathbf{p}}^C$ is also a solution (11), together with beliefs and costs $((\tilde{d}_{h,n}), \tilde{c}_h)_{h \in H}$ such that, for all

$h \in H$, $((\tilde{d}_{h,n}), \tilde{c}_h) = ((d_{\alpha(h),n}), c_{\alpha(h)})$. Moreover, note that, for all n , $\widehat{D}_n(\tilde{\mathbf{p}}^C) = \widehat{D}_n(\mathbf{p}^C)$. Hence, beliefs and costs $((\tilde{d}_{h,n}), \tilde{c}_h)_{h \in H}$ satisfy the IC constraints, and the aggregate and information constraints given $\tilde{\mathbf{p}}^C$.

Since this is true for any such permutation α , it follows that there exists $\bar{\mathbf{p}}^C \in [0, 1]^{|H|}$ and corresponding beliefs and costs $((\bar{d}_{h,n}), \bar{c}_h)_{h \in H}$, such that

- (i) for all $y \in Y$ and all $h, h' \in H(y)$, $\bar{p}_h^C = \bar{p}_{h'}^C = p_y \in [0, 1]$,
- (ii) $\frac{1}{|H|} \sum_h \bar{p}_h^C = \frac{1}{|H|} \sum_h p_h^C = \widehat{s}_{\text{comp}}$,
- (iii) $\frac{1}{\sum_{h \in H} \bar{p}_h^C} \sum_{h \in H} \bar{p}_h^C \times (\bar{d}_{h,n}) \in A(\bar{\mathbf{p}}^C)$,
- (iv) for all $h \in H$, $(\bar{d}_{h,n}) \in \text{Conv}[Z \cap I(\bar{\mathbf{p}}^C)]$.

Let $q : Y \rightarrow [0, 1]$ be such that $q(y) = p_y$, so that $\bar{\mathbf{p}}^C = \mathbf{p}(q)$. Since $\sum_{h \in H} \frac{\bar{p}_h^C}{\sum_{h \in H} \bar{p}_h^C} \times (\bar{d}_{h,n}) \in \text{Conv}[Z \cap I(\bar{\mathbf{p}}^C)] \cap A(\mathbf{p}(q))$, it follows that

$$\widehat{s}_{\text{comp}} \leq \max_{q: Y \rightarrow [0, 1]} \{ \tilde{s}_{\text{comp}}(q) \mid \text{Conv}[Z \cap I(\mathbf{p}(q))] \cap A(\mathbf{p}(q)) \neq \emptyset \}.$$

■

A.3 Proofs of Section 6

Proof of Proposition 6. To establish Proposition 6, we proceed in two steps. First, we consider the problem of finding the profile of intended bids (\hat{b}_1, \hat{b}_2) that maximize the sum of the bidders payoffs, and show that this profile of intended bids satisfies the conditions in Proposition 6 when σ is lower than some cutoff $\bar{\sigma} > 0$. Second, we show that the payoff maximizing profile of intended bids can be supported in equilibrium whenever the players' discount factor is higher than some cutoff $\bar{\delta} < 1$.

Fix a cost realization $\mathbf{c} = (c_1, c_2)$. The profile of intended bids (\hat{b}_1, \hat{b}_2) that maximizes the bidders' sum of payoffs solves

$$\begin{aligned} \max_{\hat{b}_1, \hat{b}_2 \in [0, 1-\sigma]^2} & \frac{1}{2} \mathbb{E} \left[\mathbf{1}_{\hat{b}_1 + \sigma \epsilon \leq \hat{b}_2} (\hat{b}_1 + \sigma \epsilon - c_1) \right] + (1 - \mathbf{1}_{\hat{b}_1 + \sigma \epsilon \leq \hat{b}_2}) (\hat{b}_2 - c_2) \\ & + \frac{1}{2} \mathbb{E} \left[\mathbf{1}_{\hat{b}_1 \leq \hat{b}_2 + \sigma \epsilon} (\hat{b}_1 - c_1) \right] + (1 - \mathbf{1}_{\hat{b}_1 \leq \hat{b}_2 + \sigma \epsilon}) (\hat{b}_2 + \sigma \epsilon - c_2) \end{aligned} \quad (13)$$

Note first that, for all cost realizations $\mathbf{c} = (c_1, c_2)$ with $c_1 = c_2$, the solution to program (13) is to set $\hat{b}_1 = \hat{b}_2 = 1 - \sigma$.

Consider next cost realizations \mathbf{c} with $c_1 \neq c_2$, and assume wlog that $c_1 < c_2$. Let

$$\Delta_c \equiv \min_{c_1, c_2 \in \text{sup } F, c_1 \neq c_2} |c_1 - c_2| > 0$$

denote the minimum possible difference between costs c_1 and c_2 . As a first step, note that the solution to program (13) is such that $\hat{b}_1 \leq \hat{b}_2$. To see why, suppose by contradiction $\hat{b}_1 > \hat{b}_2$, and consider an alternative bidding profile $(\hat{b}'_1, \hat{b}'_2) = (\hat{b}_2, \hat{b}_1)$. Note that the expected revenue of the cartel is the same under this alternative bidding profile, but the expected procurement costs are strictly lower (since $c_1 < c_2$). Hence, the cartel is strictly better off under this alternative bidding profile, a contradiction.

Next, we show that there exists $\bar{\sigma}_1 > 0$ such that, if $\sigma < \bar{\sigma}_1$, the solution to program (13) is such that $\hat{b}_1 = \hat{b}_2 - \sigma$. To see why, consider any intended bid $\hat{b}_1 \in (\hat{b}_2 - \sigma, \hat{b}_2]$. Note that the derivative of (13) with respect to \hat{b}_1 (evaluated at $\hat{b}_1 \in (\hat{b}_2 - \sigma, \hat{b}_2]$) is

$$\begin{aligned} & \frac{1}{2\sigma} (c_1 - c_2) \left[g \left(\frac{\hat{b}_2 - \hat{b}_1}{\sigma} \right) + g \left(\frac{\hat{b}_1 - \hat{b}_2}{\sigma} \right) \right] + \frac{1}{2} G \left(\frac{\hat{b}_2 - \hat{b}_1}{\sigma} \right) + \frac{1}{2} - \frac{1}{2} G \left(\frac{\hat{b}_1 - \hat{b}_2}{\sigma} \right) \\ & \leq \frac{-1}{\sigma} \Delta_c \underline{g} + 1, \end{aligned}$$

where the inequality follows since $|c_1 - c_2| \geq \Delta_c > 0$ (and $c_1 < c_2$), and since $g(\epsilon) \geq \underline{g} > 0$ for all $\epsilon \in [-1, 1]$. It follows that, for σ small enough, this derivative is strictly negative,

which cannot hold at an optimum. Hence, there exists $\bar{\sigma}_1 > 0$ such that, for all $\sigma < \bar{\sigma}_1$, the solution to program (13) is such that $\hat{b}_1 \leq \hat{b}_2 - \sigma$. Finally, note that if $\hat{b}_1 < b_2 - \sigma$, then increasing \hat{b}_1 slightly strictly increases the bidders' payoffs (since a small increase in \hat{b}_1 does not affect the allocation). Therefore, $\hat{b}_1 = \hat{b}_2 - \sigma$ whenever $\sigma < \bar{\sigma}_1$. Finally, note that it is optimal to set $\hat{b}_2 = 1 - \sigma$.

Next, we show that there exists $\bar{\sigma}_2 > 0$ and $\bar{\delta} < 1$ such that, if $\sigma < \min\{\bar{\sigma}_1, \bar{\sigma}_2\}$ and $\delta > \bar{\delta}$, the bidding profile that solves program (13) can be sustained in equilibrium.

Let \hat{V} be the sum of the expected discounted payoffs that players get from playing at each period the profile of bids that solve (13). Let V^{NE} be a player's expected discounted payoffs from playing the (worst) stage game Nash equilibrium of this game at each period. Note that: (i) for every δ , $\hat{V} > 2V^{NE}$; and (ii) $\hat{V} - (V_1^{NE} + V_2^{NE}) \rightarrow \infty$ as $\delta \rightarrow 1$.

Suppose $\sigma < \bar{\sigma}_1$, and consider the following strategy profile. At each period, for any cost realization \mathbf{c} , firms bid according to the solution to (13). Note that under that solution, the lowest cost firm wins with probability 1. In periods in which costs \mathbf{c} are such that $c_1 \neq c_2$, the lowest cost firm pays her opponent a transfer equal to $\delta(\hat{V}/2 - V^{NE})$ if the contract is allocated to the lowest cost firm. If the contract is allocated to the lowest cost firm and the lowest cost firm pays the required transfer, then firms continue to play the same actions in the next period. Otherwise, if either firm deviates, starting in the next period firms play the stage game Nash equilibrium delivering payoffs V^{NE} to each firm. In periods in which costs \mathbf{c} are such that $c_1 = c_2$, if the winning bid is weakly above $1 - 2\sigma$, there are no transfers and firms continue to play the same actions in the next period. If the winning bid is strictly below $1 - 2\sigma$, there are no transfers and starting next period firms play the stage game Nash equilibrium delivering payoffs V^{NE} to each firm.

Consider first cost realizations \mathbf{c} with $c_1 \neq c_2$. Note that the firm that wins the contract has exact incentives to pay transfer $\delta(\hat{V}/2 - V^{NE})$ to the other firm. Note further that the winning firm does not have an incentive to decrease her bid. We show that there exists $\bar{\sigma}_2 > 0$ such that, when $\sigma < \min\{\bar{\sigma}_1, \bar{\sigma}_2\}$, the lowest cost firm does not have an incentive to

increase her bid either. Suppose wlog that $c_1 < c_2$. Let $T = \delta(\hat{V}/2 - V^{NE})$ be the transfer that this firm pays to the other firm if she wins. Then, the payoff that this firm obtains under this strategy profile if she places intended bid $b > \hat{b}_1 = \hat{b}_2 - \sigma$ is

$$\begin{aligned} V_1(b) &= \frac{1}{2} \mathbb{E} \left[\mathbf{1}_{b+\sigma\epsilon \leq \hat{b}_2} (b + \sigma\epsilon - c_1 - T + \delta\hat{V}/2) + (1 - \mathbf{1}_{b+\sigma\epsilon \leq \hat{b}_2}) \delta V^{NE} \right] \\ &\quad + \frac{1}{2} \mathbb{E} \left[\mathbf{1}_{b \leq \hat{b}_2 + \sigma\epsilon} (b - c_1 - T + \delta\hat{V}/2) + (1 - \mathbf{1}_{b \leq \hat{b}_2 + \sigma\epsilon}) \delta V^{NE} \right] \\ &= \frac{1}{2} \mathbb{E} \left[\mathbf{1}_{b+\sigma\epsilon \leq \hat{b}_2} (b + \sigma\epsilon - c_1) \right] + \frac{1}{2} \mathbb{E} \left[\mathbf{1}_{b_1 \leq \hat{b}_2 + \sigma\epsilon} (b - c_1 - T) \right] + \delta V^{NE}, \end{aligned} \quad (14)$$

where we used $T = \delta(\hat{V}/2 - V^{NE})$. Differentiating $V_1(b)$ with respect to b yields

$$\begin{aligned} V_1'(b) &= \frac{-1}{2\sigma} \left[(\hat{b}_2 - c_1) g \left(\frac{\hat{b}_2 - b}{\sigma} \right) + (b - c_1) g \left(\frac{b - \hat{b}_2}{\sigma} \right) \right] + \frac{1}{2} G \left(\frac{\hat{b}_2 - b}{\sigma} \right) + \frac{1}{2} - \frac{1}{2} G \left(\frac{b - \hat{b}_2}{\sigma} \right) \\ &\leq \frac{-1}{2\sigma} \left[(\hat{b}_2 - c_1) \underline{g} + (b - c_1) \underline{g} \right] + 1, \end{aligned}$$

which converges to $-\infty$ as $\sigma \rightarrow 0$. Hence, there exists $\bar{\sigma}_2 > 0$ such that, if $\sigma < \bar{\sigma}_2$, under the proposed strategy profile the winning firm does not have an incentive to deviate at any period. Define $\bar{\sigma} \equiv \min\{\bar{\sigma}_1, \bar{\sigma}_2\}$.

Next, we show that there exists $\bar{\delta}_1 < 1$ such that, if $\delta > \bar{\delta}_1$, the losing firm does not have an incentive to change her bid either at periods with $c_1 \neq c_2$. Wlog, assume $c_1 < c_2$, and note that firm 2 (the highest cost firm) cannot profit from bidding strictly above $\hat{b}_2 = \hat{b}_1 + \sigma$. The expected payoff that firm 2 gets by bidding $b < \hat{b}_1 + \sigma$ is

$$\begin{aligned} V_2(b) &= \frac{1}{2} \mathbb{E} \left[\mathbf{1}_{\hat{b}_1 + \sigma\epsilon \leq b} (T + \delta\hat{V}/2) + (1 - \mathbf{1}_{\hat{b}_1 + \sigma\epsilon \leq b}) (b - c_2 + \delta V^{NE}) \right] \\ &\quad + \frac{1}{2} \mathbb{E} \left[\mathbf{1}_{\hat{b}_1 \leq b + \sigma\epsilon} (T + \delta\hat{V}/2) + (1 - \mathbf{1}_{\hat{b}_1 \leq b + \sigma\epsilon}) (b + \sigma\epsilon - c_2 + \delta V^{NE}) \right] \\ &= \frac{1}{2} \mathbb{E} \left[\mathbf{1}_{\hat{b}_1 + \sigma\epsilon \leq b} \delta(\hat{V} - 2V^{NE}) + (1 - \mathbf{1}_{b + \sigma\epsilon \leq \hat{b}_2}) (b - c_2) \right] \\ &\quad + \frac{1}{2} \mathbb{E} \left[\mathbf{1}_{\hat{b}_1 \leq b + \sigma\epsilon} \delta(\hat{V} - 2V^{NE}) + (1 - \mathbf{1}_{\hat{b}_1 \leq b + \sigma\epsilon}) (b + \sigma\epsilon - c_2) \right] + \delta V^{NE}, \end{aligned} \quad (15)$$

where again we used $T = \delta(\hat{V}/2 - V^{NE})$. Note that if $b \leq \hat{b}_1 - \sigma$, firm 2 wins with probability 1 and obtains an expected payoff of $b - c_2 + \delta V^{NE}$, which for δ large enough is strictly lower than the payoff of $\delta\hat{V}/2$ that the firm obtains from bidding \hat{b}_2 . Fix $\nu > 0$ small, and let $\bar{\delta}_1$ be the lowest discount factor δ such that $\delta(\hat{V}/2 - V^{NE}) \geq 1 + \nu$. Since $b \leq 1 - \sigma < 1 + \nu$, for $\delta > \bar{\delta}_1$ the losing firm 2 does not have an incentive to place intended bid $b \leq \hat{b}_1 - \sigma$.

Consider next bids $b \in (\hat{b}_1 - \sigma, \hat{b}_1 + \sigma) = (\hat{b}_1 - \sigma, \hat{b}_2)$, and suppose that $\delta > \bar{\delta}_1$. The derivative of payoffs (15) with respect to b , evaluated at such a bid b , is

$$\begin{aligned} V_2'(b) &= \frac{1}{2\sigma} \left[(\delta(\hat{V} - 2V^{NE}) - (b - c_2))g\left(\frac{b - \hat{b}_1}{\sigma}\right) + (\delta(\hat{V} - 2V^{NE}) - (\hat{b}_1 - c_2))g\left(\frac{\hat{b}_1 - b}{\sigma}\right) \right] \\ &\quad + \frac{1}{2}G\left(\frac{\hat{b}_1 - b}{\sigma}\right) + \frac{1}{2} - \frac{1}{2}G\left(\frac{b - \hat{b}_1}{\sigma}\right) \\ &\geq \frac{1}{2\sigma}[2\nu g] > 0, \end{aligned}$$

where we used the fact that $\delta(\hat{V}/2 - V^{NE}) \geq 1 + \nu$ for all $\delta > \bar{\delta}_1$ and that $1 = r \geq \max\{b - c_2, \hat{b}_1 - c_2\}$. Therefore, for $\delta > \bar{\delta}_1$ the high cost bidder does not have an incentive to bid lower than $\hat{b}_2 = \hat{b}_1 + \sigma$.

Lastly, we consider cost realizations \mathbf{c} with $c_1 = c_2$. Under the proposed strategy profile, the expected payoff that bidder $i = 1, 2$ obtains from placing intended bid b when her opponent bids $\hat{b}_{-i} = 1 - \sigma$ is

$$\begin{aligned} V_i(b_i) &= \delta V^{NE} + \frac{1}{2}\mathbb{E} \left[\mathbf{1}_{b_i + \sigma\epsilon \geq 1 - 2\sigma} \delta \left(\frac{\hat{V}}{2} - V^{NE} \right) \right] + \frac{1}{2}\mathbb{E} \left[\mathbf{1}_{b_i \geq 1 - 2\sigma} \delta \left(\frac{\hat{V}}{2} - V^{NE} \right) \right] \\ &\quad + \frac{1}{2}\mathbb{E} \left[\mathbf{1}_{b_i + \sigma\epsilon < \hat{b}_{-i}} (b_i + \sigma\epsilon - c_i) \right] + \frac{1}{2}\mathbb{E} \left[\mathbf{1}_{b_i < \hat{b}_{-i} + \sigma\epsilon} (b_i - c_i) \right] \end{aligned} \quad (16)$$

Note first that bidding $b_i < 1 - 2\sigma$ is never optimal: since $\hat{b}_{-i} = 1 - \sigma$, a bid if $b_i = 1 - 2\sigma$

guarantees that bidder i wins the auction. Next, note that for all $b_i \in [1 - 2\sigma, 1 - \sigma)$,

$$\begin{aligned} V'_i(b_i) &= \frac{-1}{2\sigma} \left[(\hat{b}_{-i} - c_i)g\left(\frac{\hat{b}_{-i} - b_i}{\sigma}\right) + (b_i - c_i)g\left(\frac{b_i - \hat{b}_{-i}}{\sigma}\right) \right] \\ &\quad \frac{1}{2}G\left(\frac{\hat{b}_{-i} - b_i}{\sigma}\right) + \frac{1}{2} - G\left(\frac{b_i - \hat{b}_{-i}}{\sigma}\right) + \frac{1}{2}g\left(\frac{1 - 2\sigma - b_i}{\sigma}\right) \delta \left(\frac{\hat{V}}{2} - V^{NE}\right) \\ &\geq \frac{-1}{2\sigma} 2\bar{g} + \delta \left(\frac{\hat{V}}{2} - V^{NE}\right) \underline{g}. \end{aligned}$$

Note that there exists $\bar{\delta}_2 < 1$ such that, for all $\delta > \bar{\delta}_2$, $V'_i(b_i) > 0$ for all $b_i \in [1 - 2\sigma, 1 - \sigma)$. Hence, for such $\delta > \bar{\delta}_2$, firms don't have an incentive to reduce their bids.

Letting $\bar{\delta} = \max\{\bar{\delta}_1, \bar{\delta}_2\}$, we conclude that, for $\sigma < \bar{\sigma}$ and $\delta > \bar{\delta}$, there exists an equilibrium in which the intended bids that firms place at each period are the bids that solve Program (13), and that delivers at each period a total surplus of \hat{V} . Since firms' profits cannot be larger than \hat{V} , this equilibrium attains \bar{V} . ■

References

- ASKER, J. (2010): "A study of the internal organization of a bidding cartel," *The American Economic Review*, 724–762.
- ATHEY, S. AND K. BAGWELL (2001): "Optimal collusion with private information," *RAND Journal of Economics*, 32, 428–465.
- (2008): "Collusion with persistent cost shocks," *Econometrica*, 76, 493–540.
- BAJARI, P. AND L. YE (2003): "Deciding between competition and collusion," *Review of Economics and Statistics*, 85, 971–989.
- CHASSANG, S. AND J. ORTNER (2016): "Collusion in Auctions with Constrained Bids:

Theory and Evidence from Public Procurement,” *Princeton University William S. Dietrich II Economic Theory Center Research Paper*.

CLARK, R. AND J.-F. HOUDE (2013): “Collusion with asymmetric retailers: Evidence from a gasoline price-fixing case,” *American Economic Journal: Microeconomics*, 5, 97–123.

CONLEY, T. G. AND F. DECAROLIS (2016): “Detecting bidders groups in collusive auctions,” *American Economic Journal: Microeconomics*, 8, 1–38.

ELLISON, G. (1994): “Theories of cartel stability and the joint executive committee,” *The Rand journal of economics*, 37–57.

GREEN, E. J. AND R. H. PORTER (1984): “Noncooperative collusion under imperfect price information,” *Econometrica: Journal of the Econometric Society*, 87–100.

HARRINGTON, J. E. (2008): “Detecting cartels,” *Handbook of antitrust economics*, 213, 245.

KAWAI, K. AND J. NAKABAYASHI (2014): “Detecting Large-Scale Collusion in Procurement Auctions,” *Available at SSRN 2467175*.

MASKIN, E. AND J. TIROLE (2001): “Markov perfect equilibrium: I. Observable actions,” *Journal of Economic Theory*, 100, 191–219.

PESENDORFER, M. (2000): “A study of collusion in first-price auctions,” *The Review of Economic Studies*, 67, 381–411.

PORTER, R. H. (1983): “A study of cartel stability: the Joint Executive Committee, 1880–1886,” *The Bell Journal of Economics*, 301–314.

PORTER, R. H. AND J. D. ZONA (1993): “Detection of Bid Rigging in Procurement Auctions,” *Journal of Political Economy*, 101, 518–38.

——— (1999): “Ohio School Milk Markets: An Analysis of Bidding,” *RAND Journal of Economics*, 30, 263–288.

ROTEMBERG, J. J. AND G. SALONER (1986): “A supergame-theoretic model of price wars during booms,” *The American Economic Review*, 390–407.