Robust Mechanism Design of Exchange

Pavel Andreyanov and Tomasz Sadzik, UCLA

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Abstract

We provide a robust (or detail-free) strategic foundation for the Walrasian Equilibrium: a mechanism for an exchange economy with asymmetric information and interdependent values that is ex-post individually rational, incentive compatible, generates budget surplus and is ex-post nearly Pareto Efficient, when there are many agents. The level of inefficiency is proportional to the impact a single agent has on the Walrasian price. Conversely, we show that mechanisms generating smaller efficiency losses must violate some of the constraints, and so our efficiency bound is tight. The tight robust asymptotic efficiency is achieved by $\sigma-$*Walrasian Equilibrium* mechanisms, in which the allocation is as if each agent traded knowing all the information distributed in the economy, faced with the price that increases in the quantity traded with slope σ.

Preliminary and Incomplete

1 Introduction

One way to achieve an efficient allocation in an exchange economy, when all the parameters are publicly known, is to set the right prices. The problem becomes much harder when we introduce asymmetric information, and so agents have private information about relevant parameters of the economy. While asymmetric information inhibits efficiency, there is a sizeable literature on designing mechanisms for exchange that achieve near efficiency when there are many agents and each is *informationally small*: has only little publicly relevant information that others do not.(see McLean and Postlewaite [2002], [2004] and [2015])

Besides the cases when agents have information that is relevant only for themselves ("private values"), however, the literature has focused mostly on the problem of Bayesian implementation (see the literature review section below). As is well known, this approach is predicated on pretty strong assumptions. Mechanism designer must know the distribution of agents' types, beliefs that the agents have about each other, beliefs about beliefs etc. The mechanism Bayes implementing near efficient outcome depends on the fine details of such knowledge (is not detail-free) and fails if the designer is wrong (is not robust). A more demanding task is to find a mechanism for exchange that ex-post, or pointwise, satisfies all the constraints and is near efficient. This single mechanism for exchange would "work" for any distribution of types and beliefs of the agents. As a bonus, such a mechanism, blind to the details of the informational environment, would have to share some of the simplicity of the original symmetric information solution. The goal of this paper is to provide such a mechanism. More precisely, we design a mechanism for an exchange economy with asymmetric information and interdependent values that is ex-post (for any realization of other agents' signals) individually rational, incentive compatible, generates budget surplus and is ex post nearly Pareto Efficient when there are many agents.

Let us rephrase the motivation using a concrete example. Suppose that the economy has a large number of "English" and "American" and one stock to be traded. Each agent gets a noisy signal of the per-unit value of the stock, and this value depends also on the signals of his compatriots (see Example 2). In the symmetric information setting, simply setting the right Walrasian Equilibrium (WE) price, which depends on all the signals, will solve the problem. How to design a trade mechanism when the signals are privately known? One mechanism would be a double-auction: each player submits a demand schedule specifying how much he wants to buy (or sell) at any given price, and the designer chooses the price that will clear the market. While the double-auction game mirrors the features of the real-life trading, it is not a good choice as a mechanism, as it will not lead to near efficient allocation no matter how large the economy. This is because a single price cannot aggregate all the relevant public information, the "average" signal of the Americans and the "average" signal of the English (see Rostek and Weretka [2012]). An alternative way, if the designer presumes he knows the commonly known distributions from which the signals are drawn, would be to elicit information of the agents via appropriate scoring rules ("reward a good guess about the compatriots' signal"), much in the spirit of Cremer and McLean [1985] and [1988], and then use this information to, say, set the Walrasian Price. If the designer does not know the distribution, this approach will not work. One could also try to use some version of the *Vickrey-Clark-Groves* mechanisms, which are distribution free and based on transfers that make agents internalize ex-post the externalities they impose on the others. But such mechanisms will typically run a budget deficit (e.g., McLean and Postlewaite [2015]). What else can be done?

More precisely, in the paper we are looking at an environment with N agents with quasilinear utilities over a single divisible good ("stock") and money. Each player has a one dimensional signal, and his utility depend on the whole profile of signals ("interdependent values"). We make no assumptions, in fact we do not even specify the belief hierarchies of the agents, or the mechanism designer. Regarding the utility functions, beside the rather standard assumptions of differentiability and bounded derivatives of marginal utilities (with respect to signals and quantity), as well as the single crossing property, which means that my utility responds most to my signal (e.g., Dasgupta and Maskin [2000]), we also assume that the agents are *informationally small*. This last assumption means that the effect of any agent's signal on the marginal utility of anyone else converges to zero as the number of agents grow. It is the analog of the informational smallness assumption from the literature on Bayesian strategic foundations of WE (McLean and Postlewaite [2002], [2004] and [2015]) in our distribution-free setting. Intuitively, in our asymmetric information setting, for the environment to be truly competitive there should be many competing agents and each should have a vanishing amount of publicly relevant information that others do not.

A trading mechanism in this environment consists of an allocation and transfer rule. We consider the mechanisms that satisfy ex-post individual rationality (IR), incentive compatibility (IC), generate budget surplus (BS) and clear the market (MC). Ex-post IR and IC here mean that the constraints have to be satisfied for any profile of signals of the other agents. They imply Bayesian IR and IC in any informational setting. A different implication is that such mechanism will work in fairly weak contractual situations, in which each agent can walk away from the trasaction even after seeing the suggested allocation and transfers, or, simply, faces a menu of allocation-tranfer pairs from which he is free to choose or not (Chung and Ely [2002]), as in a spot market. Ex-post BS and MC are standard feasibility constraints, where we assume that the surplus of money can be freely disposed of (donated, burned). The objective is to achieve ex-post $\varepsilon-$Pareto Efficiency, for small $\varepsilon > 0$. It means that for any signal profile there is a fully Pareto Efficient allocation - Walrasian Equilibrium allocation in our setting - in which each agent gets at most ε greater utility.

The main result of the paper is a construction of mechanisms that are robustly asymptotically $(f(N) + \frac{1}{N})$-efficient: for sufficiently large economy N they satisfy all the ex-post constraints and are $O(f(N)+\frac{1}{N})-$Pareto Efficient (Proposition 2). $f(N)$ is the bound on the effect of one agent's signal on the marginal utility of others, and so a measure of informational smallness; $f(N) \to 0$ as $N \to \infty$. Conversely, we show that a quicker convergence to Pareto Efficiency is not possible without violating the constraints (Proposition 3). In other words, our positive result is tight.

In the paper we show that the goal is achieved by relatively simple $\sigma-$ *Walrasian Equilibrium* mechanisms, for $\sigma > 0$ small. In those mechanisms, for every signal profile there is a "reference price", mirroring the role of the Walrasian Equilibrium price. Each agent's allocation is simply what he would purchase (or sell) if he knew all the signals and faced an upward sloping price schedule, with reference price as the intercept and price increasing in the quantity with slope σ. We show how to construct the transfers as the integrals of variable per-unit prices, which depend on the details of the utility functions, so that the ex-post IR and IC are satisfied. Slopes of the order $f(N)+\frac{1}{N}$ distort the allocation only slightly and result in $O(f(N)+\frac{1}{N})-$Pareto Efficiency. Crucially, we show that an appropriate choice of slopes of the same order also results in BS. On the other hand, we show that in the case when each agent would have an impact on the Walrasian Equilibrium price of order $f(N)+\frac{1}{N}$, an IR and IC mechanism that results in an $o(f(N)+\frac{1}{N})$-Pareto Efficient allocation must run a budget deficit.

Literature Review. The question of strategic foundations of Walrasian Equilibrium is of course not new. One strand of literature focused on the question of when the specific mechanism of double auction guarantees asymptotic efficiency. Important contributions include Gresik and Satterthwaite [1989], Satterthwaite and Williams [1989], Rustichini et al. [1994], Fudenberg et al. [2007] and Cripps and Swinkels [2006] for the private value case, and Reny and Perry [2006] for the case of interdependent values. Going beyond the double auctions, Gul and Postlewaite [1992], McLean and Postlewaite [2002] and [2004] construct asymptotically efficient mechanism in broader informational settings, drawing from the insights of Cremer and McLean [1985] and [1988]. In our case we require a stronger requirement of ex-post incentive compatibility and individual rationality. In particular, unlike double auctions, the mechanism neither requires any assumptions on the beliefs of the agents/distribution of signals, such as conditional independence, nor the symmetry in the payoffs (see Rostek and Weretka [2012]). On the other hand, the detail-free nature of the mechanism prevents the use of distribution-dependent scoring rules to extract private information.

A different approach has been to analyze the limits of manipulability of the mechanisms when there are many agents. Roberts and Postlewaite [1976], Jackson [1992] and

Jackson and Manelli [1997] have analyzed the limits of manipulability of the Walrasian Equilibrium allocation under certain conditions in the symmetric information setting. McLean and Postlewaite [2015] have shown that under information-smallness the benefits of ex-post deviations vanish when the agents are informationally small. We work with asymmetric information economies and require full ex-post incentive compatibility. There is also a large literature on the limits of manipulability with many agents in matching.

In the private value setting, the Vickrey-Clark-Groves (VCG) mechanism is fully efficient, satisfies the ex-post incentive compatibility and individual rationality but typically runs a budget deficit. However, in the auction context, when the seller's value of the good is commonly known, the VCG mechanism satisfies all the constraints. In the private value case the mechanism is the celebrated Vickrey (second price) auction. Dasgupta and Maskin [2000], Perry and Reny [2002] and Ausubel [2004] have shown how to extend the VCG mechanism to the interdependent case (under the assumptions of affiliation and symmetry the mechanism is simply the English auction).

In the case of an exchange when both sellers and buyers have private information, the question of robust asymptotic efficiency has been established only in the case of private values and unit demands and supplies, by Wilson [1985] and McAfee [1992]. In an ongoing work, Loertscher and Mezzetti [2013] consider an extension in which they allow for multiunit demands, while preserving private values. Kojima and Yamashita [2014] provide a mechanism that satisfies the ex-post constraints and creates a small expected Pareto loss, under the assumptions that the types of the agents are independently distributed and agents are symmetric. In our case the almost Pareto Efficiency is satisfied robustly as well, and in particular without independence or symmetry assumptions.

2 Model

The exchange economy consists of N agents and a single good (an asset) that can be traded among them. Each agent i observes a signal $s_i \in [0,1]$, and his utility of consumption depends on the whole profile of signals $\mathbf{s} = (s_1, ..., s_N) \in [0,1]^N =: S$. More precisely, we will assume that agent i has a utility function

$$U_i(q_i, t_i, \mathbf{s}) = u_i(q_i, \mathbf{s}) - t_i,$$

where $q_i \in \mathbb{R}$ is the quantity of the good the agent ends up with and t_i is the amount of money he pays. The function u_i is continuously differentiable in all the arguments and strictly concave in q_i and we normalize $u_i(0, \mathbf{s}) = 0$. Note that the utilities are indexed by the agents, and so we are not making any symmetry assumptions. We are also not specifying at all the beliefs of the agents, and so the setting is entirely distribution-free.

We make the following additional assumptions on the utility function:

A1) $mu_i(0, \mathbf{s}) \in [-M, M]$, $\forall i \forall \mathbf{s}$;

A2) $\frac{\partial mu_i(q_i,\mathbf{s})}{\partial s_i} \in [c_s, C_s]$, $C_s > c_s > 0$, $\forall i \forall q_i \forall \mathbf{s}$;

A3) $\left|\frac{\partial mu_i(q_i,\mathbf{s})}{\partial s_j}\right| \leq \phi_N$, $\phi_N < c_s$, $\phi_N \to 0$ as $N \to \infty$, $\forall i \neq j \forall q_i \forall \mathbf{s}$;

A4) $\frac{\partial mu_i(q_i,\mathbf{s})}{\partial q_i} \in [c_q, C_q]$, $c_q < C_q < 0$, $\forall i \forall q_i \forall \mathbf{s}$;

First, all the assumptions impose uniform bounds on the utility functions. Inuitively, this is necessary given our aim of providing uniform bounds on the convergence to the efficient allocation, or bounding the maximum efficiency loss across all the agents and signal realizations. The second assumption also normalizes the marginal (per-unit) utility to be increasing in own signals.

The third assumption captures the notion of *information smallness* crucial for our results: as the number of agents in the economy grows, the impact of each single agent on the utility of any other agent vanishes. It captures the intuition that as the number of agents grow, no single agent can have big informational impact on the economy.

The notion is very closely related to the version of informational smallness used in the Bayesian setting (McLean and Postlewaite [2002], [2004] and [2015]), as the following interpretation of our model illustrates. Imagine that each agent i has a private state space Θ_i and a continuously differentiable utility function $v_i(q_i, \theta_i, s_i) - t_i$ that depends on the transfers t_i, quantity q_i, own type s_i and the state of the world θ_i and satisfies assumptions A1, A2 and A4 and $\frac{\partial mu_j}{\partial \theta_i}$ bounded. Suppose also that i has a private prior $\delta_i \in \Delta(\Theta_i \times S_1 \times ... \times S_N)$ and that the agents are informationally small in the sense that[1]

$$\left|\frac{\partial \delta_i(\theta_i|\mathbf{s})}{\partial s_j}\right| \leq \phi'_N, \ \forall i \neq j \forall \theta_i \forall \mathbf{s}$$

for a function ϕ'_N such that $\phi'_N \to 0$ as $N \to 0$. In other words, any single agent j has a small impact on $i's$ beliefs about the payoff relevant state to him. It is easy to see that

[1]The definition used by Mclean and Postleweite has a common finite state space Θ and common prior δ. On the other hand, their requirement is weaker in that the bound need not hold for every $\mathbf{s}_{-j}$ but only for $\mathbf{s}_{-j}$ with probability close to one. The strengthening that we require is dictated by the stronger, ex-post version of the incentive compatibility that we use (and so the lack of prior distribution).

this model reduces to ours with

$$u_i(q_i, \mathbf{s}) = \mathbb{E}_{\delta_i}[v_i(q_i, \theta_i, s_i)|\mathbf{s}].$$

Example 1 *("Fundamental Value Model", Vives [2011], Rostek and Weretka [2012]) Suppose that the utilities are*

$$U_i(q_i, t_i, \mathbf{s}) = (\alpha s_i + \beta \overline{s})q_i - \frac{\mu}{2}q_i^2 - t_i,$$

for some constants $\alpha, \beta, \mu > 0$, where $\overline{s}$ is the arithmetic average of the signals. One can think of $(\alpha s_i + \beta \overline{s})$ as the expected per-unit value of the stock to trader i, and $\frac{\mu}{2}q_i^2$ as the cost of risk involved in holding q_i units of the asset. Indeed, it can be shown that the quadratic utility above is the certainty equivalent of holding q_i units of a risky asset, when agent i has CARA utility function, he believes that all the per-unit values θ_j are Normally distributed with a constant correlation between any θ_i and θ_j, $j \neq i$, and $s_j = \theta_j + \varepsilon_j$ for iid Normally distributed noises ε_j. Additionally, we do not need to require any symmetry between the agents, in which case the constants α, β and μ are indexed by i.

In the example the assumption A3 of information smallness is satisfied, since each signal s_j enters the utility of player $i \neq j$ with coefficient $\frac{\beta}{N-1}$. More generally, the assumption of information smallness will be satisfied in the conditional iid setting (Reny Perry '06), when each agent i believes that agents $j \neq i$ observe conditionally iid signals of his value θ_i, under the appropriate regularity condition on the prior distribution of θ_i and conditional distributions of the signals.[2]

Example 2 *("Group Model"', Rostek and Weretka [2012]) A slightly more complicated example will have the agents divided into, say, two groups (American and English), and for each agent i in group A the utility is*

$$U_i^A(q_i, t_i, \mathbf{s}) = (\alpha s_i + \beta_h \overline{s}^A + \beta_l \overline{s}^E)q_i - \frac{\mu}{2}q_i^2 - t_i,$$

and similarly for the members of group E, where $\overline{s}^A$ and $\overline{s}^E$ are the average signals in each group and $\beta_h > \beta_l > 0$. Intuitively, compared to Example 1 now the agents think that the per-unit values θ_j are more correlated among the compatriots.

While the mechanism of a double auction will guranatee almost efficiency in Example 1, this will not be true for Example 2 (Rostek and Weretka [2012]). Intuitively, the one

[2]Find a reference or crank through the Bayes formula to uncover the sufficient regularity conditions.

dimensional price cannot aggregate the relevant public information, the average signal of the Americans and the average signal of the English, which is two dimensional. When we send the numbers of agents in each group to infinity, each of them will act almost as a price-taker, and the allocation will converge to that in a non-fully revealing Rational Expectations Equilibrium, which is not approximately Pareto Efficient.

A *mechanism* in our setting is $\{(q_i(\mathbf{s}), t_i(\mathbf{s}))\}_{i\leq N, s\in S}$, where $\{q_i(\mathbf{s})\}_{i\leq N, s\in S}$ is the *allocation* and $\{t_i(\mathbf{s})\}_{i\leq N, s\in S}$ the *transfers* profile. Let us list below the properties that we might want a mechanism to satisfy. All of the properties below need to hold "ex-post", and so for every realization of the signals, rather than only "in expectation", for given beliefs of the agents:

- Market Clearing.
$$\sum_i q_i(\mathbf{s}) = 0, \ \forall \mathbf{s}.$$
- Budget Surplus.
$$\sum_i t_i(\mathbf{s}) \geq 0, \ \forall \mathbf{s}.$$

Note that the last is weaker than the "Budget Balance". We are assuming free disposal of money in our economy: surplus money can be freely burned (or donated).

- Individual Rationality
$$u_i(q_i(\mathbf{s}), t_i(\mathbf{s}), \mathbf{s}) \geq u_i(0, 0, \mathbf{s}), \ \forall i \forall \mathbf{s}.$$
- Incentive Compatibility
$$u_i(q_i(\mathbf{s}), t_i(\mathbf{s}), \mathbf{s}) \geq u_i(q_i(s_i', \mathbf{s}_{-i}), t_i(s_i', \mathbf{s}_{-i}), \mathbf{s}), \ \forall i \forall \mathbf{s}, s_i'.$$
- ε-Pareto Efficiency. For every $\mathbf{s}$ there is a Pareto Efficient profile $(q_1, ..., q_N, t_1, ..., t_N)$ such that
$$u_i(q_i(\mathbf{s}), \mathbf{s}) - t_i(\mathbf{s}) \geq u_i(q_i, \mathbf{s}) - t_i - \varepsilon, \ \forall i$$

IR says that even if agent i learns the signal realizations of everybody else, he still prefers to participate in the mechanism (assuming everybody else do es and is truthful). IC says that even if agent i learns the signal realizations of everybody else, he still finds it optimal to report his type truthfully (assuming everybody else does and is truthful).

The "ex-post" nature of the conditions is a strong requirement. The first and most important interpretation is that it implies the more standard "Bayesian" version of the constraints, for any beliefs that the agents might have about the signals of the opponents, beliefs about beliefs etc. In this sense a mechanism that satisfies those "ex-post" constraints is *robust* to the misspecification of the belief hierarchy. A different way to put it is that it is *detail-free*, as it does not depend on the fine details of the beliefs of the agents, which, realistically, are not known by the mechanism designer.

Second interpretation is that the "ex-post" nature of the constraints means that the mechanism will work under weak contractual conditions. In particular, they guarantee that the mechanism will work even if the agents cannot be prevented from "walking away from the table", or changing their mind at any stage of the transaction, even after they learn the allocation (compare to spot markets).

Likewise, ε-PE requires that the allocation is "almost" Pareto Efficient for every realization of types. It is an almost state- and agent- wise efficiency, or simply *pointwise* efficiency. It is stronger than, for example, the asymptotic average efficiency, when the average - over states or agents - distance to the optimal allocation converges to zero. This efficiency concept is immune to changing Pareto weights on the agents, according to a social preference, or weights on the states of the world, according to designer's beliefs, or his beliefs about the beliefs of the agents etc.

A particular Pareto Efficient allocation, for any $\mathbf{s}$, in our setting is the *competitive* or *Walrasian Equilibrium* allocation, $(q_1^*(\mathbf{s}), ..., q_N^*(\mathbf{s}))$, which is defined implicitly by

$$
\begin{gathered}
mu_i(q_i^*(\mathbf{s}), \mathbf{s}) = p^*(\mathbf{s}), \ \forall i \\
\sum_i q_i^*(\mathbf{s}) = 0,
\end{gathered}
$$

where $p^*(\mathbf{s})$ is the Walrasian price. The question is, of course, how to implement this allocation. Had the information been symmetric and the realized state $\mathbf{s}$ known by the mechanism designer - in other words, if we ignore the IC constraints - this would be an easy task: letting agents buy or sell as much as they want at the Walrasian price would do. The resulting allocation and the transfers would satisfy all the other constraints besides IC.

It is relatively easy to show (see Myerson and Satterthwaite [1983]) that for a fixed number of players there is typically no mechanism that implements an $\varepsilon - PE$ allocation, for small ε, and which satisfies all the other constraints (MC, BS, IR, IC). The purpose of this paper is to show that there is a class of mechanisms, which satisfies all the constraints and is almost efficient when N is sufficiently large.

More precisely, for a function $f : \mathbb{N} \to \mathbb{R}_+$, $f_N \to 0$ as $N \to 0$, we say that a family of mechanisms $\{(q_i(\mathbf{s}), t_i(\mathbf{s}))\}_{i \leq N, s \in S}$ parametrized by N is *robustly asymptotically f-efficient* if for N sufficiently large each mechanism satisfies all the constraints MC, BS, IR, IC and is $O(f_N)$-Pareto Efficient.

Fix a number $\sigma \geq 0$. We call an allocation $\{q_i(\mathbf{s})\}_{i \leq N, \mathbf{s} \in S}$ *$\sigma-$ Walrasian Equilibrium* $(\sigma - WE)$ if for every profile $\mathbf{s}$ there exists a *reference price* $p^\sigma(\mathbf{s})$ such that

$$
\begin{aligned}
mu(q_i(\mathbf{s}), \mathbf{s}) &= p^\sigma(\mathbf{s}) + \sigma \times q_i(\mathbf{s}), \ \forall i \\
\sum_i q_i(\mathbf{s}) &= 0.
\end{aligned}
\tag{1}
$$

In other words, for any signal profile $\mathbf{s}$ the allocation is as if each agent chose freely when faced with a posted price schedule, in with the price for the first (infinitesimal) unit was the reference price $p^\sigma(\mathbf{s})$ and the per-unit price was linearly increasing in the quantity purchased (decreasing in the quantity sold) with the slope σ. $0 - WE$ allocation agrees with the Walrasian Equilibrium allocation. We define a *$\sigma-$ WE mechanism* as one whose underlying allocation is $\sigma - WE$.

Lemma 1 *For any $\sigma \geq 0$ there exists exactly one $\sigma - WE$ allocation.*

The definition of a $\sigma-$WE mechanism pins down only the allocation and leaves the definition of the transfers open. The direct interpretation of the allocation in analogy with the Walrasian Equilibrium, as above, suggests transfers

$$
t_i(\mathbf{s}) = [p^\sigma(\mathbf{s}) + \sigma \times q_i(\mathbf{s})] \times q_i(\mathbf{s}). \tag{2}
$$

However, such transfers typically will *not* be (ex-post) IC. In a finite economy, faced with a mechanism that allocates WE quantities and charges WE prices each agent has incentive to misreport his signal and manipulate price in his favor.

Example 3 *Let us consider the fundamental value model from Example 1. For a fixed $\sigma \geq 0$ and any* $\mathbf{s}$ *the $\sigma - WE$ allocation is characterized by*

$$
\alpha s_i + \beta \overline{s} - \mu q_i = p^\sigma(\mathbf{s}) + \sigma q_i. \tag{3}
$$

Imposing market clearing conditions yields

$$p^{\sigma}(\mathbf{s}) = (\alpha+\beta)\overline{s},$$
$$q_i(\mathbf{s}) = \frac{\alpha}{\mu+\sigma}(s_i - \overline{s}).$$

On the other hand, for the agent i with signal s_i that knows the signal profile of the other agents $\mathbf{s}_{-i}$, faces the $\sigma - WE$ mechanism with transfers (2) maximization over the reported s_i' yields the local IC condition

$$\frac{\partial}{\partial s_i'}\left\{(\alpha s_i + \beta\overline{s})q_i(s_i', \mathbf{s}_{-i}) - \frac{\mu}{2}q_i^2(s_i', \mathbf{s}_{-i}) - \left(p^{\sigma}(s_i', \mathbf{s}_{-i}) + \sigma q_i(s_i', \mathbf{s}_{-i})\right) q_i(s_i', \mathbf{s}_{-i})\right\}\Big|_{s_i'=s_i} = 0,$$
$$\alpha s_i + \beta\overline{s} - \mu q_i(\mathbf{s}) - \frac{\partial p^{\sigma}(\mathbf{s})}{\partial s_i}q_i(\mathbf{s}) - 2\sigma q_i(\mathbf{s}) = 0,$$
$$q_i(\mathbf{s})\left(\frac{\partial p^{\sigma}(\mathbf{s})}{\partial s_i} + \sigma\right) = 0,$$

where the last line follows from (3). In other words, for any $\sigma \geq 0$ the $\sigma - WE$ mechanism with "naive" transfers (2) is not IC. Each buyer has an incentive to understate (seller to overstate) his signal, given the impact it has on the "price" he faces.[3]

Following the Vickrey logic (see Hölmstrom [1979]), there is a way to modify the transfers in a way that a $\sigma - WE$ mechanism achieves IC and IR. More precisely, the following Proposition shows that for any $\sigma - WE$ allocation the IC and IR constraints pin down the transfers.$t_i(\cdot, \mathbf{s}_{-i})$ up to a constant. The constant captures the rent of the boundary type of agent i.

Proposition 1 *For any $\sigma \geq 0$ the transfers in any IR and IC $\sigma - WE$ mechanism are characterized by*

$$t_i(\mathbf{s}) = t_i(\underline{s_i}, \mathbf{s}_{-i}) + \int_{q_i(\underline{s_i}, \mathbf{s}_{-i})}^{q_i(\mathbf{s})} \left[p^{\sigma}\left(s_i(x), \mathbf{s}_{-i}\right) + \sigma x\right] dx, \text{ if } q_i(\mathbf{s}) \geq 0, \tag{4}$$
$$t_i(\mathbf{s}) = t_i(\overline{s_i}, \mathbf{s}_{-i}) - \int_{q_i(\mathbf{s})}^{q_i(\overline{s_i}, \mathbf{s}_{-i})} \left[p^{\sigma}\left(s_i(x), \mathbf{s}_{-i}\right) + \sigma x\right] dx, \text{ if } q_i(\mathbf{s}) \leq 0,$$

[3] In this linear model IC would be achievable with "naive" transfers as in (2) with $\sigma < 0$, which would come at the expense of running a budget deficit.

where $s_i(x)$, $\underline{s_i}$ *and* $\overline{s_i}$ *are defined via*

$$\begin{aligned} x &= q_i\left(s_i\left(x\right), \mathbf{s}_{-i}\right), \\ \underline{s_i} &= \inf\{s_i' | q_i(s_i', \mathbf{s}_{-i}) > 0\}, \\ \overline{s_i} &= \sup\{s_i' | q_i(s_i', \mathbf{s}_{-i}) < 0\}, \end{aligned} \tag{5}$$

and the constants $t_i(\underline{s_i}, \mathbf{s}_{-i})$ *and* $t_i(\overline{s_i}, \mathbf{s}_{-i})$ *satisfy*

$$\begin{aligned} t_i(\underline{s_i}, \mathbf{s}_{-i}) &\leq u_i(q_i(\underline{s_i}, \mathbf{s}_{-i}), (\underline{s_i}, \mathbf{s}_{-i})), \\ t_i(\overline{s_i}, \mathbf{s}_{-i}) &\leq u_i(q_i(\overline{s_i}, \mathbf{s}_{-i}), (\overline{s_i}, \mathbf{s}_{-i})). \end{aligned} \tag{6}$$

The economic intuition behind the result is as follows. Fix a profile of signals $\mathbf{s}$ and an agent i that is a buyer, $q_i(\mathbf{s}) > 0$ In order to achieve IC the amount of money that i pays for every $x'th$ infinitesimal inframarginal unit of the good, $x < q_i(\mathbf{s})$, must equal his value for this unit had he reported the type that makes him pivotal. Given the definition of the $\sigma - WE$ allocation in (1), agent i is pivotal for $x'th$ unit of the good precisely when he reports the type s_i' such that $q_i(s_i', \mathbf{s}_{-i}) = x$, and his value for it equals $mu_i\left(x, \left(s_i\left(x\right), \mathbf{s}_{-i}\right)\right)$, which is $p^{\sigma}\left(s_i\left(x\right), \mathbf{s}_{-i}\right) + \sigma x$. Of course, in the case when the slope σ is zero and so we are incentivizing the WE allocation, this payment equals also the externality that the buyer imposes on others by getting this unit, $mu_j\left(q_j\left(s_i\left(x\right), \mathbf{s}_{-i}\right), \left(s_i\left(x\right), \mathbf{s}_{-i}\right)\right)$ for any j. Integrating over such per-unit payments gives rise to the transfer functions in (4).

The conditions (6) are just the IR constraints for the types $\underline{s_i}$ and $\overline{s_i}$. In the case when there is a type $\underline{s_i} \in (0, 1)$ that results in agent i not trading, $q_i(\underline{s_i}, \mathbf{s}_{-i}) = 0$, then $\underline{s_i} = \overline{s_i}$ and the conditions (6) boil down to $t_i(\underline{s_i}, \mathbf{s}_{-i}) \leq 0$. In the case when global IC holds, those IR for the types $\underline{s_i}$ and $\overline{s_i}$ extend to IR holding for all the types.

In order to establish full IC it is sufficient to establish the monotonicity of the allocation, i.e., show that $q_i(\cdot, \mathbf{s}_{-i})$ is weakly increasing for any i and $\mathbf{s}_{-i}$. To simplify notation let $\varphi_N = \phi_N + \frac{1}{N}$.

Lemma 2 *Fix number of agents* N, σ *and a* $\sigma - WE$ *allocation* $\{q_i(\mathbf{s})\}_{i \leq N, \mathbf{s} \in S}$. *There are* C_p, c_p, C_q *and* c_q*(independent of* σ *and* N*),* $C_{ps} > c_{ps} > 0$, $C_{qs} > c_{qs} > 0$ *such that for any profile of signals* $\mathbf{s}$ *and an agent* i *we have*

$$\frac{\partial p^{\sigma}(\mathbf{s})}{\partial s_i} = \frac{\sum_j \frac{\partial mu_j(q_j(\mathbf{s}), \mathbf{s})}{\partial s_i} \times (\sigma - \frac{\partial mu_j(q_j(\mathbf{s}), \mathbf{s})}{\partial q_j})^{-1}}{\sum_j (\sigma - \frac{\partial mu_j(q_j(\mathbf{s}), \mathbf{s})}{\partial q_j})^{-1}} \in [c_{ps}\varphi_N, C_{ps}\varphi_N],$$

as well as

$$\frac{\partial q_i(\mathbf{s})}{\partial s_i} = \frac{\frac{\partial mu_i(q_i(\mathbf{s}),\mathbf{s})}{\partial s_i} - \frac{\partial p^\sigma(\mathbf{s})}{\partial s_i}}{\sigma - \frac{\partial mu_i(q_i(\mathbf{s}),\mathbf{s})}{\partial q_i}} \in [\frac{c_{qs}}{\sigma - c_q}, C_{qs}].$$

The first part of the lemma will be crucial for our main results. It says that the effect of the change in $i's$ signal on the reference price $p^\sigma(\mathbf{s})$ is a weighted average of its effect on the marginal utilities of all the agents. Thus, given the assumption of informational smallness, the extent to which each agent can affect the reference price vanishes as the economy grows. If the slope σ of the $\sigma - WE$ allocation is small, this means that misreporting has limited effect on the "per-unit price" that he pays for each $x'th$ infinitesimal inframarginal unit, $p^\sigma\left(s_i\left(x\right), \mathbf{s}_{-i}\right) + \sigma x$. The monotonicity and so the global IC follows thus from the single-crossing assumption ($i's$ signal affects $i's$ marginal utility most).

While the Proposition shows how to design IR and IC transfers, it is silent about BS. On the negative note, the following is an easy corollary to the form of transfers in (4) and each agent's impact on prices, as in Lemma 2.

Corollary 1 *Consider an IR, IC* $0 - WE$ *mechanism* $\{q_i(\mathbf{s}), t_i(\mathbf{s})\}_{i\leq N, s\in S}$ *and a signal profile* $\mathbf{s}$ *such that*

$$q_i(\mathbf{s}) \neq 0, \ \textit{for some } i$$
$$s_i \in (0,1). \ \forall i$$

Then the mechanism runs a budget deficit at $\mathbf{s}$, $\sum_i t_i(\mathbf{s}) < 0$.

For the proof, it is enough to observe that each buyer with an interior signal must pay the WE price $p^*(\mathbf{s})$ for the marginal unit he purchases and a strictly lower price $p^*(s_i', \mathbf{s}_{-i})$ for every inframarginal unit $x < q_i(\mathbf{s})$, for some $s_i' < s_i$, and similarly for the sellers.

Pick a $\sigma - WE$ allocation $\{q_i(\mathbf{s})\}_{i\leq N, s\in S}$ together with transfers $\{t_i(\mathbf{s})\}_{i\leq N, s\in S}$ that are determined by the local IC, i.e. (4), together with

$$\begin{aligned} t_i(\underline{s_i}, \mathbf{s}_{-i}) &= \left[p^\sigma(\underline{s_i}, \mathbf{s}_{-i}) + \sigma q_i(\underline{s_i}, \mathbf{s}_{-i})\right] \times q_i(\underline{s_i}, \mathbf{s}_{-i}), \ \forall i \forall \mathbf{s}_{-i} \\ t_i(\overline{s_i}, \mathbf{s}_{-i}) &= -\left[p^\sigma(\overline{s_i}, \mathbf{s}_{-i}) + \sigma q_i(\underline{s_i}, \mathbf{s}_{-i})\right] \times q_i(\overline{s_i}, \mathbf{s}_{-i}), \ \forall i \forall \mathbf{s}_{-i} \end{aligned} \tag{7}$$

In the case when $\underline{s_i} = \overline{s_i}$ and so $q_i(\underline{s_i}, \mathbf{s}_{-i}) = 0$ we thus set the transfers $t_i(\underline{s_i}, \mathbf{s}_{-i})$ to be zero. Intuitively, those are the largest transfers consistent with IR, and so the ones most conducive to BS. In the case when, say, $\underline{s_i} = 0$ and $q_i(0, \mathbf{s}_{-i}) > 0$ the transfers leave strictly positive surplus to agent i. While IR would be consistent with larger payments larger payments would drive agent $i's$ utility strictly below that in the WE even if $q_i(0, \mathbf{s}_{-i})$ is close to the WE allocation $q_i^*(0, \mathbf{s}_{-i})$. Note that the transfers for the boundary types defined in (7) satisfy conditions (6), and so the mechanism $\{q_i(\mathbf{s}), t_i(\mathbf{s})\}_{i \leq N, s \in S}$ satisfies MC, IC and IR.

The following is the main result of the paper.

Proposition 2 *Fix $C > 0$ sufficiently big. The $C\varphi_N - WE$ mechanisms defined by (1), (4) and (7) are robustly asymptotically φ_N−efficient.*

The intuition for the result is as follows. For sufficiently large economy, and so small slope $C\varphi_N$ in the $C\varphi_N - WE$ allocation, the marginal utility $mu_i(q_i(\mathbf{s}), (\mathbf{s}))$ of any agent is close to the reference price $p^{C\varphi_N}(\mathbf{s})$, which, in turn, must be close to the WE price $p^0(\mathbf{s})$. Thus the allocation $q_i(\mathbf{s})$ must be close to the efficient WE allocation. Regarding the transfers, as we argued below Lemma 2, given the small effect of misreporting on the reference price, the average price that, say, a buyer pays must be close to the reference, and so the Walrasian price. This establishes asymptotic efficiency of the mechanism.

Regarding budget surplus, the proof establishes that, for sufficiently large C and N, the average price that any buyer pays is weakly greater than the reference price $p^{C\varphi_N}(\mathbf{s})$ and the average price that any seller gets is weakly lower than $p^{C\varphi_N}(\mathbf{s})$. Let us fix a signal profile $\mathbf{s}$, a slope $\sigma > 0$ and focus on a buyer i such that $\underline{s_i} \in (0, 1)$, meaning that $q_i(\underline{s_i}, \mathbf{s}_{-i}) = 0$. On the one hand, the "per-unit price" that i pays for the first infinitesimal unit equals the reference price given his signal that makes him pivotal, $p^\sigma(\underline{s_i}, \mathbf{s}_{-i})$, which is strictly lower than $p^\sigma(\mathbf{s})$. On the other hand, however, the strictly positive slope means that the price he pays for the last unit, $p^\sigma(\mathbf{s}) + \sigma q_i(\mathbf{s})$, exceeds the reference price. The result follows from the careful comparison of those two price differences. Given Lemma 2 the excess of the reference price over the lowest "per-unit price" is proportional to the price impact of misreporting, φ_N, and quantity purchased $q_i(\mathbf{s})$. In particular, it is independent of the slope σ. The excess of the "per-unit price" for the last unit purchased over the reference price is $\sigma q_i(\mathbf{s})$. Given the bounds on how the "per-unit prices" change with the quantity purchased (again, from Lemma 2), setting the slope $\sigma = C\varphi_N$ for sufficiently large $C > 0$ will result in the average price exceeding $p^\sigma(\mathbf{s})$, establishing BS.

Example 4 *Consider the group model from Example 2. The $\sigma - WE$ allocation and the* *reference price* $p^{\sigma}(\mathbf{s})$ *satisfy*

$$q_i(\mathbf{s}) = \frac{\alpha}{\mu+\sigma}(s_i - \overline{s}),$$
$$p^{\sigma}(\mathbf{s}) = (\alpha + \beta_h + \beta_l)\overline{s}.$$

For any signal profile $\mathbf{s}$ *and an agent* i *we have* $\underline{s_i} = \overline{s_i} = \overline{s_{-i}}$ *and* $q_i(\overline{s_{-i}}, \mathbf{s}_{-i}) = 0$*. This implies that for the IR and IC transfers* $t_i(\mathbf{s})$ *the average per-unit price equals*

$$\overline{p_i}(\mathbf{s}) := \frac{t_i(\mathbf{s})}{q_i(\mathbf{s})} = \frac{1}{2}\left((\alpha+\beta_h+\beta_l)\bar{s}_{-i} + (\alpha+\beta_h+\beta_l)\overline{s} + \sigma q_i(\mathbf{s})\right),$$

and so

$$\overline{p_i}(\mathbf{s}) - p^{\sigma}(\mathbf{s}) = \frac{1}{2}\left((\alpha+\beta_h+\beta_l)\bar{s}_{-i} - (\alpha+\beta_h+\beta_l)\overline{s} + \sigma\frac{\alpha}{\mu+\sigma}(s_i - \overline{s})\right) =$$
$$= \frac{(s_i - \overline{s})}{2}\left(\sigma\frac{\alpha}{\mu+\sigma} - \frac{(\alpha+\beta_h+\beta_l)}{n-1}\right).$$

This implies that BS is satisfied as long as

$$\sigma \geq \frac{\mu(\alpha+\beta_h+\beta_l)}{\alpha(n-1) - (\alpha+\beta_h+\beta_l)} = O\left(\frac{1}{N}\right).$$

On the other hand, picking a slope of order $\frac{1}{N}$ *guarantees asymptotic* $\frac{1}{N}-$*efficiency.*

In the case when $\beta_h = \beta_l$ *the model and the solution reduce to the fundamental value model from Example 1.*

Recall that the equilibrium of the double auction game satisfies the (ex-post) constraints and achieves the assymptotic efficiency only for the fundamental value and not the group model (RW 12). In contrast, the example shows that the mechanism design problem of constructing a $\sigma - WE$ mechanism that achieves both objectives is virtually identical for both models.

In the following we establish that the efficiency bound in Proposition 2 is actually tight: the quicker convergence to Pareto Efficiency is not possible. For that we must complement A3 with

A3') $\frac{\partial mu_i}{\partial s_j} \geq c_{\phi}\phi_N$ for $c_{\phi} > 0$, $i \neq j$;

Proposition 3 *There are $e, E > 0$ such that for sufficiently large economy N and any MC, IC, IR and $e\varphi_N-$Pareto Efficient mechanism $\{q_i(\mathbf{s}), t_i(\mathbf{s})\}_{i \leq N, s \in S}$, for any signal profile $\mathbf{s}$ such that*

$$|q_i(\mathbf{s})| \geq E\varphi_N, \text{ for some } i$$
$$s_i \in [E\varphi_N, 1 - E\varphi_N], \ \forall i$$

the mechanism generates budget deficit, $\sum_i t_i(\mathbf{s}) < 0$.

In the case of quicker convergence to Pareto Effiiciency, budget need not run a deficit for all the types, and the two qualifications as in the proposition are needed. Even the $0 - WE$ allocation together with IR, IC transfer balances the budget for the types at which it is efficient for no-one to trade; hence the first qualification. If all the agents have boundary types - lowest types for the buyers and highest for the sellers - the $0 - WE$ allocation with transfers as in (7) balances the budget; hence the second qualification.

The proof generalizes the intuition from Corollary 1 that establishes budget deficit for the exactly optimal WE allocation together with IR, IC transfers. Given the impact of own signal on the price of order φ_N and the form of IC transfers in (4), the average per-unit price a buyer faces will dip below the reference price as long as the target allocation is sufficiently close to the Walrasian Equilibrium allocation.

3 Proofs

Proof. (Lemma 1) First, assumptions A1 and A4 and the continuous differentiability of the utility functions u_i guarantee that for any signal profile $\mathbf{s}$, any agent i and any p there is a unique quantity $q_i(p, \mathbf{s})$ that satisfies

$$mu_i(q_i(p, \mathbf{s}), \mathbf{s}) = p + \sigma \times q_i(p, \mathbf{s}).$$

Second, the same assumptions guarantee that, for any $\mathbf{s}$, there is unique p such that

$$\sum_i q_i(p, \mathbf{s}) = 0.$$

■

Proof. (Lemma 2) From the first order condition we have for any agent j

$$\frac{\partial mu_j(q_j(\mathbf{s}),\mathbf{s})}{\partial q_j}\frac{\partial q_j(\mathbf{s})}{\partial s_i}+\frac{\partial mu_j(q_j(\mathbf{s}),\mathbf{s})}{\partial s_i}=\frac{\partial p^{\sigma}(\mathbf{s})}{\partial s_i}+\sigma\frac{\partial q_j(\mathbf{s})}{\partial s_i}.$$

This establishes the formula for $\frac{\partial q_i(\mathbf{s})}{\partial s_i}$. On the other hand, since

$$\sum_j q_j(\mathbf{s}')=0,$$

for all $\mathbf{s}'$, it follows that

$$0=\sum_j\frac{\partial q_j(\mathbf{s})}{\partial s_i}=\sum_j\frac{\frac{\partial mu_j(q_j(\mathbf{s}),\mathbf{s})}{\partial s_i}-\frac{\partial p^{s}(\mathbf{s})}{\partial s_i}}{\sigma-\frac{\partial mu_j(q_j(\mathbf{s}),\mathbf{s})}{\partial q_j}},$$

which establishes the formula for $\frac{\partial p^{\sigma}(\mathbf{s})}{\partial s_i}$. The bounds follow from the assumptions A2-A4.
■

Proof. (Proposition 1) Necessity. Fix a $\sigma-WE$ allocation $\{q_i(\mathbf{s})\}_{i\leq N,s\in S}$. The functions $q_i(\cdot,\mathbf{s}_{-i})$ are continuous (given the assumptions on u_i) and Lemma 2 establishes that they are also strictly increasing. We can thus define

$$t_i\left(\mathbf{s}\right)=\widetilde{t}_i\left(q_i\left(\mathbf{s}\right),\mathbf{s}_{-i}\right).$$

Fix a signal profile $\mathbf{s}$ and an agent i and consider the allocation $q_i(\cdot,\mathbf{s}_{-i})$ as a function of $i's$ signal. Suppose that $q_i(\mathbf{s})>0$. The local IC implies that, for all $s_i'\in(\underline{s_i},s_i)$,

$$mu_i\left(q_i\left(s_i',\mathbf{s}_{-i}\right),\left(s_i',\mathbf{s}_{-i}\right)\right)\frac{\partial q_i\left(s_i',\mathbf{s}_{-i}\right)}{\partial s_i'}=\frac{\partial\widetilde{t}_i\left(q_i\left(s_i',\mathbf{s}_{-i}\right),\mathbf{s}_{-i}\right)}{\partial q_i}\frac{\partial q_i\left(s_i',\mathbf{s}_{-i}\right)}{\partial s_i'},$$
$$\frac{\partial\widetilde{t}_i\left(q_i\left(s_i',\mathbf{s}_{-i}\right),\mathbf{s}_{-i}\right)}{\partial q_i}=mu_i\left(q_i\left(s_i',\mathbf{s}_{-i}\right),\left(s_i',\mathbf{s}_{-i}\right)\right),$$

and so

$$t_i(\mathbf{s}) = \widetilde{t}_i(q_i(\mathbf{s}), \mathbf{s}_{-i}) = t_i(\underline{s_i}, \mathbf{s}_{-i}) + \int_{q_i(\underline{s_i}, \mathbf{s}_{-i})}^{q_i(\mathbf{s})} \frac{\partial \widetilde{t}_i(x, \mathbf{s}_{-i})}{\partial q_i} dx =$$
$$= t_i(\underline{s_i}, \mathbf{s}_{-i}) + \int_{q_i(\underline{s_i}, \mathbf{s}_{-i})}^{q_i(\mathbf{s})} mu_i(x, (s_i(x), \mathbf{s}_{-i})) dx =$$
$$= t_i(\underline{s_i}, \mathbf{s}_{-i}) + \int_{q_i(\underline{s_i}, \mathbf{s}_{-i})}^{q_i(\mathbf{s})} [p^\sigma(s_i(x), \mathbf{s}_{-i}) + \sigma x] dx.$$

We similarly establish that the local IC implies (4) in the case when $q_i(\mathbf{s}) < 0$. The conditions (6) are implied by the IR constraints for the types $\underline{s_i}$ and $\overline{s_i}$.

Sufficiency. Lemma 2 establishes the monotonicity of the allocation, which, together with the local IC, is sufficient for the global IC. Given the global IC, global IR follows from the IR satisfied by the types $\underline{s_i}$ and $\overline{s_i}$. ■

Proof. (Proposition 2) By Proposition 1 the mechanism satisfies MC, IC and IR and so it remains to establish asymptotic φ_N-efficiency and BS. Fix $C > 0$ and consider the $C \times \varphi_N - WE$ mechanisms as in the Proposition. Pick an agent i and suppose without loss of generality that $q_i(\mathbf{s}) > 0$.

Assumptions A1 and A4 imply that there are constants $\overline{q}, \overline{mu} > 0$ that bound the quantities allocatted and the marginal utilities in any $\sigma - WE$ allocation $\{q_i(\mathbf{s})\}_{i \leq N, \mathbf{s} \in S}$:

$$|q_i(\mathbf{s})| \leq \overline{q}, \ \forall i \forall \mathbf{s} \tag{8}$$
$$|mu_i(q_i(\mathbf{s}), \mathbf{s})| \leq \overline{mu}, \ \forall i \forall \mathbf{s}$$

Assymptotic Efficiency. Since

$$p^*(\mathbf{s}) = p^0(\mathbf{s}) \in [p^{C/N}(\mathbf{s}) - C\varphi_N \overline{q}, p^{C/N}(\mathbf{s}) + C\varphi_N \overline{q}], \tag{9}$$

it follows from A4 that

$$|q_i(\mathbf{s}) - q_i^*(\mathbf{s})| \leq -\frac{C\varphi_N}{C_q} \overline{q}, \tag{10}$$

and so

$$|u_i(q_i^*(\mathbf{s}), \mathbf{s}) - u_i(q_i(\mathbf{s}), \mathbf{s})| = |\int_{q_i(\mathbf{s})}^{q_i^*(\mathbf{s})} mu_i(x, (s_i(x), \mathbf{s}_{-i})) dx| \leq \overline{mu} \times -\frac{C\varphi_N}{C_q} \overline{q}. \tag{11}$$

On the other hand for any $s_i' \in [\underline{s_i}, s_i]$ we have

$$\begin{aligned}
&|p^{\sigma}(s_i', \mathbf{s}_{-i}) + \sigma q_i(s_i', \mathbf{s}_{-i}) - p^0(\mathbf{s})| \\
&\le |p^{\sigma}(s_i', \mathbf{s}_{-i}) + \sigma q_i(s_i', \mathbf{s}_{-i}) - p^{\sigma}(\mathbf{s}) + \sigma q_i(\mathbf{s})| + |p^{\sigma}(\mathbf{s}) + \sigma q_i(\mathbf{s}) - p^0(\mathbf{s})| \le \\
&\le C_p \varphi_N + C\varphi_N (q_i(\mathbf{s}) - q_i(s_i', \mathbf{s}_{-i})) + |p^{\sigma}(\mathbf{s}) + \sigma q_i(\mathbf{s}) - p^0(\mathbf{s})| \le \\
&\le C_p \varphi_N + C\varphi_N (q_i(\mathbf{s}) - q_i(s_i', \mathbf{s}_{-i})) + C\varphi_N \overline{q} \le \varphi_N (C_p + 2C\overline{q}),
\end{aligned}$$

where the second inequality follows from Lemma 2, $s_i, s_i' \in [0,1]$ and the definition of a $C\varphi_N - WE$ allocation, and the third inequality follows from (9).

Overall, using the above, (10) and the definition of transfers by (4) and (7) we have, for $s_i\,(q_i')$ as in (5),

$$\begin{aligned}
|t_i(\mathbf{s}) - p^*(\mathbf{s}) \times q_i^*(\mathbf{s})| &\le |t_i(\mathbf{s}) - p^*(\mathbf{s}) \times q_i(\mathbf{s})| + |p^*(\mathbf{s}) \times q_i(\mathbf{s}) - p^*(\mathbf{s}) \times q_i^*(\mathbf{s})| = \\
&\le \varphi_N \left(q_i(\mathbf{s}) \times (C_p + 2C\overline{q}) + p^*(\mathbf{s}) \times -\frac{C}{C_q} \overline{q} \right).
\end{aligned} \tag{12}$$

Equations (11) and (12) establish the asymptotic φ_N-efficiency. The case when i is a seller ($q_i(\mathbf{s}) < 0$) is completely analogous.

Budget Surplus. Fix N such that $C\varphi_N < 1$. We will show that for any $\mathbf{s}$ and i the transfers satisfy $t_i(\mathbf{s}) \ge p^{C\varphi_N}(\mathbf{s}) q_i(\mathbf{s})$, which is sufficient for BS. Recall that the transfers satisfy

$$t_i(\mathbf{s}) = \left[p^{C\varphi_N}(\underline{s_i}, \mathbf{s}_{-i}) + C\varphi_N q_i(\underline{s_i}, \mathbf{s}_{-i}) \right] \times q_i(\underline{s_i}, \mathbf{s}_{-i}) + \int_{q_i(\underline{s_i}, \mathbf{s}_{-i})}^{q_i(\mathbf{s})} \left[p^{C\varphi_N}\left(s_i\left(x\right), \mathbf{s}_{-i}\right) + C\varphi_N x \right] dx,$$

with $s_i\,(x)$ as in (5). Let us first bound the distance from $p^{C\varphi_N}(\mathbf{s})$ of the lowest and the highest "per-unit prices" that i must pay, $p^{C\varphi_N}(\underline{s_i}, \mathbf{s}_{-i}) + C\varphi_N q_i(\underline{s_i}, \mathbf{s}_{-i})$ and $p^{C\varphi_N}(\mathbf{s}) + C\varphi_N q_i(\mathbf{s})$. Regarding the highest "per-unit price", by definition, we have

$$p^{C\varphi_N}(\mathbf{s}) + C\varphi_N q_i(\mathbf{s}) - p^{C\varphi_N}(\mathbf{s}) = C\varphi_N q_i(\mathbf{s}). \tag{13}$$

Regarding the lowest "per-unit price",

$$\begin{aligned}
&p^{C\varphi_N}(\underline{s_i}, \mathbf{s}_{-i}) + C\varphi_N q_i(\underline{s_i}, \mathbf{s}_{-i}) - p^{C\varphi_N}(\mathbf{s}) \\
&= [p^{C\varphi_N}(\underline{s_i}, \mathbf{s}_{-i}) + C\varphi_N q_i(\underline{s_i}, \mathbf{s}_{-i})) - p^{C\varphi_N}(\underline{s_i}, \mathbf{s}_{-i})] + [p^{C\varphi_N}(\underline{s_i}, \mathbf{s}_{-i}) - p^{C\varphi_N}(\mathbf{s})] = \\
&= C\varphi_N q_i(\underline{s_i}, \mathbf{s}_{-i}) + [p^{C\varphi_N}(\underline{s_i}, \mathbf{s}_{-i}) - p^{C\varphi_N}(\mathbf{s})].
\end{aligned}$$

From Lemma 2 we have that

$$\frac{\partial p^{C\varphi_N}(s_i', \mathbf{s}_{-i})}{\partial s_i} \leq \varphi_N \frac{C_{ps}(1-e_l)}{c_{qs}} \frac{\partial q_i(s_i', \mathbf{s}_{-i})}{\partial s_i}, \; \forall s_i' \in [\underline{s_i}, s_i]$$
$$p^{C\varphi_N}(\mathbf{s}) - p^{C\varphi_N}(\underline{s_i}, \mathbf{s}_{-i}) \leq \varphi_N \frac{C_{ps}(1-e_l)}{c_{qs}} \left(q_i(\mathbf{s}) - q_i(\underline{s_i}, \mathbf{s}_{-i})\right),$$

and so

$$\begin{aligned} p^{C\varphi_N}(\underline{s_i}, \mathbf{s}_{-i}) + C\varphi_N q_i(\underline{s_i}, \mathbf{s}_{-i})) - p^{C\varphi_N}(\mathbf{s}) &= p^{C\varphi_N}(\underline{s_i}, \mathbf{s}_{-i}) + C\varphi_N q_i(\underline{s_i}, \mathbf{s}_{-i}) - p^{C\varphi_N}(\mathbf{s}) \geq \\ &\geq C\varphi_N q_i(\underline{s_i}, \mathbf{s}_{-i}) - \varphi_N \frac{C_{ps}(1-e_l)}{c_{qs}} \left(q_i(\mathbf{s}) - q_i(\underline{s_i}, \mathbf{s}_{-i})\right). \end{aligned} \tag{14}$$

Thus, choosing C large shifts the range of "per-unit prices" to the right relative to $p^{C\varphi_N}(\mathbf{s})$. More precisely, for $s_i(\cdot)$ as in (5)

$$\begin{aligned} \frac{d(p^{C\varphi_N}(s_i(q_i), \mathbf{s}_{-i}) + C\varphi_N q_i)}{\partial q_i} &= \frac{\partial p^{C\varphi_N}(s_i(q_i), \mathbf{s}_{-i})}{\partial s_i} \frac{\partial s_i(q_i)}{\partial q_i} + C\varphi_N \in \\ &\in \varphi_N [\frac{c_{ps}}{C_{qs}} + C, \frac{C_{ps}(1-e_l)}{c_{qs}} + C], \end{aligned} \tag{15}$$

where the bounds follow from Lemma 2. The bounds (13) and (14) on the range together with the bounds (15) on the slope of the marginal utilities as a function of quantity easily establish that for every $C^* > 0$ there exists $C > 0$ such that

$$\int_{q_i(\underline{s_i}, \mathbf{s}_{-i})}^{q_i(\mathbf{s})} \left[p^{C\varphi_N}(s_i(x), \mathbf{s}_{-i}) + C\varphi_N x\right] dx \geq (p^{C\varphi_N}(\mathbf{s}) + C^*\varphi_N) \times (q_i(\mathbf{s}) - q_i(\underline{s_i}, \mathbf{s}_{-i})).$$

Thus, for sufficiently high C (and so C^*) we have

$$\begin{aligned} & t_i(\mathbf{s}) \\ & = \left[p^{C\varphi_N}(\underline{s_i}, \mathbf{s}_{-i}) + C\varphi_N q_i(\underline{s_i}, \mathbf{s}_{-i})\right] \times q_i(\underline{s_i}, \mathbf{s}_{-i}) + \int_{q_i(\underline{s_i}, \mathbf{s}_{-i})}^{q_i(\mathbf{s})} \left[p^{C\varphi_N}(s_i(x), \mathbf{s}_{-i}) + C\varphi_N x\right] dx \geq \\ & \geq \left(p^{C\varphi_N}(\mathbf{s}) - \varphi_N \frac{C_{ps}(1-e_l)}{c_{qs}} \left(q_i(\mathbf{s}) - q_i(\underline{s_i}, \mathbf{s}_{-i})\right)\right) \times q_i(\underline{s_i}, \mathbf{s}_{-i}) + (p^{C\varphi_N}(\mathbf{s}) + C^*\varphi_N) \times (q_i(\mathbf{s}) - q_i(\underline{s_i}, \mathbf{s}_{-i} \\ & = p^{C\varphi_N}(\mathbf{s}) \times q_i(\mathbf{s}) + (q_i(\mathbf{s}) - q_i(\underline{s_i}, \mathbf{s}_{-i})) \times \varphi_N \left(C^* - \frac{C_{ps}(1-e_l)\overline{q}}{c_{qs}}\right) \geq p^{C\varphi_N}(\mathbf{s}) \times q_i(\mathbf{s}). \end{aligned}$$

The case when i is a seller ($q_i(\mathbf{s}) < 0$) is completely analogous. This establishes the proof. ■

Proof. (Proposition 3) To be completed. ■

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