Type-Symmetric Randomized Equilibrium

Xiang Sun* Yeneng Sun† Haomiao Yu‡

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Abstract

We introduce the notion of *type-symmetric randomized equilibrium* (TSRE) by requiring those agents with the same type of characteristics to choose the same randomized choice. Such a notion provides a generic micro-foundation for the macro notion of equilibrium distribution, as used in the literature on games and economies with many agents. In particular, we show that if the space of agents is modeled by the classical Lebesgue unit interval, any Nash (resp. Walrasian) equilibrium distribution in a large game (resp. economy) is *uniquely* determined by one TSRE. Furthermore, we provide examples to demonstrate that this uniqueness characterization does not necessarily hold when a non-Lebesgue agent space is used.

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*Economics and Management School, Wuhan University, 299 Bayi Road, Wuhan, 430072, China. E-mail: xiangsun.econ@gmail.com

†Department of Economics, National University of Singapore, 1 Arts Link, Singapore 117570. E-mail: ynsun@nus.edu.sg

‡Department of Economics, Ryerson University, Toronto, ON M5B2K3, Canada. E-mail: haomiao@ryerson.ca

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1 Introduction

An atomless probability space, in particular the Lebesgue unit interval, has been commonly used for modeling many agents[1] in economics. There are two approaches to deal with equilibrium concepts associated with a continuum of agents: one is the individualized microeconomic notion that specifies a choice for each agent, and the other is the distributional macroeconomic notion that is formulated as a distribution on the joint space of characteristics and choices without individualistic features. In the particular settings as considered in this paper, a Nash equilibrium[2] in a large game (resp. a Walrasian allocation in a large economy) is defined as a measurable mapping—from the agent space to the action (resp. consumption)

[1] Milnor and Shapley (1961) and Aumann (1964) used the Lebesgue unit interval as an agent space that captures the negligible influence of an individual in a large society. A game/economy with a continuum of agents is also considered to be a good approximation for games/economies with large but finitely many agents; see, for example, Debreu and Scarf (1963), Hildenbrand (1974), Hammond (1979), Green (1984), Mas-Colell and Vives (1993), Khan and Sun (2002), and McLean and Postlewaite (2002, 2004).

[2] Throughout the paper, Nash equilibrium refers to pure strategy Nash equilibrium. We shall always say randomized (resp. mixed) strategy Nash equilibrium when randomized (resp. mixed) strategies are used. In the sequel, we shall also use "agent" interchangeably with "player" in the informal discussion of large games.

space—that focuses on the micro level, while a Nash equilibrium distribution in a large game (resp. a Walrasian equilibrium distribution in a large economy) is defined as a distribution—on the joint space of payoffs and actions (resp. the joint space of preferences, endowments and consumptions)—that focuses on the macro level. Both approaches have been extensively studied and applied in many areas.[3]

The distributional approach has been regarded as a reformulation of the individualized approach. An individualized equilibrium always induces an equilibrium distribution. However, a given equilibrium distribution may not be induced by any individualized equilibrium. For instance, in the large game in Example 1, the Nash equilibrium distribution $\hat{\tau}$ does not correspond to any Nash equilibrium;[4] in the large economy in Example 3, there exists a Walrasian equilibrium distribution that cannot be induced by any Walrasian allocation.[5] Hence, in general, the notion of equilibrium distribution is not merely an equivalent reformulation of the notion of individualized equilibrium. To understand the distributional approach and its advantage, one needs to find its generic microeconomic counterpart. After all, without a justifiable micro-foundation, a macro-aggregate solution concept can hardly be satisfactory.

In this paper, we introduce another notion of equilibrium via the individualized approach—a *type-symmetric randomized equilibrium*. In a large game/economy, it is defined as an equilibrium where those agents with the same characteristics choose the same randomized choice. We show that in a large game/economy, an equilibrium distribution corresponds to a type-symmetric randomized equilibrium. The

[3]See Hildenbrand (1974) and Khan and Sun (2002) for various references; also see Rauh (2007), Anderson and Raimondo (2008), Yannelis (2009), Acemoglu and Wolitzky (2011), Guesnerie and Jara-Moroni (2011), Duffie and Strulovici (2012) and Hammond (2015) for some recent applications.

[4]Note that the large game in Example 1 does have Nash equilibria f_1 and f_2. Rath *et al.* (1995) presented a rather involved example of a large game with the action space $[-1, 1]$ that has a Nash equilibrium distribution but no Nash equilibrium; see also Khan *et al.* (2013a) and Qiao and Yu (2014) for some recent examples of this kind. In consideration of the existence issue, Keisler and Sun (2009) and Khan *et al.* (2013a) showed that the saturation of the agent space is necessary and sufficient for the existence of a Nash equilibrium; see also Barelli and Duggan (2015) for a product structure on the agent space.

[5]This example is based on the example in Kannai (1970, Section 7). In contrast to the possible nonexistence of Nash equilibrium for a large game with a one-dimensional action space as mentioned in Footnote 4, a Walrasian allocation always exists in a finite-dimensional commodity space. On the other hand, when one works with an infinite-dimensional commodity space, a Walrasian allocation may not exist; see Tourky and Yannelis (2001) for example.

classical Lebesgue unit interval plays an interesting role in this characterization. If the agent space is modeled by the Lebesgue unit interval, one equilibrium distribution corresponds *uniquely* to one type-symmetric randomized equilibrium. As the latter is an equilibrium that focuses on the individual level, it indeed provides a generic micro-foundation of the macro notion of equilibrium distribution. All this reveals the fact that an equilibrium distribution is in general a reformulation of a randomized equilibrium rather than a deterministic individualized equilibrium. We may point out that several properties of the analytic sets in the descriptive set theory play the key role for the uniqueness characterization.

The rest of the paper is organized as follows. Section 2 considers large games. First, we present the set-up of a large game and two canonical solution concepts—Nash equilibrium and Nash equilibrium distribution, and provide an example showing the non-equivalence of the two concepts. Then, we introduce the notion of type-symmetric randomized equilibrium in a large game, and establish its relationship with the concept of Nash equilibrium distribution in Theorem 1. Example 2 shows that the uniqueness characterization in Theorem 1 (iii) may fail for a large game if the underlying agent space is not the Lebesgue unit interval. Section 3 presents the analogous results in the context of large economies. Section 4 provides further discussion on the notion of type-symmetric randomized equilibrium. Under the framework of a rich Fubini extension, we can then obtain a complete delineation of all the equilibrium concepts in large games, including the notion of mixed strategy equilibrium that involves a continuum of independent random variables.[6] All the proofs are given in Section 5.

2 Large Games

Unless otherwise specified, a topological space—say X—as discussed in this paper is understood to be equipped with its Borel σ-algebra $\mathscr{B}(X)$, and the measurability is defined in terms of that. For a Polish (complete separable metrizable topological) space X, $\mathscr{M}(X)$ denotes the set of (Borel) probability measures on X endowed

[6]The issues of measurability and exact law of large numbers for processes with a continuum of independent random variables are resolved in Sun (2006) by working with an extension of the usual product probability space that retains the Fubini property.

with the topology of weak convergence, χ_E denotes the indicator function of a subset E of X, and δ_x denotes the Dirac measure at the point $x \in X$. For a probability measure $\tau \in \mathscr{M}(X \times Y)$ on the product of two Polish spaces X and Y, τ_X and τ_Y denote the marginals of τ on X and Y respectively.

We next introduce some specific notation and terminology for a large game considered in this paper. Note that we work within the recent generalized framework of a large game by Khan *et al.* (2013a) that take players' traits into consideration. In this framework, a player's characteristics consist of both payoffs and traits, and a societal summary is formulated as a joint distribution on the space of traits and the action set.[7] Formally, let $(I, \mathscr{I}, \lambda)$ be an atomless probability space representing the player space, A a compact metric space representing the set of actions, and T another compact metric space representing the space of players' traits. Let $\mathscr{U}_{(A,T)}$ be the space of real-valued continuous functions on $A \times \mathscr{M}(T \times A)$, representing the space of payoff functions, metrized by the supremum norm. The characteristics of an individual player consist of a trait and a payoff function, and thus the space of characteristics is $T \times \mathscr{U}_{(A,T)}$. A *large game* is a measurable function $\mathscr{G}$ from the player space $(I, \mathscr{I}, \lambda)$ to the space of characteristics $T \times \mathscr{U}_{(A,T)}$.[8]

2.1 Nash Equilibrium and Nash Equilibrium Distribution

We are now ready to present two standard notions of equilibria that are commonly used in the literature of large games. The first one involves the player space.

Definition 1 (NE)**.** A pure strategy profile f of a large game $\mathscr{G} \colon I \to T \times \mathscr{U}_{(A,T)}$ is a measurable function from $(I, \mathscr{I}, \lambda)$ to the action set A, and is said to be a *(pure strategy) Nash equilibrium* (NE) if for λ-almost all $i \in I$,

$$u_i\left(f(i), \lambda(\alpha, f)^{-1}\right) \geq u_i\left(a, \lambda(\alpha, f)^{-1}\right) \text{ for all } a \in A,$$

with u_i abbreviated for $\mathscr{G}_2(i)$, and α abbreviated for $\mathscr{G}_1$, where $\mathscr{G}_k$ is the projection

[7]See Khan *et al.* (2013b) for its corresponding distributional approach. It is straightforward to see that when the space of traits is reduced to a singleton, a large game with traits reduces to a conventional large game (without traits) as surveyed in Khan and Sun (2002).

[8]When the trait space is a singleton, a large game can be reduced to a measurable function from the player space to to the set $\mathscr{U}_A$, where $\mathscr{U}_A$ is the space of real-valued continuous functions on $A \times \mathscr{M}(A)$.

of $\mathscr{G}$ on its k-th coordinate, $k = 1, 2$.

Given a large game $\mathscr{G}\colon I \to T \times \mathscr{U}_{(A,T)}$, one can consider its distributional form $\lambda\mathscr{G}^{-1}$. This is a Borel probability measure on $T \times \mathscr{U}_{(A,T)}$, and the corresponding solution notion can be defined as below by ignoring the player space.

Definition 2 (NED)**.** A Borel probability measure τ on $T \times \mathscr{U}_{(A,T)} \times A$ is said to be a *Nash equilibrium distribution* (NED) of a large game $\lambda\mathscr{G}^{-1}$ in its distributional form if $\tau_{T\times\mathscr{U}_{(A,T)}} = \lambda\mathscr{G}^{-1}$ and $\tau\big(Br(\tau)\big) = 1$ where

$$Br(\tau) = \big\{(t, v, a) \in T \times \mathscr{U}_{(A,T)} \times A \mid v(a, \tau_{T\times A}) \geq v(x, \tau_{T\times A}) \text{ for all } x \in A\big\}.$$

It is now well-understood that in a large game $\mathscr{G}\colon I \to T \times \mathscr{U}_{(A,T)}$, if f is an NE of $\mathscr{G}$, then $\lambda(\mathscr{G}, f)^{-1}$ is an NED of $\lambda\mathscr{G}^{-1}$. Hence, given any NE as a solution at the individual microeconomic level, its macroeconomics counterpart automatically becomes an NED of the same game in its distributional form. However, the converse often fails. This is to say, in a large game $\mathscr{G}\colon I \to T \times \mathscr{U}_{(A,T)}$, given an NED τ of $\lambda\mathscr{G}^{-1}$, it is possible that there does not exist any NE f of $\mathscr{G}$ such that $\tau = \lambda(\mathscr{G}, f)^{-1}$. Below is a simple example.[9]

Example 1. Let the player space be the Lebesgue unit interval $([0, 1], \mathscr{B}([0, 1]), \ell)$, the space of traits be a compact metric space $\hat{T}$, and the common action set be $\hat{A} = [0, 1]$. Take an element $\hat{t}$ in $\hat{T}$. Let $\hat{\mathscr{G}}\colon [0, 1] \to \hat{T} \times \mathscr{U}_{(\hat{A},\hat{T})}$ be such that for each player $i \in [0, 1]$, each action $a \in \hat{A}$ and each societal trait-action distribution $\nu \in \mathscr{M}(\hat{T} \times \hat{A})$,

$$\hat{\mathscr{G}}(i)(a, \nu) = \big(\hat{t}, -a^2 \cdot (a - i)^2\big).$$

It is clear that $\hat{\mathscr{G}}$ is a well-defined large game.

Let f_1 and f_2 be two strategy profiles of $\hat{\mathscr{G}}$ such that $f_1(i) = 0$ and $f_2(i) = i$ for each $i \in [0, 1]$. Both f_1 and f_2 are NE of the game $\hat{\mathscr{G}}$ since actions 0 and i are the best choices for player i. Let $\hat{\tau} = \frac{1}{2}\ell(\hat{\mathscr{G}}, f_1)^{-1} + \frac{1}{2}\ell(\hat{\mathscr{G}}, f_2)^{-1}$. One can easily check that $\hat{\tau}$ is an NED of $\ell(\hat{\mathscr{G}})^{-1}$.

However, there does not exist any NE f such that $\ell(\hat{\mathscr{G}}, f)^{-1} = \hat{\tau}$. Suppose not, then by the construction of $\hat{\tau}$, we must have $\ell f^{-1} = \frac{1}{2}\delta_0 + \frac{1}{2}\ell$. Furthermore, since

[9] In this example, both NE and NED exist. Note that while NED always exists, there are examples where NE does not exist at all; see Footnote 5.

f is an NE, $f(i)$ should be 0 or i for ℓ-almost all $i \in [0,1]$. Suppose that $f(i) = i$ holds on some set $C \in \mathscr{B}([0,1])$. Then we must have

$$\ell(C) = \ell f^{-1}(C) = \left(\tfrac{1}{2}\delta_0 + \tfrac{1}{2}\ell\right)(C),$$

and hence $\ell(C)$ is 0 or 1. This contradicts the fact that $\ell f^{-1} = \frac{1}{2}\delta_0 + \frac{1}{2}\ell$. □

2.2 Type-Symmetric Randomized Equilibrium in a Large Game

Instead of an individualized notion of equilibrium in pure strategies in a large game, we now consider one in randomized strategies, and a novel refinement of it.

Definition 3 (TSRE)**.** A randomized strategy[10] profile h of a large game $\mathscr{G}\colon I \to T \times \mathscr{U}_{(A,T)}$ is a measurable function from $(I, \mathscr{I}, \lambda)$ to $\mathscr{M}(A)$, and is said to be a *randomized strategy equilibrium* (RSE) if for λ-almost all $i \in I$,

$$\int_A u_i\left(a, \int_I \delta_{\alpha(j)} \otimes h(j)\,\mathrm{d}\lambda(j)\right) h(i;\mathrm{d}a) \geq \int_A u_i\left(a, \int_I \delta_{\alpha(j)} \otimes h(j)\,\mathrm{d}\lambda(j)\right) \mathrm{d}\eta(a)$$

for all $\eta \in \mathscr{M}(A)$. An RSE h of a large game $\mathscr{G}\colon I \to T \times \mathscr{U}_{(A,T)}$ is said to be a *type-symmetric randomized equilibrium* (TSRE) if $h(i) = h(j)$ whenever $\mathscr{G}(i) = \mathscr{G}(j)$.

Note that for any given randomized strategy profile h of a large game $\mathscr{G}\colon I \to T \times \mathscr{U}_{(A,T)}$, the corresponding *type-action distribution* $\int_I \delta_{\mathscr{G}(j)} \otimes h(j)\,\mathrm{d}\lambda(j)$ is a well-defined macro aggregate.

In a TSRE, intuitively, all players with the same characteristics are required to play the *same randomized action*. We next show that this new notion of equilibrium does not only provide a generic micro-foundation of NED in general, but also provide a uniqueness characterization of an NED when the player space is the Lebesgue unit interval $([0,1], \mathscr{B}([0,1]), \ell)$.

Theorem 1. *In a large game $\mathscr{G}\colon I \to T \times \mathscr{U}_{(A,T)}$, we have the following statements:*

[10] Note that we avoid using mixed strategy here due to the measurability issue associated with a continuum of independent random variables; see Sun (2006) for a detailed discussion of the issue, and Definition 7 below for a proper definition of a mixed strategy equilibrium in a large game.

(i) For any TSRE of $\mathscr{G}$, the induced type-action distribution is an NED of $\lambda\mathscr{G}^{-1}$.

(ii) For any NED of $\lambda\mathscr{G}^{-1}$, there is a TSRE of $\mathscr{G}$ that induces it.

(iii) If $(I, \mathscr{I}, \lambda)$ is the Lebesgue unit interval, then for any NED of $\lambda\mathscr{G}^{-1}$, there is a unique *TSRE of $\mathscr{G}$ that induces it.*

Statement (i) basically suggests that the type-action distribution associated with any TSRE is an NED, and statement (ii) says that every NED can be *lifted* to some TSRE at the individual level so that the TSRE induces the same aggregate as given. These characterizations themselves are new in the literature. What is surprising is statement (iii). It says that any NED τ is uniquely determined by a particular TSRE h when the player space is the Lebesgue unit interval. That is to say, for large games with the Lebesgue agent space, we have a one-to-one characterization between an equilibrium distribution at the aggregate level and an equilibrium at the individual level.

In Example 1, let $\hat{h}\colon [0,1] \to \mathscr{M}(\hat{A})$ be a randomized strategy profile of $\hat{\mathscr{G}}$ such that for each $i \in [0,1]$,

$$\hat{h}(i) = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_i. \tag{1}$$

It is clear that $\hat{h}$ is a TSRE that induces the NED $\hat{\tau}$. Furthermore, since an NE is a degenerated RSE and every player in $\hat{\mathscr{G}}$ in Example 1 has a payoff different from others, any NE of $\hat{\mathscr{G}}$ is a degenerated TSRE. Statement (iii) in Theorem 1 implies that $\hat{h}$ is the unique TSRE that induces $\hat{\tau}$. Thus, this is to show directly that there is no NE of $\hat{\mathscr{G}}$ that can induce $\hat{\tau}$, and to reemphasize that whereas NE is not suitable as a micro-foundation for NED, our new notion of equilibrium, TSRE, is indeed a generic and sharp micro counterpart of NED.

We next offer three observations related to TSRE in large games.

First, note that in the discussion above, we have used the type-symmetric idea directly to NE since an NE can be always trivially viewed as an RSE. We next make this refinement of NE explicit, and say that an NE f^s of a large game $\mathscr{G}\colon I \to T \times \mathscr{U}_{(A,T)}$ is *type-symmetric* if $f^s(i) = f^s(j)$ whenever $\mathscr{G}(i) = \mathscr{G}(j)$.

In the literature of large games, a symmetric NED is often discussed as a macro solution concept that implicitly requires that all players with the same characteristics play the same action.[11] The type-symmetric NE defined above connects explicitly the individual behavior and the societal outcome, and allows us to characterize a symmetric NED. Recall that in a large game $\mathscr{G}\colon I \to T \times \mathscr{U}_{(A,T)}$, an NED τ of $\lambda\mathscr{G}^{-1}$ is *symmetric* if there exists a measurable function $s\colon T \times \mathscr{U}_{(A,T)} \to A$ such that $\tau(\text{graph of } s) = 1$. Below is a direct corollary of Theorem 1.

Corollary 1. *In a large game $\mathscr{G}$ with the Lebesgue agent space $([0,1], \mathscr{B}([0,1]), \ell)$, if τ is a symmetric NED, then there exists a unique type-symmetric NE f such that $\ell(\mathscr{G}, f)^{-1} = \tau$.*

Second, the next observation allows us to connect an RSE to a TSRE in a large game. We say that an RSE h can be *symmetrized* if there exists a TSRE h^s such that h and h^s induce the same type-action distribution. The result below shows that at the aggregate level, there is no essential difference between an RSE and a TSRE.

Corollary 2. *Every RSE in a large game can be symmetrized.*

Finally, we conclude this section by using the example below to point out that Theorem 1 (iii) may fail to hold if a probability space other than the Lebesgue unit interval is used as the player space.

Example 2. Consider a game $\bar{\mathscr{G}}$ which differs from the game $\hat{\mathscr{G}}$ defined in Example 1 only in terms of the player space: let that of $\bar{\mathscr{G}}$ be $([0,1], \bar{\mathscr{B}}, \bar{\ell})$—an extension of the Lebesgue unit interval $([0,1], \mathscr{B}([0,1]), \ell)$ such that there is a $\bar{\mathscr{B}}$-measurable subset B of measure half and independent with $\mathscr{B}([0,1])$ under $\bar{\ell}$.[12] It is obvious that $\hat{\tau}$ defined in Example 1 is still an NED of $\bar{\ell}(\bar{\mathscr{G}})^{-1}$, and that $\hat{h}$ defined as in Equation (1) is a TSRE such that $(\bar{\mathscr{G}}, \hat{h})$ induces the NED $\hat{\tau}$.

[11] For symmetric NED, see Mas-Colell (1984) and Khan and Sun (2002, Section 4). Such an idea is used in a large economy as "the equal treatment" as well; see the references of equal treatment and the discussion of symmetric mechanism in Hammond (1979).

[12] It is well-known that there is a nonmeasurable (in the Lebesgue sense) subset B in $[0,1]$ with the inner measure zero and the outer measure one; see the construction in Section 3.4 in Royden (1988). Let $\bar{\mathscr{B}} = \{(B_1 \cap B) \cup (B_2 \setminus B) \mid B_1, B_2 \in \mathscr{B}([0,1])\}$ and $\bar{\ell}\big((B_1 \cap B) \cup (B_2 \setminus B)\big) = \frac{1}{2}\big(\ell(B_1) + \ell(B_2)\big)$. It is easy to check that $([0,1], \bar{\mathscr{B}}, \bar{\ell})$ is a probability space and the set B is of measure $\frac{1}{2}$ and independent with $\mathscr{B}([0,1])$ under $\bar{\ell}$.

Let $\bar{h}\colon [0,1] \to \mathscr{M}(\hat{A})$ be defined as follows:

$$\bar{h}(i) = \begin{cases} \delta_i, & \text{if } i \in B, \\ \delta_0, & \text{if } i \in [0,1] \setminus B. \end{cases}$$

It is clear that $\bar{h}$ is a TSRE of the game $\bar{\mathscr{G}}$ and induces the NED $\hat{\tau}$. Thus, the uniqueness characterization in Theorem 1 (iii) does not work for the game $\bar{\mathscr{G}}$. $\square$

3 Large Economies

Let $(I, \mathscr{I}, \lambda)$ be an atomless probability space representing the space of agents, $\mathbb{R}^n_+$ representing the commodity space, and $\mathscr{P}_{mo}$ representing the space of monotonic preference relations on $\mathbb{R}^n_+$. We endow the space $\mathscr{P}_{mo}$ with the metric of closed-convergence.[13] The characteristics of an agent consist of a preference as well as an endowment, and thus the space of characteristics of agents is $\mathscr{P}_{mo} \times \mathbb{R}^n_+$. A *large economy* is a measurable function $\mathscr{E}$ from the agent space $(I, \mathscr{I}, \lambda)$ to the space of characteristics $\mathscr{P}_{mo} \times \mathbb{R}^n_+$ such that for each $i \in I$, $\mathscr{E}(i) = (\succsim_i, \boldsymbol{e}(i))$, and the mean endowment $\int_I \boldsymbol{e}\, \mathrm{d}\lambda$ is finite and strictly positive.

3.1 Walrasian Allocation and Walrasian Equilibrium Distribution

Let $D(\boldsymbol{p}, \succsim, \boldsymbol{e})$ be the demand correspondence when the price vector, preference and endowment are $\boldsymbol{p}$, $\succsim$ and $\boldsymbol{e}$ respectively. That is to say, $D(\boldsymbol{p}, \succsim, \boldsymbol{e})$ is the set of all maximal elements for $\succsim$ in the budget set $\{\boldsymbol{x} \in \mathbb{R}^n_+ \colon \boldsymbol{p} \cdot \boldsymbol{x} \leq \boldsymbol{p} \cdot \boldsymbol{e}\}$. We are now ready to state two standard notions of equilibria that are commonly used in the literature of large economies. Similar to large games, we have both individual and distributional approaches to define an equilibrium in a large economy. The individualized approach, as in Aumann (1964), involves the agent space.

[13]The lemma on Hildenbrand (1974, p. 98) shows that the space $\mathscr{P}_{mo}$ with the metric of closed-convergence is a G_δ set in a compact metric space. By the classical Alexandroff's Lemma (see Aliprantis and Border (2006, p. 88)), $\mathscr{P}_{mo}$ is a Polish space.

Definition 4 (WE)**.** An allocation $\boldsymbol{f}$ of a large economy $\mathscr{E}\colon (I, \mathscr{I}, \lambda) \to \mathscr{P}_{mo} \times \mathbb{R}^n_+$ is an integrable function from $(I, \mathscr{I}, \lambda)$ to $\mathbb{R}^n_+$, and is said to be a *Walrasian allocation* under a nonzero price vector $\boldsymbol{p} \in \mathbb{R}^n_+$ if

1. for λ-almost all $i \in I$, $\boldsymbol{f}(i) \in D\big(\boldsymbol{p}, \succsim_i, \boldsymbol{e}(i)\big)$;
2. $\int_I \boldsymbol{f}(i)\,\mathrm{d}\lambda(i) = \int_I \boldsymbol{e}(i)\,\mathrm{d}\lambda(i)$.

The pair $(\boldsymbol{f}, \boldsymbol{p})$ above is also said to be a *Walrasian equilibrium* (WE) of the economy $\mathscr{E}$.

Given a large economy $\mathscr{E}\colon (I, \mathscr{I}, \lambda) \to \mathscr{P}_{mo} \times \mathbb{R}^n_+$, we can also consider its distributional form $\lambda\mathscr{E}^{-1}$. As in Hildenbrand (1974), we can also apply the distributional approach to model a macro notion of solution as below:

Definition 5 (WED)**.** A Borel probability measure τ on $(\mathscr{P}_{mo} \times \mathbb{R}^n_+) \times \mathbb{R}^n_+$ is said to be a *Walrasian equilibrium distribution* (WED) under a nonzero price vector $\boldsymbol{p} \in \mathbb{R}^n_+$ of a large economy $\lambda\mathscr{E}^{-1}$ in its distributional form if the marginal distribution of τ on the space of characteristics $\mathscr{P}_{mo} \times \mathbb{R}^n_+$ is $\lambda\mathscr{E}^{-1}$, and

1. $\tau(E_{\boldsymbol{p}}) = 1$, where $E_{\boldsymbol{p}} = \big\{(\succ, \boldsymbol{e}, \boldsymbol{x}) \in \mathscr{P}_{mo} \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \mid \boldsymbol{x} \in D(\boldsymbol{p}, \succ, \boldsymbol{e})\big\}$;
2. $\int_{\mathscr{P}_{mo} \times \mathbb{R}^n_+} \boldsymbol{e}\,\mathrm{d}\lambda\mathscr{E}^{-1} = \int_{\mathbb{R}^n_+} \boldsymbol{x}\,\mathrm{d}\nu$, where ν is the marginal distribution of τ on the space of commodity space (*i.e.*, the second $\mathbb{R}^n_+$ in $(\mathscr{P}_{mo} \times \mathbb{R}^n_+) \times \mathbb{R}^n_+$).

Note that given any large economy $\mathscr{E}$, if $(\boldsymbol{f}, \boldsymbol{p})$ is a WE of $\mathscr{E}$, then the joint (type-allocation) distribution of the economy $\mathscr{E}$ and the Walrasian allocation $\boldsymbol{f}$ is a WED under $\boldsymbol{p}$ of $\lambda\mathscr{E}^{-1}$. However, the converse may not hold: given a large economy $\mathscr{E}$ and a WED τ under $\boldsymbol{p}$ of $\lambda\mathscr{E}^{-1}$, there may not exist any Walrasian allocation $\boldsymbol{f}$ under $\boldsymbol{p}$ such that $\lambda(\mathscr{E}, \boldsymbol{f})^{-1} = \tau$. And thus, we also need to provide an alternative notion of equilibrium as the micro-foundation of the notion of equilibrium distribution of a large economy as well. In fact, we can directly exploit the same idea as in Section 2.2 to define a refinement of a randomized Walrasian allocation as below.

3.2 Type-Symmetric Randomized Equilibrium in a Large Economy

Definition 6 (TSRE)**.** A randomized allocation $\boldsymbol{h}$ of a large economy $\mathscr{E}$ is an integrable function from $(I, \mathscr{I}, \lambda)$ to $\mathscr{M}(\mathbb{R}^n_+)$, and is said to be a *randomized Walrasian allocation* under a nonzero price vector $\boldsymbol{p} \in \mathbb{R}^n_+$ if

1. for λ-almost all $i \in I$, $\boldsymbol{h}(i) \in \mathscr{M}(\mathbb{R}^n_+)$ with support in $D\big(\boldsymbol{p}, \succsim_i, \boldsymbol{e}(i)\big)$;

2. $\int_I \boldsymbol{h}\, \mathrm{d}\lambda$ is the Dirac measure at $\int_I \boldsymbol{e}\, \mathrm{d}\lambda$.

The pair $(\boldsymbol{h}, \boldsymbol{p})$ above is said to be a *type-symmetric randomized equilibrium* (TSRE) if $\boldsymbol{h}$ is a type-symmetric randomized Walrasian allocation (*i.e.* $\boldsymbol{h}(i) = \boldsymbol{h}(j)$ whenever $\mathscr{E}(i) = \mathscr{E}(j)$).

Every randomized allocation $\boldsymbol{h}$ of a large economy $\mathscr{E}$ naturally induces a type-allocation distribution $\int_I \delta_{\mathscr{G}(j)} \otimes \boldsymbol{h}(j)\, \mathrm{d}\lambda(j)$ as the corresponding macro aggregate. Analogous to the proof of Theorem 1 in the last section, we can show the following theorem:

Theorem 2. *In a large economy $\mathscr{E}\colon (I, \mathscr{I}, \lambda) \to \mathscr{P}_{mo} \times \mathbb{R}^n_+$, we have the following statements:*

(i) For any TSRE $(\boldsymbol{h}, \boldsymbol{p})$ of $\mathscr{E}$, the type-allocation distribution induced by $\boldsymbol{h}$ is a WED of $\lambda\mathscr{E}^{-1}$ under $\boldsymbol{p}$.

(ii) For any WED τ of $\lambda\mathscr{E}^{-1}$ under $\boldsymbol{p}$, there is a TSRE $(\boldsymbol{h}, \boldsymbol{p})$ of $\mathscr{E}$ such that $\boldsymbol{h}$ induces it.

(iii) If $(I, \mathscr{I}, \lambda)$ is the Lebesgue unit interval, then for any WED τ of $\lambda\mathscr{E}^{-1}$ under $\boldsymbol{p}$, there is a unique TSRE allocation $\boldsymbol{h}$ under $\boldsymbol{p}$ such that $\boldsymbol{h}$ induces it.

The large economy $\hat{\mathscr{E}}$ in the next example is taken from Kannai (1970). In such an economy, we construct a WED $\hat{\tau}$ of $\lambda(\hat{\mathscr{E}})^{-1}$ and show that $\hat{\tau}$ is corresponding to a unique (non-degenerated) TSRE of $\hat{\mathscr{E}}$ under the same price vector $\hat{\boldsymbol{p}}$. Similar to the discussion below Equation (1) for a large game, this uniqueness also implies that there does not exist any Walrasian allocation $\boldsymbol{f}$ under $\hat{\boldsymbol{p}}$ such that $\lambda(\hat{\mathscr{E}}, \boldsymbol{f})^{-1} = \hat{\tau}$.

Example 3. Consider the following economy $\hat{\mathscr{E}}$ with two goods. Let the space of agents be the Lebesgue unit interval $([0,1], \mathscr{B}([0,1]), \ell)$. Let all agents have the same preference where the corresponding indifference curves are parallel as shown in Figure 1 below. For each $k = 1, 2$, the line segment D_k is represented

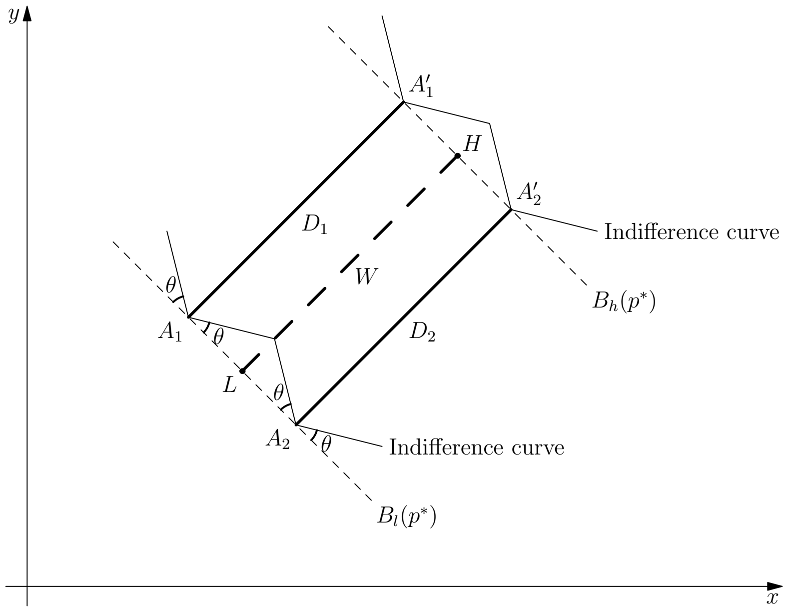

Figure 1: Indifference curves and budget lines

by $y = x + \frac{3}{2} - k$ for $x \in [\frac{2k+1}{4}, \frac{2k+5}{4}]$ with the end points $A_k = (\frac{2k+1}{4}, \frac{7-2k}{4})$ and $A_k' = (\frac{2k+5}{4}, \frac{11-2k}{4})$. The angle θ is chosen to be sufficiently small (*e.g.*, $\theta < 45$ degrees) so that the preference is monotonic. Let the endowment for every agent $i \in [0,1]$ be $\boldsymbol{e}(i) = (1+i, 1+i)$. In Figure 1, the set of all endowments is just the line segment W: $y = x$ for $x \in [1,2]$ with the end points $L = (1,1)$ and $H = (2,2)$. The economy $\hat{\mathscr{E}}$ specified by the preference and the endowment is clearly a well-defined large economy.

Let $\hat{\boldsymbol{p}} = (1,1)$. The parallel dashed lines $B_l(\hat{\boldsymbol{p}})$ and $B_h(\hat{\boldsymbol{p}})$ are perpendicular to the parallel line segments D_1 and D_2. Let $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ be two allocations of $\hat{\mathscr{E}}$ such that $\boldsymbol{f}_1(i) = (i + \frac{3}{4}, i + \frac{5}{4})$ and $\boldsymbol{f}_2(i) = (i + \frac{5}{4}, i + \frac{3}{4})$ for all $i \in [0,1]$. Both $(\boldsymbol{f}_1, \hat{\boldsymbol{p}})$ and $(\boldsymbol{f}_2, \hat{\boldsymbol{p}})$ are WE of $\hat{\mathscr{E}}$ since allocations $(i + \frac{3}{4}, i + \frac{5}{4})$ and $(i + \frac{5}{4}, i + \frac{3}{4})$ are the best choices for each agent $i \in [0,1]$. Let $\tau_1 = \ell(\hat{\mathscr{E}}, \boldsymbol{f}_1)^{-1}$, $\tau_2 = \ell(\hat{\mathscr{E}}, \boldsymbol{f}_2)^{-1}$ and $\hat{\tau} = \frac{1}{2}\tau_1 + \frac{1}{2}\tau_2$. One can check that $\hat{\tau}$ is a WED of $\ell(\hat{\mathscr{E}})^{-1}$ under $\hat{\boldsymbol{p}}$.

Let $\hat{\boldsymbol{h}}\colon [0,1] \to \mathscr{M}(\mathbb{R}^2_+)$ be a randomized allocation of $\hat{\mathscr{E}}$ such that for $i \in [0,1]$,

$$\hat{\boldsymbol{h}}(i) = \frac{1}{2}\delta_{(i+\frac{3}{4}, i+\frac{5}{4})} + \frac{1}{2}\delta_{(i+\frac{5}{4}, i+\frac{3}{4})}.$$

It is clear that $(\hat{\boldsymbol{h}}, \hat{\boldsymbol{p}})$ is a TSRE where $\hat{\boldsymbol{h}}$ induces the WED $\hat{\tau}$. By Theorem 2 (iii), $\hat{\boldsymbol{h}}$ is also the unique type-symmetric Walrasian allocation that induces $\hat{\tau}$ under $\hat{\boldsymbol{p}}$. □

We have concluded the last section with three observations for a large game. Here we simply point out that one easily does the same for a large economy: one can define a type-symmetric (pure) Walrasian allocation, show the symmetrization of any randomized Walrasian allocation, and also demonstrate the failure of the uniqueness in Theorem 2 (iii) if a non-Lebesgue agent space is used.

4 Discussion

We have demonstrated that in a large game/economy, an equilibrium distribution corresponds to a type-symmetric randomized equilibrium, and such a correspondence is unique when the agent space is the Lebesgue unit interval. On the one hand, the uniqueness characterization explains that the robustness of the existence of an equilibrium distribution simply lies in that it is essentially a notion of randomized equilibrium. On the other hand, the type-symmetric characterization might be useful in applications to explain the similar behaviors of agents with the similar type of characteristics in the macro notion of equilibrium distribution.

One more notion of equilibrium with randomization is the so-called mixed strategy equilibrium. Such an notion requires the randomization to be independent across agents. In the setting of a continuum of agents, this leads to a process with a continuum of independent random variables. In order to resolve the measurability issue of such processes and to guarantee the existence of such processes with a variety of distributions, we adopt the framework of a rich Fubini extension as in Sun (2006). Recall that a *Fubini extension* $(I \times \Omega, \mathscr{I} \boxtimes \mathscr{F}, \lambda \boxtimes P)$ is a probability space that extends the usual product space of the agent space $(I, \mathscr{I}, \lambda)$ and a sample space $(\Omega, \mathscr{F}, P)$, and retains the Fubini property. A Fubini extension is *rich* if it has a measurable process with independent random variables of uniform distribution on $[0,1]$.[14]

For simplicity, we shall consider mixed strategy equilibrium only for large games.

Definition 7 (MSE). A mixed strategy profile of a large game $\mathscr{G}$ is a $\mathscr{I} \boxtimes \mathscr{F}$-measurable function $g\colon I \times \Omega \to A$ where g is essentially pairwise independent,[15] and is said to be a *mixed strategy equilibrium* (MSE) of $\mathscr{G}$ if for λ-almost all i,

$$\int_\Omega u_i\left(g_i(\omega), \lambda(\alpha, g_\omega)^{-1}\right) \mathrm{d}P \geq \int_\Omega u_i\left(\eta(\omega), \lambda(\alpha, g_\omega)^{-1}\right) \mathrm{d}P \qquad (2)$$

for all random variables $\eta\colon \Omega \to A$. Furthermore, we say that a mixed strategy profile g of $\mathscr{G}\colon I \to T \times \mathscr{U}_{(A,T)}$ has the *ex-post Nash property* if for P-almost all $\omega \in \Omega$, g_ω is an NE for the same game.

Note that NE, RSE and MSE are equilibrium notions under the individualized approach. The next result[16] addresses the relationship of these three equilibrium notions.

Proposition 1. *In a large game $\mathscr{G}\colon I \to T \times \mathscr{U}_{(A,T)}$, we have the following statements:*

(i) For every MSE g, if h is such that $h(i) = P(g_i)^{-1}$ for all $i \in I$, then h is an RSE. For every RSE h, there exists an MSE g such that $h(i) = P(g_i)^{-1}$ for all $i \in I$;

[14] A rich Fubini extension is also called a rich product probability space in Sun (2006).

[15] This is to say, for λ-almost all $i \in I$, $g(i,\cdot)$ and $g(j,\cdot)$ are independent for λ-almost all $j \in I$; see Definition 2.7 of Sun (2006). Given that $(I, \mathscr{I}, \lambda)$ is an atomless (complete) probability space, a single point (and thus up to countably many points) has measure zero, and thus essential pairwise independence is more general than the usual pairwise and mutual independence.

[16] The result appeared in an earlier unpublished version of Khan *et al.* (2015). We state it here for the sake of completeness. The key idea for proving the result is to apply the exact law of large numbers in Corollary 2.9 of Sun (2006) to claim that for P-almost all $\omega \in \Omega$, $\lambda(\alpha, g_\omega)^{-1} = \lambda \boxtimes P(\alpha, g)^{-1}$. Thus Equation (2) is equivalent to the fact that for P-almost all $\omega \in \Omega$, $g_i(\omega) \in \arg\max_{a \in A} u_i\left(a, \lambda \boxtimes P(\alpha, g)^{-1}\right) = \arg\max_{a \in A} u_i\left(a, \lambda(\alpha, g_\omega)^{-1}\right)$, which is also equivalent to the ex-post Nash property. To represent a given RSE h by an MSE g, one can directly apply Proposition 5.3 of Sun (2006) on the universality property of a rich Fubini extension, which says that one can construct processes on a rich Fubini extension with essentially pairwise independent random variables that take any given variety of distributions. The exact law of large numbers and the optimality condition of an RSE then imply Equation (2).

(ii) A mixed strategy profile is an MSE if and only if it has the ex-post Nash property.

Together with Theorem 1 and Corollary 2, we are now ready to delineate the relations of *all* the equilibrium notions in large games in Figure 2.[17]

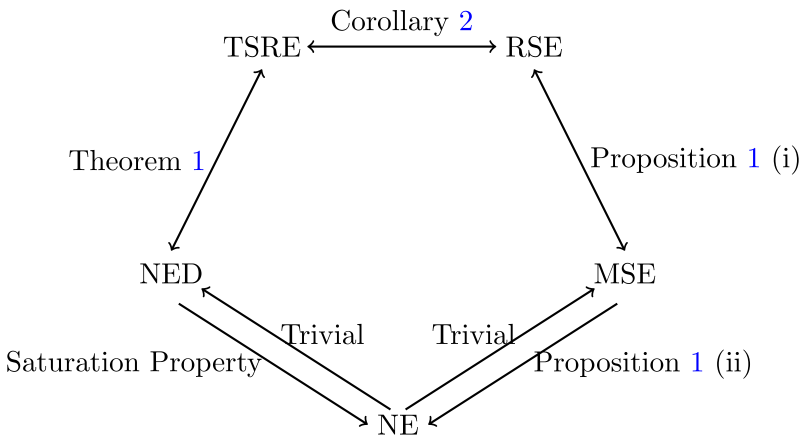

Figure 2: Classification of all equilibrium notions in large games

Sun and Zhang (2009) constructed a rich Fubini extension whose agent space extends the usual Lebesgue unit interval. Based on such a framework, Proposition 1 (i) guarantees that a mixed strategy equilibrium $\hat{g}$ for the game $\hat{\mathscr{G}}$ in Example 1 can be constructed from the TSRE $\hat{h}$ in Equation (1). It then follows from Proposition 1 (ii) that the realized pure strategy profile $\hat{g}_\omega$ is a pure strategy Nash equilibrium of the game $\hat{\mathscr{G}}$ for almost all sample realization $\omega \in \Omega$, where exactly half of the agents play action 0 and the other half play action according to their names.

5 Proof of Results

The proof of Theorem 1 uses the following properties of analytic sets.[18]

[17]It follows from Theorem 4.2 in Sun (2006) that the agent space in the framework of a rich Fubini extension will automatically have the saturation property. For the details of the saturation property and its equivalent versions, see Keisler and Sun (2009).

[18]For the definition of analytic sets and Lemma 1 (i), see Aliprantis and Border (2006, p. 446) and for Lemma 1 (ii), see Aliprantis and Border (2006, Theorem 12.28).

Lemma 1. *(i) The continuous image in a Polish space of an analytic set is analytic. (ii) A function between Polish spaces is Borel measurable if and only if its graph is analytic.*

The following lemma is also used in the proof of Theorem 1.

Lemma 2. *If h be a randomized strategy profile of a large game $\mathscr{G}\colon I \to T \times \mathscr{U}_{(A,T)}$, then h is an RSE of $\mathscr{G}$ if and only if the type-action distribution induced by h is an NED of $\lambda\mathscr{G}^{-1}$.*

Proof. Let $\tau = \int_I \delta_{\mathscr{G}(j)} \otimes h(j)\, \mathrm{d}\lambda(j)$, the type-action distribution induced by h. It is clear that

$$\int_I \delta_{\alpha(j)} \otimes h(j)\, \mathrm{d}\lambda(j) = \tau_{T\times A} \text{ and } \lambda\mathscr{G}^{-1} = \tau_{T\times\mathscr{U}_{(A,T)}}.$$

By definition, if h is an RSE of $\mathscr{G}$, then for λ-almost all $i \in I$, we have

$$\int_A u_i\left(a, \int_I \delta_{\alpha(j)} \otimes h(j)\, \mathrm{d}\lambda(j)\right) h(i; \mathrm{d}a) \geq \int_A u_i\left(a, \int_I \delta_{\alpha(j)} \otimes h(j)\, \mathrm{d}\lambda(j)\right) \eta(\mathrm{d}a)$$

for all $\eta \in \mathscr{M}(A)$. This is equivalent to that for λ-almost all $i \in I$, we have

$$\int_A u_i\left(a, \tau_{T\times A}\right) h(i; \mathrm{d}a) \geq \int_A u_i\left(a, \tau_{T\times A}\right) \eta(\mathrm{d}a)$$

for all $\eta \in \mathscr{M}(A)$. The above inequality holds if and only if for τ_A-almost all $a \in A$,

$$u_i(a, \tau_{T\times A}) = \max_{a\in A} u_i(a, \tau_{T\times A}),$$

i.e., $\tau\big(Br(\tau)\big) = 1$. The proof is now complete. □

Proof of Theorem 1. It is implied by Lemma 2 directly that (i) each TSRE of a large game induces an NED of the distributional form of the large game since any TSRE is automatically an RSE of the same game. We next show that (ii) every NED of $\lambda\mathscr{G}^{-1}$ can be "lifted" to some TSRE in $\mathscr{G}$. Let τ be an NED of $\lambda\mathscr{G}^{-1}$ and $k\colon T \times \mathscr{U}_{(A,T)} \to \mathscr{M}(A)$ the disintegration of τ with respect to $\lambda\mathscr{G}^{-1}$.[19] Let

[19] The disintegration of a probability measure τ on two polish spaces with respect to a given

$h = k \circ \mathscr{G} \colon I \to \mathscr{M}(A)$. By the change of variable theorem and the property of the disintegration, we have that

$$\int_I \delta_{\mathscr{G}(j)} \otimes h(j) \, \mathrm{d}\lambda(j) = \int_{T \times \mathscr{U}_{(A,T)}} \delta_{(t,v)} \otimes k(t,v) \, \mathrm{d}\lambda\mathscr{G}^{-1} = \tau.$$

By Lemma 2 and the construction of h, it is clear that h is a TSRE of $\mathscr{G}$.

We only need to prove statement (iii). Let $\mathscr{G}^l$ be a large game with the Lebesgue name space $([0,1], \mathscr{B}[0,1], \ell)$. Let τ be an NED of $\ell(\mathscr{G}^l)^{-1}$ and let $k^l \colon T \times \mathscr{U}_{(A,T)} \to \mathscr{M}(A)$ be the disintegration of τ with respect to $\ell(\mathscr{G}^l)^{-1}$. From the proof of statement (ii) above, we know that $h^l := k^l \circ \mathscr{G}^l$ is a TSRE of $\mathscr{G}^l$. We next show that it is also the unique TSRE.

Towards this end, let h' be another TSRE of $\mathscr{G}^l$ that induces τ. It is enough to show that for ℓ-almost all $i \in I$, $h'(i) = h^l(i)$. Consider the mapping

$$(\mathscr{G}^l, h') \colon [0,1] \to T \times \mathscr{U}_{(A,T)} \times \mathscr{M}(A).$$

Since both $\mathscr{G}^l$ and h' are Borel measurable, so is $(\mathscr{G}^l, h')$. Thus, Lemma 1 (ii) implies that $\mathrm{Graph}(\mathscr{G}^l, h')$ is analytic.

Next, fix $a_0 \in A$ and let $H \colon T \times \mathscr{U}_{(A,T)} \to \mathscr{M}(A)$ be a mapping such that

$$H(t,v) = \begin{cases} h'(i), & \text{if } (t,v) = \mathscr{G}^l(i); \\ \delta_{a_0}, & \text{otherwise.} \end{cases}$$

Let $H|_{\mathrm{range}(\mathscr{G}^l)} \colon \mathrm{range}(\mathscr{G}^l) \to \mathscr{M}(A)$ be the restriction of H on the range of $\mathscr{G}^l$. From the construction, the graph of $H|_{\mathrm{range}(\mathscr{G}^l)}$,

$$\mathrm{Graph}\left(H|_{\mathrm{range}(\mathscr{G}^l)}\right) = \mathrm{Proj}_{T \times \mathscr{U}_{(A,T)} \times \mathscr{M}(A)} \mathrm{Graph}(\mathscr{G}^l, h'),$$

the projection of the graph of $(\mathscr{G}^l, h')$ on $T \times \mathscr{U}_{(A,T)} \times \mathscr{M}(A)$. Since the projection mapping is continuous and $\mathrm{Graph}(\mathscr{G}^l, h')$ is analytic, we can now appeal

marginal $\lambda\mathscr{G}^{-1}$ on one polish space always exist and is unique ($\lambda\mathscr{G}^{-1}$-almost everywhere); see Crauel (2002, Proposition 3.6) for example. In fact, such an observation of existence has been pointed out in Khan (1989) even for a large game with a non-metrizable action set; see Khan (1989) for more details. Note that a disintegration is also known as a regular conditional distribution in probability theory; see Dudley (2002, p. 342–345) for related discussion.

to Lemma 1 (i) to assert that Graph $\left(H|_{\text{range}(\mathscr{G}^l)}\right)$ is also analytic. Thus, by Lemma 1 (ii) again, we can say that $H|_{\text{range}(\mathscr{G}^l)}$ is Borel measurable. As the Borel sets of range($\mathscr{G}^l$) are the restrictions of the Borel sets of $T \times \mathscr{U}_{(A,T)}$ to range($\mathscr{G}^l$) (see Aliprantis and Border (2006, Corollary 4.20) for example), there exists a Borel measurable mapping $H' \colon T \times \mathscr{U}_{(A,T)} \to \mathscr{M}(A)$ such that $H|_{\text{range}(\mathscr{G}^l)}$ is also the restriction of H' on the range of $\mathscr{G}^l$; see, for example, Dudley (2002, Theorem 4.2.5) and the discussion below it. Furthermore, from the construction, it is clear that $h' = H \circ \mathscr{G}^l = H|_{\text{range}(\mathscr{G}^l)} \circ \mathscr{G}^l = H' \circ \mathscr{G}^l$. Since h' is a TSRE that induces τ, we now have

$$\tau = \int_I \delta_{\mathscr{G}^l(j)} \otimes h'(j) \, \mathrm{d}\ell(j) = \int_{T \times \mathscr{U}_{(A,T)}} \delta_{(t,v)} \otimes H'(t,v) \, \mathrm{d}\ell(\mathscr{G}^l)^{-1}.$$

This is to say, H' is also a disintegration of τ with respect to $\ell(\mathscr{G}^l)^{-1}$. By the ($\ell(\mathscr{G}^l)^{-1}$-almost everywhere) uniqueness of the disintegration, H' and k^l are $\ell(\mathscr{G}^l)^{-1}$-almost everywhere the same. So, $h'(= H' \circ \mathscr{G}^l)$ and $h^l(= k^l \circ \mathscr{G}^l)$ must also be ℓ-almost everywhere the same. This completes the proof. □

Proof of Corollary 1. Let $\mathscr{G}$ be a large game with the Lebesgue unit interval as its name space, and τ^s be a symmetric NED of it. Then there exists a measurable function $s \colon T \times \mathscr{U}_{(A,T)} \to A$ such that τ^s(graph of s) $= 1$. We can construct a measurable function $f \colon [0,1] \to A$ so that $f = s \circ \mathscr{G}$. It is easy to check that f is a symmetric NE (hence a TSRE) of $\mathscr{G}$ such that $\tau^s = \ell(\mathscr{G}, f)^{-1}$. As τ^s is an NED, Theorem 1 (iii) implies that there is a unique TSRE that induces τ^s. Hence f is the unique symmetric NE that induces τ^s. □

Proof of Corollary 2. Lemma 2 implies that the type-action distribution induced by an RSE is an NED. Then by Theorem 1 (ii), we can find an TSRE that induces the same NED. □

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