A FOUNDATION FOR PROBABILISTIC BELIEFS WITH OR WITHOUT ATOMS

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Abstract

We provide sufficient conditions for a qualitative probability (Bernstein, 1917; de Finetti, 1937; Koopman, 1940; Savage, 1954) to have a unique countably-additive measure representation, generalizing Villegas (1964) to allow atoms. Instead of imposing a cancellation axiom or a solvability axiom, we propose a novel axiom of divisibility: *third-order smaller-atoms domination* requires that for each atom A, there are three pairwise disjoint events, each a union of atoms less likely than A and each at least as likely as A.

Theorem 1 states that our divisibility axiom and *monotone continuity* (Villegas, 1964; Arrow, 1970) are sufficient to guarantee a qualitative probability has a unique countably-additive measure representation, and are necessary for that representation to belong to a particular class that includes atomless measures, purely atomic measures, and hybrids. Applications include beliefs about when a delivery will arrive, intertemporal preferences over streams of indivisible goods, preferences over parts of a heterogeneous good, and the analysis of incomplete data due to limited granularity.

Keywords: smaller-atoms domination, qualitative probability, monotone continuity.

1 Introduction

1.1 Executive summary

From the doctor's choice of treatment, to the employer's choice of job applicant, to the investor's choice of portfolio, to the mortal's choice of religion, and beyond: much behavior, including much of the economic behavior we observe and strive to model, is the selection of an action with uncertain consequences. Our standard model is founded on the postulate that such choices, when made by someone who is rational, can be decomposed into (1) beliefs about the relative likelihood of events, and (2) tastes among outcomes

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(Ramsey, 1931). This article revisits a well-studied question: when can such beliefs be said to be consistent with classical probability theory?[1]

More precisely, suppose we are given a nonempty set of states S, a σ-algebra of events $\mathcal{A} \subseteq 2^S$ with $S \in \mathcal{A}$, and a *qualitative probability* $\succsim_l$ on $\mathcal{A}$: a binary relation on $\mathcal{A}$, consisting of comparisons of events on the basis of relative likelihood, satisfying minimal probabilistic requirements (Bernstein, 1917; de Finetti, 1937; Koopman, 1940; Savage, 1954). When does $\succsim_l$ admit representation by a σ-measure[2] $\mu : \mathcal{A} \to [0, 1]$?

The discrete case faces technical challenges due to *atoms*: non-null events for which each subset is either equally likely or null (Villegas, 1964). In fact, the large literature addressing our question can be classified according to which of the following is imposed:

- there are no atoms;
- the qualitative probability satisfies a "cancellation" axiom;
- the qualitative probability satisfies a "solvability" axiom; or
- there are additional primitives beyond S, $\mathcal{A}$, and $\succsim_l$.

In this article, we proceed without imposing any of the above. Instead, we impose a novel divisibility axiom and a well-studied continuity axiom.

The divisibility axiom, *third-order smaller-atoms domination* (3-SAD), states that if there are any atoms at all, then each is overwhelmed by those less likely. Formally, for each atom A, there are three[3] pairwise disjoint events, each a union of atoms less likely than A and each at least as likely as A (Figure 1). In our discussion of applications, we argue that this axiom may be interpreted as a requirement of sufficient doubt or sufficient patience, depending on context.

In addition to 3-*SAD*, we impose *monotone continuity* (Villegas, 1964; Arrow, 1970). As formulated by Arrow, a sequence of events $(C_i) \in \mathcal{A}^{\mathbb{N}}$ is *vanishing* if $C_1 \supseteq C_2 \supseteq ...$ and $\cap C_i = \emptyset$. The axiom requires that if $A \succ B$, and if (C_i) is vanishing, then there is $i \in \mathbb{N}$ such that $A \succ_l B \cup C_i$. On the appeal of this axiom, Arrow writes: "The assumption of Monotone Continuity seems, I believe correctly, to be the harmless simplification almost inevitable in the formalization of any real-life problem."

Our main result, Theorem 1, states that $\succsim_l$ is a qualitative probability satisfying 3-*SAD* and *monotone continuity* if and only if it admits a unique σ-measure representation belonging to a particular class. This class includes atomless measures, purely atomic measures, and hybrids.

[1] Machina and Schmeidler (1992) call this question the first of two lines of inquiry culminating in the modern theory of subjective probability.

[2] In this article, a *measure* is a finitely-additive probability measure, σ*-additivity* is *countable-additivity*, and a σ*-measure* is a σ-additive measure.

[3] Why three? For each natural number k, we can define an analogous k-*SAD* axiom—so can we obtain a similar result using only 1-*SAD* or 2-*SAD*? This is an open question discussed in the conclusion. We remark that *at least three* has appeared in earlier research on a similar topic: given an ordering over a product space, if there are only two "essential" factors, an additional condition is used to guarantee an additive representation (Debreu, 1959). Intuitively, while addition is a binary operation, its cancellation property is articulated with three elements.

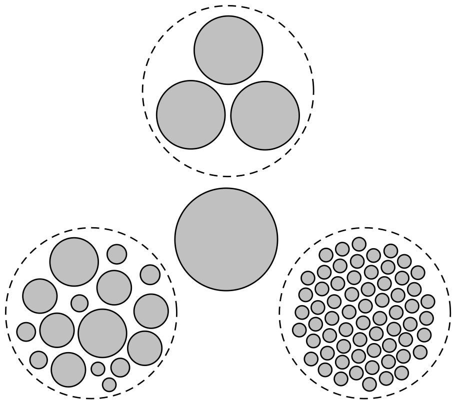

Figure 1: Third-order smaller atoms domination. In the illustration, each gray circle is an atom, and a more likely event has a greater area. For the largest atom in the center, there are three pairwise disjoint events (circled by dashed lines), each a union of less-likely atoms and each at least as likely as the center atom. The top event is a union of a few relatively large atoms, the rightmost event is a union of many relatively small atoms, and the leftmost event is a union of atoms of different sizes. Not pictured, but allowed, is a union of an infinite collection of atoms. The axiom requires that there are three such events for *each* atom.

1.2 Applications

Beliefs about when a delivery will arrive

Our first application is a concrete example of our primary interpretation of the primitives as beliefs. Suppose you are waiting for a delivery, unsure about the date and hour of arrival. If you are certain the package will arrive by the end of the week, then your beliefs do not satisfy 3-*SAD*. On the other hand, suppose the package is already late, leading you to believe that each hour is more likely than the next, and suppose that you lack conviction in the following manner: you are unwilling to wager on the package's arrival at a particular hour over its arrival some time later, regardless of the hour in question. Moreover, for each hour t, you can partition the hours after t into three groups, each of which you would bet on over t alone. In this case, your beliefs do satisfy 3-*SAD*. In fact the axiom may be satisfied even if you do not believe each hour is more likely than the next—say if delivery is deemed possible only during business hours of non-holiday weekdays, or if for a given day afternoon hours are deemed more likely than morning hours—provided the lack of conviction is adapted accordingly.

In this example, a special case is belief that there is a fixed probability $p \in (0,1)$ such that for each hour t, p is the probability of arrival at t conditional on the package having not yet arrived. In this case, 3-*SAD* is satisfied if and only if $p \leq \dfrac{1}{4}$. Our axiom may therefore be interpreted as a notion of sufficient doubt.

Intertemporal preferences

The standard model of intertemporal preference is a special case of the Savage model with an alternative interpretation of its primitives: $S = \mathbb{N}$, with its usual well-ordering, the members of S interpreted as *periods*. This literature studies preferences of an agent (or dynasty, or institution) over consumption streams, and the preferences with σ-measure subjective expected utility representations are precisely, in the language of Olea and Strzalecki (2014), the *time separable* class. This central class contains:

- *geometric discounting*, (Samuelson, 1937; Koopmans, 1960; Koopmans, 1972; Bleichrodt, Rohde, and Wakker, 2008),
- *generalized hyperbolic discounting* (Loewenstein and Prelec, 1992), and
- *quasi-hyperbolic discounting* (Laibson, 1997; Olea and Strzalecki, 2014).

Because this class uses atoms, the Savage (1954) characterization of subjective expected utility does not apply. The standard approach in the literature is to take preferences over consumption streams with a rich consumption space, such as Euclidean commodity space.

By contrast, our theorem applies in the minimal case that commodity space has two members, allowing us to handle for example the case where consumption in each period consists of some indivisible goods (or *objects*) taken from a finite set. This setting is also considered by Kochov (2013), who proposes *patience*: a weak axiom implied by 3-*SAD*. Kochov establishes that if $\succsim_l$ has a geometric representation and satisfies *patience*, then $\succsim_l$ has no other geometric representation; we revisit this result in our conclusion. Our axiom may be interpreted as a stronger patience requirement, and in the special case that beliefs admit a geometric representation, 3-*SAD* is satisfied if and only the discount factor is at least $\frac{3}{4}$.

Preferences over parts of a heterogeneous good

Economic models often take goods as infinitely divisible. If, in the background, we have in mind that a unit of such a good is physically a set in Euclidean space—such as the milk within a particular carton—then we should also have in mind additional structure preventing the trivialization of scarcity by duplication of the good through clever disassembly and reassembly (Banach and Tarski, 1924). The measure-theoretic approach is to consider a collection of parts into which the good may be divided, and to attach to each of those parts a measure that preferences respect. When all agents share a measure, the good is *homogeneous*; when the measures might differ, it is *heterogeneous*.

The classic problem of fair division, starting with Steinhaus (1948), is the problem of partitioning a heterogeneous good (or a *cake*) into parts (or *slices*) and then assigning those parts to agents according to some notion of fairness. Similarly, the model of land for urban economics proposed by Berliant (1985) treats *land* as a heterogeneous good that can be divided into *parcels*; this model has also been used in the context of fair division (Berliant, Thomson, and Dunz, 1992). The standard assumption in both settings is that each agent's preferences are represented by a measure, and the use of atomless measures has axiomatic foundations for both preferences over slices (Barbanel and Taylor, 1995) and preferences over parcels (Berliant, 1986). Our theorem is also a preference representation theorem for these settings, differing from the existing results in that it allows atoms: *crumbs* in the context of cake-cutting,[4] or parcels that cannot be subdivided (such as perhaps cities or houses) in the context of land.

[4]An example of such a *crumb* is the small plastic figurine (*la fève*) in a French king cake. According to tradition, whomever receives it with his slice is king for the day.

Analysis of incomplete data due to limited granularity

The standard assumption of *completeness*—that comparisons are always possible—has been criticized on both normative and positive grounds (von Neumann and Morgenstern, 1947; Aumann, 1962; Schmeidler, 1989; Mandler, 2005). Indecisiveness can be observed and distinguished from indifference (Eliaz and Ok, 2006), and in fact has been in a recent experiment (Cettolin and Riedl, 2013). The axiomatic investigation of this behavior spans diverse settings; for example:

- an agent may defer to the likelihood appraisals of experts, reticent whenever they disagree (Bewley, 2002; Gilboa, Maccheroni, Marinacci, and Schmeidler, 2010; Faro, 2015);
- refusal to compare may be due solely to fuzziness of tastes (Aumann, 1962; Aumann, 1964; Ok, 2002; Dubra, Maccheroni, and Ok, 2004; Evren and Ok, 2011; Ok, Ortoleva, and Riella, 2012), or may derive from the joint consideration of tastes and beliefs (Seidenfeld, Schervish, and Kadane, 1995; Nau, 2006; Galaabaatar and Karni, 2013);
- decisive individuals may exhibit collective indecisiveness as a coalition (Baucells and Shapley, 2008; Gul and Pesendorfer, 2014); and
- the researcher may be working with incomplete observational data (Nau, 2006).

Though we work with complete beliefs in this article, there is nevertheless a connection to this last example in a natural special case: comparisons are observed only with limited granularity. As an example, an investor's beliefs about the return of a stock might be observed only to the nearest dollar, the researcher for instance observing the comparison between $[2.5, 3.5)$ and $[3.5, 4.5)$ while observing nothing involving strict subevents of either. To be precise, suppose that there is a finest partition generating a σ-algebra for which observations are complete, and suppose moreover that this partition is countably infinite. Then assignment of probabilities to the associated events involves atoms, and our theorem provides conditions under which this may be done unambiguously. It may be of future interest to consider incomplete data sets for which data is completely observed up to some level of granularity, beyond which data is partially observed.

Limitations

We remark that the systematic violation of one of our axioms, *separability*, has been observed in an experiment where certain events are attached to probability appraisals while others are not (Ellsberg, 1961); the favoring of the appraised events is a phenomenon typically ascribed to *ambiguity aversion* (though not always, see Ergin and Gul (2009)).

1.3 Related literature

As mentioned above, the previous literature on our topic can be classified according to which of four assertions is imposed. Though our article does not belong to any of these four categories, it is closely related to each of them.

The literature with no atoms

The seminal contributions to the qualitative probability literature (Bernstein, 1917; de Finetti, 1937; Koopman, 1940) imposed that S can be partitioned into an arbitrarily large number of equally likely events. This implies that there is a unique measure that "almost represents" $\succsim_l$, though it may assign the same probability to two distinguished events (see Kreps, 1988).

To guarantee representation, Savage (1954) imposed a stronger axiom, *fineness-and-tightness* (see Section 2.3). Surprisingly, Savage's axioms are in fact compatible when $|S| = |\mathbb{N}|$; they are only incompatible when S is finite (see Kreps, 1988, Chapter 8, Problem 4.) That said, Savage's axioms are incompatible with atoms, and under the Continuum Hypothesis, they are incompatible with σ-additivity when $|S| \leq |\mathbb{R}|$.[5]

Savage deliberately avoided any continuity axiom implying σ-additivity, explaining: "I know of no argument leading to the requirement of countable additivity[...]it therefore seems better not to assume countable additivity outright as a postulate, but to recognize it as a special hypothesis yielding, where applicable, a large class of useful theorems." But there have since been two particularly strong such arguments: first, σ-additivity is required for avoiding money pumps (Adams, 1962; Seidenfeld and Schervish, 1983), and second, σ-additivity is required to ensure that choice always respects strict first order stochastic dominance (Wakker, 1993). Based on these observations and others, Stinchcombe (1997) argues that a measure which is only finitely-additive indicates a misspecified state space: "One summary[...]is that countably infinite constructions require countably additive probabilities."

Villegas (1964) identified the appropriate continuity axiom: if μ is a measure representation of $\succsim_l$, then μ is a σ-additive if and only if $\succsim_l$ is *monotonely continuous*. Furthermore, to guarantee representation by a σ-measure, it suffices to impose only *monotone continuity* and *atomlessness*. We refer to this important result as Theorem V and appeal to it directly in our proof. Unfortunately, these axioms are incompatible when $|S| \leq |\mathbb{N}|$.

While Kopylov (2010) does not explicitly study qualitative probabilities, it is clear from his analysis that σ-measure representation is guaranteed by imposing only *strong monotone continuity* when $\mathcal{A}$ is *countably separated* (Mackey, 1957). That there are no atoms is then implied. Because Kopylov's axioms are incompatible when $|S| \leq |\mathbb{N}|$, and because $\mathcal{A}$ can only be countably separated when $|S| \leq |\mathbb{R}|$,[6] it follows that under the Continuum Hypothesis, this approach is custom-tailored to the $|S| = |\mathbb{R}|$ case.

Our result generalizes that of Villegas (1964) by weakening *atomlessness* to 3-*SAD*. Like Savage (1954), our axioms are incompatible only when S is finite. Unlike the rest of this part of the literature, our axioms are compatible with atoms, and are compatible with σ-additivity whenever S is infinite.

The literature with cancellation

Even when S is finite, there are qualitative probabilities without measure representations (Kraft, Pratt, and Seidenberg, 1959). This was first demonstrated by means of a

[5]If $|S| = |\mathbb{R}|$, then the existence of an atomless σ-measure defined on 2^S is inconsistent with the Continuum Hypothesis (Ulam, 1930). Savage escapes this predicament by relaxing σ-additivity, as there are well-behaved atomless measures—for example when $S = [0, 1]$, there are measures that (1) agree with the Lebesgue measure on those sets where it is defined, and (2) assign the same number to any pair of congruent sets (Banach, 1932). By contrast, we escape this predicament both by allowing $\mathcal{A}$ to be any σ-algebra and by allowing atoms.

[6]$\mathcal{A}$ is *countable separated* if there is a countable collection of events $\mathcal{A}^* \subseteq \mathcal{A}$ such that for each distinct pair $s, s' \in S$, there is $A \in \mathcal{A}^*$ with $s \in A$ and $s' \notin A$. That this implies $|S| \leq |\mathbb{R}|$ can be seen from the following argument of Faris (2007): index $\mathcal{A}^*$ by the natural numbers, so that $\mathcal{A}^* = \{A_1, A_2, ...\}$, and define the mapping φ from S to the Cantor set $\mathbb{C}$ (Cantor, 1883) by defining, for each $i \in \mathbb{N}$ and each $s \in S$, $\varphi_i(s) \equiv \mathbb{1}_{A_i}(s)$. Then φ is an injection, so $|S| \leq |\mathbb{C}| = |\mathbb{R}|$.

qualitative probability $\succsim_l$ on the subsets of $\{s_1, s_2, s_3, s_4, s_5\}$ that includes the comparisons:

$$
\begin{array}{llcl}
(1) & \{\, s_1 \,\} & \succsim_l & \{\, s_2,\; s_4 \,\} \\
(2) & \{\, s_3,\; s_4 \,\} & \succsim_l & \{\, s_1,\; s_2 \,\} \\
(3) & \{\, s_2,\; s_5 \,\} & \succsim_l & \{\, s_1,\; s_4 \,\} \\
(4) & \{\, s_1,\; s_2,\; s_4 \,\} & \succ_l & \{\, s_3,\; s_5 \,\}
\end{array}
$$

A necessary and sufficient condition for measure representation is *finite cancellation* (Kraft, Pratt, and Seidenberg, 1959; Scott, 1964; Fishburn, 1970; Krantz, Luce, Suppes, and Tversky, 1971), the assertion that there is no such list of comparisons. This condition is quite strong, alone necessary and sufficient for additive representation (Fishburn, 1970). Attempts to justify this axiom generally rely on enriched models—for example, preferences over multi-sets of states, where such a multi-set is interpreted as a portfolio of Arrow-Debreu-McKenzie securities—and as such it can be difficult to interpret.

Stronger and more complex conditions have been identified that guarantee measure representation while allowing atoms when S is infinite (Domotor, 1969; Chateauneuf and Jaffray, 1984; Chateauneuf, 1985). Chateauneuf (1985) in fact provides necessary and sufficient conditions using an axiom that implies *finite cancellation*. We recommend Fishburn's excellent survey for more information (Fishburn, 1986).

By contrast, we do not impose any cancellation axiom. Rather, following Cantor and Debreu (Cantor, 1895; Debreu, 1954; Debreu, 1964), we construct an order-dense family of equivalence classes. Over our family—or more precisely, over carefully selected representatives of our family's classes—we define a binary operation. Following Peano (1889), we show that this binary operation interacts appropriately with succession—that is, addition by one—from which we derive cancellation, commutativity, associativity, and all other properties of ordinary addition.

The literature with solvability

To our knowledge, there are two articles studying qualitative probabilities with atoms that do not impose cancellation (Abdellaoui and Wakker, 2005; Chew and Sagi, 2006). Both approaches involve *solvability*:[7] for each pair of disjoint events, there is a subevent of one that is as likely as the other. Abdellaoui and Wakker (2005) allow for mosaics instead of σ-algebras and allow for measures that are not convex-ranged, while Chew and Sagi (2006) work with an ordering of events induced from preferences over acts through "exchangeability" and proceed without *monotonicity*. In both cases there may be atoms, provided any pairwise disjoint collection of atoms is finite with equally likely members.

By contrast, our axioms imply that if there are atoms, then there is a pairwise disjoint collection of atoms that is countably infinite with members that are not all equally likely. We explain the difference in approach with a visual metaphor where size corresponds to likelihood. Given two events, one larger than the other, we would often like to find a piece of the larger that is the same size as the smaller. In the *solvability* approach, the larger event is made of a fabric from which the desired piece can be cleanly cut. In our

[7]This is not quite the language used in either article. Abdellaoui and Wakker (2005) use "solvability" to refer to a stronger axiom, while Chew and Sagi (2006) use "completeness" to refer the given axiom and use "solvability" to refer to a related property for measures.

approach, we cannot always create the desired piece because the larger event may consist of blocks that cannot be cut. Nevertheless, whenever the larger event has a rich enough composition, the desired piece can be mosaicked by iteratively adding smaller and smaller blocks to better-approximate the specified size.

The literature with additional primitives

We refer to works that use the entire Savage (1954) model: a state space S, an outcome space X, and preferences $\succsim$ over *acts* $f : S \to X$. Implicitly, $\mathcal{A} = 2^S$. To uncover the embedded qualitative probability, first select a pair of outcomes x^* and x such that the act guaranteeing x^* is preferred to that guaranteeing x. Beliefs can then be defined as preferences among simple bets: $A \succsim_l B$ if and only if the act returning x^* on A and x otherwise is at least as desirable as the act returning x^* on B and x otherwise. It is imposed that this is well-defined; $\succsim_l$ does depend on the choice of x^* and x.

Savage used only $\succsim_l$ to deliver the unique measure representation of beliefs. But to handle atoms that are not equally likely without using a cancellation axiom, others have used the entire model, particularly when S is discrete. Typically this involves imposing, at a minimum, that X has cardinality of at least the continuum and that X has a rich topological structure. For example, X might be a simplex of objective "roulette lotteries" that can be mixed (as in Anscombe and Aumann, 1963 and the vast literature that followed), or an interval of dollar amounts (see Wakker, 1989; Gul, 1992; and references therein), or Euclidean commodity space (as in the literature on intertemporal preferences already mentioned).

In contrast to those who use the full Savage model under the uncertainty interpretation, we do not rely on any extraneous assumptions about the space of outcomes to deliver the measure representation, keeping our analysis firmly focused on beliefs. In contrast to most studies of intertemporal preference, as mentioned above, our pursuit of the Savage approach allows us to handle finite consumption spaces.

1.4 Outline

From here until the conclusion, the article is dedicated to technique. Section 2 introduces the model, then clarifies the relationship between our result and Savage's (Theorem S; Savage, 1954) with Proposition 2.1: Savage's axiom is equivalent to a continuity axiom, a richness axiom, and a divisibility axiom. Each of the next three sections focuses on modifying one of these axioms.

Section 3 ('Continuity') begins with Proposition 3.1: *monotone continuity* is equivalent to the usual requirement that upper and lower contour sets are closed. Thus by Theorem CD (Cantor, 1895; Debreu, 1954; Debreu, 1964), an order-dense family of equivalence classes—what we call a *quotient skeleton*—guarantees a continuous representation. We propose a stronger structure, called a *dyadic skeleton*, that guarantees a σ-measure representation (Proposition 3.2). We then show that we can construct our dyadic skeleton in pieces (Proposition 3.3), and that the atomless pieces have already been constructed (Theorem V; Villegas, 1964).

In Section 4 ('Richness'), we organize the atoms. Proposition 4.1 essentially allows us to study the atoms in isolation as though they are singletons. To simplify this task by taking the natural numbers as an auxiliary state space, we propose a tentative richness axiom, *no finite atom-catalogues*, that is implied by our final hypotheses.

Section 5 ('Divisibility') is the heart of this article. We use a problem of making exact change and an observation about real sequences (Theorem K; Kakeya, 1914) to motivate our *smaller-atoms domination* family. In particular, our axioms give the *greedy transforms*, an idea inspired by Rényi (1957), a variety of powerful properties. We then outline our central proof, the construction of the dyadic skeleton for the atoms (Proposition 5.1).

In Section 6, we state our main result, Theorem 1, then prove it using only the propositions summarized here and Theorem V. Section 7 concludes.

2 Model

2.1 Qualitative probability

A *likelihood space* $(S, \mathcal{A}, \succsim_l)$ is a triple consisting of

(1) a set of *states* S with generic members s, s',[8]

(2) a σ-algebra of *events* $\mathcal{A} \subseteq 2^S$, including S, with generic members A, B, C,[9] and

(3) a *likelihood relation* $\succsim_l$: a binary relation on $\mathcal{A}$.

We interpret $\succsim_l$ as a collection of comparisons of events on the basis of relative likelihood: $A \succsim_l B$ denotes "A is at least as likely as B." We write $A \sim_l B$ to denote $A \succsim_l B$ and $B \succsim_l A$, and write $A \succ_l B$ to denote $A \succsim_l B$ but not $A \sim_l B$. We emphasize that, for convenience, we include degenerate triples $(\emptyset, \{\emptyset\}, \succsim_l)$ as likelihood spaces.

Formally, all our results concern likelihood spaces. That said, we often implicitly take S and $\mathcal{A}$ as the set of states and σ-algebra, respectively, abusing notation by referring to the likelihood space by its likelihood relation. In certain cases, however, we are instead explicit about the entire likelihood space, particularly when comparing several likelihood spaces or when imposing structure on the set of states and σ-algebra.

The following assumptions are standard:

- Order: The relation $\succsim_l$ is complete and transitive.

- Separability: For each triple $A, B, C \in \mathcal{A}$ such that $A \cap C = B \cap C = \emptyset$,

$$A \succsim_l B \text{ if and only if } A \cup C \succsim_l B \cup C.$$

- Monotonicity: For each pair $A, B \in \mathcal{A}$, $A \subseteq B$ implies $B \succsim_l A$.

- Nondegeneracy: There are $A, B \in \mathcal{A}$ such that $A \succ_l B$.

We gather these standard assumptions in the following definition:

[8]Some study qualitative probability without specifying a set of states (Villegas, 1964; Villegas, 1967). If $\mathcal{A}$ is abstractly taken to be an algebra, then by Stone's Representation Theorem (Birkhoff, 1935; Stone, 1936) no generality is gained, while if $\mathcal{A}$ is abstractly taken to be a σ-algebra, the additional generality is made explicit by the Loomis-Sikorski Representation Theorem (Loomis, 1947; Sikorski, 1960). The former result relies on the Boolean Prime Ideal Theorem, which is weaker than the Axiom of Choice (Halpern and Lévy, 1964). The latter result does not (Buskes, de Pagter, and van Rooij, 2008), nor does the fact that no generality is gained when $\mathcal{A}$ is finite (Birkhoff, 1937).

[9]Savage (1954) is less general, asserting that $\mathcal{A} = 2^S$, while others are more general, allowing $\mathcal{A}$ to be any algebra, or even any Boolean ring (Villegas, 1967) or mosaic (Kopylov, 2007; Abdellaoui and Wakker, 2005). The generalization is motivated by the distinction between "risk" and "ambiguity" (Knight, 1921) illustrated by the Ellsberg paradox (Ellsberg, 1961): the collection of "subjectively risky events" (Epstein and Zhang, 2001) need only be a mosaic.

Definition: A *qualitative probability space* is a likelihood space that satisfies *order*, *separability*, *monotonicity*, and *nondegeneracy*. A *qualitative probability* is the likelihood relation of a qualitative probability space.[10]

Appendix 1 contains some basic lemmas about qualitative probability spaces.

2.2 Quantitative probability

We seek conditions under which our ordinal notion of qualitative probability space coincides with the standard cardinal notion of (quantitative) probability space (Kolmogoroff, 1933). The notions coincide when the likelihood relation is represented by a measure, formalized as follows:

Definition: A function $\mathcal{R} : \mathcal{A} \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a *representation of* $\succsim_l$ if for each pair $A, B \in \mathcal{A}$,

$$A \succsim_l B \text{ if and only if } \mathcal{R}(A) \geq \mathcal{R}(B).$$

In this case we say $\mathcal{R}$ *represents* $\succsim_l$.

Definition: A function $\mu : \mathcal{A} \to [0, 1]$ is a *measure* if
(1) $\mu(S) = 1$, and
(2) for each finite, pairwise disjoint collection $\{A_i\}_{i \in I} \subseteq \mathcal{A}$,

$$\mu(\cup A_i) = \sum \mu(A_i).$$

A measure μ is moreover a *σ-measure* if
(3) for each countably infinite, pairwise disjoint collection $\{A_i\}_{i \in I} \subseteq \mathcal{A}$,

$$\mu(\cup A_i) = \sum \mu(A_i).$$

Let $\mathbb{M}(\mathcal{A}) \subseteq [0, 1]^{\mathcal{A}}$ denote the set of measures and let $\mathbb{M}^\sigma(\mathcal{A}) \subsetneq \mathbb{M}(\mathcal{A})$ denote the set of σ-measures. In special cases where $|S| \leq |\mathbb{N}|$ and $\mathcal{A} = 2^S$, we favor a slight abuse of notation: because each σ-measure is determined by its assignment to singleton events, we accordingly conflate a σ-measure with its restriction to singletons, favoring $\Delta(S) \equiv \{\mu \in [0, 1]^S | \sum \mu(s) = 1\}$ over $\mathbb{M}^\sigma(\mathcal{A})$.

In this article, we seek conditions guaranteeing there is $\mu \in \mathbb{M}^\sigma(\mathcal{A})$ representing $\succsim_l$. This is in contrast to Savage (1954), whose guaranteed representation may belong to $\mathbb{M}(\mathcal{A}) \backslash \mathbb{M}^\sigma(\mathcal{A})$.

2.3 Savage's axiom and infinite divisibility

To guarantee measure representation, Savage (1954) introduces two separate conditions, *fineness* and *tightness*, which are together equivalent to a condition he calls $P6'$:

∘ Fineness-and-tightness ($P6'$): For each pair $A, B \in \mathcal{A}$ such that $A \succ_l B$, there is a finite partition of S, $\{C_i\}_{i \in I}$, such that for each $i \in I$, $A \succ_l B \cup C_i$.

[10] Savage (1954) and others give an equivalent definition using a weaker version of *monotonicity* and a stronger version of *nondegeneracy*.

Savage deduces from this axiom that there is a unique measure representation, and moreover that it satisfies the following property:

Definition: A measure $\mu \in \mathbb{M}(\mathcal{A})$ is *infinitely divisible* if for each $A \in \mathcal{A}$ and each $\lambda \in [0,1]$, there is $B \subseteq A$ such that $\mu(B) = \lambda\mu(A)$.

Theorem S (Savage, 1954):[11] *If $\succsim_l$ is a qualitative probability satisfying fineness-and-tightness, then there is a unique $\mu \in \mathbb{M}(\mathcal{A})$ representing $\succsim_l$. Moreover, μ is infinitely divisible.*

The class of qualitative probabilities satisfying these hypotheses is in one sense large: their representations need not be σ-additive. But in another sense it is narrow: *infinite divisibility* rules out atoms altogether.[12] By contrast, the representations compatible with our hypotheses are all σ-additive, but not all are atomless.

The elegance of *fineness-and-tightness* obscures how one might modify it to allow atoms. To clarify our approach, and its relationship to Savage's, we introduce three conditions that appear unnecessarily strong, but are in fact together equivalent to *fineness-and-tightness* for qualitative probabilities. These conditions involve partitions of the state space into equally likely events (Bernstein, 1917; de Finetti, 1937; Koopman, 1940):

Definition: For each $k \in \mathbb{N}$, a *k-uniform partition* is a partition of S, $\{A_k^1, A_k^2, ..., A_k^k\}$, such that for each pair $i, j \in \{1, 2, ..., k\}$, $A_k^i \sim_l A_k^j$.

∘ Uniform partition continuity: For each $A \in \mathcal{A}$, if for each $k \in \mathbb{N}$, there is k-uniform partition $\{A_k^1, A_k^2, ..., A_k^k\}$ such that $A_k^1 \succsim_l A$, then $A \sim_l \emptyset$.

∘ Uniform partition richness: For each $k \in \mathbb{N}$, there is a k-uniform partition.

∘ Event divisibility: For each pair $A, B \in \mathcal{A}$ such that $A \succ_l B$, there is $B' \subseteq A$ such that $B' \sim_l B$.

Proposition 2.1: *If $\succsim_l$ is a qualitative probability, then uniform partition richness, uniform partition continuity, and event divisibility are together equivalent to fineness-and-tightness.*

The proof is in Appendix 2. Our approach can be understood as the modification of these three conditions to obtain another continuity axiom, another richness axiom, and another divisibility axiom.

[11] If $\mathcal{A}$ is only required to be an algebra and a weaker version of *fineness-and-tightness* is imposed, then a measure representation is still guaranteed (Wakker, 1981), but its range need only be a dense subset of $[0,1]$ (Marinacci, 1993).

[12] Does *infinite divisibility* imply more than that there are no atoms? Under the Axiom of Choice, yes: there is an atomless measure that is not *infinitely divisible* (Nunke and Savage, 1952).

3 Continuity

3.1 Monotone continuity and standard continuity

As argued in Section 1.1, there are strong reasons to impose that the measure representation is σ-additive (Adams, 1962; Seidenfeld and Schervish, 1983; Wakker, 1993; Stinchcombe, 1997), and the appropriate axiom for doing so has been identified (Villegas, 1964).[13] As presented by Arrow (1970):

Definition (Arrow, 1970): An event sequence $(A_i) \in \mathcal{A}^{\mathbb{N}}$ is *vanishing* if (1) $A_1 \supseteq A_2 \supseteq$ ..., and (2) $\cap A_i = \emptyset$.

∘ Monotone continuity (Villegas, 1964):[14] For each pair $A, B \in \mathcal{A}$ and each vanishing $(C_i) \in \mathcal{A}^{\mathbb{N}}$, if for each $i \in \mathbb{N}$, $A \cup C_i \succsim_l B$, then $A \succsim_l B$.

For example, if $S = \mathbb{N}$ and for each $i \in \mathbb{N}$, $A_i = \{i, i+1, i+2, ...\}$, then (A_i) is vanishing. As Arrow writes: "Clearly, an event which is far out on a vanishing sequence is 'small' by any reasonable standard."

To compare *monotone continuity* to a more familiar axiom, consider pointwise convergence, its associated topology (under which $\mathcal{A}$ is closed), and the standard requirement that upper and lower contour sets are closed:

Definition: An event sequence $(A_i) \in \mathcal{A}^{\mathbb{N}}$ is *convergent* if for each $s \in S$, either

(1) there is $i^* \in \mathbb{N}$ such that for each $i \geq i^*$, $s \in A_i$, and

(2) there is $i^* \in \mathbb{N}$ such that for each $i \geq i^*$, $s \notin A_i$.

In this case, we say $\cup_{i=1}^{\infty}(\cap_{j=i}^{\infty} A_j) = \cap_{i=1}^{\infty}(\cup_{j=i}^{\infty} A_j)$ is the *(pointwise) limit of* (A_i), written $\lim A_i$.

∘ Continuity: For each $A \in \mathcal{A}$ and each convergent $(B_i) \in \mathcal{A}^{\mathbb{N}}$,

(1) if for each $i \in \mathbb{N}$, $B_i \succsim_l A$, then $\lim B_i \succsim_l A$, and

(2) if for each $i \in \mathbb{N}$, $A \succsim_l B_i$, then $A \succsim_l \lim B_i$.

Though *continuity* seems stronger than *monotone continuity*, in fact the two are equivalent for qualitative probabilities, a point which to our knowledge has not been made previously:

Proposition 3.1: If $\succsim_l$ is a qualitative probability, then $\succsim_l$ is *monotonely continuous* if and only if $\succsim_l$ is *continuous*.

The proof is in Appendix 3. Based on this equivalence, we are justified in writing *(monotone) continuity* in stating our results while using *continuity* in our proofs, as well as revisiting previous work involving standard *continuity*.

[13] In fact, this is the appropriate axiom even in the multiple priors model (Gilboa and Schmeidler, 1989), guaranteeing that the set of priors is a relatively weak compact set of σ-measures (Chateauneuf, Maccheroni, Marinacci, and Tallon, 2005).

[14] This definition, which is convenient for our approach, is the equivalent contrapositive of the definition in the introduction, whose informal statement we find simpler.

3.2 Quotients and dyadic skeletons

A measure is σ-additive if and only if it is a continuous function when $\mathcal{A}$ has the topology of pointwise convergence.[15] We are thus interested, in part, in conditions guaranteeing a binary relation has a continuous representation. Fortunately, this question is well-studied (see Debreu, 1964), and it is fruitful to revisit a particular answer involving standard *continuity*.

Define the *quotient* $[\mathcal{A}/{\sim_l}] \subseteq 2^{\mathcal{A}}$ to be the family of likelihood equivalence classes:

$$[\mathcal{A}/{\sim_l}] \equiv \{\{B \in \mathcal{A} | B \sim_l A\} | A \in \mathcal{A}\}.$$

A continuous representation can be guaranteed if $[\mathcal{A}/{\sim_l}]$ contains a structured subfamily $\{\mathcal{Z}_i\}_{i\in I}$ about which the representation may be constructed. We propose to suggestively refer to $\{\mathcal{Z}_i\}_{i\in I}$ a "skeleton" (using z as s is already assigned to the state space):

DEFINITION: For each qualitative probability space $(S, \mathcal{A}, \succsim_l)$, a collection of equivalence classes $\{\mathcal{Z}_i\}_{i\in I} \subseteq [\mathcal{A}/{\sim_l}]$ is a *(quotient) skeleton of* $(S, \mathcal{A}, \succsim_l)$ if

(1) $|I| \leq \mathbb{N}$, and

(2) for each pair $A, B \in \mathcal{A}$ such that $A \succ_l B$, there is $Z \in \cup \mathcal{Z}_i$ such that $A \succsim_l Z \succsim_l B$.

THEOREM CD (CANTOR, 1895; DEBREU, 1954; DEBREU, 1964):[16] If $(S, \mathcal{A}, \succsim_l)$ has a quotient skeleton and $\succsim_l$ is *continuous*, then $\succsim_l$ has a continuous representation $\mathcal{R}$.[17]

Guaranteeing a continuous representation that is moreover a measure requires a quotient skeleton with additional structure. Define the *dyadic rationals* $\mathbb{2} \subseteq [0, 1]$ by:

$$\mathbb{2} \equiv \{\sum_{i\in F} (\frac{1}{2})^i | F \subseteq \mathbb{N},\ |F| < |\mathbb{N}|\} \cup \{1\}.$$

DEFINITION: For each qualitative probability space $(S, \mathcal{A}, \succsim_l)$, a collection of equivalence classes $\{\mathcal{Z}_v\}_{v\in\mathbb{2}} \subseteq [\mathcal{A}/{\sim_l}]$ is a *dyadic (quotient) skeleton of* $(S, \mathcal{A}, \succsim_l)$ if

[DS1] $\emptyset \in \mathcal{Z}_0$ and $S \in \mathcal{Z}_1$,

[DS2] for each pair $v, v' \in \mathbb{2}$ such that $v + v' \leq 1$, there are disjoint $Z_v \in \mathcal{Z}_v$ and $Z_{v'} \in \mathcal{Z}_{v'}$ such that $Z_v \cup Z_{v'} \in \mathcal{Z}_{v+v'}$, and

[DS3] for each four monotonic[18] $(v_i), (v'_i), (w_i), (w'_i) \in \mathbb{2}^{\mathbb{N}}$ such that

(i) for each $i \in \mathbb{N}$, $v_i + w_i \leq 1$ and $v'_i + w'_i \leq 1$, and

(ii) $\lim v_i = \lim v'_i$ and $\lim w_i = \lim w'_i$,

there are convergent $(A_i), (A'_i), (B_i), (B'_i) \in \mathcal{A}^{\mathbb{N}}$ such that

(i) for each $i \in \mathbb{N}$, $A_i \in \mathcal{Z}_{v_i}$, $A'_i \in \mathcal{Z}_{v'_i}$, $B_i \in \mathcal{Z}_{w_i}$, and $B'_i \in \mathcal{Z}_{w'_i}$,

[15] Many texts on measure theory prove a version of this result using a weaker notion of continuity. Though we did not find this particular statement elsewhere, we suspect it is well-known; in any case we omit the straightforward proof as it is not essential to our main results.

[16] Cantor first proved the result under the stronger $A \succ_l Z \succ_l B$ assumption (Cantor, 1895). Nearly 60 years later, at the suggestion of Savage, Debreu attempted to prove the result under the weaker $A \succsim_l Z \succsim_l B$ assumption (Debreu, 1954). Debreu then noticed an error in his own proof, which he corrected 10 years later using his acclaimed Gap Lemma (Debreu, 1964).

[17] For narrative clarity, we have chosen a weaker statement than that in Debreu (1964). In fact $\succsim_l$ can be any binary relation and $\mathcal{A}$ need not be a σ-algebra. Moreover, without imposing *continuity*, Debreu concludes that $\mathcal{R}$ is upper semi-continuous in any topology for which upper contour sets are closed and lower semi-continuous in any topology for which lower contour sets are closed.

[18] Since these four sequences are bounded, they are also convergent.

(ii) for each $i \in \mathbb{N}$, $A_i \cap B_i = A'_i \cap B'_i = \emptyset$, and
(iii) $\lim A_i = \lim A'_i$ and $\lim B_i = \lim B'_i$.

PROPOSITION 3.2: If $(S, \mathcal{A}, \succsim_l)$ is a qualitative probability space that has a dyadic skeleton and satisfies *(monotone) continuity*, then there is a unique $\mu \in \mathbb{M}^\sigma(\mathcal{A})$ such that μ represents $\succsim_l$.

The proof is in Appendix 3. In establishing our main result, we do not assume the existence of a dyadic skeleton; we construct one using our axioms. In order for the construction to proceed in pieces, we introduce two similar notions: subspace and coarsening.

DEFINITION: For each likelihood space $(S, \mathcal{A}, \succsim_l)$ and each $S' \in \mathcal{A}$, define the *S'-subspace of* $(S, \mathcal{A}, \succsim_l)$, $(S, \mathcal{A}, \succsim_l)|_{S'}$, to be the likelihood space $(A, \mathcal{A} \cap 2^{S'}, \succsim_l|_{S'})$, where $\succsim_l|_{S'}$ is the binary relation on $\mathcal{A} \cap 2^{S'}$ defined by:

$$B \succsim_l|_{S'} C \text{ if and only if } B, C \in \mathcal{A} \cap 2^{S'} \text{ and } B \succsim_l C.$$

DEFINITION: For each likelihood space $(S, \mathcal{A}, \succsim_l)$ and each σ-algebra $\mathcal{A}' \subseteq \mathcal{A}$ with $\cup\mathcal{A}' = S$, define the *$\mathcal{A}'$-coarsening of* $(S, \mathcal{A}, \succsim_l)$, $(S, \mathcal{A}, \succsim_l)|_{\mathcal{A}'}$, to be the likelihood space $(S, \mathcal{A}', \succsim_l|_{\mathcal{A}'})$, where $\succsim_l|_{\mathcal{A}'}$ is the binary relation on $\mathcal{A}'$ defined by:

$$B \succsim_l|_{\mathcal{A}'} C \text{ if and only if } B, C \in \mathcal{A}' \text{ and } B \succsim_l C.$$

If $(S, \mathcal{A}, \succsim_l)$ is a qualitative probability space, then (i) $S' \succ_l \emptyset$ implies $(S, \mathcal{A}, \succsim_l)|_{S'}$ is a qualitative probability space, and (ii) $\mathcal{A}' \subseteq \mathcal{A}$ is a σ-algebra with $\cup\mathcal{A}' = S$ implies $(S, \mathcal{A}, \succsim_l)|_{\mathcal{A}'}$ is a qualitative probability space. We emphasize that because $\succsim_l|_{S'}$ compares only subevents of S', and because $\succsim_l|_{\mathcal{A}'}$ compares only members of $\mathcal{A}'$, both notions are distinct from conditional qualitative probability (Savage, 1954). We do not use conditional probabilities anywhere in this article, and we postulate nothing about how beliefs change in response to new information.

The following proposition allows us to construct our dyadic skeleton in pieces:

PROPOSITION 3.3: A qualitative probability space $(S, \mathcal{A}, \succsim_l)$ has a dyadic skeleton in each of the following cases:

C1: there is σ-algebra $\mathcal{A}' \subseteq \mathcal{A}$ with $\cup\mathcal{A}' = S$ such that $(S, \mathcal{A}, \succsim_l)|_{\mathcal{A}'}$ is a qualitative probability space with a dyadic skeleton,

C2: there are $S', S'' \in \mathcal{A}$ partitioning S such that $(S, \mathcal{A}, \succsim_l)|_{S'}$ is a qualitative probability space with a dyadic skeleton and $S'' \sim_l \emptyset$, and

C3: there are $S', S'' \in \mathcal{A}$ partitioning S such that $(S, \mathcal{A}, \succsim_l)|_{S'}$ and $(S, \mathcal{A}, \succsim_l)|_{S''}$ are each qualitative probability spaces with dyadic skeletons.

The proof is in Appendix 3.

4 Richness

4.1 Atom-catalogues

Loosely speaking, the dyadic skeleton pieces corresponding to atomless subspaces have already been constructed by previous work, so to complete the skeleton, it remains to construct one final piece for one final space that is not atomless. In general, however, there is not a unique choice for this final space. This section is dedicated to formalizing these ideas. We begin with our notion of an event with no smaller subevent:

DEFINITION: An *atom* is an event $A \in \mathcal{A}$ such that
(1) $A \succ_l \emptyset$, and
(2) for each $B \subseteq A$, either $B \sim_l A$ or $B \sim_l \emptyset$.
We write $\mathcal{A}^{\oplus} \subseteq \mathcal{A}$ for the collection of atoms.

Lemma 4 of Villegas (1964) incorrectly claims that for each *monotonely continuous* qualitative probability, the collection of atoms is at most countably infinite. As a counterexample, consider $S = [0,1]$ and the (*monotonely continuous*) qualitative probability on 2^S for which $\{0\} \sim_l S$. That said, the following correction to that claim involves many of the ideas already present in that work:

DEFINITION: For each index set $I \subseteq \mathbb{N}$, we say a tuple $(A_i)_{i\in I} \in (\mathcal{A}^{\oplus})^I$ is an *atom-catalogue* if
(1) for each pair $i, j \in I$, $A_i \cap A_j = \emptyset$,
(2) for each pair $i, j \in I$ such that $i < j$, $A_i \succsim_l A_j$, and
(3) for each $A \subseteq S \backslash (\cup A_i)$, $A \notin \mathcal{A}^{\oplus}$.

PROPOSITION 4.1: If $\succsim_l$ is a *(monotonely) continuous* qualitative probability, and if $\mathcal{A}^{\oplus} \neq \emptyset$, then there is an atom-catalogue.

The proof is in Appendix 4. Though there may be many atom-catalogues, our approach is to study just one and our selection is not of consequence. Given this atom-catalogue (A_i), we separately construct a dyadic skeleton for two spaces: (1) the atomless subspace $(S, \mathcal{A}, \succsim_l)|_{S\backslash \cup A_i}$, and (2) the coarsening of subspace $(S, \mathcal{A}, \succsim_l)|_{\cup A_i}$ with respect to the σ-algebra $2^{\{A_i\}} \subseteq \mathcal{A}$. The construction of the first has been completed by previous work:

THEOREM V (VILLEGAS, 1964): If $\succsim_l$ is an *atomless* and *(monotonely) continuous* qualitative probability, then it has a unique representation $\mu \in \mathbb{M}^{\sigma}(\mathcal{A})$. Moreoever, μ is *infinitely divisible.*

We thus turn our attention to the second space, which for convenience we refer to as *the coarsened subspace.*

4.2 Richness and atom-standard spaces

Our investigation of the coarsened subspace is greatly simplified through the introduction of an auxiliary state space. Anticipating an implication of our divisibility axiom, we tentatively impose the following richness axiom:

∘ No finite atom-catalogues: If $I \subseteq \mathbb{N}$ and $(A_i)_{i\in I}$ is an atom-catalogue, then $|I| = |\mathbb{N}|$.

Under *no finite atom-catalogues*, it is without loss of generality to assume our index set is $\mathbb{N}$ and our atom-catalogue is $(A_i)_{i\in\mathbb{N}}$. In this case, our coarsened subspace has state space $\cup_{\mathbb{N}} A_i$ and σ-algebra $\{\cup_J A_i | J \subseteq \mathbb{N}\}$. To simplify, we view $\mathbb{N}$ as an auxiliary state space, and we identify each atom A_i with the singleton event $\{i\}$ taken from the auxiliary σ-algebra $2^{\mathbb{N}}$. We propose the following definition to factor these details out of the ensuing discussion:

Definition: We say the triple $(S, \mathcal{A}, \succsim_l)$ is *atom-standard* if
(1) $S = \{1, 2, ...\}$,
(2) $\mathcal{A} = 2^S$,
(3) $\succsim_l$ is a *(monotonely) continuous* qualitative probability,
(4) for each $s \in S$, $\{s\} \succ_l \emptyset$, and
(5) for each pair $s, s' \in S$, $s < s'$ implies $\{s\} \succsim_l \{s'\}$.

The following section, which is the heart of this article, is exclusively concerned with atom-standard spaces.

5 Divisibility

5.1 Bus fares and greedy transforms

To motivate our approach, consider the following situation. You would like to take a bus, and in order to do so, you must pay the fare with exact change. On your person you have various coins in different denominations. Can you make exact change, and if so, how?

One possible approach is to use a so-called "greedy" algorithm: consider your coins, one at a time, in non-increasing order of value. If the coin under consideration puts your running total over the fare, then return it to your pocket; otherwise, place it into the growing pile that you hope will pay your way. Once you have considered every coin, present the pile to the bus driver.

It is easy to see the greedy algorithm can unnecessarily fail, for example if the fare is 30¢ when you have a quarter and three dimes. This failure is an implication of a broader principle: determining whether or not you can make exact change is NP-complete.[19] But there is a particular class of problems for which the greedy algorithm always works: when you have a countably infinite collection of coins whose value tends to zero, your total wealth exceeds the fare, and the value of each coin is less than the value of all smaller coins:

Theorem K (Kakeya 1914, Kakeya 1915):[20] For each $(\mu_i) \in [0, 1]^{\mathbb{N}}$, if
(i) (μ_i) is non-increasing,
(ii) $\lim \mu_i = 0$, and

[19] This is true even if it is given that all coin values and the fare are integers. This is the *subset sum problem*, a special case of the *knapsack problem*. See for example Garey and Johnson (1979).

[20] For narrative clarity, we have chosen a slightly weaker statement. In fact Kakeya observes that for each $(\mu_i) \in \mathbb{R}^{\mathbb{N}}$ such that $\sum |\mu_i|$ is finite, (iii) is necessary and sufficient for $\{\sum_{i\in A} \mu_i | A \subseteq \mathbb{N}\}$ to be convex.

(iii) for each $i \in \mathbb{N}$, $\sum_{j>i} \mu(j) \geq \mu(i)$,
then for each $v \in [0, \sum \mu_i]$, there is $A \subseteq \mathbb{N}$ such that

$$\sum_A \mu(i) = v.$$

Theorem K raises a series of related questions:

- What if coins are not assigned numerical values, but you can compare any two piles of coins, and moreoever the fare is given by a pile of coins?
- What if you have to take a series of buses that may have different fares?
- If a sequence of fares converges to some limit, does the corresponding sequence of coin piles generated by the greedy algorithm converge as well?

We investigate these questions and others, only instead of a collection of coins and a fare, we have a collection of states and a target event.

Formally, for each event A, we define the greedy transform $\mathcal{G}^A : \mathcal{A} \to \mathcal{A}$, which associates each event B with another event $\mathcal{G}^A(B) \subseteq A$. This transform is an adaptation of the *greedy expansion* from the mathematics literature on β-expansions (Rényi, 1957).[21] In the analogy, A is the collection of coins in your wallet while B is the fare:

DEFINITION:[22] If $(S, \mathcal{A}, \succsim_l)$ is atom-standard, then for each pair $A \in \mathcal{A}$, we define the *greedy transform* $\mathcal{G}^A : \mathcal{A} \to \mathcal{A}$ as follows. For each $B \in \mathcal{A}$, define $\mathcal{G}^A(B) \subseteq A$ by:

- Define $\mathcal{G}_0^A(B) \equiv \emptyset$.
- For each $i \in \mathbb{N}$, define

$$\mathcal{G}_i^A(B) \equiv \begin{cases} \mathcal{G}_{i-1}^A(B) \cup \{i\}, & i \in A \text{ and } B \succsim_l \mathcal{G}_{i-1}^A(B) \cup \{i\} \\ \mathcal{G}_{i-1}^A(B), & \text{else.} \end{cases}$$

- Clearly $(G_i^A(B)) \in \mathcal{A}^{\mathbb{N}}$ is convergent. Define

$$\mathcal{G}^A(B) \equiv \lim \mathcal{G}_i^A(B).$$

By construction and by *continuity*, $B \succsim_l \mathcal{G}^A(B)$. Provided $A \succsim_l B$, when is equivalence guaranteed?

5.2 Smaller-atoms domination

In order to discipline the greedy transforms, we introduce a parametric family of divisibility axioms based on Kakeya's observation, with higher-parameter assumptions logically stronger than lower-parameter ones. For each $k \in \mathbb{N}$, we define:

[21] An earlier version of this work used a second transformation based on the *quasi-greedy expansion*, which has interesting properties (de Vries and Komornik, 2009).

[22] To be completely formal, the notation should be $\mathcal{G}^{A|\succsim_l}$ as the definition relies on $\succsim_l$. But since $\succsim_l$ will always be clear from context, we suppress it in the notation.

$\circ$ kTH-ORDER SMALLER-ATOMS DOMINATION (k-SAD): For each $A \in \mathcal{A}^{\oplus}$, there are $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_k \subseteq \{A' \in \mathcal{A}^{\oplus} | A \succ_l A'\}$ such that

(1) for each pair $i, j \in \{1, 2, ..., k\}$, $(\cup \mathcal{A}_i) \cap (\cup \mathcal{A}_j) = \emptyset$, and

(2) for each $i \in \{1, 2, ..., k\}$, $(\cup \mathcal{A}_i) \succsim_l A$.

These axioms have powerful implications for the greedy transforms. For example, returning to the bus fare example, we have the following ordinal analogue of Theorem K: under k-*SAD*, you can make exact change for k consecutive fares, provided the sum of the fares does not exceed what's in your wallet, even if coins are not assigned numerical values. Two lemmas are essential to outlining our approach, and we state them formally here; for the precise statements and proofs of the others, see Appendix 5.

FIRST HALVING LEMMA: If $(S, \mathcal{A}, \succsim)$ is atom-standard, then for each $A \in \mathcal{A}$ such that $A \succ_l \emptyset$ and $\succsim_l|_A$ satisfies 1-*SAD*, there is $H \subseteq A$ such that $H \sim_l A \backslash H$ and $A \succ_l H \succ_l \emptyset$.

SECOND HALVING LEMMA: If $(S, \mathcal{A}, \succsim)$ is atom-standard and $\succsim_l$ satisfies 2-*SAD*, then for each $A \in \mathcal{A}$ such that $A \succ_l \emptyset$, there are disjoint $H(A), H'(A) \in \mathcal{A}$ such that

(1) $A \sim_l H(A) \cup H'(A)$, and

(2) $A \succ_l H(A) \sim_l H'(A) \succ_l \emptyset$.

The Second Halving Lemma states that under 2-*SAD*, we can construct the pairs of equally likely events that are fundamental to the approach outlined by Ramsey (1931) and pursued by others, such as Gul (1992). But it is the proof technique for its own lemma, the First Halving Lemma, that merits comment: by viewing the collection of events ordered lexicographically as the canonical Cantor set (Cantor, 1883), we gain access to order-topological properties of a bounded subset of the real line. In particular, each closed collection of events has a lexicographic maximum, which under 1-*SAD* implies the existence of our desired "half." Informally, it is in this way that we first glimpse cardinality in our ordinal relation.

5.3 The atom skeleton

Our objective is to construct a dyadic skeleton $\{\mathcal{Z}_v\}_{v \in \mathbb{2}}$. As each $\mathcal{Z}_v$ is an equivalence class, it suffices to construct, for each $v \in \mathbb{2}$, a representative event $A_v \in \mathcal{Z}_v$.

Each $v \in \mathbb{2}$ can be written $\frac{p}{2^q}$, where $p, q \in \{0, 1, ...\}$ with $p \leq 2^q$. Our first step is to construct, for each such pair p, q, an event A_q^p. We then go on to show that if $\frac{p}{2^q} = \frac{p'}{2^{q'}} = v$, then $A_q^p = A_{q'}^{p'}$, so that constructing $A_v \equiv A_q^p$ is well-defined.

The construction involves two ingredients: "halving" and "addition by one." First, for each $q \in \{0, 1, ...\}$, define $A_q^0 \equiv \emptyset$. Second, define the $\{A_q^1\}$ recursively:

- Define $A_0^1 \equiv S$.
- For each $q \in \{0, 1, ...\}$, using the Second Halving Lemma, identify disjoint H and H' such that $A_q^1 \sim_l H \cup H'$. Define $A_{q+1}^1 \equiv \mathcal{G}^S(H)$; this, it turns out, is well-defined.

Finally, for each $q \in \{0, 1, ...\}$, define $\{A_q^p\}$ recursively:

- For each $p \in \{0, 1, ..., 2^q - 1\}$, define $A_q^{p+1} \equiv \mathcal{G}^S(A_q^p \cup \mathcal{G}^{S \backslash A_q^p}(A_q^1))$.

This process can be understood as the construction of a sequence of progressively finer rulers (Figure 2).

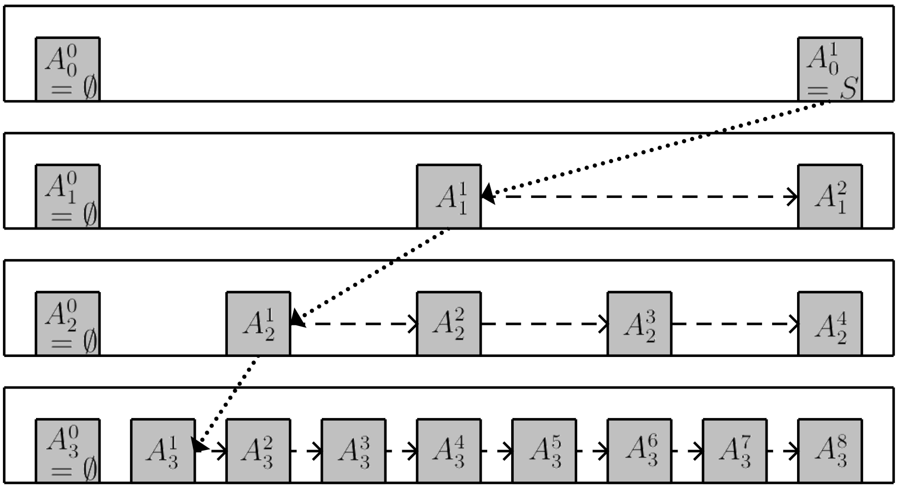

Figure 2: Each row is one member in our sequence of progressively finer rulers. The leftmost point on each is identified by $\emptyset$. The diagonal dotted lines represent construction through "halving." The horizontal dashed lines represent construction through "addition by one." It is then proved that any two events aligned vertically are in fact the same event.

In order to derive cancellation, we generalize "addition by one." For each $q \in \{0, 1, ...\}$ and each pair $p, p' \in \{0, 1, ..., 2^q\}$ such that $p + p' \leq 2^q$, define $A_q^p \biguplus A_q^{p'} \in \mathcal{A}$ by:

$$A_q^p \biguplus A_q^{p'} \equiv \mathcal{G}^S\big(A_q^p \cup \mathcal{G}^{S\setminus A_q^p}(A_q^{p'})\big).$$

To prove that this binary operation is in fact addition—that is, $A_q^p \biguplus A_q^{p'} = A_q^{p+p'}$—we follow Peano's axiomatization of addition (Peano, 1889); the essential step is establishing that $A_q^{p+p'} = A_q^p \biguplus A_q^{p'}$ implies $A_q^{p+(p'+1)} = A_q^p \biguplus A_q^{p'+1}$, for which we use 3-$SAD$.

After verifying that $\{\mathcal{Z}_v\}_{v \in \mathbb{2}}$ satisfies all the requirements of a dyadic skeleton, we establish:

Proposition 5.1: If $(S, \mathcal{A}, \succsim_l)$ is atom-standard and satisfies 3-SAD, then $(S, \mathcal{A}, \succsim_l)$ has a dyadic skeleton.

The proof, which is the central proof of this article, is in Appendix 6.

6 Main result

In this section, we state and prove the primary theorem of our article. The proof relies on several key results described in the body of this article, but visiting the appendix should not be necessary.

Theorem 1: A likelihood relation $\succsim_l$ is a qualitative probability satisfying *(monotone) continuity* and 3-SAD if and only if it has a representation $\mu \in \mathbb{M}^\sigma(\mathcal{A})$ such that for each $A \in \mathcal{A}$,

(i) if A is an atom, $\mu(\cup\{A' \in \mathcal{A}^{\oplus} | A \succ_l A'\}) \geq 3\mu(A) > 0$, and

(ii) if A contains no atoms, for each $\lambda \in [0,1]$, there is $B \subseteq A$ such that $\mu(B) = \lambda\mu(A)$.
Moreover, $\mu \in \mathbb{M}^{\sigma}(\mathcal{A})$ is unique and S is infinite.

PROOF: It is trivial that our axioms are necessary for such a representation.

If $\mathcal{A}^{\oplus} = \emptyset$, then we are done by Theorem V, so assume $\mathcal{A}^{\oplus} \neq \emptyset$. Then by Proposition 4.1, there is an atom-catalogue $(A_i)_{i \in I}$. Define $S' \equiv \cup A_i$ and $\mathcal{A}' \equiv \mathcal{A} \cap 2^{S'}$. Then the subspace $(S, \mathcal{A}, \succsim_l)|_{S'} = (S', \mathcal{A}', \succsim_l|_{S'})$ is a qualitative probability space; by the definition of atom-catalogue and by *separability*, it satisfies 3-*SAD*. By 1-*SAD*, $|I| = \mathbb{N}$; without loss of generality, $I = \mathbb{N}$. Define $\mathcal{A}^* \equiv 2^{\{A_i\}}$. Using $\mathbb{N}$ as an auxiliary state space, the coarsening $(S^{\oplus}, \mathcal{A}^{\oplus}, \succsim_l|_{S'})|_{\mathcal{A}^*}$ is an atom-standard qualitative probability space; by the definition of atom-catalogue, *separability*, and *continuity*, it satisfies 3-*SAD*. Thus by Proposition 5.1, this coarsening has a dyadic skeleton, so by Proposition 3.3 C1, $(S', \mathcal{A}', \succsim_l |_{S'})$ has a dyadic skeleton.

If $S \backslash S' \sim_l \emptyset$, then by Proposition 3.3 C2, $(S, \mathcal{A}, \succsim_l)$ has a dyadic skeleton. If $S \backslash S' \succ_l \emptyset$, then by the definition of atom-catalogue and Theorem V, $(S, \mathcal{A}, \succsim_l)|_{S \backslash S'}$ has a dyadic skeleton, so by Proposition 3.3 C3, $(S, \mathcal{A}, \succsim_l)$ has a dyadic skeleton.

Since $(S, \mathcal{A}, \succsim_l)$ has a dyadic skeleton, by Proposition 3.2, there is a unique $\mu \in \mathbb{M}^{\sigma}(\mathcal{A})$ such that μ represents $\succsim_l$. Conclusion (i) follows from 3-*SAD*, while Conclusion (ii) follows from Theorem V applied to $(S, \mathcal{A}, \succsim_l)|_{S \backslash S'}$. ■

7 Conclusion

We began by observing that Savage's *finite-and-tightness* is equivalent to *uniform partition continuity*, *uniform partition richness*, and *event divisibility*. We modified these, respectively, to *monotone continuity*, *no finite atom-catalogues* (which is implied by the other axioms), and 3-*SAD*, hypotheses that generalize those of Villegas as 3-*SAD* is weaker than *atomlessness*.

An open question is whether 3-*SAD* can be weakened to 1-*SAD* or 2-*SAD*. Some steps have been taken in this direction (Mackenzie, working paper A). In particular, *monotone continuity*, 1-*SAD*, and *finite cancellation* (Kraft, Pratt, and Seidenberg, 1959; Scott, 1964; Fishburn, 1970; Krantz, Luce, Suppes, and Tversky, 1971) imply that there is a σ-measure representation, which is moreover unique under 2-*SAD*. This article then completes the progression: 3-*SAD* allows us to relax *finite cancellation*. That said, it is not even known if 2-*SAD* is required for uniqueness, and there is some evidence it is not: a corollary of Kochov (2013) is that a geometric representation is unique under 1-*SAD*.

The natural next step is embedding these hypotheses about beliefs into Savage's grander model of decision under uncertainty, a topic we pursue in the sequel (Mackenzie, working paper B). We pursue both standard subjective expected utility and the more general *probabilistically sophisticated* preferences that were introduced by Machina and Schmeidler (1992) to address the Allais paradox (Allais, 1953; Allais, 1979).

Appendix 1

In this appendix, we state and prove (or provide a proof reference for) our primary lemmas about qualitative probabilities: the Complement Lemma, the Domination Lemma, the Half-Equivalence Lemma, and the Limit-Order Lemma.

The Complement Lemma states that order reverses under complements. This is a slight extension of exercise 3 on page 32 of Savage (1972), and thus the proof is omitted.

• Complement Lemma (Savage, 1954): If $\succsim_l$ is a qualitative probability, then for each $A \in \mathcal{A}$ and each pair $B, B' \subseteq A$, $B \succsim_l B'$ if and only if $A \backslash B' \succsim_l A \backslash B$.

The Domination Lemma states that for any two pairs such that the first is disjoint and dominates the second in likelihood, the union of the first is at least as likely as the union of the second. Moreover, strict domination implies the union of the first is more likely than the union of the second. This is a slight extension of exercise 5a on page 32 of Savage (1972), and thus the proof is omitted.

• Domination Lemma (Savage, 1954): If $\succsim_l$ is a qualitative probability, then for each four $A, A', B, B' \in \mathcal{A}$, if
(1) $A \cap A' = \emptyset$,
(2) $A \succsim_l B$, and
(3) $A' \succsim_l B'$,
then $A \cup A' \succsim_l B \cup B'$. Moreover, if $A \succ_l B$, then $A \cup A' \succ_l B \cup B'$.

The Half-Equivalence Lemma states that for any two disjoint pairs whose unions are equally likely such that each pair's members are equally likely, all four events are equally likely.

• Half-Equivalence Lemma: If $\succsim_l$ is a qualitative probability, then for each disjoint pair $A, A' \in \mathcal{A}$ and each disjoint pair $B, B' \in \mathcal{A}$, if
(1) $A \sim_l A'$,
(2) $B \sim_l B'$, and
(3) $A \cup A' \sim_l B \cup B'$,
then $A \sim_l B$.

Proof: For convenience, label the components of the Euler diagram for A, A', B, B' according to Figure 3:

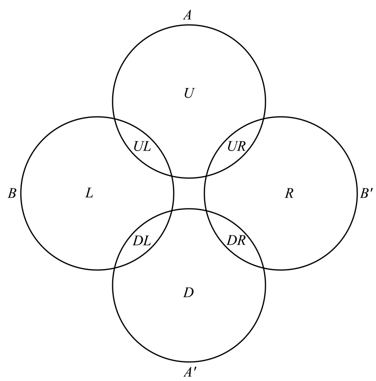

Figure 3: Euler diagram for A, B, A', B'. For example, $UL \equiv A \cap B$ and $D \equiv A' \backslash (B \cup B')$.

Assume, by way of contradiction, $A \not\sim_l B$; without loss of generality, assume $A \succ_l B$.

We claim $DL \cup D \succ_l UR \cup R$. Otherwise, by *separability*,

$$\begin{aligned} B &\sim_l B' \\ &= DR \cup (UR \cup R) \\ &\succsim_l DR \cup (DL \cup D) \\ &= A' \\ &\sim_l A, \end{aligned}$$

contradicting $A \succ_l B$.

We claim $L \cup R \succsim_l U \cup D$. Otherwise, by *separability*,

$$\begin{aligned} A \cup A' &= (UL \cup UR \cup DL \cup DR) \cup (U \cup D) \\ &\succ_l (UL \cup UR \cup DL \cup DR) \cup (L \cup R) \\ &= B \cup B', \end{aligned}$$

contradicting $A \cup A' \sim_l B \cup B'$. Similarly, $U \cup D \succsim_l L \cup R$, so $L \cup R \sim_l U \cup D$.

But then by *separability*,

$$\begin{aligned} (L \cup R) \cup (UL \cup UR) &\sim_l (U \cup D) \cup (UL \cup UR) \\ &= A \cup D \\ &\succ_l B \cup D \\ &= (UL \cup L) \cup (DL \cup D) \\ &\succ_l (UL \cup L) \cup (UR \cup R), \end{aligned}$$

contradicting $L \cup R \cup UL \cup UR \sim_l L \cup R \cup UL \cup UR$. ■

The Limit-Order Lemma states that for each pair of convergent sequences, if each member of the first sequence is at least the corresponding member of the second, then the limit of the first sequence is at least the limit of the second:

• Limit-Order Lemma:[23] If $\succsim_l$ satisfies *order* and *continuity*, then for each pair of convergent sequences (A_i), $(B_i) \in \mathcal{A}^{\mathbb{N}}$ such that for each $j \in \mathbb{N}$, $A_j \succsim_l B_j$, we have $\lim(A_i) \succsim_l \lim(B_i)$.

Proof: Let $(A_i), (B_i) \in \mathcal{A}^{\mathbb{N}}$ satisfy the hypothesis. Since $\succsim_l$ is complete, by a standard argument[24] there is $M \subseteq \mathbb{N}$ such that that $(B'_i) \equiv (B_i)|_M$ is a $\succsim_l$-monotonic sequence. Define $(A'_i) \equiv (A_i)|_M$. Necessarily $\lim(A_i) = \lim(A'_i)$ and $\lim(B_i) = \lim(B'_i)$.

Case 1: (B'_i) is non-decreasing. Then for each pair $j, k \in \mathbb{N}$ with $k \geq j$,

$$\begin{aligned} A'_k &\succsim_l B'_k \\ &\succsim_l B'_j, \end{aligned}$$

so by *continuity* $\lim(A'_i) \succsim_l B'_j$. Thus by *continuity*, $\lim(A'_i) \succsim_l \lim(B'_i)$, so $\lim(A_i) \succsim_l \lim(B_i)$.

[23] Variants of this result appear in Villegas (1964) and Arrow (1970). This particular result does not require $\mathcal{A}$ to be a σ-algebra; any Hausdorff space will do, as can be seen from the proof.

[24] A common proof of the Bolzano-Weierstrass Theorem includes a lemma stating that each real sequence has a monotonic subsequence; the standard proof of that lemma suffices here.

CASE 2: (B'_i) is non-increasing. Then for each pair $j, k \in \mathbb{N}$ with $k \geq j$,

$$\begin{aligned} A'_j &\succsim_{\mathcal{H}} B'_j \\ &\succsim_{\mathcal{H}} B'_k, \end{aligned}$$

so by *continuity* $A'_j \succsim_l \lim(B'_i)$. Thus by *continuity*, $\lim(A'_i) \succsim_l \lim(B'_i)$, so $\lim(A_i) \succsim_l \lim(B_i)$. ■

Appendix 2

In this appendix, we prove Proposition 2.1.

• PROPOSITION 2.1: If $\succsim_l$ is a qualitative probability, then *uniform partition richness*, *uniform partition continuity*, and *event divisibility* are together equivalent to *fineness-and-tightness*.

PROOF: That any qualitative probability satisfying *fineness-and-tightness* necessarily satisfies the other axioms is an immediate corollary of Theorem S.

Suppose $\succsim_l$ is a qualitative probability satisfying *uniform partition richness*, *uniform partition continuity*, and *event divisibility*, and let $A, B \in \mathcal{A}$ such that $A \succ_l B$. By *event divisibility*, there is $B' \subseteq A$ such that $A \succ_l B \sim_l B'$. Then $(A \backslash B') \succ_l \emptyset$, else by *separability* $B' \succsim_l B' \cup (A \backslash B') = A$, contradicting $A \succ_l B'$.

By *uniform partition richness*, for each $k \in \mathbb{N}$, there is a k-uniform partition $\{A_k^1, A_k^2, ..., A_k^k\}$. If for each $k \in \mathbb{N}$, $A_k^1 \succsim_l (A \backslash B')$, then by *uniform partition continuity*, $(A \backslash B') \sim_l \emptyset$, contradicting $(A \backslash B') \succ_l \emptyset$. Thus there is $k \in \mathbb{N}$ such that for each $i \in \{1, 2, ..., k\}$, $(A \backslash B') \succ_l A_k^i$. By *monotonicity*, $(S \backslash B') \succsim_l (A \backslash B') \succ_l A_k^1$, so by the Complement Lemma, $(S \backslash B) \sim_l (S \backslash B') \succ_l A_k^1$. By *event divisibility*, there is $C \subseteq (S \backslash B)$ such that $C \sim_l A_k^1$.

Since $B' \succsim_l B$, $(A \backslash B') \succ_l A_k^1 \sim_l C$, and $B' \cap (A \backslash B') = \emptyset$, by the Domination Lemma, $A = B' \cap (A \backslash B') \succ_l B \cup C$. For each $i \in \{1, 2, ..., k\}$, since $B \succsim_l B$, $C \succsim_l A_k^i$, and $B \cap C = \emptyset$, by the Domination Lemma, $B \cup C \succsim_l B \cup A_k^i$. Thus for each $i \in \{1, 2, ..., k\}$, $A \succ_l B \cup A_k^i$, as desired. ■

Appendix 3

In this appendix, we prove Proposition 3.1, Proposition 3.2, and Proposition 3.3.

• PROPOSITION 3.1: If $\succsim_l$ is a qualitative probability, then $\succsim_l$ is *monotonely continuous* if and only if $\succsim_l$ is *continuous*.

PROOF: Clearly *continuity* implies *monotone continuity*, so suppose $\succsim_l$ is a qualitative probability satisfying *monotone continuity*.

Let $A \in \mathcal{A}$ and let $(B_i) \in \mathcal{A}^{\mathbb{N}}$ be convergent such that for each $i \in \mathbb{N}$, $B_i \succsim_l A$. Define $B_\infty \equiv \lim B_i$. For each $i \in \mathbb{N}$, define $C_i \in \mathcal{A}$ by:

$$C_i \equiv \bigcup_{j=i}^{\infty} (B_j \backslash B_\infty).$$

For each $s \in B_\infty$ and each $i \in \mathbb{N}$, $s \notin C_i$. For each $s \in S \backslash B_\infty$, since $B_\infty = \lim B_i$, there is $i^* \in \mathbb{N}$ such that for each $i \geq i^*$, $s \notin B_i$; thus for each $i \geq i^*$, $s \notin C_i$. Thus $\lim C_i = \emptyset$, and clearly $C_1 \subseteq C_2 \subseteq ...$, so (C_i) is vanishing. For each $i \in \mathbb{N}$, $B_i \subseteq B_\infty \cup C_i$, so by *monotonicity* $B_\infty \cup C_i \succsim_l B_i \succsim_l A$. Thus by *monotone continuity*, $B_\infty \succsim_l A$.

Let $A \in \mathcal{A}$ and let $(B_i) \in \mathcal{A}^{\mathbb{N}}$ be convergent such that for each $i \in \mathbb{N}$, $A \succsim_l B_i$. Define $B_\infty = \lim B_i$. Then $B_\infty^c = \lim B_i^c$. By *separability*, for each $i \in \mathbb{N}$, $B_i^c \succsim_l A^c$. Thus by the above argument, $B_\infty^c \succsim_l A^c$, so by *separability*, $A \succsim_l B_\infty$. ■

• PROPOSITION 3.2: If $(S, \mathcal{A}, \succsim_l)$ is a qualitative probability space that has a dyadic skeleton and satisfies *(monotone) continuity*, then there is a unique $\mu \in \mathbb{M}^\sigma(\mathcal{A})$ such that μ represents $\succsim_l$.

PROOF: Let $\{\mathcal{Z}_v\}_{v \in \mathbb{2}} \subseteq [\mathcal{A}/\sim_l]$ be a dyadic skeleton. We sometimes apply [DS3] using only two sequences in $\mathbb{2}^{\mathbb{N}}$, implicitly taking the other two to be constantly 0.

∘ STEP 1: For each $k \in \{0, 1, ...\}$, there is $Z \in \mathcal{Z}_{(\frac{1}{2})^k}$ such that $Z \succ_l \emptyset$.

We proceed by induction on k. By [DS1], $S \in \mathcal{Z}_1$, and by *nondegeneracy* and *monotonicity*, $S \succ_l \emptyset$.

For the inductive hypothesis, assume $k \in \{0, 1, ...\}$ is such that there is $Z \in \mathcal{Z}_{(\frac{1}{2})^k}$ such that $Z \succ_l \emptyset$. By [DS2], there are disjoint $Z', Z'' \in \mathcal{Z}_{(\frac{1}{2})^{k+1}}$ such that $Z' \cup Z'' \sim_l Z$. By *separability*, $Z' \succ_l \emptyset$, else $\emptyset \succsim_l Z' \cup Z'' \sim_l Z$, contradicting $Z \succ_l \emptyset$. By induction we are done. □

∘ STEP 2: For each $v \in \mathbb{2}$ such that $v > 0$, there is $Z \in \mathcal{Z}_v$ such that $Z \succ_l \emptyset$.

Let $v \in \mathbb{2}$ such that $v > 0$. Since $\mathbb{2}$ is dense in $[0, 1]$ there is $k \in \mathbb{2}$ such that $v > (\frac{1}{2})^k$. Since $v - (\frac{1}{2})^k \in \mathbb{2}$, by [DS2] there are disjoint $Z \in \mathcal{Z}_{(\frac{1}{2})^k}$ and $Z' \in \mathcal{Z}_{v-(\frac{1}{2})^k}$ such that $Z \cup Z' \in \mathcal{Z}_v$. By Step 1, $Z \succ_l \emptyset$, and by *monotonicity*, $Z' \succsim_l \emptyset$, so by *separability*, $Z \cup Z' \succ_l \emptyset$. □

∘ STEP 3: For each pair $v, v^* \in \mathbb{2}$ such that $v^* > v$, each $A \in \mathcal{Z}_{v^*}$, and each $B \in \mathcal{Z}_v$, $A \succ_l B$.

Let $v, v^* \in \mathbb{2}$ such that $v^* > v$, let $A \in \mathcal{Z}_{v^*}$, and let $B \in \mathcal{Z}_v$. Since $v^* - v \in \mathbb{2}$, by [DS2] there are disjoint $Z \in \mathcal{Z}_v$ and $Z' \in \mathcal{Z}_{v^*-v}$ such that $Z \cup Z' \in \mathcal{Z}_{v^*}$. Since $v^* - v > 0$ (and since $\mathcal{Z}_{v^*-v} \in [\mathcal{A}/\sim_l]$), by Step 2, $Z' \succ_l \emptyset$. Thus by *separability*, $A \sim_l Z \cup Z' \succ_l Z \sim_l B$. □

∘ STEP 4: Define $\mu : \mathcal{A} \to [0, 1]$.

For each $v \in \mathbb{2}$ and each $A \in \mathcal{Z}_v$, define $\mu(A) \equiv v$.

Let $A \in \mathcal{A} \backslash \cup_{v \in \mathbb{2}} \mathcal{Z}_v$. Define

$$\begin{aligned} \mathbb{2}^- &\equiv \{v \in \mathbb{2} | B \in Z_v \text{ implies } A \succsim_l B\}, \text{ and} \\ \mathbb{2}^+ &\equiv \{v \in \mathbb{2} | B \in Z_v \text{ implies } B \succsim_l A\}. \end{aligned}$$

By *monotonicity*, $S \succsim_l A \succsim_l \emptyset$, so by [DS1], $0 \in \mathbb{2}^-$ and $1 \in \mathbb{2}^+$. By Step 3, for each pair $v, v^* \in \mathbb{2}$ such that $v^* > v$, (1) $v^* \in \mathbb{2}^-$ implies $v \in \mathbb{2}^- \backslash \mathbb{2}^+$, and (2) $v \in \mathbb{2}^+$ implies

$v^* \in \mathbb{2}^+ \backslash \mathbb{2}^-$. Thus $\inf(\mathbb{2}^+), \sup(\mathbb{2}^-)$ are defined, and moreover $\inf(\mathbb{2}^+) \geq \sup(\mathbb{2}^-)$. Since $\mathbb{2}^- \cup \mathbb{2}^+ = \mathbb{2}$ and $\mathbb{2}$ is dense in $[0,1]$, $\inf(\mathbb{2}^+) = \sup(\mathbb{2}^-)$. Define

$$\begin{aligned} \mu(A) &\equiv \inf(\mathbb{2}^+) \\ &= \sup(\mathbb{2}^-). \ \square \end{aligned}$$

∘ Step 5: For each pair $A, B \in \mathcal{A}$, $\mu(A) > \mu(B)$ implies $A \succ_l B$.

Let $A, B \in \mathcal{A}$ such that $\mu(A) > \mu(B)$. Since $\mathbb{2}$ is dense in $[0,1]$, there are $v', v'' \in \mathbb{2}$ such that $\mu(A) > v'' > v' > \mu(B)$. Let $Z_{v'} \in \mathcal{Z}_{v'}$ and $Z_{v''} \in \mathcal{Z}_{v''}$. By Step 3, $Z_{v''} \succ_l Z_{v'}$.

If $A \in \cup \mathcal{Z}_v$, then by construction, $A \in \mathcal{Z}_{\mu(A)}$, so by Step 3, $A \succ_l Z_{v''}$. If $A \notin \cup \mathcal{Z}_v$, then by construction, $\inf\{v \in \mathbb{2} | C \in \mathcal{Z}_v \text{ implies } C \succsim_l A\} > v''$, so $A \succ_l Z_{v''}$.

If $B \in \cup \mathcal{Z}_v$, then by construction, $B \in \mathcal{Z}_{\mu(B)}$, so by Step 3, $Z_{v'} \succ_l B$. If $B \notin \cup \mathcal{Z}_v$, then by construction, $v' > \sup\{v \in \mathbb{2} | C \in \mathcal{Z}_v \text{ implies } B \succsim_l C\}$, so $Z_{v'} \succ_l B$.

Thus $A \succ_l Z_{v''} \succ_l Z_{v'} \succ_l B$. $\square$

∘ Step 6: For each pair $A, B \in \mathcal{A}$, $\mu(A) = \mu(B)$ implies $A \sim_l B$.

Let $A, B \in \mathcal{A}$ such that $\mu(A) = \mu(B)$. We proceed with three cases whose proofs are similar.

Case 1: $\mu(A) = 0$. Since $\mathbb{2}$ is dense in $[0,1]$, there is decreasing $(v_i^+) \in \mathbb{2}^{\mathbb{N}}$ such that $\lim v_i^+ = 0$. For each $i \in \mathbb{N}$, define $v_i^- \equiv 0$. Since $\lim v_i^+ = \lim v_i^-$, by [DS3] there are convergent $(A_i^+), (A_i^-) \in \mathcal{A}^{\mathbb{N}}$ such that

(i) for each $i \in \mathbb{N}$, $A_i^+ \in \mathcal{Z}_{v_i^+}$ and $A_i^- \in \mathcal{Z}_{v_i^-}$, and

(ii) $\lim A_i^+ \sim_l \lim A_i^-$.

By construction, for each $i \in \mathbb{N}$, $\mu(A_i^+) > 0 = \mu(A)$, so by Step 5, $A_i^+ \succ_l A$. Thus by *continuity*, $\lim A_i^+ \succsim_l A$. By [DS1], for each $i \in \mathbb{N}$, $A_i^- \sim_l \emptyset$, so by *continuity*, $\lim A_i^- \sim_l \emptyset$. Altogether, $\emptyset \succsim_l A$, so by *monotonicity*, $A \sim_l \emptyset$. By a symmetric argument, $B \sim_l \emptyset$, so $A \sim_l B$.

Case 2: $\mu(A) = 1$. Since $\mathbb{2}$ is dense in $[0,1]$, there is increasing $(v_i^-) \in \mathbb{2}^{\mathbb{N}}$ such that $\lim v_i^- = 1$. For each $i \in \mathbb{N}$, define $v_i^+ \equiv 1$. Since $\lim v_i^+ = \lim v_i^-$, by [DS3] there are convergent $(A_i^+), (A_i^-) \in \mathcal{A}^{\mathbb{N}}$ such that

(i) for each $i \in \mathbb{N}$, $A_i^+ \in \mathcal{Z}_{v_i^+}$ and $A_i^- \in \mathcal{Z}_{v_i^-}$, and

(ii) $\lim A_i^+ \sim_l \lim A_i^-$.

By construction, for each $i \in \mathbb{N}$, $\mu(A) = 1 > \mu(A_i^-)$, so by Step 5, $A \succ_l A_i^-$. Thus by *continuity*, $A \succsim_l \lim A_i^-$. By [DS1], for each $i \in \mathbb{N}$, $A_i^+ \sim_l S$, so by *continuity*, $\lim A_i^+ \sim_l S$. Altogether, $A \succsim_l S$, so by *monotonicity*, $A \sim_l S$. By a symmetric argument, $B \sim_l S$, so $A \sim_l B$.

Case 3: $\mu(A) \in (0,1)$. Since $\mathbb{2}$ is dense in $[0,1]$, there are decreasing $(v_i^+) \in \mathbb{2}^{\mathbb{N}}$ such that $\lim v_i^+ = \mu(A)$ and increasing $(v_i^-) \in \mathbb{2}^{\mathbb{N}}$ such that $\lim v_i^- = \mu(A)$. Since $\lim v_i^+ = \lim v_i^-$, by [DS3] there are convergent $(A_i^+), (A_i^-) \in \mathcal{A}^{\mathbb{N}}$ such that

(i) for each $i \in \mathbb{N}$, $A_i^+ \in \mathcal{Z}_{v_i^+}$ and $A_i^- \in \mathcal{Z}_{v_i^-}$, and

(ii) $\lim A_i^+ \sim_l \lim A_i^-$.

By construction, for each $i \in \mathbb{N}$, $\mu(A_i^+) > \mu(A) > \mu(A_i^-)$, so by Step 5, $A_i^+ \succ_l A \succ_l A_i^-$. Thus by *continuity*, $\lim A_i^+ \succsim_l A \succsim_l \lim A_i^- \sim_l \lim A_i^+$, so $A \sim_l \lim A_i^+$. By a symmetric argument, $B \sim_l \lim A_i^+$, so $A \sim_l B$. □

∘ STEP 7: μ represents $\succsim_l$.

Immediate from Step 5 and Step 6. □

∘ STEP 8: For each disjoint pair $A, B \in \mathcal{A}$, $\mu(A) + \mu(B) \leq 1$.

Let $A, B \in \mathcal{A}$ be disjoint. Assume, by way of contradiction, $\mu(A) + \mu(B) > 1$. Since $\mathbb{2}$ is dense in $[0,1]$, there are $v, v' \in \mathbb{2}$ such that $\mu(A) > v$, $\mu(B) > v'$, and $v + v' > 1$. Then $v > 1 - v'$. By [DS2], there are disjoint $Z_{v'} \in \mathcal{Z}_{v'}$ and $Z_{1-v'} \in \mathcal{Z}_{1-v'}$ such that $Z_{v'} \cup Z_{1-v'} \in \mathcal{Z}_1$. Then $Z_{v'} \cup Z_{1-v'} \sim_l S$. Since $\mu(A) > v > 1 - v' = \mu(Z_{1-v'})$ and $\mu(B) > v' = \mu(Z_{v'})$, by Step 7, $A \succ_l Z_{1-v'}$ and $B \succ_l Z_{v'}$. But then by the Domination Lemma, $A \cup B \succ_l Z_{1-v'} \cup Z_{v'} \sim_l S$, contradicting *monotonicity*. □

∘ STEP 9: For each disjoint pair $A, B \in \cup\mathcal{Z}_v$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Let $A, B \in \cup\mathcal{Z}_v$ be disjoint. By construction, $A \in \mathcal{Z}_{\mu(A)}$ and $B \in \mathcal{Z}_{\mu(B)}$, and by Step 8, $\mu(A) + \mu(B) \leq 1$, so by [DS2] and the Domination Lemma, $A \cup B \in \mathcal{Z}_{\mu(A)+\mu(B)}$. Thus by construction, $\mu(A \cup B) = \mu(A) + \mu(B)$. □

∘ STEP 10: For each disjoint pair $A, B \in \mathcal{A}$, $\mu(A \cup B) = \mu(A) + \mu(B)$.

Let $A, B \in \mathcal{A}$ be disjoint. By Step 8, $\mu(A) + \mu(B) \leq 1$. We proceed with three cases.

CASE 1: $\mu(A) + \mu(B) = 0$. Then necessarily $\mu(A) = 0$ and $\mu(B) = 0$, so by Step 7, $A \sim_l \emptyset$ and $B \sim_l \emptyset$. By *separability*, $A \cup B \sim_l \emptyset$, so by Step 7, $\mu(A \cup B) = \mu(\emptyset) = 0 = \mu(A) + \mu(B)$.

CASE 2: $\mu(A)+\mu(B) = 1$. Since $\mathbb{2}$ is dense in $[0,1]$, there are monotonic $(v_i^-), (w_i^-) \in \mathbb{2}^{\mathbb{N}}$ such that
(1) for each $i \in \mathbb{N}$, $\mu(A) \geq v_i^-$ and $\mu(B) \geq w_i^-$, and
(2) $\lim v_i^- = \mu(A)$ and $\lim w_i^- = \mu(B)$.
By [DS3], there are convergent $(A_i^-), (B_i^-) \in \mathcal{A}^{\mathbb{N}}$ such that
(1) for each $i \in \mathbb{N}$, $A_i^- \in \mathcal{Z}_{v_i^-}$ and $B_i^- \in \mathcal{Z}_{w_i^-}$, and
(2) for each $i \in \mathbb{N}$, $A_i^- \cap B_i^- = \emptyset$.
Define $A_\infty \equiv \lim A_i^-$ and $B_\infty \equiv \lim B_i^-$.

Let $\epsilon > 0$. Since $\lim v_i^- = \mu(A)$ and $\lim w_i^- = \mu(B)$, there is $i^* \in \mathbb{N}$ such that $i \geq i^*$ implies $\mu(A) - v_i^- < \frac{\epsilon}{2}$ and $\mu(B) - w_i^- < \frac{\epsilon}{2}$. By construction, $i \geq i^*$ implies $\mu(A_i^-) + \mu(B_i^-) = v_i^- + w_i^- > \mu(A) + \mu(B) - \epsilon = 1 - \epsilon$. By Step 9, $i \geq i^*$ implies $\mu(A_i^- \cup B_i^-) > 1 - \epsilon$.

Thus for each $\epsilon > 0$, there is $i \in \mathbb{N}$ such that $\mu(A_i^- \cup B_i^-) > 1 - \epsilon$. Since for each $i \in \mathbb{N}$, $A_i^- \cap B_i^- = \emptyset$, thus $A_\infty \cap B_\infty = \emptyset$. By the Domination Lemma, for each $i \in \mathbb{N}$, $A_\infty \cup B_\infty \succsim_l A_i^- \cup B_i^-$, so by Step 7, $\mu(A_\infty \cup B_\infty) \geq \mu(A_i^- \cup B_i^-)$. Thus $\mu(A_\infty \cup B_\infty) = 1$, so by Step 7, $A_\infty \cup B_\infty \sim_l S$.

By construction and by Step 7, for each $i \in \mathbb{N}$, $A \succsim_l A_i^-$ and $B \succsim_l B_i^-$, so by *continuity*, $A \succsim_l A_\infty$ and $B \succsim_l B_\infty$. By the Domination Lemma, $A \cup B \succsim_l A_\infty \cup B_\infty \sim_l S$, so by *monotonicity*, $A \cup B \sim_l S$. Thus by Step 7, $\mu(A \cup B) = 1 = \mu(A) + \mu(B)$.

CASE 3: $\mu(A) + \mu(B) \in (0, 1)$. Since $\mathbb{2}$ is dense in $[0, 1]$, there are monotonic $(v_i^+), (v_i^-), (w_i^+), (w_i^-) \in \mathbb{2}^{\mathbb{N}}$ such that

(1) for each $i \in \mathbb{N}$, $v_i^+ \geq \mu(A) \geq v_i^-$ and $w_i^+ \geq \mu(B) \geq w_i^-$,

(2) for each $i \in \mathbb{N}$, $v_i^+ + w_i^+ \leq 1$, and

(3) $\lim v_i^+ = \lim v_i^- = \mu(A)$ and $\lim w_i^+ = \lim w_i^- = \mu(B)$.

By [DS3], there are convergent $(A_i^+), (A_i^-), (B_i^+), (B_i^-) \in \mathcal{A}^{\mathbb{N}}$ such that

(1) for each $i \in \mathbb{N}$, $A_i^+ \in \mathcal{Z}_{v_i^+}$, $A_i^- \in \mathcal{Z}_{v_i^-}$, $B_i^+ \in \mathcal{Z}_{w_i^+}$, and $B_i^- \in \mathcal{Z}_{w_i^-}$,

(2) for each $i \in \mathbb{N}$, $A_i^+ \cap B_i^+ = A_i^- \cap B_i^- = \emptyset$, and

(3) $\lim A_i^+ = \lim A_i^-$ and $\lim B_i^+ = \lim B_i^-$.

Define $A_\infty \equiv \lim A_i^+ = \lim A_i^-$ and $B_\infty \equiv \lim B_i^+ = \lim B_i^-$. Let $\epsilon > 0$.

Since $\lim v_i^+ = \mu(A)$ and $\lim w_i^+ = \mu(B)$, there is $i^* \in \mathbb{N}$ such that $i \geq i^*$ implies $v_i^+ - \mu(A) < \frac{\epsilon}{2}$ and $w_i^+ - \mu(B) < \frac{\epsilon}{2}$. By construction, $i \geq i^*$ implies $\mu(A_i^+) + \mu(B_i^+) = v_i^+ + w_i^+ < \mu(A) + \mu(B) + \epsilon$. By Step 9, $i \geq i^*$ implies $\mu(A_i^+ \cup B_i^+) < \mu(A) + \mu(B) + \epsilon$.

Since $\lim v_i^- = \mu(A)$ and $\lim w_i^- = \mu(B)$, there is $i^* \in \mathbb{N}$ such that $i \geq i^*$ implies $\mu(A) - v_i^- < \frac{\epsilon}{2}$ and $\mu(B) - w_i^- < \frac{\epsilon}{2}$. By construction, $i \geq i^*$ implies $\mu(A_i^-) + \mu(B_i^-) = v_i^- + w_i^- > \mu(A) + \mu(B) - \epsilon$. By Step 9, $i \geq i^*$ implies $\mu(A_i^- \cup B_i^-) > \mu(A) + \mu(B) - \epsilon$.

Thus for each $\epsilon > 0$, there is $i \in \mathbb{N}$ such that $\mu(A_i^+ \cup B_i^+) < \mu(A) + \mu(B) + \epsilon$ and $\mu(A_i^- \cup B_i^-) > \mu(A) + \mu(B) - \epsilon$. Since for each $i \in \mathbb{N}$, $A_i^+ \cap B_i^+ = \emptyset$, thus $A_\infty \cap B_\infty = \emptyset$. By the Domination Lemma, for each $i \in \mathbb{N}$, $A_i^+ \cup B_i^+ \succsim_l A_\infty \cup B_\infty \succsim_l A_i^- \cup B_i^-$, so by Step 7, $\mu(A_i^+ \cup B_i^+) \geq \mu(A_\infty \cup B_\infty) \geq \mu(A_i^- \cup B_i^-)$. Thus $\mu(A_\infty \cup B_\infty) = \mu(A) + \mu(B)$.

By construction and by Step 7, for each $i \in \mathbb{N}$, $A_i^+ \succsim_l A \succsim_l A_i^-$ and $B_i^+ \succsim_l B \succsim_l B_i^+$. Thus by *continuity*, $A_\infty \succsim_l A \succsim_l A_\infty$ and $B_\infty \succsim_l B \succsim_l B_\infty$, so $A \sim_l A_\infty$ and $B \sim_l B_\infty$. By the Domination Lemma, $A \cup B \sim_l A_\infty \cup B_\infty$, so $\mu(A \cup B) = \mu(A_\infty \cup B_\infty) = \mu(A) + \mu(B)$. □

∘ STEP 11: $\mu \in \mathbb{M}^\sigma(\mathcal{A})$.

Since $\mu(S) = 1$, by Step 10 and induction, $\mu \in \mathbb{M}(\mathcal{A})$. By Step 7, μ represents $\succsim_l$, so by Theorem V, $\mu \in \mathbb{M}^\sigma(\mathcal{A})$. □

∘ STEP 12: If $\mu' \in \mathbb{M}^\sigma(\mathcal{A})$ represents $\succsim_l$, then $\mu' = \mu$.

If $\mu' \in \mathbb{M}^\sigma(\mathcal{A})$ represents $\succsim_l$, then by [DS1] and [DS2], it is immediate that μ' must be defined as in Step 4. ■

• PROPOSITION 3.3: A qualitative probability space $(S, \mathcal{A}, \succsim_l)$ has a dyadic skeleton in each of the following cases:

C1: there is σ-algebra $\mathcal{A}' \subseteq \mathcal{A}$ with $\cup\mathcal{A}' = S$ such that $(S, \mathcal{A}, \succsim_l)|_{\mathcal{A}'}$ is a qualitative probability space with a dyadic skeleton,

C2: there are $S', S'' \in \mathcal{A}$ partitioning S such that $(S, \mathcal{A}, \succsim_l)|_{S'}$ is a qualitative probability space with a dyadic skeleton and $S'' \sim_l \emptyset$, and

C3: there are $S', S'' \in \mathcal{A}$ partitioning S such that $(S, \mathcal{A}, \succsim_l)|_{S'}$ and $(S, \mathcal{A}, \succsim_l)|_{S''}$ are each qualitative probability spaces with dyadic skeletons.

Proof: We handle the cases in sequence.

C1: Let $\{\mathcal{Z}'_v\}_{v\in 2}$ be a dyadic skeleton for $(S, \mathcal{A}, \succsim_l)|_{\mathcal{A}'}$. By [DS2], for each $v \in 2$, there is $Z_v \in \mathcal{Z}'_v$. For each $v \in 2$, define

$$\mathcal{Z}_v \equiv \{A \in \mathcal{A} | A \sim_l Z_v\}.$$

It is straightforward to verify $\{Z_v\}_{v\in 2}$ is a dyadic skeleton for $(S, \mathcal{A}, \succsim_l)$.

C2: Let $\{Z'_v\}_{v\in 2}$ be a dyadic skeleton for $(S, \mathcal{A}, \succsim_l)|_{S'}$. By [DS2], for each $v \in 2$, there is $Z_v \in \mathcal{Z}'_v$. For each $v \in 2$, define

$$\mathcal{Z}_v \equiv \{A \in \mathcal{A} | A \sim_l Z_v\}.$$

It is straightforward to verify $\{Z_v\}_{v\in 2}$ is a dyadic skeleton for $(S, \mathcal{A}, \succsim_l)$.

C3: Let $\{\mathcal{Z}^*_v\}_{v\in 2}$ be a dyadic skeleton of $(S, \mathcal{A}, \succsim_l)|_{S'}$ and let $\{\mathcal{Z}^{**}_v\}_{v\in 2}$ be a dyadic skeleton of $(S, \mathcal{A}, \succsim_l)|_{S''}$. By [DS2], for each $v \in 2$, there are $Z'_v \in \mathcal{Z}^*_v$ and $Z''_v \in \mathcal{Z}^{**}_v$. For each $v \in 2$, define

$$\mathcal{Z}_v \equiv \{A \in \mathcal{A} | A \sim_l Z'_v \cup Z''_v\}.$$

We claim $\{\mathcal{Z}_v\}_{v\in 2}$ is a dyadic skeleton for $(S, \mathcal{A}, \succsim_l)$.

[DS1]: Since $\emptyset \in \mathcal{Z}^*_0$ and $\emptyset \in \mathcal{Z}^{**}_0$, thus $\emptyset \in \mathcal{Z}_0$. Since $S' \in \mathcal{Z}^*_1$ and $S'' \in \mathcal{Z}^{**}_1$, thus $S = S' \cup S'' \in \mathcal{Z}_1$.

[DS2]: Let $v, v' \in 2$ such that $v + v' \leq 1$. Then there are disjoint $Z^*_v \in \mathcal{Z}^*_v$ and $Z^*_{v'} \in \mathcal{Z}^*_{v'}$ such that $Z^*_v \cup Z^*_{v'} \in \mathcal{Z}^*_{v+v'}$, and there are disjoint $Z^{**}_v \in \mathcal{Z}^{**}_v$ and $Z^{**}_{v'} \in \mathcal{Z}^{**}_{v'}$ such that $Z^{**}_v \cup Z^{**}_{v'} \in \mathcal{Z}^{**}_{v+v'}$. Then we have disjoint $Z^*_v \cup Z^{**}_v \in \mathcal{Z}_v$ and $Z^*_{v'} \cup Z^{**}_{v'} \in \mathcal{Z}_{v'}$ such that $(Z^*_v \cup Z^{**}_v) \cup (Z^*_{v'} \cup Z^{**}_{v'}) = (Z^*_v \cup Z^*_{v'}) \cup (Z^{**}_v \cup Z^{**}_{v'}) \in \mathcal{Z}_{v+v'}$.

[DS3]: Let $(v_i), (v'_i), (w_i), (w'_i) \in 2^{\mathbb{N}}$ be convergent such that
(i) for each $i \in \mathbb{N}$, $v_i + w_i \leq 1$ and $v_i + w'_i \leq 1$, and
(ii) $\lim v_i = \lim v'_i$ and $\lim w_i = \lim w'_i$.
Then we have $(A^*_i), (A'^*_i), (B^*_i), (B'^*_i), (A^{**}_i), (A'^{**}_i), (B^{**}_i), (B'^{**}_i) \in \mathcal{A}^{\mathbb{N}}$ such that
(i) for each $i \in \mathbb{N}$, $A^*_i \in \mathcal{Z}^*_{v_i}$, $A'^*_i \in \mathcal{Z}^*_{v'_i}$, $B^*_i \in \mathcal{Z}^*_{w_i}$, $B'^*_i \in \mathcal{Z}^*_{w'_i}$, $A^{**}_i \in \mathcal{Z}^{**}_{v_i}$, $A'^{**}_i \in \mathcal{Z}^{**}_{v'_i}$, $B^{**}_i \in \mathcal{Z}^{**}_{w_i}$, and $B'^{**}_i \in \mathcal{Z}^{**}_{w'_i}$,
(ii) for each $i \in \mathbb{N}$, $A^*_i \cap B^*_i = A'^*_i \cap B'^*_i = A^{**}_i \cap B^{**}_i = A'^{**}_i \cap B'^{**}_i = \emptyset$, and
(iii) $\lim A^*_i = \lim A'^*_i$, $\lim B^*_i = \lim B'^*_i$, $\lim A^{**}_i = \lim A'^{**}_i$, and $\lim B^{**}_i = \lim B'^{**}_i$.
For each $i \in \mathbb{N}$, define

$$\begin{aligned} A_i &\equiv A^*_i \cup A^{**}_i \\ A'_i &\equiv A'^*_i \cup A'^{**}_i, \\ B_i &\equiv B^*_i \cup B^{**}_i, \text{ and} \\ B'_i &\equiv B'^*_i \cup B'^{**}_i. \end{aligned}$$

That $(A_i), (A'_i), (B_i), (B'_i) \in \mathcal{A}^{\mathbb{N}}$ are as desired follows from construction and the definition of pointwise convergence. ■

Appendix 4

In this appendix, we prove Proposition 4.1.

• PROPOSITION 4.1: If $\succsim_l$ is a *(monotonely) continuous* qualitative probability, and if $\mathcal{A}^\oplus \neq \emptyset$, then there is an atom-catalogue.

PROOF: We show that each collection of atoms has a most-likely member, use this fact to construct the atom-catalogue, and verify it satisfies the requirements.

◦ STEP 1: There is no increasing sequence of atoms.

Assume, by way of contradiction, there is $(A_i) \in (\mathcal{A}^\oplus)^{\mathbb{N}}$ such that $A_1 \prec_l A_2 \prec_l \ldots$. Then for each pair $i, j \in \mathbb{N}$ with $i > j$, by *monotonicity*, $A_i \succ_l A_j \succsim_l A_i \cap A_j$, so since A_i is an atom, thus $A_i \cap A_j \sim_l \emptyset$. For each $i \in \mathbb{N}$, define $B_i \equiv A_i \backslash (\bigcup_{j<i} A_j) = A_i \backslash (\bigcup_{j<i} A_j \cap A_i)$; by *separability*, $(\bigcup_{j<i} A_i \cap A_j) \sim_l \emptyset$, so by *separability* $B_i \sim_l A_i$. By construction, the B_i are pairwise disjoint. But then for each $i \in \mathbb{N}$, $B_i \sim_l A_i \succsim_l A_1$, so by *continuity*, $\emptyset = \lim B_i \succsim_l A_1$, contradicting that A_1 is an atom. □

◦ STEP 2: Construct the atom-catalogue (A_i).

We construct inductively. For the base step, by Step 1, there is a $\succsim_l$-maximum A_1 in $\mathcal{A}^\oplus$, and for each $B \in \mathcal{A}^\oplus$ such that $B \subseteq S \backslash A_1$, $A_1 \succsim_l B$. For the inductive step, assume that $k \in \mathbb{N}$ and we have pairwise disjoint $A_1, A_2, \ldots, A_k$ such that

(1) $A_1 \succsim_l A_2 \succsim_l \ldots \succsim_l A_k$, and

(2) for each $B \in \mathcal{A}^\oplus$ such that $B \subseteq S \backslash \bigcup_{i=1}^k A_i$, $A_k \succsim_l B$.

Define $\mathcal{A}_k^\oplus \subseteq \mathcal{A}^\oplus$ by:

$$\mathcal{A}_k^\oplus \equiv \{B \in \mathcal{A}^\oplus | B \subseteq S \backslash \bigcup_{i=1}^{k} A_i\}.$$

If $\mathcal{A}_k^\oplus$ is empty, then we are done. If $\mathcal{A}_k^\oplus$ is nonempty, then by Step 1 it has $\succsim_l$-maximum A_{k+1}. Since $A_{k+1} \in \mathcal{A}^\oplus$ and $A_{k+1} \subseteq S \backslash \bigcup_{i=1}^k A_i$, thus by the inductive hypothesis, $A_1, A_2, \ldots, A_{k+1}$ are pairwise disjoint and $A_1 \succsim_l A_2 \succsim_l \ldots \succsim_l A_{k+1}$. For each $B \in \mathcal{A}^\oplus$ such that $B \subseteq S \backslash \bigcup_{i=1}^{k+1} A_i$, $B \in \mathcal{A}_k^\oplus$, so by construction $A_k \succsim_l B$. □

◦ STEP 3: Conclude.

By construction, we have $(A_i) \subseteq \mathcal{A}^\oplus$ satisfying the first two requirements for an atom-catalogue, and if $\{A_i\}$ is finite, then it satisfies the third. Assume, by way of contradiction, $\{A_i\}$ is countably infinite and there is $A \subseteq S \backslash (\cup A_i)$ such that $A \in \mathcal{A}^\oplus$. By construction, for each $i \in \mathbb{N}$, $A_i \succsim_l A$. But then by *continuity*, $\emptyset = \lim A_i \succsim_l A$, contradicting that A is an atom. ■

Appendix 5

In this appendix, we prove our primary lemmas about *smaller atoms domination* axioms and greedy transforms: the Idempotence Lemma, the 1-SAD Lemma, the Greedy Removal Lemma, the First Halving Lemma, the Second Halving Lemma, and the Convergence Lemma. We abuse language in our informal summaries of these lemmas, writing that an event satisfies k-SAD instead of writing that its associated subspace does.

The Idempotence Lemma states that each greedy transform is idempotent, and for convenience includes the easy corollary that two images of a greedy transform that are equally likely are in fact equivalent.

• IDEMPOTENCE LEMMA: For each pair $A, B \in \mathcal{A}$, $\mathcal{G}^A(\mathcal{G}^A(B)) = \mathcal{G}^A(B)$. Moreover, for each $A \in \mathcal{A}$ and each pair $B, B' \in \mathcal{A}$, $\mathcal{G}^A(B) \sim_l \mathcal{G}^A(B')$ implies $\mathcal{G}^A(B) = \mathcal{G}^A(B')$.

PROOF: We proceed by induction on s, covering the base case with our inductive hypothesis. Assume $s \in S$ is such that for each $s' \in S$,

$$\mathcal{G}^A(\mathcal{G}^A(B)) \cap \{s' \in S | s' < s\} = \mathcal{G}^A(B) \cap \{s' \in S | s' < s\}.$$

If $s \in \mathcal{G}^A(B)$, then by construction,

$$\begin{aligned}\mathcal{G}^A(B) &\succsim_l [\mathcal{G}^A(B) \cap \{s' \in S | s' < s\}] \cup \{s\} \\ &= [\mathcal{G}^A(\mathcal{G}^A(B)) \cap \{s' \in S | s' < s\}] \cup \{s\},\end{aligned}$$

so by construction, $s \in \mathcal{G}^A(\mathcal{G}^A(B))$.

If $s \notin \mathcal{G}^A(B)$, then by construction,

$$\begin{aligned}[\mathcal{G}^A(\mathcal{G}^A(B)) \cap \{s' \in S | s' < s\}] \cup \{s\} &= [\mathcal{G}^A(B) \cap \{s' \in S | s' < s\}] \cup \{s\} \\ &\succ_l B \\ &\succsim_l \mathcal{G}^A(B),\end{aligned}$$

so by construction, $s \notin \mathcal{G}^A(\mathcal{G}^A(B))$. Thus

$$\mathcal{G}^A(\mathcal{G}^A(B)) \cap \{s' \in S | s' < s+1\} = \mathcal{G}^A(B) \cap \{s' \in S | s' < s+1\}.$$

By induction, $\mathcal{G}^A(\mathcal{G}^A(B)) = \mathcal{G}^A(B)$.

Now assume $A \in \mathcal{A}$ and $B, B' \in \mathcal{A}$ are such that $\mathcal{G}^A(B) \sim_l \mathcal{G}^A(B')$. Then $\mathcal{G}^A(B) = \mathcal{G}^A(\mathcal{G}^A(B)) = \mathcal{G}^A(\mathcal{G}^A(B')) = \mathcal{G}^A(B')$, as desired. ■

The 1-SAD Lemma is the ordinal analogue of Theorem K: 1-*SAD* of A guarantees that applying A's greedy transform to an event B that is no larger than A yields an event as likely as B.

• 1-SAD LEMMA: If $(S, \mathcal{A}, \succsim_l)$ is atom-standard, then for each $A \in \mathcal{A}$ such that $\succsim_l|_A$ satisfies *1-SAD* and each $B \in \mathcal{A}$ such that $A \succsim_l B$, $\mathcal{G}^A(B) \sim_l B$.

PROOF: If $A = \emptyset$ the result is trivial, so assume $A \neq \emptyset$. Then by *1-SAD*, $|A| = |\mathbb{N}|$. For convenience, re-index S so that (1) $A = \mathbb{N}$, (2) index order is preserved for A, and (3) members of $S \backslash A$ are not indexed by natural numbers. Since $B \succsim_l \mathcal{G}^A(B)$, it suffices to show $\mathcal{G}^A(B) \succsim_l B$.

CASE 1: $|A \backslash \mathcal{G}^A(B)| = 0$. Then $\mathcal{G}^A(B) = A \succsim_l B$.

CASE 2: $0 < |A \backslash \mathcal{G}^A(B)| < |\mathbb{N}|$. Define $s^* \equiv \max(A \backslash \mathcal{G}^A(B))$. By construction, $[\mathcal{G}^A(B) \backslash \{s^*+1, s^*+2, ...\} \cup \{s^*\}] \succ_l B$. Since $\succsim_l |_A$ satisfies 1-*SAD*, thus by *separability*

$$\begin{aligned}\mathcal{G}^A(B) &= [\mathcal{G}^A(B) \backslash \{s^*+1, s^*+2, ...\}] \cup \{s^*+1, s^*+2, ...\} \\ &\succsim_l [\mathcal{G}^A(B) \backslash \{s^*+1, s^*+2, ...\}] \cup \{s^*\} \\ &\succ_l B.\end{aligned}$$

But this contradicts $B \succsim_l \mathcal{G}^A(B)$, so in fact Case 2 is impossible.

CASE 3: $|A \backslash \mathcal{G}^A(B)| = |\mathbb{N}|$. By construction, for each $s \in A \backslash \mathcal{G}^A(B)$, $[\mathcal{G}^A(B) \backslash \{s+1, s+2, ...\}] \cup \{s\} \succ_l B$. Thus by *continuity*,

$$\begin{aligned}\mathcal{G}^A(B) &= \lim_{s \in S' \backslash \mathcal{G}^A(B)} [\mathcal{G}^A(B) \backslash \{s+1, s+2, ...\}] \cup \{s\} \\ &\succsim_l B. \blacksquare\end{aligned}$$

The Greedy Removal Lemma states that if A is $(k+1)$-*SAD*, then removing an image of its greedy transform yields a subevent that is k-*SAD*.

• GREEDY REMOVAL LEMMA: If $(S, \mathcal{A}, \succsim_l)$ is atom-standard, then for each $k \in \mathbb{N}$, each $A \in \mathcal{A}$ such that $\succsim_l|_A$ satisfies $(k+1)$-*SAD*, and each $B \in \mathcal{A}$,

$$\succsim_l|_{A \backslash \mathcal{G}^A(B)} \text{ satisfies } k\text{-SAD}.$$

PROOF: Let $k \in \mathbb{N}$ and assume $\succsim_l|_A$ satisfies k-*SAD*. If $A = \emptyset$ or $B \succsim_l A$, the result is trivial, so assume $A \succ_l B \succsim_l \emptyset$. By *1-SAD*, $|A| = |\mathbb{N}|$. For convenience, re-index S so that (i) $A = \mathbb{N}$, (2) index order is preserved for A, and (iii) members of $S \backslash A$ are not indexed by natural numbers.

∘ STEP 1: If $k' < k+1$ greedy transform sets G_i are iteratively removed from A, and if $A \backslash \cup G_i$ is nonempty, then $\succsim_l |_{A \backslash \cup G_i}$ satisfies 1-*SAD*.

Let $k' \in \mathbb{N}$ such that $k' < k+1$, and let $B_1, ..., B_{k'} \in \mathcal{A}$. Define $A_1 \equiv A$, and for each $i \in \{1, 2, ..., k'\}$, define:

- $G_i \equiv \mathcal{G}^{A_i}(B_i)$,
- $A_{i+1} \equiv A_i \backslash G_i$.

By construction, the G_i are pairwise disjoint. We claim $A \backslash \cup G_i \neq \emptyset$ implies $\succsim_l |_{A \backslash \cup G_i}$ satisfies 1-*SAD*. Indeed, let $s \in A \backslash \cup G_i$. Since $\succsim_l |_A$ satisfies $(k+1)$-*SAD*, there are pairwise disjoint $B_1, ..., B_{k+1} \subseteq \{s+1, s+2, ...\}$ such that for each $i \in \{1, 2, ..., k+1\}$, $B_i \succsim_l \{s\}$. By construction, for each $i \in \{1, 2, ..., k'\}$, $\{s\} \succsim_l G_i \cap \{s+1, s+2, ...\}$. Thus for each $i \in \{1, 2, ..., k'\}$, $B_i \succsim_l G_i \cap \{s+1, s+2, ...\}$. By repeated application of the

Domination Lemma,

$$\begin{aligned} B_1 &\succsim_l G_1 \cap \{s+1, s+2, ...\}, \\ B_1 \cup B_2 &\succsim_l \big(G_1 \cap \{s+1, s+2, ...\}\big) \cup \big(G_2 \cap \{s+1, s+2, ...\}\big), \\ &\vdots \\ \bigcup_{i=1}^{k'} B_i &\succsim_l \bigcup_{i=1}^{k'} \big(G_i \cap \{s+1, s+2, ...\}\big) \end{aligned}$$

Thus if $\{s\} \succ_l [\{s+1, s+2, ...\} \backslash \cup G_i]$, then $B_{k'+1} \succ_l [\{s+1, s+2, ...\} \backslash \cup G_i]$, so by the Domination Lemma,

$$\begin{aligned} \bigcup_{i=1}^{k'+1} B_i &\succ_l \bigcup_{i=1}^{k'} \big(G_i \cap [A \cap \{s+1, s+2, ...\}]\big) \cup [\{s+1, s+2, ...\} \backslash \cup G_i] \\ &= \{s+1, s+2, ...\}, \end{aligned}$$

contradicting *monotonicity*. Thus $[\{s+1, s+2, ...\} \backslash \cup G_i] \succsim_l \{s\}$. Since $s \in A \backslash G_i$ was arbitrary, thus $\succsim_l |_{A \backslash \cup G_i}$ satisfies 1-*SAD*. □

∘ Step 2: Conclude.

Let $B \in \mathcal{A}$ such that $A \succ_l B$ and let $G_1 = \mathcal{G}^A(B)$. Then there is $s \in A \backslash G_1$, else by *monotonicity* $B \succsim_l G_1 = A \succ_l B$, contradicting $B \sim_l B$. By Step 1, $\succsim_l |_{A \backslash G_1}$ satisfies 1-*SAD*.

Let $s \in A \backslash G_1$ and define $A_1 \equiv \{s+1, s+2, ...\}$. Then $\succsim_l |_{A_1}$ satisfies $(k+1)$-*SAD* and $\succsim_l |_{A_1 \backslash G_1}$ satisfies 1-*SAD*. Thus there are pairwise disjoint $B_1, ..., B_{k+1} \subseteq A_1$ such that for each $i \in \{1, 2, ..., k+1\}$, $B_i \succsim_l \{s\}$.

Define $A_2 \equiv A_1 \backslash G_1$, and for each $i \in \{2, 3, ..., k+1\}$, define:

- $G_i \equiv \mathcal{G}^{\leq}(A_i, \{s\})$,
- $A_{i+1} \equiv A_i \backslash G_i$.

By construction, the G_i are pairwise disjoint. Assume, by way of contradiction, there is $i \in \{2, 3, ..., k+1\}$ such that $\{s\} \succ_l A_i$. Let i^* be the least such i. Then $B_{i^*} \succsim_l \{s\} \succ_l A_{i^*}$, and for each $i \in \{1, ..., i^*-1\}$, $B_i \succsim_l \{s\} \succsim_l G_i$. But then by repeated application of the Domination Lemma as in Step 1,

$$\begin{aligned} \bigcup_{i=1}^{i^*-1} B_i &\succ_l \bigcup_{i=1}^{i^*-1} G_i \cup A_{i^*} \\ &= A_1, \end{aligned}$$

contradicting *monotonicity*.

Thus for each $i \in \{2, 3, ..., k+1\}$, $A_i \succsim_l \{s\}$, so by *monotonicity* A_i is nonempty, so by Step 1 $\succsim_l |_{A_i}$ satisfies 1-*SAD*, so by the 1-*SAD Lemma* $B_i \sim_l \{s\}$. Since $s \in A \backslash G_1$ was arbitrary, $\succsim_l |_{A \backslash G_1}$ satisfies k-*SAD*. ■

The First Halving Lemma states any event satisfying 1-*SAD* can be associated with two disjoint subevents analogous to its halves:

• First Halving Lemma: If $(S, \mathcal{A}, \succsim)$ is atom-standard, then for each $A \in \mathcal{A}$ such that $A \succ_l \emptyset$ and $\succsim_l|_A$ satisfies 1-*SAD*, there is $H \subseteq A$ such that $H \sim_l A \backslash H$ and $A \succ_l H \succ_l \emptyset$.

Proof: Since $A \neq \emptyset$, by 1-*SAD*, $|A| = |\mathbb{N}|$. Assume, without loss of generality, $A = S = \mathbb{N}$. Let $\mathbb{C} \subseteq [0,1]$ be the canonical Cantor set (Cantor, 1883). Define $\Psi : \mathcal{A} \to \mathbb{C}$ by

$$\Psi(B) = \sum_{s \in B} \frac{2s}{3^s}.$$

It is well-known that Ψ is an order-preserving homeomorphism when $\mathcal{A}$ has the lexicographic order $>_{\text{LEX}}$ and $\mathbb{C}$ has the usual order $>$; thus each closed collection $\mathcal{A}' \subseteq \mathcal{A}$ contains its $>_{\text{LEX}}$-supremum. Furthermore, for each $B \in \mathcal{A}$, $\Psi(B^c) = 1 - \Psi(B)$.

Define the collection of events $\mathcal{A}^- \subseteq \mathcal{A}$ by:

$$\mathcal{A}^- \equiv \{B \in \mathcal{A} | B^c \succsim_l B\}$$

We claim $\mathcal{A}^-$ is closed. Indeed, let $(B_i) \in (\mathcal{A}^-)^{\mathbb{N}}$ be convergent and define $B \equiv \lim(B_i)$. Then

$$\begin{aligned}
\lim(B_i^c) &= \lim(\Psi^{-1}(\Psi(B_i^c))) \\
&= \lim(\Psi^{-1}(1 - \Psi(B_i))) \\
&= \Psi^{-1}(\lim(1 - \Psi(B_i))) \\
&= \Psi^{-1}(1 - \lim \Psi(B_i)) \\
&= \Psi^{-1}(1 - \Psi(B)) \\
&= B^c.
\end{aligned}$$

Thus $(B_i), (B_i^c)$ are convergent such that for each $i \in \mathbb{N}$, $B_i^c \succsim_l B_i$, so by the Limit-Order Lemma,

$$\begin{aligned}
\lim(B)^c &= B^c \\
&= \lim(B_i^c) \\
&\succsim_l \lim(B_i),
\end{aligned}$$

and hence $\lim(B_i) \in \mathcal{A}^-$. Thus $\mathcal{A}^-$ is closed, so it contains its $>_{\text{LEX}}$-supremum H.

Since $H = S$ implies $H \succ_l H^c$, contradicting $H \in \mathcal{A}^-$, thus $H^c \neq \emptyset$. Similarly, $H^c \neq S$. Assume, by way of contradiction, $|H^c| < |\mathbb{N}|$. Define $s^* \equiv \max H^c$. By 1-*SAD* and *separability*,

$$\begin{aligned}
H &= [H \backslash \{s^*+1, s^*+2, ...\}] \cup \{s^*+1, s^*+2, ...\} \\
&\succsim_l [H \backslash \{s^*+1, s^*+2, ...\}] \cup \{s^*\}.
\end{aligned}$$

Then by the Complement Lemma and *separability*,

$$\begin{aligned}
([H \backslash \{s^*+1, s^*+2, ...\}] \cup \{s^*\})^c &\succsim_l H^c \\
&\succsim_l H \\
&\succsim_l [H \backslash \{s^*+1, s^*+2, ...\}] \cup \{s^*\}.
\end{aligned}$$

But then $[H \backslash \{s^*+1, s^*+2, ...\}] \cup \{s^*\} \in \mathcal{A}^-$, contradicting the $>_{\text{LEX}}$-maximality of H. Thus $|H^c| = |\mathbb{N}|$.

Finally, for each $s \in H^c$, $H \cup \{s\} \notin \mathcal{A}^-$, else $H \cup \{s\}$ would contradict the $>_{\text{LEX}}$-maximality of H in $\mathcal{A}^-$. Thus by the Limit-Order Lemma,

$$\begin{aligned} H &= \lim_{s \in H^c} (H \cup \{s\}) \\ &\succsim_l \lim_{s \in H^c} (H \cup \{s\})^c \\ &= H^c, \end{aligned}$$

and thus $H \sim_l H^c$.

Since $A \succ_l \emptyset$, necessarily $A \succ_l H$, else $H \succsim_l A$ and $H^c \sim_l H \succsim_l A \succ_l \emptyset$, so by the Domination Lemma, $A \sim_l H \cup H^c \succ_l A$, contradicting $A \sim_l A$. Necessarily $H \succ_l \emptyset$, else $\emptyset \succsim_l H$ and $\emptyset \succsim_l H \sim_l H^c$, so by the Domination Lemma, $\emptyset \succsim_l H \cup H^c \sim_l A$, contradicting $A \succ_l \emptyset$. Thus $A \succ_l H \succ_l \emptyset$. ■

The Second Halving Lemma states that under 2-*SAD*, each event can be associated with two disjoint events analogous to its halves:

• Second Halving Lemma: If $(S, \mathcal{A}, \succsim)$ is atom-standard and $\succsim_l$ satisfies 2-*SAD*, then for each $A \in \mathcal{A}$ such that $A \succ_l \emptyset$, there are disjoint $H(A), H'(A) \in \mathcal{A}$ such that
(1) $A \sim_l H(A) \cup H'(A)$, and
(2) $A \succ_l H(A) \sim_l H'(A) \succ_l \emptyset$.

Proof: Let $A \in \mathcal{A}$ such that $A \succ_l \emptyset$. Since $\succsim_l$ satisfies 1-*SAD* and, by *monotonicity*, $S \succsim_l S \backslash A$, thus by the 1-SAD Lemma, $\mathcal{G}^S(S \backslash A) \sim_l S \backslash A$. By the Complement Lemma,
(1) $S \succ_l S \backslash A \sim_l \mathcal{G}^S(S \backslash A)$, and
(2) $S \backslash \mathcal{G}^S(S \backslash A) \sim_l S \backslash (S \backslash A) = A \succ_l \emptyset$.
Since $\succsim_l$ satisfies 2-*SAD*, by the Greedy Removal Lemma, $\succsim_l|_{S \backslash \mathcal{G}^S(S \backslash A)}$ satisfies 1-*SAD*. By the First Halving Lemma, there are disjoint $H(A), H'(A) \subseteq S \backslash \mathcal{G}^S(S \backslash A)$ such that
(1) $A \sim_l S \backslash \mathcal{G}^S(S \backslash A) = H(A) \cup H'(A)$, and
(2) $A \sim_l S \backslash \mathcal{G}^S(S \backslash A) \succ_l H(A) \sim_l H'(A) \succ_l \emptyset$,
as desired. ■

The Convergence Lemma concerns generalized greedy transforms, each of which takes as input a list of events and outputs the same number of events:

Definition: For each $A \in \mathcal{A}$ and each $k \in \mathbb{N}$, the *greedy transform* $\mathcal{G}^A : \mathcal{A}^k \to \mathcal{A}^k$ is defined as follows. For each $(B_1, B_2, ..., B_k) \in \mathcal{A}^k$, $\mathcal{G}^A(B_1, B_2, ..., B_k) \in \mathcal{A}^k$ is defined recursively by:

- $\mathcal{G}_0^A(B_1, B_2, ..., B_k) \equiv \emptyset$, and
- for each $i \in \{1, 2, ..., k\}$,

$$\mathcal{G}_i^A(B_1, B_2, ..., B_k) \equiv \mathcal{G}^{S \backslash (\cup_{j<i} \mathcal{G}_j^A(B_1, B_2, ..., B_k))}(A_i).$$

The lemma states that for nice sequences of event lists, the associated sequence of event lists output by a generalized greedy transform converges.

• Convergence Lemma: If $(S, \mathcal{A}, \succsim)$ is atom-standard, $k \in \mathbb{N}$, and $(B_i^1), (B_i^2), ..., (B_i^k) \in \mathcal{A}^{\mathbb{N}}$ are such that

(1) for each $j \in \{1,2,...,k\}$, (B_i^j) is monotonic, and
(2) for each $i \in \mathbb{N}$, $B_i^1, B_i^2, ..., B_i^k$ are pairwise disjoint,
then for each $A \in \mathcal{A}$ and each $j \in \{1,2,...,k\}$, $(\mathcal{G}_j^A(B_i^1, B_i^2, ..., B_i^k)) \in \mathcal{A}^{\mathbb{N}}$ is convergent.

Proof: Let $A \in \mathcal{A}$, let $k \in \mathbb{N}$, and let $(B_i^1), (B_i^2), ..., (B_i^k) \in \mathcal{A}^{\mathbb{N}}$ satisfy the hypotheses. For each $i \in \mathbb{N}$ and each $j \in \{1,2,...,k\}$, define $G_i^j \in \mathcal{A}$ by:

$$G_i^j \equiv \mathcal{G}_j^A(B_i^1, B_i^2, ..., B_i^k).$$

To prove $(G_i^1), (G_i^2), ..., (G_i^k)$ are convergent, we proceed by induction. We cover the base step with our inductive hypothesis on j^*: assume $j^* \in \{1,2,...,k-1\}$ is such that for each $j \in \{1,2,...,k\}$ such that $j < j^*$, (G_i^j) is convergent. We claim $(G_i^{j^*})$ is convergent.

Within the current inductive argument, we make a second inductive argument. We cover the base step with our inductive hypothesis on s: assume $s \in S$ is such that for each $s' < s$, there is $i^* \in \mathbb{N}$ such that for each $j \in \{1,2,...,j^*\}$, either
(1) $i \geq i^*$ implies $s' \in G_i^j$, or
(2) $i \geq i^*$ implies $s' \in G_i^j$.
By the inductive hypothesis on j^*, there is $i^* \in \mathbb{N}$ such that for each $j \in \{1,2,...,k\}$ such that $j < j^*$, either
(1) $i \geq i^*$ implies $s \in G_i^j$, or
(2) $i \geq i^*$ implies $s \in G_i^j$.
Assume, by way of contradiction, that for each $i^{**} \in \mathbb{N}$ there are $i_1 \geq i^{**}$ and $i_2 \geq i^{**}$ such that $s \in G_{i_1}^{j^*}$ and $s \notin G_{i_2}^{j^*+1}$. Then there are $i_1, i_2, i_3 \in \mathbb{N}$ with $i_3 > i_2 > i_1 > i^*$ such that $s \in G_{i_1}^{j^*}$, $s \notin G_{i_2}^{j^*}$, and $s \in G_{i_3}^{j^*}$. By definition of i^*, $G_{i_1}^{j^*} \cap \{s' \in S | s' < s\} = G_{i_2}^{j^*} \cap \{s' \in S | s' < s\} = G_{i_3}^{j^*} \cap \{s' \in S | s' < s\}$. Thus by construction, $B_{i_1}^{j^*} \succsim_l G_{i_1}^{j^*} \cap \{s' \in S | s' < s\} \cup \{s\} \succ_l B_{i_2}^{j^*}$, so $(B_i^{j^*})$ is non-increasing. But then $B_{i_3}^{j^*} \succsim_l G_{i_1}^{j^*} \cap \{s' \in S | s' < s\} \cup \{s\} \succ_l B_{i_2}^{j^*}$, contradicting that $(B_i^{j^*})$ is non-increasing.

By induction on s, $(G_i^{j^*})$ is convergent. By induction on j^*, $(G_i^1), (G_i^2), ..., (G_i^k)$ are convergent. ■

Appendix 6

In this appendix, we prove Proposition 5.1.

• Proposition 5.1: If $(S, \mathcal{A}, \succsim_l)$ is atom-standard and satisfies 3-*SAD*, then $(S, \mathcal{A}, \succsim_l)$ has a dyadic skeleton.

Proof: The only notation carried from one step to the next is the notation in the step's statement.

∘ Step 1: Define $\{A_q^1\}_{q \in \{0,1,...\}}, \{H(A_q^1)\}_{q \in \{0,1,...\}}, \{H'(A_q^1)\}_{q \in \{0,1,...\}} \subseteq \mathcal{A}$ such that for each $q \in \{0,1,...\}$,
(1) $H(A_q^1) \cap H'(A_q^1) = \emptyset$,
(2) $A_q^1 \sim_l H(A_q^1) \cup H'(A_q^1)$, and
(3) $A_q^1 \succ_l A_{q+1}^1 \sim_l H(A_q^1) \sim_l H'(A_q^1) \succ_l \emptyset$.

We proceed recursively. Define $A_0^1 \equiv \mathcal{G}^S(S) = S$; by *monotonicity* and *nondegeneracy*, $A_0^1 \succ_l \emptyset$.

Suppose we have $A_q^1 \in \mathcal{A}$ such that $A_q^1 \succ_l \emptyset$. Since $\succsim_l$ satisfies 2-*SAD*, by the Second Halving Lemma, there are disjoint $H(A_q^1), H'(A_q^1)$ such that

(1) $A_q^1 \sim_l H(A_q^1) \cup H'(A_q^1)$, and

(2) $A_q^1 \succ_l H(A_q^1) \sim_l H'(A_q^1) \succ_l \emptyset$.

Define $A_{q+1}^1 \equiv \mathcal{G}^S(H(A_q^1))$; by the Half-Equivalence Lemma this is well-defined. Since $\succsim_l$ satisfies 1-*SAD* and, by *monotonicity*, $S \succsim_l H(A_q^1)$, thus by the 1-SAD Lemma, $A_{q+1}^1 \sim_l H(A_q^1)$. □

∘ Step 2: For each $q \in \{0, 1, ...\}$, define $\{A_q^p\}_{q \in \{0,1,...,2^q\}} \subseteq \mathcal{A}$ such that for each $p \in \{0, 1, ..., 2^q - 1\}$,

$$A_q^{p+1} \sim_l A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^1).$$

We proceed recursively. For each $q \in \mathbb{N}$, define $A_q^0 \equiv \emptyset$. For each $q \in \{0, 1, ...\}$ and each $p \in \{0, 1, ..., 2^q - 1\}$, define

$$A_q^{p+1} \equiv \mathcal{G}^S\big(A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^1)\big).$$

Since $\succsim_l$ satisfies 1-*SAD* and, by *monotonicity*, $S \succsim_l A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^1)$, thus by the 1-SAD Lemma, $A_q^{p+1} \sim_l A_q^p \cup \mathcal{G}^{S \setminus A_q^p}(A_q^1)$.

By the Idempotence Lemma, this definition gives the same $\{A_q^1\}_{q \in \{0,1,...\}}$ defined before. □

∘ Step 3: For each $q \in \{0, 1, ...\}$ and each $p \in \{0, 1, ..., 2^q\}$, $\succsim_l|_{S \setminus A_q^p}$ satisfies 2-*SAD*.

Let $q \in \{0, 1, ...\}$ and $p \in \{0, 1, ..., 2^q\}$ be such that $A_q^p \neq S$. Since $\succsim_l$ satisfies 3-*SAD*, thus by the Greedy Removal Lemma and the Idempotence Lemma, $\succsim_l|_{S \setminus A_q^p} = \succsim_l|_{S \setminus \mathcal{G}^S(A_q^p)}$ satisfies 2-*SAD*. □

∘ Step 4: For each $q \in \{0, 1, ...\}$,

(1) $p \in \{0, 1, ..., 2^q - 1\}$ implies $\mathcal{G}^{S \setminus A_q^p}(A_q^1) \sim_l A_q^1$, and

(2) $p \in \{0, 1, ..., 2^q\}$ implies $A_q^p = A_{q+1}^{2p}$.

We proceed by induction on q. For the base step, let $q = 0$. Then $\mathcal{G}^{S \setminus A_q^0}(A_q^1) = \mathcal{G}^S(S) = S = A_q^1$. For the inductive hypothesis, assume $q \in \{0, 1, ...\}$ is such that for each $p \in \{0, 1, ..., 2^q - 1\}$, $\mathcal{G}^{S \setminus A_q^p}(A_q^1) \sim_l A_q^1$.

Within the current inductive argument, we make a second inductive argument, on p. For the base step, $A_q^0 = \emptyset = A_{q+1}^0$. For the inductive hypothesis, assume $p \in \{0, 1, ..., 2^q - 1\}$ is such that $A_q^p = A_{q+1}^{2p}$. For convenience, define $G_1, G_2, G_2' \in \mathcal{A}$ by

- $G_1 \equiv \mathcal{G}^{S \setminus A_{q+1}^{2p}}(A_{q+1}^1)$,
- $G_2 \equiv \mathcal{G}^{S \setminus A_{q+1}^{2p+1}}(A_{q+1}^1)$, and
- $G_2' \equiv \mathcal{G}^{S \setminus [A_{q+1}^{2p} \cup G_1]}(A_{q+1}^1)$.

We make three claims, which we prove in sequence:

Claim 1: $G_1 \sim_l A^1_{q+1}$,
Claim 2: $G_2 \sim_l A^1_{q+1}$, and
Claim 3: $A^{p+1}_q = A^{2(p+1)}_{q+1}$.

Proof of Claim 1: By the inductive hypothesis on p, $A^p_q = A^{2p}_{q+1}$, so by Step 1, *monotonicity*, and the inductive hypothesis on q,

$$\begin{aligned} S\backslash A^{2p}_{q+1} &= S\backslash A^p_q \\ &\succsim_l \mathcal{G}^{S\backslash A^p_q}(A^1_q) \\ &\sim_l A^1_q \\ &\succ_l A^1_{q+1}. \end{aligned}$$

Since by Step 3, $\succsim_l|_{S\backslash A^{2p}_{q+1}} = \succsim_l|_{S\backslash A^p}$ satisfies 2-*SAD*, thus by the 1-SAD Lemma, $G_1 = \mathcal{G}^{S\backslash A^{2p}_{q+1}}(A^1_{q+1}) \sim_l A^1_{q+1}$.

Proof of Claim 2: Since by Claim 1, $H(A^1_q) \sim_l A^1_{q+1} \sim_l G_1$, necessarily $S\backslash[A^{2p}_{q+1} \cup G_1] \succsim_l H'(A^1_q)$, else by the Domination Lemma, the hypothesis on p, and the hypothesis on q,

$$\begin{aligned} A^1_q &\sim_l H(A^1_q) \cup H'(A^1_q) \\ &\succ_l G_1 \cup S\backslash[A^{2p}_{q+1} \cup G_1] \\ &= S\backslash A^{2p}_{q+1} \\ &= S\backslash A^p_q \\ &\succsim_l \mathcal{G}^{S\backslash A^p_q}(A^1_q) \\ &\sim_l A^1_q, \end{aligned}$$

contradicting $A^1_q \sim_l A^1_q$.

By Step 2, $A^{2p+1}_{q+1} \sim_l A^{2p}_{q+1} \cup G_1$. By the Complement Lemma, $S\backslash A^{2p+1}_{q+1} \sim_l S\backslash[A^{2p}_{q+1} \cup G_1] \succsim_l H'(A^1_q) \sim_l A^1_{q+1}$. By Step 3, $\succsim_l|_{S\backslash A^{2p+1}_{q+1}}$ satisfies 2-*SAD*, so by the 1-SAD Lemma, $G_2 = \mathcal{G}^{S\backslash A^{2p+1}_{q+1}}(A^1_{q+1}) \sim_l A^1_{q+1}$.

Proof of Claim 3: As argued in the proof of Claim 2, $S\backslash[A^{2p}_{q+1} \cup G_1] \succsim_l H'(A^1_q) \sim_l A^1_{q+1}$. By Step 3, $\succsim_l|_{S\backslash A^{2p}_{q+1}}$ satisfies 2-*SAD*, so by the Greedy Removal Lemma, $\succsim_l|_{(S\backslash A^{2p}_{q+1})\backslash G_1} = \succsim_l|_{S\backslash[A^{2p}_{q+1} \cup G_1]}$ satisfies 1-*SAD*. Thus by the 1-SAD Lemma, $G'_2 \sim_l A^1_{q+1}$.

Since $H(A^1_q) \sim_l G_1$ and $H'(A^1_q) \sim_l G'_2$, thus by the Domination Lemma, $A^1_q \sim_l H(A^1_q) \cup H'(A^1_q) \sim_l G^1 \cup G'_2$. By the hypothesis on q, $\mathcal{G}^{S\backslash A^p_q}(A^1_q) \sim_l A^1_q \sim_l G^1 \cup G'_2$. By Step 2, the hypothesis on p, and *separability*,

$$\begin{aligned} A^{p+1}_q &\sim_l A^p_q \cup \mathcal{G}^{S\backslash A^p_q}(A^1_q) \\ &= A^{2p}_{q+1} \cup \mathcal{G}^{S\backslash A^p_q}(A^1_q) \\ &\sim_l A^{2p}_{q+1} \cup (G_1 \cup G'_2) \\ &= [A^{2p}_{q+1} \cup G_1] \cup G'_2 \end{aligned}$$

By Step 2, $A^{2p}_{q+1} \cup G_1 \sim_l A^{2p+1}_{q+1}$. By the first paragraph of this claim's proof and Claim 2, $G'_2 \sim_l A^1_{q+1} \sim_l G_2$. Thus by the Domination Lemma, $[A^{2p}_{q+1} \cup G_1] \cup G'_2 \sim_l A^{2p+1}_{q+1} \cup G_2$. By Step 2, $A^{2p+1}_{q+1} \cup G_2 \sim_l A^{2p+2}_{q+1}$. Altogether, $A^{p+1}_q \sim_l A^{2p+2}_{q+1}$.

By induction on p, we conclude the following: if $q \in \{0, 1, ...\}$ is such that for each $p \in \{0, 1, ..., 2^q - 1\}$, $\mathcal{G}^{S \backslash A^p_q}(A^1_q) \sim_l A^1_q$, then

(1) for each $p \in \{0, 1, ..., 2^{q+1} - 1\}$, $\mathcal{G}^{S \backslash A^p_{q+1}}(A^1_{q+1}) \sim_l A^1_{q+1}$, and

(2) for each $p \in \{0, 1, ..., 2^q\}$, $A^p_q = A^{2p}_{q+1}$.

By induction on q, we are done. $\square$

∘ Step 5: For each $q \in \{0, 1, ...\}$ and each $p \in \{0, 1, ..., 2^q\}$, $A^p_q \sim_l S \backslash A^{2^q - p}_q$.

Let $q \in \{0, 1, ...\}$. For each $p \in \{0, 1, ..., 2^q\}$, define $B^p_q \equiv S \backslash A^{2q-p}_q$. We proceed by induction on p.

For the base step, let $p = 0$. By Step 4, $B^p_q = S \backslash A^{2q}_q = S \backslash A^1_0 = S \backslash S = \emptyset$, so $A^0_q \sim_l B^0_q$. For the inductive hypothesis, assume $p \in \{0, 1, .., 2^q - 1\}$ is such that $A^p_q \sim_l B^p_q$.

Define $A, B, C', B', A' \in \mathcal{A}$ by:

$$
\begin{aligned}
A &\equiv A^p_q, \\
B &\equiv \mathcal{G}^{S \backslash A}(A^1_q), \\
C' &\equiv A^{2^q - (p+1)}_q \\
B' &\equiv \mathcal{G}^{S \backslash C'}(A^1_q) \\
A' &\equiv S \backslash (C' \cup B')
\end{aligned}
$$

By Step 2, $C' \cup B' \sim_l A^{2q-p}_q$. By the Complement Lemma, $A' = S \backslash (C' \cup B') \sim_l S \backslash A^{2q-p}_q = B^p_q$. By the hypothesis on p, $A' \sim_l A^p_q = A$. By Step 4, $B \sim_l A^1_q \sim_l B'$. Thus by the Domination Lemma, $A \cup B \sim_l A' \cup B'$.

By Step 2, $A \cup B \sim_l A^{p+1}_q$, and by definition, $A' \cup B' = S \backslash A^{2^q - (p+1)}_q = B^{p+1}_q$, so altogether $A^{p+1}_q \sim_l B^{p+1}_q$. By induction on p, we are done. $\square$

∘ Step 6: Define the binary operation $\uplus$.

For each $q \in \{0, 1, ...\}$ and each pair $p, p' \in \{0, 1, ..., 2^q\}$ such that $p + p' \leq 2^q$, define $A^p_q \uplus A^{p'}_q \in \mathcal{A}$ by:

$$A^p_q \biguplus A^{p'}_q \equiv \mathcal{G}^S\big(A^p_q \cup \mathcal{G}^{S \backslash A^p_q}(A^{p'}_q)\big). \ \square$$

∘ Step 7: For each $q \in \{0, 1, ...\}$ and each pair $p, p' \in \{0, 1, ..., 2^q\}$ such that $p + p' \leq 2^q$,

$$A^p_q \biguplus A^{p'}_{q'} = A^{p+p'}_q.$$

Let $q \in \{0, 1, ...\}$ and let $p \in \{0, 1, ..., 2^q\}$. We proceed by induction on p'. For the base step, let $p' = 0$. Then by the Idempotence Lemma, $A^p_q \uplus A^{p'}_q = \mathcal{G}^S(A^p_q) = A^p_q$.

For the inductive hypothesis, assume $p' \in \{0, 1, ..., [2^q - 1] - p\}$ is such that $A_q^p \uplus A_q^{p'} = A_q^{p+p'}$. Define $A, B, C \in \mathcal{A}$ by:

$$\begin{aligned} &A \equiv A_q^p, \\ &B \equiv \mathcal{G}^{S\setminus A}(A_q^{p'}), \text{ and} \\ &C \equiv \mathcal{G}^{S\setminus(A\cup B)}(A_q^1). \end{aligned}$$

By hypothesis, $A_q^{p+p'} = A_q^p \uplus A_q^{p'} = \mathcal{G}^S(A \cup B)$. Since $\succsim_l$ satisfies 1-*SAD* and, by *monotonicity*, $S \succsim_l A \cup B$, thus by the 1-SAD Lemma, $\mathcal{G}^S(A \cup B) \sim_l A \cup B$. Altogether, $A_q^{p+p'} \sim_l A \cup B$.

By *monotonicity* and Step 4, $S\setminus A_q^{p+p'} \succsim_l \mathcal{G}^{S\setminus A_q^{p+p'}}(A_q^1) \sim_l A_q^1$, so by the Complement Lemma, $S\setminus(A \cup B) \sim_l S\setminus A_q^{p+p'} \succsim_l A_q^1$. By Step 3, $\succsim_l|_{S\setminus A}$ satisfies 2-*SAD*, so by the Greedy Removal Lemma, $\succsim_{l(S\setminus A)\setminus B} = \succsim_{lS\setminus(A\cup B)}$ satisfies 1-*SAD*. Thus by the 1-SAD Lemma, $A_q^1 \sim_l C$. By Step 4, $\mathcal{G}^{S\setminus A_q^{p+p'}}(A_q^1) \sim_l A_q^1$, so $\mathcal{G}^{S\setminus A_q^{p+p'}}(A_q^1) \sim_l C$.

Since $A_q^{p+p'} \sim_l A \cup B$ and $\mathcal{G}^{S\setminus A_q^{p+p'}}(A_q^1) \sim_l C$, thus by Step 2 and the Domination Lemma,

$$\begin{aligned} A_q^{p+p'+1} &\sim_l A_q^{p+p'} \cup \mathcal{G}^{S\setminus A_q^{p+p'}}(A_q^1) \\ &\sim_l (A \cup B) \cup C. \end{aligned}$$

By Step 5, $S\setminus A_q^p \sim_l A_q^{2^q-p}$. Since $2^q - p \geq p' + 1$, by Step 2 and *monotonicity*, $A_q^{2^q-p} \succsim_l A_q^{p'+1}$, so $S\setminus A_q^p \succsim_l A_q^{p'+1}$. By Step 3, $\succsim_l|_{S\setminus A_q^p}$ satisfies 2-*SAD*, so by the 1-SAD Lemma, $\mathcal{G}^{S\setminus A_q^p}(A_q^{p'+1}) \sim_l A_q^{p'+1}$.

Since $S\setminus A_q^p \succsim_l A_q^{p'+1}$, by Step 2 and *monotonicity*, $S\setminus A_q^p \succsim_l A_q^{p'}$. By Step 3, $\succsim_l|_{S\setminus A_q^p}$ satisfies 2-*SAD*, so by the 1-SAD Lemma, $A_q^{p'} \sim_l B$. By Step 4, $\mathcal{G}^{S\setminus A_q^{p'}}(A_q^1) \sim_l A_q^1$, and as argued above, $A_q^1 \sim_l C$, so $\mathcal{G}^{S\setminus A_q^{p'}}(A_q^1) \sim_l C$. Thus by Step 2 and the Domination Lemma, $A_q^{p'+1} \sim_l A_q^{p'} \cup \mathcal{G}^{S\setminus A_q^{p'}}(A_q^1) \sim_l B \cup C$.

Altogether, $\mathcal{G}^{S\setminus A_q^p}(A_q^{p'+1}) \sim_l B \cup C$. Since $\succsim_l$ satisfies 1-*SAD* and, by *monotonicity*, $S \succsim_l A_q^p \cup \mathcal{G}^{S\setminus A_q^p}(A_q^{p'+1})$, thus by the 1-SAD Lemma and *separability*,

$$\begin{aligned} A_q^p \uplus A_q^{p'+1} &= \mathcal{G}^S\big(A_q^p \cup \mathcal{G}^{S\setminus A_q^p}(A_q^{p'+1})\big) \\ &\sim_l A_q^p \cup \mathcal{G}^{S\setminus A_q^p}(A_q^{p'+1}) \\ &\sim_l A_q^p \cup (B \cup C) \\ &= A \cup (B \cup C). \end{aligned}$$

Since $(A\cup B)\cup C = A\cup(B\cup C)$, thus $A_q^{p+p'+1} \sim_l A_q^p \uplus A_q^{p'+1}$. By the Idempotence Lemma, $A_q^{p+p'+1} = A_q^p \uplus A_q^{p'+1}$.

By induction on p', for each $p' \in \{0, 1, ..., 2^q - 1\}$, $A_q^p \uplus A_q^{p'} = A_q^{p+p'}$. Since $q \in \{0, 1, ...\}$ and $p \in \{0, 1, ..., 2^q\}$ were arbitrary, we are done. $\square$

∘ Step 8: $\lim A_q^1 = \emptyset$.

By Step 2, *monotonicity*, and the Convergence Lemma, $(A_q^1) \in \mathcal{A}^{\mathbb{N}}$ is convergent. Assume, by way of contradiction, there is $s \in \lim A_q^1$.

We claim for each $i \in \mathbb{N}$ and each $k \in \mathbb{N}$, $A_i^1 \succ_l \{s, s+1, ..., s+k\}$. We proceed by induction on k. For the base step, let $k = 0$. Then by *continuity* and *monotonicity*, for each $i \in \mathbb{N}$, $A_i^1 \succsim_l \lim A_q^1 \succsim_l \{s\}$, so by Step 1, $A_i^1 \succ_l A_{i+1}^1 \succsim_l \{s\}$.

For the inductive hypothesis, assume $k \in \mathbb{N}$ is such that for each $i \in \mathbb{N}$, $A_i^1 \succ_l \{s, s+1, ..., s+k\}$. Let $i \in \mathbb{N}$. By Step 1, $H(A_i^1) \sim_l A_{i+1}^1$. By Step 4 and Step 1, $H'(A_{i+1}^1) \sim_l A_{i+1}^1 \sim_l \mathcal{G}^{S \setminus A_{i+1}^1}(A_{i+1}^1)$. By the inductive hypothesis, $A_{i+1}^1 \succ_l \{s, s+1, ..., s+k\}$. By Step 4 and the inductive hypothesis, $\mathcal{G}^{S \setminus A_{i+1}^1}(A_{i+1}^1) \sim_l A_{i+1}^1 \succ_l \{s\} \succsim_l \{s+k+1\}$. Thus by Step 1 and two applications of the Domination Lemma,

$$\begin{aligned} A_i^1 &\sim_l H(A_i^1) \cup H'(A_i^1) \\ &\sim_l A_{i+1}^1 \cup \mathcal{G}^{S \setminus A_{i+1}^1}(A_{i+1}^1) \\ &\succ_l \{s, s+1, ..., s+k\} \cup \{s+k+1\}. \end{aligned}$$

Since $i \in \mathbb{N}$ was arbitrary, for each $i \in \mathbb{N}$, $A_i^1 \succ_l \{s, s+1, ..., s+k+1\}$. By induction on k, we are done.

Thus by *continuity*, for each $i \in \mathbb{N}$, $A_i^1 \succsim_l \lim\{s, s+1, ..., s+k\} = \{s, s+1, ...\}$. If $s = 1$, then by Step 1, $S = A_0^1 \succ_l A_1^1 \succsim_l \{1, 2, ...\} = S$, contradicting $S \sim_l S$. Thus $s > 1$, so by 2-*SAD*, for each $i \in \mathbb{N}$, $A_i^1 \succ_l \{s, s+1, ...\} \succsim_l \{s-1\}$.

Thus for each $s \in S$, if for each $i \in \mathbb{N}$, $A_i^1 \succ_l \{s\}$, then (1) $s \neq 1$, and (2) for each $i \in \mathbb{N}$, $A_i^1 \succ_l \{s-1\}$. Then there can be no such s, contradicting that there is. □

∘ Step 9: Define $\{A_v\}_{v \in \mathbb{2}} \subseteq \mathcal{A}$ and $\{\mathcal{Z}_v\}_{v \in \mathbb{2}} \subseteq [\mathcal{A}/\sim_l]$ such that for each pair $v, v' \in \mathbb{2}$ such that $v' > v$, $A_{v'} \succ_l A_v$.

Let $v \in \mathbb{2}$. Then there are $p, q \in \{0, 1, ...\}$ such that $p \leq 2^q$ and $v = \dfrac{p}{2^q}$. Define

$$\begin{aligned} A_v &\equiv A_q^p, \text{ and} \\ \mathcal{Z}_v &\equiv \{A \in \mathcal{A} | A \sim_l A_v\}. \end{aligned}$$

By Step 4, this is well-defined.

Let $v, v' \in \mathbb{2}$ such that $v' > v$. Since $v' - v > 0$, by Step 1, Step 2, and *monotonicity*, $A_{v'-v} \succ_l \emptyset$. Similarly, $A_{1-v} \succ_l \emptyset$. By Step 5, $S \setminus A_v \sim_l A_{1-v} \succ_l \emptyset$, so $\mathcal{G}^{S \setminus A_v}(A_{v'-v}) \succ_l \emptyset$. Since $\succsim_l$ satisfies 1-*SAD* and, by *monotonicity*, $S \succsim_l A_v \cup \mathcal{G}^{S \setminus A_v}(A_{v'-v})$, thus by the 1-SAD Lemma, $\mathcal{G}^S\big(A_v \cup \mathcal{G}^{S \setminus A_v}(A_{v'-v})\big) \sim_l A_v \cup \mathcal{G}^{S \setminus A_v}(A_{v'-v})$. Thus by Step 6, $A_{v'} = \mathcal{G}^S\big(A_v \cup \mathcal{G}^{S \setminus A_v}(A_{v'-v})\big) \sim_l A_v \cup \mathcal{G}^{S \setminus A_v}(A_{v'-v})$, so by *separability*, $A_{v'} \succ_l A_v$. □

∘ Step 10: For each convergent pair $(v_i), (v'_i) \in \mathbb{2}^{\mathbb{N}}$ such that $\lim v_i = \lim v_i$, if $(A_{v_i}), (A_{v'_i}) \in \mathcal{A}^{\mathbb{N}}$ are convergent, then $\lim A_{v_i} \sim_l \lim A_{v'_i}$.

Define $v_\infty \equiv \lim v_i = \lim v'_i$, $A_\infty \equiv \lim A_{v_i}$, and $A'_\infty \equiv \lim A_{v'_i}$. Let $v \in \mathbb{2}$. If $v > v_\infty$, then there is $v' \in \mathbb{2}$ such that $v > v' > v_\infty$. Since $\lim v_i = v_\infty$, there is $i^* \in \mathbb{N}$ such that for each $i \geq i^*$, $v' > v_i$. Thus for each $i \geq i^*$, $A_{v'} \succ_l A_{v_i}$, so by Step 9 and *continuity*, $A_v \succ_l A_{v'} \succsim_l A_\infty$. By the same argument, $v > v_\infty$ implies $A_v \succ_l A'_\infty$.

Similarly, for each $v \in \mathbb{2}$, $v_\infty > v$ implies $A_\infty \succ_l A_v$ and $A'_\infty \succ_l A_v$.

Assume, by way of contradiction, $A_\infty \not\sim_l A'_\infty$. Assume, without loss of generality, $A_\infty \succ_l A'_\infty$. Define $G \equiv \mathcal{G}^S(A'_\infty)$. Since $\succsim_l$ satisfies 1-*SAD* and, by *monotonicity*, $S \succsim_l A'_\infty$, thus by the 1-SAD Lemma, $A_\infty \succ_l A'_\infty \sim_l G$.

Necessarily $S\backslash G \succ_l \emptyset$, else by the Complement Lemma and *monotonicity*, $G \succsim_l S \succsim_l A_\infty$, contradicting $A_\infty \succ_l G$. Then there is $q \in \mathbb{N}$ such that $S\backslash G \succ_l A_q^1$, else by Step 8 and *continuity*, $\emptyset = \lim A_q^1 \succsim_l S\backslash G$, contradicting $S\backslash G \succ_l \emptyset$.

Since $\succsim_l$ satisfies 3-*SAD*, thus by the Greedy Removal Lemma, $\succsim_l|_{S\backslash G}$ satisfies 2-*SAD*, so by Step 1 and the 1-SAD Lemma, for each $q' \in \mathbb{N}$ such that $q' \geq q$, $\mathcal{G}^{S\backslash G}(A_{q'}^1) \sim_l A_{q'}^1$. Then by Step 1, $(\mathcal{G}^{S\backslash G}(A_{q'}^1)) \in \mathcal{A}^{\mathbb{N}}$ is monotonic, so by the Convergence Lemma, it is convergent. By the Limit-Order Lemma and Step 8, $\lim \mathcal{G}^{S\backslash G}(A_{q'}^1) \sim_l \lim A_{q'}^1 = \emptyset$, so $\lim \mathcal{G}^{S\backslash G}(A_{q'}^1) = \emptyset$. Then $\lim G \cup \mathcal{G}^{S\backslash G}(A_{q'}^1) = G$. Thus there is $q^* \in \mathbb{N}$ such that (1) $q^* \geq q$ and thus $\mathcal{G}^{S\backslash G}(A_{q^*}^1) \sim_l A_{q^*}^1$, and (2) $A_\infty \succ_l G \cup \mathcal{G}^{S\backslash G}(A_{q^*}^1)$, else by *continuity* $G \succsim_l A_\infty$, contradicting $A_\infty \succ_l G$.

We proceed by reaching a contradiction in three cases:

Case 1: $v_\infty = 0$. Then since $\frac{1}{2^{q^*}} > v_\infty$, by *monotonicity*, $A_{q^*}^1 \succ_l A_\infty \succ_l G \cup \mathcal{G}^{S\backslash G}(A_{q^*}^1) \succsim_l \mathcal{G}^{S\backslash G}(A_{q^*}^1) \sim_l A_{q^*}^1$, contradicting $A_{q^*}^1 \sim_l A_{q^*}^1$.

Case 2: $v_\infty = 1$. Since $v_\infty > \frac{2^{q^*}-1}{2^{q^*}}$, $G \sim_l A'_\infty \succ_l A_{q^*}^{2^{q^*}-1}$. Since $\mathcal{G}^{S\backslash G}(A_{q^*}^1) \sim_l A_{q^*}^1 \succsim_l \mathcal{G}^{S\backslash A_{q^*}^{2^{q^*}-1}}(A_{q^*}^1)$, thus by the Domination Lemma, $A_\infty \succ_l G \cup \mathcal{G}^{S\backslash G}(A_{q^*}^1) \succ_l A_{q^*}^{2^{q^*}-1} \cup \mathcal{G}^{S\backslash A_{q^*}^{2^{q^*}-1}}(A_{q^*}^1)$. Since $\succsim_l$ satisfies 1-*SAD* and, by *monotonicity*, $S \succsim_l A_{q^*}^{2^{q^*}-1} \cup \mathcal{G}^{S\backslash A_{q^*}^{2^{q^*}-1}}(A_{q^*}^1)$, thus by the 1-SAD Lemma, $A_{q^*}^{2^{q^*}-1} \cup \mathcal{G}^{S\backslash A_{q^*}^{2^{q^*}-1}}(A_{q^*}^1) \sim_l A_{q^*}^{2^{q^*}-1} \uplus A_{q^*}^1$. But then by Step 7, Step 4, and Step 1, $A_\infty \succ_l A_{q^*}^{2^{q^*}-1} \uplus A_{q^*}^1 = A_{q^*}^{2^{q^*}} = S$, contradicting $S \succsim_l A_\infty$.

Case 3: $v_\infty \in (0,1)$. Define $\epsilon^* \equiv \frac{1}{2^{q^*}}$. Since $\mathbb{2}$ is dense in $[0,1]$, there is $v^* \in \mathbb{2}$ such that $1 \geq v^* + \epsilon^* > v_\infty > v^*$. Then $A_{v^*+\epsilon^*} \succ_l A_\infty \succ_l G \sim_l A'_\infty \succ_l A_{v^*}$.

Since $1 - v^* \geq \epsilon^*$, thus by Step 5, $S\backslash A_{v^*} \sim_l A_{1-v^*} \succsim_l A_{\epsilon^*} = A_{q^*}^1$. Then by Step 3 and the 1-SAD Lemma, $\mathcal{G}^{S\backslash G}(A_{q^*}^1) \sim_l A_{q^*}^1 \sim_l \mathcal{G}^{S\backslash A_{v^*}}(A_{q^*}^1)$. Since $G \succ_l A_{v^*}$, thus by the Domination Lemma, $A_\infty \succ_l G \cup \mathcal{G}^{S\backslash G}(A_{q^*}^1) \succ_l A_{v^*} \cup \mathcal{G}^{S\backslash A_{v^*}}(A_{q^*}^1)$.

Since $\succsim_l$ satisfies 1-*SAD* and, by *monotonicity*, $S \succsim_l A_{v^*} \cup \mathcal{G}^{S\backslash A_{v^*}}(A_{q^*}^1)$, thus by the 1-SAD Lemma, $A_{v^*} \uplus A_{\epsilon^*} = A_{v^*} \cup \mathcal{G}^{S\backslash A_{v^*}}(A_{q^*}^1)$. But then by Step 7, $A_\infty \succ_l A_{v^*} \uplus A_{\epsilon^*} = A_{v^*+\epsilon^*}$, contradicting $A_{v^*+\epsilon^*} \succ_l A_\infty$. $\square$

$\circ$ Step 11: Conclude.

We verify that $\{\mathcal{Z}_v\}_{v \in \mathbb{2}}$ satisfies [DS1], [DS2], and [DS3].

DS1: By Step 1 and Step 9, $\emptyset = A_0 \in \mathcal{Z}_0$ and $S = A_1 \in \mathcal{Z}_1$.

DS2: Let $v, v' \in \mathbb{2}$ such that $v + v' \leq 1$. Then there are $p, p', q \in \{0, 1, ...\}$ such that $v = \frac{p}{2^q}$ and $v' = \frac{p'}{2^q}$, and $p + p' \leq 2^q$.

By construction, $A_q^p \in \mathcal{Z}_v$, $A_q^{p'} \in \mathcal{Z}_{v'}$, and $A_q^{p+p'} \in \mathcal{Z}_{v+v'}$. By Step 5, $S\backslash A_q^p \sim_l A_q^{2^q-p}$. Since $p + p' \leq 2^q$, thus by *monotonicity* and Step 2, $S\backslash A_q^p \succsim_l A_q^{p'}$. By Step 3, $\succsim_l|_{S\backslash A_q^p}$

satisfies 2-SAD, so by the 1-SAD Lemma,

$$\mathcal{G}^{S\backslash A_q^p}(A_q^{p'}) \sim_l A_q^{p'}.$$

Thus we have disjoint $A_q^p \in \mathcal{Z}_v$ and $\mathcal{G}^{S\backslash A_q^p}(A_q^{p'}) \in \mathcal{Z}_{v'}$.

Since $\succsim_l$ satisfies 1-SAD and, by *monotonicity*, $S \succsim_l A_q^p \cup \mathcal{G}^{S\backslash A_q^p}(A_q^{p'})$, thus by the 1-SAD Lemma, $\mathcal{G}^S\big(A_q^p \cup \mathcal{G}^{S\backslash A_q^p}(A_q^{p'})\big) \sim_l A_q^p \cup \mathcal{G}^{S\backslash A_q^p}(A_q^{p'})$. Thus by Step 7,

$$\begin{aligned} A_q^{p+p'} &= A_q^p \biguplus A_q^{p'} \\ &= \mathcal{G}^S\big(A_q^p \cup \mathcal{G}^{S\backslash A_q^p}(A_q^{p'})\big) \\ &\sim_l A_q^p \cup \mathcal{G}^{S\backslash A_q^p}(A_q^{p'}), \end{aligned}$$

so $A_q^p \cup \mathcal{G}^{S\backslash A_q^p}(A_q^{p'}) \in \mathcal{Z}_{v+v'}$.

DS3: Let $(v_i), (v_i'), (w_i), (w_i') \in 2^{\mathbb{N}}$ be monotonic such that
(i) for each $i \in \mathbb{N}$, $v_i + w_i \leq 1$ and $v_i' + w_i' \leq 1$, and
(ii) $\lim v_i = \lim v_i'$ and $\lim w_i = \lim w_i'$.
For each $i \in \mathbb{N}$, define $A_i, A_i', B_i, B_i' \in \mathcal{A}$ by:

$$\begin{aligned} A_i &\equiv A_{v_i}, \\ A_i' &\equiv A_{v_i'}, \\ B_i &\equiv \mathcal{G}^{S\backslash A_i}(A_{w_i}), \text{ and} \\ B_i' &\equiv \mathcal{G}^{S\backslash A_i'}(A_{w_i'}). \end{aligned}$$

By Step 5, for each $i \in \mathbb{N}$, $S\backslash A_i \sim_l A_{1-v_i}$, so since $1 - v_i \geq w_i$, by Step 9, $S\backslash A_{v_i} \succsim_l A_{w_i}$. Thus by Step 3 and the 1-SAD Lemma, for each $i \in \mathbb{N}$, $B_i \sim_l A_{w_i}$. Similarly, for each $i \in \mathbb{N}$, $B_i' \sim_l A_{w_i'}$. Thus for each $i \in \mathbb{N}$,
(i) $A_i \in \mathcal{Z}_{v_i}$, $A_i' \in \mathcal{Z}_{v_i'}$, $B_i \in \mathcal{Z}_{w_i}$, and $B_i' \in \mathcal{Z}_{w_i'}$, and
(ii) $A_i \cap B_i = A_i' \cap B_i' = \emptyset$.
Since $\succsim_l$ satisfies 2-SAD and, by Step 2 and *monotonicity*, $(A_i), (A_i'), (B_i), (B_i')$ are monotonic, thus by the Convergence Lemma, $(\mathcal{G}_1^S(A_i, B_i))$, $(\mathcal{G}_2^S(A_i, B_i))$, $(\mathcal{G}_1^S(A_i', B_i'))$, and $(\mathcal{G}_2^S(A_i', B_i'))$ are convergent. By the Idempotence Lemma, these are, respectively $(A_i), (B_i), (A_i'), (B_i')$; thus $(A_i), (B_i), (A_i'), (B_i')$ are convergent. By Step 10, $\lim A_i \sim_l \lim A_i'$. By the Limit-Order Lemma and Step 10, $\lim B_i \sim_l \lim A_{w_i} = \lim A_{w_i'} \sim_l \lim B_i'$. Altogether:
(iii) $\lim A_i = \lim A_i'$ and $\lim B_i = \lim B_i'$. ■

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