Delay in Bargaining with Outside Options

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Abstract

A seller negotiates price with a buyer who has an outside option that arrives at a random time during the negotiation. Both the buyer's valuation of the good and the value of the outside option are unknown to the seller. We show that the interplay between information asymmetry and outside options is a source of delay in bargaining. In the seller-optimal bargaining mechanism, the seller and the buyer delay in reaching an agreement with positive probability. A delay occurs even in the limit as the arrival rate of the outside option goes to infinity. If the seller cannot commit to the seller-optimal bargaining mechanism, the same outcome is approximately achieved in a perfect Bayesian equilibrium of the bargaining game in which the seller makes all offers.

Keywords: asymmetric information, bargaining, commitment, delay, outside option

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1 Introduction

In many real-world negotiations, parties often experience a delay. Wage negotiations can become deadlocked as workers hold out or even strike. Plea bargaining typically involves a delay, and litigants often end up in court even after long negotiations. Delays are also commonplace in business-to-business negotiations. For example, Apple reportedly delayed launching its live TV service in 2015 as price negotiations with content providers stalled.[1] Such delays in negotiations often entail immense costs for both sides. Workers and employers suffer lost income and profit during a strike action, and litigants also could save a vast amount of legal fees by reaching an earlier settlement.

Why are negotiation parties unable to settle earlier despite the costs of disputes? Economists have attempted to explain it with asymmetric information. For example, the buyer in price negotiations may hold out to signal that he or she is a tough negotiator in the sense of having a low reservation value. However, the *no-haggling result* (Riley and Zeckhauser, 1983) and the *Coase conjecture* (Coase, 1972) have identified the fundamental difficulty with this program; the two observations, which are applicable for the situations with and without the seller's commitment respectively, show that no delay occurs even with asymmetric information.[2]

This paper proposes and analyzes a new source of delays in bargaining—*outside options*. We consider a price negotiation during which an outside option randomly arrives to the buyer. In each period, the seller first quotes a price. Then, the buyer can accept the seller's offer or exercise the outside option if it has already arrived; otherwise, the buyer delays her decision by one period. The arrival time of the outside option is private information of the buyer, as are the buyer's valuation of the seller's good and the value of the outside option. We will focus on the continuous-time limit as the time between two consecutive offers becomes arbitrarily small.

We consider negotiations both with and without the seller's commitment and show that a delay in reaching agreement is present in both cases.[3] Notably, a delay is beneficial to the seller, and the seller's most profitable equilibrium necessarily involves a delay. Both results contrast the bargaining outcome without outside options, where either the seller cannot benefit from a delay (no-haggling result) or the seller cannot credibly delay his agreement even if it is beneficial (Coase conjecture).

A comparative static further highlights the importance of the outside option. We compare the *frictionless case* and the *frictional case*. In the frictionless case, the outside option is available from the beginning to the end. In the frictional case, however, the arrival rate is finite,

[1] Burrows, Shaw, and Smith (2015), "Apple Said to Delay Live TV Service to 2016 as Negotiations Stall," *Bloomberg Business*, `http://bloom.bg/1Pan9aH` (accessed October 9, 2015).

[2] The no-haggling result shows that the seller's profit-maximizing scheme is to commit to a fixed price, and the Coase conjecture proposes no delay occurs even if the seller cannot commit to a fixed price. The Coase conjecture has been confirmed for many bargaining situations by Gul, Sonnenschein, and Wilson (1986), Fudenberg, Levine, and Tirole (1987), among many others.

[3] We treat the buyer exercising the outside option as another way to resolve the bargaining situation. Accordingly, *delay of agreement* and *delay of settlement* refer to the case in which the buyer continues to decline the seller's offers while also not exercising her outside option.

hence the buyer has no outside option in the initial periods. When the seller has commitment, the friction in the outside option's arrival is found to be irrelevant. A delay is present in an equilibrium with or without friction. Moreover, the sets of the seller's equilibrium profits in two cases asymptotically coincide with one another as the arrival rate for the frictional case grows to infinity.

This equivalence breaks down once we turn to the non-commitment case. No delay ever occurs in the frictionless case without commitment, while a delay is still present in the frictional case. Equilibrium profits for the frictional case and the frictionless case also no longer agree, as the seller can achieve a higher profit level with friction. Moreover, this gap persists even if the outside option's arrival rate in the frictional case goes to infinity. The discontinuity between the frictional and frictionless cases has interesting practical significance, as it provides one rationale for a "launching-date war." The interests of a seller with no commitment make it important to launch its product and approach buyers as soon as possible, before any potential competitors who will serve as outside options to buyers.

Outside options are omnipresent in real-world bargaining situations, and economists have accordingly studied various bargaining models with outside options. Our model has two distinctive features that drive the main results. First, the value of the outside option is unknown to the seller. Second, the buyer has a withdrawal right. The buyer can always walk away to pursue her outside option in the middle of negotiations (which we refer to as the autarky strategy), and parties cannot sign a contract that forfeits the withdrawal right. In the language of the mechanism design literature, any equilibrium has to satisfy the buyer's ex post, type-dependent participation constraints.

The two-type example is helpful in understanding why a delay is beneficial to the seller even with commitment. Let the high-type refer to the type with the wider gap between the valuation of the good and the outside option, and suppose there is no friction in the outside option's arrival to make the argument simple. Also suppose the prior probability that the buyer is of the high-type is sufficiently large, so that the low-type has to be rationed in the seller's profit-maximizing mechanism (hereafter, optimal mechanism).

If the outside options are type-independent, the optimal mechanism completely excludes the low-type. If the mechanism ever attempts to trade with the low-type as well, the high-type could obtain a positive payoff from acting as if she were of the low-type. Hence, the incentive-compatibility constraint for the high-type requires the seller to shade the price offered to the high-type. However, as the buyer is more likely to be of the high-type, the loss from reducing the price for the high-type would surpass the gain from inviting the low-type. The seller is therefore satisfied by the mechanism trading with only the high-type.

With type-dependent outside options, by contrast, the seller can invite the low-type without conceding the profit from the high-type by committing to delaying the transaction with the low-type. The terms of trade offered to the the low-type (time and price) have to be such that the high-type is indifferent between accepting the offer made for her and acting as if she were the low-type. As long as the high-type's outside option exceeds the payoff from mimicking the low-type, this scheme is incentive compatible and improves on the mechanism involving no delay.

The buyer's withdrawal right is crucial for a delay to occur in bargaining with type-dependent outside options. Without the withdrawal right, all the buyer types obtain zero profit if the mechanism decides not to trade after the buyer participates in. In this case, the randomization over the two outcomes, trade-at-time-zero and no-trade, is as effective in rationing the low-type as a delayed trade. Hence, the seller can achieve the maximum profit with the randomization scheme instead of delay.

If the buyer has the withdrawal right as assumed in this paper, on the contrary, the delayed offer is more effective than the randomization scheme. Note that, with the withdrawal right, the buyer will still be able to exercsie the outside option as soon as the randomization scheme is resolved as "no-trade." However, under the delayed-offer scheme, the buyer has to suspend the outside option to subsequently accept the seller's delayed offer. Ex post participation constraints for the buyer is therefore more relaxed under the delayed-offer scheme, and hence the optimal mechanism has to involve a delay rather than randomization.

In the frictional case, as the arrival rate goes to infinity, the seller can asymptotically attain all of the profit from the optimal mechanism even without commitment. Moreover, a delay is present in the seller's most profitable equilibrium, as in the case with commitment. The basic idea is that we can use the *Coasian equilibrium*, one in which the Coase conjecture holds, as the punishment scheme for any deviation by the seller. To understand why the Coasian equilibrium is guaranteed to exist, note that the buyer and the seller can bargain as if there were no outside option in a very narrow time interval when there is friction in the outside option's arrival. The logic of the Coase conjecture for negotiation without outside options is then operative during this narrow time window.

However, we cannot employ the Coasian equilibrium to construct an equilibrium for the frictionless case because it does not exit in the first place. Indeed, as Board and Pycia (2014) note, there exists no equilibrium on whose path a delay is present; all buyer types either immediately accept the seller's offer or exercise the outside option. Moreover, the seller's profit from the most profitable equilibrium is strictly lower than the profit under the optimal mechanism with commitment.

1.1 Related Literature

This article is based on the literature on bargaining with asymmetric information. Most papers in this literature focus on the case without outside options. When the seller (uninformed party) has commitment power, Riley and Zeckhauser (1983) and Samuelson (1984) show that the uninformed party's profit-maximizing bargaining mechanism commits to a take-it-or-leave-it offer; hence, no delay occurs. The *Coase conjecture* predicts that the no-haggling result also holds without commitment. Stokey (1981), Sobel and Takahashi (1983), Fudenberg, Levine, and Tirole (1985), Gul, Sonnenschein, and Wilson (1986), and Ausubel and Deneckere (1989b) demonstrate the existence of at least one equilibrium for which the Coase conjecture holds, which is often also a unique equilibrium.

Several papers have attempted to counter the no-haggling result and the Coase conjecture using different sources of delays such as interdependent values (Deneckere and Liang, 2006; Evans, 1989; Hörner and Vieille, 2009; Vincent, 1989) and reputation building (Abreu and Gul, 2000; Myerson, 1991). Perry and Admati (1987) consider a bargaining game with asymmetric information in which players can endogenously choose their response time to the counterparty's offer and identify a sequential equilibrium involving delayed agreement.

Economists have considered the strategic role of outside options in bargaining at least since the seminal work by Nash (1953), which was followed by Shaked and Sutton (1984), Binmore (1985), Binmore, Shaked, and Sutton (1989), Muthoo (1995), and Chatterjee and Lee (1998), among many others. A few recent papers incorporate the buyer's (informed party's) type-dependent outside option into the classical asymmetric information bargaining model. Board and Pycia (2014) and Hwang (2015) are the two papers most similar to the current article. Combined, the findings of the two papers demonstrate the existence of an equilibrium for which the Coase conjecture in its strongest sense fails without the seller's commitment power; the seller's equilibrium profit is higher than the Coase conjecture's prediction. However, in terms of the bargaining dynamics, no delay occurs in either paper's models when the friction in the arrival process for the outside option is either very small or absent.[4]

Some other papers consider different types of outside options in bargaining. Rubinstein and Wolinsky (1985), Gale (1986), Fudenberg, Levine, and Tirole (1987), Bester (1988), Fuchs and Skrzypacz (2010), Atakan and Ekmekci (2014), and Chang (2015), among many others, study bargaining models in which outside options are endogenously formed in equilibrium, whereas this paper assumes that outside options are exogenously given from outside the model. Lee and Liu (2013) study a repeated bargaining game between an informed long-run player and a sequence of uninformed short-run players, where the long-run player has a stochastic outside option that is implemented conditional on failing to reach an agreement. Lee and Liu focus on the incentive of the informed party (long-run player) to develop a reputation by gambling with the outside option, while this paper focuses on how the uninformed party can use the informed party's outside option for screening.

This paper is also related to mechanism design papers devoting particular attention to informed agents' outside options. Jullien (2000) and Rochet and Stole (2002) consider the optimal nonlinear pricing scheme of a monopoly when buyers have type-dependent outside options. Whereas these two papers consider the case in which the buyer lacks a withdrawal right, the current paper assumes that the buyer can withdraw in the middle of negotiations. The mechanism design problem with a withdrawal right is also recently analyzed by Krähmer and Strausz (2015).

Finally, we would like to emphasize that this article analyzes bargaining situations both with

[4]In Hwang (2015), there is an equilibrium with a delay when the friction level is intermediate, but it vanishes as the friction becomes arbitrarily small.

and without commitment power of the uninformed party. Mechanism design largely neglects sequential rationality by focusing on the commitment case, while the sequential bargaining literature, following Rubinstein (1982), focuses on the case without commitment. Among related works, Ausubel and Deneckere (1989a) and Gerardi, Hörner, and Maestri (2014) consider bargaining situations with private valuation and interdependent valuation, respectively, and both of them compare the bargaining outcomes with and without commitment. Although all three papers, including the current article, have a different context and focus, one may derive the common message that the uninformed party's commitment may not make a decisive difference in bargaining outcomes, which contrasts with common sense.

The paper is organized as follows. Section 2 formally sets up the bargaining situation studied in this article. Sections 3 and 4 constitute the main sections and analyze bargaining with and without commitment, respectively. Section 5 discusses alternative assumptions concerning the outside option's arrival process, and Section 6 concludes the article. Appendix A contains proofs omitted from the main text.

2 Environment

A seller attempts to sell a durable good to a buyer in periods $n = 1, 2, \ldots$.[5] The interval between two consecutive periods is $\Delta > 0$. The seller's valuation of the good is normalized to be zero without loss of generality. The buyer's valuation of the good depends on her type $\theta \in \Theta$ and is denoted by $v_\theta \in [\underline{v}, \bar{v}] \subset \mathbb{R}_+$. The buyer also has a type-dependent outside option, the value of which is $w_\theta > 0$. The buyer's type is private information, and the seller's prior probability of type θ is commonly known to be $q_\theta \in [0, 1]$. We assume $|\Theta| < \infty$ and $v_\theta \geq w_\theta > 0$ for any $\theta \in \Theta$. Moreover, there is at least one buyer type θ such that $v_\theta > w_\theta$.[6]

The buyer's outside option (randomly) arrives during negotiations. The history of the outside option's availability is denoted $a = (a_1, a_2, \ldots, a_n, \ldots) \in A := \{0, 1\}^{\mathbb{N}}$ where $a_n = 1$ if and only if the outside option is available in period $n \geq 1$. We will consider both frictional and frictionless cases.

1. The frictional case:

$$\mathbb{P}\{a_1 = 1\} = \mathbb{P}\{a_{n+1} = 1 | a_n = 0\} = 1 - e^{-\lambda\Delta} \quad \forall n \geq 1, \quad \lambda \in (0, \infty) \tag{1}$$

2. The frictionless case:

$$\mathbb{P}\{a_n = 1\} = 1 \quad \forall n \geq 1, \quad \text{or equivalently} \quad \lambda = \infty. \tag{2}$$

[5] Throughout the paper, we use male pronouns for the seller and female pronouns for the buyer.

[6] The model is mathematically equivalent to a durable good monopolist facing a continuum of consumers with outside options.

$\lambda \in (0, \infty]$ is referred to as the *arrival rate (of outside options)*. We interpret a finite arrival rate as friction in the buyer's locating her outside option. Once the outside option arrives, it remains perpetually available:

$$\mathbb{P}\{a_k = 1 | a_n = 1\} = 1 \quad \forall k \geq n \geq 1. \tag{3}$$

At the beginning of each period $n \geq 1$, the buyer privately observes $a_n \in \{0, 1\}$, and the seller then offers a price $p_n \geq 0$. We consider two cases that differ in the seller's commitment power.

- Bargaining with commitment (discussed in Section 3): p_n is determined by a mechanism that the seller committed to prior to the negotiation. Two parties can freely communicate in mechanisms, and p_n is contingent on communication.
- Bargaining without commitment (discussed in Section 4): The seller cannot bind himself to any mechanism. The seller also cannot communicate with the buyer. p_n is therefore contingent only on a history of prices rejected in the past.

In either case, the buyer then chooses whether to accept p_n, exercise her outside option (this action is available only if $a_n = 1$), or delay her decision. If the buyer accepts p_n or exercises her outside option, the game ends immediately; if the buyer delays, the game moves on to the next period and the same procedure repeats. Note that the buyer has a withdrawal right; the buyer can exercise her outside option in any period after its arrival, which stands in contrast to the standard mechanism design environment.

A bargaining outcome is a triplet $(b, p, n) \in O := \{T, W\} \times [0, \bar{v}] \times \mathbb{N}$. b denotes the nature of the agreement between two parties, $b = T$ if they trade the seller's good, and $b = W$ if the buyer walks away to take the outside option. p and n represent the payment from the buyer to the seller and the period in which the negotiation terminates, respectively. The payoff of buyer type θ from the outcome $(b, p, n) \in O$ is

$$U_\theta^B(b, p, n) = e^{-r(n-1)\Delta}\Big(v_\theta \cdot \mathbb{1}\{b = T\} + w_\theta \cdot \mathbb{1}\{b = W\} - p\Big) \tag{4}$$

and the seller's payoff is

$$U^S(b, p, n) = e^{-r(n-1)\Delta} p \tag{5}$$

where $r > 0$ is the common discount rate and $\mathbb{1}\{\cdot\}$ is the indicator function.

One of the buyer's feasible strategies, which we refer to as the "autarky strategy," is to reject all offers from the seller and then exercise her outside option as soon as possible. The buyer's expected payoff in this case is

$$\pi_\theta^B = \begin{cases} \sum_{k \geq 0} e^{-r\Delta} e^{-\lambda k \Delta}(1 - e^{-\lambda\Delta}) w_\theta = \frac{1 - e^{-\lambda\Delta}}{1 - e^{-(\lambda + r)\Delta}} w_\theta & \text{if } \lambda < \infty \\ w_\theta & \text{if } \lambda = \infty \end{cases}$$

and constitutes a lower bound for the buyer's equilibrium payoff. Because $v_\theta \geq w_\theta > 0$ for all $\theta \in \Theta$,

$$U^B_\theta(T, p, 1) + U^S(T, p, 1) = v_\theta \geq \underline{\pi}^B_\theta \quad \forall \theta \in \Theta, \quad \forall p \geq 0 \tag{6}$$

with strict inequality at least for one θ. Hence, immediate trade with the seller is a unique first-best outcome.

In the following sections, both outcomes T and W are considered an *agreement* or *settlement* between two negotiation parties. *Delay of agreement* and *delay of settlement* refer to the case in which the buyer and the seller cannot make any agreement in the first period. The observation (6) indicates that any delay of agreement results in an inefficient outcome.

The model has two sources of bargaining friction. First, the seller can make only one offer in every period; hence, he has to wait $\Delta > 0$ for the next opportunity to revise his offer. Second, unless λ is exactly infinity, there is also friction in the buyer's locating her outside option. In the following sections, we focus on the case in which the two sources of friction vanish. Formally, we will focus on the limiting case in which $\Delta > 0$ approaches zero and λ goes to infinity or λ is exactly infinity. Whenever the order of limits matters, we first take Δ to zero and then λ to infinity.

3 Delay in Bargaining with Commitment

3.1 Mechanisms

Suppose that the seller can commit to any mechanism prior to the negotiation. Formally, a mechanism is a probability transition

$$\mu : \bigcup_{n \geq 1} \left[\left(\prod_{k=1}^{n} M_k \right) \times [0, \bar{v}]^{n-1} \right] \to \triangle[0, \bar{v}]$$

where M_n represents the set of messages that the buyer can report in period $n \geq 1$. For any $n \geq 1$ and $(m^n, p^{n-1}) \in \prod_{k=1}^n M_k \times [0, \bar{v}]^{n-1}$, $\mu(m^n, p^{n-1})$ is the probability distribution of the mechanism's offer p_n in period n, conditional on a history of the buyer's messages m^n and mechanism's past offers p^{n-1}. For any Borel set $B \subset [0, \bar{v}]$, the probability of $p_n \in B$ is denoted by $\mu(B|m^n, p^{n-1}) \in [0, 1]$.[7] By the revelation principle, we can focus on direct mechanisms such that $M_1 = \Theta \times \{0, 1\}$ and $M_n = \{0, 1\}$ for all $n \geq 2$, where "0" (respectively "1") indicates that the outside option is not yet available (respectively, available).

After the seller commits to a mechanism μ and the buyer privately learns her type $\theta \in \Theta$, the following events occur in order in each period $n \in \mathbb{N}$, conditional on history (m^{n-1}, p^{n-1}) and the buyer's private information (θ, a^{n-1}).

[7] Let a superscript "0" stand for a null history. That is, p^0, m^0, and a^0 represent a null history of offers, a null history of messages, and a null history of the outside option's availability, respectively. Finally, abusing notation, let $[0, \bar{v}]^0 \equiv \{p^0\}$.

1. The buyer privately learns a_n and then chooses $m_n \in M_n$ according to a probability distribution $\eta_\theta(m^{n-1}, p^{n-1}; a^n) \in \triangle M_n$.

2. The mechanism chooses its offer $p_n \in \text{supp } \mu(m^n, p^{n-1})$.

3. The buyer accepts the mechanism's offer p_n or exercises the outside option with probabilities $\chi_\theta(p_n|m^n, p^{n-1}; a^n)$ and $\xi_\theta(p_n|m^n, p^{n-1}; a^n) \in [0, 1]$, respectively.

The mechanism terminates when the buyer accepts an offer or exercises the outside option.

Let $\mathcal{M}$ denote the set of direct mechanisms, and let $\mathcal{D}$ denote the buyer's set of feasible decision rules. Each mechanism in $\mathcal{M}$ is generically denoted by μ, and each decision rule in $\mathcal{D}$ is generically denoted by $\sigma_B \equiv (\sigma_\theta)_\theta$, where σ_θ refers to the decision rule of buyer type θ, which consists of η_θ, χ_θ, and ξ_θ as described above. A mechanism $\mu \in \mathcal{M}$, together with the decision rule $\sigma_B \in \mathcal{D}$ of the buyer, gives rise to a probability measure $F^{\mu,\sigma_B}(\cdot|\theta, k) \in \triangle O$ over negotiation outcomes conditional on the buyer type being θ and the outside option arriving in period $k \geq 1$. The expected payoff of the seller and the expected payoff of buyer type θ are respectively

$$\pi_S(\mu, \sigma_B; \lambda, \Delta) := \sum_{\theta \in \Theta} \sum_{k \geq 1} q_\theta \psi_k(\lambda, \Delta) \int_O U^S(b, p, n) \mathrm{d}F^{\mu,\sigma_B}(b, p, n|\theta, k)$$

and

$$\pi_\theta(\mu, \sigma_B; \lambda, \Delta) := \sum_{k=1}^{\infty} \psi_k(\lambda, \Delta) \int_O U^B_\theta(b, p, n) \mathrm{d}F^{\mu,\sigma_B}(b, p, n|\theta, k) \quad \forall \theta \in \Theta.$$

where

$$\psi_k(\lambda, \Delta) := \begin{cases} e^{-\lambda(k-1)\Delta}(1 - e^{-\lambda\Delta}) & \text{if } \lambda < \infty \\ \mathbb{1}\{k = 1\} & \text{if } \lambda = \infty \end{cases}$$

is the probability mass function for the outside option's arrival time. $\sigma_B = (\sigma_\theta)_{\theta \in \Theta}$ is called *admissible for* μ if each σ_θ maximizes $\sum_{k=1}^\infty \psi_k(\lambda, \Delta) \int_O U^B_\theta(b, p, n) \mathrm{d}F^{\mu,\sigma_B}(b, p, n|\theta, k)$ over all possible decision rules.

Let $\Pi^{\lambda,\Delta} \subset \mathbb{R}_+^{|\Theta|+1}$ be the set of payoff profiles attainable by a direct mechanism for any $\lambda \in (0, \infty]$ and $\Delta \in (0, \infty)$:

$$\Pi^{\lambda,\Delta} = \left\{ (\tilde{\pi}_S, (\tilde{\pi}_\theta)_{\theta \in \Theta}) \in \mathbb{R}_+^{|\Theta|+1} \,\middle|\, \begin{array}{c} \tilde{\pi}_S = \pi_S(\mu, \sigma_B; \lambda, \Delta), \\ \tilde{\pi}_\theta = \pi_\theta(\mu, \sigma_B; \lambda, \Delta) \quad \forall \in \Theta, \\ \mu \in \mathcal{M} \quad \text{and} \quad \sigma_B \text{ is admissible for } \mu \end{array} \right\}$$

The next lemma characterizes $\Pi^{\lambda,\Delta}$ when $\lambda = \infty$, in which case information asymmetry regarding the outside option's arrival time is eliminated. The seller is only uncertain about (v_θ, w_θ), and hence, the bargaining problem degenerates into a multi-dimensional screening problem.

Lemma 1. $(\tilde{\pi}_S, (\tilde{\pi}_\theta)_{\theta\in\Theta}) \in \Pi^{\infty,\Delta}$ *if and only if there exists* $(x_\theta, y_\theta, p_\theta)_{\theta\in\Theta} \geq 0$ *such that*

$$\tilde{\pi}_S = \sum_{\theta\in\Theta} q_\theta p_\theta, \tag{7}$$

$$\tilde{\pi}_\theta = x_\theta v_\theta + y_\theta w_\theta - p_\theta \geq w_\theta \tag{8}$$

$$x_\theta v_\theta + y_\theta w_\theta - p_\theta \geq x_{\theta'} v_\theta + y_{\theta'} w_\theta - p_{\theta'} \tag{9}$$

$$0 \leq y_\theta \leq 1 - x_\theta \leq 1 \tag{10}$$

for any $\theta, \theta' \in \Theta$.

Note that the program (7)-(10) in Lemma 1 is related to the problem of a multi-product monopolist studied by McAfee and McMillan (1988), Thanassoulis (2004), Manelli and Vincent (2006, 2007), Pycia (2006), and Pavlov (2011), among others. In this vein, the constraints (8) and (9) correspond to individual-rationality and incentive-compatibility constraints in these studies. What distinguishes the program (7)-(10) from the multi-product monopolist's problem is the particular way in which the incentive-compatibility and individual-rationality constraints are connected through the buyer's type-dependent outside option. Here, the value of the outside option w_θ affects both the buyer's payoff from withholding from the transaction and the payoff from mimicking other buyer types, whereas the literature has focused on the case in which all buyer types' outside options are zero.

The program (7)-(10) is also connected with the nonlinear pricing problem with the buyer's type-dependent outside option. Again, the appearance of w_θ in both the incentive-compatibility and individual-rationality constraints contrasts the program with the nonlinear pricing literature. In models studied by Jullien (2000) and Rochet and Stole (2002), for example, the buyer cannot opt out for the outside option once she requests a quote from the seller; hence, the buyer's type-dependent outside option does not appear in the incentive-compatibility constraints.

To prove Lemma 1, first define for any mechanism μ, decision rule $\sigma_B \equiv (\sigma_\theta)_{\theta\in\Theta}$, and $\theta \in \Theta$,

$$\begin{aligned} x_\theta^{\mu,\sigma_B} &= \sum_{k=1}^{\infty} \psi_k(\lambda,\Delta) \int_{O_T} e^{-r(n-1)\Delta} \mathrm{d}F^{\mu,\sigma_B}(b,p,n|\theta,k) \\ y_\theta^{\mu,\sigma_B} &= \sum_{k=1}^{\infty} \psi_k(\lambda,\Delta) \int_{O_W} e^{-r(n-1)\Delta} \mathrm{d}F^{\mu,\sigma_B}(b,p,n|\theta,k) \end{aligned} \tag{11}$$

and

$$p_\theta^{\mu,\sigma_B} = \sum_{k=1}^{\infty} \psi_k(\lambda,\Delta) \int_{O} p \cdot e^{-r(n-1)\Delta} \mathrm{d}F^{\mu,\sigma_B}(b,p,n|\theta,k)$$

where $O_T := \{(b,n,p) \in O : b = T\}$ and $O_W := \{(b,n,p) \in O : b = W\}$, and hence

$$\pi_\theta(\mu,\sigma_B;\lambda,\Delta) = x_\theta^{\mu,\sigma_B} v_\theta + y_\theta^{\mu,\sigma_B} w_\theta - p_\theta^{\mu,\sigma_B} \quad \forall\theta\in\Theta.$$

Because each buyer type $\theta \in \Theta$ can always act as if she is another type $\theta' \neq \theta$,

$$x_\theta^{\mu,\sigma_B} v_\theta + y_\theta^{\mu,\sigma_B} w_\theta - p_\theta^{\mu,\sigma_B} \geq x_{\theta'}^{\mu,\sigma_B} v_\theta + y_{\theta'}^{\mu,\sigma_B} w_\theta - p_{\theta'}^{\mu,\sigma_B}$$

for any $\theta, \theta' \in \Theta$ if $\sigma_B = (\sigma_\theta)_{\theta\in\Theta}$ is admissible for μ. Hence, any $(\tilde{\pi}_S, (\tilde{\pi}_\theta)_{\theta\in\Theta}) \in \Pi^{\infty,\Delta}$ satisfies all the conditions in the last lemma with $(x_\theta, y_\theta, p_\theta)_{\theta\in\Theta} = (x_\theta^{\mu,\sigma_B}, y_\theta^{\mu,\sigma_B}, p_\theta^{\mu,\sigma_B})_{\theta\in\Theta}$.

Conversely, suppose that $(\tilde{\pi}_S, (\tilde{\pi}_\theta)_{\theta\in\Theta})$ and $(x_\theta, y_\theta, p_\theta)_{\theta\in\Theta}$ satisfy all constraints in Lemma 1. Let $p_\theta^* \in \mathbb{R}_+$, $n_\theta^* \in \mathbb{N}$, and $x_\theta^* \in [0,1]$ be the solution of the following system of equations:

$$x_\theta = x_\theta^* e^{-r(n_\theta^*-1)\Delta}, \quad y_\theta = (1-x_\theta^*)e^{-r(n_\theta^*-1)\Delta}, \quad \text{and} \quad p_\theta = p_\theta^* x_\theta^* e^{-r(n_\theta^*-1)\Delta} \quad \forall \theta \in \Theta$$

Now, consider a mechanism μ that offers p_n in period n, where

$$p_n = \begin{cases} p_\theta^* & \text{with probability } x_\theta^* \\ \bar{v} & \text{with probability } 1 - x_\theta^* \end{cases}$$

if (i) the buyer reports her type to be θ in period 1, (ii) $n = n_\theta^*$, (iii) the two negotiation parties fail to reach an agreement in periods $k \leq n_\theta^*$. Otherwise,

$$p_n = \bar{v} \quad \text{with probability 1.}$$

Each buyer type θ can guarantee the payoff $\tilde{\pi}_\theta = x_\theta v_\theta + y_\theta w_\theta - p_\theta$ in mechanism μ by reporting her type truthfully in the first period. It is optimal for each buyer type, once she truthfully reports θ in period 1, to wait until period n_θ^* and accept $p_{n_\theta^*}$ if and only if $p_{n_\theta^*} = p_\theta^*$; buyer type θ will exercise the outside option immediately if $p_{n_\theta^*} = \bar{v}$. Hence, the expected payoff of buyer type θ from the truthful revelation of her type is

$$e^{-r(n_\theta^*-1)\Delta}\left[x_\theta^*(v_\theta - p_\theta^*) + (1-x_\theta^*)w_\theta\right] = x_\theta v_\theta + y_\theta w_\theta - p_\theta = \tilde{\pi}_\theta.$$

On the other hand, conditional on having reported her type as θ in the first period, it is never a best response for any buyer type $\theta' \in \Theta$ (not necessarily θ) to exercise the outside option in any period k such that $1 < k < n_\theta^*$. This means that any buyer type $\theta' \in \Theta$ cannot achieve a payoff strictly higher than $\max\{w_{\theta'}, v_{\theta'}x_\theta - w_{\theta'}y_\theta - p_\theta\}$ by misreporting her type in period one; hence, any buyer type θ will obtain exactly $\tilde{\pi}_\theta$ in mechanism μ. Finally, it is straightforward that the seller's expected payoff in μ is

$$\sum_{\theta\in\Theta} q_\theta e^{-r(n_\theta^*-1)\Delta} x_\theta^* p_\theta^* = \sum_{\theta\in\Theta} q_\theta p_\theta = \tilde{\pi}_S.$$

The characterization of $\Pi^{\lambda,\Delta}$ is more challenging with $\lambda < \infty$. The main difficulty arises from a structural change due to the arrival of the outside option in the middle of negotiation. A mechanism suffices to guarantee each buyer type a continuation payoff no lower than $\underline{\pi}_\theta^B$ while the outside option is still not available. However, once the outside option arrives, the buyer can always opt out for the outside option, which yields w_θ as the final payoff; hence, a mechanism now has to guarantee a continuation payoff of at least $w_\theta > \underline{\pi}_\theta^B$. This also means that the seller has to track the arrival time of the outside option, providing the buyer with an incentive to report it truthfully. Together with the multidimensionality that already exists whether λ is finite or infinite, the seller now faces a dynamic multidimensional screening problem that is, in general, difficult to solve.

However, the next lemma shows that the program (7)-(10) still provides an approximate characterization of $\Pi(\lambda, \Delta)$ with a finite but large arrival rate $\lambda > 0$. Let X and Y be arbitrary non-empty subsets of the real line $\mathbb{R}^{|\Theta|+1}$. The Hausdorff distance, or Hausdorff metric, between X and Y, $d_H(X, Y)$, is defined by

$$d_H(X, Y) := \inf\{\epsilon \geq 0 : X \subset Y_\epsilon, Y \subset X_\epsilon\}$$

where

$$X_\epsilon := \bigcup_{x \in X} \{s \in \mathbb{R}^{|\Theta|+1} : d(s, x) \leq \epsilon\}$$

for any $X \subset \mathbb{R}^{|\Theta|+1}$ and $\epsilon > 0$. Here, d_E stands for the Euclidean distance. The following lemma shows that $\Pi^{\lambda,\Delta}$ converges to $\Pi^{\infty,\Delta}$ in terms of the Hausdorff distance as $\Delta \to 0$ and $\lambda \to \infty$.

Lemma 2. $\lim_{\lambda \to \infty} \lim_{\Delta \to \infty} \Pi^{\lambda,\Delta} = \Pi^{\infty,\Delta}$ *in terms of the Hausdorff distance.*[8]

Proof. In Appendix. ■

The order of limits matters in Lemma 2. We first take $\Delta \to 0$ and then $\lambda \to \infty$. That is, we first characterize the set of achievable payoff profiles in the continuous-time limit $\lim_{\Delta \to 0} \Pi^{\lambda,\Delta}$ for any arrival rate $\lambda > 0$; in the continuous-time limit, the outside option arrives according to a Poisson rate with arrival rate $\lambda > 0$, and the seller (or mechanism) can revise its offer at any moment in continuous time. The lemma shows that $\lim_{\Delta \to 0} \Pi^{\lambda,\Delta}$ converges to $\Pi^{\infty,\Delta}$ as the arrival rate of the outside option goes to infinity.[9]

The proof can be found in the Appendix, though the basic idea for the lemma is simple. The gap between $\underline{\pi}_\theta^B = \frac{1-e^{-\lambda\Delta}}{1-e^{-(r+\lambda)\Delta}} w_\theta$ and w_θ shrinks in the limiting case; hence, the structural change due to the arrival of the outside option has only a negligible effect in the limit. Moreover, at time zero, both the buyer and the seller expect that the buyer's outside option will arrive very quickly. As a result, the buyer and the seller can approximately achieve any payoff profile in $\Pi^{\lambda,\Delta}$ with a mechanism that assumes $\lambda = \infty$.

[8] $\Pi^{\infty,\Delta}$ does not depend on $\Delta > 0$. $\Pi^{\infty,\Delta'} = \Pi^{\infty,\Delta''}$ for any $\Delta', \Delta'' > 0$.

[9] However, we can show that $\lim_{\lambda \to \infty} \Pi^{\lambda,\Delta} = \Pi^{\infty,\Delta}(\lambda, \Delta)$ for any $\Delta > 0$, and hence

$$\Pi^{\infty,\Delta}(\lambda, \Delta) = \lim_{\Delta \to 0} \lim_{\lambda \to \infty} \Pi^{\lambda,\Delta} = \lim_{\lambda \to \infty} \lim_{\Delta \to 0} \Pi^{\lambda,\Delta}.$$

3.2 Delay in Optimal Mechanisms

In this section, we apply Lemmas 1 and 2 to show that the seller's profit-maximizing mechanism often involves a delay in agreement. For any $\lambda \in (0, \infty]$ and $\Delta > 0$, define $\bar{\pi}_S(\lambda, \Delta)$ by the upper bound of profit levels that the seller can achieve with a mechanism:

$$\bar{\pi}_S(\lambda, \Delta) = \sup\{\pi_S \geq 0 : (\pi_S, (\pi_\theta)_{\theta\in\Theta}) \in \Pi^{\infty,\Delta}\}$$

For any $\mu \in \mathcal{M}$, the seller's (maximum) expected profit from μ is

$$\pi_S^\mu(\lambda, \Delta) = \sup\left\{\pi_S(\mu, \sigma_B; \lambda, \Delta) : \sigma_B \in \mathcal{D} \text{ is admissible for } \mu\right\}.$$

For any $\epsilon \geq 0$, a mechanism $\mu \in \mathcal{M}$ such that $\pi_S^\mu(\lambda, \Delta) \geq \bar{\pi}_S - \epsilon$ is called called ϵ-optimal. We will call a 0-optimal mechanism simply an *optimal mechanism.*

For any $\mu \in \mathcal{M}$ and admissible $\sigma_B \in \mathcal{D}$, $\tau_\theta^{\mu,\sigma_B}(\lambda, \Delta)$ is the (expected) delay in bargaining between the seller and buyer type $\theta \in \Theta$, and $\tau^{\mu,\sigma_B}(\lambda, \Delta)$ is the average delay across all buyer types:

$$\tau_\theta^{\mu,\sigma_B}(\lambda, \Delta) = \sum_{k\geq 1} \psi_k(\lambda, \Delta) \int_{(b,n,p)\in O} (n-1)\Delta \, \mathrm{d}F^{\mu,\sigma_B}(b, n, p; \theta, k)$$

$$\tau^{\mu,\sigma_B}(\lambda, \Delta) = \sum_{\theta\in\Theta} \tau_\theta^{\mu,\sigma_B}(\lambda, \Delta) q_\theta.$$

Finally, for any $\mu \in \mathcal{M}$, the expected delay of agreement in $\mu \in \mathcal{M}$ is denoted

$$\tau^\mu(\lambda, \Delta) = \inf\left\{\tau^{\mu,\sigma_B}(\lambda, \Delta) : \sigma_B \in \mathcal{D} \text{ is admissible for } \mu\right\}.$$

3.2.1 Binary-type Case

We first consider the case in which the buyer's type space Θ is binary. Without loss of generality, let $\Theta = \{H, L\}$ and suppose that either of the following two conditions holds.

$$\begin{array}{ll} (i) & v_H - w_H > v_L - w_L \\ (ii) & v_H - w_H = v_L - w_L \quad \text{and} \quad v_H > v_L. \end{array} \tag{12}$$

Type H and type L are referred to as "high-type" and "low-type," respectively.

The following proposition shows that the optimal mechanism for the frictionless case involves a delay in agreement. The delay does not vanish, even in the continuous-time limit.

Proposition 1. *Suppose that* $\Theta = \{H, L\}$ *and* $\lambda = \infty$*. Then,*

$$\bar{\pi}_S(\lambda, \Delta) = \begin{cases} q_H\left(v_H - w_H\right) + q_L \frac{v_L w_H - w_L v_H}{v_H - v_L} & \text{if } \frac{w_H}{w_L} > \frac{v_H}{v_L} \geq \frac{1}{q_H} \\ \max\left\{q_H\left(v_H - w_H\right), v_L - w_L\right\} & \text{otherwise.} \end{cases} \tag{13}$$

Moreover, for the optimal mechanism $\mu \in \mathcal{M}$ for each case

$$\tau^{\mu}(\lambda, \Delta) = \begin{cases} \frac{q_L}{r} \log \frac{v_H - v_L}{w_H - w_L} + O(\Delta)^{10} & \text{if } \frac{w_H}{w_L} > \frac{v_H}{v_L} \geq \frac{1}{q_H} \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Proof. In Appendix. ■

The proof for (13) proceeds by examining every case and can be found in the Appendix. Here, we describe the optimal mechanisms that exactly achieve (13). First note that a high-type buyer (respectively, low-type) will not accept any price higher than $v_H - w_H$ (respectively, $v_L - w_L$). Therefore, the total surplus extractable from the buyer is larger when she is of the high-type. As in the standard screening problem, the seller's optimal mechanism will seek to minimize the information rent of a high-type by acting as he is facing a low-type buyer.

If one of the inequalities in $\frac{w_H}{w_L} > \frac{v_H}{v_L} \geq \frac{1}{q_H}$ fails, the seller can earn $\bar{\pi}^S(\lambda, \Delta)|_{\lambda=\infty}$ with a take-it-or-leave-it offer that offers p_n in each period n, where

$$p_n = p^{\dagger} := \begin{cases} v_H - w_H & \text{if } q_H \geq \frac{v_L - w_L}{v_H - w_H} \\ v_L - w_L & \text{otherwise.} \end{cases} \quad \forall n \geq 1. \tag{15}$$

However, if the inequality

$$\frac{w_H}{w_L} > \frac{v_H}{v_L} \geq \frac{1}{q_H} \tag{16}$$

holds, then the seller can improve his profit by delaying his offer to a low-type buyer.[11] Specifically, the optimal mechanism's offer p_n in each period $n \geq 1$ is as follows:

$$\begin{aligned} p_1 &= v_H - w_H \quad \text{with probability } 1 \\ p_{n_L^{\ddagger}} &= \begin{cases} p^{\ddagger} := \frac{v_L w_H - v_H w_L}{w_H - w_L} & \text{with probability } x_L^{\ddagger} \\ \bar{v} & \text{with probability } 1 - x_L^{\ddagger}. \end{cases} \\ p_n &= \bar{v} \quad \text{with probability 1 for any } n \notin \{1, n_L^{\ddagger}\} \end{aligned}$$

which are independent of messages from the buyer, where $n_L^{\ddagger}$ and $x_L^{\ddagger}$ are the solutions of the following equations:

$$w_H = e^{-r(n_L^{\ddagger}-1)\Delta} x_L^{\ddagger}(v_H - p_L^{\ddagger}) \quad \text{and} \quad w_L = e^{-r(n_L^{\ddagger}-1)\Delta}\left[x_L^{\ddagger}(v_L - p_L^{\ddagger}) + (1 - x_L^{\ddagger})w_L\right].$$

For the future reference, let $\mu^{\dagger}$ refer to the mechanism with a take-or-leave-it-offer $p^{\dagger}$. Further, let $\mu^{\ddagger}$ refer to the mechanism with the offer $p_H^{\ddagger}$ in the first period and then the delayed offer $p_L^{\ddagger}$ in period $n_L^{\ddagger}$.

[10] Following the Bachmann-Landau notation, for any function $f : [0, \infty] \times (0, \infty) \to \mathbb{R}$, $f(\lambda, \Delta) = O(\Delta)$ means that $|f(\lambda, \Delta)| < M\Delta$ for some constant M and all values λ and Δ.

[11] The condition (16) implies $v_H - w_H > v_L - w_L$, $v_H > v_L$, and $w_H > w_L$.

There is a decision rule of the buyer that serves as an essentially unique admissible decision rule for both $\mu^\dagger$ and $\mu^\ddagger$; buyer type θ will accept p_n if and only if $p_n \leq v_\theta - w_\theta$ and exercises the outside option immediately in period n if $p_k > v_\theta - w_\theta$ for all $k \geq n$. Combined with this decision rule of the buyer, there is no delay in $\mu^\dagger$, and the expected profit from mechanism $\mu^\dagger$ is

$$\pi_S^{\mu^\dagger}(\lambda, \Delta)|_{\lambda=\infty} = \max\{q_H(v_H - w_H), q_L(v_L - w_L)\}.$$

However, the seller's profit in $\mu^\ddagger$ is

$$\pi_S^{\mu^\ddagger}(\lambda, \Delta)|_{\lambda=\infty} = q_H p_H^\ddagger + q_L e^{-rn_L^\ddagger \Delta} x_L^\ddagger p_L^\ddagger = q_H(v_H - w_H) + q_L \frac{v_L w_H - v_H w_L}{v_H - w_L}. \tag{17}$$

and the expected delay in $\mu^\ddagger$ is

$$\tau^{\mu^\ddagger}(\lambda, \Delta)|_{\lambda=\infty} = \frac{q_L}{r} \log \frac{v_H - v_L}{w_H - w_L} + O(\Delta)$$

as stated in Proposition 1, and it remains strictly positive even in the limit $\Delta \to 0$

It is important to note that the delay of agreement in $\mu^\ddagger$ cannot be replaced by a lottery. This observation contrasts with the standard mechanism design environment with type-independent outside options and no withdrawal right. Suppose that the seller in mechanism $\mu^\ddagger$ provides a low-type buyer with a lottery, instead of the delaying his offer to low-type buyer until period $n_L^\ddagger$, such that low-type buyer purchases the good in period 1 at price p_1, where

$$p_1 = \begin{cases} p_L^\ddagger & \text{with probability } e^{-r(n_L^\ddagger - 1)\Delta} x_L^\ddagger \\ \bar{v} & \text{with probability } 1 - e^{-r(n_L^\ddagger - 1)\Delta} x_L^\ddagger. \end{cases} \tag{18}$$

If the buyer has no withdrawal right and thus has to abandon the outside option once she requests a randomized quote p_1, a low-type buyer will accept p_1 if and only if $p_1 = p_L^\ddagger$. The seller could achieve in this case the profit exactly equal to (17), and hence the delay and the lottery are truly equivalent.

With the withdrawal right, however, the lottery and the delayed offer are not equivalent. A low-type buyer still can exercise her outside option once the lottery outcome turns out to be unfavorable (that is, once $p_1 = \bar{v}$); hence, a low-type buyer's expected payoff with the lottery (18) is

$$e^{-r(n_L^\ddagger - 1)\Delta} x_L^\ddagger (v_L - p^\ddagger) + (1 - e^{-r(n_L^\ddagger - 1)\Delta} x_L^\ddagger) \left[\mathbb{1}\{a_1 = 1\} \cdot w_\theta + \mathbb{1}\{a_1 = 0\} \cdot e^{-r\Delta} \underline{\pi}_L^B\right]$$

which is strictly larger than low-type buyer's expected payoff in the orignal mechanism with delay. Similarly, a high-type buyer's payoff from mimicking a low-type buyer is also strictly higher with the lottery. As a result, the seller could reduce a high-type buyer's information rent more effectively with delay, which results in a higher profit for the seller.

Lemma 2 guarantees that $\mu^\dagger$ and $\mu^\ddagger$ are approximately optimal mechanisms for either case, as long as we focus on the limiting case. Hence, we obtain the following corollary.

Corollary 1. *Suppose that* $\Theta = \{H, L\}$ *and* $\frac{w_H}{w_L} > \frac{v_H}{v_L} \geq \frac{1}{q_H}$. *For any* $\epsilon \in (0, v_H - w_H - v_L + w_L)$, *there are* $\lambda^* > 0$ *and* $\{\Delta^*_\lambda > 0 | \lambda^* < \lambda \leq \infty\} \subset (0, \infty)$ *such that the following statement is true whenever* $\lambda \in (\lambda^*, \infty]$ *and* $\Delta \in (0, \Delta^*_\lambda)$*:*

$$\exists \mu \in \mathcal{M} \quad \textit{such that} \quad \mu \textit{ is } \epsilon\textit{-optimal} \quad \textit{and} \quad \tau^\mu(\lambda, \Delta) = \frac{q_L}{r} \log \frac{v_H - v_L}{w_H - w_L} + O(\Delta).$$

The above proposition and corollary show that there exists an ϵ-optimal mechanism (which is also 0-optimal if $\lambda = \infty$) in which a delay in agreement is present. However, it does not guarantee that all ϵ-optimal mechanisms involve a delay. The next result shows that a delay of agreement is *inevitable* in any ϵ-optimal mechanism when the condition (16) holds. To this end, define

$$\mathcal{M}^{\epsilon|\lambda,\Delta} := \left\{ \mu \in \mathcal{M} \middle| \ \sigma_B \text{ is admissible for } \mu \implies \tau_\theta^{\mu,\sigma_B}(\lambda, \Delta) < \epsilon \quad \forall \theta \in \Theta \right\}$$

by all mechanisms in which at most ϵ delay is present for any buyer type. Moreover, let $\bar{\pi}^S_\epsilon(\lambda, \Delta)$ be the upper bound of the seller's profit levels achievable with a mechanism in $\mathcal{M}^{\lambda,\Delta}_\epsilon$:

$$\bar{\pi}^\epsilon_S(\lambda, \Delta) = \sup \left\{ \pi_S(\mu, \sigma_B | \lambda, \Delta) : \mu \in \mathcal{M}^{\epsilon|\lambda,\Delta} \text{ and } \sigma_B \text{ is admissible} \right\}$$

The next lemma identifies $\bar{\pi}^\epsilon_S(\lambda, \Delta)$ when $\Delta > 0$ is small (i.e., in the continuous-time limit).

Lemma 3. *Suppose that* $\Theta = \{H, L\}$, *and that there exists* $\lambda_0 \in (0, \infty]$, $\Delta_0 \in (0, \infty)$, *and* $\epsilon_0 \in (0, \infty)$ *such that* $\mathcal{M}^{\epsilon|\lambda,\Delta} \neq \varnothing$ *for any* $\Delta \in (0, \Delta_0)$, $\epsilon \in [0, \epsilon_0]$, *and* $\lambda \in [\lambda_0, \infty]$. *Then,*

$$\lim_{\Delta \to 0} \bar{\pi}^\epsilon_S(\lambda, \Delta)|_{\epsilon=0} = \lim_{\epsilon \to 0} \lim_{\Delta \to 0} \bar{\pi}^\epsilon_S(\lambda, \Delta) = \max\{q_H(v_H - w_H), v_L - w_L\}.$$

for any $\lambda \geq \lambda_0$.

Proof. In the Appendix. ■

Under the condition (16), the seller's expected profit from $\mu^\ddagger$ is strictly larger than $\max\{q_H(v_H - w_H), q_L(v_L - w_L)\}$. Hence, the last lemma shows that a delay in agreement is inevitable in the optimal mechanism.

Corollary 2. *Suppose that* $\Theta = \{H, L\}$ *and* $\frac{w_H}{w_L} > \frac{v_H}{v_L} \geq \frac{1}{q_H}$. *For any* $\epsilon \geq 0$, *there is* $\delta > 0$ *such that*

$$\tau^\mu(\lambda, \Delta) > \delta + O(\Delta)$$

for any ϵ*-optimal mechanism* μ.

Finally, note that mechanism $\mu^\ddagger$ randomizes its offer in period $n^\ddagger_L$ between $p^\ddagger_L$ and $\bar{v}$, which can be considered practically unappealing.[12] We conclude the binary-type case by highlighting

[12] For example, Radner and Rosenthal (2007) note that randomization has limited appeal in many practical situations.

that the seller can earn approximately the same profit with a deterministic mechanism. Note that $x_L^\ddagger$, the probability of $p_n|_{n=n_L^\ddagger} = p_L^\ddagger$, converges to one as $\Delta \to \infty$, and hence the seller can obtain a profit close to $\bar{\pi}_S(\lambda, \Delta)$ by simply committing to $p_n|_{n=n_L^\ddagger} = p_L^\ddagger$ with probability 1.

3.2.2 General Type Space

In this section, we will identify conditions under which a delayed agreement is present in the optimal mechanism with more than two buyer types. We begin by ordering buyer types. For any buyer type $\theta \in \Theta$, define *ex ante net-valuation*

$$u_\theta^{ex-ante} := v_\theta - \underline{\pi}_\theta^B$$

by the gap between two surplus levels, one generated from immediate trade and the autarky strategy. Also, define *ex post net-valuation* by the difference between v_θ and w_θ:

$$u_\theta^{ex-post} := v_\theta - w_\theta.$$

$u_\theta^{ex-ante}$ is the gain from trading in the ex ante situation in which the buyer remains uncertain of when the outside option will arrive. $u_\theta^{ex-post}$ represents the gain from trading in the ex post situation with the outside option already having arrived. In respective cases, the buyer would reject any price higher than $u_\theta^{ex-ante}$ and $u_\theta^{ex-post}$ respectively.

We will order buyer types according to ex post net-valuation and ex ante net-valuation. For any finite type space $\Theta = \{0, 1, 2, \ldots, |\Theta| - 1\}$, we label each buyer type so that

$$v_k - w_k = u_k^{ex-post} \geq u_\ell^{ex-post} = v_\ell - w_\ell \qquad \Longleftrightarrow \qquad k \geq \ell. \tag{19}$$

If two buyer types ℓ and k have the same ex post net-valuations, we order them so as to have $v_k > v_\ell$ if $k > \ell$. Generally, this ordering does not coincide with the ordering by ex ante net-valuation. However, there is $\tilde{\lambda} > 0$ such that

$$\lim_{\Delta \to 0} u_i^{ex-ante} \geq \lim_{\Delta \to 0} u_j^{ex-ante} \quad \text{for any } i \text{ and } j \text{ in } \Theta \text{ such that } u_i^{ex-post} \geq u_j^{ex-post} \tag{20}$$

whenever $\lambda > \tilde{\lambda}$. Unless noted otherwise, we maintain $\lambda > \tilde{\lambda}$ and Δ is sufficiently small that two orders are equivalent to one another.

The next lemma extends Lemma 3 to general type spaces.

Lemma 4. *Suppose there exist $\lambda_0 \in (0, \infty]$, $\Delta_0 \in (0, \infty)$, and $\epsilon_0 \in (0, \infty)$ such that $\mathcal{M}^{\epsilon|\lambda,\Delta} \neq \varnothing$ for any $\Delta \in (0, \Delta_0)$, $\epsilon \in [0, \epsilon_0]$, and $\lambda \in [\lambda_0, \infty]$. Then*

$$\lim_{\Delta \to 0} \bar{\pi}_S^\epsilon(\lambda, \Delta)|_{\epsilon=0} = \lim_{\epsilon \to 0} \lim_{\Delta \to 0} \bar{\pi}_S^\epsilon(\lambda, \Delta) = \begin{cases} \max_{n \in \{0,1,2,\ldots,|\Theta|-1\}} \sum_{k=n}^{|\Theta|-1} q_k \left(v_n - \frac{\lambda}{\lambda + r} w_n \right) & \text{if } \lambda < \infty \\ \max_{n \in \{0,1,2,\ldots,|\Theta|-1\}} \sum_{k=n}^{|\Theta|-1} q_k \left(v_n - w_n \right) & \text{if } \lambda = \infty. \end{cases}$$

for any $\lambda \geq \lambda_0$.

Proof. In the Appendix. ■

Thanks to the last lemma, we can show that the optimal mechanism involves a delay once we identify a mechanism from which the seller's profit is strictly higher than $\lim_{\Delta \to 0} \bar{\pi}_S^0(\lambda, \Delta)$. This is indeed true under the following two conditions

$$v_{n+1} \geq v_n \quad \text{and} \quad w_{n+1} \geq w_n \quad \forall n = 0, 1, 2, \ldots, |\Theta| - 2 \tag{21}$$

$$\frac{w_{n+1}}{v_{n+1}} > \frac{w_n}{v_n} \quad \forall n = 0, 1, 2, \ldots, |\Theta| - 2. \tag{22}$$

Assumption (21) requires that both the buyer's valuation of the seller's good and her outside option are positively correlated in the strongest sense, while (22) supposes a sort of convexity; for example, it holds for $(v_n, w_n)_{n \in \Theta}$ if they are on the graph of a convex function f on a (v, w)-plane such that $f(0) = 0$. Note that Condition (16) is a special case of (21) and (22) when $|\Theta| = 2$. Finally, notice that these assumptions allow two buyer types with the same valuations to have different outside options.

In order to describe a mechanism with delayed agreement that yields a profit strictly larger than $\lim_{\Delta \to 0} \bar{\pi}_S^0(\lambda, \Delta)$ to the seller, define $k^\star$ by the largest element in

$$\arg \max_{k \in \{0,1,2,\ldots,|\Theta|-1\}} \sum_{j=k}^{|\Theta|-1} q_j \left(v_k - \frac{\lambda}{\lambda + r} w_k \right) \quad \text{if } \lambda < \infty$$

or

$$\arg \max_{k \in \{0,1,2,\ldots,|\Theta|-1\}} \sum_{j=k}^{|\Theta|-1} q_j \left(v_k - w_k \right) \quad \text{if } \lambda = \infty,$$

depending on $\lambda > 0$, and suppose that $k^\star > 1$. We will show that a bargaining mechanism analogous to $\mu^\ddagger$ will generate a higher profit. For any $\Delta > 0$, let $n^\star_\Delta \in \mathbb{N}$, $x^\star_\Delta \in [0, 1]$, and $p^\star$ be such that

$$e^{-r(n^\star_\Delta - 1)\Delta} \geq \frac{w_{k^\star} - w_{k^\star - 1}}{v_{k^\star} - v_{k^\star - 1}} > e^{-rn^\star_\Delta \Delta}, \quad e^{-r(n^\star_\Delta - 1)\Delta} x^\star_\Delta = \frac{w_{k^\star} - w_{k^\star - 1}}{v_{k^\star} - v_{k^\star - 1}},$$

and

$$p^\star = \frac{v_{k^\star - 1} w_{k^\star} - v_{k^\star} w_{k^\star - 1}}{w_{k^\star} - w_{k^\star - 1}}$$

respectively. The following mechanism $\mu^\star$ generalizes $\mu^\ddagger$ for the binary-type case.

- Conditional on the buyer revealing that she is of type $\theta \geq k^\star$ in the first period, the mechanism offers $p_1 = u_{k^\star}^{ex-post}, 1$ in period one and then never trades once this offer is rejected by the buyer.

- Conditional on the buyer revealing that she is of type $\theta = k^\star - 1$ in the first period, the mechanism delays until period $n_\Delta^\star$ and then offers $p_{n_z^\star} = p^\star$ and then never trades once this offer is rejected by the buyer.

- The mechanism excludes $\theta < k^\star - 1$ (if any) and never sells the good to them.

Note that $\mu^\star$ makes a delayed offer to a buyer of type $k^\star - 1$ but trades immediately with buyer types $\theta \geq k^\star$; all other buyer types are excluded. There is again essentially unique admissible decision rule of the buyer in $\mu^\star$. Any buyer type reveals θ truthfully, and accepts any offer other than $\bar{v}$ for sure. The buyer exercises her outside option if only and only if $\mu^\star$ only offers $\bar{v}$ in the future. Let $\sigma_B^\star$ denote the buyer's unique admissible decision rule.

The delay of agreement with $k^\star - 1$ does not vanish even in the continuous-time limit

$$\lim_{\Delta \to 0} \tau_\theta^{\mu^\star, \sigma_B^\star}(\lambda, \Delta)|_{\theta = k^\star - 1} = \frac{1}{r} \log \frac{v_{k^\star} - v_{k^\star - 1}}{w_{k^\star} - w_{k^\star - 1}} > 0$$

and the seller's profit from $\mu^\star$ is

$$\lim_{\Delta \to 0} \pi_S(\mu^\star, \sigma^\star|\lambda, \Delta) = \lim_{\epsilon \to 0} \lim_{\Delta \to 0} \pi_\epsilon^S(\lambda, \Delta) + \underbrace{q_{k^\star - 1} \frac{v_{k^\star - 1} w_{k^\star} - v_{k^\star} w_{k^\star - 1}}{v_{k^\star} - v_{k^\star - 1}}}_{>0}$$

which is strictly higher than the seller's profit without delay, hence the proposition below follows.

Proposition 2. *Suppose that* (21) *and* (22) *hold and* $k^\star > 1$. *Then a delay in agreement is present in the optimal mechanism.*

The last proposition shows that the optimality of delayed agreement is observed for a large set of parameters, beyond cases with a binary type space. This result contrasts with the optimal selling scheme without the buyer's outside option. Without the buyer's outside option, or more precisely, if the buyer's outside option is independent of her type, a delayed agreement is never present in the optimal bargaining mechanism; all types either trade with the seller immediately or never. See, for example, Riley and Zeckhauser (1983) and Samuelson (1984).

4 Delay in Bargaining with No Commitment

We identified in Section 3 the seller's benefit from delaying an agreement while bargaining. However, such a delay often demands that the seller be able to commit to a delayed offer. In bargaining mechanism $\mu^\ddagger \in \mathcal{M}$, for example, the asymmetric information regarding the buyer's type is immediately resolved by the buyer's initial message, but $\mu^\ddagger$ still defers the transaction with the buyer without any learning or change in the situation. Hence, the seller is likely tempted to step in and advance the transaction with the buyer, and the successful implementation of $\mu^\ddagger$ hinges on the extent to which the seller can restrain himself from pursuing such an intervention. Of course, such commitment power is not always available in real-life bargaining situations.

The next natural question is therefore whether delayed agreement will also be present even in the absence of commitment on the seller's side. We will approach to this task by analyzing the set of equilibria for a bargaining game in which the seller can revise his offer in every period. The lack of commitment is embodied in the assumption that the seller's equilibrium strategy and belief have to fulfill sequential rationality. We are interested in both whether there is any equilibrium with a delay of agreement on its path and how close to the optimal bargaining mechanism's profit the seller can obtain even without commitment.

4.1 Strategies and Equilibrium

The bargaining environment is identical to what we introduced in Section 2. Here, we develop the notation for each player's strategy, belief system, and payoffs, as well as the notion of equilibrium. A history of the first $n \geq 1$ rejected offers and the history of the outside option's availability in the first n periods are denoted $h^n = (p_1, \dots, p_n) \in H^n := \mathbb{R}_+^n$ and $a^n = (a_1, \dots, a_n) \in \{0,1\}^n$, respectively. Note that h^n is observed by both parties, while a^n is only revealed to the buyer; hence, we often refer to h^n and a^n as the *public history* and *private history*, respectively. Let $H := \cup_{n\geq 0} H^n$ be the collection of all public histories, where $H^0 := \{h^0\}$ is (the set of) null history, and $A := \cup_{n\geq 1} A^n$ be the collection of all private histories. For any $n \geq 1$, let $H_B^n := H^{n-1} \times A^n$ be the collection of the buyer's observations up to the beginning of period n (including a_n) and $H_B := \cup_{n\geq 1} H_B^n$.

A seller's behavior strategy $\sigma : H \to \triangle\mathbb{R}_+$ is a probability transition such that $\sigma(h^{n-1})$ maps a public history $h^{n-1} \in H^{n-1}$ onto the probability distribution over offers in period $n \geq 1$; let $\sigma(p; h^{n-1})$ be the probability of the seller's offering p at h^{n-1}. The type-θ buyer's behavioral strategy is generically denoted by $(\chi_\theta, \xi_\theta) : \mathbb{R}_+ \times H_B \to [0,1]^2$, where $\chi_\theta(p; h_B^n)$ and $\xi_\theta(p; h_B^n)$ are the probability of a buyer of type θ accepting the seller's offer p and the probability of exercising the outside option, respectively, conditional on the seller's having offered $p \geq 0$ at history $h_B^n \in H_B^n$. We assume without loss that

$$\chi_\theta(p; h_B^n) + \xi_\theta(p; h_B^n) \in [0,1] \quad \text{and} \quad \xi_\theta(p; h_B^n) = 0 \quad \text{if } a_n = 0.$$

Let $(\chi, \xi) = (\chi_\theta, \xi_\theta)_{\theta\in\Theta}$ denote a profile of all buyer types' strategies. For any $n \geq 1$, $h^{n-1} \in H^{n-1}$, and $B \subset \Theta \times A^n$, let $q(B; h^{n-1})$ be the seller's subjective belief (probability) that the buyer's private information belongs to B. For any $h_B^n = (h^{n-1}, a^n) \in H_B^n$, let $\bar{\chi}$, $\bar{\xi}$, and $\bar{q}$ refer to marginal probabilities averaging over the buyer's private histories. That is,

$$\bar{\chi}_\theta(p; h) := \sum_{h_B \in H_B : \text{proj}_H(h_B) = h} q(h_B, \theta; h) \chi_\theta(p; h_B)$$

$$\bar{\xi}_\theta(p; h) := \sum_{h_B \in H_B : \text{proj}_H(h_B) = h} q(h_B, \theta; h) \xi_\theta(p; h_B)$$

$$\bar{q}(\Theta'; h) := \sum_{h_B \in H_B : \text{proj}_H(h_B) = h} q(\Theta' \times \{h_B\}; h)$$

for all $h \in H$, $p \geq 0$, $\theta \in \Theta$, and $\Theta' \subset \Theta$. [13]

We use the *perfect Bayesian equilibrium* (hereafter, *PBE* or *equilibrium*) as defined by Definition 8.2 in Fudenberg and Tirole (1991),[14] with one more restriction: We require that the support of the seller's belief about the buyer's type only decreases over time. That is,

$$\text{supp } \bar{q}(\cdot; h) \supset \text{supp } \bar{q}(\cdot; h')$$

for any two public histories h and its successor h'. For any $\Delta > 0$, $\lambda \in (0, \infty]$, and the seller's prior $(q_\theta)_{\theta \in \Theta}$, let $\mathcal{E}(\lambda, \Delta, (q_\theta)_{\theta \in \Theta})$ or often simply $\mathcal{E}(\lambda, \Delta)$ be the set of all PBE assessments, the generic element of which is denoted $\alpha = (\sigma, \chi, \xi, q)$. For any equilibrium $\alpha \in \mathcal{E}(\lambda, \Delta)$, $V^S(h; \alpha)$ and $V^B_\theta(h_B; \alpha)$ are the seller's expected profit at history $h \in H$ and the type-θ buyer's expected profit at history $h_B \in H_B$ (just before the seller makes an offer), respectively. In particular, let $V^S(\alpha)$ and $V^B_\theta(\alpha)$ be the (ex ante) expected equilibrium profits in $\alpha \in \mathcal{E}(\lambda, \Delta)$.

For any given equilibrium, we follow standard convention in calling a seller's offer *serious* if it will be accepted by the buyer with positive probability in equilibrium. In particular, we will call an offer *winning* if it is accepted with probability 1. An offer is *losing* if it is not serious.

4.2 Frictional Case

In this section, we present a folk theorem stating that the set of the seller's equilibrium profit levels converges to the set of the seller's profit levels with commitment. In light of Proposition 2, this result also implies there is an equilibrium in which a delay in agreement is present on the path.[15] Throughout this section we will maintain the assumption $\lambda \in (\tilde{\lambda}, \infty)$, hence buyer types are ordered so that (19) and (20) hold. Let "0" denote the buyer type with the lowest ex ante and ex post net-valuation levels.

4.2.1 Coasian Equilibrium

The folk theorem involves the construction of the effective punishment strategies against any deviation. We will use an equilibrium that satisfies the properties in the next lemma as the punishment in any continuation game followed by a deviation by the seller.

Lemma 5. *Fix $\lambda \in (0, \infty)$. There exist $\Delta_c \in (0, \infty)$ and $N_c \in \mathbb{N}$ for which the following statement holds: for any $\Delta \in (0, \Delta_c)$ there exists $\alpha \in \mathcal{E}(\lambda, \Delta, (q_\theta)_{\theta \in \Theta})$ such that*

[13] For any Cartesian product $A \times B$, $\text{proj}_A : A \times B \to A$ is the projection mappting onto A.

[14] Formally, Fudenberg and Tirole define perfect Bayesian equilibria for finite games of incomplete information only. Its generalization to this setting is straightforward and is omitted here. Notably, as in the original definition by Fudenberg and Tirole, we also impose the condition "no signaling what you don't know." That is, the seller's actions, even zero-probability actions, do not change his belief concerning the buyer's type.

[15] Here, we follow the literature's convention in referring to the result that any seller's profit levels below one from the optimal mechanism as the folk theorem; see, for instance, Ausubel and Deneckere (1989b) and Sobel (1991). This contrasts with the convention in the literature on repeated games in which the folk theorem usually states that any feasible and individual rational payoff profile of *all players* can be realized as an equilibrium payoff profile.

(i) the bargaining game concludes within N_c periods with probability 1, either by the buyer's opting for the outside option or trading with the seller;

(ii) the seller never offers a price higher than $v_0 - e^{-r\Delta}\underline{\pi}_0^B + (N_c - 1)(\lambda + r)\Delta$ on its path; hence,

$$v_0 - e^{-r\Delta}\underline{\pi}_0^B \le V^S(\alpha) \le v_0 - e^{-r\Delta}\underline{\pi}_0^B + (N_c - 1)(\lambda + r)\Delta.$$

Proof. See the Appendix. ∎

We refer to an equilibrium that satisfies the properties in the last lemma as *Coasian equilibrium*, motivated by the observation that it satisfies key contents of the Coase Conjecture; most notably, the seller trades (almost) immediately with all buyer types with probability very close to 1 (Coase, 1972). Let $\mathcal{E}_c(\lambda, \Delta, (q_\theta)_{\theta\in\Theta}) \subset \mathcal{E}(\lambda, \Delta, (q_\theta)_{\theta\in\Theta})$ or simply $\mathcal{E}_c(\lambda, \Delta)$ denote the set of all Coasian equilibria. One implication of Lemma 5 is

$$\lim_{k\to\infty} V^S(\alpha_k) = \lim_{\Delta\to 0} v_0 - e^{-r\Delta}\underline{\pi}_0^B + (N_c - 1)(\lambda + r)\Delta = v_0 - \frac{\lambda}{\lambda + r} w_0 \tag{23}$$

for any sequence of equilibria $(\alpha_k)_{k\geq 0} \in \prod_{k\geq 0} \mathcal{E}_c(\lambda, \Delta_k, (q_\theta)_{\theta\in\Theta})$ such that $\lim_{k\to\infty} \Delta_k = 0$.

To capture the intuition for the existence of Coasian equilibrium, note first that the Coase conjecture holds true for the asymmetric information bargaining model with no outside option (Gul, Sonnenschein, and Wilson, 1986); that is, there exists an equilibrium for which a statement similar to *(i)* holds if there is no outside option. We may naturally conjecture that a similar result holds when the outside option is available to the buyer with only a very small probability. It suffices for Lemma 5 to have all players *believe* that the bargaining concludes with probability close to 1 before the arrival of the outside option, and this belief will be self-fulfilled in equilibrium.

Formally, suppose that all players believe that the bargaining game ends within $N > 0$ periods (or earlier). Under this hypothesis, all players also expect that no outside option arrives with probability

$$\mathbb{P}\{a_1 = a_2 = \ldots = a_{N-1} = 0\} = e^{-\lambda(N-1)\Delta}$$

during the negotiation, which is close to 1 with a small $\Delta > 0$. Hence, we can naturally conjecture that all players are under pressure to rapidly conclude the bargaining as predicted by Coase, and the initial hypothesis shared by players concerning the speed of the negotiation is self-fulfilled in equilibrium.

One may still worry about the possibility that a low-probability event can have a dramatic effect on equilibrium play. This concern is especially justifiable if the buyer is willing and able to indicate that this event indeed occurs. However, the buyer in our model has neither such an ability nor willingness. The buyer can reveal the outside option's availability only by actually exercising it; such an action concludes the game and hence has no influence on the bargaining outcome. The rigorous proof that formalizes this idea can be found in the Appendix.

4.2.2 The Folk Theorem in the Limit as $\lambda \to \infty$

In this section, we formally state the folk theorem and sketch its proof. We begin by introducing some definitions. For any $\lambda \in (0, \infty)$ and $\Delta \in (0, \infty)$, let

$$\mathcal{V}^S(\lambda, \Delta) := \left\{V^S(\alpha; \lambda, \Delta) | \alpha \in \mathcal{E}(\lambda, \Delta)\right\} \quad \text{and} \quad \Pi^S(\lambda, \Delta) := [0, \bar{\pi}_S(\lambda, \Delta)]$$

be the set of equilibrium profit levels and the set of direct incentive-compatible mechanism profit levels.[16] Then, the folk theorem (Proposition 3) states that

$$\lim_{\lambda \to \infty} \lim_{\Delta \to 0} d_H \left(\mathcal{V}^S(\lambda, \Delta), \Pi^S(\lambda, \Delta)\right) = 0 \tag{24}$$

under the following assumption:

$$\Theta = \{0, 1, 2\} \quad \text{and} \quad u_2^{ex-post} \geq u_1^{ex-post} \geq u_0^{ex-post} = v_0 - w_0 = 0. \tag{25}$$

PROPOSITION 3 (Folk Theorem). *If* (25) *holds,* $\lim_{\lambda \to \infty} \lim_{\Delta \to 0} d_H \left(\mathcal{V}^S(\lambda, \Delta), \Pi^S(\lambda, \Delta)\right) = 0.$

Before we sketch the proof of the proposition, we first discuss the implications of assumption (25), particularly the assumption that $v_0 - w_0 = 0$. The role of this assumption is twofold. First, $v_0 - w_0 = 0$ implies that the seller's profit in a Coasian equilibrium (precisely, its continuous-time limit) becomes arbitrarily small as $\lambda \to \infty$. Indeed, for any sequence of equilibria $(\alpha_k)_{k \geq 0} \in \prod_{k \geq 0} \mathcal{E}_c(\lambda, \Delta_k, (q_\theta)_{\theta \in \Theta})$ such that $\lim_{k \to \infty} \Delta_k = 0$,

$$\lim_{k \to \infty} V^S(\alpha_k) = \lim_{\Delta \to 0} \left(v_0 - e^{-r\Delta} \underline{\pi}_0^B + (N_c - 1)(\lambda + r)\Delta\right) = v_0 - \frac{\lambda}{\lambda + r} w_0 = \frac{r}{\lambda + r} v_0$$

This property of Coasian equilibria allows us to use a Coasian equilibrium strategy profile as an effective punishment scheme for any deviation by the seller.

Second, the assumption that $v_0 - w_0 = 0$ excludes rather less interesting cases. Our primary interest is whether the seller can (approximately) earn the optimal mechanism profit when the optimal mechanism involves a delay. This is actually impossible for a set of parameters. One can show that the seller never offers a price lower than $v_0 - w_0 \geq 0$ in *any* equilibrium, and hence a 0-type buyer exercises the outside option as soon as it arrives. When $\lambda > 0$ is large, therefore, the seller can never earn a profit that he could earn from a mechanism in which (i) a delay is present, and (ii) the seller trades with a 0-type with positive probability. The assumption that $v_0 - w_0 = 0$ guarantees that the optimal mechanism never trades with a 0-type and hence

[16] Note that any profit level below $\bar{\pi}_S(\lambda, \Delta)$ can be achieved in a bargaining mechanism that begins by paying a certain amount of money to the buyer (irrespective of buyer type) and then follows the optimal mechanism; hence,

$$\{\pi \in [0, \infty) : \pi < \bar{\pi}_S(\lambda, \Delta)\} \subset \Pi^S(\lambda, \Delta) \subset \{\pi \in [0, \infty) : \pi \leq \bar{\pi}_S(\lambda, \Delta)\}$$

and

$$\overline{\Pi^S(\lambda, \Delta)} = \{\pi \in [0, \infty) : \pi \leq \bar{\pi}_S(\lambda, \Delta)\}$$

where $\overline{A}$ denotes the closure of A for any subset A of the real line.

excludes the trivial case in which the folk theorem necessarily fails.

Finally note that even under Assumption $u_0^{ex-post} = 0$, the zero type's ex ante net valuation $u_0^{ex-ante} = v_0 - \frac{e^{-r\Delta}(1-e^{-\lambda\Delta})}{1-e^{-(\lambda+r)\Delta}} w_0$ is still strictly positive; hence, it is common knowledge that the gain from trading is strictly positive. Our folk theorem stands in contrast to the classical bargaining game without outside options in which the Coase conjecture holds in all equilibria with the commonly known gain from trading.

To prove Proposition 3, it suffices to show

$$\lim_{\lambda\to\infty} \lim_{\Delta\to 0} d_H \left(V^S(\lambda, \Delta), \Pi^S(\infty, \Delta) \right) = 0. \tag{26}$$

which implies, together with Lemma 2 and the triangle inequality, (24). The proof of (26) is by construction. Specifically, for any profit $\pi_S \in \lim_{\Delta\to 0} \Pi^S(\infty, \Delta)$, we can construct a (double) sequence of equilibria $(\alpha_{\lambda,\Delta})_{\lambda,\Delta\in(0,\infty)} \subset \prod_{\lambda,\Delta\in(0,\infty)} \mathcal{E}(\lambda, \Delta)$ such that $V^S(\alpha_{\lambda,\Delta}; \lambda, \Delta) \to \pi_S$ in the limit. The full proof can be found in the Appendix, and here we only sketch the equilibrium that approximates the highest profit level under the assumption that $\frac{w_2}{w_1} > \frac{v_2}{v_1} \geq \frac{q_1+q_2}{q_2}$, namely $q_2 u_2^{ex-post} + q_1 \frac{v_1 w_2 - v_2 w_1}{v_2 - v_1}$. Recall that the bargaining mechanism $\mu^{\ddagger}$ that achieves this profit level with commitment involves a positive delay of agreement. It is thus an interesting question whether there is an equilibrium with a delay on its path even without commitment.

The equilibrium consists of five phases, beginning at Skimming Phase I and then proceeding to Impasse Phase I, Skimming Phase II, Impasse Phase II, and the Coasian Phase in sequence unless the seller deviates. Once the seller deviates, the equilibrium play immediately proceeds to the Coasian Phase. Here, we will describe everything as if the bargaining game were played in continuous-time. $\Delta > 0$ is small but positive; hence, all statements hereafter are only approximately true. Although doing so sacrifices rigor, we can render the statement in a much simpler and cleaner way, which facilitates exposition. A more precise description of the equilibrium assessment can be found in the Appendix. In the following, t generically refers to time in continuous-time.

- Skimming Phase I ($t = 0$): At $t = 0$, the seller offers $p_H = u_2^{ex-post} = v_2 - w_2$ and this is accepted by a buyer of type $2 \in \Theta$ and rejected by those of other types.

- Impasse Phase I ($0 < t < \frac{1}{r} \log \frac{v_2 - v_1}{w_2 - w_1}$): The seller insists on p_H. The type-1 buyer neither accepts p_H nor exercises the outside option through the phase. The type-0 buyer continues to reject p_H and exercises the outside option as soon as it arrives.

- Skimming Phase II ($t = \frac{1}{r} \log \frac{v_H - v_L}{w_H - w_L}$): The seller offers $p_L = \frac{v_1 w_2 - v_2 w_1}{w_2 - w_1}$ at $t = t_L := \frac{1}{r} \log \frac{v_2 - v_1}{w_2 - w_1}$, which is instantly accepted by a type-1 buyer with probability $b_L \in (0, 1)$.

- Impasse Phase II ($t > \frac{1}{r} \log \frac{v_H - v_L}{w_H - w_L}$): The seller continues to insist on p_L, conceding to playing a Coasian equilibrium (see the description of the Coasian phase below) at the rate $y_L > 0$. Meanwhile, a type-1 buyer concedes to accepting p_L at rate λ, whereas she never exercises the outside option. A type-0 buyer never accepts p_L and exercises the outside option as soon as it arrives.

- Coasian Phase: Once the investor deviates from equilibrium play, or once he offers other than p_L in Impasse Phase II, all players begin to play a Coasian equilibrium in which the seller only offers prices arbitrarily close to $v_0 + \frac{\lambda}{\lambda+r} w_0$, and all buyer types accept a seller's offer almost immediately.

To complete the description of the equilibrium, we need to determine y_L, the rate at which the seller concedes to playing a Coasian equilibrium, and b_L, the probability that a type-1 buyer accepts p_L at $t = \frac{1}{r} \log \frac{v_2 - v_1}{w_2 - w_1}$. First, b_L must justify the seller's randomization in Impasse Phase II. A type-0 buyer always exercises the outside option as soon as possible in Skimming Phase and Impasse Phase I; hence, the seller's posterior immediately after the buyer rejects p_L at the beginning of Impasse Phase II is

$$q(\theta) = \begin{cases} 0 & \text{if } \theta = 2 \\ \frac{(1-b_L)q_1}{(1-b_L)q_1 + q_0 e^{-\frac{\lambda}{r} \log \frac{v_2-v_1}{w_2-w_1}}} & \text{if } \theta = 1 \\ \frac{e^{-\frac{\lambda}{r} \log \frac{v_2-v_1}{w_2-w_1}} q_0}{(1-b_L)q_1 + q_0 e^{-\frac{\lambda}{r} \log \frac{v_2-v_1}{w_2-w_1}}} & \text{if } \theta = 0 \end{cases}$$

With this posterior belief, the seller's expected profit from insisting on p_L (which is accepted by a type-1 buyer at rate λ) is

$$p_L \frac{\lambda}{\lambda + r} \frac{(1-b_L)q_1}{(1-b_L)q_1 + e^{-\frac{\lambda}{r} \log \frac{v_2-v_1}{w_2-w_1}} q_0}$$

while the profit from conceding to the Coasian equilibrium immediately is

$$v_0 - \frac{\lambda}{\lambda + r} w_0 = \frac{r}{\lambda + r} v_0$$

By equating two expected profits

$$b_L = 1 - \frac{r}{\lambda} \frac{v_0}{p_L} \left[1 + \frac{q_0}{q_1} e^{-\frac{\lambda}{r} \log \frac{v_2-v_1}{w_2-w_1}} \right]$$

and it is between 0 and 1 and hence a legitimate probability for all sufficiently large λs.

y_L must justify a type-1 buyer's randomization between accepting p_L immediately and waiting for the Coasian equilibrium to be played. The payoff from the first option is

$$v_1 - p_L$$

while the payoff from the second option is

$$\frac{y_L}{y_L + r}\left[v_1 - \left(v_0 - \frac{\lambda}{\lambda + r}w_0\right)\right]$$

hence

$$y_L = \frac{r(v_1 - p_L)}{p_1 - \left(v_0 - \frac{\lambda}{\lambda+r}w_0\right)}.$$

It is straightforward to see that the seller has no incentive to deviate, as long as λ is sufficiently high. Most importantly, the seller's equilibrium profit is

$$q_2 u_2^{ex-post} + e^{-r\frac{1}{r}\log\frac{v_2-v_1}{w_2-w_1}}\left[q_1 y_L p_L + (1 - q_2 - q_1(1-y_L))\left(v_0 - \frac{\lambda}{\lambda + r}w_0\right)\right].$$

As $\lambda \to \infty$,

$$y_L \to 1 \quad \text{and} \quad v_0 - \frac{\lambda}{\lambda + r}w_0 \to 0$$

and hence the investor's equilibrium profit converges to

$$q_2 p_H + q_1 \frac{w_2 - v_1}{v_2 - v_1} p_L = q_2 u_2^{ex-post} + q_1 \frac{v_1 w_2 - v_2 w_1}{v_2 - v_1}$$

as desired.

On the equilibrium path, the seller makes three offers in sequence, p_H, p_L and then $v_0 - \frac{\lambda}{\lambda+r}w_0$ with an impasse between two consecutive offers. In the limiting case of $\lambda \to \infty$, the outside option arrives and a type-0 buyers opts out almost immediately; hence, the negotiation concludes with probability approaching 1 before $v_0 - \frac{\lambda}{\lambda+r}w_0$ is offered by the seller. Further note that the impasse in Impasse Phase I does not shrink as $\lambda \to \infty$, while the second impasse disappears in the limit. Similar offer dynamics are observed in the equilibria characterized by Abreu and Gul (2000) and Deneckere and Liang (2006), although the driving forces differ.

4.3 Frictionless Case

We conclude this section by noting that no delay ever occurs in the frictionless case without commitment by the seller. The following proposition is due to Board and Pycia (2014).[17]

Proposition 4 (Board and Pycia, 2014). *Suppose that $\lambda = \infty$. For any $\Delta > 0$ and for any equilibrium in $\mathcal{E}(\lambda, \Delta)$, all buyer types accept the seller's offer or exit to take the outside option in the first period.*

Board and Pycia also show that there is essentially a unique equilibrium[18] such that the seller maintains one price in equilibrium, and buyer types with high ex post net-valuation levels

[17] Precisely, Proposition 1 of Board and Pycia (2014) states that all players' equilibrium payoffs are identical across all equilibria and is silent with respect to the equilibrium strategies. The statement (and its proof) that all buyer types buy or exit in the first period can be found in the proof of this proposition.

[18] All equilibria have the same equilibrium payoff profile.

accept it immediately, while types with low ex post net-valuation levels opt for their outside options in period zero. Note that this equilibrium achieves $\bar{\pi}_0^S(\lambda, \Delta)$, but it is strictly lower than the optimal mechanism's profit level $\bar{\pi}_S(\lambda, \Delta)$ under Conditions (21) and (22).

Due to Proposition 2 (the second inequality), the proposition implies that the folk theorem also generally fails in the frictionless case.

Corollary 3. *Suppose $\lambda = \infty$, and*

$$\lim_{\Delta \to 0} \bar{\pi}_S(\lambda, \Delta)|_{\lambda=\infty} > \lim_{\Delta \to 0} \lim_{\epsilon \to 0} \bar{\pi}_S^{\epsilon}(\lambda, \Delta)|_{\lambda=\infty}.$$

There is $\Delta^* > 0$ and $M > 0$ such that

$$\sup \mathcal{V}^S(\lambda, \Delta) < \sup \Pi^S(\lambda, \Delta) - M \quad \forall \Delta \in (0, \Delta^*).$$

The failure of the folk theorem stems from the nonexistence of an effective punishment scheme such as Coasian equilibrium. Recall that the Coasian equilibrium could exist in the frictional case because two negotiation parties can bargain as if there were no outside option in the presence of some frictions, at least over a short time interval; this small time interval is sufficient for the logic of the Coase conjecture to come into effect. However no time window for which the seller is relieved from the buyer's outside option is allowed in the frictionless case because it is common knowledge that the outside option is available throughout the negotiation.

As noted in the introduction, the last proposition highlights the discontinuity of the bargaining outcome in terms of the friction present in the arrival of the buyer's outside option. If the seller can ensure that he is the buyer's first bargaining counterpart, he can achieve a far better profit relative to the case in which there is a chance that he is the second counterpart. This discontinuity has some interesting implications for the competition between sellers observable in real-life situations. For example, in shopping malls, sellers sometimes aggressively bid for places adjoining main elevators, which will increase the probability of being the first seller to random buyers. It also explains some observed launching-date wars for new products or services.

5 Discussion

5.1 General Class of Arrival Processes

All results in this paper maintain either of two Assumptions (1) and (2). In both cases, the outside option never expires once it arrives to the buyer. However, the outside option may naturally expire after a certain duration in certain contexts. Alternatively, the outside option could randomly switch from being available and unavailable over time, which is the case if the outside option comes from offers from a sequence of short-lived outside sellers whose arrival timings are random. One might wonder how far we can generalize our results to cover at least some of those alternative arrival processes for the outside option.

We can show that all results for the frictional case remain true for a more general class of arrival processes for the outside option that satisfy the following condition

$$\exists\lambda \in (0,\infty) \quad \text{such tat} \quad \mathbb{P}\{a_1 = 1\} = \mathbb{P}\{a_n = 1 | a_k = 0 \ \ \forall k < n\} = 1 - e^{-\lambda\Delta} \quad \forall n \geq 2 \tag{27}$$

where λ is interpreted as the *rate of the first arrival of the outside option.* This condition allows, for example, an outside option that is randomly available in each period (with a probability that may not be stationary) after it becomes to the buyer for the first time. Note that (1) is a special case of (27).

One can easily verify that all results for bargaining with commitment holds for all arrival processes for which (27) holds. To verify that the results for bargaining without commitment also hold, note first that the following refined version of Lemma 5 is true, the proof of which can be found in the Appendix.

Lemma 6. *Suppose Assumptions* (1) *and* (25)*. For any small* $\Delta > 0$*, there exists a Coasian equilibrium* $\alpha \in \mathcal{E}(\lambda, \Delta, (q_\theta)_{\theta\in\Theta})$ *for which the following statement holds: there is a partition* Θ_1*,* Θ_2 *of* Θ *such that* $\Theta = \Theta_1 \cup \Theta_2$*,* $\Theta_1 \cap \Theta_2 = \emptyset$*, and*

$$\xi_\theta(p_n; h^{n-1}, a^n) = \begin{cases} \mathbb{1}\{a_n = 1, p_n > v_\theta - w_\theta\} & \text{if } \theta \in \Theta_1 \\ 0 & \text{if } \theta \in \Theta_2. \end{cases} \tag{28}$$

for any $(h^{n-1}, a^n) \in H_B^n$*,* $p_n \geq 0$*, and* $n \geq 1$*.*

Note that the last lemma maintains Assumption (1), not Assumption (27). The lemma states that we can construct a Coasian equilibrium in which all buyer types either exercise their outside option as soon as it becomes available (if $\theta \in \Theta_1$), or never exercise otherwise (if $\theta \in \Theta_2$).

Now recall the assessment provided for the illustration of Proposition 3, and note that we can replace the equilibrium play in the Coasian Phase by one described in the last lemma without loss. One can also easily verify that (28) also holds in all three other phases. Hence, the presence of the buyer in period $n \geq 1$ always indicates that either $\theta \in \Theta_2$ or $a_n = 0$. The only relevant consideration for the seller regarding the outside option is, therefore, (i) whether it has arrived and (ii) the probability of arriving in the next period conditional on $a_n = 0$. In other words, the arrival rate of the *first outside option* is the only parameter that matters to the seller. All arrival processes with (27) share the common property regarding the arrival rate of the first outside option, which suggests that the same assessment still constitutes an equilibrium for the general class of arrival processes for the outside option that satisfies (27).

5.2 Observable Outside Options

We have focused on the case in which the arrival of the outside option is not observable to the seller and is hence the buyer's private information. Another possibility is that both parties commonly observe the availability of the outside option in each period. It is straightforward that all our results with commitment hold true with an observable outside option. The more interesting question is what happens without commitment.

The bargaining game without commitment and with observable outside options is a special case of the environment considered in Fuchs and Skrzypacz (2010). They consider a bargaining game in which the negotiation stochastically breaks down in the middle, yielding prescribed payoffs to the buyer and the seller that depend on the buyer's true type. To see this equivalence, note that once two parties commonly observe the arrival of the outside option, the continuation game is equivalent to the frictionless case analyzed in Section 4.3. In this continuation game, due to Proposition 4, there is essentially a unique equilibrium in which the bargaining concludes immediately with all players' payoffs depending on the true type of the buyer.

However, the results in Fuchs and Skrzypacz (2010) cannot directly apply to this case due to the difference in technical assumptions. For instance, Fuchs and Skrzypacz assume that the buyer's type space is a continuum without atom, while this article maintains the assumption that it is finite. The case with two buyer types and arriving observable outside options is studied by Hwang and Li (2014). They show that a generalized version of the Coase conjecture holds with the observable arriving outside option; in our terminology, there is essentially a unique equilibrium that satisfies the statement in Lemma 5. Their observation again reinforces the importance of the nature of the outside option's arrival process in determining the bargaining outcome with outside options.

6 Conclusion

This article considered an asymmetric information bargaining game in which the buyer (informed party) has a type-dependent outside option. The article showed that the outside option has stark effects on the bargaining outcome. Most notably, the outside option leads to a delay in agreement either with or without commitment by the seller (uninformed party) when there is a friction in the outside option's arrival. A delay is found to be beneficial to the seller; hence an equilibrium with a delay likely constitutes a focal point among the model's multiple equilibria, especially when the seller can take the initiative in bargaining. The outside option's arrival process and the outside option itself are decisive in determining the bargaining outcome. A small change in the arrival process may result in a stark difference in outcomes hence, a careful study of it appears necessary for a better understanding of bargaining.

In addition to these theoretical contributions, the article has immediate practical implications. For example, the seller can achieve a higher equilibrium profit if there is slight chance that he has arrived ahead of the buyer's outside option, relative to the case in which it is certain that the outside option came ahead of the seller. This observation highlights the benefit of moving first relative to other competing sellers and offers a rationale for the launching-date wars that are often observed in newly emerging industries.

Let us conclude the paper by with some suggestions for further extensions of the model. The value of the outside option in this article's model was exogenously given as one of the model's parameters, and the origin of the option was not explicitly modeled. It would be an interesting direction for the future research to endogenize the buyer's outside option. Another possibility would be to enrich the model by incorporating strategic moves to reduce the value of the counterparty's outside option or shore up one's own. The majority of business discussions on bargaining center on such tactics; hence, incorporating strategic moves could improve on our understanding of contemporary business activities.

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Appendix A Omitted Proofs

A.1 Proof of Lemma 2

We begin by making a preliminary observation. With $\lambda < \infty$, $\mu \in \mathcal{M}$, and $\sigma_B = (\sigma_\theta)_{\theta\in\Theta}$ (which is not necessarily admissible for μ), define x_θ^{μ,σ_B}, y_θ^{μ,σ_B}, and p_θ^{μ,σ_B} as (11). Then,

$$y_\theta^{\mu,\sigma_B} \leq \max_{\substack{\nu_1,\nu_2\in[0,1] \\ x_\theta^{\mu,\sigma_B}=\mathbb{P}\{a_1=0\}\cdot\nu_0+\mathbb{P}\{a_1=1\}\cdot\nu_1}} \mathbb{P}\{a_1=1\}(1-\nu_1)+\mathbb{P}\{a_1=0\}\cdot(1-\nu_0)\cdot\frac{e^{-r\Delta}(1-e^{-\lambda\Delta})}{1-e^{-(\lambda+r)\Delta}}$$

or equivalently,

$$y_\theta^{\mu,\sigma_B} \leq \mathcal{B}^{\lambda,\Delta}(x_\theta^{\mu,\sigma_B}) := \begin{cases} \frac{1-e^{-\lambda\Delta}}{1-e^{-(\lambda+r)\Delta}} - x_\theta^{\mu,\sigma_B} & \text{if } x_\theta^{\mu,\sigma_B} \leq 1-e^{-\lambda\Delta} \\ (1-x_\theta^{\mu,\sigma_B})\frac{e^{-r\Delta}(1-e^{-\lambda\Delta})}{1-e^{-(\lambda+r)\Delta}} & \text{otherwise} \end{cases} \tag{29}$$

Conversely, for any $(x_\theta, y_\theta)_{\theta\in\Theta}$ such that $y_\theta \leq \mathcal{B}^{\lambda,\Delta}(x_\theta)$ for any $\theta\in\Theta$, there is $\mu\in\mathcal{M}$ and σ_B such that $x_\theta^{\mu,\sigma_B} = x_\theta$ and $y_\theta^{\mu,\sigma_B} = y_\theta$ for any $\theta \in \Theta$.

To obtain the limiting characterization of $\Pi^{\lambda,\Delta}$, consider an arbitrary positive real number $\epsilon > 0$, and choose sequences $(\lambda_m)_{m\geq1} \subset \mathbb{R}_+$ $(\Delta_{m,k})_{m,k\geq1} \subset \mathbb{R}_+$, $(K_m)_{m\geq1}\subset\mathbb{N}$, and a positive integer $M > 0$ such that

$$\lambda_m \to \infty \quad \text{as} \quad m\to\infty$$
$$\Delta_{m,k} \to 0 \quad \text{as} \quad k\to\infty \quad \forall m\in\mathbb{N}$$
$$1-e^{-\lambda_m\Delta_{m,k}} < \frac{\epsilon^2}{5(|\Theta|+1)\bar{v}} \quad \text{and} \quad |\pi_\theta^B - w_\theta| = \left|\frac{e^{-\lambda_m\Delta_{m,k}}(1-e^{-r\Delta_{m,k}})}{1-e^{-(r+\lambda_m)\Delta_{m,k}}}w_\theta\right| < \frac{\epsilon^2}{2(|\Theta|+1)} \tag{30}$$

for any $k > K_m$, $m > M$, and $\theta\in\Theta$. Without loss, we may assume $0 < \epsilon < \sqrt{\bar{v}(|\Theta|+1)}$. In the following paragraphs, fix any m and k such that $k > K_m$, $m > M$.

We first show

$$\Pi^{\lambda_m,\Delta_{m,k}} \subset \bigcup_{\substack{(\hat{\pi}_S,(\hat{\pi}_\theta)_{\in\Theta}) \\ \in\ \Pi^{\infty,\Delta_{m,k}}}} \left\{(\tilde{\pi}_S,(\tilde{\pi}_\theta)_{\theta\in\Theta}) \in \mathbb{R}_+^{|\Theta|+1} : d_E\big((\hat{\pi}_S,(\hat{\pi}_\theta)_{\theta\in\Theta}),(\tilde{\pi}_S,(\tilde{\pi}_\theta)_{\theta\in\Theta})\big) < \epsilon\right\} \tag{31}$$

where d_E stands for the Euclidean metric. Consider a mechanism μ, admissible decision rule $\sigma_B = (\sigma_\theta)_{\theta\in\Theta}$. Define x_θ^{μ,σ_B}, y_θ^{μ,σ_B}, and p_θ^{μ,σ_B} as (11), so that

$$\pi_S(\mu,\sigma_B;\lambda_m,\Delta_{m,k}) = \sum_{\theta\in\Theta} q_\theta p_\theta^{\mu,\sigma_B}, \quad \pi_\theta(\mu,\sigma_B;\lambda_m,\Delta_{m,k}) = x_\theta^{\mu,\sigma_B}v_\theta + y_\theta^{\mu,\sigma_B}w_\theta - p_\theta^{\mu,\sigma_B} \quad \forall\theta\in\Theta.$$

We will identify a point in $\Pi^{\infty\Delta_{m,k}}$ whose distance to $(\pi_S(\mu,\sigma_B;\lambda_m,\Delta_{m,k}),(\pi_\theta(\mu,\sigma_B;\lambda_m,\Delta_{m,k}))_{\theta\in\Theta})$ is less than ϵ. First suppose $\pi_S(\mu,\sigma;\lambda_m,\Delta_{m,k}) \geq \frac{\epsilon^2}{|\Theta|+1}$, and define

$$x_\theta = x_\theta^{\mu,\sigma_B}, \quad y_\theta = y_\theta^{\mu,\sigma_B}, \quad p_\theta = p_\theta^{\mu,\sigma_B} + \frac{\epsilon^2}{2(|\Theta|+1)} \quad \forall\theta\in\Theta.$$

The constraint clearly (10) holds for any $\theta \in \Theta$. By (30),

$$x_\theta v_\theta + y_\theta w_\theta - p_\theta = \pi_\theta(\mu, \sigma_B; \lambda_m, \Delta_{m,k}) + \frac{\epsilon^2}{2(|\Theta|+1)} \geq \underline{\pi}_\theta^B + \frac{\epsilon^2}{2(|\Theta|+1)} \geq w_\theta \quad \forall \theta \in \Theta$$

Also, as each buyer type $\theta \in \Theta$ always can act as if another type $\theta' \neq \theta$ in μ,

$$x_\theta v_\theta + y_\theta w_\theta - p_\theta \geq x_{\theta'} v_\theta + y_{\theta'} w_\theta - p_{\theta'}.$$

Hence, $\left(\pi_S(\mu, \sigma_B; \lambda_m, \Delta_{m,k}) - \frac{\epsilon^2}{2(|\Theta|+1)}, \left(\pi_\theta(\mu, \sigma_B; \lambda_m, \Delta_{m,k}) - \frac{\epsilon^2}{2(|\Theta|+1)}\right)_{\theta \in \Theta}\right) \in \Pi^{\infty, \Delta_{m,k}}$ and its Euclidean distance to $(\pi_S(\mu, \sigma_B; \lambda_m, \Delta_{m,k}), (\pi_\theta(\mu, \sigma_B; \lambda_m, \Delta_{m,k}))_{\theta \in \Theta})$ is less than ϵ.

Now suppose $\pi_S(\mu, \sigma_B; \lambda_m, \Delta_{m,k}) < \frac{\epsilon^2}{|\Theta|+1}$. Then x_θ^{μ, σ_B} is necessarily smaller than $\frac{\epsilon^2}{v_\theta(|\Theta|+1)}$, and

$$\begin{aligned}
\pi_\theta(\mu, \sigma_B; \lambda_m, \Delta_{m,k}) &< \frac{\epsilon^2}{|\Theta|+1} + \left(1 - \frac{\epsilon^2}{v_\theta(|\Theta|+1)}\right) \frac{e^{-r\Delta_{m,k}}(1 - e^{-\lambda_m \Delta_{m,k}})}{1 - e^{-(\lambda_m + r)\Delta_{m,k}}} w_\theta \\
&\leq \frac{\epsilon^2}{|\Theta|+1} \left(1 - \frac{e^{-r\Delta_{m,k}}(1 - e^{-\lambda_m \Delta_{m,k}})}{1 - e^{-(\lambda_m + r)\Delta_{m,k}}}\right) + \frac{e^{-r\Delta_{m,k}}(1 - e^{-\lambda_m \Delta_{m,k}})}{1 - e^{-(\lambda_m + r)\Delta_{m,k}}} w_\theta \\
&< \frac{\epsilon^2}{|\Theta|+1} + \frac{1 - e^{-\lambda_m \Delta_{m,k}}}{1 - e^{-(\lambda_m + r)\Delta_{m,k}}} w_\theta
\end{aligned}$$

for any $\theta \in \Theta$. Moreover, because σ_B is admissible,

$$\pi_\theta(\mu, \sigma_B; \lambda_m, \Delta_{m,k}) \geq \frac{1 - e^{-\lambda_m \Delta_{m,k}}}{1 - e^{-(\lambda_m + r)\Delta_{m,k}}} w_\theta = \underline{\pi}_\theta^B \quad \text{for all } \theta \in \Theta,$$

and hence

$$\pi_\theta(\mu, \sigma_B; \lambda_m, \Delta_{m,k}) \in \left(\underline{\pi}_\theta^B, \underline{\pi}_\theta^B + \frac{\epsilon^2}{|\Theta|+1}\right)$$

The distance between $(\pi_S(\mu, \sigma_B; \lambda_m, \Delta_{m,k}), (\pi_\theta(\mu, \sigma_B; \lambda_m, \Delta_{m,k}))_{\theta \in \Theta})$ and $\left(0, (\underline{\pi}_\theta^B)_{\theta \in \Theta}\right)$ is less than ϵ, where the latter is clearly in $\Pi^{\infty, \Delta_{m,k}}$.

We next show

$$\Pi^{\infty, \Delta_{m,k}} \subset \bigcup_{\substack{(\hat{\pi}_S, (\hat{\pi}_\theta)_{\in \Theta}) \\ \in\ \Pi^{\lambda_m, \Delta_{m,k}}}} \left\{(\tilde{\pi}_S, (\tilde{\pi}_\theta)_{\theta \in \Theta}) \in \mathbb{R}_+^{|\Theta|+1} : d_E\big((\hat{\pi}_S, (\hat{\pi}_\theta)_{\theta \in \Theta}), (\tilde{\pi}_S, (\tilde{\pi}_\theta)_{\theta \in \Theta})\big) < \epsilon\right\} \tag{32}$$

which, together with (32), will complete the proof. Choose a point $(\tilde{\pi}_S, (\tilde{\pi}_\theta)_{\theta \in \Theta}) \in \Pi^{\infty, \Delta_{m,k}}$ such that (7)-(10) hold for some $(x_\theta, y_\theta, p_\theta)_{\theta \in \Theta} \geq 0$. As

$$\lim_{m \to \infty} \lim_{k \to \infty} \{(x_\theta, y_\theta) : 0 \leq y_\theta \leq \mathcal{B}^{\lambda, \Delta}(x_\theta)\} = \{(x_\theta, y_\theta) : 0 \leq y_\theta \leq 1 - x_\theta \leq 1\}$$

in Hausdorff metric, we may assume without loss

$$y_\theta \leq \mathcal{B}^{\lambda_m, \Delta_{m,k}}\left(1 - \max\left\{0, x_\theta - \frac{\epsilon^2}{5(|\Theta|+1)\bar{v}}\right\}\right)$$

Let $p_\theta^* \in \mathbb{R}_+$, $n_\theta^* \in \mathbb{N}$, and $x_\theta^* \in [0,1]$ be the solution of the following system of equations:

$$\max\left\{0, x_\theta - \frac{\epsilon^2}{5(|\Theta|+1)\bar{v}}\right\} = x_\theta^* e^{-r(n_\theta^*-1)\Delta_{m,k}}, \quad y_\theta = (1-x_\theta^*)e^{-r(n_\theta^*-1)\Delta_{m,k}},$$

and

$$p_\theta - \frac{7\epsilon^2}{10(|\Theta|+1)} = p_\theta^* x_\theta^* e^{-r(n_\theta^*-1)\Delta_{m,k}} \quad \forall \theta \in \Theta$$

As $y_\theta = (1-x_\theta^*)e^{-r(n_\theta^*-1)\Delta_{m,k}} \leq \mathcal{B}^{\lambda_m,\Delta_{m,k}}(x_\theta^* e^{-r(n_\theta^*-1)\Delta_{m,k}})$ for any $\theta \in \Theta$, we can find a mechanism μ and decision rule σ_B such that $x_\theta^{\mu,\sigma} = x_\theta^* e^{-r(n_\theta^*-1)\Delta_{m,k}}$ and $y_\theta^{\mu,\sigma} = (1-x_\theta^*)e^{-r(n_\theta^*-1)\Delta_{m,k}}$ for all θ. Indeed, consider the following mechanism μ: if the buyer has reported her type as $\theta \in \Theta$ in the first period, μ offers p_n in period each period n where

$$p_n = \begin{cases} p_\theta^* & \text{with probability } x_\theta^* \\ \bar{v} & \text{with probability } 1-x_\theta^* \end{cases}$$

if $n = n_\theta^*$ and the two negotiation parties fail in reaching an agreement in periods $k \leq n = n_\theta^*$, and

$$p_n = \bar{v} \quad \text{with probability 1,} \quad \text{otherwise.}$$

The mechanism does not discriminate the buyer based on her report about the outside option's arrival time.

Each buyer's optimal decision rule facing μ is as follows: truthfully report θ in period 1, and then wait until period n_θ^* and accept $p_{n_\theta^*}$ if and only if $p_{n_\theta^*} = p_\theta^*$; if μ offers $p_{n_\theta^*} = \bar{v}$, the buyer rejects it and exercise the outside option immediately.

$$e^{r(n_\theta^*-1)} x_\theta^*(v_\theta - p_\theta^*) > x_\theta v_\theta - p_\theta + \frac{\epsilon^2}{2(|\Theta|+1)} > w_\theta$$

hence every buyer type has no incentive to exercise her outside option in period such that $1 < k < n_\theta^*$. Under this decision rule, the seller's expected profit in μ is

$$\sum_{\theta\in\Theta} q_\theta \left(p_\theta - \frac{7\epsilon^2}{10(|\Theta|+1)}\right) < \pi_S(\mu, \sigma_B; \lambda_m, \Delta_{m,k}) \leq \sum_{\theta\in\Theta} q_\theta p_\theta$$

and each buyer type's payoff is

$$\tilde{\pi}_\theta + \frac{\epsilon^2}{2(|\Theta|+1)} < \pi_\theta(\mu, \sigma; \lambda_m, \Delta_{m,k}) < \tilde{\pi}_\theta + \frac{7\epsilon^2}{10(|\Theta|+1)}.$$

Hence the distance between $(\tilde{\pi}_S, (\tilde{\pi}_\theta)_{\theta\in\Theta})$ and $(\pi_S(\mu, \sigma_B; \lambda_m, \Delta_{m,k})(\pi_\theta(\mu, \sigma_B; \lambda_m, \Delta_{m,k}))_{\theta\in\Theta})$ is less than $\epsilon > 0$.

A.2 Proof of Lemma 3 and Lemma 4

We first identify $\lim_{\Delta\to 0} \bar{\pi}_S^\epsilon(\lambda, \Delta)|_{\epsilon=0}$. Because any mechanism in $\mathcal{M}^{\epsilon|\lambda,\Delta}|_{\epsilon=0}$ never offers to trade in period $n > 1$, any buyer types necessarily exercise their outside options as soon as they arrive, once they fail to trade with the seller in period 0. Hence

$$\bar{\pi}_S^\epsilon|_{\epsilon=0} = \max_{(x_\theta, y_\theta, p_\theta)_{\theta=H,L}} q_H p_H + q_L p_L \tag{33}$$

subject to

$$
\begin{aligned}
& v_\theta x_\theta - p_\theta + w_\theta(1-x_\theta)\frac{1-e^{-\lambda\Delta}}{1-e^{-(\lambda+r)\Delta}} \geq v_\theta x_{\theta'} - p_{\theta'} + w_\theta(1-x_{\theta'})\frac{1-e^{-\lambda\Delta}}{1-e^{-(\lambda+r)\Delta}} \\
& v_\theta x_\theta - p_\theta + w_\theta(1-x_\theta)\frac{1-e^{-\lambda\Delta}}{1-e^{-(\lambda+r)\Delta}} \geq \frac{1-e^{-\lambda\Delta}}{1-e^{-(\lambda+r)\Delta}} w_\theta \\
& x_\theta \in [0,1], p_\theta \geq 0
\end{aligned}
$$

for any $\theta, \theta' \in \Theta$. The first two constraints are equivalent to

$$
\begin{aligned}
& \left(v_\theta - \frac{1-e^{-\lambda\Delta}}{1-e^{-(\lambda+r)\Delta}} w_\theta\right) x_\theta - p_\theta \geq \left(v_\theta - \frac{1-e^{-\lambda\Delta}}{1-e^{-(\lambda+r)\Delta}} w_\theta\right) x_{\theta'} - p_{\theta'} \\
& \left(v_\theta - \frac{1-e^{-\lambda\Delta}}{1-e^{-(\lambda+r)\Delta}} w_\theta\right) x_\theta - p_\theta \geq 0
\end{aligned}
$$

respectively. Then the seller's problem (33) is equivalent to one with no type-dependent outside option of the buyer which is already thoroughly studied by Riley and Zeckhauser (1983) and Samuelson (1984), except that the buyer's valuation of the good is replaced by $v_\theta - \frac{1-e^{-\lambda\Delta}}{1-e^{-(\lambda+r)\Delta}} w_\theta$ for $\theta = H, L$. Applying the main results in the above cited papers, one can easily see the statement of the lemma holds.

Note that the objective function and constraints for the seller's problem (33) linear and hence continuous with respect to all choice variables. Hence, invoking the continuity, we can actually show that

$$
\lim_{\epsilon \to 0} \lim_{\Delta \to 0} \left| \bar{\pi}_S^{\epsilon}(\lambda, \Delta) - \bar{\pi}_S^{\epsilon=0}(\lambda, \Delta) \right| = 0
$$

for all large $\lambda > 0$ with which $\mathcal{M}^{\epsilon|\lambda,\Delta}$ is nonempty.

A.3 Proof of Lemma 5 and Lemma 6

A.3.1 Preliminaries

Lemma 6 is more general proposition which has Lemma 5 as a special case. We will prove Lemma 6 by constructing an equilibrium where the seller only offers a price $p > 0$ such that

$$
p = \underline{u}^\lambda + O(\Delta) := \min \left\{ v_{\theta'} - \frac{\lambda}{\lambda + r} w_{\theta'} : \theta' \in \Theta \right\} + O(\Delta) \quad \text{for } \Delta \to 0
$$

both on and off the equilibrium path.[19] In such an equilibrium (if any) all buyer types in

$$
\Theta_0 := \{\theta \in \Theta : w_\theta > v_\theta - \underline{u}^\lambda\}
$$

will exercise the outside option once it arrives, while buyer types in

$$
\Theta_1 := \{\theta \in \Theta : w_\theta \leq v_\theta - \underline{u}^\lambda\}
$$

will never opt out on the equilibrium path.

[19] For any $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$, we writhe

$$
a(\Delta) = b(\Delta) + O(\Delta) \quad \text{for } \Delta \to 0
$$

if and only if there are $M > 0$ and $\epsilon > 0$ such that $|a(\Delta) - b(\Delta)| < M\Delta$ for all Δ such that $|\Delta| < \delta$.

We begin by relabeling (reordering) buyer's types. First, relabel the buyer type $\theta \in \Theta$ such that $v_\theta - \frac{\lambda}{\lambda+r} w_\theta = \underline{u}^\lambda$ as 0, and for any $\theta \in \Theta$, define

$$\rho(\theta) = \begin{cases} v_\theta & \text{if } \theta \in \Theta_0 \\ v_\theta - \frac{\lambda}{r}\left[v_0 - \frac{\lambda}{\lambda+r} w_0\right] & \text{if } \theta \in \Theta_1 \end{cases}$$

and reorder and relabel each buyer type as one in $\{0, 1, \ldots, |\Theta| - 1\}$ so that

$$\theta \geq \theta' \iff \rho(\theta) \geq \rho(\theta').$$

Finally, let

$$\Theta^* := \{0, 1, \ldots, |\Theta| - 1\}, \quad \Theta_0^* := \left\{\theta \in \Theta^* : \underline{u}^\lambda > v_\theta - w_\theta\right\}, \quad \text{and} \quad \Theta_1^* := \left\{\theta \in \Theta^* : \underline{u}^\lambda \leq v_\theta - w_\theta\right\}.$$

In the equilibrium constructed below, the equilibrium play of the buyer with $\theta = 0$ is as follows:

$$\chi_0(p; h^{n-1}, a^n) = \begin{cases} \mathbb{1}\left\{p \leq v_0 - \frac{e^{-r\Delta}(1-e^{-\lambda\Delta})}{1-e^{-(\lambda+r)\Delta}} \} w_0\right\} & \text{if } a_n = 0 \\ \mathbb{1}\left\{p \leq v_0 - w_0\right\} & \text{if } a_n = 1 \end{cases} \tag{34}$$

and

$$\xi_0(p; h^{n-1}, a^n) = 1 \tag{35}$$

for any $(h^{n-1}, a^n) = (p_1, \ldots, p_{n-1}; a_1, \ldots, a_n) \in H_B^n \subset H_B$, $p \geq 0$, and $n \in \mathbb{N}$. For $\theta = 1$ or 2,

$$\xi_\theta(p; h^{n-1}, a^n) = \mathbb{1}\{\theta \in \Theta_0^* \text{ and } a_n = 1\}. \tag{36}$$

Below, we will specify the seller's equilibrium offer strategy σ and his belief q, also complete acceptance decision rules χ_θ for all buyer types other than 0.

For any $\theta \in \Theta^*$, let

$$\zeta_\theta := e^{-\lambda\Delta} + (1 - e^{-\lambda\Delta})(1 - \mathbb{1}\{\theta \in \Theta_0^*\}) = \begin{cases} e^{-\lambda\Delta} & \text{if } \theta \in \Theta_0^* \\ 1 & \text{if } \theta \in \Theta_1^*. \end{cases}$$

and let $\delta := e^{-r\Delta}$ be the common discounting factor. Finally, for any $x \in \mathbb{R}$, let

$$\gamma(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Finally, for any two public histories h^{n-1} and h^{m-1} such that $m \geq n$, we write $h^{m-1} \succeq h^{n-1}$ if h^{m-1} is a continuation history of h^{n-1}. That is, either $h^{m-1} = h^{n-1}$ or there is $(p_n, p_{n+2}, \ldots, p_{m-1})$ such that $h^{m-1} = (h^{n-1}, p_n, p_{n+2}, \ldots, p_{m-1})$. Similarly, $a^m \succeq a^n$ if a^m is the continuation history of a^n, and $h_B^m = (h^{m-1}, a^m) \succeq h_B^n = (h^{n-1}, a^n)$ if both $h^{m-1} \succeq h^{n-1}$ and $a^m \succeq a^n$ hold.

A.3.2 Equilibrium Strategy Profile

The equilibrium consists of phase I and phase II. The bargaining begins in phase II and then enters in phase I under the condition we will specify soon; and then the bargaining stays in phase I indefinitely until the game concludes. We will first describe phase I, and then phase II will be discussed afterward.

Phase I: Let $(q_k^I)_{k\geq 0} \in [0,1]^{\mathbb{N}_0}$ be a sequence of positive numbers such that

$$0 = q_0^I \leq q_k^I < q_{k+1}^I \quad \text{for any nonnegative integer } k \in \mathbb{N} \quad \text{and} \lim_{k\to\infty} q_k^I = 1. \tag{37}$$

Also define

$$p_k^I = \left(1-(\zeta_1\delta)^k\right)\left(v_1 - \frac{\delta(1-\zeta_1)}{1-\zeta_1\delta}w_1\right) + (\zeta_1\delta)^k\left(v_0 - \frac{\delta(1-\zeta_0)}{1-\zeta_0\delta}w_0\right) \quad \forall k \in \mathbb{N}$$

and

$$p_\infty^I = v_1 - \frac{\delta(1-\zeta_1)}{1-\zeta_1\delta}w_1,$$

The bargaining enters in phase one, once the seller puts zero probability to the buyer being type 2. That is, for any histories (h^{n-1}, a^n) and $(h^{n-2}, a^{n-1}) \in H_B$ such that $n \geq 2$,

$$(h^{n-1}, a^n) \succ (h^{n-2}, a^{n-1}), \quad \text{and} \quad \bar{q}(2; h^{n-2}) > \bar{q}(2; h^{n-1}) = 0,$$

the bargaining is in phase II at (h^{n-2}, a^{n-1}) and in phase I at (h^{n-1}, a^n). The following completely describes the equilibrium strategy profile in the continuation game follows (h^n, a^{n-1}).

- *Strategy of the buyer type* $\theta = 1$: Consider any history $(h^{m-1}, a^m) \succeq (h^{n-1}, a^n)$.

$$\chi_1(p; h^m, a^m) = \mathbb{1}\{p \leq v_1 - w_1\} \quad \text{if } a_m = 1 \text{ and } 1 \in \Theta_0^*$$

  In all other cases,

$$\chi_1(p; h^{m-1}, a^m) = \begin{cases} 1 & \text{if } p \leq p_0^I \\ \gamma\left(1 - \frac{(1-\bar{q}(1;h^{m-1}))q_{k-1}^I}{(1-q_{k-1}^I)\bar{q}(1;h^{m-1})}\frac{\zeta_0}{\zeta_1}\right) & \text{if } p \in (p_{k-1}^I, p_k^I] \text{ for any } k \in \mathbb{N} \\ 0 & \text{if } p \geq p_\infty^I \end{cases}$$

- *Strategy of the buyer type* $\theta = 2$: Consider any history $(h^{m-1}, a^m) \succeq (h^{n-1}, a^n)$.

$$\chi_2(p; h^m, a^m) = \mathbb{1}\{p \leq v_2 - w_2\} \quad \text{if } a_m = 1 \text{ and } 2 \in \Theta_0^*$$

  In all other cases,

$$\chi_1(p; h^{m-1}, a^m) = \mathbb{1}\left\{p \leq (1-\delta)v_2 + \delta\zeta_2 p_k^I \text{ and } p \leq v_2 - w_2\delta(1-\zeta_2)/(1-\delta\zeta_2)\right\}$$

  for p such that $\bar{q}(1; h^{m-1}, p) \in [q_k^I, q_{k+1}^I)$ for an integer $k \geq 0$.[20]

[20] The buyer's strategy completely and uniquely pins down the seller's belief after any public history in the continuation game following (h^{n-1}, a^n), excepts for a history (h^{m-1}, a^m) such that $h^{m-1} = (p_1, \ldots, p_{m-1})$, $p_{m-1} \leq p_1^I$, and $m \geq 2$. For completeness, let

$$\bar{q}(\theta; h^{m-1}) = \begin{cases} \bar{q}(\theta; p_1, \ldots, p_{m-2}) & \text{if } m \geq 3 \\ \bar{q}(\theta; h^0) & \text{otherwise} \end{cases} \quad \forall \theta \in \Theta^*.$$

in this case. In other words, the seller does not change his belief conditional the buyer's rejection of p.

- *Strategy of the seller:* After any public history $h^{m-1} \in H$ such that $h^{m-1} \succeq h^{n-1}$,

$$\sigma(p_k^I; h^{m-1}) = 1 \qquad \text{if } \bar{q}(1; h^{m-1}) \in (q_k^I, q_{k+1}^I)$$

$$\left.\begin{array}{r}\sigma(p_k^I; h^{m-1}) = \alpha_k^I(p_{m-1}) \\ \sigma(p_{k-1}^I; h^{m-1}) = 1 - \alpha_k^I(p_{m-1})\end{array}\right. \quad \text{if } \bar{q}(1; h^{m-1}) = q_k^I$$

  for an integer $k \geq 0$, where

$$\alpha_k^I(p_{m-1}) = \begin{cases} \gamma\left(\frac{p_{m-1} - \delta\zeta_1 p_{k-1}^I - (1-\delta\zeta_1)v_1 + \delta(1-\zeta_1)w_1}{\delta\zeta_1(p_k^I - p_{k-1}^I)}\right) & \text{if } h^{m-1} \neq h^0 \\ 1 & \text{if } h^{m-1} = h^0 \end{cases}$$

There is a real number $\Delta^I > 0$ such that the following statement holds whenever $\Delta < \Delta^I$:

$$(h^{m-1}, a^m) \succeq (h^{n-1}, a^n)\bar{q}(1; h^{m-1}) > q_k^I, \quad \text{and} \quad p \in (p_k^I, p_{k+1}^I]$$
$$\Longrightarrow \quad \chi_1(p; h^{m-1}, a^m) = 1 - \frac{(1-\bar{q}(1; h^{m-1}))q_k^I}{(1-q_k^I)\bar{q}(1; h^{m-1})}\frac{\zeta_0}{\zeta_1} \in (0,1) \quad \text{and} \quad \bar{q}(1; p, h^{m-1}) = q_k^I,$$

In the following period (conditional on p is being rejected), the seller offers randomizes p_k^I and p_{k-1}^I with probability $\sigma(p_k^I; h^{m-1}, p)$ and $1 - \sigma(p_k^I; h^{m-1}, p)$ respectively, where the probabilities are chosen so that the buyer with $\theta = 1$ is indifferent between accepting p and not.

The sequence of beliefs $(q_k^I)_{k\geq 0}$ has to justify the seller's randomization with his belief that the buyer is of type-1 with probability q_k^I for each $k \geq 0$. For example, for any public history $(h_1^I, a) \in H_B$ such that $(h_1^I, a) \succeq (h^{n-1}, a^n)$ and $\bar{q}(1; h_1^I) = q_1^I$,

$$V^S(h_1^I) = p_1^I q_1^I \zeta_1 + \delta(1-q_1^I)\zeta_0^2 p_0^I = (q_1^I\zeta_1 + (1-q_1^I)\zeta_0)p_0^I \tag{38}$$

which implies

$$q_1^I = \frac{v_0 - \frac{\delta(1-\zeta_0)}{1-\zeta_0\delta}w_0}{\frac{\zeta_1(1-\zeta_1\delta)}{\zeta_0(1-\zeta_0\delta)}\left(v_1 - \frac{\delta(1-\zeta_1)}{1-\zeta_1\delta}w_1\right) + \left(1 - \frac{\zeta_1(1-\zeta_1\delta)}{\zeta_0(1-\zeta_0\delta)}\right)\left(v_0 - \frac{\delta(1-\zeta_0)}{1-\zeta_0\delta}w_0\right)} \in (0,1)$$

In general, for any public history $(h_k^I, a) \in H_B$ such that $(h_k^I, a) \succeq (h^{n-1}, a^n)$ and $\bar{q}(1; h_k^I) = q_k^I$,

$$V^S(h_k^I) = p_k^I q_k^I \zeta_1\left[1 - \frac{(1-q_k^I)q_{k-1}^I}{q_k^I(1-q_{k-1}^I)}\frac{\zeta_0}{\zeta_1}\right] + \delta\zeta_0\frac{1-q_k^I}{1-q_{k-1}^I}V^S(h_k^I, p_k^I) \tag{39}$$
$$= p_{k-1}^I q_k^I \zeta_1\left[1 - \frac{(1-q_k^I)q_{k-2}^I}{q_k^I(1-q_{k-2}^I)}\frac{\zeta_0}{\zeta_1}\right] + \delta\zeta_0\frac{1-q_k^I}{1-q_{k-2}^I}V^S(h_k^I, p_{k-1}^I) \tag{40}$$

for any $k \geq 2$ The system of equations (38)- uniquely determines $(q_k^I)_{k\geq 0}$. Moreover, we can show that there are $\Delta^I > 0$ and M^I such that the solution of the system of the difference equations satisfies

$$\frac{q_{k+1}^I}{1-q_{k+1}^I} - \frac{q_k^I}{1-q_k^I} \geq M^I \quad \forall k \geq 0, \quad \forall \Delta \in (0, \Delta^I).$$

Let $\nu^I(h^{n-1})$ be the integer such that

$$\bar{q}(1; h^{n-1}) \in [q_{\nu^I(h^{n-1})}^I, q_{\nu^I(h^{n-1})+1}^I).$$

Abusing notation, for any nonnegative integer $n \geq 0$, let $\nu^I(n)$ be equal to $h^n \in H$ such that (i) $\bar{q}(2; h^n) = 0$, and (ii) all prices in h^n are supposed to have been rejected in equilibrium by both buyer types 0 and 1. That is,

$$\frac{\bar{q}(1; h^0)\zeta_1^n}{\bar{q}(0; h^0)\zeta_0^n + \bar{q}(1; h^0)\zeta_1^n} \in [q^I_{\nu^I(n)}, q^I_{\nu^I(n)+1}).$$

Phase II: The bargaining begins in the second phase. Define

$$p^{II}_{m,k} := (1 - (\zeta_2\delta)^k)\left(v_2 - \frac{\delta(1-\zeta_2)}{1-\zeta_2\delta} w_2\right) + (\zeta_2\delta)^k p^I_{\nu^I(m+k)} \quad \text{and} \quad p^{II}_{m,\infty} := v_2 - \frac{\delta(1-\zeta_2)}{1-\zeta_2\delta} w_2$$

for all nonnegative integers m and k, and fix a double sequence $(q^{II}_{m,k})_{m,k\geq 0} \in [0,1]^{\mathbb{N}_0 \times \mathbb{N}_0}$ such that

$$q^{II}_{m,0} = 0 \quad \text{and} \quad 0 < q^{II}_{m,k} < 1 \quad \text{for any nonnegative integers } m \geq 0 \text{ and } k \geq 1.$$

The complete description of the equilibrium strategy profile is as follows.

- *Strategy of the buyer with $\theta = 2$:* Consider any history $(h^{m-1}, a^m) = (p_1, \ldots, p_{m-1}; a_1, \ldots a_m)$ such that $\bar{q}(2; h^{m-1}) > 0$.

  $$\chi_2(p; h^{m-1}, a^m) = \mathbb{1}\{p \leq v_2 - w_2\} \quad \text{if } a_m = 1 \textit{ and } 2 \in \Theta_0^*.$$

  In all other cases,

  $$\chi_2(p; h^{m-1}, a^m) = \begin{cases} 1 & \text{if } p \leq p^{II}_{m,0} \\ \gamma\left(1 - \frac{q^{II}_{m+1,k-1}\left(\bar{q}(0;h^{m-1})\zeta_0 + \bar{q}(1;h^{m-1})\zeta_1\right)}{(1-q^{II}_{m+1,k-1})\bar{q}(2;h)\zeta_2}\right) & \text{if } \exists k \text{ such that } p \in (p^{II}_{m,k-1}, p^{II}_{m,k}] \\ 0 & \text{if } p \geq p^{II}_{m,\infty} \end{cases}$$

- *Strategy of the buyer with $\theta = 1$:* Consider any history $(h^{m-1}, a^m) = (p_1, \ldots, p_{m-1}; a_1, \ldots a_m)$ such that $\bar{q}(2; h^{m-1}) > 0$.

  $$\chi_1(p; h^{m-1}, a^m) = \mathbb{1}\{p \leq v_2 - w_2\} \quad \text{if } a_m = 1 \textit{ and } 1 \in \Theta_0^*.$$

  In all other cases,

  $$\chi_1(p; h^{m-1}, a^m) = \begin{cases} 1 & \text{if } p \leq p^I_0 \\ \gamma\left(1 - \frac{(1-\bar{q}(1;h^{m-1}))q^I_{k-1}}{(1-q^I_{k-1})\bar{q}(1;h^{m-1})}\frac{\zeta_0}{\zeta_1}\right) & \text{if } \exists k \leq \nu^I(h^{m-1}) \text{ such that } p \in (p^I_{k-1}, p^I_k] \\ 0 & \text{if } p > p^I_{\nu^I(h^{m-1})} \end{cases}$$

- After any public history $h^{m-1} \in H$ such that $\bar{q}(2; h^{m-1}) > 0$,

  $$\sigma(p^{II}_{m,k}; h^{m-1}) = 1 \qquad \text{if } \bar{q}(2; h^{m-1}) \in (q^{II}_{m,k}, q^{II}_{m,k+1})$$

  $$\left.\begin{array}{r} \sigma(p^{II}_{m,k}; h^{m-1}) = \alpha^{II}_{m,k}(p_{m-1}) \\ \sigma(p^{II}_{m,k-1}; h^{m-1}) = 1 - \alpha^{II}_{m,k}(p_{m-1}) \end{array}\right. \quad \text{if } \bar{q}(2; h^{m-1}) = q^{II}_{m,k}$$

for any integers $m, k \geq 0$, where

$$\alpha^{II}_{m,k}(p_{m-1}) = \begin{cases} \gamma\left(\frac{p_{m-1}-\delta\zeta_2 p^{II}_{m,k-1}-(1-\delta\zeta_2)v_2+(1-\zeta_2)w_2}{\delta\zeta_2(p^{II}_{m,k}-p^{II}_{m,k-1})}\right) & \text{if } h^{m-1} \neq h^0 \\ 1 & \text{if } h^{m-1} = h^0. \end{cases}$$

There is a real number $\Delta^{II} > 0$ such that

$$\begin{aligned} &\forall (h^{m-1}, a^m) \quad \text{such that} \quad \bar{q}(2; h^{m-1}) > q^{II}_{m+1,k-1} \\ &\implies \quad \chi_2(p; h^{m-1}, a^m) \in (0,1) \quad \text{and} \quad \bar{q}(2; p, h^{m-1}) = q^{II}_{m+1,k-1} \quad \forall p \in (p^{II}_{m,k}, p^{II}_{m,k+1}] \end{aligned}$$

for any $m \geq 0$ and $k \geq 0$. If $p \in (p^{II}_{m,k}, p^{II}_{m,k+1})$, the seller randomizes $p^{II}_{m+1,k}$ and $p^{II}_{m+1,k-1}$ in the next period; if $p = p^{II}_{m,k+1}$, the seller offers $p^{II}_{m+1,k-1}$ with probability 1 in the next period. In the continuation game that follows (h^{m-1}, a^m) such that $\bar{q}(2; p, h^{m-1}) > 0$, the seller offers on its path

$$p^{II}_{m,\nu^{II}(h^{m-1})}, p^{II}_{m+1,\nu^{II}(h^{m-1})-1}, \ldots, p^{II}_{m+\nu^{II}(h^{m-1}),0} = p^{I}_{\nu^I(m+\nu^{II}(h^{m-1}))}$$

in order, where $\nu^{II}(h^{n-1}) \in \mathbb{N}_0$ will be defined soon. $p^I_{\nu^I(n+\nu^{II}(h^{n-1}))}$ will be accepted by type-2 with probability 1, hence the phase one begins afterward. Notice that the first and the second subscripts m and k of $p^{II}_{m,k}$ represent the number of remaining periods in phase II and the number of passed periods since the beginning of the negotiation, respectively.

The sequence of beliefs $(q^{II}_{m,k})_{m,k\geq 0}$ has to justify the seller's randomization off the equilibrium path. That is, for any $(h^{II}_{m,k}, a^m) \in H_B$ such that $\bar{q}(2; h^{II}_{m,k}) = q^{II}_{m,k}$,

$$\begin{aligned} V^S(h^{II}_{m,k}) &= p^{II}_{m,k} q^{II}_{m,k} \zeta_2 \chi_2(p^{II}_{m,k}; h^{II}_{m,k}, a) + \delta \frac{\sum_{\theta=0,1} \bar{q}(\theta; h^0)\zeta_\theta^m}{\sum_{\theta=0,1} \bar{q}(\theta; h^0)\zeta_\theta^{m-1}} \frac{1 - q^{II}_{m,k}}{1 - q^{II}_{m+1,k-1}} V^S(h^{II}_{m+1,k-1}) && (41) \\ &= p^{II}_{m,k-1} q^{II}_{m,k} \zeta_2 \chi_2(p^{II}_{m,k-1}; h^{II}_{m,k}, a) + \delta \frac{\sum_{\theta=0,1} \bar{q}(\theta; h^0)\zeta_\theta^m}{\sum_{\theta=0,1} \bar{q}(\theta; h^0)\zeta_\theta^{m-1}} \frac{1 - q^{II}_{m,k}}{1 - q^{II}_{m+1,k-2}} V^S(h^{II}_{m+1,k-2}) && (42) \end{aligned}$$

for any $m \geq 1$ and $k \geq 2$, and

$$\begin{aligned} V^S(h^{II}_{m,1}) &= p^{II}_{m,1} q^{II}_{m,1} \zeta_2 + \delta \frac{\sum_{\theta=0,1} \bar{q}(\theta; h^0)\zeta_\theta^m}{\sum_{\theta=0,1} \bar{q}(\theta; h^0)\zeta_\theta^{m-1}} \frac{1 - q^{II}_{m,1}}{1 - q^{II}_{m+1,0}} V^S(h^{II}_{m+1,0}) && (43) \\ &= p^{II}_{m,0} q^{II}_{m,1} \zeta_2 + \frac{1 - q^{II}_{m,1}}{\sum_{\theta=0,1} \bar{q}(\theta; h^0)\zeta_\theta^{m-1}} \begin{bmatrix} p^{II}_{m,0}\bar{q}(1; h^0)\zeta_1^m \chi_1(p^{II}_{m,0}; h^{II}_{m,1}, a^m) \\ +\delta \frac{\bar{q}(0; h^0)\zeta_0^m}{1 - q^I_{\nu^I(m-1)-1}} V^S(h^I_{\nu^I(m-1)-1}) \end{bmatrix} && (44) \end{aligned}$$

for all $m \geq 1$, where Applying equations (41)-(44) repeatedly, we can uniquely identifies $(q^{II}_{m,k})_{m,k\geq 0}$ as the solution for the system of equations. Similar to the phase I, we can show that there is M^{II} such that the phase II concludes within in M^{II} periods. Hence the real-time duration of the phase II is at most ΔM^{II} which converges to as $\Delta \to 0$.

The entire strategy profile can be constructed backward by repeatedly applying the same idea.

A.4 Proof of Proposition 1

Fix $\lambda = (0, \infty]$ and $\Delta > 0$, define the program

$$\hat{\pi}^S \equiv \max_{(x_\theta, y_\theta, p_\theta)_{\theta\in\Theta}} \sum q_\theta p_\theta \qquad (R)$$

subject to

$$
\begin{aligned}
&v_\theta x_\theta + w_\theta y_\theta - p_\theta \geq v_\theta x_{\theta'} + w_\theta y_{\theta'} - p_{\theta'} \\
&v_\theta x_\theta + w_\theta y_\theta - p_\theta \geq \underline{\pi}_\theta^B \\
&0 \leq p_\theta \leq \max\{v_\theta : \theta \in \Theta\}, \quad x_\theta \in [0,1] \\
&0 \leq 1 - x_\theta \leq y_\theta \leq 1
\end{aligned}
$$

for any $\theta, \theta' \in \Theta$. Let Γ be the set of all feasible variables (sextuplet when $\Theta = \{H, L\}$) and let g denote its generic element. For any $g = (x_\theta, y_\theta, p_\theta)_{\theta=H,L}$,

$$
\begin{aligned}
\hat{\pi}_H^B(g) &:= x_H v_H + y_H w_H - p_H, \quad \hat{\pi}_L^B(g) := x_L v_L + y_L w_L - p_L \\
\hat{\pi}_{L|H}^B(g) &:= x_L v_H + y_L w_H - p_L, \quad \hat{\pi}_{H|L}^B(g) := x_H v_L + y_H w_L - p_H
\end{aligned}
$$

Then $g \in \Gamma$ if and only if the feasibility constraints

$$
0 \leq p_H, p_L \leq \max\{v_H, v_L\}, \quad 0 \leq x_H, x_L \leq 1,
$$

$$
0 \leq 1 - x_\theta \leq y_\theta \leq 1 \tag{45}
$$

and *relaxed-incentive compatibility* and *relaxed-individual rationality* constraints for each type

$$
\hat{\pi}_H^B(g) \geq \hat{\pi}_{L|H}^B(g), \quad \hat{\pi}_L^B(g) \geq \hat{\pi}_{H|L}^B(g), \quad \hat{\pi}_H^B(g) \geq \underline{\pi}_H^B, \quad \text{and} \quad \hat{\pi}_L^B(g) \geq \underline{\pi}_L^B \tag{46}
$$

holds. For any $g \in \Gamma$, let $\hat{\pi}^S(g) := q_H p_H + q_L p_L$.

LEMMA A.1. *There is a solution $g^* = (x_\theta^*, y_\theta^*, p_\theta^*)_{\theta=H,L} \in \Gamma$ for (R) such that*

$$
x_H^* = 1, \quad y_H^* = 0, \quad \textit{and} \quad \hat{\pi}_L^B(g^*) = \underline{\pi}_L^B. \tag{47}
$$

Proof. For contradiction, suppose there is a solution $g = (x_\theta, y_\theta, p_\theta)_{\theta=H,L} \in \Gamma$ of (R) such that (47) does not hold.

Some Preliminaries: Without loss we may assume $y_H = 0$. Otherwise, we may consider $\hat{g} = (\hat{x}_\theta, \hat{y}_\theta, \hat{p}_\theta)_{\theta \in \Theta}$ another solution of (R) such that

$$
\hat{x}_H = x_H + y_H, \quad \hat{y}_H = 0, \quad \hat{p}_H = p_H + y_H(v_H - w_H).
$$

and

$$
\hat{x}_L = x_L, \quad \hat{y}_L = y_L, \quad \text{and} \quad \hat{p}_L = p_L.
$$

Clearly, $\hat{g} \in \Gamma$ and $\pi^S(\hat{g}) \geq \pi^S(g)$. We also may assume $x_H > 0$; if $x_H = 0$ and $y_H = 0$ and hence $p_H = 0$. Then $\tilde{\mu} = (\tilde{x}_\theta, \tilde{y}_\theta, \tilde{p}_\theta)_{\theta \in \Theta} \in \Gamma$ such that

$$
\begin{aligned}
&\tilde{x}_H = 0, \quad \tilde{y}_H = 1, \quad \tilde{p}_H = 0 \\
&\tilde{x}_L = 1, \quad \tilde{y}_L = 0, \quad \tilde{p}_L = v_L - w_L
\end{aligned}
$$

will give a higher profit than g and satisfies (47).

Case I: $v_H \geq v_L$ or $\hat{\pi}_H^B(g) - \underline{\pi}_H^B = \hat{\pi}_L^B(g) - \underline{\pi}_L^B = 0$ or $x_H = 1$. Consider $g' = (x'_\theta, y'_\theta, p'_\theta)_{\theta \in \Theta} \in \Gamma$ such that

$$
\begin{aligned}
&x'_H = 1, \quad y'_H = 0,\\
&p'_H = p_H + (1 - x_H)v_H + \min\{\hat{\pi}^B_H(g) - \underline{\pi}^B_H, \hat{\pi}^B_L(g) - \underline{\pi}^B_L\} \geq p_H\\
&x'_L = x_L, \quad y'_L = y_L, \quad p'_L = p_L + \min\{\hat{\pi}^B_H(g) - \underline{\pi}^B_H, \hat{\pi}^B_L(g) - \underline{\pi}^B_L\} \geq p_L
\end{aligned}
$$

Clearly, g' satisfies (45) and $\hat{\pi}^S(g') \geq \hat{\pi}^S(g)$. To verify the relaxed-individual rationality and relaxed-incentive compatibility constraints for the high type, note that

$$
\begin{aligned}
\hat{\pi}^B_H(g') &= v_H - p_H - (1 - x_H)v_H - \min\{\hat{\pi}^B_H(g) - \underline{\pi}^B_H, \hat{\pi}^B_L(g) - \underline{\pi}^B_L\}\\
&= \hat{\pi}^B_H(g) - \min\{\hat{\pi}^B_H(g) - \underline{\pi}^B_H, \hat{\pi}^B_L(g) - \underline{\pi}^B_L\} \geq \underline{\pi}^B_H
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\pi}^B_H(g') &= \hat{\pi}^B_H(g) - \min\{\hat{\pi}^B_H(g) - \underline{\pi}^B_H, \hat{\pi}^B_L(g) - \underline{\pi}^B_L\}\\
&\geq \hat{\pi}^B_{H|L}(g) - \min\{\hat{\pi}^B_H(g) - \underline{\pi}^B_H, \hat{\pi}^B_L(g) - \underline{\pi}^B_L\} = \hat{\pi}^B_{H|L}(g').
\end{aligned}
$$

For the low-type's relaxed-individual rationality, note

$$
\hat{\pi}^B_L(g') = \hat{\pi}^B_L(g) - \min\{\hat{\pi}^B_H(g) - \underline{\pi}^B_H, \hat{\pi}^B_L(g) - \underline{\pi}^B_L\} \geq \underline{\pi}^B_L \tag{48}
$$

For the relaxed-incentive compatibility:

- Subcase 1: Suppose $\hat{\pi}^B_H(g) - \underline{\pi}^B_H = \hat{\pi}^B_L(g) - \underline{\pi}^B_L = 0$. Then p_H is necessarily $v_H x_H$ from the high-type's relaxed individual rationality hence $p'_H = v_H$.

$$
\hat{\pi}^B_{H|L}(g') = v_L - v_H \leq \underline{\pi}^B_L - \underline{\pi}^B_H < \underline{\pi}^B_L = \hat{\pi}^B_L(g').
$$

- Subcase 2: On the other hand, if $v_H \geq v_L$,

$$
\begin{aligned}
\hat{\pi}^B_{H|L}(g') &= v_L - p_H - (1 - x_H)v_H - \min\{\hat{\pi}^B_H(g) - \underline{\pi}^B_H, \hat{\pi}^B_L(g) - \underline{\pi}^B_L\}\\
&= \hat{\pi}^B_{H|L}(g) - (1 - x_H)(v_H - v_L) - \min\{\hat{\pi}^B_H(g) - \underline{\pi}^B_H, \hat{\pi}^B_L(g) - \underline{\pi}^B_L\}\\
&\leq \hat{\pi}^B_{H|L}(g) - \min\{\hat{\pi}^B_H(g) - \underline{\pi}^B_H, \hat{\pi}^B_L(g) - \underline{\pi}^B_L\}\\
&\leq \hat{\pi}^B_L(g) - \min\{\hat{\pi}^B_H(g) - \underline{\pi}^B_H, \hat{\pi}^B_L(g) - \underline{\pi}^B_L\} = \hat{\pi}^B_L(g')
\end{aligned}
$$

- Subcase 3: Finally suppose $x_H = 1$.

$$
\begin{aligned}
\hat{\pi}^B_{H|L}(g') &= v_L - p_H - \min\{\hat{\pi}^B_H(g) - \underline{\pi}^B_H, \hat{\pi}^B_L(g) - \underline{\pi}^B_L\}\\
&\leq \hat{\pi}^B_{H|L}(g) - \min\{\hat{\pi}^B_H(g) - \underline{\pi}^B_H, \hat{\pi}^B_L(g) - \underline{\pi}^B_L\}\\
&\leq \hat{\pi}^B_L(g) - \min\{\hat{\pi}^B_H(g) - \underline{\pi}^B_H, \hat{\pi}^B_L(g) - \underline{\pi}^B_L\} = \hat{\pi}^B_L(g')
\end{aligned}
$$

In conclusion, $g' \in \Gamma$. The proof is complete if we can show $\hat{\pi}^B_H(g) - \underline{\pi}^B_H \geq \hat{\pi}^B_L(g) - \underline{\pi}^B_L$ in which case, $\hat{\pi}^B_L(g') = \underline{\pi}^B_L$. Otherwise,

$$
0 \leq \hat{\pi}^B_H(g) - \underline{\pi}^B_H < \hat{\pi}^B_L(g) - \underline{\pi}^B_L \implies \hat{\pi}^B_L(g') > \underline{\pi}^B_L
$$

hence the relaxed-individuals rationality of the low-type does not bind in g'. Also,

$$\hat{\pi}_H^B(g') = \hat{\pi}_H^B(g) - \min\{\hat{\pi}_H^B(g) - \underline{\pi}_H^B, \hat{\pi}_L^B(g) - \underline{\pi}_L^B\} = \underline{\pi}_H^B$$

hence p'_H is necessarily $v_H - \underline{\pi}_H^B$ which in turn implies

$$\hat{\pi}_{H|L}^B(g') = v_L - v_H + \underline{\pi}_H^B \leq \underline{\pi}_L^B < \hat{\pi}_L^B(g')$$

where the weak inequality comes from

$$v_H - \underline{\pi}_H^B = u_H^{ex-ante} \geq u_L^{ex-ante} = v_L - \underline{\pi}_L^B$$

Hence neither incentive compatibility constraint nor individual rational constraint for the low type binds in g'. Therefore we can easily find $g'' \in \Gamma$ that yields a higher $\hat{\pi}^S(g)$ by decreasing p'_L until either constraints becomes binding, which contradicts our hypothesis that g constitutes a solution of (R).

Case II: $v_H < v_L$ and $\min\{\hat{\pi}_H^B(g) - \underline{\pi}_H^B, \hat{\pi}_L^B(g) - \underline{\pi}_L^B\} > 0$ and $x_H \in (0,1)$: We first show that we may assume $x_L \in (0,1)$ as well. Suppose $x_L = 0$. The relaxed-individual rationality constraint for the low type requires $p_L = 0$ and hence $\hat{\pi}^S(g)$ is at most $q_H(v_H - \underline{\pi}_H^B)$. Noting this, consider $g' = (x'_\theta, y'_\theta, p'_\theta)_{\theta\in\Theta}$ such that

$$\begin{aligned} x'_H = 1, \quad y'_H = 0, \quad p'_H = v_H - \underline{\pi}_H^B \\ x'_L = 0, \quad y'_H = B, \quad p'_L = 0 \end{aligned}$$

which is clearly in Γ and yields $\hat{\pi}^S(g') = \hat{q}_H(v_H - w_H)$ as the investor's profit, but satisfies all conditions stated in the Lemma.

On the other hand, if $x_L = 1$, the low type's individual rationality requires $p_L \leq v_L - \underline{\pi}_L^B$. The relaxed-incentive compatibility constraints for both types respectively imply

$$\pi_L^B(g) = v_L - p_L \geq \pi_{H|L}^B(g) = v_L x_H - p_H$$

and

$$\pi_H^B(g) = x_H v_H - p_H \geq \pi_{L|H}^B(g) = v_H - p_L$$

Combining two inequalities,

$$\frac{v_H - p_L + p_H}{v_H} \leq x_H \leq \frac{v_L - p_L + p_H}{v_L}$$

which implies $p_L \geq p_H$ when $v_H < v_L$, hence

$$\hat{\pi}^S(g) \leq v_L - \underline{\pi}_L^B.$$

But then $g' \in \Gamma = (x'_\theta, y'_\theta, p'_\theta)_{\theta\in\Theta}$ such that

$$x'_H = x'_L = 1, \quad y'_H = y'_L = 0, \quad p'_H = p'_L = v_L - \underline{\pi}_L^B \tag{49}$$

satisfies the lemma's statement and still yields the profit not lower than $\hat{\pi}^S(g)$.

Note that $\min\{\hat{\pi}_H^B(g) - \underline{\pi}_H^B, \hat{\pi}_L^B(g) - \underline{\pi}_L^B\} > 0$ means both types' relaxed-individual rational constraints do not bind. First suppose

$$\frac{v_H w_L - v_L w_H}{v_L - v_H} \geq 0$$

noting that the left-hand side is well-defined fraction because $v_L > v_H$ by assumption. Consider the alternative mechanism $g' \in (x'_\theta, y'_\theta, p'_\theta)_{\theta\in\Theta}$ such that

$$x'_H = x_H + \epsilon\frac{v_H - v_L - \alpha(w_H - w_L)}{v_L - v_H}, \quad y'_H = y_H, \quad p'_H = p_H + \epsilon\alpha\frac{v_H w_L - v_L w_H}{v_L - v_H} + \epsilon$$
$$x'_L = x_L - \epsilon, \quad y'_L = y_L + \alpha\epsilon, \quad p'_L = p_L + \epsilon.$$

which satisfies (45) as long as both $\alpha > 0$ and $\epsilon > 0$ are sufficiently small due to the assumption $x_H, x_L \in (0,1)$. To check the relaxed-incentive compatibility of g', note

$$\begin{aligned}&\hat{\pi}^B_H(g') - \hat{\pi}^B_{L|H}(g')\\ &\quad= \hat{\pi}^B_H(g') - \hat{\pi}^B_{L|H}(g') + \epsilon\underbrace{\left[v_H\frac{v_H - v_L - \alpha(w_H - w_L)}{v_L - v_H} - \alpha\frac{v_H w_L - v_L w_H}{v_L - v_H} + v_H - \alpha w_H\right]}_{=0} \geq 0\end{aligned}$$

and

$$\begin{aligned}&\hat{\pi}^B_L(g') - \hat{\pi}^B_{H|L}(g')\\ &\quad= \hat{\pi}^B_L(g) - \hat{\pi}^B_{H|L}(g) + \epsilon\underbrace{\left[-v_L + \alpha w_L - v_L\frac{v_H - v_L - \alpha(w_H - w_L)}{v_L - v_H} + \alpha\frac{v_H w_L - v_L w_H}{v_L - v_H}\right]}_{=0} \geq 0.\end{aligned}$$

Also relaxed-individual constraints for both types also hod as long as $\epsilon > 0$ and $\alpha > 0$ are small. Hence $g' \in \Gamma$ but $\hat{\pi}^S(g') > \hat{\pi}^S(g)$ which contradicts our assumption that g solves the relaxed problem.

Now suppose

$$\frac{v_H w_L - v_L w_H}{v_L - v_H} < 0.$$

consider $g' \in (x'_\theta, y'_\theta, p'_\theta)_{\theta\in\Theta}$ such that

$$x'_H = x_H + \epsilon\frac{v_H - v_L - \alpha(w_H - w_L)}{v_L - v_H}, \quad y'_H = y_H = 0, \quad p'_H = p_H + \epsilon$$
$$x'_L = x_L - \epsilon, \quad y'_L = y_L + \alpha\epsilon, \quad p'_L = p_L - \epsilon\alpha\frac{v_H w_L - v_L w_H}{v_L - v_H} + \epsilon$$

which is indeed in Γ and $\hat{\pi}^S(g') > \hat{\pi}^S(g)$ as long as both $\alpha > 0$ and $\epsilon > 0$ are sufficiently small, which contradicts our assumption that g solves the relaxed problem. ■

Define $\bar{\Gamma} \subset \Gamma$ by

$$\{(x_\theta, y_\theta, p_\theta)_{\theta=H,L} : x_H = 1, y_H = 0, v_L x_L + w_L y_L - p_L = \underline{\pi}^B_L\}$$

and let $g = (x_L, y_L, p_H)$ identifies $(x_\theta, y_\theta, p_\theta)_{\theta=H,L}$ such that $x_H = 1$, $y_H = 0$, and $p_L = v_L x_L + w_L y_L$ hence a point in $\bar{\Gamma}$. Thanks to the last lemma, (R) is equivalent to

$$\max_{g\in\bar{\Gamma}} \hat{\pi}^S(g) = \max_{(x_L, y_L, p_H)} q_H p_H + (1 - q_H)(v_L x_L + w_L y_L - \underline{\pi}^B_L) \tag{50}$$

subject to

$$v_L - \pi_L^B \le p_H \le \min\{v_H - \pi_H^B, \underbrace{v_H - (v_H - v_L)x_L - (w_H - w_L)y_L - \pi_L^B}_{:=R(x_L,y_L)}\}$$

where the constraint of the maximization problem summarizes relaxed-incentive compatibility constraints for both types and relaxed-individual constraint for the low-type. Note that $R(x,y) \le v_H - w_H$ if and only if $(v_H - v_L)x \ge (w_H - w_L)(1-y)$. Also, $R(x,y) \ge v_L - Bw_L$ if and only if $(v_H - v_L)(1-x) \ge (w_H - w_L)y$. Some properties of solutions for the program (53) are immediately. For example, we must have $p_H = \min\{v_H - w_H, R(x_L, y_L)\}$.

Actually we can simplify this constraint even further: any solution $g = (x_L, y_L, p_H)$ of the above program such that

$$p_H = R(x_L, y_L) \le v_H - \pi_H^B. \tag{51}$$

For contradiction, suppose a solution $g = (x_L, y_L, p_H)$ does not satisfies (51). That is, $p_H = v_H - \pi_H^B < R(x_L, y_L)$. If $x_L = 1$, y_L is necessarily 0 and $p_H = v_L - \pi_L^B$ hence (51) trivially holds. Without loss, therefore, let $x_L < 1$. Now consider $g' = (x'_L, y'_L, p'_H) \in \bar{\Gamma}$ such that

$$\begin{array}{ll} x'_L = x_L + \epsilon, \quad y'_L = y_L - \epsilon, \quad p'_H = p_H & \text{if } y_L > 0 \\ x'_L = x_L + \epsilon, \quad y'_L = y_L, \quad p'_H = p_H & \text{if } y_L = 0. \end{array} \tag{52}$$

In both cases, as long as $\epsilon > 0$ is small, all constraints still hold for g' but $\hat{\pi}^S(g') > \hat{\pi}^S(g)$, contradiction. In conclusion, (R) is equivalent to

$$\max_{g \in \bar{\Gamma}} \hat{\pi}^S(g) = \max_{(x_L, y_L, p_H)} q_H p_H + (1 - q_H)(v_L x_L + w_L y_L - \pi_L^B) \tag{53}$$

subject to

$$v_L - \pi_L^B \le p_H = \underbrace{v_H - (v_H - v_L)x_L - (w_H - w_L)y_L - \pi_L^B}_{:=R(x_L,y_L)} \le v_H - \pi_H^B$$

LEMMA A.2. *There is a solution $g^* = (x^*_\theta, y^*_\theta, p^*_\theta)_{\theta=H,L} \in \Gamma$ for (R) such that*

$$y^*_L \in \{0, 1\}.$$

Proof. For contradiction, suppose all solutions $g = (x_L, y_L, p_H)$ of (53) are such that $y_L \in (0,1)$. Without loss, we may assume $x_L \in (0,1)$. The hypothesis $y_L \in (0,1)$ itself requires $x_L < 1$. On the other hand, if $x_L = 0$, a contradiction follows in all three subcases.

- Subcase 1: If $w_H > w_L$,

  $$p_H = R(0, y_L) < v_H - \pi_L^B$$

  Moreover, $p_H > v_L - \pi_L^B$; otherwise, the constraint of (53) requires $p_H = v_L - \pi_L^B$ which in turn implies

  $$R(0, y_L) = v_H - (w_H - w_L)y_L - \pi_L^B = v_L - \pi_L^B \implies y_L = \frac{v_H - v_L}{w_H - w_L} \ge B$$

  contradiction.

- Subcase 2: Suppose $w_H = w_L$ in which case

$$v_L - \pi_L^B \leq R(0, y_L) = v_H - \pi_L^B = v_H - \pi_H^B$$

for any $y_L \in [0, 1]$ and hence

$$\hat{\pi}^S(g) = q_H(v_H - \pi_L^B) + (1 - q_H)(w_L y_L - \pi_L^B).$$

In particular, the seller can increase his profit without affecting the constraint of (53) by increasing y_L.

- Subcase 3: Suppose $w_H < w_L$ in which case

$$R(0, y_L) > v_H - \pi_L^B$$

for all $y_L \in [0, 1]$ and hence

$$\hat{\pi}^S(g) = q_H R(0, y_L) + (1 - q_H)(w_L y_L - \pi_L^B)$$

Note that $v_H - \pi_L^B < v_H - \pi_H^B$ and $R(0, y_L)$ is increasing in y_L hence $R(0, y_L)$ is necessarily equal to $v_H - \pi_H^B$ for g being a solution of (53). But then

$$R(0, y_L) = v_H - (w_H - w_L)y_L - \pi_L^B = v_H - \pi_H^B \implies y_L = \frac{\pi_H^B - \pi_L^B}{w_H - w_L} = 1$$

again contradiction.

We may also assume $y_L < 1 - x_L$ without loss, because we already know from Lemma 4 that the mechanism that satisfies the lemma's statement is optimal among any mechanisms such that $x_H = 1$, $y_H = 0$, and $y_L = 1 - x_L$.

Case I: $\frac{v_L w_H - v_H w_L}{v_H - v_L} \geq 0$ and $v_H > v_L$. Consider $g' = (p'_H, x'_L, y'_L)$ such that

$$x'_L = x_L + \frac{w_H - w_L}{v_H - v_L} y_L \quad y'_L = 0, \quad p'_H = R(x'_L, y'_L)$$

which satisfies all constraints of the investor's problem (53), and $y'_L = 0$. Hence, we necessarily have $\hat{\pi}^S(g') < \pi^S(g)$ by hypothesis. However,

$$\begin{aligned}
\hat{\pi}^S(g') - \hat{\pi}^S(g) &= (1 - q_H) \cdot y_L \cdot \left[\frac{v_L(w_H - w_L)}{v_H - v_L} - w_L \right] \\
&\geq (1 - q_H) \cdot y_L \cdot \frac{v_L w_H - v_H w_L}{v_H - v_L} \\
&\geq 0
\end{aligned}$$

Case II: $\frac{v_L w_H - v_H w_L}{v_H - v_L} < 0$ and $v_H > v_L$. Consider $g' = (p'_H, x'_L, y'_L)$ such that

$$x'_L = x_L - \frac{w_H - w_L}{v_H - v_L} \epsilon \quad y'_L = y_L + \epsilon, \quad p'_H = R(x'_L, y'_L)$$

As long as $\epsilon > 0$ is sufficiently small (because $x_L \in (0,1)$ and $y_L < 1 - x_L$) (p'_H, x'_L, y'_L) satisfies all constraints of (53). However,

$$\begin{aligned}
\hat{\pi}^S(g') - \hat{\pi}^S(g) &= -(1-q_H) \cdot \epsilon \cdot \left[\frac{v_L(w_H - w_L)}{v_H - v_L} - w_L\right] \\
&\geq -(1-q_H) \cdot \epsilon \cdot \frac{v_L w_H - v_H w_L}{v_H - v_L} \\
&> 0
\end{aligned}$$

contradiction.

Case III: $v_L \geq v_H$ (hence $w_L \geq w_H$). Suppose

$$q_H(v_H - w_H) \leq v_L - w_L$$

and consider $g' = (x'_L, y'_L, p'_H)$

$$x'_L = 1, \quad y'_L = 0, \quad p'_H = v_L - w_L$$

which clearly satisfies the constraints of the program (53). Then

$$\begin{aligned}
\hat{\pi}^S(g') - \hat{\pi}^S(g) &= (1 - x_L)(v_L - q_H v_H) - y_L(w_L - q_H w_H) \\
&\geq (1 - x_L)(v_L - q_H v_H) - (1 - x_L)(w_L - q_H w_H) \\
&= (1 - x_L)(v_L - w_L - q_H(v_H - w_H)) \\
&\geq 0
\end{aligned}$$

which cannot be true under our hypothesis. On the other hand, if

$$q_H(v_H - w_H) > v_L - w_L$$

consider $g' = (x'_L, y'_L, p'_H)$ such that

$$x'_L = 0, \quad y'_L = 1, \quad p'_H = v_H - w_H \tag{54}$$

which clearly satisfies the constraints of the program (53). Then

$$\begin{aligned}
\hat{\pi}^S(g') - \hat{\pi}^S(g) &= x_L(q_H v_H - v_L) - (1 - y_L)(q_H w_H - w_L) \\
&> x_L(q(v_H - w_H) - (v_L - w_L)) \\
&> 0
\end{aligned}$$

which cannot be true under our hypothesis. ■

Now we are ready to prove the proposition. Due to the last two lemmas,

$$\hat{\pi}^S = \max_{x_L, y_L, p_H} q_H p_H + q_L p_L$$

subject to

$$\begin{aligned}
&v_H - p_H \geq x_L(v_H - v_L) + y_L(w_H - w_L) \\
&v_H - p_H \geq \pi_H^B \\
&\pi_L^B \geq v_L - p_H
\end{aligned}$$

and

$$(x_L, y_L) \in \{(x,y) : x \in (0,1], y = 0\} \cup \{(0,1)\}.$$

Here the first two inequalities are relaxed-incentive compatibility and relaxed-individual rationality constraints for the high-type, and the third inequality stands for the relaxed-individual rationality constraint for the low-type. The last inequality, which is due to the last lemma, shows that we need to focus on one dimensional subset of whole feasible (x_L, y_L) space which effectively reduces to the problem into the simple one-dimensional screening problem.

A.5 Proof of Proposition 3

Fix a small number $\epsilon > 0$, so that

$$v_0 - \frac{\lambda}{\lambda + r} w_0 + \epsilon < v_1 - \frac{w_1}{1-\epsilon} < v_2 - w_2$$

for all sufficiently large $\lambda > 0$. For any profit level π^S achievable with commitment in the limit, we will construct an equilibrium $\alpha^{\lambda,\Delta} = (\sigma^{\lambda,\Delta}, \chi^{\lambda,\Delta}, \xi^{\lambda,\Delta}, q^{\lambda,\Delta}) \in \mathcal{E}(\lambda, \Delta)$ for each $\Delta > 0$ and $\lambda > 0$ such that $\lim_{\lambda\to\infty} \lim_{\Delta\to 0} V^S(\alpha^{\lambda,\Delta}; \lambda, \Delta)$ lies in $(\pi^S - \epsilon, \pi^S + \epsilon)$. The equilibrium construction is done by using a Coasian equilibrium for the punishment for the seller's deviation. For any seller's belief $(q_\theta)_{\theta\in\Theta} \in [0,1]^3$, choose one Coasian equilibrium $\alpha^{c|\lambda,\Delta,(q_\theta)_{\theta\in\Theta}} \in \mathcal{E}^c(\lambda, \Delta, (q_\theta)_{\theta\in\Theta})$, and let $p^{1st}(\alpha^{c|\lambda,\Delta,(q_\theta)_{\theta\in\Theta}})$ be the seller's first offer in $\alpha^{c|\lambda,\Delta,(q_\theta)_{\theta\in\Theta}}$; choose an arbitrary one if the seller randomizes over multiple prices. We may choose Coasian equilibria so that

$$V^S(\alpha^{c|\lambda,\Delta,(q_0,q_1,q_2)})\Big|_{q_2=0, q_0=1-q_1}$$

is strictly increasing in q_1.

Case I: $\frac{w_2}{w_1} \geq \frac{v_2}{v_1} \geq \frac{q_1+q_2}{q_2}$. We construct $\alpha_1^{p_H|\lambda,\Delta} \in \mathcal{E}(\lambda, \Delta)$ for each $p_H \geq (v_1 - w_1, v_2 - w_2]$, $\lambda > 0$, and $\Delta > 0$. First, take $p_H \in (v_1 - w_1, v_2 - w_2)$. The equilibrium consists of five phases. We will omit to describe the seller's belief because the Bayes rule uniquely pins down it. Let $x_L \in [0,1)$ and $p_L \geq 0$ be such that

$$x_L(v_1 - p_L) = w_1 \quad \text{and} \quad v_2 - p_H = x_L(v_2 - p_L)$$

or equivalently

$$x_L = \frac{v_2 - w_1 - p_H}{v_2 - v_1} \in \left(\frac{w_2 - w_1}{v_2 - v_1}, 1\right) \quad \text{and} \quad p_L = \frac{v_2(v_1 - w_1) - v_1 p_H}{v_2 - w_1 - p_H} > 0.$$

Let n_L be the smallest positive integer such that

$$e^{-r(n_L - 1)\Delta} \geq x_L > e^{-r n_L \Delta}$$

Note that

$$e^{-r(n_L+1)\Delta} \to x_L \quad \text{and} \quad n_L\Delta \to t_L := \frac{1}{r}\log\frac{1}{x_L} \quad \text{as } \Delta \to 0$$

Coasian Phase: The equilibrium play enters the Coasian phase if either (i) the seller deviates from the equilibrium play described below, or the seller offers $p^{1st}(\alpha^{c|\lambda,\Delta,(\bar{q}(\theta;h))_{\theta\in\Theta}})$ at history $h_B = (h,a) \in H_B$. Suppose the equilibrium play just has entered the Coasian phase by the seller's offer $p \geq 0$ at $h \in H$. Then each buyer type θ accepts/rejects p or exercises her outside option as specified in $\alpha^{c|\lambda,\Delta,(\bar{q}(\theta;h))_{\theta\in\Theta}}$ indefinitely. That is, each type buyer $\theta \in \Theta$ reacts to the seller's offer p according to

$$\chi_\theta^{c|\lambda,\Delta,(\bar{q}(\theta;h))_{\theta\in\Theta}}(p;h,a) \quad \text{and} \quad \xi_\theta^{c|\lambda,\Delta,(\bar{q}(\theta;h))_{\theta\in\Theta}}(p;h,a)$$

and then in the next period the seller offers according to $\sigma^{c|\lambda,\Delta,(\bar{q}(\theta;h))_{\theta\in\Theta}}(p) \in \triangle\mathbb{R}_+$, and so on.

Skimming Phase I: At the null public history h^0, the seller offers p_H. 2-type buyer accepts it with probability 1 and rejected by all other types: for any $a_1 \in \{0,1\}$,

$$\begin{aligned}
&\sigma^{\lambda,\Delta}(p_H;h^0) = 1 \\
&\chi_2^{\lambda,\Delta}(p_H;h^0,a_1) = 1, \quad \xi_2^{\lambda,\Delta}(p_H;h^0,a_1) = 0 \\
&\chi_1^{\lambda,\Delta}(p_H;h^0,a_1) = 0, \quad \xi_1^{\lambda,\Delta}(p_H;h^0,a_1) = 0 \\
&\chi_0^{\lambda,\Delta}(p_H;h^0,a_1) = 0, \quad \xi_0^{\lambda,\Delta}(p_H;h^0,a_1) = \mathbb{1}\{a_1 = 1\}
\end{aligned}$$

Impasse Phase I: The equilibrium play enters this phase at the beginning of the second period conditional on no deviation by the seller in Skimming Phase. In this phase, the seller insists on p_H which is supposed to be rejected by all buyer types other than $\theta = 2$. 0-type buyer exercises her outside option once it arrives.

$$\begin{aligned}
&\sigma^{\lambda,\Delta}(p_H;p_H^{n-1}) = 1 \\
&\chi_2^{\lambda,\Delta}(p_H;p_H^{n-1},a^n) = 1, \quad \xi_2^{\lambda,\Delta}(p_H;p_H^{n-1},a^n) = 0 \\
&\chi_1^{\lambda,\Delta}(p_H;p_H^{n-1},a^n) = 0, \quad \xi_1^{\lambda,\Delta}(p_H;p_H^{n-1},a^n) = 0 \\
&\chi_0^{\lambda,\Delta}(p_H;p_H^{n-1},a^n) = 0, \quad \xi_0^{\lambda,\Delta}(p_H;p_H^{n-1},a^n) = \mathbb{1}\{a_n = 1\}
\end{aligned}$$

in any period $1 < n \leq n_L$ where $p_H^{n-1} = \underbrace{(p_H, p_H, \ldots, p_H)}_{n-1 \text{ times}}$ for each integer n.

Skimming Phase II: The equilibrium enters the second skimming phase in period $n_L + 1$ conditional on no seller's deviation in previous periods. Note that at the beginning of the skimming phase II (before the seller makes an offer) the seller believes that the buyer is 1-type with probability

$$\frac{q_1}{q_1 + e^{-\lambda n_L} q_0}$$

and 0-type with the complementary probability. In this phase which lasts only one period, the seller offers p_L and 1-type accepts with $b_L \in (0,1)$, a unique zero of the following equation:

$$\frac{(1-b_L)q_1}{(1-b_L)q_1 + e^{-\lambda(n_L+1)\Delta}q_0} = \frac{1-e^{-(\lambda+r)\Delta}}{1-e^{-\lambda\Delta}} \frac{V^S(\alpha^{c|\lambda,\Delta,(1-q_1^*,q_1^*,0)})}{p_L}.$$

where q_1^* is the unique zero of

$$V^S(\alpha^{c|\lambda,\Delta,(1-q_1^*,q_1^*,0)}) = p_L \frac{1-e^{-\lambda\Delta}}{1-e^{-(\lambda+r)\Delta}} q_1^*.$$

0-type rejects p_L. 1-type does not exercise her outside option even if available, while 0-type opts out if possible. Formally,

$$\begin{array}{ll}\sigma(p_L; p_H^{n_L}) = 1 & \\ \chi_2^{\lambda,\Delta}(p_L; p_H^{n_L}, a^{n_L}) = 1, & \xi_2^{\lambda,\Delta}(p_L; p_H^{n_L}, a^{n_L}) = 0 \\ \chi_1^{\lambda,\Delta}(p_L; p_H^{n_L}, a^{n_L}) = b_L, & \xi_1^{\lambda,\Delta}(p_L; p_H^{n_L}, a^{n_L}) = 0 \\ \chi_0^{\lambda,\Delta}(p_L; p_H^{n_L}, a^{n_L}) = 0, & \xi_0^{\lambda,\Delta}(p_L; p_H^{n_L}, a^{n_L}) = \mathbb{1}\{a_{n_L} = 1\}\end{array}$$

Impasse Phase II: The equilibrium enters the second skimming phase in period $n_L + 2$ conditional on no seller's deviation in previous periods. Note that at the beginning of the skimming phase II (before the seller makes an offer) the seller believes that the buyer is 1-type with probability q_1^* and and 0-type with the complementary probability. In this period, the seller randomizes over p_L and $p^{1st}(\alpha^{c|\lambda,\Delta,(q_1^*,1-q_1^*,0)})$ with probability

$$c_L := \frac{p_L - (1-e^{-r\Delta})v_1 - e^{-r\Delta} p^{1st}(\alpha^{c|\lambda,\Delta,(q_1^*,1-q_1^*,0)})}{e^{-r\Delta}(p_L - p^{1st}(\alpha^{c|\lambda,\Delta,(q_1^*,1-q_1^*,0)}))} \in (0,1)$$

and the complementary probability. 1-type buyer accepts p_L with probability $1-e^{-\lambda\Delta}$ which is exactly the same probability that 0-type opts out. Formally, for any $k \geq 1$,

$$\begin{array}{ll}\sigma(p_L; p_H^{n_L+k}) = c_L, & \sigma(p^{1st}(\alpha^{c|\lambda,\Delta,(q_1^*,1-q_1^*,0)}); p_H^{n_L+k}) = 1-c_L \\ \chi_2^{\lambda,\Delta}(p_L; (p_H^{n_L}, p_L^k), a^{n_L+k+1}) = 1, & \xi_2^{\lambda,\Delta}(p_L; (p_H^{n_L}, p_L^k), a^{n_L}) = 0 \\ \chi_1^{\lambda,\Delta}(p_L; (p_H^{n_L}, p_L^k), a^{n_L+k+1}) = 1-e^{-\lambda\Delta}, & \xi_1^{\lambda,\Delta}(p_L; (p_H^{n_L}, p_L^k), a^{n_L+k+1}) = 0 \\ \chi_0^{\lambda,\Delta}(p_L; (p_H^{n_L}, p_L^k), a^{n_L+k+1}) = 0, & \xi_0^{\lambda,\Delta}(p_L; (p_H^{n_L}, p_L^k), a^{n_L+k+1}) = \mathbb{1}\{a_{n_L+k+1} = 1\}\end{array}$$

The equilibrium play remains in Impasse Phase II unless the seller deviates or offers $p^{1st}(\alpha^{c|\lambda,\Delta,(q_1^*,1-q_1^*,0)})$.

Let $\alpha_I^{p_H|\lambda,\Delta}$ denote the above assessment. One can easily compute

$$\lim_{\Delta\to 0} V^S(\alpha_I^{p_H|\lambda,\Delta}) = q_2 p_H + e^{-rt_L}\left[q_1\left(1 - \frac{(1-q_1)rv_0}{q_1(\lambda p_L - rv_0)}\right)p_L + \left(\frac{(1-q_1)rv_0}{\lambda p_L - rv_0} + q_0 e^{-\lambda t_L}\right)\frac{r}{\lambda + r}v_0\right]$$

There is $\bar{\lambda} > 0$ such that $\alpha_I^{p_H|\lambda,\Delta} \in \mathcal{E}(\lambda,\Delta)$ and $\lim_{\Delta\to 0} V^S(\alpha_I^{p_H|\lambda,\Delta})$ is increasing in λ over $\lambda \in (\bar{\lambda}, \infty)$. Hence, there is $\bar{\epsilon}_I > 0$ such that

$$\begin{aligned}&\left((v_1-w_1)(q_1+q_2), (v_2-w_2)q_2 + \frac{v_1w_2 - v_2w_1}{v_2-v_1}q_1 - \bar{\epsilon}\right) \\ &\subset \lim_{\lambda\to\infty}\lim_{\Delta\to 0}\{V^S(\alpha_I^{p_H|\lambda,\Delta}) : v_1 - w_1 < p_H < v_2 - w_2\} \\ &\subset \left((v_1-w_1)(q_1+q_2) - \bar{\epsilon}, (v_2-w_2)q_2 + \frac{v_1w_2 - v_2w_1}{v_2-v_1}q_1\right)\end{aligned} \tag{55}$$

whenever $\bar{\epsilon} \in (0, \bar{\epsilon}_I)$

Now set $p_H = v_2 - w_2$ and

$$x_L = 1 - \epsilon \quad \text{and} \quad p_L \in \left(v_0 - \frac{\lambda}{\lambda + r} w_0 + \epsilon, v_1 - \frac{w_1}{1-\epsilon} \right) \tag{56}$$

where ϵ is a small positive real number, and consider $\alpha_{II}^{p_L|\lambda,\Delta}$ which is same as one described above except for that now 2-type buyer will reject p_H in Skimming Phase I because delaying until the seller offers p_L will yield a higher profit to her:

$$\chi_2^{p_L|\lambda,\Delta}(p_H; h^0, a_1) = 0, \quad \xi_2^{p_L|\lambda,\Delta}(p_H; h^0, a_1) = 0.$$

Again one can find $\bar{\epsilon}_{II} > 0$ such that

$$\begin{aligned} & \Big(\bar{\epsilon}, (q_1 + q_2)(v_1 - w_1) + \bar{\epsilon} \Big) \\ & \subset \lim_{\lambda \to \infty} \lim_{\Delta \to 0} \left\{ \{ V^S(\alpha_{II}^{p_L|\lambda,\Delta}) : p_L \in \left(v_0 - \frac{\lambda}{\lambda + r} w_0 + \epsilon, v_1 - \frac{w_1}{1-\epsilon} \right) \right\} \\ & \subset \Big(0, (q_1 + q_2)(v_1 - w_1) + \bar{\epsilon} \Big) \end{aligned} \tag{57}$$

whenever $\bar{\epsilon} \in (0, \bar{\epsilon}_{II})$. Combining (55) and (57), there is $\bar{\epsilon}^*$ such that

$$\begin{aligned} & \left(\bar{\epsilon}, (v_2 - w_2) q_2 + \frac{v_1 w_2 - v_2 w_1}{v_2 - v_1} q_1 - \bar{\epsilon} \right) \\ & \qquad \subset \lim_{\lambda \to \infty} \lim_{\Delta \to 0} \mathcal{V}^S(\lambda, \Delta) \subset \left(0, (v_2 - w_2) q_2 + \frac{v_1 w_2 - v_2 w_1}{v_2 - v_1} q_1 \right) \end{aligned}$$

whenever $\bar{\epsilon} \in (0, \bar{\epsilon}^*)$.

Case II: $\frac{w_2}{w_1} \geq \frac{v_2}{v_1} \geq \frac{q_1 + q_2}{q_2}$ *fails.* Suppose, without loss,

$$(q_1 + q_2)(v_1 - w - 1) < q_2(v_2 - w_2) \iff \frac{q_2}{q_1 + q_2} > \frac{v_1 - w_1}{v_2 - w_2}$$

Let

$$p_H \in \left(\frac{q_1 + q_2}{q_2} (v_1 - w_1), v_2 - w_2 \right)$$

and choose n_L be an integer such that

$$e^{-r(n_L - 1)\Delta} \geq \frac{v_2 - p_H}{v_2 - \frac{q_2}{q_1 + q_2}(v_1 - w_1)} > e^{-r n_L \Delta}$$

and p_L be such that

$$e^{-r n_L \Delta}(v_2 - p_L) = v_2 - p_H.$$

Moreover, n_L^* be the smallest integer such that

$$e^{-r(n_L + 1 - n_L^*)\Delta}(v_1 - p_L) \geq w_1$$

Now consider an equilibrium $\alpha_{III}^{p_H|\lambda,\Delta}$ which is constructed similar to $\alpha_I^{p_H|\lambda,\Delta}$ except for Skimming Phase I and Impasse Phase I.

Skimming Phase I: At the null public history h^0, the seller offers p_H. 2-type buyer accepts it with probability 1 and rejected by all other types: for any $a_1 \in \{0, 1\}$,

$$\begin{aligned}
&\sigma^{\lambda,\Delta}(p_H; h^0) = 1 - \epsilon \\
&\chi_2^{\lambda,\Delta}(p_H; h^0, a_1) = 1, \quad \xi_2^{\lambda,\Delta}(p_H; h^0, a_1) = 0 \\
&\chi_1^{\lambda,\Delta}(p_H; h^0, a_1) = 0, \quad \xi_1^{\lambda,\Delta}(p_H; h^0, a_1) = 0 \\
&\chi_0^{\lambda,\Delta}(p_H; h^0, a_1) = 0, \quad \xi_0^{\lambda,\Delta}(p_H; h^0, a_1) = \mathbb{1}\{a_1 = 1\}
\end{aligned}$$

Impasse Phase I: The equilibrium play enters this phase at the beginning of the second period conditional on no deviation by the seller in Skimming Phase. In this phase, the seller insists on p_H which is supposed to be rejected by all buyer types. Formally,

$$\begin{aligned}
&\sigma^{\lambda,\Delta}(p_H; p_H^{n-1}) = 1 \\
&\chi_2^{\lambda,\Delta}(p_H; p_H^{n-1}, a^n) = 0, \quad \xi_2^{\lambda,\Delta}(p_H; p_H^{n-1}, a^n) = 0 \\
&\chi_1^{\lambda,\Delta}(p_H; p_H^{n-1}, a^n) = 0, \quad \xi_1^{\lambda,\Delta}(p_H; p_H^{n-1}, a^n) = \mathbb{1}\{n_L^* \le n \le n_L^*, a_n = 1\} \\
&\chi_0^{\lambda,\Delta}(p_H; p_H^{n-1}, a^n) = 0, \quad \xi_0^{\lambda,\Delta}(p_H; p_H^{n-1}, a^n) = \mathbb{1}\{a_n = 1\}
\end{aligned}$$

in any period $1 < n \le n_L$ where $p_H^{n-1} = \underbrace{(p_H, p_H, \ldots, p_H)}_{n-1 \text{ times}}$ for each integer n.

All other phases are constructed similar to $\alpha_1^{p_H|\lambda,\Delta}$. Again, one can show $\lim_{\Delta \to 0} V^S(\alpha_{III}^{p_H|\lambda,\Delta})$ is increasing in λ for all sufficiently large arrival rates, and

$$\lim_{\lambda \to \infty} \lim_{\Delta \to 0} V^S(\alpha_{III}^{p_H|\lambda,\Delta}) = q_2 \left[p_H - \epsilon \frac{p_H - \frac{q_2}{q_1+q_2}(v_1 - w_1)}{v_2 - \frac{q_2}{q_1+q_2}(v_1 - w_1)} \right]$$

Hence, there is $\bar{\epsilon}_{III} > 0$ such that

$$\begin{aligned}
&\Big((q_1 + q_2)(v_1 - w_1), q_2(v_2 - w_2) - \bar{\epsilon}\Big) \\
&\subset \lim_{\lambda \to \infty} \lim_{\Delta \to 0} \left\{ V^S(\alpha_{III}^{p_H|\lambda,\Delta}) : p_H \in \left(\frac{q_1 + q_2}{q_2}(v_1 - w_1), v_2 - w_2 \right) \right\} \\
&\subset \Big((q_1 + q_2)(v_1 - w_1) + \bar{\epsilon}, q_2(v_2 - w_2)\Big)
\end{aligned} \tag{58}$$

whenever $\bar{\epsilon} \in (0, \bar{\epsilon}_{III})$. Note that (57) holds for this case. Combining (57) and (58), we complete the proof.