Robust Mechanisms Under Common Valuation

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Abstract

We study robust mechanisms to sell a common-value good. We assume that the mechanism designer knows the prior distribution of the buyers' common value but is unsure about the buyers' information structure. We use linear programming duality to derive mechanisms that guarantee a good revenue among all information structures and all equilibria. When there are two buyers and a uniform $[0, 1]$ distribution of common value, our mechanism guarantees a revenue of at least 0.27 for any information structure and any equilibrium, which is $0.27/0.5 = 54\%$ of the best possible revenue. When there is a single buyer, we obtain the optimal mechanism that maximizes the revenue guarantee among all information structures and all equilibria.

1 Introduction

In this paper we study robust mechanism design for selling a common-value good. A robust mechanism is one that works well under a variety of circumstances, in particular under weak assumptions about participants' information structure. The goal of robust mechanism design is to reduce the "base of common knowledge required to conduct useful analyses of practical problems," as envisioned by Wilson (1987).

The literature on robust mechanism design has so far largely focused on private value settings.[1] Common value is of course important in many real-life markets (particularly

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[1] See Chung and Ely (2007), Frankel (2014), Li and Chen (2015), Yamashita (2015, 2016), Carroll (2015, 2016), Carrasco, Farinha Luz, Monteiro, and Moreira (2015), among others; we follow this literature by adopting a max-min approach for robust mechanisms.

financial markets) and has a long tradition in auction theory. Robustness with respect to information structure is especially relevant in a common-value setting, since it is hard in practice to pinpoint exactly what is a signal (or a set of signals) for a participant and to quantify the correlation between the signal and the common value, not to mention specifying the joint distribution of signals for all participants that correctly captures their beliefs and higher order beliefs about the common value.

We are inspired by a recent paper of Bergemann, Brooks, and Morris (2015) which, among other results, shows that in a first price auction of a common value good and for a fixed prior distribution of value, there is a strictly positive lower bound such that for any information/signal structure that is consistent with the prior the equilibrium revenue is always above that lower bound. For example, if there are two buyers and a uniform $[0, 1]$ prior distribution for the common value, then the seller can guarantee a minimum revenue of $1/6$ for any information structure consistent with the prior and any equilibrium from the information structure. Notice that the maximum revenue that the seller can possibly achieve in equilibrium is $1/2$ which is the expected common value, so the minimum revenue guarantee of $1/6$ is already 33% of the best case scenario. This is quite an attractive prospect for a ambiguity averse seller (in the sense of Gilboa and Schmeidler (1989)) who has large uncertainty about buyers' information structure.

The natural followup question is whether we can achieve an even better revenue guarantee with alternative mechanisms. To answer this question we take a max-min approach, where we first minimize the revenue over the set of information structures and equilibria for a given mechanism, and then maximize the minimized revenue over the set of mechanisms. Bergemann and Morris (2016) points out that we can combine information structure and equilibrium into a single entity called *Bayes Correlated Equilibrium*, which is a joint distribution over actions and value subject to incentive and consistency constraints. We note that minimizing revenue over Bayes correlated equilibria for any fixed mechanism is a linear programming problem, and we can equivalently solve the dual of that problem which is a maximization problem over the dual variables of constraints associated with Bayes correlated equilibrium. These dual variables have the interpretation as transition probability rates for a continuous-time Markov process over discrete states. Moreover, we can combine the maximization over these dual variables with the maximization over the mechanism design variables, so we have a single maximization problem which is equivalent to but more tractable than the original max-min problem.

This dual approach yields new mechanisms with better revenue guarantee than the first price auction. Here we describe the simplest mechanism given by our framework: suppose two buyers, a single good to sell, and a uniform $[0,1]$ distribution of common value for the good. Consider a binary mechanism: each buyer simultaneously says Yes or No, with the following allocations and payments:

1. If one says No, then he pays nothing and does not get the good.
2. If one says Yes and the other says No, the Yes buyer pays 3/16 and gets the good.
3. If both say Yes, then each still pays 3/16 individually and each gets the good with probability 1/2.

We claim that this mechanism guarantees a minimum revenue (over all information structures and equilibria) of 3/16, which is higher than 1/6 from a first price auction. We leave the proof of this claim and how we derive this mechanism to Section 3.1. Let us sketch here some examples of information structure and illustrate the performance of this mechanism. First, take the information structure in which both buyers know exactly the common value v, for each $v \in [0,1]$. Consider the symmetric mixed-strategy equilibrium in which a buyer says Yes with probability $\rho(v)$. For $v \in [3/16, 3/8]$ each buyer is indifferent between Yes and No, so

$$(1-\rho(v))v + \rho(v)v/2 = 3/16. \tag{1}$$

In this mixed-strategy equilibrium, the expected revenue is

$$\int_{3/16}^{3/8} \left(3/16 \cdot 2\rho(v)(1-\rho(v)) + 3/8 \cdot \rho(v)^2\right) dv + (1-3/8)\cdot 3/8 \approx 0.277526, \tag{2}$$

which is indeed higher than the lower bound of 3/16.

Consider another information structure with two signals (high and low): conditional on $v \le 1/4$, both buyers get the low signals; conditional on $1/4 \le v \le 3/4$, with probability 1/2 buyer 1 gets the high signal and buyer 2 gets the low signal (and vice versa with probability 1/2); conditional on $v \ge 3/4$, both buyers get the high signals. Suppose that each buyer says Yes if he gets the high signal and says No if he gets the low signal. This is an equilibrium: conditional on a buyer receiving the low signal, his expected payoff from saying Yes is:

$$\frac{\int_0^{1/4} v\,dv + \frac{1}{2}\int_{1/4}^{3/4} v/2\,dv}{1/2} - \frac{3}{16} = 0,$$

which is the same as his payoff from saying No. And conditional on a buyer receiving a high signal, his expected payoff from saying Yes is:

$$\frac{\frac{1}{2}\int_{1/4}^{3/4} v\,dv + \int_{3/4}^{1} v/2\,dv}{1/2} - \frac{3}{16} > 0,$$

which is strictly better than saying No. Finally, it is easy to see that the expected revenue from this equilibrium is 3/16, exactly hitting the lower bound.

We generalize the previous mechanism by allowing for more messages. As the number of messages increases, the revenue that the generalized mechanism guarantees (over all information structures and all equilibria) becomes better. In the limit as the number of messages tends to infinity, the generalized mechanism guarantees a revenue of 0.27 when there are two buyers and a uniform $[0, 1]$ distribution of common value. Moreover, we apply the same methodology to the case of one buyer and show that the resulting mechanism achieves the optimal revenue guarantee. Here we make a connection to Roesler and Szentes (2016), who study the optimal information structure for a buyer when the seller is best responding to this information structure. The information structure and revenue given by Roesler and Szentes (2016) yield a subtle upper bound on the revenue that any mechanism can guarantee with one buyer, and we show that our mechanism exactly achieves this upper bound.

Our paper is also related to Yamashita (2016), who studies revenue guarantee in mechanisms with private values. Besides the distinction between common vs. private value, the two papers use different notions of guarantee. Our paper considers the minimum revenue over every information structure and every equilibrium of the information structure, while Yamashita (2016) considers the minimum revenue over every information structure and some equilibrium of the information structure. Thus we use a more demanding notion of guarantee.

2 Model

Information

Suppose the mechanism designer has a single good to sell, and there are a finite number of buyers (let I be the set of buyers, with $1 \leq |I| < \infty$). The buyers have a common value $v \in V = \{\underline{v}, \underline{v} + \nu, \underline{v} + 2\nu, \ldots, \overline{v}\}$ for the good and have quasi-linear utility. Let $p \in \Delta(V)$ be the prior distribution of value; the prior p is known by the designer as well as by the buyers. (The designer only knows the prior p about the value.) Without loss suppose that $p(v) > 0$

for every $v \in V$.

Each buyer i *may* possess some additional information $t_i \in T_i$ about the common value beyond the prior, where $\tilde{p} \in \Delta(V \times \prod_{i \in I} T_i)$ such that $\operatorname{marg}_V \tilde{p} = p$,[2] so his information about the common value is informed by $\tilde{p}(\cdot \mid t_i)$. As discussed in the introduction, the information structure $(T_i, \tilde{p})_{i \in I}$ is *not* known by the designer.

Mechanism

A mechanism is a set of allocation rules $q_i : M \to [0,1]$ and payment rules $P_i : M \to \mathbb{R}$ satisfying $\sum_{i \in I} q_i(m) \leq 1$, where M_i is the message space of buyer i, and $M = \prod_{i \in I} M_i$ the space of message profile. We assume that a mechanism always has an opt-out option for each buyer i: there exists a message $m_i \equiv 0 \in M_i$ such that $q_i(0, m_{-i}) = P_i(0, m_{-i}) = 0$ for every $m_{-i} \in M_{-i}$.

Equilibrium

Given a mechanism $(q_i, P_i)_{i \in I}$ and an information structure $(T_i, \tilde{p})_{i \in I}$, we have a game of incomplete information. A *Bayes Nash Equilibrium* (BNE) of the game is defined by strategy $\sigma_i : T_i \to \Delta(M_i)$ for each buyer i such that for every $t_i \in T_i$, the support of $\sigma_i(t_i)$ are best responding to others' strategies:

$$\operatorname{supp} \sigma_i(t_i) \in \operatorname*{argmax}_{m_i \in M_i} \sum_{(v, t_{-i}) \in V \times T_{-i}} (v q_i(m_i, \sigma_{-i}(t_{-i})) - P_i(m_i, \sigma_{-i}(t_{-i})) \tilde{p}(v, t_{-i} \mid t_i). \tag{3}$$

The ex ante distribution $\mu \in \Delta(V \times M)$ generated by any BNE $(\sigma_i)_{i \in I}$ of any information structure $(T_i, \tilde{p})_{i \in I}$ satisfies the following two conditions:

$$\sum_{m \in M} \mu(v, m) = p(v), \quad v \in V, \tag{4}$$

$$\sum_{(v, m_{-i}) \in V \times M_{-i}} \mu(v, m) \big(v_i(q_i(m_i, m_{-i}) - q_i(m_i', m_{-i})) - P_i(m_i, m_{-i}) + P_i(m_i', m_{-i}) \big) \geq 0,$$
$$i \in I, (m_i, m_i') \in M_i \times M_i. \tag{5}$$

A distribution $\mu \in \Delta(V \times M)$ that satisfies the above two conditions is called a *Bayes Correlated Equilibrium* (BCE) of the mechanism $(q_i, P_i)_{i \in I}$. For any BCE μ, there exists

[2] Let $\operatorname{marg}_V \tilde{p}$ be the marginal distribution of $\tilde{p}$ over V.

an information structure and a BNE of that information structure that generates μ. See Bergemann and Morris (2016) for more details.

Designer's problem

Since the mechanism designer does not know the buyers' information structure, a natural way for him to design his mechanism is to be cautious and maximize the minimum revenue among all information structure and equilibrium.

The mechanism designer wants to solve:

$$\max_{(q_i,P_i)_{i\in I}} \min_{\mu\in\Delta(V\times M)} \sum_{(v,m)} \sum_i \mu(v,m)P_i(m) \tag{6}$$
$$\text{such that } \mu \text{ is a BCE of } (q_i,P_i)_{i\in I}$$

2.1 Minimum-Revenue BCE

As a prerequisite to solve the designer's problem, we first study the minimum revenue generated by its BCE for a fixed mechanism $(P_i, q_i)_{i\in I}$. We define

$$U_i(v,m) \equiv vq_i(m) - P_i(m), \quad m \in M, v \in V. \tag{7}$$

The BCE that minimizes revenue can be found by the following primal problem:

$$\min_{\mu} \sum_{(v,m)} \sum_i P_i(m)\mu(v,m) \tag{8}$$

subject to:

$$\sum_{(v,m_{-i})} (U_i(v,m) - U_i(v,(m_i',m_{-i})))\mu(v,m) \geq 0, \quad i \in I, (m_i,m_i') \in M_i \times M_i,$$
$$\sum_m \mu(v,m) = p(v), \quad v \in V,$$
$$\mu(v,m) \geq 0, \quad v \in V, m \in M.$$

For notational brevity, we omit the set of which a summation variable belongs when it is obvious; for example, summing over m means summing over $m \in M$.

The dual problem is:

$$\max_{(\alpha_i,\gamma)_{i\in I}} \sum_v p(v)\gamma(v) \tag{9}$$

subject to:

$$\gamma(v) + \sum_i \sum_{m_i'} (U_i(v,m) - U_i(v,(m_i',m_{-i})))\alpha_i(m_i' \mid m_i) \le \sum_i P_i(m), \quad v \in V, m \in M,$$

$$\alpha_i(m_i' \mid m_i) \ge 0, \quad i \in I, (m_i, m_i') \in M_i \times M_i,$$

where $\alpha(m_i' \mid m_i)$ is the dual variable for the obedience or incentive constraint of not playing m_i' when advised to play m_i in (8), and $\gamma(v)$ is the dual variable for the consistency constraint of $\sum_m \mu(v,m) = p(v)$.

The dual problem can be succinctly written as:

$$\max_{(\alpha_i)_{i\in I}} \sum_v p(v) \cdot \min_m \mathrm{Rev}(v,m), \tag{10}$$

where

$$\mathrm{Rev}(v,m) \equiv \sum_i \left(P_i(m) + \sum_{m_i'} (U_i(v,m_i',m_{-i}) - U_i(v,m))\alpha_i(m_i' \mid m_i) \right). \tag{11}$$

We note that $\mathrm{Rev}(v,m)$ is independent of the value of $\alpha_i(m_i \mid m_i)$ for every m_i.

Here is an interesting representation of $\mathrm{Rev}(v,m)$ by matrices when there are two buyers: suppose buyer 1's messages M_1 are the rows, and buyer 2's messages M_2 are the columns, $\mathrm{Rev}(v)$, P_i and $U_i(v)$ are $M_1 \times M_2$ matrices,

$$\mathrm{Rev}(v) = P_1 + \alpha_1 \cdot U_1(v) + P_2 + U_2(v) \cdot \alpha_2, \tag{12}$$

where α_1 is a $M_1 \times M_1$ transition rate matrix (or Q matrix): the off-diagonal entries of α_1 are non-negative, each row of α_1 sums to 0; and α_2 is a $M_2 \times M_2$ transition rate matrix: the off-diagonal entries of α_2 are non-negative, each column of α_2 sums to 0. Transition rate matrix is the analogue of the transition probability matrix for a continuous-time Markov process over discrete states (see Stroock (2013), Chapter 5).

The Problem (9) is bounded, by the following lemma:

Lemma 1. *For any $\{\alpha_i(m_i' \mid m_i)\}$ such that $\alpha_i(m_i' \mid m_i) \geq 0$ for $m_i' \neq m_i$, we have*

$$\min_m \mathrm{Rev}(v, m) \leq v, \tag{13}$$

for every $v \in V$.

Proof. Fix an arbitrary $v \in V$. Consider the problem:

$$\max_{\gamma,(\alpha_i)_{i\in I}} \gamma \tag{14}$$

subject to:

$$\gamma + \sum_i \sum_{m_i'} (U_i(v, m) - U_i(v, (m_i', m_{-i})))\alpha_i(m_i' \mid m_i) \leq \sum_i P_i(m), \quad m \in M,$$

$$\alpha_i(m_i' \mid m_i) \geq 0, \quad i \in I, (m_i, m_i') \in M_i \times M_i.$$

The dual to the above problem is:

$$\min_\mu \sum_m \mu(m) \sum_i P_i(m) \tag{15}$$

subject to:

$$\sum_{m_{-i}} \mu(m)(U_i(v, m) - U_i(v, (m_i', m_{-i}))) \geq 0, \quad i \in I, (m_i, m_i') \in M_i \times M_i,$$

$$\sum_m \mu(m) = 1,$$

$$\mu(m) \geq 0, \quad m \in M,$$

which is minimizing the revenue over *complete-information* correlated equilibria μ (for the fixed v). For any μ satisfying the constraints, we have $\sum_{m_{-i}} \mu(m)U_i(v, m) = \sum_{m_{-i}} \mu(m)(vq_i(m) - P_i(m)) \geq 0$ for every $i \in I$ and $m_i \in M_i$ because of the presence of the opt-out message $0 \in M_i$. Therefore, $\sum_m \mu(m) \sum_i (vq_i(m) - P_i(m)) \geq 0$, and $\sum_m \mu(m) \sum_i P_i(m) \leq \sum_m \mu(m) \sum_i vq_i(m) \leq v$. Thus the optimal solution of (14) is bounded above by v. $\square$

3 Revenue Guarantee with Two Buyers

Given the previous derivations, mechanism designer's problem in (6) can be written as:

$$\max_{(P_i,q_i,\alpha_i)_{i\in I}} \sum_v p(v)\cdot \min_m \mathrm{Rev}(v,m) \tag{16}$$

subject to:

$$\begin{aligned}
&\mathrm{Rev}(v,m)\\
&=\sum_i P_i(m)+\sum_i\sum_{m_i'}(v(q(m_i',m_{-i})-q(m))-P(m_i',m_{-i})+P(m))\alpha_i(m_i'\mid m_i), \quad v\in V, m\in M,\\
&q_i(m)\ge 0,\quad i\in I,\quad \sum_i q_i(m)\le 1,\quad m\in M,\\
&q_i(0,m_{-i})=P_i(0,m_{-i})=0,\quad m_{-i}\in M_{-i},\quad \alpha_i(m_i'\mid m_i)\ge 0,\quad (m_i,m_i')\in M_i\times M_i,\quad i\in I.
\end{aligned}$$

where we label the opt-out message as $0\in M_i$.

The advantage of problem (16) over the equivalent problem (6) is that we minimize over a finite set M instead of an infinite set in $\Delta(V\times M)$. Moreover, (16) is a bilinear programming problem: fixing the mechanism $(P_i,q_i)_{i\in I}$ the maximization problem over $(\alpha_i)_{i\in I}$ is linear; and fixing the dual variables $(\alpha_i)_{i\in I}$ the maximization problem over $(P_i,q_i)_{i\in I}$ is also linear.

In this section we restrict to the case of two buyers ($I=\{1,2\}$) and symmetric mechanism and dual variables in the sense of Definition 1. We abbreviate (q_1,P_1,α_1) to (q,P,α). We also allow the possibility of $\nu\to 0$ (the set of values becomes a continuum); as $\nu\to 0$, the sum in (16) can be approximated by an integral, since the allocations and prices that we obtain in the following sections are bounded.

Definition 1. Suppose there are two buyers $I=\{1,2\}$. The tuple $(q_i,P_i,\alpha_i)_{i\in I}$ is *symmetric* if $M_1=M_2$, $q_1(m_1,m_2)=q_2(m_2,m_1)$, $P_1(m_1,m_2)=P_2(m_2,m_1)$, and $\alpha_1(m_2\mid m_1)=\alpha_2(m_2\mid m_1)$ hold for every $m_1,m_2\in M_1=M_2$.

3.1 Two Messages

Consider first a mechanism with two messages: $M_1=M_2=\{0,1\}$ (where 0 is the opt-out message).

We write the objective of Problem (16) as:

$$\Pi=\int \min(\mathrm{Rev}(v,(0,0)),\mathrm{Rev}(v,(1,0)),\mathrm{Rev}(v,(1,1)))\,p(dv), \tag{17}$$

which we want to maximize subject to feasibility constraints: $0 \le q(1,0) \le 1$ and $0 \le q(1,1) \le 1/2$ (recall that $q(0,1) = q(0,0) = 0 = P(0,0) = P(0,1)$). By symmetry we have $\mathrm{Rev}(v,(0,1)) = \mathrm{Rev}(v,(1,0))$.

Suppose $\alpha(0 \mid 1) = 0$ and $\alpha(1 \mid 0) = a > 0$. Under these assumptions,

$$\frac{\partial \,\mathrm{Rev}(v,m)}{\partial q(1,0)} = \begin{cases} 2av & m = (0,0) \\ 0 & m \neq (0,0) \end{cases} \tag{18}$$

Similarly for the derivative with respect to $q(1,1)$. Thus we have

$$\frac{\partial \Pi}{\partial q(1,0)} \geq 0 \text{ and } \frac{\partial \Pi}{\partial q(1,1)} \geq 0, \tag{19}$$

and to maximize Π we set $q(1,0) = 1$ and $q(1,1) = 1/2$.

We compute:

$$\begin{aligned} \mathrm{Rev}(v,(0,0)) &= 2av - 2aP(1,0), \\ \mathrm{Rev}(v,(1,0)) &= av/2 - aP(1,1) + P(1,0), \\ \mathrm{Rev}(v,(1,1)) &= 2P(1,1). \end{aligned} \tag{20}$$

Thus, we solve

$$\Pi^* \equiv \max_{a \geq 0,\, P(1,0),\, P(1,1)} \int \min\left(2av - 2aP(1,0),\, \frac{av}{2} - aP(1,1) + P(1,0),\, 2P(1,1)\right) p(dv). \tag{21}$$

Suppose p is the uniform distribution on $[0,1]$. The optimal solution to the above problem is $P(1,0) = P(1,1) = 3/16$ and $a = 1$, and they give the maximum $\Pi^* = 3/16$.

Proposition 1. *Suppose there are two buyers and the prior p is the uniform distribution on $[0,1]$. The symmetric mechanism of $M_1 = M_2 = \{0,1\}$ and*

$$\begin{aligned} &q(0,0) = q(0,1) = 0,\ q(1,0) = 1,\ q(1,1) = 1/2, \\ &P(0,0) = P(0,1) = 0,\ P(1,0) = P(1,1) = 3/16, \end{aligned} \tag{22}$$

guarantees a minimum revenue (among all BCE) of $3/16$.

For comparison we note that for the same setting (two buyers, uniform distribution of common value) the first price auction guarantee a minimum revenue (among all BCE) of

1/6; see Bergemann, Brooks, and Morris (2015). And Lemma 1 implies that the revenue that any mechanism can guarantee among BCE is bounded above by $\mathbb{E}[v] = 1/2$ (since any BCE revenue in any mechanism is less or equal to 1/2), although this bound is unlikely to be tight.

For the mechanism in Proposition 1, the dual variables of $\alpha(1 \mid 0) = 1$ and $\alpha(0 \mid 1) = 0$ satisfies the complementary slackness conditions for Problems (8) and (9) with respect to primal variables $\mu(v, m)$ such that

$$\mu(\cdot \mid v) = \begin{cases} \mathbf{1}_{(0,0)} & v < 1/4 \\ \frac{1}{2} \cdot \mathbf{1}_{(0,1)} + \frac{1}{2} \cdot \mathbf{1}_{(1,0)} & 1/4 \leq v < 3/4 \\ \mathbf{1}_{(1,1)} & v \geq 3/4 \end{cases}, \tag{23}$$

$$\operatorname{marg}_V \mu = \text{uniform distribution on } [0, 1].$$

Recall that $\mu(v, m)$ is the BCE described in the introduction. From the complementary slackness conditions we conclude that α solves Problem (9) and μ solves Problem (8), so μ is the BCE that minimizes revenue for this mechanism.

3.2 General Messages

We can do better with more messages. In general we want to maximize:

$$\Pi = \int \min_{m \in M} (\operatorname{Rev}(v, m))\, p(dv), \tag{24}$$

subject to the feasibility constraints. Thus, it makes sense to make $\operatorname{Rev}(v, m)$ over m as redundant as possible, to minimize the number of things in $\min(\cdot)$ inside the integral.

Consider a mechanism with $k + 1$ messages: $M_1 = M_2 = \{0, 1, \ldots, k\}$. Our usual assumption on the mechanism is:

$$q(0, j) = 0 = P(0, j), \quad q(j, l) \geq 0, \quad q(j, l) + q(l, j) \leq 1, \quad (j, l) \in \{0, 1, \ldots, k\}^2. \tag{25}$$

We make the following additional assumptions:

$$\begin{aligned}
\operatorname{Rev}(v,(0,0)) &= \operatorname{Rev}(v,(j,l)), \quad v \in V,\ (j,l) \in \{0,1,\ldots,k-1\}^2 \\
\operatorname{Rev}(v,(0,k)) &= \operatorname{Rev}(v,(j,k)), \quad v \in V,\ j \in \{0,1,\ldots,k-1\}
\end{aligned} \tag{26}$$

$$\alpha(j' \mid j) = \begin{cases} a & j' = j+1 \\ 0 & j' \neq j+1 \end{cases}, \quad (j,j') \in \{0,1,\ldots,k\}^2, \tag{27}$$

where a is a positive constant.[3]

Condition (26) attempts to make $\operatorname{Rev}(v,m)$ as redundant as possible. Condition (27) is for tractability and is inspired by binding local incentive constraints.

Under (27), Equation (26) holds if and only if

$$\begin{aligned}
&2q(1,0) = q(j+1,l) - q(j,l) + q(l+1,j) - q(l,j), \quad (j,l) \in \{0,1,\ldots,k-1\}^2, \\
&q(1,k) = q(j+1,k) - q(j,k), \quad j \in \{0,1,\ldots,k-1\},
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
-2aP(1,0) &= P(j,l) + P(l,j) - a(P(j+1,l) - P(j,l)) - a(P(l+1,j) - P(l,j)), \\
&\qquad\qquad (j,l) \in \{0,1,\ldots,k-1\}^2, \\
P(k,0) - aP(1,k) &= P(j,k) + P(k,j) - a(P(j+1,k) - P(j,k)), \quad j \in \{0,1,\ldots,k-1\}.
\end{aligned} \tag{29}$$

Since

$$\frac{\partial \operatorname{Rev}(v,m)}{\partial (q(j+1,l) - q(j,l))} = \begin{cases} av & m = (j,l) \text{ or } (l,j),\ \text{and } j \neq l \\ 2av & m = (j,l),\ \text{and } j = l \\ 0 & \text{otherwise} \end{cases}, \tag{30}$$

in the optimum the feasibility constraint $q(j,k) + q(k,j) \leq 1$ should bind (if not, we can increase $q(k,j)$ without decreasing any $\operatorname{Rev}(v,m)$ or violating any feasibility constraint):

$$q(j,k) + q(k,j) = 1, \quad j \in \{0,1,\ldots,k\}. \tag{31}$$

Lemma 2. *For every $k \geq 1$, there exists $\{q(j,l) : 1 \leq j \leq k, 0 \leq l \leq k\}$ that satisfies*

[3] By symmetry, the second line of (26) implies that $\operatorname{Rev}(v,(0,k)) = \operatorname{Rev}(v,(k,0)) = \operatorname{Rev}(v,(k,j)) = \operatorname{Rev}(v,(j,k))$, $j = 0,1,\ldots,k-1$.

Conditions (25), (28) and (31). For any such q we have $q(1,0) = (3k+1)/(4k^2)$ and $q(1,k) = 1/(2k)$.

Lemma 3. *For any given values of $a > 0$, $k \geq 1$, $P(1,0)$, $P(1,k)$ and $P(k,0)$, there exists $\{P(j,l) : 1 \leq j \leq k, 0 \leq l \leq k, (j,l) \notin \{(1,0),(1,k),(k,0)\}\}$ that satisfies Condition (29). For any such P we have*

$$P(k,k) = \left((1+1/a)^k - 1\right)^2 aP(1,0) + ((1+1/a)^k - 1)(aP(1,k) - P(k,0)). \tag{32}$$

Using the above two lemmas and Equation (27), we compute:

$$\begin{aligned}
\mathrm{Rev}(v,(0,0)) &= \frac{3k+1}{2k^2}av - 2aP(1,0), \\
\mathrm{Rev}(v,(0,k)) &= \frac{av}{2k} - aP(1,k) + P(k,0), \\
\mathrm{Rev}(v,(k,k)) &= 2\left((1+1/a)^k - 1\right)^2 aP(1,0) + 2((1+1/a)^k - 1)(aP(1,k) - P(k,0)).
\end{aligned} \tag{33}$$

Define,

$$X \equiv aP(1,0), \qquad Y \equiv aP(1,k) - P(k,0) \tag{34}$$

We thus solve:

$$\Pi^* \equiv \max_{k\geq 1,\, a \geq 0,\, X,\, Y} \int \min\left(\frac{3k+1}{2k^2}av - 2X,\ \frac{av}{2k} - Y,\ 2\left((1+1/a)^k - 1\right)^2 X + 2((1+1/a)^k - 1)Y\right) p(dv). \tag{35}$$

Proposition 2. *Suppose there are two buyers. There exists a symmetric mechanism that guarantees a minimum revenue (among all BCE) of Π^* defined in (35).*

Proof. The proof is given by the construction above. $\square$

We now elaborate on the allocation and payment rules for the mechanism in Proposition 2. Let us fix k, a, X, Y from solving (35).

Consider the following allocation rule:

$$\begin{aligned}
q(j+1,l) - q(j,l) &= \begin{cases} (2k+1)/(4k^2) & j < l \\ (3k+1)/(4k^2) & j = l \\ (4k+1)/(4k^2) & j > l \end{cases}, \qquad (j,l) \in \{0,1,\ldots,k-1\}^2, \\
q(j+1,k) - q(j,k) &= 1/(2k), \qquad j \in \{0,1,\ldots,k-1\},
\end{aligned} \tag{36}$$

where the first and second lines above are constructed to satisfy the first and second lines of Condition (28), respectively. Because of Condition (31), we must have $q(1,0) = q(l+1,l) - q(l,l) = (3k+1)/(4k^2)$ by Lemma 2. In the proof to Lemma 2 we show that the above q is a legitimate allocation rule (it satisfies the feasibility constraint (25)) and satisfies Condition (31) as well.

Since $q(0,l) = 0$ for every l, Equation (36) is equivalent to:

$$q(j,l) = \begin{cases} j(2k+1)/(4k^2) & j \le l, l < k \\ [l(2k+1) + 3k + 1 + (j-l-1)(4k+1)]/(4k^2) & j > l, l < k \\ j/2k & l = k \end{cases}. \tag{37}$$

Among the manifold of possibilities, we choose the following simplest solution to (29):

$$P(j,l) - a(P(j+1,l) - P(j,l)) = \begin{cases} -aP(1,0) & 0 \le l < k, \\ -aP(1,k) & l = k, \end{cases}, \quad j \in \{0,1,\ldots,k-1\}. \tag{38}$$

The case of $0 \le l < k$ in (38) clearly satisfies the first line of (29) and implies that $P(j,0) = P(j,1) = \cdots = P(j,k-1)$ for every j. And given $P(k,0) = P(1,0) = \cdots = P(k,k-1)$, the case of $l = k$ in (38) implies the second line of (29).

Since we have $P(j+1,l) - P(j,l) = (1+1/a)(P(j,l) - P(j-1,l))$ in (38), Equation (38) is equivalent to:

$$P(j,l) = \begin{cases} ((1+1/a)^j - 1)aP(1,0) & 0 \le l < k \\ ((1+1/a)^j - 1)aP(1,k) & l = k \end{cases}, \qquad (j,l) \in \{0,1,\ldots,k\}^2. \tag{39}$$

In Equation (39) we take (cf. (34)):

$$\begin{aligned} &P(1,0) = X/a, \\ &P(1,k) = (Y + P(k,0))/a = \left(Y + \left((1+1/a)^k - 1\right)X\right)/a. \end{aligned} \tag{40}$$

3.3 Uniform Distribution

Suppose the prior p is the uniform distribution on $[0,1]$. In the Appendix we solve (35) and get:

Corollary 1. *Suppose there are two buyers and the prior p is the uniform distribution on $[0,1]$. There exists a symmetric mechanism that guarantees a minimum revenue (among all BCE) of $\Pi^* \approx 0.273$. This is given by $k \to \infty$, $a/k \to 0.7367$, $X = 0.102496$ and $Y = -0.201343$ in (35).*

For variables j and l in $q(j,l)$ and $P(j,l)$, let us re-parameterize them as $x_1 \equiv j/k$ and $x_2 \equiv l/k$. As $k \to \infty$, the allocation rule for the mechanism of Corollary 1, as described in (37), becomes:

$$q(x_1, x_2) = \begin{cases} x_1/2 & x_1 \le x_2 \\ x_1 - x_2/2 & x_1 > x_2 \end{cases}, \qquad (x_1, x_2) \in M_1 \times M_2 = [0,1]^2. \tag{41}$$

As $k \to \infty$, the P defined in (39) is unfortunately discontinuous at $x_2 = 1$. The discontinuity is driven by the fact that we have solved the two lines of (29) separately to get (39).

Let

$$A \equiv \lim_{k \to \infty} a/k = 0.7367. \tag{42}$$

As $k \to \infty$, Condition (29) is implied by the following ordinary differential equation:

$$\begin{aligned} P(x_1, x_2) - A \cdot P_1(x_1, x_2) + P(x_2, x_1) - A \cdot P_1(x_2, x_1) &= C, \qquad (x_1, x_2) \in [0,1]^2, \\ P(0, x_2) = 0, \quad P_1(1, x_2) &= D, \qquad x_2 \in [0,1], \end{aligned} \tag{43}$$

for constants C and D, where we abuse the notation by letting P_1 be the partial derivative of P with respect to the first parameter. We insist on $P_1(1, x_2) = D$ so that both lines of (29) are simultaneously implied by $P(x_1, x_2) - A \cdot P_1(x_1, x_2) + P(x_2, x_1) - A \cdot P_1(x_2, x_1) = C$. We conjecture that there is a continuous solution to (43).

4 Revenue Guarantee with a Single Buyer

In this section we consider the case of one buyer $I = \{1\}$ for comparison. In this case the buyer does not necessarily know his own value, and each BCE corresponds to an information structure about value and the associated equilibrium play. In the case of one buyer we know sharp upper bound on the BCE revenue that the designer can guarantee. For example, if the prior is the uniform distribution on $[0,1]$, the designer can guarantee (among all BCE) a revenue of at most $1/4$: fix any mechanism, there is a BCE corresponding to the buyer knowing his value (i.e., the classical private-value information structure), and its revenue must be less than 1/4 which is obtained by the private-value optimal mechanism (a posted price of 1/2).[4] In fact, a result of Roesler and Szentes (2016) gives a tighter and more subtle upper bound: the designer can guarantee (among all BCE) a revenue of at most 0.2036 for uniform $[0, 1]$ distribution. We will show that for the case of one buyer the analogues of Assumptions (26) and (27) give a mechanism that guarantees exactly the Roesler and Szentes upper bound.

As before we abbreviate (q_1, P_1, α_1) to (q, P, α). Consider $k+1$ messages: $M_1 = \{0, 1, \ldots, k\}$. Our usual assumption on the mechanism is

$$q(0) = P(0) = 0, \qquad 0 \le q(j) \le 1,\ j = 0, 1, \ldots, k. \tag{44}$$

Using our previous methodology (making $\mathrm{Rev}(v, m)$ as redundant as possible), we consider (q, P, α) such that:

$$\mathrm{Rev}(v, 0) = \mathrm{Rev}(v, 1) = \cdots = \mathrm{Rev}(v, k-1), \quad v \in V, \tag{45}$$

$$\alpha(j' \mid j) = \begin{cases} a & j' = j+1 \\ 0 & j' \neq j+1 \end{cases}, \quad (j, j') \in \{0, 1, \ldots k\}^2, \tag{46}$$

for a constant $a > 0$.

Under Condition (46), Equation (45) holds if and only if for every $j \in \{1, 2, \ldots, k-1\}$,

$$\begin{aligned} q(j+1) - q(j) &= q(j) - q(j-1), \\ P(j+1) - P(j) &= \left(1 + \frac{1}{a}\right)(P(j) - P(j-1)). \end{aligned}$$

As before, the optimum must have $q(k) = 1$; thus we are led to the following mechanism:

$$q(j) = j/k, \qquad P(j) = P(1)a\left(\left(1 + \frac{1}{a}\right)^j - 1\right), \qquad 0 \le j \le k. \tag{47}$$

[4] A posted price of 1/2 guarantees a BCE revenue of 0: if the buyer has no information beyond the prior, then not buying the good is an equilibrium. While this is not the only equilibrium, we can construct information structure in which the unique equilibrium has very little revenue: for example, when the buyer's information about the value is the partition $\{[0, 0.99), [0.99, 1]\}$, the only equilibrium is that the buyer chooses not to buy if $v \in [0, 0.99)$ and to buy if $v \in [0.99, 1]$.

By our construction, Problem (16) becomes

$$\Pi^* \equiv \max_{k\geq 1,\, a\geq 0,\, P(1)} \int \min\left(\frac{av}{k} - aP(1),\ P(1)a\left(\left(1+\frac{1}{a}\right)^k - 1\right)\right) p(dv). \tag{48}$$

Proposition 3. *Suppose there is one buyer. There exists a mechanism that guarantees a minimum revenue (among all BCE) of* Π^* *defined in* (48).

Proposition 4. *Suppose there is one buyer and the prior* p *admits a density on* $[0,1]$*. If* $\rho \geq 0$ *and* $A \geq 0$ *satisfy*

$$\int_0^{\rho\exp(1/A)} v\, p(dv) = \rho/A, \quad \int_{\rho\exp(1/A)}^1 p(dv) = \exp(-1/A), \quad \rho\exp(1/A) \leq 1, \tag{49}$$

then as $k \to \infty$*,* $a/k \to A$ *and* $P(1)k \to \rho$*, the mechanism in* (47) *guarantees a minimum revenue (among all BCE) of* ρ*. Moreover, there exists* $\rho \geq 0$ *and* $A \geq 0$ *that satisfy* (49) *such that* ρ *is the best BCE revenue that the designer can guarantee, i.e., it is a solution to Problem* (16).

As $k \to \infty$, $a/k \to A$ and $P(1)k \to \rho$, the mechanism in (47) becomes:

$$q(x) = x, \qquad P(x) = \rho A(\exp(x/A) - 1), \qquad x \in M_1 = [0,1], \tag{50}$$

where we re-parameterize $x \equiv j/k$ for $q(j)$ and $P(j)$.

The first part of Proposition 4 comes from solving (48) and taking the first order condition. The second part is due to Roesler and Szentes (2016). Roesler and Szentes (2016) study the optimal information structure for the buyer given the seller is best responding to this information structure. Such information structure has the following cumulative distribution function for signals:

$$G_\pi^B(s) = \begin{cases} 1 & s \geq B \\ 1 - \pi/s & s \in [\pi, B) \\ 0 & s < \pi \end{cases}, \tag{51}$$

where $s \in [0,1]$ is an unbiased signal of the buyer for his value ($\mathbb{E}[v \mid s] = s$), $0 < \pi \leq B$ are two free parameters, and there is an atom of size π/B at $s = B$. If the buyer has this distribution of signals (and observes the realization of the signal), then the seller is clearly indifferent between every posted price in $[\pi, B]$ and has an optimal revenue of π.

Clearly, $G_\pi^B(s)$ is a distribution of unbiased signal of v if and only if the prior p is a mean-preserving spread of $G_\pi^B(s)$, which holds if and only if

$$\int_0^1 s\, dG_\pi^B(s) = \int_0^1 v\, p(dv), \text{ and } \int_0^x G_\pi^B(s)\, ds \le \int_0^x p(v \le s)\, ds \text{ for every } x \in [0,1], \quad (52)$$

i.e., G_π^B has the same mean as p and second-order stochastically dominates p.

Roesler and Szentes (2016) prove that the best information structure for the buyer is $G_{\pi^*}^B$, where π^* is the smallest π such that Condition (52) holds for some B (such B must be unique). We note that if the buyer has information structure $G_{\pi^*}^B$, then in any mechanism and any BCE of that mechanism the seller's revenue is at most π^*, since the best mechanism for the seller given a fixed information structure is a posted price mechanism (any posted price in $[\pi^*, B]$ generates a revenue of π^* by construction)[5].

Simple algebra in the appendix shows that $\rho = \pi^*$ satisfies (49) with some A, which proves the second part of Proposition 4.

[5]The order of quantifiers is important: suppose p is the uniform distribution. For every posted price mechanism, there exists an information structure for the buyer and an equilibrium of that information structure in which the seller gets at most $1/8$; $1/8$ is given by $P(1) = 1/4$ and $a = 1$ which maximize (48) when $k = 1$. At the same time $\pi^* = 0.2036$.

Appendix

A Proofs

Proof of Lemma 2. We first prove the second part of the lemma. By (31) we have $q(k,k) = 1/2$. By the second line of (28) this implies that $q(j,k) = j/(2k)$ and $q(k,j) = 1 - j/(2k)$, $j = 0, 1, \dots, k$. Then we have

$$k - \frac{(k-1)k}{4k} = \sum_{j=0}^{k-1} q(k,j) = \sum_{j=0}^{k-1}\sum_{l=0}^{k-1} q(l+1,j) - q(l,j) = k^2 q(1,0) \tag{53}$$

where the last equality follows from the first line of (29). Thus, $q(1,0) = (3k+1)/(4k^2)$.

For the first part, we claim that q defined by Equation (36) satisfies Conditions (28), (25) and (31).

Condition (28) is obvious.

We clearly have $q(j,k) = j/(2k)$ for every $j = 0, 1, \dots, k$ from the second line of (36). The third line of (36) implies that $q(k,0) = [(3k+1) + (k-1)(4k+1)]/(4k^2) = 1$ and that $q(k,l) - q(k,l+1) = (4k-2k)/(4k^2) = 1/(2k)$, $l = 0, 1, \dots, k-1$. This proves Condition (31).

Finally, for Condition (25), we need to show that $q(j,l) + q(l,j) \leq 1$. The case of $j = k$ is shown in the previous paragraph, and when $j = l \leq k-1$ this follows from the fact that $q(j,j) = j(2k+1)/(4k^2) < 1/2$ from the third line of (36). So suppose $l < j \leq k-1$. We have $q(j,l) - q(j,l-1) = -2k/(4k^2)$ and $q(l,j) - q(l-1,j) = (2k+1)/(4k^2)$ by the third line of (36). This implies that $q(j,l) + q(l,j) > q(j,l-1) + q(l-1,j)$ for every $l < j$. Thus, it suffices to show that $q(j-1,j) + q(j,j-1) \leq 1$ for every $j = 1, 2, \dots, k-1$. Since $q(j-1,j) = q(j,j) - (2k+1)/(4k^2)$ and $q(j,j-1) = q(j,j) - (2k+1)/(4k^2) + (3k+1)/(4k^2)$, we have $q(j-1,j) + q(j,j-1) < 2q(j,j) < 1$.

□

Proof of Lemma 3. We first prove the existence part of the lemma. There exists

$$\{P(j,l) : 1 \leq l \leq k-1, 1 \leq j \leq l+1\}$$

that satisfies the first line of (29), for any given values of

$$\{P(j,l) : 1 \leq j \leq k, 0 \leq l \leq k-1\} \setminus \{P(j,l) : 1 \leq l \leq k-1, 1 \leq j \leq l+1\}.$$

For the second line of (29), we can rewrite it as $\mathrm{Rev}(v,(j-1,k)) = \mathrm{Rev}(v,(j,k))$, i.e.,

$$P(j+1,k) - P(j,k) = (1+1/a)(P(j,k) - P(j-1,k)) + (P(k,j) - P(k,j-1))/a, \quad (54)$$

for $j = 1, 2, \ldots, k-1$. Equation (54) implies that

$$P(j+1,k) - P(j,k) = (1+1/a)^j P(1,k) + \sum_{j'=1}^{j} (1+1/a)^{j-j'} (P(k,j') - P(k,j'-1))/a, \quad (55)$$

and as a consequence, for any $j = 0, 1, \ldots, k$:

$$P(j,k) = a((1+1/a)^j - 1)P(1,k) + \sum_{j'=1}^{j-1} ((1+1/a)^{j-j'} - 1)(P(k,j') - P(k,j'-1)). \quad (56)$$

Clearly, the above equation can be satisfied for any given values of $P(1,k)$ and $P(k,j)$, $j = 1, 2, \ldots, k-1$. This proves the first part of the lemma.

Now fix an arbitrary P that satisfies Condition (29).

We claim that

$$\begin{aligned} X(l) &\equiv \sum_{j=1}^{l-1} (1+1/a)^{l-j} (P(l,j) - P(l,j-1)) \\ &= P(l,l-1) + a((1+1/a)^l - 1)^2 P(1,0) - (1+1/a)^l P(l,0), \end{aligned} \quad (57)$$

for every $l = 1, 2, \ldots, k$. Equation (57) for $l = k$ and Equation (56) together imply Equation (32), which proves the second part of the lemma.

Clearly, (57) is true for $l = 1$. Suppose (57) is true for $l = \kappa < k$ as an induction hypothesis; we prove that this implies (57) is true for $l = \kappa + 1$.

From $\mathrm{Rev}(v,(\kappa,j-1)) = \mathrm{Rev}(v,(\kappa,j))$ we have:

$$\begin{aligned} &P(\kappa+1,j) - P(\kappa+1,j-1) \\ = &(1+1/a)(P(\kappa,j) - P(\kappa,j-1)) + (1+1/a)(P(j,\kappa) - P(j-1,\kappa)) \\ &- (P(j+1,\kappa) - P(j,\kappa)), \end{aligned} \quad (58)$$

summing the above equation across $j = 1, 2, \ldots, \kappa - 1$ gives:

$$\begin{aligned}
&\sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+1-j}(P(\kappa+1,j)-P(\kappa+1,j-1)) \qquad (59)\\
=&\sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+2-j}(P(\kappa,j)-P(\kappa,j-1))+\sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+2-j}(P(j,\kappa)-P(j-1,\kappa))\\
&-\sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+1-j}(P(j+1,\kappa)-P(j,\kappa))\\
=&\sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+2-j}(P(\kappa,j)-P(\kappa,j-1))+(1+1/a)^{\kappa+1}P(1,\kappa)-(1+1/a)^2(P(\kappa,\kappa)-P(\kappa-1,\kappa)).
\end{aligned}$$

That is,

$$\begin{aligned}
&X(\kappa+1) \qquad (60)\\
=&\sum_{j=1}^{\kappa-1}(1+1/a)^{\kappa+2-j}(P(\kappa,j)-P(\kappa,j-1))+(1+1/a)^{\kappa+1}P(1,\kappa)\\
&-(1+1/a)^2(P(\kappa,\kappa)-P(\kappa-1,\kappa))+(1+1/a)(P(\kappa+1,\kappa)-P(\kappa+1,\kappa-1))\\
=&(1+1/a)^2[P(\kappa,\kappa-1)+a((1+1/a)^\kappa-1)^2P(1,0)-(1+1/a)^\kappa P(\kappa,0)]+(1+1/a)^{\kappa+1}P(1,\kappa)\\
&-(1+1/a)^2(P(\kappa,\kappa)-P(\kappa-1,\kappa))+(1+1/a)(P(\kappa+1,\kappa)-P(\kappa+1,\kappa-1)),
\end{aligned}$$

where in the last equality we have used the induction hypothesis (57) for $l = \kappa$.

From $\mathrm{Rev}(v,(\kappa,0)) = \mathrm{Rev}(v,(1,0))$ we have $(1+1/a)P(\kappa,0) - P(1,\kappa) = P(\kappa+1,0) - 2P(1,0)$. Therefore, the previous equation is equivalent to:

$$\begin{aligned}
&X(\kappa+1) \qquad (61)\\
=&(1+1/a)^2P(\kappa,\kappa-1)+a(1+1/a)^2((1+1/a)^\kappa-1)^2P(1,0)-(1+1/a)^{\kappa+1}(P(\kappa+1,0)-2P(1,0))\\
&-(1+1/a)^2(P(\kappa,\kappa)-P(\kappa-1,\kappa))+(1+1/a)(P(\kappa+1,\kappa)-P(\kappa+1,\kappa-1))\\
=&(1+1/a)^2P(\kappa,\kappa-1)+[a(1+1/a)^2((1+1/a)^\kappa-1)^2+2(1+1/a)^{\kappa+1}]P(1,0)-(1+1/a)^{\kappa+1}P(\kappa+1,0)\\
&-(1+1/a)^2(P(\kappa,\kappa)-P(\kappa-1,\kappa))+(1+1/a)(P(\kappa+1,\kappa)-P(\kappa+1,\kappa-1))
\end{aligned}$$

From $\mathrm{Rev}(v,(\kappa,\kappa)) = \mathrm{Rev}(v,(1,0))$ we have $(1+1/a)P(\kappa,\kappa) - P(\kappa+1,\kappa) = -P(1,0)$.

Therefore, the previous equation is equivalent to:

$$\begin{aligned}
&X(\kappa+1) \qquad (62)\\
=&(1+1/a)^2P(\kappa,\kappa-1)+[a(1+1/a)^2((1+1/a)^\kappa-1)^2+2(1+1/a)^{\kappa+1}+(1+1/a)]P(1,0)\\
&-(1+1/a)^{\kappa+1}P(\kappa+1,0)+(1+1/a)^2P(\kappa-1,\kappa)-(1+1/a)P(\kappa+1,\kappa-1).
\end{aligned}$$

From $\mathrm{Rev}(v,(\kappa-1,\kappa))=\mathrm{Rev}(v,(1,0))$ we have $(1+1/a)P(\kappa,\kappa-1)+(1+1/a)P(\kappa-1,\kappa)-P(\kappa+1,\kappa-1)=P(\kappa,\kappa)-2P(1,0)$, Therefore, the previous equation is equivalent to:

$$\begin{aligned}
&X(\kappa+1) \qquad (63)\\
=&[a(1+1/a)^2((1+1/a)^\kappa-1)^2+2(1+1/a)^{\kappa+1}-(1+1/a)]P(1,0)\\
&-(1+1/a)^{\kappa+1}P(\kappa+1,0)+(1+1/a)P(\kappa,\kappa).
\end{aligned}$$

Finally, using $(1+1/a)P(\kappa,\kappa)-P(\kappa+1,\kappa)=-P(1,0)$ again we get:

$$\begin{aligned}
&X(\kappa+1) \qquad (64)\\
=&[a(1+1/a)^2((1+1/a)^\kappa-1)^2+2(1+1/a)^{\kappa+1}-(1+1/a)-1]P(1,0)\\
&-(1+1/a)^{\kappa+1}P(\kappa+1,0)+P(\kappa+1,\kappa).
\end{aligned}$$

Since $a(1+1/a)^2((1+1/a)^\kappa-1)^2+2(1+1/a)^{\kappa+1}-(1+1/a)-1=a((1+1/a)^{\kappa+1}-1)^2$, this proves (57) when $l=\kappa+1$. □

Proof of Corollary 1. Suppose p is the uniform $[0,1]$ distribution. We want to maximize:

$$\Pi=\int\min\left(\frac{3k+1}{2k^2}av-2X,\ \frac{av}{2k}-Y,\ 2\left((1+1/a)^k-1\right)^2X+2((1+1/a)^k-1)Y\right)dv. \qquad (65)$$

Fixing a and k, it is easy to see that $\frac{\partial\Pi}{\partial X}=\frac{\partial\Pi}{\partial Y}=0$ have a unique solution in (X,Y); substituting such (X,Y) into (65) gives:

$$\Pi^*(a,k)=\frac{a((1+1/a)^k-1)^2\left(1+3k-2(1+1/a)^k(1+k)+(1+1/a)^{2k}(1+3k)\right)}{4k^2(1+1/a)^{4k}}. \qquad (66)$$

We numerically verify that $\max_a\Pi^*(a,k)$ is strictly increasing in k. Let us take $k\to\infty$ and choose a sequence of $\{a(k)\}$ such that $A\equiv\lim_{k\to\infty}a(k)/k$ is a well-defined limit (potentially infinity). Applying Taylor's theorem, we have

$$\lim_{k\to\infty} k\log(1+1/a(k)) = k/a(k) = A,$$

thus

$$\lim_{k\to\infty} \Pi^*(a(k),k) = \frac{A(\exp(1/A)-1)^2(3-2\exp(1/A)+3\exp(2/A))}{4\exp(4/A)} \tag{67}$$

The above function is maximized at $A = 0.7367$, yielding $\Pi^* = 0.272651$. Finally, we substitute $A = 0.7367$ back to (X, Y) that solves $\frac{\partial \Pi}{\partial X} = \frac{\partial \Pi}{\partial Y} = 0$.

□

Proof of Proposition 4. As $k \to \infty$, $a/k \to A$ and $kP(1) \to \rho$,

$$\int_0^1 \min(av/k, aP(1)(1+1/a)^k)\, p(dv) - aP(1) \to \int_0^1 \min(Av, \rho A\exp(1/A))\, p(dv) - \rho A = \Pi. \tag{68}$$

We maximize Π over A and ρ; the first order condition is:

$$\begin{aligned} \frac{\partial \Pi}{\partial \rho} &= \int_{\rho\exp(1/A)}^1 A\exp(1/A)\, p(dv) - A = 0, \qquad (69)\\ \frac{\partial \Pi}{\partial A} &= \int_0^{\rho\exp(1/A)} v\, p(dv) + \int_{\rho\exp(1/A)}^1 \rho(\exp(1/A) - \exp(1/A)/A)\, p(dv) - \rho = 0, \end{aligned}$$

which is equivalent to (49). Moreover, we have $\Pi = \rho$ under (69).

We now relate Roesler and Szentes (2016) to Equation (69), which simultaneously proves the existence of ρ and A that satisfy (69) as well as their optimality in solving (16); see the discussion about Roesler and Szentes (2016) following Proposition 4. Let $\mu \equiv \int_0^1 v\, p(dv)$, the requirement that G_π^B has the same mean as p is equivalent to:

$$\mu = \int_0^1 s\, dG_\pi^B(s) = B - \int_\pi^B G_\pi^B(s)\, dv = \pi + \int_\pi^B \pi/s\, ds = \pi + \pi\log B - \pi\log\pi, \tag{70}$$

i.e.,

$$\log(B) = \frac{\mu - \pi + \pi\log\pi}{\pi}. \tag{71}$$

Clearly, $B \geq \pi$ if and only if $\pi \leq \mu$.

The requirement that G_π^B second-order stochastic dominates p is equivalent to:

$$\int_\pi^x (1 - \pi/s)\, ds = x - \pi - \pi\log x + \pi\log\pi \leq \int_0^x p(v \leq s)\, ds, \tag{72}$$

for every $x \in [\pi, B]$.

Define

$$f(x,\pi) \equiv \int_0^x p(v \le s)\, ds - (x - \pi - \pi \log x + \pi \log \pi). \tag{73}$$

Let π^* be the minimum π such that $\min_{x\in[\pi,B]} f(x,\pi) \ge 0$, and let B^* be the corresponding B from (71). Let x^* be an arbitrary selection from $\operatorname{argmin}_{x\in[\pi^*,B^*]} f(x,\pi^*)$.

For $\pi \le \mu$, B (defined by (71)) decreases as π increases. Moreover, $\frac{\partial f}{\partial \pi}(x,\pi) > 0$ for every $x > \pi$. Thus, $\min_{x\in[\pi,B]} f(x,\pi)$ is a continuously increasing function of π. So we must have $f(x^*,\pi^*) = 0$.

If $B^* < 1$, we must have $f(B^*,\pi^*) > 0$, for otherwise we would have $\int_0^1 G_{\pi^*}^{B^*}(s)\, ds > \int_0^1 p(v \le s)\, ds$, which would contradict the fact that $G_{\pi^*}^{B^*}$ has the same mean as p. If $B^* = 1$, then we have $\frac{\partial f}{\partial x}(B^*,\pi^*) > 0$. In any case $B^* \ne x^*$. Since $f(\pi^*,\pi^*) > 0$, we also have $\pi^* \ne x^*$. Thus we must have $\frac{\partial f}{\partial x}(x^*,\pi^*) = 0$.

Therefore, we have (the first line is $\frac{\partial f}{\partial x}(x^*,\pi^*) = 0$, and the second line is $f(x^*,\pi^*) = 0$):

$$\begin{aligned}
&p(v \le x^*) - 1 + \pi^*/x^* = 0, \qquad (74)\\
&\int_0^{x^*} p(v \le s)\, ds - (x^* - \pi^* - \pi^* \log x^* + \pi^* \log \pi^*) = -\int_0^{x^*} v\, p(dv) + \pi^* \log x^* - \pi^* \log \pi^* = 0,
\end{aligned}$$

where in the second equality of the second line we use integration by parts and substitute in the first line. Clearly, the above equations are (49) with $x^* = \rho \exp(1/A)$ and $\pi^* = \rho$.

□

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