The Demand and Supply for Favours in Dynamic Relationships

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PRELIMINARY AND INCOMPLETE

Abstract

We characterise the optimal demand and supply for favours in a dynamic principal-agent model of joint production, in which heterogenous project opportunities are stochastically generated and publicly observed upon arrival. No transfers are available to distribute the utility from joint projects across periods and there is limited commitment, so that expected future production must provide incentives for current production decisions. Our results characterise those subgame perfect equilibria that maximise the principals payoffs, and we establish that the principal's supply of favours (the production of projects that benefit the agent but not the principal) is backloaded, while the principal's demand for favours (the production of projects that benefit the principal but not the agent) is frontloaded. We give an exact characterisation of principal-optimal equilibria when project opportunities follow a Markov process.

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1 Introduction

Many long-running relationships are not based on explicit contractual engagements but rather depend on an implicit web of mutual obligations generated by the exchange of favours. In this paper, we study a dynamic relationship between a principal and and agent in which (a) heterogenous opportunities for joint production and trade arrive exogenously, (b) utility from these projects is non-transferable, and (c) there is limited commitment to future production. Our results characterise those subgame perfect equilibria that maximise the principal's payoffs.

Because potential projects are differentiated and utility is non-transferable, a critical decision in this relationship concerns the selection of those project opportunities that are actually produced. We interpret the demand for a favour as the production of a project that benefits the principal but not the agent, and the supply of a favour as the production of a project that benefits the agent but not the principal. By demanding a favour, the principal incurs a utility debt towards the agent whose level is determined by the agent's loss from producing that particular project. The principal's repays this debt through taking losses, denominated in her own utility units, on the production of future projects. However, because future opportunities are stochastic and outside the principal's control, her ability to return utility to the agent is constrained.

Because optimal relationships are Pareto-efficient, mutually beneficial projects are produced and mutually disagreeable projects are not produced. Therefore, optimal relationships are completely characterised by their associated demand and supply for favours. Our main results establish that the supply for favours is backloaded, while the demand for favours is front-loaded. Both these properties imply that the agent is better off as the relationship progresses, and hence our results are in line with well-known backloading results for dynamic relationships (e.g., Ray (2002)). However, our results are not driven by the standard calculus through which promising high future rewards to the agent optimally provides incentives for his current actions. Rather, backloading the supply of favours is part of the principal's optimal distribution of project production over future opportunities. Specifically, some projects are more efficient as tools for the principal to return utility to the agent, in that the agent's gains from trade are high relative to the principal's loss. Our backloading result is driven by the fact that the principal will not promise to supply a favour to the agent through a less efficient project if some future opportunity with a more efficient project is passed over. Our frontloading result for the demand for favours is driven by similar incentives: if the principal ever passes over asking for a favour at a project that has high efficiency in terms of extracting utility from the agent, then any future opportunity at lower-efficiency projects must also be passed over.

That is, the principal's ability to extract utility from the agent is conditioned on the availability of projects with which to return this utility in the future. From Proposition 1, we know that the set of such project becomes more constrained over the relationship's lifetime, so that the principal must reduce her demands on the agent over time: a favour that is passed over at some history is never asked in any future history (Part 2).

Our paper has close links to the literature on dynamic risk-sharing studies the transfer of stochastic income between players as a tool for self-insurance (e.g., Kocherlakota (1996), Ligon et al. (2002) and Thomas and Worrall (1988)). These models typically feature a stationary environment in which income shocks are iid, while we allow for an arbitrary stochastic process generating project opportunities (Dixit et al. (2000), who focus on a Markov process for income shocks, is an important exception). Also, in our model both the principal and the agent are risk-neutral in every stage game, although utility is non-transferable and their payoffs, because they depend on the current project opportunity, are time-varying. Therefore, risk-sharing plays no role in our results, and our focus is on the selection of those joint projects that are produced. A subsequent literature has studied the case in which the (risk-neutral) players' ability to transfer income, interpreted as providing a favour, is privately observed (e.g., Abdulkadiroglu and Bagwell (2012), Hauser and Hopenhayn (2008) and Möbius (2001). Related to these is the literature on dynamic contracts with and without commitment, e.g., Garrett and Pavan (2012), Guo and Hörner (2014) and Lipnowski and Ramos (2015)). These environments give rise to equilibria described by "chips mechanisms" in which truth-telling constraints ration players' demands for favours as a function of their previous supply. In this paper, project opportunities and their production are publicly observed. By abstracting from informational asymmetries between the principal and the agent, we allow for a rich space of possible project opportunities, and we obtain detailed results on the dynamics of project selection and production.

Another closely related literature studies related models in which the agent and/or the principal take actions in the relationship (e.g., Albuquerque and Hopenhayn (2004), Kovrijnykh (2013), Thomas and Worrall (1994, 2014)), which has close connections to the literature on relational contracts (e.g., Doornik (2006), Halac (2014), Levin (2003)). These models typically exploit transferability of utility to focus on stationary equilibria, while optimal relationships are not stationary in our environment, even if we assume that project opportunities are generated by an iid, or Markov, stochastic process. A notable exception is Board (2011), who characterises optimal supply contracts when potential suppliers have stochastic costs (Fong and Li (2013) also consider non-stationary dynamics in relational contracts).

2 Model

A principal and an agent participate in a long-lived relationship in which opportunities for joint projects may accrue in each period $t = 1, 2, ...$ Specifically, let $\mathcal{U} \subset \mathbb{R}^2$ be a finite set and let $u = \{u_t\}_{t \geq 1}$ be a $\mathcal{U}$-valued stochastic process that describes the arrival of projects over time. Let $u^t = (u_1, ..., u_t)$ denote a project history at t. In any period, any fraction of a project can be produced if the principal and the agent unanimously agree to do so. Specifically, the principal declares $0 \leq k_P \leq 1$, the agent declares $0 \leq k_A \leq 1$, and the project is produced with intensity

$$k = \begin{cases} k_P & \text{if } k_P = k_A \\ 0 & \text{otherwise.} \end{cases}$$

The payoffs to the principal and the agent if the project at t is produced with intensity k are $ku_t = (ku_{P,t}, ku_{A,t})$. Both project opportunities and production decisions, and hence all players' payoffs, are publicly observed. Because each player receives a payoff of 0 if no project is produced, it follows that player i prefers to produce the project if $u_{i,t} > 0$ and not to produce it if $u_{i,t} < 0$. For simplicity, we assume that if a joint project is available, then the players' preferences over production are strict. Specifically, if $u \neq 0$, then $u_A \neq 0$ and $u_P \neq 0$. Finally, the players discount future payoffs with common factor $0 \leq \delta < 1$.

A *relationship process* $\kappa = \{\kappa_t\}_{t \geq 1}$ is a stochastic process such that $0 \leq \kappa_t \leq 1$

for all t. We interpret κ_t as the recommended intensity with which the project at t is produced, with this recommendation conditioned on the project history at t. Formally, we impose that all relationship processes κ are adapted to the filtration generated by u. In words, a relationship process specifies a complete plan for what projects should be produced by the principal and the agent in all contingencies that can arise during their interaction. Let $\mathcal{K}$ denote the set of all relationship processes.

Given a relationship process κ and time t, let

$$U_{i,t} = \mathbb{E}_t \sum_{t'=t}^{\infty} \delta^{t'-t} \kappa_{t'} u_{i,t'},$$

denote the associated discounted sum of payoffs to player i starting from t, where $\mathbb{E}_t$ stands for the expectation conditional on the information available at t, which resides in project histories u^t. Note that we can rewrite

$$U_{i,t} = \kappa_t u_{i,t} + \delta \mathbb{E}_t U_{i,t+1}.$$

A relationship process κ is *player i-feasible at t* if $U_{i,t} \geq 0$, and simply *player-i feasible* if it is player-i feasible at all t. A relationship process κ^* is *optimal* if (i) it is both principal and agent-feasible, and (ii) $\mathbb{E}_0 U^*_{P,1} \geq \mathbb{E}_0 U_{P,1}$ for all relationship processes κ that are both principal and agent-feasible. Optimal relationship processes are those feasible relationship process that are preferred by the principal, and they always exist, as we establish in Lemma 2 in the Appendix. Our most general results are obtained in the model in which the principal can commit to her production decisions in the relationship. In that case, we require only that relationship processes are agent-feasible.

Any feasible relationship process can be supported by a subgame perfect equilibrium of the game between the principal and the agent. First, note that 0 is a subgame perfect equilibrium payoff for both the principal and the agent: this payoff is attained by the *no-production* strategy profile in which both players refuse to undertake any project after any history. Second, note that no subgame perfect equilibrium can deliver a payoff lower than 0 to any player: both players can secure the payoff 0 unilaterally following any history by refusing to undertake any future projects. By definition, a relationship process is feasible if, following all project histories, both players obtain at least their payoff from the no-production equilibrium which describes the worst possible outcome of their interactions. Therefore, by standard arguments, feasible relationship processes characterise all subgame perfect equilibrium payoffs of this game.

Optimal relationship processes must be Pareto efficient. In particular, this implies that if the preferences of the principal and the agent over the project at t are aligned, then an optimal relationship process implements jointly optimal decisions.

Lemma 1. *If the relationship process κ^* is optimal, then*

1. *if $u_{P,t}, u_{A,t} > 0$, then $\kappa^*_t = 1$, and*
2. *if $u_{P,t}, u_{A,t} < 0$, then $\kappa^*_t = 0$.*[1]

This simple result has the important implication that an optimal relationship process can be identified with the production decisions it prescribes for those projects on which the two players disagree. To this end, define the sets $\mathcal{D} = \{v \in \mathcal{U} : v_P > 0 > v_A\}$ and $\mathcal{S} = \{w \in \mathcal{U} : w_A > 0 > w_P\}$. Given a relationship process κ, we say that the principal *demands a favour with intensity* κ_t *at* t if $v_t \in \mathcal{D}$, and conversely that the principal *supplies a favour with intensity* κ_t *at* t if $w_t \in \mathcal{S}$. Given some relationship process κ, define the demand process $D = \{D_t\}_{t\geq 1}$ such that $\Delta D_t = \kappa_t$ if and only if $v_t \in \mathcal{D}$. Similarly, define the supply process $S = \{S_t\}_{t\geq 1}$ such that $\Delta S_t = \kappa_t$ if and only $w_t \in \mathcal{S}$. Lemma 1 implies that any optimal relationship process κ^* is completely characterised by its corresponding demand and supply processes (D^*, S^*).

3 Commitment by the Principal

It turns out that the decomposition of an optimal relationship process into demand and supply processes is quite fruitful. In this section, we assume one-sided commitment by the principal and characterise optimal relationship processes in two steps. First, we fix an optimal demand process and provide a characterisation of the associated optimal supply process. Second, we fix an optimal supply process and provide a characterisation of the associated optimal demand process. These results hold generally: in particular, we impose no restriction on the underlying project process u. In our third set of results in this section, we impose additional structure on the process driving project arrivals

[1] The proofs of all results are in the Appendix.

to refine our results: we assume that u is a Markov process and provide a complete characterisation of optimal relationship process.

For all these results, a first task is to determine those projects that the principal would prefer to use to demand favours from the agent, or to supply favours to the agent, irrespective of any dynamic incentive considerations. To this end, we define an ordering of projects such that $u \succ u'$ if and only $|{}^{u_P}/_{u_A}| > |{}^{u'_P}/_{u'_A}|$. In words, if $v, v' \in \mathcal{D}$ and $v \succ v'$, then project v has a comparative advantage over project v' when it is used by the principal to demand favours from the agent: in this case the ratio ${}^{v_P}/_{|v_A|}$ measures the efficiency of project v, from the principal's perspective, as a tool for extracting utility from the agent. Conversely, if $w, w' \in \mathcal{S}$ and $w' \succ w$, then project w has a comparative advantage over project w' when it is used by the principal to supply favours to the agent: in this case the ratio ${}^{w_A}/_{|w_P|}$ measures the efficiency of project w, from the principal's perspective, as a tool for providing utility to the agent. For simplicity, we assume that the ordering $\succ$ is complete on $\mathcal{D} \cup \mathcal{S}$ (i.e., that all project pairs are ranked strictly by comparative advantage).

3.1 Optimal Supply of Favours

Our first main result shows backloading the supply of favours is optimal.

Proposition 1. *Suppose that the demand and supply processes (D^*, S^*) are optimal. Then (without loss of generality for optimal payoffs), for all $w \in \mathcal{S}$, there exists a $\mathbb{R}_+$-valued process $T^w = \{T^w\}_{t \geq 1}$ such that, for all t,*

$$\Delta S_t^* = \begin{cases} 1 & \text{if } T_t^{w_t} \leq t, \\ t + 1 - T_t^{w_t} & \text{if } t < T_t^{w_t} < t + 1, \\ 0 & \text{if } T_t^{w_t} \geq t + 1. \end{cases} \tag{1}$$

Given any time t and project w, the process T^w has the following properties.

1. $T_{t'}^w = T_t^w$ *for all* $t' > t$ *such that* $D_{t'}^* = D_t^*$.
2. T^w *is non-increasing.*
3. $T_t^{w'} = \infty$ *for all* $w' \in \mathcal{S}$ *if and only if* $D_t^* = 0$.

4. If $T_t^w < T_{t-1}^w$, then $V_{A,t}^* = 0$.

5. If $T_t^w < \infty$, then $T_t^{w'} \leq t$ for all $w \succ w'$, and if $T_t^w > t$, then $T^{w'} = \infty$ for all $w' \succ w$.

Optimal supply processes can be characterised by simple time-threshold rules. The collection of threshold processes $\{T^w\}_{w \in \mathbb{S}}$ identifies those projects that are used to supply favours by the principal: any project with $T_t^{w_t} \leq t$ is produced with full intensity at t; no project with $T_t^{w_t} \leq t+1$ is produced with any intensity at t; and any project with $t < T_t^{w_t} < t+1$ is produced with interior intensity. We interpret $\lfloor T_t^w \rfloor$ as the time at which the principal plans to use project w to supply favours to the agent, conditional the relationship's status at time t. As we will also establish for demand processes below, the linearity of stage payoffs in production intensity implies that optimal supply processes are typically bang-bang. Interior project intensities are needed to overcome rounding issues due to the discreteness of time periods: project intensities would be 0 or 1 with probability 1 in the continuous-time limit of our model.

Optimal supply processes are constant when the relationship is in between two favours demanded by the principal (Part 1). That is, the principal adjusts her plan for returning utility to the agent only after she has asked for a new favour, for which she incurs a utility debt. Also, asking for a new favour never leads the principal to enact a less generous supply process: the time T_t^w at which she starts to use project w to supply favours to the agent can only move forward (Part 2). Notice the associated backloading property exhibited by optimal supply processes, which we will discuss in detail below: by agreeing to use some project to supply a favour to the agent following some history, the principal also commits to supplying a favour to the agent in all future occurrences of this project. Not surprisingly, it is never optimal for the principal to supply any favours to the agent before she has demanded any favours. However, as soon as the principal demands a first favour, then in exchange she commits to supplying favours infinitely often in the future (Part 3). Now while the principal can adjust her supply of favours whenever she demands a new favour, she only does this if failure to do so violates agent-feasibility. In particular, if an optimal supply process becomes more generous when the principal demands a new favour, the agent must be indifferent between enacting the project and quitting the relationship (Part 4). In particular, by Part 3 this implies that the agent's feasibility constraint always binds when a first favour is demanded. This does not imply that the agent's feasibility constraint always binds in a relationship process. In fact, the agent's constraint must be slack for some histories following a demand for a favour: if not, then the agent's continuation payoff following the favour would be 0, and because providing a favour is costly, the agent's feasibility constraint would fail. Therefore, the principal's incentives to smooth her supply of favours while no intervening favours are demanded does benefit the agent, who obtains utility when not strictly necessitated by incentives. However, the principal delays adjusting thresholds until doing so is necessary.

The principal's selection of projects to use to supply favours is driven by their rank in comparative advantage: at most one project w^* has $t < T_t^{w^*} < \infty$; for all projects $w^* \succ w$ we have $T_t^w \leq t$ (and hence $\Delta S_t^* = 1$); and for all projects $w \succ w^*$ we have $T_t^w = \infty$ (and hence $\Delta S_t^* = 0$) (Part 5). That the time thresholds $\{T^w\}_{w \in \mathbb{S}}$ are non-increasing implies that this threshold project w^* is increasing (with respect to $\succ$) over time. Put differently, the principal transitions to supplying favours with less advantageous projects as the relationship matures and her demands for favours accumulate.

The key insights for our backloading result comes from the arguments that establish Parts 2 and 5 of Proposition 1. These rely on intertemporal smoothing, for which the driving force is the principal's incentive to concentrate the supply of future favours on those projects with a comparative advantage for delivering utility to the agent (i.e., that are lower ranked under $\succ$). More precisely, fix a project history u^t, times $t', t'' \geq t$, projects $\underline{w} \succ \overline{w}$ and histories $u^{t'}$ and $u^{t''}$ with $u_{t'} = \overline{w}$ and $u_{t''} = \underline{w}$. If the principal supplies a favour at $u^{t''}$ but not at $u^{t'}$, then she can gain by increasing her supply at $u^{t'}$ and deceasing it at $u^{t'}$. This can be done while keeping the agent indifferent at u^t, so that no agent-feasibility constraint is violated at any time $r \leq t$. The difficulty is to do this in a way that does not violate any feasibility constraints between times t and either t' or t''. Clearly, the agent is better off at $u^{t'}$. The agent is worse off at $u^{t''}$, but his feasibility constraint must still be satisfied. The reason is that when a favour is supplied, the agent always achieves a stage payoff that is higher than under the no-production process, so that supplying fewer favours at $u^{t''}$ cannot lead to a failure of agent-feasibility. The problem, however, is that offering a lower payoff at $u^{t''}$ can lead to the failure of a feasibility constraint between t and t''. However, this concern is only relevant if, between t and t'', there is some history at which the principal demands a favour. If not, then by the same argument as detailed above, supplying a favour with lower intensity at $u^{t''}$ cannot lead to a failure of agent-feasibility prior to t''. That is, the principal is free to transfer the supply of favours from $\underline{w}$ to $\overline{w}$ while no further favours have been asked. When can the principal transfer the supply of favours from $\underline{w}$ to $\overline{w}$ without regard for whether or not intervening favours have been asked? This is possible when history $u^{t'}$ follows history $u^{t''}$. In that case, the smoothing operation described above increases the agent's payoff in the future, so that the incentives that support the asking of intervening favours are strengthened. Together, these steps imply that the principal cannot supply favour $\underline{w}$ in the current period if future opportunities to supply favour $\overline{w}$ are unused.

3.2 Optimal Demand for Favours

Our second main result shows that frontloading the demand for favours is optimal. Here, as with the optimal supply of favours, the ranking of two projects v_t and $v_{t'}$ with respect to $\succ$ will be critical to determining whether the principal demands a favour at histories u^t and $u^{t'}$. However, an important remark is that while their ranking in comparative advantage orders the principal's marginal benefit from demanding these favours, the principal's marginal cost to demanding these favours is endogenous. Specifically, the principal's accumulated commitments to supply favours can differ at these two histories. The marginal cost of asking for an additional favour is measured by the project w that has a comparative advantage in providing utility to the agent, but only among those projects that have not yet been committed to supplying favours.

We need to introduce some notation before stating our result. Let $\overline{W}_{t-1} = \min_{\succ}\{w \in \mathcal{S} : T^w_{t-1} > t+1\}$ and $\underline{W}_{t-1} = \max_{\succ}\{w \in \mathcal{S} : T^w_{t-1} < t+2\}$. To interpret these two projects, suppose that the principal is in a position to demand a favour from the agent on some project v_t at t. The project $\overline{W}_{t-1}$ is the principal's preferred project among those that, if the principal does not demand a favour at t, would not be used to supply favours to the agent at $t+1$, given the principal's supply commitments $\{T^w_{t-1}\}_{w\in\mathcal{S}}$ inherited from $t-1$. Similarly, the project $\underline{W}_{t-1}$ is the principal's least preferred project among those that, if the principal does not demand a favour at t, would be used to supply favours to the agent with some intensity at $t+1$, given the principal's supply commitments inherited from $t-1$. Note that from Proposition 1, we have $\overline{W}_{t-1} \succ \underline{W}_{t-1}$ whenever $\overline{W}_{t-1} \neq \underline{W}_{t-1}$.

Proposition 2. *Suppose that the demand and supply processes (D^*, S^*) are optimal. Then (without loss of generality for optimal payoffs), for all $v \in \mathcal{D}$ there exists a $\mathbb{R}_+$-valued process $T^v = \{T^v\}_{t\geq 1}$ such that, for all t,*

$$\Delta D_t^* = \begin{cases} 1 & \text{if } T_t^{v_t} \geq t+1, \\ T_t^{v_t} - t & \text{if } t < T_t^{v_t} < t+1, \\ 0 & \text{if } T_t^{v_t} \leq t. \end{cases} \tag{2}$$

Given any time t and project v, the process T^v has the following properties.

1. *T^v is non-increasing.*
2. *If $v \succ \overline{W}_{t-1}$, then $T_t^v > t$, and if $\underline{W}_{t-1} \succ v$, then $T_t^v \leq t$.*

As with the supply of favours, optimal demand processes can be characterised by simple time-threshold processes $\{T^v\}_{v\in\mathbb{S}}$: any project with $T_t^{v_t} \geq t+1$ is produced with full intensity at t; no project with $T_t^{v_t} \leq t$ is produced with any intensity at t; and any project with $t < T_t^{v_t} < t+1$ is produced with interior intensity. Here, we interpret $\lfloor T_t^v \rfloor$ as the time at which the principal plans to stop demanding the production of project v, conditional on the relationship's status at time t. That T^v is non-increasing implies a frontloading property for optimal demand processes: if the principal ever passes on the opportunity to demand a favour following some history, then the principal also commits to never demanding any favour in all future occurrences of this project (Part 1).

To decide whether to demand a favour at some history, the principal compares the rank, in comparative advantage for extracting utility from the agent, of this favour with the rank, in comparative advantage for returning utility to the agent, of the favour she would need to supply to the agent in exchange (Part 2). The marginal cost of demanding a favour at time t is determined by $\{T_{t-1}^w\}_{w\in\mathcal{S}}$, which describe the supply commitments accumulated in the relationship's history up to t. Specifically, any marginal increase in supply commitments at t will be delivered in future occurrences of project $\overline{W}_{t-1}$, so that the principal must demand a favour with some intensity at t if $v \succ \overline{W}_{t-1}$. Conversely, any marginal reduction in supply commitments at t will reduce the production of future occurrences of projects that rank no better than $\underline{W}_{t-1}$ in comparative advantage, so that the principal cannot demand a favour with any intensity if $\underline{W}_{t-1} \succ v$. Note that the fact that supply thresholds $\{T^w\}_{w \in \mathcal{S}}$ are non-increasing implies that both $\overline{W}$ and $\underline{W}$ are non-decreasing processes with respect to $\succ$. In this sense, the frontloading of the demand for favours is the natural complement to the backloading of their supply: as the principal accumulates supply commitments over time, her marginal cost for asking new favours increases, choking off the principal's ability demand additional favours.

By highlighting the relationship between the principal's demand for favours at t and her accumulated supply commitments at $t-1$, we can also predict dynamics properties of optimal demands. Specifically, given history u^t, let v^* denote the project with $v^* \succ \overline{W}_{t-1}$ that is worst-ranked by comparative advantage. Then we know that this project must be demanded with some intensity at time t (again, Part 2). But it must also be the case that all projects $v \succ v^*$ are also demanded with some intensity at t. Hence, those projects demanded as favours by the principal satisfy a threshold property in comparative advantage. Similarly, those projects that will never be demanded in the future also satisfy a threshold property in comparative advantage determined by the relationship's supply history. Let v_* denote the project with $\underline{W}_{t-1} \succ v_*$ that is best-ranked by comparative advantage. Then no project with $v_* \succ v$ is ever again demanded with any intensity. As for our key claims from Proposition 1, the dynamic results of Proposition 2 are established through intertemporal smoothing arguments.

3.3 Markov Project Processes

In this section we impose additional structure on the process driving joint project opportunities: we assume that u is a Markov process. This allows us to sharpen our results considerably, and in fact we provide a complete characterisation of optimal relationship processes in this case. An important note is that a Markov project process u does not generate optimal demand and supply processes that are themselves stationary, that is, that specify the same production decisions following two histories with the same current project opportunity. In fact, optimal relationship processes are history-dependent even in the special case of iid project processes. There are two related explanations for this fact. First, as we established in Propositions 1 and 2, the backloading of the supply of favours and the frontloading of the demand for favours naturally induces non-stationarity in project production. Second, the history of agent-feasibility constraints matters for optimal relationship processes, and two occurrences of the same project can be treated differently if they are preceded by different histories of production decisions.

Recall that agent-feasibility constraints can only bite if the principal demands a favour. The key to Proposition 3 below to all projects at which the principal can demand a favour (projects $v \in \mathcal{D}$), it associates a relationship process that extends a minimal level of generosity of the principal towards the agent. The critical implication of the process u being Markov is that these minimal processes are history-independent. In turn, these minimal processes are used to construct the time-thresholds $\{T^v\}_{v\in\mathcal{D}}$ and $\{T^w\}_{w\in\mathcal{S}}$ that characterise optimal demand and supply processes in Propositions 1 and 2 through an explicit, recursive procedure.

Proposition 3. *For all $v, v' \in \mathcal{D}$ and all $w \in \mathcal{W}$, there exist $\tau^{vv'}, \tau^{vw} \geq 0$ that recursively define the $\mathbb{R}_+$-valued processes $T^v = \{T^v_t\}_{t\geq 1}$ and $T^w = \{T^w_t\}_{t\geq 1}$ as follows.*

1. *$T^v_1 = T^w_1 = \infty$ for all $v \in \mathcal{D}$ and $w \in \mathcal{S}$.*

2. *Given any $t > 1$,*

$$T^v_t = \begin{cases} t + \tau^{v_t v} & \text{if } t + \tau^{v_t v} < T^v_{t-1}, \\ T^v_{t-1} & \text{otherwise,} \end{cases} \quad \text{for all } v \in \mathcal{D}, \text{ and}$$

$$T^w_t = \begin{cases} t + \tau^{v_t w} & \text{if } t + \tau^{v_t w} < T^w_{t-1}, \\ T^w_{t-1} & \text{otherwise,} \end{cases} \quad \text{for all } w \in \mathcal{S}.$$

In turn, the collection of processes $(\{T^v\}_{v\in\mathcal{D}}, \{T^w\}_{w\in\mathcal{S}})$ define the optimal supply process S^* through (1) and the optimal demand process D^* through (2).

Given an opportunity for the principal to demand favour v_t at time t, $\lfloor \tau^{v_t v'} \rfloor$ is interpreted as the number of periods following t during which the principal would demand favour v', and $\lfloor \tau^{v_t w} \rfloor$ is interpreted as the number of periods following t during which the principal would supply favour w. Whether or not the principal ever commits to a relationship process described by $\{\tau^{v_t v'}\}_{v' \in \mathcal{D}}$ and $\{\tau^{v_t w}\}_{w \in \mathcal{S}}$ following some demand for favour v_t depends on the project history. If v_t is the the principal's first opportunity to demand a favour, then the principal will temporarily commit to time thresholds $\{\tau^{v_t v'}\}_{v' \in \mathcal{D}}$ and $\{\tau^{v_t w}\}_{w \in \mathcal{S}}$ from t on. These commitments are revisited whenever an opportunity for a new favour ${v'}_{t'}$ arises at $t' > t$, in which case there are two possibilities. First, if the inherited demand and supply processes are sufficiently generous to ensure agent-feasibility, then the principals' commitments from t are extended from t' on. Note that commitments $\{\tau^{v_t v'}\}_{v' \in \mathcal{D}}$ and $\{\tau^{v_t w}\}_{w \in \mathcal{S}}$ may have initially been less generous than $\{\tau^{v'_{t'} v''}\}_{v'' \in \mathcal{D}}$ and $\{\tau^{v'_{t'} w}\}_{w \in \mathcal{S}}$. However, at t' the time remaining before the principal stops demanding favour v'' is $t + \tau^{v_t v''}$ and the time remaining before she starts to supply favour w is $t + \tau^{v_t w}$, so that inherited commitments will always be more generous than new commitments if enough time has elapsed since an original demand for a favour. Second, if inherited commitments fall short of those set by $\{\tau^{v'_{t'} v''}\}_{v'' \in \mathcal{D}}$ and $\{\tau^{v'_{t'} w}\}_{w \in \mathcal{S}}$, then the relationship process is updated to these more generous commitments, which are themselves revisited the next time a project arrives at which a favour can be demanded by the principal.

Proposition 3 reproduces the properties of optimal relationship processes for general project processes derived in Propositions 1 and 2, and adds an exact description of how the history of binding agent-feasibility constraints shapes the current state of the relationship between the principal and the agent under Markov project processes. We can go further: a key step in the proof of Proposition 3 is the construction of a ranking of projects at which the principal can demand a favour in terms of the stringency of their corresponding agent-feasibility constraint.

Corollary 1. *Given any projects $\overline{v}, \underline{v} \in \mathcal{D}$, if either*

$$\tau^{\overline{v} v'} \leq \tau^{\underline{v} v'} \text{ for some } v' \in \mathcal{D} \text{ or } \tau^{\overline{v} w'} \leq \tau^{\underline{v} w'} \text{ for some } w' \in \mathcal{S},$$

then both

$$\tau^{\overline{v} v} \leq \tau^{\underline{v} v} \text{ for all } v \in \mathcal{D} \text{ and } \tau^{\overline{v} w} \leq \tau^{\underline{v} w} \text{ for all } w \in \mathcal{S}.$$

Given two projects $\overline{v}$ and $\underline{v}$ at which the principal can demand a favour, the resulting time thresholds $(\{\tau^{\overline{v} v'}\}_{v' \in \mathcal{D}}, \{\tau^{\overline{v} w}\}_{w \in \mathcal{S}})$ and $(\{\tau^{\underline{v} v'}\}_{v' \in \mathcal{D}}, \{\tau^{\underline{v} w}\}_{w \in \mathcal{S}})$ can be ranked uniformly in terms of their generosity to the agent. This yields a concrete sense in which project $\overline{u}$ is more difficult for the principal to demand then project $\underline{u}$: in return, the principal must commit (at least provisionally) to demand less, and supply more, future favours to the agent. Intuitively, this ranking of projects $v \in \mathcal{D}$ depends on two factors: the cost to the agent associated with favour v (indexed by $|v_A|$), and the value to the agent of future project opportunities conditional on having reached project v. This last factor depends on the project process u. However, if the project process is iid, then the value to the agent of future project opportunities is history-independent. In that case, the stringency of agent-feasibility constraints for favours that the principal can demand are ranked solely by their stage costs to the agent.

Corollary 2. *If the project process u is iid, then, given any projects $\overline{v}, \underline{v} \in \mathcal{D}$,*

$$\tau^{\overline{v}v} \leq \tau^{\underline{v}v} \text{ for all } v \in \mathcal{D} \text{ and } \tau^{\overline{v}w} \leq \tau^{\underline{v}w} \text{ for all } w \in \mathcal{S} \text{ if and only if } |\overline{v}_A| \geq |\underline{v}_A|.$$

This provides a comparative statics result of sorts, which shows how optimal relationship processes vary with the properties of the project process u. As noted above, even if the process u is iid, optimal demand and supply processes are not stationary and depend on the relationship's history of transitions to favours that are more costly to demand for the principal. However, in this case the ranking of favours in terms of their cost to the principal is independent of u. If u is Markov and displays some persistence, then this ranking of favours by their cost, while stationary, depends on the details of the process u.

The proof of Proposition 3 constructs optimal relationship processes through an inductive sequence of reduced problems. To this end, fix any project $v \in \mathcal{D}$ and suppose that $u_1 = v$. We define the reduced problem

$$\max_{\kappa \in \mathcal{K}} U_{P,1} \text{ subject to } U_{A,1} \geq 0, \tag{3}$$

which corresponds to the problem of finding an optimal relationship process conditional on $u_1 = v$, but in which agent-feasibility is only required to hold at $t = 1$. This problem has a solution characterised by fixed time thresholds $\left(\{\tau^{vv'}\}_{v' \in \mathcal{D}}, \{\tau^{vw}\}_{w \in \mathcal{S}}\right)$: contrary to the corresponding history-dependent thresholds of Propositions 1 and 2, these need not be adjusted at times $t > 1$ because no future agent-feasibility constraints need to be accommodated. Nevertheless, this solution has properties that are expected given our results for general project processes. First, if $\tau^{v\underline{v}} > 0$, then $\tau^{v\overline{v}} = \infty$ for all $\overline{v} \succ \underline{v}$. In words, an optimal demand process for problem (3) has the principal select a threshold project, with all projects ranked higher in comparative advantage always demanded, and all projects ranked lower in comparative advantage never demanded. Second, if $\tau^{v\underline{w}} < \infty$, then $\tau^{v\overline{w}} = 0$ for all $\underline{w} \succ \overline{w}$. In words, an optimal supply process for problem (3) has the principal select a threshold project, with all projects ranked higher in comparative advantage never supplied, and all projects ranked lower in comparative advantage always supplied. Third, if $v' \succ w'$ and $\tau^{vv'} < \infty$, then $\tau^{vw'} = 0$. In words, the lowest-ranked project (in comparative advantage) among those that are demanded by the principal must be succeeded (in comparative advantage) by the highest-ranked project (in comparative advantage) among those that are supplied by the principal. This last point implies the following comparison of solutions to the reduced problem 3 for different initial $u_1 = v$: if the principal demands less favours following $u_1 = \overline{v}$ than following $u_1 = \underline{v}$, then she must also supply more favours following $u_1 = \overline{v}$ than following $u_1 = \underline{v}$. In words, solutions to (3) following $u_1 = \overline{v}$ and $u_1 = \underline{v}$ are ordered by their generosity to the agent.

Now consider the project v^1 with the most generous solution to (3), which we denote by (D^{1*}, S^{1*}). Because the process u is Markov and this relationship process becomes more generous over time, then it must be agent-feasible whenever v^1 occurs at times $t > 1$. Furthermore, the pair (D^{1*}, S^{1*}) must also be agent-feasible whenever any other project v occurs at times $t > 1$. This follows because, again, u is Markov, and also because (D^{1*}, S^{1*}) is the most generous solution to (3) among all initial projects $u_1 = v$. Finally, because the the solution to the reduced problem (3) is agent-feasible, it follows that no feasible relationship process can yield higher payoffs to the principal following any history at which project v^1 occurs.

With project v^1 assigned as the costliest project for the principal in our ordering of projects from $\mathcal{D}$, we proceed inductively to define the second project in this ordering.

Given any $v \neq v^1$, we define the reduced problem

$$\max_{\kappa \in \mathcal{K}} U_{1,t} \text{ subject to } \begin{aligned}[t] & U_{A,1} \geq 0, \\ & U_{A,t} \geq 0 \text{ at each } t > 1 \text{ at which } u_t = v^1, \\ & (D,S) = (D^{1*}, S^{1*}) \text{ at each } t > 1 \text{ with } u_t = v^1 \text{ and } U_{A,t} = 0, \end{aligned} \tag{4}$$

which corresponds to the problem of finding an optimal relationship process conditional on (i) $u_1 = v$, (ii) on agent-feasibility being required at $t = 1$, and (iii) on agent-feasibility being required at all $t > 1$ with $u_t = v^1$, with the processes (D^{1*}, S^{1*}) being specified whenever this constraint binds at such histories. For the same reasons as above, the solution to problem (4) is such that (i) either the relationship has transitioned to (D^{1*}, S^{1*}) following some occurrence of v^1, or otherwise (ii) this solution characterised by fixed time thresholds $\left(\{\tau^{vv'}\}_{v' \in \mathcal{D}}, \{\tau^{vw}\}_{w \in \mathcal{S}}\right)$. Furthermore, these time thresholds are ranked by their generosity to the agent. The second project v^2 in our ordering of projects from $\mathcal{D}$, interpreted as the second-costliest favour for the principal to demand, is therefore the project for which the solution to (4) is the most generous to the agent, and we can define the corresponding processes (D^{2*}, S^{2*}). Furthermore, for the same reasons as above, the pair (D^{2*}, S^{2*}) is agent-feasible at all $t > 1$, so that no feasible relationship process can yield higher payoffs to the principal following any history at which project v^2 occurs. This inductive process can be repeated to rank all favours in $\mathcal{D}$ in terms of how costly they are for the principal to demand and to complete the construction of the optimal relationship process if u is Markov.

Note the critical feature of this construction: the project v^2 is determined by anticipating transitions to more generous relationship process generated by demanding the costliest favour v^1 at some future time. Demands of less costly favour are also anticipated, but in these cases the adjustments to (D^{2*}, S^{2*}) are not necessary to maintain agent-feasibility. Furthermore, anticipating costlier favours in the future allows the principal to demand more of less costly favours than she could do otherwise. To see this, note that from above, we know that the solution to problem (3) given $u_1 = v^2$ is not, in general, agent-feasible if v^1 occurs at $t > 1$. Also, the processes (D^{1*}, S^{1*}) are not optimal for the principal given $u_1 = v \neq v^1$, because they are too generous towards the agent. Therefore, by anticipating that the relationship may transition to the more generous (D^{1*}, S^{1*}) following some histories, the solution to problem (4) for $u_1 = v^2$

can be less generous to the agent before such a transition than the solution to problem (3) for $u_1 = v^2$ would be.

4 Conclusion

This project is not complete, and we are still pursuing further results. We are investigating how to relax our assumption that the principal can commit to her production decisions in a relationship process. There are two avenues for doing this. The first is more straightforward: derive conditions under which optimal relationship processes under commitment are also principal-feasible. For example, if the principal is sufficiently patient and common-interest projects arrive often enough under the process u, then the relationship will generate enough surplus to provide incentives to the principal at all histories. The second approach is more ambitious, and more difficult: try to characterise optimal relationship processes without commitment directly, by modifying our existing arguments to integrate the principal's feasibility constraints. The model in which the project process u is Markov offers a tractable environment in which to apply this approach.

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A Appendix

Existence of Optimal Relationship Processes. A simple adaptation of a proof from Dixit et al. (2000) establishes the existence of optimal relationship processes.

Lemma 2. *There exists an optimal relationship process.*

Proof. Note that because $\mathcal{U}$ is finite and time is discrete, any relationship process can be identified with a point in $[0,1]^\infty$, a compact set in the product topology. The set of feasible relationship processes $\mathcal{K}^f$ is a closed subset of this space, and hence it is also compact. Furthermore, the set of feasible relationship processes is nonempty because it contains the no-production relationship process. Therefore, because the principal's utility $U_{P,0} : [0,1]^\infty \to \mathbb{R}$ is continuous, it follows that the problem

$$\max_{\kappa \in \mathcal{K}^f} \mathbb{E}_0 U_{P,1}$$

has a solution, which is an optimal relationship process. □

Refinement of Optimal Relationship Processes. There is indeterminacy in optimal relationships that is due solely to zero-probability events. To refine our results, we consider a sequence $\{u^n\}_{n\geq 1}$ of perturbed versions of the process u, which is such that $u^n \stackrel{a.s.}{\to} u$ and, for all times t and project histories u^t, $\mathbb{P}_0(u^{n,t} = u^t) > 0$ (such a sequence is easy to contruct). In what follows, we drop references to the perturbed versions of u, but we use extensively, and without special reference, the fact that, for all project histories u^t, $\mathbb{P}_0(u^t) > 0$. Therefore, our results describe those optimal processes κ^* for the process u that are selected by the limits of the optimal processes for the processes $\{u^n\}$.

Proof of Lemma 1. Suppose, towards a contradiction, that κ^* is optimal and that, for some project history u^t such that $u_P, u_A > 0$, we have that $\kappa^*_t < 1$. Fix a relationship process $\tilde{\kappa}$ that is identical to κ^* except that $\tilde{\kappa}_t = 1$ at u^t. It follows that $\tilde{\kappa}$ is feasible because κ^* is feasible. Furthermore, $\tilde{U}_{P,t} > U^*_{P,t}$, yielding the desired contradiction. The proof for the case of u such that $u_P, u_A < 0$ is similar, and is omitted. □

Proof of Proposition 1. We proceed in a number of steps.
Step 1. Fix optimal processes (D^*, S^*), project history u^t, its superhistories $u^{t'}$ and $u^{t''}$, and projects $\underline{w} \succ \overline{w}$. Suppose that (i) $u_{t'} = \overline{w}$ and $D^*_{t'} = D^*_t$ at $u^{t'}$, and that (ii) $u_{t''} = \underline{w}$ and $D^*_{t''} = D^*_t$ at $u^{t''}$. We show that

$$\text{if } \Delta S^*_{t'} < 1, \text{ then } \Delta S^*_{t''} = 0.$$

To see this suppose, towards a contradiction, that $\Delta S^*_{t'} < 1$ at $u^{t'}$ and that $\Delta S^*_{t''} > 0$ at $u^{t''}$. Now consider an alternative supply process $\tilde{S}$, identical to S^* except that (i) $\Delta S^*_{t'} < \Delta \tilde{S}_{t'} \leq 1$ at $u^{t'}$, (ii) $0 \leq \Delta \tilde{S}_{t''} < \Delta S^*_{t''}$ at $u^{t''}$ and (iii)

$$\tilde{U}_{A,t} - U^*_{A,t} = \delta^{t'-t}\mathbb{P}_t(u^{t'})[\tilde{S}_{t'} - S^*_{t'}]\overline{w}_A - \delta^{t''-t}\mathbb{P}_t(u^{t''})[S^*_{t''} - \tilde{S}_{t''}]\underline{w}_A = 0. \tag{5}$$

Such a process $\tilde{S}$ always exists. Also, note that the pair $(D^*, \tilde{S})$ is agent-feasible for all times $r \leq t$. To show that the pair $(D^*, \tilde{S})$ is agent-feasible for times $r > t$, we proceed recursively. First note that, for all r such that either (i) $D^*_r > D^*_t$ or (ii) $r > \max\{t', t''\}$, we have that $\tilde{U}_{A,r} = U^*_{A,r} \geq 0$, where the inequality follows because the pair (D^*, S^*) is agent-feasible. Second, whenever $D^*_{\max\{t',t''\}} = D^*_t$, we have that $\tilde{\kappa}_{\max\{t',t''\}} u_{A,\max\{t',t''\}} \geq 0$ and therefore

$$\begin{aligned}\tilde{U}_{A,\max\{t',t''\}} &\geq \delta \mathbb{E}_{\max\{t',t''\}} U^*_{A,\max\{t',t''\}+1} \\ &\geq 0,\end{aligned}$$

where the final inequality follows because the pair (D^*, S^*) is agent-feasible. Third, the two previous remarks ensure that, for all $\min\{t', t''\} < r \leq \max\{t', t''\}$ such that $D^*_r = D^*_t$, we have that $\tilde{U}_{A,r} \geq 0$. Fourth, whenever $D^*_{\min\{t',t''\}} = D^*_t$,

$$\begin{aligned}\tilde{U}_{A,\min\{t',t''\}} &\geq \delta \mathbb{E}_{\min\{t',t''\}} \tilde{U}_{A,\min\{t',t''\}+1} \\ &\geq 0.\end{aligned}$$

Finally, the previous remarks together ensure that, for all $t < r \leq \min\{t', t''\}$ such that $D^*_r = D^*_t$, we have that $\tilde{U}_{A,r} \geq 0$.

It remains only to note that, by (5), we have

$$\begin{aligned}\tilde{U}_{P,t} - U^*_{P,t} &= -\delta^{t'-t}\mathbb{P}_t(u^{t'})[\tilde{S}_{t'} - S^*_{t'}]|\overline{w}_P| + \delta^{t''-t}\mathbb{P}_t(u^{t''})[S^*_{t''} - \tilde{S}_{t''}]|\underline{w}_P| \\ &= \delta^{t''-t}\mathbb{P}_t(u^{t''})[S^*_{t''} - \tilde{S}_{t''}]|\underline{w}_P| \left[1 - \frac{|\overline{w}_P|/\overline{w}_A}{|\underline{w}_P|/\underline{w}_A}\right] \\ &> 0,\end{aligned}$$

where the inequality follows because $\underline{w} \succ \overline{w}$, contradicting the optimality of (D^*, S^*).

Step 2. Fix optimal processes (D^*, S^*), project history u^t, its superhistories $u^{t'}$ and $u^{t''}$, and projects $\underline{w} \succ \overline{w}$. Suppose that (i) $u_{t'} = \overline{w}$ and $D^*_{t'} > D^*_t$ at $u^{t'}$, and that (ii)

$u_{t''} = \underline{w}$ and $D^*_{t''} = D^*_t$ at $u^{t''}$. We show that

$$\text{if } \Delta S^*_{t'} < 1, \text{ then } \Delta S^*_{t''} = 0.$$

To see this suppose, towards a contradiction, that $\Delta S^*_{t'} < 1$ at $u^{t'}$ and that $\Delta S^*_{t''} > 0$ at $u^{t''}$. Now consider an alternative supply process $\tilde{S}$, identical to S^* except that (i) $\Delta S^*_{t'} < \Delta \tilde{S}_{t'} \leq 1$ at $u^{t'}$, (ii) $0 \leq \Delta \tilde{S}_{t''} < \Delta S^*_{t''}$ at $u^{t''}$ and (iii)

$$\tilde{U}_{A,t} - U^*_{A,t} = \delta^{t'-t}\mathbb{P}_t(u^{t'})[\tilde{S}_{t'} - S^*_{t'}]\overline{w}_A - \delta^{t''-t}\mathbb{P}_t(u^{t''})[S^*_{t''} - \tilde{S}_{t''}]\underline{w}_A = 0. \tag{6}$$

Note that the pair $(D^*, \tilde{S})$ is agent-feasible for all times $r \leq t$. To show that the pair $(D^*, \tilde{S})$ is agent-feasible for times $r > t$, we proceed recursively. First note that, for all r such that either (i) $D^*_r > D^*_{t'}$, (ii) $D^*_r = D^*_{t'}$ and $r > t'$, or (iii) $r > \max\{t', t''\}$, we have that $\tilde{U}_{A,r} = U^*_{A,r} \geq 0$. Second, if $D^*_{t'} > D^*_t$, then we have that $\tilde{\kappa}_{A,t'} u_{A,t'} \geq \kappa^*_{A,t'} u_{A,t'}$, and therefore $\tilde{U}_{A,t'} \geq U^*_{A,t'} \geq 0$. Third, the previous remarks imply that if (i) $D^*_r = D^*_{t'}$ and $r < t'$, or if (ii) $D^*_r = D^*_t$ and $r > t''$, then $\tilde{U}_{A,r} \geq U^*_{A,r} \geq 0$. Fourth, if $D^*_r < D^*_{t'}$, then given the previous remark we can assume that $r \leq t''$. Because $\tilde{\kappa}_{A,t''} u_{A,t''} \geq 0$ we have that

$$\begin{aligned}\tilde{U}_{A,r} &\geq \delta \mathbb{I}_{t'' \geq t'} \mathbb{E}_r U^*_{A,r+1} + \delta \mathbb{I}_{t'' < t'} \mathbb{E}_r \tilde{U}_{A,r+1} \\ &\geq 0,\end{aligned}$$

where the final inequality follows because the pair (D^*, S^*) is agent-feasible and, by the previous remarks, the pair $(D^*, \tilde{S})$ is agent feasible for those $t'' < r+1 \leq t'$. Fifth, the previous remark implies that if $D^*_r = D^*_t$ and $t < r < t''$, we have that $\tilde{U}_{A,r} \geq 0$. Finally, an argument as in Step 1 shows that (6) and the fact that $\underline{w} \succ \overline{w}$ imply that $\tilde{U}_{P,t} - U^*_{P,t} > 0$, yielding the desired contradiction.

Step 3. Fix optimal processes (D^*, S^*), project history u^{t-1}, its superhistories u^t and $u^{t'}$, a superhistory $u^{t''}$ of u^t, and projects $\underline{w} \succ \overline{w}$. Suppose that (i) $u_{t'} = \overline{w}$ and $D^*_{t'} = D^*_{t-1}$ at $u^{t'}$, and that (ii) $u_{t''} = \underline{w}$ and $D^*_{t''} = D^*_t$ at $u^{t''}$. We show that, if $U^*_{A,t} > 0$, then

$$\text{if } \Delta S^*_{t'} < 1, \text{ then } \Delta S^*_{t''} = 0.$$

To see this suppose, towards a contradiction, that $\Delta S^*_{t'} < 1$ at $u^{t'}$ and that $\Delta S^*_{t''} > 0$ at $u^{t''}$. Now consider an alternative supply process $\tilde{S}$, identical to S^* except that (i) $\Delta S^*_{t'} < \Delta \tilde{S}_{t'} \leq 1$ at $u^{t'}$, (ii) $0 \leq \Delta \tilde{S}_{t''} < \Delta S^*_{t''}$ at $u^{t''}$ and (iii)

$$\tilde{U}_{A,t-1} - U^*_{A,t-1} = \delta^{t'-(t-1)}\mathbb{P}_{t-1}(u^{t'})[\tilde{S}_{t'} - S^*_{t'}]\overline{w}_A - \delta^{t''-(t-1)}\mathbb{P}_t(u^{t''})[S^*_{t''} - \tilde{S}_{t''}]\underline{w}_A = 0. \tag{7}$$

Note that the pair $(D^*, \tilde{S})$ is agent-feasible at all times $r \leq t-1$. Also, let $\tilde{S}^*$ be such that $\tilde{U}_{A,t} \geq 0$, so that the pair $(D^*, \tilde{S})$ is agent-feasible at t. The proof that the pair $(D^*, \tilde{S})$ is agent-feasible at all times $r > t$ follows from arguments very close to those in Steps 1 and 2, and is omitted. Finally, an argument as in Step 1 shows that (7) and the fact that $\underline{w} \succ \overline{w}$ imply that $\tilde{U}_{P,t-1} - U^*_{P,t-1} > 0$, yielding the desired contradiction.
Step 4. The results of Steps 1 and 2 imply that to any optimal processes (D^*, S^*) corresponds a $\mathcal{S}$-valued threshold process $W = \{W_t\}_{t \geq 1}$ such that, for all t,

$$\Delta S^*_t = \begin{cases} 1 & \text{if } W_t \succ w_t, \\ 0 & \text{if } w_t \succ W_t, \end{cases}$$

and where W is non-decreasing (with respect to $\succ$), and such that, given any history u^t and its superhistory $= u^{t'}$, $W_{t'} = W_t$ if $D_{t'} = D_t$. The threshold is given by

$$W_t = \max_{\succ} \left\{ w \in \mathcal{S} : \mathbb{P}_t(\Delta S^*_{t'} > 0, u_{t'} = w, D^*_{t'} = D^*_t, t' \geq t) > 0 \right\},$$

if this is well-defined, and by

$$W_t = \min_{\succ} \mathcal{S},$$

otherwise.
Step 5. Steps 1 and 2 do not determine optimal relationship processes at time t if $w_t = W_t$. We now show that, without loss of generality for optimal payoffs, we can restrict attention to relationship processes with the property that, for all times $t' \geq t$ at which $D^*_{t'} = D^*_t$, there exists time T^{W_t} such that $\Delta S^*_{t'} = 1$ if and only if $t' \geq T^{W_t}$. More precisely, fix optimal processes (D^*, S^*) and history u^t, and consider an alternative supply process $\hat{S}$, identical to S^* except that, at all superhistories $u^{t'}$ of u^t with $D^*_{t'} = D^*_t$ and $u_{t'} = W_t$,

$$\Delta \hat{S}_{t'} = \begin{cases} 1 & \text{if } \hat{T} \leq t', \\ t' + 1 - T & \text{if } t' < \hat{T} < t' + 1 \\ 0 & \text{if } \hat{T} \geq t' + 1. \end{cases}$$

Note that $\hat{U}_{A,t} \geq U^*_{A,t}$ if $\hat{T} = t$. Also, $\lim_{\hat{T} \to \infty} \hat{U}_{A,t} \leq U^*_{A,t}$. By continuity of $\hat{U}_{A,t}$ in $\hat{T}$, there exists some $\tilde{T} \geq t$ such that $\tilde{U}_{A,t} = U^*_{A,t}$. Also, note that

$$\begin{aligned}\tilde{U}_{P,t} - U^*_{P,t} &= |W_{P,t}|\,\mathbb{E}_t\left[\sum_{t'\geq t}\delta^{t'-t}\left[\Delta S^*_{t'} - \Delta S^{\tilde{T}}_{t'}\right]\right]\\ &= \frac{|W_{P,t}|}{W_{A,t}}\left[U^*_{A,t} - \tilde{U}_{A,t}\right]\\ &= 0.\end{aligned}$$

To verify that the pair $(D^*, \tilde{S})$ is agent-feasible, first note that $\tilde{U}_{A,r} = U^*_{A,r} \geq 0$ if either (i) $r \leq t$ or (ii) $r > t$ and $D^*_r > D^*_t$. Second, if $r \geq \tilde{T}$ and $D^*_r = D^*_t$, then because $\tilde{\kappa}_r u_{A,r} \geq \kappa^*_r u_{A,r}$, it follows that $\tilde{U}_{A,r} \geq U^*_{A,r} \geq 0$. Third, if $t < r < \tilde{T}$ and $D^*_r = D^*_t$, then by the previous point and because $\tilde{\kappa}_{A,r} u_{A,r} \geq 0$, it follows recursively that

$$\begin{aligned}\tilde{U}_{A,r} &\geq \delta\mathbb{E}_r\tilde{U}_{A,r+1}\\ &\geq 0.\end{aligned}$$

The last point is to establish that the procedure above, which modifies S^* at a single history in a payoff-invariant way, can extended simultaneously to all histories. We do this in Step 9 below.

Step 6. We show that, without loss of generality for optimal payoffs, the supply process S^* defined in Step 5 can be taken such that, given any time t, project history u^t and any superhistory $u^{t'}$ such that $W_t = W_{t'}$, we have that $T^{W_t}_t \geq T^{W_t}_{t'}$. Notice that by Step 5, if $\mathbb{P}_t(\Delta S^*_{t'} > 0, u_{t'} = W_t, D^*_{t'} = D^*_t, t' \geq t) = 0$, then $T^{W_t}_t = \infty$, and the claim is true. Therefore, we assume in what follows that $\mathbb{P}_t(\Delta S^*_{t'} > 0, u_{t'} = W_t, D^*_{t'} = D^*_t, t' \geq t) > 0$.

Fix time t and history u^t, and let $\overline{T} = \sup\{t' \geq t : u_{t'} = W_t = W_{t'}, T^{W_t}_{t'} \geq t'\}$. It is without loss of generality for optimal payoffs to assume that $\overline{T} < \infty$. To see this, consider the alternative supply process $\hat{S}$, identical to S^* except that $\hat{T}^{W_t}_t = \hat{T}$ and $\hat{T}^{W_t}_{t'} = \min\{T^{W_t}_{t'}, \hat{T}\}$ at all superhistories of u^t with $u_{t'} = W_t = W_{t'}$ and $D^*_{t'} > D^*_t$. We have that $\hat{U}_{A,t} \geq U^*_{A,t}$ if $\hat{T} = t$, and $\lim_{\hat{T}\to\infty}\hat{U}_{A,t''} < U^*_{A,t}$ because $\mathbb{P}_t(\Delta S^*_{t'} > 0, u_{t'} = W_t, D^*_{t'} = D^*_t, t' \geq t) > 0$. By continuity, there exists $\tilde{T} < \infty$ such that $\tilde{U}_{A,t} = U^*_{A,t}$, as well as $\tilde{U}_{P,t} = U^*_{P,t}$. To verify that the pair $(D^*, \tilde{S})$ is agent-feasible, first note that $\tilde{U}_{A,r} = U^*_{A,r} \geq 0$ if either (i) $r \leq t$ or (ii) $r > t$ and either $W_r \neq W_t$ or $u_r \neq W_r = W_t$. Second, if $r \geq \tilde{T}$ and $u_r = W_r = W_t$, then because $\tilde{\kappa}_r u_{A,r} \geq \kappa^*_r u_{A,r}$, it follows that $\tilde{U}_{A,r} \geq U^*_{A,r} \geq 0$. Third, if $t < r < \tilde{T}$, $u_r = W_r = W_t$ and $D^*_r > D^*_t$, then by the previous point it follows recursively that $\tilde{U}_{A,r} \geq U^*_{A,r} \geq 0$. Fourth, if $t < r < \tilde{T}$ and $D^*_r = D^*_t$, then by the previous points and because $\tilde{\kappa}_r u_{A,r} \geq 0$, it follows recursively that

$$\begin{aligned}\tilde{U}_{A,r} &\geq \delta \mathbb{E}_r \tilde{U}_{A,r+1} \\ &\geq 0.\end{aligned}$$

Now consider the alternative relationship process S^α, identical to S^* except that (i) $T^{\alpha,W_t}_t = (1-\alpha)\overline{T} + \alpha t$ and (ii) $T^{\alpha,W_t}_{t'} = (1-\alpha)T^{W_t}_{t'} + \alpha t$ for all $t' > t$ with $u_{t'} = W_{t'} = W_t$. This process is well-defined because $\overline{T} < \infty$. Notice that $U^{\alpha=0}_{A,t} \leq U^*_{A,t}$ and that $U^{\alpha=1}_{A,t} \geq U^*_{A,t}$. By continuity, there exists $\tilde{\alpha} \in [0,1]$ such that $U^{\tilde{\alpha}}_{A,t} = U^*_{A,t}$, as well as $U^{\tilde{\alpha}}_{P,t} = U^*_{P,t}$. Note that, by construction, $S^{\tilde{\alpha}}$ is such that $T^{\tilde{\alpha},W_t}_t \geq T^{\tilde{\alpha},W_t}_{t'}$ whenever $t' > t$ and $u_{t'} = W_{t'} = W_t$. The proof that the pair $(D^*, S^{\tilde{\alpha}})$ is agent-feasible is almost identical to that of the previous paragraph for the process $\tilde{S}$, and is omitted.

Step 7. We show that it is without loss of generality for optimal payoffs to restrict attention to supply processes S^* such that, given any time t, if either (i) $W_t \succ W_{t-1}$ or (ii) $W_t = W_{t-1}$ and $T^{W_t}_t < T^{W_{t-1}}_{t-1}$, then $U^*_{A,t} = 0$. To see part (i) of this claim, suppose that there exist project history u^{t-1} and its superhistory u^t such that $W_t \succ W_{t-1}$ and $U^*_{A,t} > 0$. By Step 3, there cannot exists any project $w \in \mathcal{S}$ such that $W_t \succ w \succ W_{t-1}$. Also because $W_t \succ W_{t-1}$, Step 3 implies that

$$\begin{aligned}&\mathbb{P}_{t-1}\big(\Delta S^*_{t'} < 1, u_{t'} = W_{t-1}, D^*_{t'} = D^*_{t-1}, t' \geq t\big) = 0, \text{ and} \\ &\qquad \mathbb{P}_t\big(\Delta S^*_{t'} > 0, u_{t'} = W_t, D^*_{t'} = D^*_t, t' \geq t\big) > 0.\end{aligned}$$

Now consider the alternative supply process $S^{\alpha,\beta}$, identical to S^* except that (i) $T^{\alpha,\beta,W_t}_t = \beta \geq t$ following u^t and (ii) $T^{\alpha,\beta,W_{t-1}}_{t'} = \alpha \geq t$ for all superhistories $u^{t'} \neq u^t$ of u^{t-1} with $u_{t'} = W_t$ and $D^*_{t'} = D^*_{t-1}$. We have that $U^{t,t}_{A,r} \geq U^*_{A,r}$ and $\lim_{\alpha,\beta\to\infty} U^{\alpha,\beta}_{A,r} \leq U^*_{A,r}$ for $r = t-1, t$. By continuity, there exist $\tilde{\beta} \leq \tilde{\alpha} < \infty$ such that $V^{\tilde{\alpha},\tilde{\beta}}_{A,t-1} = U^*_{A,t-1}$ and $U^{\tilde{\alpha},\tilde{\beta}}_{P,t-1} = U^*_{P,t-1}$, and either (a) $U^{\tilde{\alpha},\tilde{\beta}}_{A,t} = 0$ and $\tilde{\alpha} \geq \tilde{\beta}$ or (b) $U^{\tilde{\alpha},\tilde{\beta}}_{A,t} > 0$ and $\tilde{\alpha} = \tilde{\beta}$. In particular, the pair $(D^*, S^{\tilde{\alpha},\tilde{\beta}})$ is agent-feasible at all times $r \leq t$. The proof that the pair $(D^*, S^{\tilde{\alpha},\tilde{\beta}})$ is agent-feasible at all times $r > t$ is similar to those of Steps 5 and 6, and is omitted. Finally, the proof of part (ii) of the claim is similar, and is omitted.

Step 8. The payoff-equivalent modifications operated on some optimal supply process S^* described in Steps 5-7 were constructed history by history. Note that any relationship process can be identified with a point in $[0,1]^\infty$, a compact set in the product topology. Therefore, given some optimal relationship process $\kappa^{*,1}$, we can construct a sequence $\{\kappa^{*,n}\}_{n\geq 1}$ in $[0,1]^\infty$ such that (i) for each n, $\kappa^{*,n+1}$ is obtained from $\kappa^{*,n}$ by some operation from Steps 5-7 at some history and (ii) given any time t, there exists N such that, for all $n \geq N$, $\kappa^{*,n}_{t'} = \kappa^{*,N}_{t'}$ for all $t' \leq t$. This sequence must then have a subsequence converging to κ^*, some optimal relationship process satisfying all the properties of Steps 5-7.

Step 9. Given any time t and associated threshold project W_t as defined in Step 5, we have defined in Steps 5-7 a time threshold process T^{W_t} that respects the conditions of Proposition 1. Now given any $w \succ W_t$, define $T^w_t = \infty$, and given any $W_t \succ w$, define $T^w_t = \min\{t' \leq t : w_t = w \text{ and } \Delta S^*_{t'} = 0\}$. Note that because, by Step 4, W is non-decreasing (with respect to $\succ$), and because, by Steps 6 and 8, T^{W_t} is non-increasing, our construction ensures that, for each $w \in \mathcal{S}$, the process T^w is non-increasing.

Step 10. Let $\underline{w} = \min_{\succ} \mathcal{S}$. We show that given an optimal processes (D^*, S^*), we have that if and only if $D^*_t = 0$. To see this, first suppose that there exists history u^t with $D^*_t = 0$ and, towards a contradiction, $T^{\underline{w}}_t < \infty$. Consider an alternative supply process $\tilde{S}$, identical to S^* except that $\Delta\tilde{S}_{t'} = 0$ at all histories $u^{t'}$ with $D_{t'} = 0$. To see that the pair $(D^*, \tilde{S})$ is agent-feasible, first note that if $D^*_r > 0$, then $\tilde{U}_{A,t} = U^*_{A,t} \geq 0$. Second, if $D^*_r = 0$, then because $\tilde{\kappa}_r u_{A,r} \geq 0$, it follows recursively by the previous point that we have $\tilde{U}_{A,t} \geq \delta\mathbb{E}_{t'} U^*_{A,t+1} \geq 0$. Finally, we have that $\tilde{U}_{P,1} > U^*_{P,1}$, yielding the desired contradiction. Second, suppose that there exists history u^t with $\Delta D^*_t > 0$ and $T^{\underline{w}}_t = \infty$. Then for any $t' \geq t$ at which $T^{\underline{w}}_t < \infty$ for the first time, we have that $U^*_{A,t'} = 0$ by Step 3. Because, by Steps 1 and 2, we have that $\Delta S_{t'} = 0$ for all $t' \geq t$ with $T^{\underline{w}}_{t'} = \infty$, It follows that $U^*_{A,t} = \Delta D^*_t w_{A,t} + \mathbb{E}_t U^*_{A,t+1} < 0$, yielding the desired contradiction.

□

Proof of Proposition 2. We proceed in a number of steps.

Step 1. Fix optimal processes (D^*, S^*), project history u^t, its superhistory $u^{t'}$, and projects $\overline{v} \succ \underline{v}$. Suppose that (i) $u_t = \overline{u}$ at u^t and that (ii) $u_{t'} = \underline{u}$ at $u^{t'}$. We show that

$$\text{if } \Delta D^*_t < 1, \text{ then } \Delta D^*_{t'} = 0.$$

To see this suppose, towards a contradiction, that $\Delta D^*_t < 1$ at u^t and that $\Delta D^*_{t'} > 0$ at $u^{t'}$. Now consider an alternative demand process $\tilde{D}$, identical to D^* except that (i)

$\Delta D^*_t < \Delta\tilde{D}_t \leq 1$ at u^t, that (ii) $0 \leq \tilde{D}_{t'} < D^*_{t'}$ at $u^{t'}$, and that (iii)

$$\tilde{U}_{A,t} - U^*_{A,t} = -\left[\Delta\tilde{D}_t - \Delta D^*_t\right]|\overline{v}_A| + \delta^{t'-t}\mathbb{P}_t(u^{t'})\left[\Delta D^*_{t'} - \Delta\tilde{D}_{t'}\right]|\underline{v}_A| = 0. \tag{8}$$

Note that the pair $(\tilde{D}, S^*)$ is agent-feasible for all times $r \leq t$. To show that the pair $(\tilde{D}, S^*)$ is agent-feasible for times $r > t$, we proceed recursively. First note that, for all $r > t'$, we have that $\tilde{U}_{A,r} = U^*_{A,r} \geq 0$. Second, if $r = t'$, we have that $\tilde{\kappa}_{t'}u_{A,t'} \geq \kappa^*_{t'}u_{A,t'}$, so that, by the previous point, $\tilde{U}_{A,r} \geq U^*_{A,r} \geq 0$. Third, if $t < r < t'$, then the previous points ensure that $\tilde{U}_{A,r} \geq 0$. Finally, an argument as in Step 1 of Proposition 1 shows that (8) and the fact that $\overline{v} \succ \underline{v}$ imply that $\tilde{U}_{P,t} - U^*_{P,t} > 0$, yielding the desired contradiction.

Step 2. Step 1 does not restrict optimal relationship processes at history u^t and its superhistory $u^{t'}$ if $v_t = v_{t'}$. We now show that, without loss of generality for optimal payoffs, we can restrict attention to relationship processes with the property that, for such histories, if $\Delta D^*_{t'} > 0$, then $\Delta D^*_t = 1$. To see this, fix optimal processes (D^*, S^*), along with history u^t, and suppose that $\Delta D^*_t < 1$. Now consider an alternative demand process $\tilde{D}$, identical to D^* except that

$$\Delta\tilde{D}_t = \begin{cases} 1 & \text{if } 1 - \Delta D^*_t \leq \mathbb{E}_t\left[\sum_{t'\geq t}\delta^{t'-t}\mathbb{I}_{v_{t'}=v_t}\Delta D^*_{t'}\right], \\ \Delta D^*_t + \mathbb{E}_t\left[\sum_{t'\geq t}\delta^{t'-t}\mathbb{I}_{v_{t'}=v_t}\Delta D^*_{t'}\right] & \text{otherwise,} \end{cases}$$

and that, for all $t' > t$ with $v_{t'} = v_t$,

$$\Delta\tilde{D}_{t'} = \begin{cases} 0 & \text{if } 1 - \Delta D^*_t \geq \mathbb{E}_t\left[\sum_{t'\geq t}\delta^{t'-t}\mathbb{I}_{v_{t'}=v_t}\Delta D^*_{t'}\right], \\ \frac{\mathbb{E}_t\left[\sum_{t'\geq t}\delta^{t'-t}\mathbb{I}_{v_{t'}=v_t}\Delta D^*_{t'}\right] - [1-\Delta D^*_t]}{\mathbb{E}_t\left[\sum_{t'\geq t}\delta^{t'-t}\mathbb{I}_{v_{t'}=v_t}\right]} & \text{otherwise,} \end{cases}$$

Note that such a process always exists, and that, by construction,

$$\begin{aligned}\tilde{U}_{A,t} - U^*_{A,t} &= |v_{A,t}|\left[-\left[\Delta\tilde{D}_t - D^*_t\right] + \mathbb{E}_t\left[\sum_{t'\geq t}\delta^{t'-t}\mathbb{I}_{v_{t'}=v_t}\left[\Delta D^*_{t'} - \Delta\tilde{D}_{t'}\right]\right]\right] \\ &= 0 \\ &= \tilde{U}_{P,t} - U^*_{P,t}.\end{aligned}$$

Furthermore, we have that either (i) $\tilde{D}_t = 1$ and $\tilde{D}_{t'} \geq 0$ for all $t' \geq t$ with $v_{t'} = v_t$, or that (ii) $\tilde{D}_t < 1$ and $\tilde{D}_{t'} = 0$ for all $t' \geq t$ with $v_{t'} = v_t$. Note that the pair $(\tilde{D}, S^*)$ is agent-feasible for all $r \leq t$. To see that the pair $(\tilde{D}, S^*)$ is agent-feasible for all $r > t$, note that $\tilde{\kappa}_r u_{A,r} \geq \kappa_r^* u_{A,r}$, so that $\tilde{V}_{A,r} \geq V_{A,r}^* \geq 0$.

Step 3. The procedure from Step 2, which modifies demand process D^* at a single history in a payoff-invariant way, can extended simultaneously to all histories as in Step 8 of the proof of Proposition 1.

Step 4. Given optimal processes (D^*, S^*) along with any $v \in \mathcal{D}$ and any history u^t, define

$$\begin{aligned}\underline{t} &= \sup\{t' \leq t : v_{t'} = v \text{ and } \Delta D_{t'}^* > 0\},\\ \bar{t} &= \sup\{t' \geq t : v_{t'} = v \text{ and } \Delta D_{t'}^* > 0\},\end{aligned}$$

as well as

$$T_t^v = \begin{cases} \underline{t} + \Delta D_{\underline{t}}^* & \text{if } \underline{t} < t, \\ \bar{t} + \Delta D_{\bar{t}}^* & \text{if } t \leq \bar{t} < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

By construction, the resulting threshold processes $\{T^v\}_{v \in \mathcal{D}}$ are non-increasing. Furthermore, by the results of Steps 1-3, it follows that, for all t,

$$\Delta D_t^* = \begin{cases} 1 & \text{if } T_t^{v_t} \geq t+1, \\ T_t^{v_t} - t & \text{if } t < T_t^{v_t} < t+1, \\ 0 & \text{if } T_t^{v_t} \leq t. \end{cases}$$

Step 5. Fix optimal processes (D^*, S^*) and project history u^t such that $v_t \succ \overline{W}_{t-1}$. We show that $\Delta D_t^* > 0$. Suppose, towards a contradiction, that $\Delta D_t^* = 0$. Recall that, by Part 2 of Proposition 1, it follows that $T_{t+1}^{\overline{W}_{t-1}} = T_{t-1}^{\overline{W}_{t-1}} > t+1$, so that $\Delta S_{t+1}^* < 1$ if $w_{t+1} = \overline{W}_{t-1}$. Now consider alternative processes $(\tilde{D}, \tilde{S})$, identical to (D^*, S^*) except that (i) $\Delta \tilde{D}_t > 0$, (ii) $\Delta S_{t+1}^* < \Delta \tilde{S}_{t+1} \leq 1$ if $w_{t+1} = \overline{W}_{t-1}$, and (iii) $\tilde{U}_{A,t} = U_{A,t}^*$. Such a pair $(\tilde{D}, \tilde{S})$ always exists, and that it is agent-feasible follows by arguments similar to those of Step 1 in the proof of Proposition 1. By (iii), we have that

$$\begin{aligned}\tilde{U}_{A,t} - U_{A,t}^* &= -\Delta \tilde{D}_t |v_{A,t}| + \delta \overline{W}_{A,t-1} \mathbb{E}_t \left[\mathbb{I}_{w_{t+1} = \overline{W}_{t-1}} \left[\Delta \tilde{S}_{t+1} - \Delta S_{t+1}^* \right] \right] \\ &= 0. \end{aligned} \tag{9}$$

But then, it follows that

$$\begin{aligned}
\tilde{U}_{P,t} - U^*_{P,t} &= \Delta\tilde{D}_t v_{P,t} - \delta\left|\overline{W}_{P,t-1}\right| \mathbb{E}_t\left[\mathbb{I}_{w_{t+1}=\overline{W}_{t-1}}\left[\Delta\tilde{S}_{t+1} - \Delta S^*_{t+1}\right]\right] \\
&= \Delta\tilde{D}_t|v_{A,t}|\left[\frac{v_{P,t}}{|v_{A,t}|} - \frac{\left|\overline{W}_{P,t-1}\right|}{\overline{W}_{A,t-1}}\right] \\
&\succ 0,
\end{aligned}$$

yielding the desired contradiction. The second equality follows from substituting (9) and the inequality follow because $v_t \succ \overline{W}_{t-1}$.

Step 6. Fix optimal processes (D^*, S^*) and project history u^t such that $\overline{W}_{t-1} \succ v_t$. We show that $\Delta D^*_t = 0$. Suppose, towards a contradiction, that $\Delta D^*_t > 0$. Recall that, by Part 2 of Proposition 1, it follows that $T^{\underline{W}_{t-1}}_{t+1} \leq T^{\underline{W}_{t-1}}_{t-1} < t+2$, to that $\Delta S^*_{t+1} > 0$ if $w_{t+1} = \underline{W}_{t-1}$. Now consider alternative processes $(\tilde{D}, \tilde{S})$, identical to (D^*, S^*) except that (i) $0 \leq \Delta\tilde{D}_t < \Delta D^*_t$, (ii) $0 \leq \Delta\tilde{S}_{t+1} < \Delta S^*_{t+1}$ if $w_{t+1} = \underline{W}_{t-1}$, and (iii) $\tilde{U}_{A,t} = U^*_{A,t}$. Such a pair $(\tilde{D}, \tilde{S})$ always exists, and that it is agent-feasible follows by arguments similar to those of Step 1 in the proof of Proposition 1. By (iii), we have that

$$\begin{aligned}
\tilde{U}_{A,t} - U^*_{A,t} &= \left[\Delta D^*_t - \Delta\tilde{D}_t\right]|v_{A,t}| - \delta\underline{W}_{A,t-1}\mathbb{E}_t\left[\mathbb{I}_{w_{t+1}=\underline{W}_{t-1}}\left[\Delta S^*_{t+1} - \Delta\tilde{S}_{t+1}\right]\right] \\
&= 0. \qquad (10)
\end{aligned}$$

But then, it follows that

$$\begin{aligned}
\tilde{U}_{P,t} - U^*_{P,t} &= -\left[\Delta D^*_t - \Delta\tilde{D}_t\right]v_{P,t} + \delta\left|\underline{W}_{P,t-1}\right|\mathbb{E}_t\left[\mathbb{I}_{w_{t+1}=\underline{W}_{t-1}}\left[\Delta S^*_{t+1} - \Delta\tilde{S}_{t+1}\right]\right] \\
&= \left[\Delta D^*_t - \Delta\tilde{D}_t\right]|v_{A,t}|\left[\frac{\left|\underline{W}_{P,t-1}\right|}{\underline{W}_{A,t-1}} - \frac{v_{P,t}}{|v_{A,t}|}\right] \\
&\succ 0,
\end{aligned}$$

yielding the desired contradiction. The second equality follows from substituting (10) and the inequality follow because $\underline{W}_{t-1} \succ v_t$.

□

Proof of Proposition 3. We proceed in a number of steps.

Step 1. Fix project $v \in \mathcal{D}$ and suppose that $u_1 = v$. We define the reduced problem

$$\max_{\kappa \in \mathcal{K}} U_{P,1} \text{ subject to } U_{A,1} \geq 0. \tag{11}$$

First, note that a standard argument establishes that $U_{A,1} = 0$ at any solution to (11). Second, note that problem (11) only requires agent-feasibility at $t = 1$. Therefore, if the solution to (11) is also agent-feasible at all times $t > 1$, then it must be part of an optimal relationship process conditional on $u_1 = v$.

Step 2. We show that there exists a solution to (11) with demand and supply processes (D^*, S^*) of the following threshold type: for each $v \in D$, there exists $T^v \geq 0$ such that, given any $t \geq 1$,

$$\Delta D_t^* = \begin{cases} 1 & \text{if } T^{v_t} \geq t+1, \\ T^{v_t} - t & \text{if } t < T^{v_t} < t+1, \\ 0 & \text{if } T^{v_t} \leq t, \end{cases}$$

and for each $w \in S$, there exists $T^w \geq 0$ such that, given any $t \geq 1$,

$$\Delta S_t^* = \begin{cases} 1 & \text{if } T^{w_t} \leq t, \\ t+1-T^{w_t} & \text{if } t < T^{w_t} < t+1, \\ 0 & \text{if } T^{w_t} \geq t+1, \end{cases}$$

The critical difference with the corresponding expressions with a general process u from Propositions 1 and 2 is that the time thresholds $(\{T^v\}_{v\in\mathcal{D}}, \{T^w\}_{w\in\mathcal{S}})$ are fixed and independent of histories. The proof of this claim follows from arguments closely mirroring those of Steps 5-7 of Proposition 1 and Step 2 of Proposition 2, and is omitted. In fact, these arguments are simplified in this case because the only agent-feasibility constraint for the agent in problem (11) is for the initial history.

Finally, we normalise these time thresholds so that (i) $T^v = 0$ if and only if $\Delta D_t^* = 0$ for all $t \geq 1$ with $u_t = v$, that is, if and only if the principal never demands favour v under D^*, and that (ii) $T^w = 0$ if and only if $\Delta S_t = 1$ for all $t \geq 1$ with $u_t = w$, that is, if the principal always supplies favour w under S^*.

Step 3. We show that there exists a solution to (11) with the following properties: there exist $v^* \in \mathcal{D}$ and $w^* \in \mathcal{S}$ with $v^* \succ w^*$ such that

1. $T^{v^*} \neq 0$, and furthermore $T^v = 0$ if $v^* \succ v$ and $T^v = \infty$ if $v \succ v^*$.

2. $T^{w^*} \neq \infty$, and furthermore $T^w = 0$ if $w^* \succ w$ and $T^w = \infty$ if $w \succ w^*$.

3. Given any $v \in \mathcal{D}$, if $T^v < \infty$, then $T^w = 0$ for all $v \succ w$. Also, if $T^v > 0$ then $T^w = \infty$ for all $w \succ v$.

Project v^* is the lowest project in the order $\succ$ at which the principal ever demands a favour, and project w^* is the highest project in that order in which the principal ever supplies a favour. Note that Item 3 implies that if $T^{v^*} < \infty$, then $T^{w^*} = 0$, and that if $T^{w^*} > 0$, then $T^{v^*} = \infty$. The proof of this claim follows from arguments closely mirroring those of Steps 1-3 of Proposition 1 and Step 1 of Proposition 2, and is omitted. Again, these arguments are simplified in this case because the only agent-feasibility constraint for the agent in problem (11) is for the initial history.

Step 4. Given $\overline{v}, \underline{v} \in \mathcal{D}$, consider the associated solutions $(\overline{D}^*, \overline{S}^*)$ and $(\underline{D}^*, \underline{S}^*)$ to the problem (11) with $u_1 = \overline{v}$ and $u_1 = \underline{v}$, respectively. We show that if either

$$(i)\ \overline{v}^* \succ \underline{v}^* \text{ or } (ii)\ \overline{v}^* = \underline{v}^* \text{ and } \overline{T}^{\overline{v}^*} < \underline{T}^{\underline{v}^*},$$

then either

$$(i)\ \overline{w}^* \succ \underline{w}^* \text{ or } (ii)\ \overline{w}^* = \underline{w}^* \text{ and } \overline{T}^{\overline{w}^*} \leq \underline{T}^{\underline{w}^*}.$$

To see this, suppose that either (i) $\overline{v}^* \succ \underline{v}^*$ or (ii) $\overline{v}^* = \underline{v}^*$ and $\overline{T}^{\overline{v}^*} < \underline{T}^{\underline{v}^*}$. Note that, by Item 1 in Step 3, we have in both cases (i) and (ii) that $\overline{T}^v \leq \underline{T}^v$ for all $v \in \mathcal{D}$, with at least one inequality strict, so that, in words, the demand process $\overline{D}^*$ is strictly more generous to the agent then $\underline{D}^*$. Now suppose, towards a contradiction, that either (i) $\underline{w}^* \succ \overline{w}^*$ or (ii) $\underline{w}^* = \overline{w}^*$ and $\underline{T}^{\underline{w}^*} < \overline{T}^{\overline{w}^*}$. Note that, by Item 2 in Step 3, we have that $\overline{T}^w \geq \underline{T}^w$ for all $w \in \mathcal{S}$, with at least one inequality strict, so that, in words, the supply process $\overline{S}^*$ is strictly more generous to the agent then $\underline{S}^*$. First, let $\tilde{v}$ be such that $\overline{T}^{\tilde{v}} < \underline{T}^{\tilde{v}}$, which by assumption must exist. Second, because it must be that $\overline{T}^{\tilde{v}} < \infty$, Item 3 of of Step 3 implies that $\overline{T}^w = 0$ for all $\tilde{v} \succ w$. Third, fix $\tilde{w}$ such that $\overline{T}^{\tilde{w}} > \underline{\tilde{w}}$, which by (our contradiction) assumption must exist. Fourth, because this implies that $\overline{T}^{\tilde{w}} > 0$, Item 3 of Step 3 implies that $\overline{T}^v = \infty$ for all $v \succ \tilde{w}$. Fifth, along with the fact that $\overline{T}^{\tilde{v}} < \infty$ the last point implies that $\tilde{w} \succ \tilde{v}$. Sixth, because $\underline{T}^{\tilde{v}} > 0$, the last point along with Item 3 of Step 3 implies that $\underline{T}^{\tilde{w}} = \infty$, which contradicts the fact that $\overline{T}^{\tilde{w}} > \underline{\tilde{w}}$, as desired.

Step 5. The previous point allows us to rank the solutions to (11) for various $v \in \mathcal{D}$ for which $u_1 = v$ in terms of how generous they are to the agent. Specifically, fix $\overline{v}, \underline{v} \in \mathcal{D}$ and consider the associated solutions $(\overline{D}^*, \overline{S}^*)$ and $(\underline{D}^*, \underline{S}^*)$ to the problem (11) with $u_1 = \overline{v}$ and $u_1 = \underline{v}$, respectively. If either (i) $\overline{v}^* \succ \underline{v}^*$ or (ii) $\overline{v}^* = \underline{v}^*$ and $\overline{T}^{\overline{v}^*} < \underline{T}^{\underline{v}^*}$, then we say that processes $(\overline{D}^*, \overline{S}^*)$ are *more generous to the agent* than processes $(\underline{D}^*, \underline{S}^*)$. In words, Step 4 says that when these conditions are met, then $\overline{D}^*$ demands less of every project than $\underline{D}^*$, and $\overline{S}^*$ supplies more of every project than $\underline{S}^*$. Fix any project u such that $u_1 = u$, and let $\overline{U}_{i,1}$ denote the payoff to i from processes $(\overline{D}^*, \overline{S}^*)$, and $\underline{U}_{i,1}$ denote the payoff to i from processes $(\underline{D}^*, \underline{S}^*)$. It follows that if processes $(\overline{D}^*, \overline{S}^*)$ are more generous to the agent than processes $(\underline{D}^*, \underline{S}^*)$, then we have that $\overline{U}_{A,1} \geq \underline{U}_{A,1}$. An implication is that the pair $(\overline{D}^*, \overline{S}^*)$ is agent-feasible if $u_1 = \underline{v}$, but that the pair $(\underline{D}^*, \underline{S}^*)$ is not agent-feasible if $u_1 = \overline{v}$.

Step 6. Let $v^1 \in \mathcal{D}$ be the project for which the solution (D^{1*}, S^{1*}) to problem (11) with $u_1 = v^1$ is the most generous among all solutions to (11) with $u_1 = v$ for some $v \in \mathcal{D}$.

First, we show that we cannot have $v^* \succ v^1$. In words, it must be that, conditional on $u_1 = v$, the principal demands a favour with positive intensity at $t = 1$ under (D^{1*}, S^{1*}). To see this suppose, towards a contradiction, that $v^* \succ v^1$. For any history u^t, let $U^*_{A,t}$ be the payoff to the agent, conditional on u^t, to (D^{1*}, S^{1*}), and let $U^1_{A,t}$ be the payoff to the agent if this solution was implemented starting from t. If $v^* \succ v^1$, then $\kappa^1_1 = 0$ and therefore

$$\begin{aligned} U^*_{A,1} &= \delta \mathbb{E}_1 U^*_{A,2} \\ &> \mathbb{E}_1 U^1_{A,2} \\ &\geq 0, \end{aligned}$$

contradicting the fact that $U^1_{A,1}$, as desired. The first inequality follows by the fact that u is a Markov process and because the pair (D^{1*}, S^{1*}) becomes more generous between times $t = 1$ and $t = 2$. The second inequality follows, again, by the fact that u is a Markov process and because the pair (D^{1*}, S^{1*}) is more generous than any solution with $u_1 = v$ for some v, and therefore, by Step 5, it is agent-feasible for all $u_2 = v$.

Second, we show that following any history at which $u_t = v^1$, the pair (D^{1*}, S^{1*}) is agent-feasible, and furthermore no agent-feasible relationship process delivers a higher payoff to the principal. To see this, for any $t' \geq t$ let $U^*_{A,t'}$ denote the agent's payoff from this solution, and first note that because u is a Markov process, we have that given $u_t = v^1$,

$$\begin{aligned} U^*_{A,t} &= U^1_{A,t} \\ &\geq 0. \end{aligned}$$

Also, for any $t' > t$, we have that

$$\begin{aligned} U^*_{A,t'} &> U^1_{A,t'} \\ &\geq 0, \end{aligned}$$

where the inequalities follow for the same reasons as the corresponding inequalities in the previous paragraph. That no agent-feasible relationship process delivers a higher payoff to the principal following u^t follows by the construction of problem (11).

Step 7. Define the set of projects $V^1 = \{v^1\}$ with associated set of processes $R^1 = \{(D^{1*}, S^{1*})\}$. Now, inductively, fix a set of projects $V^{n-1} = \{v^1, ..., v^{n-1}\}$ and associated set of processes $R^{n-1} = \{(D^{1*}, S^{1*}), ..., (D^{n-1*}, S^{n-1*})\}$. Assume that (D^{i*}, S^{i*}) is such that, following any history with $u_t = v^i$, the pair (D^{i*}, S^{i*}) is agent-feasible, and furthermore no agent-feasible relationship process delivers a higher payoff to the principal than (D^{i*}, S^{i*}). Assume further that, following any history with $u_t \in \mathcal{D} \setminus V^{n-1}$, the pair (D^{i*}, S^{i*}) is agent-feasible. Fix any project $v \in \mathcal{D} \setminus V^{n-1}$ and suppose that $u_1 = v$. We define the reduced problem

$$\begin{aligned} \max_{\kappa \in \mathcal{K}} U_{1,t} \;\text{ subject to } \quad & U_{A,1} \geq 0, \\ & U_{A,t} \geq 0 \text{ at each } t > 1 \text{ at which } u_t \in V^{n-1}, \qquad (12) \\ & (D, S) = (D^{i*}, S^{i*}) \text{ at each } t > 1 \text{ with } u_t = v^i \text{ and } U_{A,t} = 0. \end{aligned}$$

This problem corresponds closely to the problem (11): the goal is to find an optimal relationship process while ignoring all agent-feasibility constraints other than (i) the constraint at time $t = 1$ but also (ii) all feasibility constraints associated with the first arrival of an opportunity to demand favour $v^i \in V^{n-1}$. In the latter case, the problem

(12) prescribes that processes (D^{i*}, S^{i*}) be adopted at this history if the agent-feasibility constraint binds.

Step 8. As in Step 2 for problem (11), we show that there exists a solution to problem (12) with demand and supply processes (D^*, S^*) of the following threshold type: for each $v \in \mathcal{D} \setminus V^{n-1}$, there exists $T^v \geq 0$ such that, given any $t \geq 1$ with $U_{A,t'} > 0$ for all $1 \leq t' \leq t$ for which $u_{t'} \in V^{n-1}$,

$$\Delta D_t^* = \begin{cases} 1 & \text{if } T^{v_t} \geq t+1, \\ T^{v_t} - t & \text{if } t < T^{v_t} < t+1, \\ 0 & \text{if } T^{v_t} \leq t, \end{cases}$$

and for each $w \in S$, there exists $T^w \geq 0$ such that, given any $t \geq 1$ with $U_{A,t'} > 0$ for all $1 \leq t' \leq t$ for which $u_{t'} \in V^{n-1}$,

$$\Delta S_t^* = \begin{cases} 1 & \text{if } T^{w_t} \leq t, \\ t+1-T^{w_t} & \text{if } t < T^{w_t} < t+1, \\ 0 & \text{if } T^{w_t} \geq t+1, \end{cases}$$

In words, the thresholds above are valid until the relationship process transitions to (D^{i*}, S^{i*}) for some $i = 1, ..., n-1$. As for Step 2, the proof of this claim follows from arguments closely mirroring those of Steps 5-7 of Proposition 1 and Step 2 of Proposition 2, and is omitted. Also as in Step 2, we normalise these time thresholds so that (i) $T^v = 0$ if and only if $\Delta D_t^* = 0$ for all $t \geq 1$ with $u_t = v$, and that (ii) $T^w = 0$ if and only if $\Delta S_t = 1$ for all $t \geq 1$ with $u_t = w$.

Step 9. As in Steps 3-5 for problem 11, we show that solutions to (12) can be ranked according to how generous they are to the agent. TO BE COMPLETED.

□

Proof of Corollary 2. □

TO BE COMPLETED