

# OPTIMAL ADAPTIVE TESTING: INFORMATIVENESS AND INCENTIVES

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**ABSTRACT.** We introduce a learning framework in which a principal seeks to determine the ability of a strategic agent. The principal assigns a test consisting of a finite sequence of questions or tasks. The test is adaptive: each question that is assigned can depend on the agent's past performance. The probability of success on a question is jointly determined by the agent's privately known ability and an unobserved action that he chooses to maximize the probability of passing the test. We identify a simple monotonicity condition under which the principal always employs the most (statistically) informative question in the optimal adaptive test. Conversely, whenever the condition is violated, we show that there are cases in which the principal strictly prefers to use less informative questions.

## 1. INTRODUCTION

A classic problem in statistics is the “sequential choice of experiments” (see, for instance, Chapter 14 of [DeGroot, 2005](#)). In this problem, a researcher who wants to learn about an unknown parameter has at her disposal a collection of “experiments,” each of which is associated with a different distribution of signals about the parameter. In one formulation, the principal can run a fixed number of experiments, and chooses each one only after observing the outcomes of the preceding ones. A key result in this literature pertains to the case in which one experiment is more informative, in the sense of [Blackwell \(1953\)](#), than all others available to the researcher. In this case, the optimal strategy is independent of the history and simply involves repeatedly drawing from the most informative experiment. We refer to this as Blackwell's result (see Corollary 4.4 in [DeGroot, 1962](#)).

A similar form of learning can be found in numerous strategic settings in which a principal seeks to determine the unknown type of an agent. Examples include interviewing to determine whether a candidate is suitable for a job opening, standardized testing with the aim of uncovering a student's ability, and assigning tasks to an employee to assess whether he deserves a promotion. In these scenarios, information is obtained by observing the agent's performance over a sequence of questions or tasks, and the principal's choice of which task to assign may depend on the agent's past performance. A critical difference between these examples and the statistical environment is that the agent can, through unobservable actions, affect the information the principal receives. For instance, the incentive for an employee seeking promotion to exert effort (or strategically shirk) on

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any given task depends on how performance affects subsequent task assignment, and ultimately his probability of promotion (see, for example, DeVaro and Gürtler, 2015). How does strategic behavior by the agent affect the optimal assignment of tasks? Does Blackwell’s result carry over?

In this paper, we introduce a model of “adaptive testing.”<sup>1</sup> For ease of exposition, we describe our model using the language of a standardized test or interview. The principal has a fixed number of time periods over which to evaluate the agent and a finite collection of questions. The agent’s probability of success on a particular question depends on his ability (or type) and his choice of action, which is not directly observable to the principal. For instance, the agent need not choose his most informed response and may even choose to answer incorrectly if doing so leads to questions that are likely to make him look better. Higher actions correspond to a greater probability of success and are more informative (in a sense that we make precise).

The principal first commits to a test. The test begins by assigning the agent a question. Upon seeing the assigned question, the agent chooses his action. Depending on the realized success or failure on the first question, the test assigns another question to the agent in the next period, and the agent again chooses an action. The test continues in this way, with the assigned question in each period possibly depending on the entire history of previous successes and failures. At the end of a fixed number of periods, the test issues a verdict indicating whether the agent passes or fails given the history of questions and the agent’s performance. The principal’s goal is to pass the agent if and only if his type belongs to a particular set (which we refer to as the set of “good types”). The agent seeks to maximize the probability with which he passes the test.

Our main goal is to understand the effect of the agent’s strategic action choice on learning. To abstract away from cost-saving incentives, we assume that all actions have the same cost for the agent. A natural benchmark is the optimal test under the assumption that the agent always chooses the most informative action. Given this strategy, designing the optimal test is essentially a special case of the sequential choice of experiments problem, which can in principle be solved by backward induction (although qualitative properties of the solution are hard to obtain except in the simplest of cases). We refer to this benchmark solution as the *optimal non-strategic test* (ONST).

In our strategic environment, Blackwell’s result does not hold in general (see Example 2). Our main result (Theorem 2) shows that it does hold if a property we refer to as “group monotonicity” is satisfied, namely, if there does not exist a question at which some bad type has higher ability than some good type. If group monotonicity holds, then it is optimal for the principal always to assign the most informative question and for the agent always to choose the most informative action (in particular, the optimal test coincides with the ONST). We provide a partial converse (Theorem 3) to this result, which indicates that whenever a question violates group monotonicity, there is an environment that includes that question in which always assigning the most informative question is not optimal for the principal.

In a static setting, the intuition behind our main result is straightforward. Since all types can choose not to succeed on the assigned question, the principal can learn about the agent’s type only if success is rewarded with a higher probability of passing the test. In that case, all types choose the most informative action since doing so maximizes the probability of success. Group

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<sup>1</sup>We borrow this term from the literature on standardized testing.

monotonicity then ensures that good types have a higher probability of passing than do bad types. Since strategic behavior plays no role, assigning the most informative question is optimal for the principal.

The dynamic setting is complicated by the fact that the agent must consider how his performance on each question affects the subsequent questions that will be assigned; he may have an incentive to perform poorly on a question if doing so makes the remainder of the test easier, and thereby increases the ultimate probability of passing. For instance, despite it reflecting badly on him, an interviewee may deliberately feign ignorance on a topic fearing that the line of questioning that would otherwise follow would be more damaging. In our model, even with group monotonicity, there are cases in which some types choose not to succeed on certain questions in the optimal test (see Example 4). If, however, there is one question  $q$  that is more informative than the others, then this turns out not to be an issue. Given any test that, at some histories, assigns questions other than  $q$ , we show that one can recursively replace each of those questions with  $q$  together with a randomized continuation test in a way that does not make the principal worse off. While this procedure resembles Blackwell garbling in the statistical problem, in our case one must be careful to consider how each such change affects the agent’s incentives; group monotonicity ensures that any change in the agent’s strategy resulting from these modifications to the test can only improve the principal’s payoff.

In Section 6, we consider optimal testing when questions are not comparable in terms of informativeness. We show that, under group monotonicity, the ONST is optimal when the agent has only two types (Theorem 4). However, when there are more than two types, this result does not hold: Example 4 shows that even if the most informative action is always optimal for the agent in the ONST, the principal may be able to do better by inducing some types to muddle. Appendix B describes other examples demonstrating a wealth of possibilities (even with group monotonicity). Section 7 shows that our main result continues to hold if the principal can offer the agent a menu of tests (Theorem 5), and if she lacks the power to commit to a test.

Our model and results are related to several distinct strands of the literature. The literature on career concerns (beginning with [Holmstrom, 1999](#)) is similar in spirit to our model in that the market is trying to learn about an agent’s unknown ability by observing his output. Like our model, standard “signal jamming” models feature moral hazard; however, unlike our model, there is no asymmetric information between the agent and the market regarding the agent’s ability, and monetary incentives are provided using contracts. In addition, these models typically do not involve task assignment by a principal. Perhaps the closest related work in this literature is [Dewatripont, Jewitt, and Tirole \(1999\)](#). They provide conditions under which the market may prefer a less informative monitoring technology (relating the agent’s action to performance variables) to a more informative one, and vice versa.

More broadly, while more information is always beneficial in a non-strategic single agent setting, it can sometimes be detrimental in multi-agent environments. Examples include oligopoly ([Mirman, Samuelson, and Schlee, 1994](#)), exchange economies with risk ([Schlee, 2001](#)), and elections ([Ashworth, de Mesquita, and Friedenber, 2015](#)). While more information is never harmful to the principal in our setting (since she could always choose to ignore it), our focus is on whether

less informative questions can be used to alter the agent's strategy in a way that generates more information.

Our model provides a starting point for studying how managers assign tasks when they benefit from learning about workers' abilities (for instance, to determine their suitability for important projects). Dynamic contracting is often modeled with pure moral hazard, where the principal chooses bonus payments in order to generate incentives to exert costly effort (see, for instance, [Rogerson, 1985](#); [Holmstrom and Milgrom, 1987](#)). However, there is evidence that suggests that promotions are often used instead of bonuses as a means of providing incentives and that wages frequently remain constant between promotions ([Baker, Jensen, and Murphy, 1988](#)). As our model does not contain transfers, it can be thought of as corresponding to this constant wage case. In future work, we plan to build on the results in this paper by studying a task assignment problem in which the principal's payoff depends not only on whether she makes the correct promotion decision, but also on the agent's output on each task.

Finally, our work is related to the literature on multi-armed bandit problems (an overview can be found in [Bergemann and Välimäki, 2006](#)), in which a principal chooses in each period which arm to pull—just as, in our model, she chooses which question to assign—and learns from the resulting outcome. The main trade-off is between maximizing short-term payoffs and the long-term gains from learning. Our model can be thought of as a first step toward understanding bandit problems in which a strategic agent can manipulate the information received by the decision-maker.

## 2. MODEL

A principal (she) is trying to learn the private type of an agent (he) by observing his performance on a sequence of questions over  $T$  periods. At each period  $t \in \{1, \dots, T\}$ , she assigns the agent a question  $q_t$  from a finite set  $Q$  of available questions. We interpret two identical questions  $q_t = q_{t'}$  assigned at time periods  $t \neq t'$  as two different questions of the same difficulty; the agent being able to succeed on one of the questions does not imply that he is sure to be able to succeed on the other. Faced with a question  $q_t \in Q$ , the agent chooses an action  $a_t \in [0, 1]$ . All actions have the same cost, which we normalize to zero. We refer to  $a_t = 1$  as the most informative action, and any  $a_t < 1$  as 'muddling'.<sup>2</sup> Depending on the agent's ability and action choice, he may either succeed ( $s$ ) or fail ( $f$ ) on a given question. This outcome is observed by both the principal and the agent.

*Type Space:* The agent's ability (which stays constant over time) is captured by his privately known type  $\theta_i : Q \rightarrow (0, 1)$ , which belongs to a finite set  $\Theta = \{\theta_1, \dots, \theta_I\}$ .<sup>3</sup> In period  $t$ , the probability of a success on a question  $q_t$  when the agent chooses action  $a_t$  is  $a_t \theta_i(q_t)$ .

The type determines the highest probability of success on each question, obtained when the agent chooses the most informative action. The zero action implies sure failure.<sup>4</sup> Note that, as is

<sup>2</sup>This terminology is consistent with our formal definition of informativeness (see Section 4): fixing the same action for all types, increasing the action always makes a question strictly more Blackwell informative.

<sup>3</sup>The restriction that  $\theta_i(q) \neq 0$  or 1 simplifies some arguments but is not necessary for any of our results.

<sup>4</sup>The agent's ability to fail for sure is not essential as none of our results are affected by making the lowest possible action strictly positive.

common in dynamic moral hazard models, the agent's probability of success on a given question is independent of events that occur before  $t$  (such as him having faced the same question before).

Before period 1, the principal announces and *commits to* an (adaptive) test. The test determines which question is assigned in each period depending on the agent's performance so far, and the final verdict given the history at the end of period  $T$ .

*Histories:* At the beginning of period  $t$ ,  $h_t$  denotes a nonterminal public history (or simply a history) up to that point. Such a history lists the questions faced by the agent and the corresponding successes or failures in periods  $1, \dots, t-1$ . The set of (nonterminal) histories is denoted by  $\mathcal{H} = \bigcup_{t=1, \dots, T} (Q \times \{s, f\})^{t-1}$ . We write  $\mathcal{H}_{T+1} = (Q \times \{s, f\})^T$  for the set of terminal histories.

Similarly,  $h_t^A$  denotes a history for the agent describing his information before choosing an action in period  $t$ . In addition to the information contained in the history  $h_t$ ,  $h_t^A$  also contains the question he currently faces.<sup>5</sup> Thus the set of all histories for the agent is given by  $\mathcal{H}^A = \bigcup_{t=1, \dots, T} (Q \times \{s, f\})^{t-1} \times Q$ .

For example,  $h_3 = \{(q_1, s), (q_2, f)\}$  is the history at the beginning of period 3 in which the agent succeeded on question  $q_1$  in the first period and failed on question  $q_2$  in the second. The corresponding history  $h_3^A = \{(q_1, s), (q_2, f), q_3\}$  also includes the question in period 3.

*Deterministic Test:* A deterministic test  $(\mathcal{T}, \mathcal{V})$  consists of functions  $\mathcal{T} : \mathcal{H} \rightarrow Q$  and  $\mathcal{V} : \mathcal{H}_{T+1} \rightarrow \{0, 1\}$ . Given a history  $h_t$  at the beginning of period  $t$ , the question  $q_t$  assigned to the agent is  $\mathcal{T}(h_t)$ . The probability that the agent passes the test given any terminal history  $h_{T+1}$  is  $\mathcal{V}(h_{T+1})$ .

*Test:* A (random) test  $\rho$  is a distribution over deterministic tests.

As mentioned above, the principal commits to the test in advance. Before period 1, a deterministic test is drawn according to  $\rho$  and assigned to the agent. The agent knows  $\rho$  but does not observe which deterministic test is realized. He can, however, update as the test proceeds based on the sequence of questions that have been assigned so far.

Note that even if the agent is facing a deterministic test, since the questions he will face can depend on his stochastic performance so far in the test, he may not be able to perfectly predict which question he will face in subsequent periods.

*Strategies:* A strategy for type  $\theta_i$  is given by a mapping  $\sigma_i : \mathcal{H}^A \rightarrow [0, 1]$  from histories for the agent to action choices; given a history  $h_t^A$  in period  $t$ , the action in period  $t$  is  $a_t = \sigma_i(h_t^A)$ . We denote the profile of strategies by  $\sigma = (\sigma_1, \dots, \sigma_I)$ .

*Agent's Payoff:* Regardless of the agent's type, his goal is to pass the test. Accordingly, faced with a deterministic test  $(\mathcal{T}, \mathcal{V})$ , the payoff of the agent at any terminal history  $h_{T+1}$  is the probability with which he passes, which is given by the verdict  $\mathcal{V}(h_{T+1})$ . Given a test  $\rho$ , we denote by  $u_i(h; \rho, \sigma_i)$  the expected payoff of type  $\theta_i$  when using strategy  $\sigma_i$  conditional on reaching history  $h \in \mathcal{H}$ .

<sup>5</sup>By not including the agent's actions in  $h_t^A$  we are implicitly excluding the possibility that the agent conditions his actions on his own past action choices. Allowing for this would only complicate the notation and make no difference for our results.

*Principal's Beliefs:* The principal's prior belief about the agent's type is given by  $(\pi_1, \dots, \pi_I)$ , with  $\pi_i$  being the probability the principal assigns to type  $\theta_i$  (thus  $\pi_i \geq 0$  and  $\sum_{i=1}^I \pi_i = 1$ ). Similarly, for any  $h \in \mathcal{H} \cup \mathcal{H}_{T+1}$ ,  $\pi(h) = (\pi_1(h), \dots, \pi_I(h))$  denotes the principal's belief at history  $h$ . We assume that each of these beliefs is consistent with Bayes' rule given the agent's strategy; in particular, at the history  $h_1 = \emptyset$ ,  $(\pi_1(h_1), \dots, \pi_I(h_1)) = (\pi_1, \dots, \pi_I)$ .

*Principal's Payoff:* The principal partitions the set of types  $\Theta$  into disjoint subsets of good types  $\{\theta_1, \dots, \theta_{i^*}\}$  and bad types  $\{\theta_{i^*+1}, \dots, \theta_I\}$ , where  $i^* \in \{1, \dots, I-1\}$ . At any terminal history  $h_{T+1}$ , she gets a payoff of 1 if the agent passes and has a good type,  $-1$  if the agent passes and has a bad type, and 0 if the agent fails. Therefore, her expected payoff from a deterministic test  $(\mathcal{T}, \mathcal{V})$  is given by  $\mathbb{E}_{h_{T+1}} \left[ \sum_{i=1}^{i^*} \pi_i(h_{T+1}) \mathcal{V}(h_{T+1}) - \sum_{i=i^*+1}^I \pi_i(h_{T+1}) \mathcal{V}(h_{T+1}) \right]$ , where the distribution over terminal histories depends on both the test and the agent's strategy.

One might expect the principal to receive different payoffs depending on the exact type of the agent, not only whether the type is good or bad. All of our results extend to the more general model in which the receives a payoff of  $\gamma_i$  from passing type  $\theta_i$ , and a payoff normalized to 0 from failing any type. Assuming without loss generality that the types are ordered so that  $\gamma_i \geq \gamma_{i+1}$  for each  $i$ , the cutoff  $i^*$  dividing good and bad types then satisfies  $\gamma_i \geq 0$  if  $i \leq i^*$  and  $\gamma_i \leq 0$  if  $i > i^*$ . The principal's problem with these more general payoffs and prior  $\pi$  is equivalent to the original problem with prior  $\pi'$  given by  $\pi'_i = \gamma_i \pi_i / \sum_{j=1}^I \gamma_j \pi_j$ . Since our results are independent of the prior, this transformation allows us to reduce the problem to the simple binary payoffs for passing the agent described above.

*Optimal Test:* The principal chooses and commits to a test that maximizes her payoff subject to the agent choosing his strategy optimally. Facing a test  $\rho$ , we write  $\sigma_i^*$  to denote an optimal strategy for type  $\theta_i$ , that is, a strategy satisfying

$$\sigma_i^* \in \operatorname{argmax}_{\sigma_i} u_i(h_1; \rho, \sigma_i).$$

Note that this implicitly requires the agent to play optimally at all histories occurring with positive probability given the strategy.

Given her prior, the principal solves

$$\max_{\rho} \mathbb{E}_{h_{T+1}} \left[ \mathcal{V}(h_{T+1}) \left( \sum_{i=1}^{i^*} \pi_i(h_{T+1}) - \sum_{i=i^*+1}^I \pi_i(h_{T+1}) \right) \right],$$

where the expectation is taken over terminal histories (the distribution of which depend on the test,  $\rho$ , and the strategy  $\sigma^* = (\sigma_1^*, \dots, \sigma_I^*)$ ), and the beliefs are updated from the prior using Bayes' rule (wherever possible). To keep the notation simple, we do not explicitly condition the principal's beliefs  $\pi$  on the agent's strategy.

An equivalent and convenient way to represent the principal's problem is to state it in terms of the agent's payoffs as

$$\max_{\rho} \left[ \sum_{i=1}^{i^*} \pi_i v_i(\rho) - \sum_{i=i^*+1}^I \pi_i v_i(\rho) \right], \quad (1)$$

where  $v_i(\rho) := u_i(h_1; \rho, \sigma_i^*)$  is the expected payoff type  $\theta_i$  receives from choosing an optimal strategy in the test  $\rho$ . Note in particular that whenever some type of the agent has multiple optimal strategies, the principal is indifferent about which one he employs.

### 3. BENCHMARK: THE OPTIMAL NON-STRATEGIC TEST

Our main goal is to understand how muddling by the agent affects the principal's ability to learn his type. Thus a natural benchmark is the statistical problem in which the agent is assumed to choose the most informative action at every history.

Formally, in this benchmark, the principal solves the problem

$$\max_{\mathcal{T}, \mathcal{V}} \mathbb{E}_{h_{T+1}} \left[ \mathcal{V}(h_{T+1}) \left( \sum_{i=1}^{i^*} \pi_i(h_{T+1}) - \sum_{i=i^*+1}^I \pi_i(h_{T+1}) \right) \right],$$

where the distribution over terminal histories is determined by the test  $(\mathcal{T}, \mathcal{V})$  together with the most informative strategy

$$\sigma_i^N(h^A) = 1 \text{ for all } h^A \in \mathcal{H}^A$$

for every  $i$ . We refer to the solution  $(\mathcal{T}^N, \mathcal{V}^N)$  to this problem as the optimal non-strategic test (ONST). Notice that we have restricted attention to deterministic tests; we argue below that this is without loss.

In principle, it is straightforward to solve for the ONST by backward induction. The principal can first choose the optimal question at all period  $T$  histories and beliefs along with the optimal verdicts corresponding to the resulting terminal histories. Formally, consider any history  $h_T$  at the beginning of period  $T$  with belief  $\pi(h_T)$ . The principal chooses the question  $\mathcal{T}(h_T)$  and verdicts  $\mathcal{V}(\{h_T, (\mathcal{T}(h_T), s)\})$  and  $\mathcal{V}(\{h_T, (\mathcal{T}(h_T), f)\})$  so that

$$\begin{aligned} & (\mathcal{T}(h_T), \mathcal{V}(\{h_T, (\mathcal{T}(h_T), s)\}), \mathcal{V}(\{h_T, (\mathcal{T}(h_T), f)\})) \\ & \in \underset{(q_T, v^s, v^f)}{\operatorname{argmax}} \left\{ \begin{aligned} & v^s \left( \sum_{i=1}^{i^*} \theta_i(q_T) \pi_i(h_T) - \sum_{i=i^*+1}^I \theta_i(q_T) \pi_i(h_T) \right) \\ & + v^f \left( \sum_{i=1}^{i^*} (1 - \theta_i(q_T)) \pi_i(h_T) - \sum_{i=i^*+1}^I (1 - \theta_i(q_T)) \pi_i(h_T) \right) \end{aligned} \right\}. \end{aligned}$$

The terms in the maximization are the expected payoffs to the principal when the agent succeeds and fails respectively at question  $q_T$ . The probability of success is based on all types choosing action  $a_T = 1$ . Note that the payoff is linear in the verdicts, so that even if randomization of verdicts is allowed, the optimal choice can always be taken to be either 0 or 1. Moreover, there is no benefit in randomizing questions: if two questions yield the same expected payoffs, the principal can choose either one.

Once questions in period  $T$  and verdicts have been determined, it remains to derive the questions in period  $T - 1$  and earlier. At any history  $h_{T-1}$ , the choice of question will determine the beliefs corresponding to success and failure respectively. In either case, the principal's payoff as a function of those beliefs has already been determined above. Hence the principal simply chooses the question that maximizes her expected payoff. This process can be continued all the way to period 1 to determine the optimal sequence of questions. At each step, by the same argument as in period  $T$ , there is no benefit from randomization. Since the principal may be indifferent between



questions at some history and between verdicts at some terminal history, the ONST need not be unique.

This problem is an instance of the general sequential choice of experiments problem from statistics that we describe in the introduction. The same backward induction procedure can be applied to (theoretically) solve this more general problem. However, it is typically very difficult to explicitly characterize or to describe qualitative properties of the solution, even in relatively simple special cases that fit within our setting (Bradt and Karlin, 1956).

#### 4. INFORMATIVENESS

Although the sequential choice of experiments problem is difficult to solve in general, there is a prominent special case that allows for a simple solution: the case in which one question is more Blackwell informative than the others.

*Blackwell Informativeness:* We say that a question  $q$  is more Blackwell informative than another question  $q'$  if there are numbers  $\alpha_s, \alpha_f \in [0, 1]$  such that

$$\begin{bmatrix} \theta_1(q) & 1 - \theta_1(q) \\ \vdots & \vdots \\ \theta_I(q) & 1 - \theta_I(q) \end{bmatrix} \begin{bmatrix} \alpha_s & 1 - \alpha_s \\ \alpha_f & 1 - \alpha_f \end{bmatrix} = \begin{bmatrix} \theta_1(q') & 1 - \theta_1(q') \\ \vdots & \vdots \\ \theta_I(q') & 1 - \theta_I(q') \end{bmatrix}. \quad (2)$$

This is the classic notion of informativeness. Essentially, it says that  $q$  is more informative than  $q'$  if the latter can be obtained by adding noise to—or garbling—the former. Note that Blackwell informativeness is a partial order; it is possible for two questions not to be ranked in terms of Blackwell informativeness.

A seminal result due to Blackwell (1953) is that, in any static decision problem, regardless of the decision-maker’s preferences, she is always better off with information from a more Blackwell informative experiment than from a less informative one. This result carries over to the sequential setting: if there is one experiment that is more Blackwell informative than every other, then it is optimal for the decision maker always to use that experiment (see Section 14.17 in DeGroot, 2005). Since the ONST is a special case of this more general problem, if there is a question  $q$  that is the most Blackwell informative, then  $\mathcal{T}^N(h) = q$  at all  $h \in \mathcal{H}$ . The following is the formal statement of Blackwell’s result applied to our context.

**Theorem 1 (Blackwell 1953).** *Suppose there is a question  $q$  that is more Blackwell informative than all other questions  $q' \in Q$ . Then there is an ONST in which the question  $q$  is assigned at every history.*

In our setting, it is possible to strengthen this result because the principal’s payoff takes a special form; Blackwell informativeness is a stronger property than what is needed to guarantee that the ONST features only a single question. We use the term “informativeness” (without the additional “Blackwell” qualifier) to describe the weaker property appropriate for our setting.

*Informativeness:* Let  $\theta_G(q, \pi) = \frac{\sum_{i \leq i^*} \pi_i \theta_i(q)}{\sum_{i \leq i^*} \pi_i}$  be the probability, given belief  $\pi$ , that success is observed on question  $q$  conditional on the agent being a good type, under the assumption that the agent chooses the most informative action. Similarly, let  $\theta_B(q, \pi) = \frac{\sum_{i > i^*} \pi_i \theta_i(q)}{\sum_{i > i^*} \pi_i}$  be the corresponding



probability of success conditional on the agent being a bad type. We say that a question  $q$  is more informative than another question  $q'$  if, for all beliefs  $\pi$ , there are numbers  $\alpha_s(\pi), \alpha_f(\pi) \in [0, 1]$  such that

$$\begin{bmatrix} \theta_G(q, \pi) & 1 - \theta_G(q, \pi) \\ \theta_B(q, \pi) & 1 - \theta_B(q, \pi) \end{bmatrix} \begin{bmatrix} \alpha_s(\pi) & 1 - \alpha_s(\pi) \\ \alpha_f(\pi) & 1 - \alpha_f(\pi) \end{bmatrix} = \begin{bmatrix} \theta_G(q', \pi) & 1 - \theta_G(q', \pi) \\ \theta_B(q', \pi) & 1 - \theta_B(q', \pi) \end{bmatrix}. \quad (3)$$

To see that Blackwell informativeness is the stronger of these two notions, note that any  $\alpha_s$  and  $\alpha_f$  that satisfy (2) must also satisfy (3) for every belief  $\pi$ . The following example consisting of three types and two questions shows that the converse need not hold.

**Example 1.** Suppose there are three types ( $I = 3$ ), and two questions,  $Q = \{q, q'\}$ . Success probabilities if the agent chooses the most informative action are as follows:

$$\begin{array}{cc} & q & q' \\ \theta_1 & .9 & .4 \\ \theta_2 & .8 & .2 \\ \theta_3 & .2 & .1 \end{array} \quad (4)$$

The first column corresponds to the probability  $\theta_i(q)$  of success on question  $q$ , and the second column to that on question  $q'$ . If  $i^* = 2$  (so that types  $\theta_1$  and  $\theta_2$  are good types),  $q$  is more informative than  $q'$ . Intuitively, this is because the performance on question  $q$  is better at differentiating  $\theta_3$  from  $\theta_1$  and  $\theta_2$ . However, if  $i^* = 1$ , then  $q$  is no longer more informative than  $q'$ . This is because performance on question  $q'$  is better at differentiating  $\theta_1$  from  $\theta_2$ . Thus, if the principal's belief assigns high probabilities to  $\theta_1$  and  $\theta_2$ , she can benefit more from question  $q'$ , whereas if her belief assigns high probability to types  $\theta_1$  and  $\theta_3$ , she can benefit more from  $q$ . Since Blackwell informativeness is independent of the cutoff  $i^*$ , neither  $q$  nor  $q'$  is more Blackwell informative than the other.

Although weaker than Blackwell's condition (2), informativeness is still a partial order, and in many cases no element of  $Q$  is more informative than all others. However, when there exists a most informative question, our main result shows that Blackwell's result continues to hold for the design of the optimal test in our setting, even when the agent is strategic, provided that a natural monotonicity condition is satisfied. A key difficulty in extending the result is that informativeness is defined independently of the agent's actions and, as the examples in Appendix B demonstrate, in some cases the principal can benefit from strategic behavior by the agent.

## 5. INFORMATIVENESS AND OPTIMALITY

### 5.1. The Optimal Test

The following example shows that strategic behavior by the agent can cause Blackwell's result to fail in our model.

**Example 2.** Suppose there are three types ( $I = 3$ ) and one period ( $T = 1$ ), with  $i^* = 2$ . There are two questions,  $Q = \{q, q'\}$ , with success probabilities given by the following matrix:

	$q$	$q'$
$\theta_1$	.5	.35
$\theta_2$	.2	.5
$\theta_3$	.4	.4

The principal's prior belief is

$$(\pi_1, \pi_2, \pi_3) = (.3, .2, .5).$$

Note that question  $q$  is more Blackwell informative than  $q'$ .<sup>6</sup> If the agent was not strategic, the optimal test would assign question  $q$  and verdicts  $\mathcal{V}\{(q, s)\} = 0$  and  $\mathcal{V}\{(q, f)\} = 1$ . In this case, all types would choose  $a_1 = 0$ , yielding the principal a payoff of 0 (which is the same payoff she would get from choosing either question and  $\mathcal{V}\{(q, s)\} = \mathcal{V}\{(q, f)\} = 0$ ).

Can the principal do better? Assigning question  $q$  and reversing the verdicts makes  $a_1 = 1$  a best response for all types of the agent but would result in a negative payoff for the principal. Instead, it is optimal for the principal to assign question  $q'$  along with verdicts  $\mathcal{V}\{(q', s)\} = 1$  and  $\mathcal{V}\{(q', f)\} = 0$ . The most informative action is a best response for all types and this yields a positive payoff.

Notice that in the last example, the types are not ordered in terms of their ability on the questions the principal can assign. In particular, for each question, the bad type can succeed with higher probability than some good type. This feature turns out to play an important role in determining whether Blackwell's result holds; our main theorem shows that the following condition is sufficient for Blackwell's result to carry over to our model.

*Group Monotonicity:* We say that group monotonicity holds if, for every question  $q \in Q$ ,  $\theta_i(q) \geq \theta_j(q)$  whenever  $i \leq i^* < j$ .

This assumption says that the two groups are ordered in terms of ability in a way that is independent of the question: good types are always at least as likely to succeed as bad ones when the most informative action is chosen.

The proof of our main result builds on two key lemmas. The first of these provides, under the assumption of group monotonicity, a simple characterization of informativeness that dispenses with the unknowns  $\alpha_s(\cdot)$  and  $\alpha_f(\cdot)$ , and is typically easier to verify than the original definition.

**Lemma 1.** *Suppose group monotonicity holds. Then a question  $q$  is more informative than  $q'$  if and only if*

$$\frac{\theta_i(q)}{\theta_j(q)} \geq \frac{\theta_i(q')}{\theta_j(q')} \quad \text{and} \quad \frac{1 - \theta_j(q)}{1 - \theta_i(q)} \geq \frac{1 - \theta_j(q')}{1 - \theta_i(q')} \quad \text{for all } i \leq i^* \text{ and } j > i^*.$$

Intuitively, a question is more informative if there is a higher relative likelihood that the agent has a good type conditional on a success, and a bad type conditional on a failure. Using this lemma, it is now straightforward to verify that  $q$  is more informative than  $q'$  in the type space (4) when  $i^* = 2$  but not when  $i^* = 1$ .

<sup>6</sup>The corresponding values of  $\alpha_s$  and  $\alpha_f$  in equation (2) are .1 and .6, respectively.

The second key lemma identifies the optimal test in the special case where the principal has only a single question (i.e.,  $|Q| = 1$ ). In this case, the test is no longer be adaptive since the principal has no choice but to assign the same question at all histories. However, it could, in principle, be optimal for the principal to induce muddling through an appropriate choice of verdicts. For instance, the verdicts could treat early successes differently from later ones. The following lemma shows that doing so is never advantageous for the principal.

**Lemma 2.** *Suppose that  $|Q| = 1$  and group monotonicity holds. Then any ONST is an optimal test, and the most informative strategy  $\sigma^N$  is optimal for the agent.*

With a single question, the principal neither benefits nor is hurt by the fact that the agent is strategic. This is not the case in Examples 4 and 5 in the appendix, where the ONST is not the optimal test. Lemma 2 implies that this feature of the examples stems from the adaptive (history dependent) nature of the question assignment.

We are now in a position to state our main result.

**Theorem 2.** *Suppose that there is a question  $q$  that is more informative than every other question  $q' \in Q$ , and group monotonicity holds. Then any ONST is an optimal test. In particular, it is optimal for the principal to assign question  $q$  at all histories and the most informative strategy  $\sigma^N$  is optimal for the agent.*

This result states that principal cannot enhance learning by inducing strategic muddling through the choice of questions, a strategy that helps her in Examples 4 and 5. If the principal assigns only the most informative question, it follows from Lemma 2 that she should assign the same verdicts as in the ONST, and the most informative strategy is optimal for the agent.

While superficially similar, there are critical differences between Theorem 2 and Blackwell's result (Theorem 1). In the latter, where the agent is assumed to always choose the most informative strategy, the optimality of using the most Blackwell informative question  $q$  can be shown constructively by garbling. To see this, suppose that at some history  $h$  in the ONST, the principal assigns a question  $q' \neq q$ , and let  $\alpha_s$  and  $\alpha_f$  denote the corresponding values solving equation (2). In this case, the principal can replace question  $q'$  with  $q$  and appropriately randomize the continuation tests to achieve the same outcome. More specifically, at the history  $\{h, (q, s)\}$ , she can choose the continuation test following  $\{h, (q', s)\}$  with probability  $\alpha_s$  and, with the remaining probability  $1 - \alpha_s$ , choose the continuation test following  $\{h, (q', f)\}$ . A similar randomization using  $\alpha_f$  can be done at history  $\{h, (q, f)\}$ .

This construction is not sufficient to yield the result when the agent is strategic. In this case, replacing the question  $q'$  by  $q$  and garbling can alter incentives in a way that changes the agent's optimal strategy, and consequently, the principal's payoff. To see this, suppose that action 1 is optimal for some type  $\theta_i$  at  $h^A = (h, q')$ . This implies that the agent's expected probability of passing the test is higher in the continuation test following  $\{h, (q', s)\}$  than in the continuation test following  $\{h, (q', f)\}$ . Now suppose the principal replaces question  $q'$  by  $q$  and garbles the continuation tests as described above. Type  $\theta_i$  may no longer find the most informative action to be optimal. In particular, if  $\alpha_f > \alpha_s$ , then action 0 will be optimal after the change since failure on question  $q$  gives a higher likelihood of obtaining the continuation test that he is more likely to pass. Therefore, the simple garbling argument does not imply Theorem 2. Instead, the proof

exploits the structure of informativeness in our particular context captured by Lemma 1, which, when coupled with a backward induction argument, enables us to verify that the continuation tests can be garbled in a way that does not adversely affect incentives.

In the non-strategic benchmark model, Blackwell's result can be strengthened to eliminate less informative questions even if there is no most informative question. More precisely, if  $q, q' \in Q$  are such that  $q$  is more informative than  $q'$ , then there exists an ONST in which  $q'$  is not assigned at any history (and thus any ONST for the set of questions  $Q \setminus \{q'\}$  is also an ONST for the set of questions  $Q$ ). The intuition behind this result is essentially the same as for Blackwell's result: whenever a test assigns question  $q'$ , replacing it with  $q$  and suitably garbling the continuation tests yields the same joint distribution of types and verdicts.

In the strategic setting, this more general result does not hold. For example, there exist cases with one bad type in which muddling is optimal for the bad type in the first period and the most informative action is strictly optimal for at least one good type; one such case is described in Example 7 in Appendix B. Letting  $q$  denote the question assigned in the first period, adding any question  $\tilde{q}$  to the set  $Q$  that is easier than  $q$  and assigning  $\tilde{q}$  instead of  $q$  does not change the optimal action for any type; doing so only increases the payoff of any type that strictly prefers the most informative action. Since only good types have this preference, such a change increases the principal's payoff. If, in addition,  $q$  is more informative than  $\tilde{q}$ , then the optimal test for the set of questions  $Q \cup \{\tilde{q}\}$  is strictly better for the principal than that for the set  $Q$ , which implies that  $\tilde{q}$  must be assigned with positive probability at some history, and the generalization of Blackwell's result fails.

## 5.2. On the Structure of the Model

While Theorem 2 may seem intuitive, as Example 2 indicates, it does rely on group monotonicity. The following partial converse to Theorem 2 extends the logic of Example 2 to show that, in a sense, group monotonicity is necessary for Blackwell's result to hold in the strategic setting.

**Theorem 3.** *Suppose  $q$  is such that  $\theta_{\bar{i}}(q) < \theta_{\bar{j}}(q)$  for some  $\bar{i}$  and  $\bar{j}$  such that  $\bar{i} \leq i^* < \bar{j}$ . Then there exist  $q'$  and  $\pi$  such that  $q$  is more Blackwell informative than  $q'$ , and for each test length  $T$ , if  $Q = \{q, q'\}$ , no optimal test assigns question  $q$  at every history  $h \in \mathcal{H}$ .*

The idea behind this result is that, if  $\theta_{\bar{i}}(q) < \theta_{\bar{j}}(q)$  and the test always asks  $q$ , type  $\bar{j}$  can pass with at least as high a probability as can type  $\bar{i}$ . When the principal assigns high prior probability to these two types, she is better off asking a question  $q'$  (at least at some histories) for which  $\theta_{\bar{i}}(q') > \theta_{\bar{j}}(q')$  (and such a less Blackwell informative  $q'$  always exists) in order to advantage the good type.

The next example demonstrates that, even if group monotonicity holds, Blackwell's result can also break down if we alter the structure of the agent's payoffs. When all types choose the most informative action, success on a question increases the principal's belief that the type is good. Not surprisingly, if some types prefer to fail the test, this can give them an incentive to muddle in a way that overturns Blackwell's result.

**Example 3.** Suppose there are two types ( $I = 2$ ), one good and one bad, and one period ( $T = 1$ ). The principal has two questions,  $Q = \{q, q'\}$ , with success probabilities given by the following matrix:

	$q$	$q'$
$\theta_1$	.9	.8
$\theta_2$	.9	.1

The principal's prior belief is

$$(\pi_1, \pi_2) = (.5, .5).$$

Compared to the main model, suppose that the principal's payoffs are the same, but the agent's are type-dependent: type  $\theta_1$  prefers a verdict of 1 to 0, while type  $\theta_2$  has the opposite preference. One interpretation is that verdicts represent promotions to different departments. The principal wants to promote type  $\theta_1$  to the position corresponding to verdict 1 and type  $\theta_2$  to the position corresponding to verdict 0, a preference that the agent shares.

Question  $q'$  is trivially more Blackwell informative than question  $q$  since the performance on question  $q$  (conditional on the most informative action) conveys no information.<sup>7</sup> Faced with a nonstrategic agent, the optimal test would assign question  $q'$  and verdicts  $\mathcal{V}\{(q', s)\} = 1$  and  $\mathcal{V}\{(q', f)\} = 0$ . Faced with a strategic agent, the optimal test is to assign question  $q$  and verdicts  $\mathcal{V}\{(q, s)\} = 1$  and  $\mathcal{V}\{(q, f)\} = 0$ . In each of these tests, type  $\theta_1$  will choose  $a_1 = 1$  and type  $\theta_2$  will choose  $a_1 = 0$ . Thus the probability with which  $\theta_2$  gets verdict 0 remains the same but the probability with which  $\theta_1$  gets verdict 1 is higher with the easier question  $q$ .

## 6. NON-COMPARABLE QUESTIONS

In many cases, questions cannot be ordered by informativeness. What can we say about the design of the optimal test and its relationship to the ONST in general?

The next result shows that, when group monotonicity holds, any ONST is an optimal test when there are only two types ( $I = 2$ ); for strategic actions to play an important role, there must be at least three types.

**Theorem 4.** *Suppose group monotonicity holds. If  $I = 2$ , any ONST is an optimal test and makes the most informative strategy  $\sigma^N$  optimal for the agent.*

To see why the strategy  $\sigma^N$  is optimal for the agent in some optimal test, suppose there is an optimal test in which the good type strictly prefers to muddle at some history  $h^A$ . This implies that his expected payoff following a failure on the current question at  $h^A$  is higher than that following a success. Now suppose the principal altered the test by replacing the continuation test following a failure with that following a success (including replacing the corresponding verdicts). This would make action 1 optimal for both types since the continuation tests no longer depend on success or failure at  $h^A$ . Since the good type chose to muddle before the change, there is no effect on his payoff. Similarly, the bad type's payoff cannot increase: if he strictly preferred action 1 before the change then he is made worse off, and otherwise his payoff is also unchanged. Therefore, this change cannot lower the principal's payoff. A similar argument applies to histories where the bad

<sup>7</sup>The corresponding  $\alpha_s$  and  $\alpha_f$  in equation (2) are both .9.

type prefers to muddle (in which case we can replace the continuation test following a failure with that following a success). Such a construction can be used inductively at all histories where there is muddling.<sup>8</sup>

Given this argument, Theorem 4 follows if  $\sigma^N$  is optimal in every ONST. This can be seen using a similar argument to that above, except for the case in which both types strictly prefer to muddle at some history. However, it turns out that this case cannot happen when the continuation tests after both outcomes are chosen optimally.

When there are more than two types, even if group monotonicity holds, there need not be an optimal test in which the fully informative strategy is optimal. The following example shows that, even if the most informative strategy  $\sigma^N$  is optimal in some ONST, the optimal test may differ; the principal can sometimes benefit from distorting the test relative to the ONST so as to *induce* muddling by some types.

**Example 4.** Suppose there are three types ( $I = 3$ ) and three periods ( $T = 3$ ), with  $i^* = 2$  (so that types  $\theta_1$  and  $\theta_2$  are good types). There are two questions,  $Q = \{q, q'\}$ , and the success probabilities are given by the following matrix:

	$q$	$q'$
$\theta_1$	1	.5
$\theta_2$	.5	.5
$\theta_3$	.5	.4

Note that the types are ranked in terms of ability (in particular, group monotonicity holds), and the questions are ranked in terms of difficulty. The principal's prior belief is

$$(\pi_1, \pi_2, \pi_3) = (.06, .44, .5).$$

The ONST  $(\mathcal{T}^N, \mathcal{V}^N)$  is represented by the tree in Figure 1. The ONST always assigns the question  $q'$ . The agent passes the test if he succeeds at least twice in the three periods. Intuitively, the principal assigns a low prior probability to type  $\theta_1$ , and so designs the test to distinguish between types  $\theta_2$  and  $\theta_3$ , for which  $q'$  is better than  $q$ . Given that only a single question is used, group monotonicity implies that the optimal verdicts feature a cutoff number of successes required to pass.<sup>9</sup>

If the principal commits to this test, then the most informative strategy is optimal for the agent: failure on the question assigned in any period has no effect on the questions assigned in the future, and merely decreases the probability of passing.

Is this test optimal when the agent is strategic? Consider instead the deterministic test  $(\mathcal{T}', \mathcal{V}')$  described by the tree in Figure 2. This alternate test differs from the ONST in several ways. The agent now faces question  $q$  instead of  $q'$  both in period 1 and at the period 2 history following a success. In addition, the agent can pass only at two of the terminal histories. We will argue that

<sup>8</sup>The discussion has ignored the effect of a change following a given period  $t$  history on the action choices at all periods  $t' < t$ ; indeed, earlier actions might change. However, it is straightforward to argue that if a type's payoff goes down at a given history after such a change, the (optimal) payoff is also lower at the beginning of the test.

<sup>9</sup>Note that the ONST is not unique in this case since the principal can assign either of the two questions (keeping the verdicts the same) at histories  $\{(q', s), (q', s)\}$  and  $\{(q', f), (q', f)\}$ .

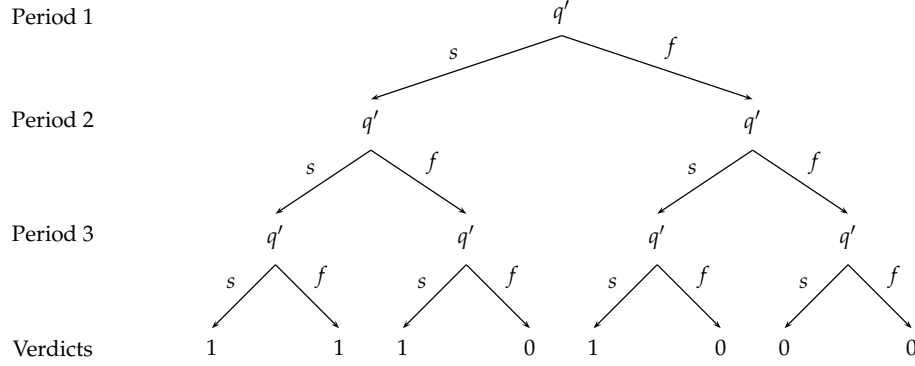


FIGURE 1. An ONST for Example 4. The level of a node corresponds to the time period. Inner nodes indicate the question assigned at the corresponding history, while the leaves indicate the verdicts. For instance, the rightmost node at level 3 corresponds to the period 3 history  $h_3 = \{(q', f), (q', f)\}$  and the question assigned by the test at this history is  $\mathcal{T}^N(h_3) = q'$ . The verdicts following this history are 0 whether he succeeds or fails at this question.

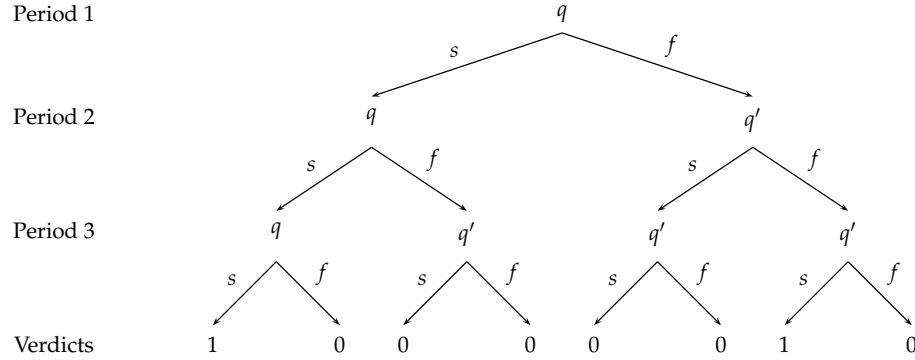


FIGURE 2. An optimal deterministic test for Example 4.

this test yields a higher payoff to the principal despite  $\sigma^N$  being an optimal strategy for the agent in test  $(\mathcal{T}^N, \mathcal{V}^N)$ .

By definition,  $(\mathcal{T}', \mathcal{V}')$  can only yield a higher payoff for the principal than does  $(\mathcal{T}^N, \mathcal{V}^N)$  if at least one type of the agent chooses to muddle at some history. This is indeed the case. Since type  $\theta_1$  succeeds at question  $q$  for sure conditional on choosing the most informative action, he will choose  $a_t = 1$  in each period and pass with probability 1. However, types  $\theta_2$  and  $\theta_3$  both prefer  $a_t = 0$  in periods  $t = 1, 2$ . Following a success in period 1, two further successes are required at question  $q$  to get a passing verdict. In contrast, by choosing the action 0 in the first two periods, the history  $\{(q, f), (q', f)\}$  can be reached with probability 1, after which the agent needs only a single success at question  $q'$  to pass. Consequently, this muddling strategy yields a higher payoff for types  $\theta_2$  and  $\theta_3$ .

The difference in payoffs for the three types in  $(\mathcal{T}', \mathcal{V}')$  relative to  $(\mathcal{T}^N, \mathcal{V}^N)$  are

$$\Delta v_1 = v_1(\mathcal{T}', \mathcal{V}') - v_1(\mathcal{T}^N, \mathcal{V}^N) = 1 - [.5 * .75 + .5 * .25] = .5,$$



$$\Delta v_2 = v_2(\mathcal{T}', \mathcal{V}') - v_2(\mathcal{T}^N, \mathcal{V}^N) = .5 - [.5 * .75 + .5 * .25] = 0,$$

and  $\Delta v_3 = v_3(\mathcal{T}', \mathcal{V}') - v_3(\mathcal{T}^N, \mathcal{V}^N) = .4 - [.4 * .64 + .6 * .16] = .048.$

The change in the principal's payoff is

$$\sum_{i=1}^2 \pi_i \Delta v_i - \pi_i \Delta v_3 = .06 * .5 - .5 * .048 > 0,$$

which implies that  $(\mathcal{T}^N, \mathcal{V}^N)$  is not the optimal test. In particular, the principal can benefit from the fact that the agent can choose his actions strategically.

The examples in Appendix B illustrate a range of possibilities for the both the optimal test and the ONST. Perhaps surprisingly, the most informative strategy need not be optimal in the ONST (see Example 5). Group monotonicity implies that, under the assumption that the agent chooses the most informative strategy, success on each question raises the principal's belief that the agent's type is good. Nonetheless, because of the adaptive nature of the test, failure on a question can make the remainder of the test easier for some types. Relative to choosing  $\sigma^N$ , strategic behavior by the agent can either help the principal (as in Example 5) or hurt her (as in Example 6). Further, in some cases the most informative strategy is optimal in the optimal deterministic test but not in the ONST.

Finally, unlike the ONST, for which it suffices to restrict to deterministic tests, there are cases in which there is no deterministic optimal test for the principal when the agent is strategic. Example 7 illustrates a case in which randomizing one verdict strictly benefits the principal.

## 7. DISCUSSION

### 7.1. Menus of Tests

We have so far ignored the possibility that the principal can offer a menu of tests and allow the agent to choose which test to take. While this is not typically observed in the applications we mentioned in the introduction, it may seem natural from a theoretical perspective. Formally, in this case, the principal offers a menu of  $M$  tests  $\{\rho_k\}_{k=1}^M$  and each type  $\theta_i$  of the agent chooses a test  $\rho_k$  that maximizes his expected payoff  $v_i(\rho_k)$ . Although a nontrivial menu could in principle help to screen the different types, our main result still holds.

**Theorem 5.** *Suppose there is a question  $q$  that is more informative than every other question  $q' \in Q$ . Then for any ONST, there is an optimal menu consisting only of that test.*

*Proof.* In the proof of Theorem 2, we show that any test can be replaced by one where the most informative question  $q$  is assigned at all histories and appropriate verdicts can be chosen so that the payoffs of the good types (weakly) increase and those of the bad types (weakly) decrease. Applying this change to every test in a menu must also increase good types' payoffs while decreasing those of bad types. Thus we can restrict attention to menus in which every test assigns question  $q$  at every history. But then the proof of Lemma 2 shows that replacing any test that is not an ONST with an ONST makes any good type that chooses that test better off and any bad type worse off.

Therefore, by the expression for the principal's payoff in (1), replacing every test in the menu with any given ONST cannot make the principal worse off.  $\square$

If there is no most informative question, it can happen that offering a nontrivial menu is strictly better for the principal than any single test, as Example 8 in Appendix B shows.

It appears to be very difficult to characterize the optimal menu in general since it involves constructing tests that are themselves complex objects that are challenging to compute. However, without identifying the optimal menu, the following result provides an upper bound on the number of tests that are required: it is always sufficient to restrict to menus containing only as many tests as there are good types. One implication is that nontrivial menus are never beneficial when there is a single good type.

**Theorem 6.** *There exists an optimal menu containing at most  $i^*$  elements. In particular, if there is a single good type ( $i^* = 1$ ), then there is an optimal menu that consists of a single test.*

*Proof.* Suppose the principal offers a menu  $\mathcal{M}$ , and let  $\mathcal{M}'$  denote the subset of  $\mathcal{M}$  consisting of the elements chosen by the good types  $\theta_1, \dots, \theta_{i^*}$  (so that  $\mathcal{M}'$  contains at most  $i^*$  elements). If instead of  $\mathcal{M}$  the principal offered the menu  $\mathcal{M}'$ , each good type would continue to choose the same test (or another giving the same payoff), and hence would receive the same payoff as from the menu  $\mathcal{M}$ . However, the payoff to all bad types must be weakly lower since the set of tests is smaller. Therefore, the menu  $\mathcal{M}'$  is at least as good for the principal as  $\mathcal{M}$  since it does not affect the probability that any good type passes and weakly decreases the probability that any bad type passes.  $\square$

## 7.2. The Role of Commitment

Throughout the preceding analysis, we have assumed that the principal can commit in advance to both the history-dependent sequence of questions and the mapping from terminal histories to verdicts. When the principal cannot commit, her choice of question at each history is determined in equilibrium as a best response to the agent's strategy given the principal's belief. Similarly, the verdicts are chosen optimally at each terminal history depending on the principal's belief (which is also shaped by the agent's strategy). Commitment power benefits the principal (at least weakly) since she can always commit to any equilibrium strategy she employs in the game without commitment (in which case it would be optimal for the agent to choose his equilibrium strategy in response).

If there is a most informative question and group monotonicity holds, then the optimal test can be implemented even without commitment. More precisely, the principal choosing any ONST together with the agent using the strategy  $\sigma^N$  constitutes a sequential equilibrium strategy profile of the game where the principal cannot commit to a test. To understand why, note first that the verdicts in this case must correspond directly to the principal's posterior belief at each terminal node, with the agent passing precisely when the principal believes it is more likely that his type is good. Given these verdicts, the fully informative action is optimal in the last period regardless of what question is assigned, and hence by Blackwell's original result assigning the most informative question is optimal at every history in period  $T$ . Given that the same question is assigned at every

history in period  $T$ , there is no benefit to muddling in period  $T - 1$ , which implies that assigning the most informative question is again optimal. Working backward in this way yields the result.

In general, optimal tests may not be implementable in the absence of commitment: Example 9 shows how the optimal test may fail to be sequentially rational.

## APPENDIX A. PROOFS

We require some additional notation for the proofs. The *length* of a history  $h_t$  at the beginning of period  $t$  is  $|h_t| = t - 1$ . We use  $S(h_{T+1})$  to denote the number of successes in the terminal history  $h_{T+1} \in \mathcal{H}_{T+1}$ . Given a history  $h$ , the set of all histories of the form  $(h, h') \in \mathcal{H}$  is denoted by  $\Lambda(h)$  and is referred to as the *subtree* at  $h$ . Similarly, we write  $\Lambda^A(h)$  for the set of all histories for the agent of the form  $(h, h') \in \mathcal{H}^A$ . The set of all terminal histories  $(h, h') \in \mathcal{H}_{T+1}$  that include  $h$  is denoted by  $\Gamma(h)$ . The length of  $\Gamma(h)$  is defined to be  $T - |h|$ .

For some of the proofs, it is useful to consider tests in which verdicts may be randomized but question assignment is not. A *deterministic test with random verdicts*  $(\mathcal{T}, \mathcal{V})$  consists of functions  $\mathcal{T} : \mathcal{H} \rightarrow Q$  and  $\mathcal{V} : \mathcal{H}_{T+1} \rightarrow [0, 1]$  (as opposed to the range of  $\mathcal{V}$  being  $\{0, 1\}$ ). Note that one can think of any test  $\rho$  as randomizing over deterministic tests with random verdicts by combining any tests in the support of  $\rho$  that share the same question assignment function  $\mathcal{T}$  and defining the randomized verdict function to be the expected verdict conditional on  $\mathcal{T}$ . In the proofs that follow, we do not distinguish between deterministic tests with or without random verdicts; the meaning will be clear from the context.

Given a test  $\rho$  and a history  $(h_t, q_t)$  for the agent, we write  $\text{supp}(h_t, q_t)$  to denote the set of deterministic tests with random verdicts in the support of  $\rho$  that generate the history  $(h_t, q_t)$  with positive probability if the agent chooses the most informative strategy.

The following observation is useful for some of the proofs.

**Observation 1.** *Given a test  $\rho$ , an optimal strategy  $\sigma^*$  for the agent, and a history  $h$ , consider an alternate test  $\hat{\rho}$  that differs only in the distribution of questions assigned in the subtree  $\Lambda(h)$  and the distribution of verdicts at terminal histories in  $\Gamma(h)$ . Let  $\hat{\sigma}^*$  be an optimal strategy in the test  $\hat{\rho}$ . Then, for each  $i$ ,  $u_i(h; \hat{\rho}, \hat{\sigma}_i^*) \geq u_i(h; \rho, \sigma_i^*)$  implies  $v_i(\hat{\rho}) \geq v_i(\rho)$ , and similarly,  $u_i(h; \hat{\rho}, \hat{\sigma}_i^*) \leq u_i(h; \rho, \sigma_i^*)$  implies  $v_i(\hat{\rho}) \leq v_i(\rho)$ .*

In words, this observation states that if we alter a test at a history  $h$  or its subtree  $\Lambda(h)$  in a way that the expected payoff of a type increases at  $h$ , then the expected payoff also increases at the beginning of the test. This observation is immediate. Consider first the case where  $u_i(h; \hat{\rho}, \hat{\sigma}_i^*) \geq u_i(h; \rho, \sigma_i^*)$ . Suppose the agent plays the strategy  $\sigma'_i$  such that  $\sigma'_i(h') = \sigma_i^*(h')$  at all histories  $h' \notin \Lambda^A(h)$  and  $\sigma'_i(h') = \hat{\sigma}_i^*(h')$  at all histories  $h' \in \Lambda^A(h)$  on test  $\hat{\rho}$ . If history  $h$  is reached with positive probability, this must yield a weakly higher payoff than playing strategy  $\sigma_i^*$  on test  $\rho$ . If history  $h$  is reached with probability 0, the payoff of the agent remains the same. Thus the agent can guarantee himself a payoff  $u_i(h_1; \hat{\rho}, \sigma'_i) \geq u_i(h_1; \rho, \sigma_i^*)$ , which in turn implies that optimal strategy  $\hat{\sigma}_i^*$  on  $\hat{\rho}$  must yield a payoff at least as high.

The opposite inequality follows from a similar argument. In that case, the agent could only raise his payoff by altering his actions at some histories  $h' \notin \Lambda^A(h)$ . But if this yielded him a higher payoff, it would contradict the optimality of the strategy  $\sigma_i^*$ .

This observation has a simple implication that we will use in what follows: any alteration in a subtree  $\Lambda(h)$  that raises the payoffs of good types and lowers the payoffs of bad types leads to a higher payoff for the principal. Formally, if  $\hat{\rho}$  differs from  $\rho$  only after history  $h$ , and  $u_i(h; \hat{\rho}, \hat{\sigma}_i^*) \geq u_i(h; \rho, \sigma_i^*)$  for all  $i \leq i^*$  and  $u_j(h; \hat{\rho}, \hat{\sigma}_j^*) \leq u_j(h; \rho, \sigma_j^*)$  for all  $j > i^*$ , then  $\hat{\rho}$  yields the principal at

least as high a payoff as does  $\rho$  (this follows from Observation 1 together with the expression (1) for the principal's payoff).

*Proof of Lemma 1*

We prove this lemma in two parts. First, we show that  $q$  is more informative than  $q'$  if and only if, for every  $\pi$ ,

$$\frac{\theta_G(q, \pi)}{\theta_B(q, \pi)} \geq \frac{\theta_G(q', \pi)}{\theta_B(q', \pi)} \quad \text{and} \quad \frac{1 - \theta_B(q, \pi)}{1 - \theta_G(q, \pi)} \geq \frac{1 - \theta_B(q', \pi)}{1 - \theta_G(q', \pi)}. \quad (5)$$

Then we show that the latter condition is equivalent to

$$\frac{\theta_i(q)}{\theta_j(q)} \geq \frac{\theta_i(q')}{\theta_j(q')} \quad \text{and} \quad \frac{1 - \theta_j(q)}{1 - \theta_i(q)} \geq \frac{1 - \theta_j(q')}{1 - \theta_i(q')}$$

for all  $i \leq i^*$  and  $j > i^*$ .

Recall that  $q$  is more informative than  $q'$  if there is a solution to

$$\begin{bmatrix} \theta_G(q, \pi) & 1 - \theta_G(q, \pi) \\ \theta_B(q, \pi) & 1 - \theta_B(q, \pi) \end{bmatrix} \begin{bmatrix} \alpha_s(\pi) & 1 - \alpha_s(\pi) \\ \alpha_f(\pi) & 1 - \alpha_f(\pi) \end{bmatrix} = \begin{bmatrix} \theta_G(q', \pi) & 1 - \theta_G(q', \pi) \\ \theta_B(q', \pi) & 1 - \theta_B(q', \pi) \end{bmatrix}$$

that satisfies  $\alpha_s(\pi), \alpha_f(\pi) \in [0, 1]$ . Note that group monotonicity implies that  $\theta_G(q, \pi) \geq \theta_B(q, \pi)$ . If, for some  $\pi$ ,  $\theta_G(q, \pi) = \theta_B(q, \pi)$ , then this last condition is satisfied if and only if  $\theta_G(q', \pi) = \theta_B(q', \pi)$ . On the other hand, since  $\theta_G(q', \pi) \geq \theta_B(q', \pi)$ , (5) also holds (for the given  $\pi$ ) if and only if  $\theta_G(q', \pi) = \theta_B(q', \pi)$ , and therefore the two conditions are equivalent.

Now suppose  $\theta_G(q, \pi) > \theta_B(q, \pi)$ . Solving for  $\alpha_s(\pi)$  and  $\alpha_f(\pi)$  gives

$$\alpha_s(\pi) = \frac{\theta_G(q', \pi)(1 - \theta_B(q, \pi)) - \theta_B(q', \pi)(1 - \theta_G(q, \pi))}{\theta_G(q, \pi) - \theta_B(q, \pi)}$$

and 
$$\alpha_f(\pi) = \frac{\theta_B(q', \pi)\theta_G(q, \pi) - \theta_G(q', \pi)\theta_B(q, \pi)}{\theta_G(q, \pi) - \theta_B(q, \pi)}.$$

Hence the condition that  $\alpha_s(\pi) \geq 0$  is equivalent to

$$\frac{\theta_G(q', \pi)}{\theta_B(q', \pi)} \geq \frac{1 - \theta_G(q, \pi)}{1 - \theta_B(q, \pi)},$$

which holds because the left-hand side is at least 1 and the right-hand side is less than 1. The condition that  $\alpha_f(\pi) \leq 1$  is equivalent to

$$\frac{\theta_B(q, \pi)}{\theta_G(q, \pi)} \leq \frac{1 - \theta_B(q', \pi)}{1 - \theta_G(q', \pi)},$$

which holds because the left-hand side is less than 1 and the right-hand side is at least 1. Finally,  $\alpha_s(\pi) \leq 1$  is equivalent to

$$\frac{1 - \theta_B(q, \pi)}{1 - \theta_G(q, \pi)} \geq \frac{1 - \theta_B(q', \pi)}{1 - \theta_G(q', \pi)},$$

and  $\alpha_f(\pi) \geq 0$  is equivalent to

$$\frac{\theta_G(q, \pi)}{\theta_B(q, \pi)} \geq \frac{\theta_G(q', \pi)}{\theta_B(q', \pi)},$$

which completes the first part of the proof.

We now show the second part. If (5) holds for every  $\pi$ , then given any  $i \leq i^*$  and  $j > i^*$ , taking  $\pi_i = \pi_j = \frac{1}{2}$  in (5) gives

$$\frac{\theta_i(q)}{\theta_j(q)} \geq \frac{\theta_i(q')}{\theta_j(q')} \quad \text{and} \quad \frac{1 - \theta_j(q)}{1 - \theta_i(q)} \geq \frac{1 - \theta_j(q')}{1 - \theta_i(q')}.$$

For the converse, observe that

$$\frac{\theta_G(q, \pi)}{\theta_B(q, \pi)} \geq \frac{\theta_G(q', \pi)}{\theta_B(q', \pi)} \iff \sum_{i \leq i^*, j > i^*} \pi_i \pi_j \theta_i(q') \theta_j(q) \left( \frac{\theta_i(q)}{\theta_j(q)} \frac{\theta_j(q')}{\theta_i(q')} - 1 \right) \geq 0,$$

which holds if  $\frac{\theta_i(q)}{\theta_j(q)} \geq \frac{\theta_i(q')}{\theta_j(q')}$  whenever  $i \leq i^* < j$ . Similarly,

$$\begin{aligned} \frac{1 - \theta_B(q, \pi)}{1 - \theta_G(q, \pi)} &\geq \frac{1 - \theta_B(q', \pi)}{1 - \theta_G(q', \pi)} \\ &\iff \sum_{i \leq i^*, j > i^*} \pi_i \pi_j (1 - \theta_i(q))(1 - \theta_j(q')) \left( \frac{1 - \theta_j(q)}{1 - \theta_i(q)} \frac{1 - \theta_i(q')}{1 - \theta_j(q')} - 1 \right) \geq 0, \end{aligned}$$

which holds if  $\frac{1 - \theta_j(q)}{1 - \theta_i(q)} \geq \frac{1 - \theta_j(q')}{1 - \theta_i(q')}$  whenever  $i \leq i^* < j$ .  $\square$

#### Proof of Lemma 2

Since there is only a single question  $q \in Q$ , a test in this case is simply a deterministic test with random verdicts, which we denote by  $(\mathcal{T}, \mathcal{V})$ . We begin by stating an observation that is useful for the proof.

**Observation 2.** Suppose that  $|Q| = 1$ . Consider a history  $h$  and the associated subtree  $\Lambda(h)$ . If there exists a number of successes  $k^* \in \{0, \dots, T\}$  such that for all terminal histories  $h_{T+1} \in \Gamma(h)$ , the verdicts satisfy  $\mathcal{V}(h_{T+1}) = 1$  whenever  $S(h_{T+1}) > k^*$  and  $\mathcal{V}(h_{T+1}) = 0$  whenever  $S(h_{T+1}) < k^*$ , then the most informative action is optimal for all types at all histories in this subtree.

*Proof.* This result holds trivially if the length of  $\Gamma(h)$  is 1; accordingly, suppose the length is at least 2. First, observe that if this property holds in  $\Lambda(h)$ , then it also holds in all subtrees of  $\Lambda(h)$ . Now take any history  $h' \in \Lambda(h)$ . Consider a terminal history  $\{h', (q, f), h''\} \in \Gamma(h)$  following a failure at  $h'$ . By the cutoff property, the verdict at the terminal history  $\{h', (q, s), h''\} \in \Gamma(h)$  must be weakly higher. Since this is true for all  $h''$ , it implies that any strategy following a failure at  $h'$  must yield a weakly lower payoff than if the corresponding strategy was employed after a success. This implies that the most informative action is optimal at  $h'$ .  $\square$

We prove the lemma by induction. The induction hypothesis states that any test  $(\mathcal{T}, \mathcal{V})$  of length  $T - 1$  that induces muddling (at some history) can be replaced by another test  $(\mathcal{T}, \mathcal{V}')$  of the same length in which (i) the most informative strategy is optimal for every type, (ii) every good type passes with at least as high a probability as in  $(\mathcal{T}, \mathcal{V})$ , and (iii) every bad type passes with probability no higher than in  $(\mathcal{T}, \mathcal{V})$ . Therefore, the principal's payoff from test  $(\mathcal{T}, \mathcal{V}')$  is at least as high as from  $(\mathcal{T}, \mathcal{V})$ .

As a base for the induction, consider  $T = 1$ . If muddling is optimal for some type, it must be that  $\mathcal{V}(\{(q, f)\}) \geq \mathcal{V}(\{(q, s)\})$ . But then muddling is an optimal action for every type. Changing

the verdict function to  $\mathcal{V}'(\{(q, s)\}) = \mathcal{V}'(\{(q, f)\}) = \mathcal{V}(\{(q, f)\})$  makes the most informative action optimal and does not affect the payoff of the agent or the principal.

The induction hypothesis implies that we only need to show that inducing muddling is not strictly optimal for the tester in the first period of a  $T$  period test. This induction step is now shown in two separate parts.

*Step 1:* Consider the two subtrees  $\Lambda(\{(q, s)\})$ ,  $\Lambda(\{(q, f)\})$  following success and failure in the first period. For each  $\omega \in \{s, f\}$ , there exists a number of correct answers  $k_\omega^* \in \{0, \dots, T\}$  such that there are optimal verdicts in the subtree  $\Lambda(\{(q, \omega)\})$  satisfying  $\mathcal{V}(h) = 1$  whenever  $S(h) > k_\omega^*$  and  $\mathcal{V}(h) = 0$  whenever  $S(h) < k_\omega^*$  for all  $h \in \Gamma(\{(q, \omega)\})$ .

*Proof.* Recall that the induction hypothesis states that it is optimal for all types to choose the most informative action in the subtrees  $\Lambda(\{(q, s)\})$  and  $\Lambda(\{(q, f)\})$ . We will prove the result for the subtree  $\Lambda(\{(q, s)\})$ ; an identical argument applies to  $\Lambda(\{(q, f)\})$ .

Suppose the statement does not hold. Consider a history  $h \in \Lambda(\{(q, s)\})$  such that the subtree  $\Lambda(h)$  is minimal among those in which the statement does not hold (meaning that the statement holds for every proper subtree of  $\Lambda(h)$ ).

Given any optimal verdict function  $\mathcal{V}$ , let  $\underline{k}_s$  and  $\bar{k}_s$  denote, respectively, the smallest and largest values of  $k_s^*$  for which the statement holds in  $\Lambda(\{(h, (q, s))\})$ . We define  $\underline{k}_f$  and  $\bar{k}_f$  analogously. If for some optimal  $\mathcal{V}$ ,  $\underline{k}_s \leq \bar{k}_f$  and  $\underline{k}_f \leq \bar{k}_s$ , then there exists  $k^*$  for which the statement holds in  $\Lambda(\{(h, (q, s))\})$  and in  $\Lambda(\{(h, (q, f))\})$ , implying that it holds in  $\Lambda(h)$ . Therefore, for each optimal  $\mathcal{V}$ , either  $\underline{k}_s > \bar{k}_f$  or  $\underline{k}_f > \bar{k}_s$ .

Suppose  $\underline{k}_s > \bar{k}_f$ .<sup>10</sup> Let the terminal history  $h_{T+1}^s \in \Gamma(\{(h, (q, s))\})$  be such that  $S(h_{T+1}^s) = \underline{k}_s$  and  $\mathcal{V}(h_{T+1}^s) < 1$ , and let  $h_{T+1}^f \in \Gamma(\{(h, (q, f))\})$  be such that  $S(h_{T+1}^f) = \bar{k}_f$  and  $\mathcal{V}(h_{T+1}^f) > 0$ . Note that such terminal histories exist by the minimality and maximality of  $\underline{k}_s$  and  $\bar{k}_f$ , respectively. Let  $r = \underline{k}_s - \bar{k}_f$ . Let  $\tilde{i} \in \arg \min_{i \leq i^*} \theta_i(q)$ , and let  $\Delta > 0$  be such that

$$\mathcal{V}'(h_{T+1}^s) := \mathcal{V}(h_{T+1}^s) + \Delta \leq 1$$

and

$$\mathcal{V}'(h_{T+1}^f) := \mathcal{V}(h_{T+1}^f) - \frac{\theta_{\tilde{i}}(q)^r}{(1 - \theta_{\tilde{i}}(q))^r} \Delta \geq 0,$$

with one of these holding with equality. Letting  $\mathcal{V}'(h) = \mathcal{V}(h)$  for every terminal history  $h \notin \{h_{T+1}^s, h_{T+1}^f\}$ , changing the verdict function from  $\mathcal{V}$  to  $\mathcal{V}'$  does not affect the cutoff property in either subtree  $\Lambda(\{(h, (q, s))\})$  or  $\Lambda(\{(h, (q, f))\})$ . Therefore, by Observation 2, the most informative action is optimal at all histories in each of these subtrees. In addition, the most informative action remains optimal at  $h$  for all types after the change to  $\mathcal{V}'$  because the payoff of all types have gone up in the subtree  $\Lambda(\{(h, (q, s))\})$  and down in the subtree  $\Lambda(\{(h, (q, f))\})$ , and, by the induction hypothesis, action 1 was optimal at  $h$  before the change.

The difference between the expected payoffs for type  $i$  at history  $h$  (given that the agent follows the most informative strategy in the subtree  $\Lambda(h)$ ) due to the change in verdicts has the same sign

<sup>10</sup>Note that  $\underline{k}_s > \bar{k}_f + 1$  implies that it would be optimal for all types to muddle at  $h$ , contrary to the induction hypothesis.



as

$$\Delta \left( \theta_i(q)^r - (1 - \theta_i(q))^r \frac{\theta_i(q)^r}{(1 - \theta_i(q))^r} \right).$$

By group monotonicity, this last expression is nonnegative if  $i \leq i^*$  and nonpositive otherwise. In other words, changing the verdict function to  $\mathcal{V}'$  (weakly) raises the payoffs of good types and lowers those of bad types at history  $h$ . Therefore, by Observation 1, the principal's payoff does not decrease as a result of this change. Iterating this process at other terminal histories  $h_{T+1}^s \in \Gamma(\{h, (q, s)\})$  such that  $S(h_{T+1}^s) = \underline{k}_s$  and  $\mathcal{V}'(h_{T+1}^s) < 1$  and  $h_{T+1}^f \in \Gamma(\{h, (q, f)\})$  such that  $S(h_{T+1}^f) = \bar{k}_f$  and  $\mathcal{V}(h_{T+1}^f) > 0$  eventually leads to an optimal verdict function for which  $\underline{k}_s = \bar{k}_f$ , as needed.

If  $\underline{k}_f > \bar{k}_s$ , then all types strictly prefer action 1 at  $h$ . To see this, note that for all  $\{h, (q, f), h'\} \in \Gamma(\{h, (q, f)\})$ , it must be that  $\mathcal{V}(\{h, (q, s), h'\}) \geq \mathcal{V}(\{h, (q, f), h'\})$ , and this inequality must be strict for all terminal histories where  $S(\{h, (q, f), h'\}) = \underline{k}_f - 1$ . A similar adjustment to that above can now be done. Let  $h_{T+1}^s \in \Gamma(\{h, (q, s)\})$  be such that  $S(h_{T+1}^s) = \bar{k}_s$  and  $\mathcal{V}(h_{T+1}^s) > 0$ , and let  $h_{T+1}^f \in \Gamma(\{h, (q, f)\})$  be such that  $S(h_{T+1}^f) = \underline{k}_f$  and  $\mathcal{V}(h_{T+1}^f) < 1$ . Once again, such terminal histories exist by the maximality and minimality of  $\bar{k}_s$  and  $\underline{k}_f$  respectively. Let  $r = \underline{k}_f - \bar{k}_s$ , and let  $\Delta > 0$  be such that

$$\mathcal{V}'(h_{T+1}^f) = \mathcal{V}(h_{T+1}^f) + \Delta \leq 1$$

and

$$\mathcal{V}'(h_{T+1}^s) = \mathcal{V}(h_{T+1}^s) - \frac{\theta_i(q)^r}{(1 - \theta_i(q))^r} \Delta \geq 0,$$

with one of these holding with equality. As before, this manipulation does not affect the cutoff property at either subtree  $\Lambda(\{h, (q, s)\})$  or  $\Lambda(\{h, (q, f)\})$  and therefore by Observation 2, action 1 is optimal at all histories in each of these subtrees.

Once again, the difference between the expected payoffs for type  $i$  at history  $h$  (given that the agent follows the most informative strategy in the subtree  $\Lambda(h)$ ) due to this adjustment has the same sign as

$$\Delta \left( \theta_i(q)^r - (1 - \theta_i(q))^r \frac{\theta_i(q)^r}{(1 - \theta_i(q))^r} \right),$$

which is nonnegative if  $i \leq i^*$  and nonpositive if  $i > i^*$ . Therefore, the principal's payoff does not decrease from these changes, and iterating leads to optimal verdicts satisfying  $\underline{k}_f = \bar{k}_s$ .  $\square$

*Step 2:* Suppose the verdicts at terminal histories  $\Gamma(\{(q, s)\})$  and  $\Gamma(\{(q, f)\})$  satisfy the above cutoff property, with cutoffs  $k_s^*$  and  $k_f^*$ , respectively. Then if one type has an incentive to muddle in the first period, so do all other types. Consequently, if all types choose  $a_1 = 1$ , the proposition follows or, if all types want to muddle, the proposition follows by replacing the test after  $\{(q, s)\}$  with the test after  $\{(q, f)\}$ .

*Proof.* This step is straightforward and can be shown by examining the three possible cases. Suppose  $k_s^* \leq k_f^*$ . Then the verdict at every terminal history  $\{(q, s), h\} \in \Gamma(\{(q, s)\})$  is weakly higher than  $\{(q, f), h\} \in \Gamma(\{(q, f)\})$  and hence  $a_1 = 1$  must be optimal for all types. When  $k_s^* > k_f^* + 1$ ,  $a_1 = 0$  is optimal for all types. Finally, when  $k_s^* = k_f^* + 1$  type  $i$  wants to muddle if and only if the sum of the verdicts at terminal histories  $\{(q, f), h\} \in \Gamma(\{(q, f)\})$  with  $S(\{(q, f), h\}) = k_f^*$  is higher

than the sum of the verdicts at terminal histories  $\{(q, s), h\} \in \Gamma(\{(q, s)\})$  with  $S(\{(q, s), h\}) = k_s^*$  (since each such history occurs with equal probability). This comparison does not depend on  $i$ .  $\square$

### Proof of Theorem 2

In this proof, we proceed backwards from period  $T$  altering each deterministic test with random verdicts in the support of  $\rho$  in a way that only question  $q$  is assigned without reducing the payoff of the principal. The result then follows from Lemma 2.

Consider first a period  $T$  history  $h_T$  together with an assigned question  $q_T$ . Let

$$v^\omega := \mathbb{E}[\mathcal{V}(\{h_T, (q_T, \omega)\}) \mid (\mathcal{T}, \mathcal{V}) \in \text{supp}(h_T, q_T)]$$

be the expected verdict following the outcome  $\omega \in \{s, f\}$  taken with the respect to the set of possible deterministic tests with random verdicts that the agent could be facing.

Suppose first that  $v^f \geq v^s$ . Then every type finds muddling optimal at  $(h_T, q_T)$  and gets expected verdict  $v^f$ . Replacing each deterministic test with random verdicts  $(\mathcal{T}, \mathcal{V}) \in \text{supp}(h_T, q_T)$  with another  $(\mathcal{T}', \mathcal{V}')$  that is identical except that  $\mathcal{T}'(h_T) = q$  and  $\mathcal{V}'(\{h_T, (q, s)\}) = \mathcal{V}'(\{h_T, (q, f)\}) = v^f$  does not alter the principal's or the agent's payoff and makes action 1 optimal at  $h_T$ .

Now suppose that  $v^s > v^f$ , so that action 1 is optimal for all types of the agent. Let  $\beta_1 := \max_{i' \leq i^*} \frac{\theta_{i'}(q_T)}{\theta_{i'}(q)}$ . If  $\beta_1 \leq 1$ , we replace each  $(\mathcal{T}, \mathcal{V}) \in \text{supp}(h_T, q_T)$  with  $(\mathcal{T}', \mathcal{V}')$  that is identical except that  $\mathcal{T}'(h_T) = q$ ,  $\mathcal{V}'(\{h_T, (q, s)\}) = \beta_1 v^s + (1 - \beta_1) v^f$  and  $\mathcal{V}'(\{h_T, (q, f)\}) = v^f$ . The change in expected payoff at history  $h_T$  is given by

$$\begin{aligned} & \theta_i(q) \left( \beta_1 v^s + (1 - \beta_1) v^f \right) + (1 - \theta_i(q)) v^f - \left( \theta_i(q_T) v^s + (1 - \theta_i(q_T)) v^f \right) \\ &= \theta_i(q_T) (v^s - v^f) \left( \frac{\theta_i(q)}{\theta_i(q_T)} \beta_1 - 1 \right) \\ &= \theta_i(q_T) (v^s - v^f) \left( \frac{\theta_i(q)}{\theta_i(q_T)} \max_{i' \leq i^*} \left\{ \frac{\theta_{i'}(q_T)}{\theta_{i'}(q)} \right\} - 1 \right). \end{aligned} \quad (6)$$

Since  $v^s - v^f > 0$ , it follows from Lemma 1 that the above is non-negative for  $i \leq i^*$  and non-positive for  $i > i^*$ .

Now suppose  $\beta_1 > 1$ . Let  $\beta_2 := 1 - \max_{i' \leq i^*} \frac{\theta_{i'}(q_T) - \theta_{i'}(q)}{1 - \theta_{i'}(q)}$  and observe that  $0 \leq \beta_2 \leq 1$  (with the latter inequality following from the assumption that  $\beta_1 > 1$ ). In this case, we replace each  $(\mathcal{T}, \mathcal{V}) \in \text{supp}(h_T, q_T)$  with  $(\mathcal{T}', \mathcal{V}')$  that is identical except that  $\mathcal{T}'(h_T) = q$ ,  $\mathcal{V}'(\{h_T, (q, s)\}) = v^s$  and  $\mathcal{V}'(\{h_T, (q, f)\}) = \beta_2 v^f + (1 - \beta_2) v^s$ . The change in expected payoff at history  $h_T$  is given by

$$\begin{aligned} & \theta_i(q) v^s + (1 - \theta_i(q)) \left( \beta_2 v^f + (1 - \beta_2) v^s \right) - \left( \theta_i(q_T) v^s + (1 - \theta_i(q_T)) v^f \right) \\ &= (\theta_i(q_T) - \theta_i(q)) (v^s - v^f) \left( \frac{1 - \theta_i(q)}{\theta_i(q_T) - \theta_i(q)} \max_{i' \leq i^*} \frac{\theta_{i'}(q_T) - \theta_{i'}(q)}{1 - \theta_{i'}(q)} - 1 \right). \end{aligned} \quad (7)$$

Note that for any  $i$  and  $i'$ ,  $\frac{1 - \theta_i(q_T)}{1 - \theta_i(q)} \geq \frac{1 - \theta_{i'}(q_T)}{1 - \theta_{i'}(q)}$  implies that  $\frac{\theta_i(q_T) - \theta_i(q)}{1 - \theta_i(q)} \leq \frac{\theta_{i'}(q_T) - \theta_{i'}(q)}{1 - \theta_{i'}(q)}$ , and so it follows from Lemma 1 that the above is non-negative for  $i \leq i^*$  and non-positive for  $i > i^*$ .

Repeating the above construction at all period  $T$  histories  $h_T \in \mathcal{H}_T$  yields a test such that all deterministic tests with random verdicts in its support assign question  $q$  at period  $T$  and the most

informative action is optimal for all types of the agent at all period  $T$  histories. Moreover, since this (weakly) raises the payoffs of good types and lowers those of bad types at all period  $T$  histories, it does not lower the principal's payoff.

We now proceed inductively backwards from period  $T - 1$ . For a given period  $1 \leq t \leq T - 1$ , we assume as the induction hypothesis that it is optimal for all types of the agent to choose the most informative action at all histories  $h_{t'} \in \mathcal{H}_{t'}$  for  $t < t' \leq T$  in all deterministic tests with random verdicts  $(\mathcal{T}, \mathcal{V})$  that are in the support of  $\rho$ . Additionally, we assume as part of the induction hypothesis that  $\mathcal{T}(h_{t'}) = q$  at all  $h_{t'} \in \mathcal{H}_{t'}$  for  $t < t' \leq T$ .

Now consider each period  $t$  history  $h_t \in \mathcal{H}_t$  and assigned question  $q_t$ . A consequence of the induction hypothesis is that it is without loss to assume that each  $(\mathcal{T}, \mathcal{V}) \in \text{supp}(h_t, q_t)$  (if nonempty), has the same verdict at each terminal history in  $\Gamma(h_t)$ . This follows because, as per the induction hypothesis, only question  $q$  is assigned in periods  $t + 1$  onwards in the subtree  $\Lambda(h_t)$ , and so the agent learns nothing further as the test progresses. In other words, it is equivalent to set the verdicts of each  $(\mathcal{T}, \mathcal{V}) \in \text{supp}(h_t, q_t)$  to be  $\mathcal{V}(h_{T+1}) = \mathbb{E}[\mathcal{V}'(h_{T+1}) \mid (\mathcal{T}', \mathcal{V}') \in \text{supp}(h_t, q_t)]$  for all  $h_{T+1} \in \Gamma(h_t)$ .

We now alter each  $(\mathcal{T}, \mathcal{V}) \in \text{supp}(h_t, q_t)$  so that question  $q$  is assigned at  $h_t$  and change the verdicts so that the most informative action is optimal for the agent at all histories in  $\Lambda(h_t)$ . First, observe that following the argument of Step 1 of Lemma 2, we can assume that the verdicts  $\mathcal{V}$  at terminal histories  $\Gamma(\{h_t, (q_t, s)\})$  and  $\Gamma(\{h_t, (q_t, f)\})$  satisfy the cutoff property of Observation 2.

Recall that a consequence of the above argument (Step 2 of Lemma 2) is that all types have the same optimal action at  $h_t$  since the same question  $q$  is assigned at all histories from  $t + 1$  onwards in the subtree  $\Lambda(h_t)$  and the verdicts satisfy the cutoff property. If the agent finds it optimal to muddle at  $h_t$ , then we can construct  $(\mathcal{T}', \mathcal{V}')$  which is identical to  $(\mathcal{T}, \mathcal{V})$  except that the verdicts at terminal histories  $\{h_t, (q_t, s), h'\} \in \Gamma(\{h_t, (q_t, s)\})$  are reassigned to those in  $\Gamma(\{h_t, (q_t, f)\})$  by setting  $\mathcal{V}'(\{h_t, (q_t, s), h'\}) = \mathcal{V}(\{h_t, (q_t, f), h'\})$ . This would make all types indifferent among all actions and would not change their payoffs or the payoff of the principal. Moreover, this replacement of verdicts makes the question at  $h_t$  irrelevant, so that we can replace  $q_t$  with  $q$  at  $h_t$  (and reassign the verdicts accordingly).

Now consider the case in which action 1 is optimal for all types at  $h_t$ . We now replace each  $(\mathcal{T}, \mathcal{V}) \in \text{supp}(h_t, q_t)$  by another test  $(\mathcal{T}', \mathcal{V}')$ . As in the argument for period  $T$  above, we consider two separate cases.

Let  $\beta'_1 := \max_{i' \leq i^*} \frac{\theta_{i'}(q_t)}{\theta_{i'}(q)}$ . First, suppose  $\beta'_1 \leq 1$ . Then, we take the test  $(\mathcal{T}', \mathcal{V}')$  to be identical to  $(\mathcal{T}, \mathcal{V})$  except that  $\mathcal{T}'(h_t) = q$  and the verdicts at the terminal histories  $\{h_t, (q, s), h'\} \in \Gamma(\{h_t, (q, s)\})$  are  $\mathcal{V}'(\{h_t, (q, s), h'\}) = \beta'_1 \mathcal{V}(\{h_t, (q_t, s), h'\}) + (1 - \beta'_1) \mathcal{V}(\{h_t, (q_t, f), h'\})$ . In words, we are replacing the verdicts following a success at  $h_t$  with a weighted average of the verdicts following a success and failure before the change.

For brevity, we define

$$u_i^s := u_i(\{h_t, (q_t, s)\}; \mathcal{T}, \mathcal{V}, \sigma_i^*) \quad \text{and} \quad u_i^f := u_i(\{h_t, (q_t, f)\}; \mathcal{T}, \mathcal{V}, \sigma_i^*)$$

to be the expected payoffs following success and failure, respectively, at  $h_t$  in test  $(\mathcal{T}, \mathcal{V})$ .

We now show that this change (weakly) raises payoffs of good types and lowers those of bad types. Since the most informative action is optimal in the modified test, the payoff of type  $i$  at  $h_t$  from  $(\mathcal{T}', \mathcal{V}')$  is

$$\theta_i(q) \left( \beta'_1 u_i^s + (1 - \beta'_1) u_i^f \right) + (1 - \theta_i(q)) u_i^f.$$

Following the same argument as for (7) (with  $u_i^s$  and  $u_i^f$  in place of  $v^s$  and  $v^f$ ), the change in expected payoff at history  $h_t$  is given by

$$\theta_i(q_t) \left( u_i^s - u_i^f \right) \left( \frac{\theta_i(q)}{\theta_i(q_t)} \max_{i' \leq i^*} \left\{ \frac{\theta_{i'}(q_t)}{\theta_{i'}(q)} \right\} - 1 \right),$$

which is non-negative for  $i \leq i^*$  and non-positive for  $i > i^*$ .

A similar construction can be used for the second case where  $\beta'_1 > 1$ . In this case, we take the test  $(\mathcal{T}', \mathcal{V}')$  to be identical to  $(\mathcal{T}, \mathcal{V})$  except that  $\mathcal{T}'(h_t) = q$  and the verdicts at the terminal histories  $\{h_t, (q, f), h'\} \in \Gamma(\{h_t, (q, f)\})$  are  $\mathcal{V}'(\{h_t, (q, f), h'\}) = \beta'_2 \mathcal{V}(\{h_t, (q, f), h'\}) + (1 - \beta'_2) \mathcal{V}(\{h_t, (q, s), h'\})$ , where  $\beta'_2 := 1 - \max_{i' \leq i^*} \frac{\theta_{i'}(q_t) - \theta_{i'}(q)}{1 - \theta_{i'}(q)}$ . In words, we are replacing the verdicts following a failure at  $h_t$  with a weighted average of the verdicts following a success and failure before the change.

As before, the difference in payoffs is

$$(\theta_i(q_t) - \theta_i(q)) \left( u_i^s - u_i^f \right) \left( \frac{1 - \theta_i(q)}{\theta_i(q_t) - \theta_i(q)} \max_{i' \leq i^*} \frac{\theta_{i'}(q_t) - \theta_{i'}(q)}{1 - \theta_{i'}(q)} - 1 \right),$$

which is non-negative for  $i \leq i^*$  and non-positive for  $i > i^*$ .

Repeating this construction at all period  $t$  histories completes the induction step, and therefore also the proof.  $\square$

### Proof of Theorem 3

Suppose that  $\pi_{\bar{i}} = \pi_{\bar{j}} = 0.5$ . Let  $\rho$  be a test for which  $\mathcal{T}(h) \equiv q$  for every  $(\mathcal{T}, \mathcal{V})$  in the support of  $\rho$ . Since  $\theta_{\bar{i}}(q) > \theta_{\bar{j}}(q)$ , for any strategy of type  $\bar{i}$ , there exists a strategy of type  $\bar{j}$  that generates the same distribution over terminal histories. In particular, it must be that  $v_{\bar{j}}(\rho) \geq v_{\bar{i}}(\rho)$ , which in turn implies that the principal's expected payoff is nonpositive.

Let  $q'$  be such that  $\theta_i(q') = 1 - \theta_i(q)$  for every  $i$ . Notice that  $q$  is more Blackwell informative than  $q'$  since (2) is satisfied with  $\alpha_s = 0$  and  $\alpha_f = 1$ .<sup>11</sup> Consider the test  $(\mathcal{T}', \mathcal{V}')$  such that  $\mathcal{T}'(h) \equiv q'$  and  $\mathcal{V}'(h) = 1$  if and only if  $h = ((q', s), \dots, (q', s))$ ; in words, the test always assigns  $q'$  and passes the agent if and only if she succeeds in every period. Given this test, the most informative strategy is optimal for the agent, and  $v_{\bar{i}}(\mathcal{T}', \mathcal{V}') > v_{\bar{j}}(\mathcal{T}', \mathcal{V}')$  since  $\theta_{\bar{i}}(q') > \theta_{\bar{j}}(q')$ . Therefore, the principal's expected payoff

$$0.5v_{\bar{i}}(\mathcal{T}', \mathcal{V}') - 0.5v_{\bar{j}}(\mathcal{T}', \mathcal{V}')$$

is positive, which in turn implies that this test is strictly better than any test that assigns  $q$  at every history.  $\square$

<sup>11</sup>The comparison between  $q$  and  $q'$  is weak in the sense that  $q'$  is also more Blackwell informative than  $q$ . An identical argument applies if instead  $q'$  solves (2) for some  $\alpha_s$  and  $\alpha_f$  satisfying  $0 < \alpha_s < \alpha_f < 1$ , in which case  $q$  is strictly more Blackwell informative than  $q'$ .

*Proof of Theorem 4*

We first show that the most informative strategy  $\sigma^N$  is optimal for the agent in some optimal test. Then we show that  $\sigma^N$  is also optimal for the agent in the ONST.

We show the first part by contradiction. Suppose  $\rho$  is an optimal test where there is at least one history where the most informative action is not optimal for the agent. We proceed backwards from period  $T$ , altering each deterministic test with random verdicts in the support of  $\rho$  in a way that both types find it optimal to choose the most informative action without reducing the payoff of the principal.

Consider first a period  $T$  history  $h_T$  together with an assigned question  $q_T$ . Let

$$v^\omega := \mathbb{E}[\mathcal{V}(\{h_T, (q_T, \omega)\}) \mid (\mathcal{T}, \mathcal{V}) \in \text{supp}(h_T, q_T)]$$

be the expected verdict following the outcome  $\omega \in \{s, f\}$  taken with the respect to the set of possible deterministic tests with random verdicts that the agent could be facing.

Suppose that muddling is optimal for some type  $\theta_i$ . Then it must be that  $v^f \geq v^s$ , which in turn implies that both types find muddling optimal and thereby get expected verdict  $v^f$ . Replacing each deterministic test with random verdicts  $(\mathcal{T}, \mathcal{V}) \in \text{supp}(h_T, q_T)$  with another  $(\mathcal{T}', \mathcal{V}')$  that is identical except that  $\mathcal{V}'(\{h_T, (q_T, s)\}) = v^f$  does not alter the payoffs of the principal or the agent and makes the most informative action optimal at  $h_T$ .

We now proceed inductively backwards from period  $T - 1$ . For a given period  $1 \leq t \leq T - 1$ , we assume as the induction hypothesis that it is optimal for all types of the agent to choose the most informative action at all histories  $h_{t'} \in \mathcal{H}_{t'}$  for  $t < t' \leq T$  in all deterministic tests with random verdicts  $(\mathcal{T}, \mathcal{V})$  that are in the support of  $\rho$ .

Now consider each period  $t$  history  $h_t \in \mathcal{H}_t$  and a question  $q_t$  such that the most informative action is not optimal for at least one type of the agent. If no such period  $t$  history exists, the induction step is complete. We now alter each  $(\mathcal{T}, \mathcal{V}) \in \text{supp}(h_t, q_t)$  so that  $a_t = 1$  is optimal for the agent at all histories in  $\Lambda(h_t)$ . We consider two separate cases:

- (1) Muddling is optimal for the good type, i.e.,  $\sigma_1^*(h_t) = 0$ .
- (2) Muddling is optimal for the bad type and the most informative action is optimal for the good type, i.e.,  $\sigma_2^*(h_t) = 0$  and  $\sigma_1^*(h_t) = 1$ .

In case (1), we replace each  $(\mathcal{T}, \mathcal{V}) \in \text{supp}(h_t, q_t)$  by  $(\mathcal{T}', \mathcal{V}')$  where the continuation test following the success is replaced by that following a failure. Formally,  $(\mathcal{T}', \mathcal{V}')$  is identical to  $(\mathcal{T}, \mathcal{V})$  except for the questions and verdicts in the subtree  $\Lambda(\{h_t, (q_t, s)\})$ . For each history  $\{h_t, (q_t, s), h'\} \in \Lambda(\{h_t, (q_t, s)\})$  in this subtree, the question assigned becomes  $\mathcal{T}'(\{h_t, (q_t, s), h'\}) = \mathcal{T}(\{h_t, (q_t, f), h'\})$ , and the verdict at each terminal history  $\{h_t, (q_t, s), h'\} \in \Gamma(\{h_t, (q_t, s)\})$  becomes  $\mathcal{V}'(\{h_t, (q_t, s), h'\}) = \mathcal{V}(\{h_t, (q_t, f), h'\})$ . Note that if we alter each  $(\mathcal{T}, \mathcal{V}) \in \text{supp}(h_t, q_t)$  in this way, the performance of the agent at  $h_t$  does not affect the expected verdict and so  $a_t = 1$  is optimal for both types. By the induction hypothesis, action 1 remains optimal for both types at all histories in the subtree  $\Lambda(h_t)$ . Finally, such an alteration does not affect the payoff of the good type and weakly decreases the payoff of the bad type at  $h_t$ , and therefore weakly increases the principal's payoff.

In case (2), we do the opposite and replace each  $(\mathcal{T}, \mathcal{V}) \in \text{supp}(h_t, q_t)$  by  $(\mathcal{T}', \mathcal{V}')$  where the continuation test following the failure is replaced by that following a success. Formally,  $(\mathcal{T}', \mathcal{V}')$  is identical to  $(\mathcal{T}, \mathcal{V})$  except for the questions and verdicts in the subtree  $\Lambda(\{h_t, (q_t, f)\})$ . For each history  $\{h_t, (q_t, f), h'\} \in \Lambda(\{h_t, (q_t, f)\})$  in this subtree, the question assigned becomes  $\mathcal{T}'(\{h_t, (q_t, f), h'\}) = \mathcal{T}(\{h_t, (q_t, s), h'\})$ , and the verdict at each terminal history  $\{h_t, (q_t, f), h'\} \in \Gamma(\{h_t, (q_t, f)\})$  becomes  $\mathcal{V}'(\{h_t, (q_t, f), h'\}) = \mathcal{V}(\{h_t, (q_t, s), h'\})$ . Once again, the performance of the agent at  $h_t$  does not affect the expected verdict and so  $a_t = 1$  is optimal for both types. By the induction hypothesis, action 1 remains optimal both types at all histories in the subtree  $\Lambda(h_t)$ . Finally, such an alteration neither increases the payoff of the bad type nor decreases the payoff of the good type at  $h_t$ , and therefore weakly increases the principal's payoff. This completes the induction step.

Finally, we show that  $\sigma^N$  is optimal for the agent in the ONST  $(\mathcal{T}^N, \mathcal{V}^N)$ . We prove the result by induction on  $T$ . The base case is trivial since  $\mathcal{V}^N(\{h_T, (\mathcal{T}^N(h_T), s)\}) \geq \mathcal{V}^N(\{h_T, (\mathcal{T}^N(h_T), f)\})$  for any history  $h_T \in \mathcal{H}_T$ , and so action 1 is optimal in the last period of the ONST (which is the only period when  $T = 1$ ).

As the induction hypothesis, we assume that the most informative action is always optimal for the agent when faced with the ONST and when the length of the test is  $T - 1$  or less. Thus, for the induction step, we need to argue that the most informative action is optimal for the agent in period 1 when the length of the test is  $T$ .

Accordingly, suppose the agent has a strict preference to muddle in period 1. We consider three separate cases:

- (1) The good type strictly prefers to muddle while the most informative action is optimal for the bad type; thus  $\sigma_1^*(h_t) = 0$  and  $\sigma_2^*(h_t) = 1$ .
- (2) The bad type strictly prefers to muddle while the most informative action is optimal for the good type; thus  $\sigma_2^*(h_t) = 0$  and  $\sigma_1^*(h_t) = 1$ .
- (3) Both types strictly prefer to muddle; thus  $\sigma_1^*(h_t) = \sigma_2^*(h_t) = 0$ .

Cases (1) and (2) can be handled in the same way as cases (1) and (2) from the first part of the proof. In case (1), the continuation test following a success is replaced by that following a failure. Given the strategy  $\sigma^N$ , this change strictly increases the payoff of the good type and weakly decreases the payoff of the bad type, contradicting the optimality of the ONST. For case (2), the continuation test following the failure can be replaced by that following a success providing the requisite contradiction.

Now consider case (3). Let  $h_2^s = \{(\mathcal{T}^N(h_1), s)\}$  and  $h_2^f = \{(\mathcal{T}^N(h_1), f)\}$ , and let  $\pi_i^N(h)$  denote the belief the principal assigns to the agent's type being  $\theta_i$  following history  $h$  under the assumption that the agent uses the most informative strategy  $\sigma^N$ . Note that group monotonicity implies that  $\pi_1(h_2^s) \geq \pi_1(h_2^f)$  (and equivalently,  $\pi_2(h_2^s) \leq \pi_2(h_2^f)$ ). If  $\pi_1(h_2^s) = \pi_1(h_2^f)$  then it must be that there is no question  $q$  satisfying  $\theta_1(q) \neq \theta_2(q)$ , for otherwise the ONST would assign such a question in the first period and  $\pi_1(h_2^s)$  would differ from  $\pi_1(h_2^f)$ . In that case, the result holds trivially. Thus we may assume that  $\pi_1(h_2^s) > \pi_1(h_2^f)$  and  $\pi_2(h_2^s) < \pi_2(h_2^f)$ .

By the optimality of the continuation test following a success, we have

$$\begin{aligned} & \pi_1^N(h_2^s)u_1(h_2^s; (\mathcal{T}^N, \mathcal{V}^N), \sigma_1^N) - \pi_2^N(h_2^s)u_2(h_2^s; (\mathcal{T}^N, \mathcal{V}^N), \sigma_2^N) \\ & \geq \pi_1^N(h_2^s)u_1(h_2^f; (\mathcal{T}^N, \mathcal{V}^N), \sigma_1^N) - \pi_2^N(h_2^s)u_2(h_2^f; (\mathcal{T}^N, \mathcal{V}^N), \sigma_2^N), \end{aligned}$$

since otherwise the principal would be better off replacing the continuation test after a success with that after a failure. Rearranging gives

$$\begin{aligned} & \pi_2^N(h_2^s)[u_2(h_2^f; (\mathcal{T}^N, \mathcal{V}^N), \sigma_2^N) - u_2(h_2^s; (\mathcal{T}^N, \mathcal{V}^N), \sigma_2^N)] \\ & \geq \pi_1^N(h_2^s)[u_1(h_2^f; (\mathcal{T}^N, \mathcal{V}^N), \sigma_1^N) - u_1(h_2^s; (\mathcal{T}^N, \mathcal{V}^N), \sigma_1^N)]. \end{aligned}$$

Similarly, by the optimality of the continuation test following a failure, we have

$$\begin{aligned} & \pi_1^N(h_2^f)[u_1(h_2^f; (\mathcal{T}^N, \mathcal{V}^N), \sigma_1^N) - u_1(h_2^s; (\mathcal{T}^N, \mathcal{V}^N), \sigma_1^N)] \\ & \geq \pi_2^N(h_2^f)[u_2(h_2^f; (\mathcal{T}^N, \mathcal{V}^N), \sigma_2^N) - u_2(h_2^s; (\mathcal{T}^N, \mathcal{V}^N), \sigma_2^N)]. \end{aligned}$$

Since  $\pi_1^N(h_2^s) > \pi_1^N(h_2^f)$  and  $u_1(h_2^f; (\mathcal{T}^N, \mathcal{V}^N), \sigma_1^N) > u_1(h_2^s; (\mathcal{T}^N, \mathcal{V}^N), \sigma_1^N)$  (since type  $\theta_1$  strictly prefers to muddle), the above two inequalities imply that

$$\begin{aligned} & \pi_2^N(h_2^s)[u_2(h_2^f; (\mathcal{T}^N, \mathcal{V}^N), \sigma_2^N) - u_2(h_2^s; (\mathcal{T}^N, \mathcal{V}^N), \sigma_2^N)] \\ & \geq \pi_2^N(h_2^f)[u_2(h_2^f; (\mathcal{T}^N, \mathcal{V}^N), \sigma_2^N) - u_2(h_2^s; (\mathcal{T}^N, \mathcal{V}^N), \sigma_2^N)]. \end{aligned}$$

Since  $u_2(h_2^f; (\mathcal{T}^N, \mathcal{V}^N), \sigma_2^N) > u_2(h_2^s; (\mathcal{T}^N, \mathcal{V}^N), \sigma_2^N)$  (since type  $\theta_2$  also strictly prefers to muddle), this inequality implies that  $\pi_2^N(h_2^s) \geq \pi_2^N(h_2^f)$ , a contradiction.  $\square$

## APPENDIX B. ADDITIONAL EXAMPLES

**Example 5.** This example shows that the most informative strategy is not always optimal in an ONST. In response, the principal may be able to improve on the ONST with a different test, even one that induces the same strategy for the agent.

Suppose there are three types ( $I = 3$ ) and three periods ( $T = 3$ ), with  $i^* = 2$ . The principal has two different questions,  $Q = \{q, q'\}$ , and the success probabilities are as follows:

	$q$	$q'$
$\theta_1$	1	.2
$\theta_2$	.2	.15
$\theta_3$	.1	.01

The principal's prior belief is

$$(\pi_1, \pi_2, \pi_3) = (.5, .1, .4).$$

Figure 3 depicts an ONST  $(\mathcal{T}^N, \mathcal{V}^N)$  for this environment. The intuition for the optimality of this test is as follows. The principal has a low prior probability that the agent's type is  $\theta_2$ . Question  $q$  is effective at distinguishing between types  $\theta_1$  and  $\theta_3$  as, loosely speaking, their ability difference is larger on that question. If there is a success on  $q$ , it greatly increases the belief that the type is  $\theta_1$ , and the principal will assign  $q$  again. Conversely, if there is a failure on question  $q$  (in any period),



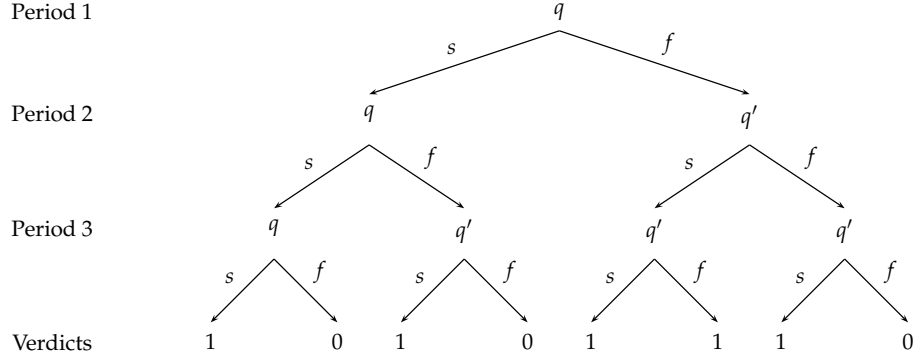


FIGURE 3. An ONST for Example 5.

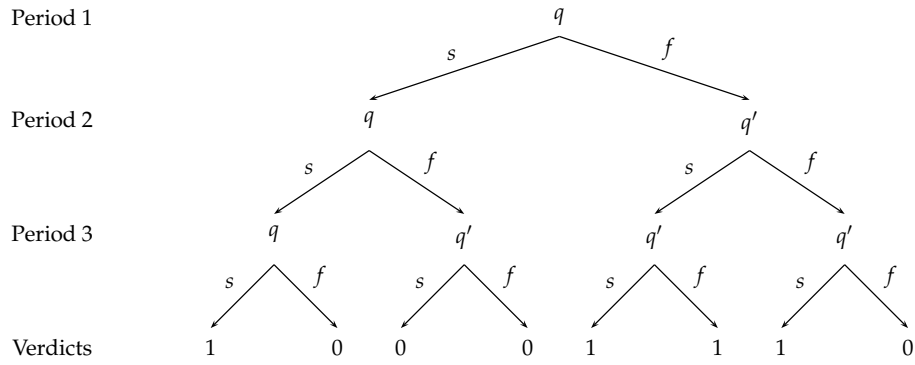


FIGURE 4. An optimal deterministic test for Example 5.

then the belief assigns zero probability to the agent having type  $\theta_1$ . The principal then instead switches to question  $q'$ , which is more effective than  $q$  at distinguishing between types  $\theta_2$  and  $\theta_3$ . Since  $\theta_3$  has very low ability on  $q'$ , a success on this question is a strong signal that the agent's type is not  $\theta_3$ , in which case the test issues a pass verdict.

Note that the most informative strategy  $\sigma^N$  is not optimal for type  $\theta_2$ : he prefers to choose action 0 in period 1 and action 1 thereafter. This is because his expected payoff at history  $h_2 = \{(q, s)\}$  is  $u_2(h_2; \mathcal{T}^N, \mathcal{V}^N, \sigma_2^N) = .2 * .2 + .8 * .15 = .16$ , which is lower than his expected payoff  $u_2(h'_2; \mathcal{T}^N, \mathcal{V}^N, \sigma_2^N) = 1 - .85 * .85 = .2775$  at the history  $h'_2 = \{(q, f)\}$ . Therefore, this example demonstrates that the most informative strategy is not always be optimal for the agent in an ONST.<sup>12</sup> The ability of the agent to behave strategically benefits the principal since  $\theta_2$  is a good type.

An optimal deterministic test  $(\mathcal{T}', \mathcal{V}')$  is depicted in Figure 4. Note that this test is identical to  $(\mathcal{T}^N, \mathcal{V}^N)$  except that the verdict at terminal history  $\{(q, s), (q, f), (q', s)\}$  is 0 as opposed to 1. In this test, types  $\theta_1$  and  $\theta_3$  choose the most informative strategy and type  $\theta_2$  chooses action 0 in period 1 and action 1 subsequently. Note that the expected payoff of type  $\theta_1$  remains unchanged relative to the ONST but that of type  $\theta_3$  is strictly lower. The payoff of type  $\theta_2$  is identical to

<sup>12</sup>Although the ONST is not unique, there is no ONST in this case for which  $\sigma^N$  is optimal.

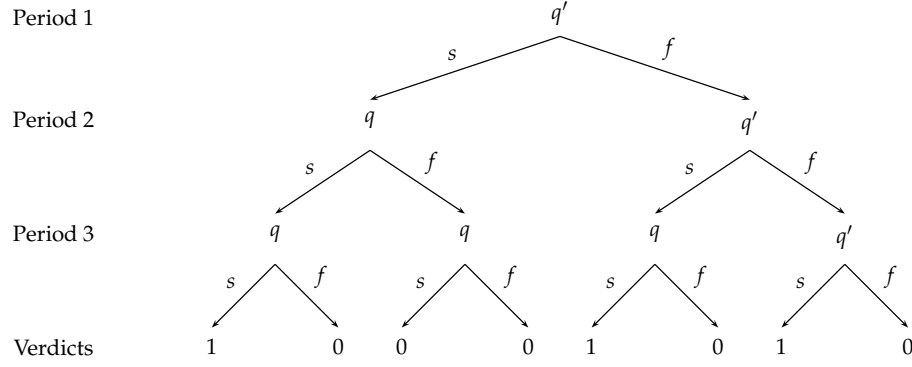


FIGURE 5. An ONST for Example 6.

what he receives from optimal play in  $(\mathcal{T}^N, \mathcal{V}^N)$ . Thus the payoff for the principal from the test  $(\mathcal{T}', \mathcal{V}')$  is higher than that from  $(\mathcal{T}^N, \mathcal{V}^N)$ .

**Example 6.** This example demonstrates that (i) strategic behavior by the agent can be harmful to the principal and yield her a lower payoff than when the agent chooses  $\sigma^N$  in the ONST, and (ii) the optimal deterministic test may differ from the ONST even if  $\sigma^N$  is optimal for the agent in the former (but not in the latter).

Suppose there are three types ( $I = 3$ ) and three period ( $T = 3$ ), with  $i^* = 1$  (so that type  $\theta_1$  is the only good type). The principal has two different questions,  $Q = \{q, q'\}$ , and the success probabilities are as follows:

	$q$	$q'$
$\theta_1$	1	.9
$\theta_2$	.85	.8
$\theta_3$	.8	0

The principal's prior belief is

$$(\pi_1, \pi_2, \pi_3) = (.4, .1, .5).$$

Figure 5 depicts an ONST  $(\mathcal{T}^N, \mathcal{V}^N)$ . The intuition for this ONST is as follows. The prior probability is such that type  $\theta_2$  is unlikely, and question  $q'$  is more effective at differentiating between types  $\theta_1$  and  $\theta_3$ . However, type  $\theta_3$  never succeeds at question  $q'$ , so as soon as a success is observed, the principal concludes that the agent's type must be either  $\theta_1$  or  $\theta_2$  and switches to question  $q$  (which is better at differentiating between these types).

Note that the most informative action is not optimal for the agent in this test: type  $\theta_2$  prefers to choose action 0 in period 1 because his expected payoff  $u_2(h_2; \mathcal{T}^N, \mathcal{V}^N, \sigma_2^N) = .85 * .85 = .7225$  at history  $h_2 = \{(q', s)\}$  is lower than  $u_2(h'_2; \mathcal{T}^N, \mathcal{V}^N, \sigma_2^N) = .8 * .85 + .2 * .8 = .84$  at the history  $h'_2 = \{(q', f)\}$ . This muddling lowers the principal's payoff since it increases the payoff of a bad type.

An optimal deterministic test  $(\mathcal{T}', \mathcal{V}')$  is depicted in Figure 6. In this test,  $\sigma^N$  is an optimal strategy for the agent. By definition, since the agent chooses the most informative strategy, this test must yield a lower payoff to the principal than she would obtain if the agent chose  $\sigma^N$  in the ONST.

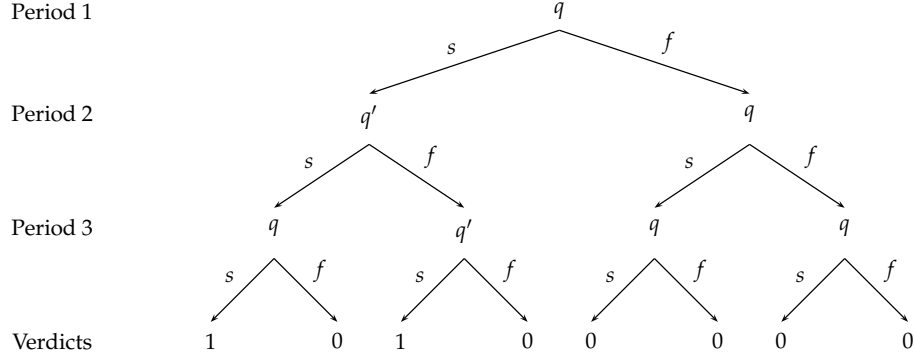


FIGURE 6. An optimal deterministic test for Example 6.

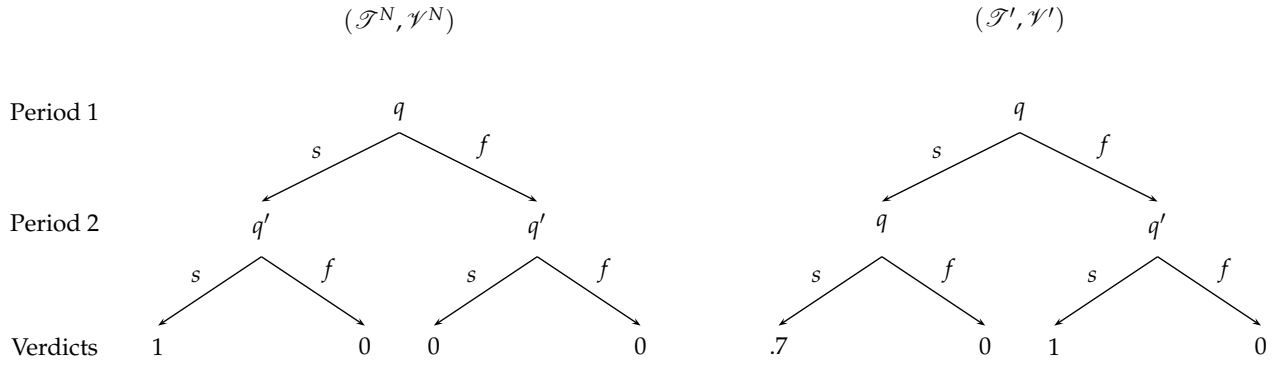


FIGURE 7. Tests for Example 7. The test on the left is an ONST and that on the right is an optimal test.

**Example 7.** The main purpose of this example is to demonstrate that the optimal test may employ a less informative question even if group monotonicity holds. In other words, Theorem 1 cannot be strengthened to state that less informative questions are not used in the optimal test when there does not exist a single most informative question. This example also shows that the principal can sometimes benefit from randomization: the optimal deterministic test in this case gives the principal a lower payoff than does the optimal test.

This example features three types ( $I = 3$ ) and two periods ( $T = 2$ ), with  $i^* = 2$  (so that type  $\theta_3$  is the only bad type). Suppose first that the principal has two different questions,  $Q = \{q, q'\}$ , with the following success probabilities:

	$q$	$q'$
$\theta_1$	.9	.5
$\theta_2$	.4	.35
$\theta_3$	.3	.21

The principal's prior belief is

$$(\pi_1, \pi_2, \pi_3) = (.02, .4, .58).$$

Figure 7 depicts, on the left, an ONST  $(\mathcal{T}^N, \mathcal{V}^N)$  (which is also an optimal deterministic test), and, on the right, an optimal test  $(\mathcal{T}', \mathcal{V}')$ . The test  $(\mathcal{T}', \mathcal{V}')$  differs from  $(\mathcal{T}^N, \mathcal{V}^N)$  in two ways: question  $q'$  at history  $\{(q, s)\}$  is replaced by question  $q$ , and the verdicts at both terminal histories involving a success in period 2 are changed. Note that, in period 1, types  $\theta_1$  and  $\theta_2$  strictly prefer actions  $a_1 = 1$  and  $a_1 = 0$ , respectively, whereas type  $\theta_3$  is indifferent.

The following simple calculations demonstrate why  $(\mathcal{T}', \mathcal{V}')$  yields the principal a higher payoff than does  $(\mathcal{T}^N, \mathcal{V}^N)$ . In  $(\mathcal{T}', \mathcal{V}')$ , the payoff of all three types is higher than in  $(\mathcal{T}^N, \mathcal{V}^N)$ . The differences in payoffs are

$$\begin{aligned}\Delta v_1 &= v_1(\mathcal{T}', \mathcal{V}') - v_1(\mathcal{T}^N, \mathcal{V}^N) = .9 * .9 * .7 + .1 * .5 - .9 * .5 = .167, \\ \Delta v_2 &= v_2(\mathcal{T}', \mathcal{V}') - v_2(\mathcal{T}^N, \mathcal{V}^N) = .35 - .4 * .35 = .21, \\ \text{and } \Delta v_3 &= v_3(\mathcal{T}', \mathcal{V}') - v_3(\mathcal{T}^N, \mathcal{V}^N) = .21 - .3 * .21 = .147.\end{aligned}$$

The change in the principal's payoff is

$$\sum_{i=1}^2 \pi_i \Delta v_i - \pi_3 \Delta u_3 = .02 * .167 + .4 * .21 - .58 * .147 > 0,$$

which implies that  $(\mathcal{T}', \mathcal{V}')$  is better than  $(\mathcal{T}^N, \mathcal{V}^N)$  for the principal.

Proving that  $(\mathcal{T}', \mathcal{V}')$  is optimal is more challenging; we provide a sketch of the argument here. Whenever there is a single bad type, there is an optimal test that satisfies at least one of the following two properties: (i) there is no randomization of questions in period two, or (ii) the bad type is indifferent among all actions in period 1. To see this, suppose, to the contrary, that the bad type has a strictly optimal action in period 1, and that the principal assigns probability  $\beta \in (0, 1)$  to  $q$  and  $1 - \beta$  to  $q'$  at one of the histories in period 2. Observe that, for a fixed strategy of the agent, the principal's payoff is linear in this probability  $\beta$ . Hence the principal can adjust  $\beta$  without lowering her payoff until either  $\theta_3$  becomes indifferent in period 1 or  $\beta$  becomes 0 or 1; any change in the strategies of types  $\theta_1$  and  $\theta_2$  resulting from this adjustment only benefits the principal more. Establishing that the optimal test must satisfy (i) or (ii) makes it possible to show that  $(\mathcal{T}', \mathcal{V}')$  is optimal by comparing the principal's payoffs from tests having one of these properties.

Now suppose the principal has at her disposal another question  $q''$  that satisfies

$$\theta_i(q'') = \theta_i(q) + \alpha(1 - \theta_i(q))$$

for all  $i \in \{1, 2, 3\}$  and some  $\alpha \in (0, 1]$ . Question  $q$  is more Blackwell informative than  $q''$  (one can take  $\alpha_s = 1$  and  $\alpha_f = \alpha$  in (2)).

The principal can now increase her payoff relative to  $(\mathcal{T}', \mathcal{V}')$  by using the less informative question  $q''$ . To see this, suppose the principal assigns  $q''$  instead of  $q$  in the first period, without changing questions and verdicts in period two. This change will not affect the payoffs or optimal strategies of types  $\theta_2$  and  $\theta_3$ ; the former still chooses  $a_1 = 0$ , and the latter remains indifferent among all actions. However, this change does increase the payoff of type  $\theta_1$  since this type strictly prefers the subtree after a success in period one to that after a failure, and question  $q''$  gives a

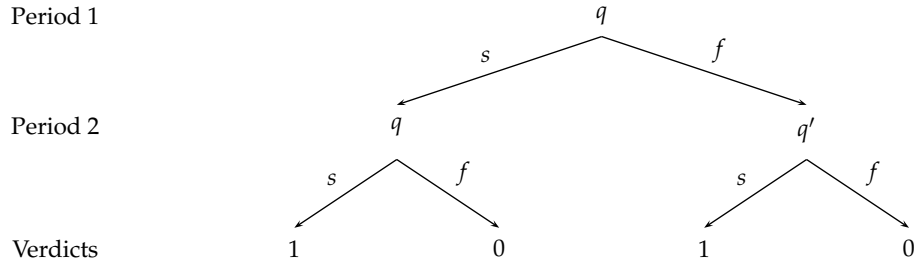


FIGURE 8. The optimal test for Example 8.

higher probability of reaching this subtree than does  $q$ . Therefore, this change increases the principal's payoff and demonstrates that any optimal test with the set of questions  $\{q, q', q''\}$  must employ  $q''$ .

**Example 8.** This example extends Example 5 by allowing the principal to offer a menu of tests. Recall that the success probabilities are

	$q$	$q'$
$\theta_1$	1	.2
$\theta_2$	.2	.15
$\theta_3$	.1	.01

and the prior is

$$(\pi_1, \pi_2, \pi_3) = (.5, .1, .4).$$

Suppose that there are only two periods ( $T = 2$ ).

The test depicted in Figure 8 is the optimal deterministic test (and also the ONST). Observe that, in this test, a failure in period 1 results in a harder question and that a success in period 2 is required to pass. Types  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$  pass with probabilities 1,  $.2 * .2 + .8 * .15 = .16$ , and  $.1 * .1 + .9 * .01 = .019$ , respectively.

Now suppose the principal instead offers the two-test menu  $\{(\mathcal{T}_1, \mathcal{V}_1), (\mathcal{T}_2, \mathcal{V}_2)\}$  depicted in Figure 9. Note that the test  $(\mathcal{T}_1, \mathcal{V}_1)$  only assigns the easier question,  $q$ , and two successes are required to pass. In contrast, test  $(\mathcal{T}_2, \mathcal{V}_2)$  assigns only the harder question,  $q'$ , but a single success in either period is sufficient to pass. It is optimal for type  $\theta_1$  to choose  $(\mathcal{T}_1, \mathcal{V}_1)$  and then use the most informative strategy as doing so enables him to pass with probability 1. Types  $\theta_2$  and  $\theta_3$  prefer to choose  $(\mathcal{T}_2, \mathcal{V}_2)$  and then use the most informative strategy. For types  $\theta_2$  and  $\theta_3$ , the passing probabilities are  $.2 * .2 = .04$  and  $.1 * .1 = .01$ , respectively, in test  $(\mathcal{T}_1, \mathcal{V}_1)$ , which are lower than the corresponding passing probabilities  $.15 + .85 * .15 = .2775$  and  $.01 + .99 * .01 = .0199$  in test  $(\mathcal{T}_2, \mathcal{V}_2)$ .

Note that in this menu, the payoffs of types  $\theta_2$  and  $\theta_3$  go up relative to what they obtain in the optimal test. However, the gain for type  $\theta_2$  is much larger than for  $\theta_3$ , making the principal better

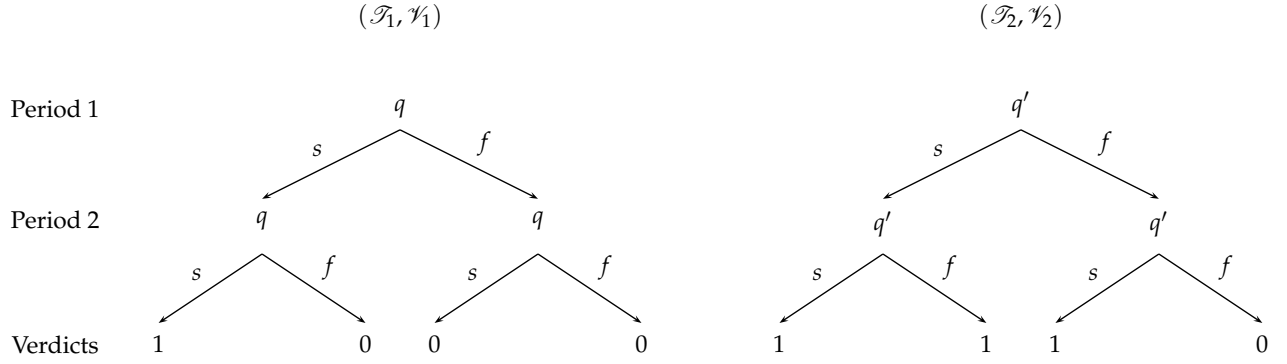


FIGURE 9. Menu of tests for Example 8.

off overall. In other words, the menu strictly increases the principal's payoff above that from the optimal test.

**Example 9.** Consider the following minor modification of the success probabilities from Example 4:

	$q$	$q'$
$\theta_1$	.999	.5
$\theta_2$	.5	.5
$\theta_3$	.5	.4

Note that the only change is that we have replaced  $\theta_1(q) = 1$  by  $\theta_1(q) = .999$ . The prior remains unchanged. Since the payoffs are continuous in these probabilities, this minor modification affects neither the ONST nor the optimal test.

Suppose the optimal test could be implemented without commitment. Recall that type  $\theta_1$  chooses the most informative strategy, whereas types  $\theta_2$  and  $\theta_3$  choose  $a_t = 0$  in periods 1 and 2. This implies that the terminal histories  $\{(q, s), (q, f), (q', s)\}$  and  $\{(q, s), (q, f), (q', f)\}$  are never reached by  $\theta_2$  and  $\theta_3$  in equilibrium. However, there is a positive (albeit small) probability that these terminal histories are reached by type  $\theta_1$ . Therefore, a sequentially rational principal would assign verdicts 1 (instead of 0) at both of these terminal histories, which would in turn make the most informative action optimal for types  $\theta_2$  and  $\theta_3$  in the first period.

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