Waiting for my neighbors

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Abstract

We study a waiting game on a network where the payoff of taking an action increases each time a neighbor takes the action. We show that the dynamic evolution of the network starkly depends on initial parameters and can take the form of either a shrinking network, where players at the edges take the action first or a fragmenting network where over time the network splits up in smaller ones. We find that, in addition to the coordination inefficiency standard in waiting games, the network structure gives rise to a spatial inefficiency. The model applies in particular to the adoption of new technologies by firms organized in a network and in this context we study the welfare impact of different subsidy programs aimed at encouraging adoption and show how their benefits depend on the network characteristics.

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1 Introduction

There is growing evidence that the decision to adopt a new technology is affected by the decisions of neighbors, i.e those close either geographically or in terms of social or technological distance (Foster and Rosenzweig 1995, Conley and Udry 2010, Bandiera and Rasul 2006, Atkin et al. 2015). One explanation is that adoption creates spillovers for neighbors that decrease their own adoption costs. These spillovers can be informational or technological. For instance, the initial adopter trains employees or suppliers with this new technology and the mobility of workers or the sharing of suppliers spreads the expertise to connected firms.

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Such environments create incentives for players to wait for their neighbors to adopt. In this paper we study a class of problems, *waiting games on networks*, that encompasses the adoption problem described above. To the best of our knowledge, this is the first paper to study a strategic timing game on a network. In fact, as Jackson and Zenou (2014) point out, there are very few papers that study dynamic games on networks. Applications are numerous: consider for instance industry shakeouts where only one firm can survive in a neighbourhood and firms wait in the hope that neighbors exit first. Such war of attrition games have been extensively studied, but the network structure has to this point been ignored.

Specifically, we consider an infinite horizon timing game played on a network. Players have to decide when to take an action, we call "stop". The benefit of the action for an individual at date t depends on the neighbors actions. Specifically, when a player stops, he increases the payoff of stopping of all his neighbors. This creates incentives for all players to wait in the hope that their neighbors stop before them, i.e gives rise to the structure of a waiting game.

We make two assumptions on the structure of the network as well as on the information structure. First, as in Jackson and Yariv (2005, 2007) or Galeotti et al (2010), we assume that each player knows her own degree (the number of her neighbors) but has incomplete information on the degree of her neighbors. Second, we assume that, for any player, the probability that two of her neighbors are mutually connected is zero.

The second assumption is for instance satisfied for players organized on a line. We derive the initial results in the line example, to illustrate the main dynamics. Each player observes his number of neighbors but does not know how many neighbors his neighbor has. There are two possible types for active players: types 1, those who have one neighbor only (i.e are at the end of the line) and types 2 (inside the line) who have two neighbors.

As is standard in waiting games there is no symmetric pure strategy equilibrium and at least some players must be mixing between waiting and stopping. The first result we obtain is that, generically, in a symmetric equilibrium of the game when the two types are still present, only one type of player will be mixing between stopping and waiting an extra unit of time while players of the other type will strictly prefer to wait. The tradeoff faced by players who mix is between delaying the benefits of stopping in the hope that the neighbor(s) stops in the short time interval, versus stopping immediately. Since the beliefs about the neighbor are independent of the own type, all players assign the same probability to the event that the neighbor stops in the time interval. Thus, since the benefits of stopping differ across types, only one type has an incentive to mix at any point in time.

This initial results gives rise to two very different dynamic evolutions of the network based on parameters of the model. First, what we call the case of shrinking networks, where the players of type 1 initially have more incentives to stop and hence the network shrinks over time, and second the case of fragmenting networks where players of type 2 initially have more incentives to stop, which leads to a fragmentation of the network in smaller networks over time.

Consider first shrinking networks. Initially, players of type 1 are mixing. As time passes, and their unique neighbor has not stopped, the beliefs about her type evolve. Two countervailing forces affect this belief. First, there is the classic updating of beliefs: since players of type 1 are more likely to stop, as time passes, the player becomes more confident that the neighbor is of type 2. However, there is a second effect, purely linked to the dynamic evolution of the network structure. Even if the neighbor started off as a type 2, her other neighbor might have stopped in the meantime, making it possible that she now turned into a type 1. Remarkably, we show that these two effects perfectly balance each other, so that for shrinking networks on a line, the beliefs that the player is of type 1 stays constant through time. As a consequence, throughout the game, only players at the extremity of the line mix and do so at a constant rate, as if they were playing a classic war of attrition with a single player of a given type.

For fragmenting networks where the players of type 2 initially have more incentives to stop, both the effects affecting beliefs mentioned above go in the same direction. As time passes and a neighbor has not stopped, players become more confident that she is of type 1. In addition, even if she started as a type 2, her own neighbor might have stopped, changing her into a type 1. Thus overall, as time passes the belief that the neighbor is of type 2 decreases. Over time the network splits up in smaller networks. At some date, all players of type 2 will have entered and only isolated pairs will remain. These pairs will then play a classical war of attrition.

As previously mentioned, one of the key applications of the model is the adoption of technologies by firms organized in a network. An extremely robust finding of the empirical literature on the topic is that adoption is typically slow, even for what are apparently profitable technologies (Geroski, Atkin et. al.).[1] Many policies based on subsidies for adoption have in fact been put in place to speed up the process (World Bank 2007). We show in our model that the benefit of these subsidies critically depend on the neighbourhood structure.

[1]As expressed in Geroski, "the central feature of most discussions of technology diffusion is the apparently slow speed at which firms adopt new technologies."

We evaluate and compare two types of subsidy programs used in practice: temporary subsidies, i.e in our model paid only for adoption at time zero, and permanent subsidies. We first show that temporary subsidies are welfare enhancing only if the expected size of the line is sufficiently small. The first order cost of the subsidy is that players with no neighbors, who would have adopted in any case, obtain a subsidy. The costs for types 1 are second order since the probability with which they accept the subsidy is proportional to its size. The marginal benefit on the other hand are higher for those at the extremities of the line (types 1) who directly benefit, than those inside the line. Overall, smaller expected neighbourhoods make temporary subsidies more likely to be attractive. Second, when comparing the two types of policies, the permanent subsidy program turns out to be more costly, as more players will obtain the subsidy, but also brings higher benefits as it speeds up entry of all players. We show that a larger expected size of the line will make permanent subsidy more attractive compared to the temporary program.

Introducing a network structure also has other possible consequences. The coordination failure induces a timing inefficiency that is a standard result in a war of attrition games. We highlight two other possible coordination inefficiencies linked to the network structure. The first is what we call a *spatial inefficiency.* In the case of a fragmenting network, the final distribution of isolated players that remain at the end of the game could be relevant. Consider for instance the application to the exit decisions by firm. The final spatial distribution of firms might matter for social welfare. You might think for instance that it should be socially optimal to have equally spaced firms if customers are uniformly distributed and pay transport costs. When we compute the total fractions of firms that remain at the end of the game, we find that it is in fact strictly less that 1/2. We refer to this as a spatial inefficiency. The second possible coordination failure relates to the order of exit, that matters for total welfare. We discuss this at the end of the paper.

We extend our analysis to large networks, where each player can have more than two neighbors. We show that the fragmenting network case is qualitatively similar to the line example, but the shrinking networks case exhibits new features. In the beginning equilibrium path looks similar: only type 1 is mixing while those with higher degree wait. However, in an infinite network with a sufficiently high average degree, type 1 will disappear altogether at some point. Intuitively, removing all the players with one neighbor one by one does not wipe out the rest of the network as it necessarily does in the case of a line network. We show that once type 1 disappears, type 2 starts to randomize. When type 2 disappears, type 3 starts to randomize, and so on.

When type $k \geq 2$ randomizes, the network exhibits *cascades*. This is because whenever a player becomes type $k-1$, she stops immediately. Every stopping decision starts a chain-reaction: some neighbors of the stopping player may become type $k-1$ and these will immediately stop and spread the cascade further. We show that as time goes on, these cascades become more predominant until at some point the network approaches a critical condition where cascades would become infinitely long. We show that the players' strategic delay will prevent that condition ever to be reached. Instead, the equilibrium path reverses and lower types return to the network ensuring a smooth evolution of the network until all the players have stopped.

To the best of our knowledge, this is the first paper studying a timing game on a network. In fact, as Jackson and Zenou (2014) point out, there is still limited work on strategic dynamic games on networks. Most interest has in fact focused on repeated games (Raub and Weesie (1990), Ali and Miller (2009, 2012) and Jackson et al (2011) among others). The core of the mechanism is that punishment of deviations by one neighbor will also impact the payoff of the other neighbors and contagion of bad behavior can thus occur.

2 Model

We consider an infinite horizon timing game played on a network. Players have to decide when to take an action, we call "stop". The benefit of the action for an individual at date t depends on how many neighbors she has at that date. We denote B_k the time invariant benefit of stopping for a player with k neighbors. We are interested in the general class of waiting games, so that B_k is a decreasing sequence ($B_k < B_{k-1} < ... < B_0$). We present foundations for this payoff structure in the next section.

The shape of the network evolves dynamically. As soon as a player takes the action, she exits the game. We represent this as a deletion of all her links. Consider a player with initially k neighbors, so that initially her payoff if she decided to stop would be B_k. If one of her neighbor stops, she is left with $k-1$ neighbors, and her payoff of stopping increases to B_{k-1}. This creates incentives for all players to wait in the hope that their neighbors stops before them.

We make two assumptions on the structure of the network as well as on the information structure. First, as in Galeotti et al (2010), we assume that each player knows her own degree (the number of her neighbors) but has incomplete information on the degree of her neighbors. All players share a common prior on the degree distribution at date 0. This degree distribution has full support on $(0, N)$ where $N \geq 2$ is the maximum number of neighbors. Second, we assume that, for any player, the probability that two of her neighbors are connected is zero. This will be the case if for instance the network is organized as an infinite tree. In sections 3 to 4 we consider the special case of the line where $k \in \{0, 1, 2\}$. We study the general case in section 5 and give more details on the networks that satisfy the assumptions above.

The only heterogeneity across players is their degree k that we call their type. This type determines their benefit of stopping and is going to evolve dynamically during the game. We will focus on symmetric perfect bayesian equilibria where the strategy depends only on the type.

We introduce some notation that will be key in the resolution. Some measures are relative to random members of the network while others are relative to neighbors of a random member of the network:

- $F(t)$ is the probability that any single *neighbor* stops in the interval $[0, t]$. This distribution captures both the expected type and strategy of the neighbor.
- $p_k(t)$ is the belief that a random *neighbor* is of type k at time t.
- $q_k(t)$ is the probability that a random *member* of the network is of type k at time t.
- $\lambda_k(t)$ is the equilibrium rate of stopping of a random *member* of type k at time t
- $\gamma(t)$ is the expected rate of stopping of a random *neighbor*: it depends both on beliefs about the neighbor's type and equilibrium strategies.

2.1 Applications

We provide in this section more details on particular applications of this model. Our leading application concerns the adoption of new technologies by firms in a context with spillovers among neighbors. The action "stop" represents here adopt the technology. The fact a neighbor adopts can decrease the cost of adoption through either technological spillovers or informational spillovers.

Consider first technological spillovers, so that a link represents technological proximity between two members of the network.[2] Upon adoption, the adopting firm trains employees and potentially trains suppliers if the new technology affects the interactions with suppliers. We know that there is large mobility of skilled labor across firms in the

[2] Informational spillovers, due for instance to the fact that firms can observe the adoption techniques used by their neighbors, are formalized in Appendix B1.

same technological areas and that firms situated close to each other often share suppliers, so that adoption by one firm may reduce the adoption costs of its neighbors (Jaffe et. al. 1993, Almeida and Kogut 1999).

Suppose the time invariant benefit of adopting the technology is given by B and denote c_a the cost of adoption for a player who does not benefit from spillovers. If a neighbor adopts, the cost of adoption will be reduced by a factor σ_1. The next adoption will reduce the cost further by another factor σ_2, and so on. Overall, the benefits of stopping are thus given by:

$$\begin{aligned} B_N &= B - c_a \\ B_{N-1} &= B - \sigma_1 c_a \\ &\dots \\ B_0 &= B - \left(\prod_{i=1}^{N} \sigma_i \right) c_a \end{aligned}$$

We make no assumption on the relative size of the series $(\sigma_i)_{i \in \{1,..,N\}}$. It might be the case that $(\sigma_i)_{i \in \{1,..,N\}}$ is a decreasing sequence if for instance spillovers are due to worker mobility and if later workers who move have less marginal contributions to make. On the other hand, we might also consider that in other instances it could be increasing if for instance the spillovers comes from suppliers and a sufficient mass of firms needs to adopt to give incentives for the supplier to also invest in the new technology. We will see that the relative size of the σ_i will matter for the pattern of adoption.

In the model, there is one state variable at time t: the number of neighbors a player has at that date. To fit even more closely to the application, there would be a priori a need to keep track of two state variables that would describe the types of the players: a the number of active neighbors, i.e those who have not yet adopted, and i the number of inactive neighbors, those who were neighbors and adopted in the past. We reduce to a single state variable by assuming that all players start out with the same number of neighbors, i.e $a + i = N$. We show in Appendix B2 that the equilibrium structure that we identify in Section 3 will be preserved if we do not impose this restriction and consider the general case with two state variables.[3]

[3]Appendix B2 should be read after having gone through Section3.

3 Waiting for my neighbors: the case of the line

We first derive a number of results in the case where the network is organized as a line, i.e players have types k in $\{0, 1, 2\}$. Results are generalized for larger networks in section 5. In our model, the heterogeneity between players is due to the network characteristics, specifically players'number of neighbors. To understand the role of the network structure, it is essential to examine the pattern of waiting in a model with heterogenous types, where the source of heterogeneity is not linked to a particular network structure. We thus start with a benchmark model with no network in the following section.

3.1 Benchmark with no network structure

We consider a game between two players who can have one of two possible types, that differ in terms of payoffs: type 1 who makes benefit B_1 if she stops first and B_0 if she stops after the other player and type 2 who gets benefit B_2 if first and B_1 if second ($B_0 > B_1 > B_2$). Both players know their type and share a common prior that the other player is of type $j \in \{1, 2\}$ with probability p_j. Consistent with our model with network structure, the belief about the other player's type is independent of the own type. We derive the symmetric equilibrium of this game. The shape of the equilibrium depends on the comparison between μ_1 and μ_2 where $\mu_j = \frac{rB_j}{B_{j-1}-B_j}$

Proposition 1 *If $\mu_j > \mu_k$ (either $j = 1$ and $k = 2$ or the reverse), then there exits a date t_b^j such that:*

- *For $t < t_b^j$ only players of type j mix between the actions stop and wait. Both players expect the other to stop at a rate μ_j*
- *At date t_b^j both players are certain that the other is of type k if she has not stopped yet: the posterior belief that the other player is of type j is such that $p_j(t_b^j) = 0$.*
- *For $t \geq t_b^j$ players of type k mix at constant rate μ_k.*

One of the key properties that stands out in Proposition 1 is that, in a symmetric equilibrium, only one single type mixes at any point in time. Indeed, when a player of a given type $l \in \{1, 2\}$ is mixing, she needs to be indifferent between the cost of waiting, equal to rB_l and the expected gain if the other player stops, equal to $(B_{l-1} - B_l)$ that accrues with probability μ where μ is the rate of entry of the other player. The key fact is that μ is independent of the own type, since types are not correlated. Thus generically only one type can satisfy the indifference condition.

$$\mu\left(B_l - B_{l-1}\right) = rB_l \tag{1}$$

Proposition 1 then characterizes the timing of actions. Consider the case where $\mu_1 \equiv \frac{rB_1}{B_0 - B_1} > \mu_2 \equiv \frac{rB_2}{B_1 - B_2}$. Players of type 1 have more incentives to stop and initially they are the only types to mix. The equilibrium mixing rate, as can be seen in equation (1), has to be such that all players share the belief that the other player will stop at rate $\mu = \mu_1$. Note that μ_1 is both a function of the belief that the other player is of type 1 and of the mixing rate λ_1 of players of type 1. We have specifically $\mu_1 = p_1(t)\lambda_1(t)$. As time passes and the other player has not stopped, the posterior $p_1(t)$ that he is of type 1 decreases. At some date t_b^1 all types 1 will have stopped. If the two are players are still active, they are then certain that the other is of type 2. Players of type 2 then start mixing at a constant rate μ_2 as in a classical war of attrition.[4]

3.2 Network structure

We now explicitly introduce the network structure and the heterogeneity between players is then due to the position in the line, which also affects the payoffs. Types differ in the number of neighbors they have (as a reminder type k has k neighbors) and thus in terms of payoff when stopping. The payoffs when stopping are the same as for the benchmark studied above: type 1 who makes benefit B_1 if she stops first and B_0 if she stops after the other player and type 2 who gets benefit B_2 if first and B_1 if second.

The extra difference compared to the benchmark model is that type 2 has two neighbors. We will see that this will imply two key differences. First, for types 2, the fact of having two neighbors doubles the chances of at least one of them stopping and thus affects the strategic choices. Second, and most importantly, the types evolve dynamically: if the neighbor of a given player is a type 2 and her other neighbor stops, she becomes a type 1. This change in type of the neighbor is not observed by the player, but the possibility of such a dynamic evolution affects the beliefs about the neighbor's type.

It will turn out to be important to distinguish two cases depending on the respective sizes of

$$\overline{\gamma}_1 := \frac{rB_1}{B_0 - B_1}.$$

[4] There is a reinforcing effect that accelerates the decrease in $p_1(t)$. Since $p_1(t)$ decreases, to keep the belief at μ_1, it needs to be the case that players of type 1 increase their rate of entry $\lambda_1(t)$. This in turn leads to further decrease of the belief $p_1(t)$.

and

$$\overline{\gamma}_2 := \frac{rB_2}{2\left(B_1 - B_2\right)}$$

We will see that the case $\overline{\gamma}_1 > \overline{\gamma}_2$ is one where the players of type 1 mix first. This gives rise to what we call "shrinking networks" since only the players at the extremities of the line mix and over time the line gets shorter. On the contrary, in the case $\overline{\gamma}_2 > \overline{\gamma}_1$, players of type 2 have more incentives to mix first. This gives rise to what we call "fragmenting networks". The initial line will be cut at some point into two smaller networks and this process will repeat itself over time.

Recall that in the benchmark model of section 3.1, two cases were distinguished based on the respective value of μ_1 and μ_2, which determined which type was mixing first (here we have $\overline{\gamma}_1 = \mu_1$ but $\overline{\gamma}_2$ is different from μ_2 since it integrates the fact that a type 2 has two neighbors in our current setup). However, both cases were perfectly symmetric in the benchmark. In the case with a network structure, the two cases will turn out to be radically different, due to the dynamic evolution of the network structure.

3.3 Shrinking networks

We start by considering the case $\overline{\gamma}_1 > \overline{\gamma}_2$. As in the benchmark model, only one type of player can be mixing at any point in time. In this case, players of type 1 have more incentives to mix and stop first.

However, the key difference with the benchmark case is that, as players of type 1 are mixing, two forces affect beliefs, as reflected in the following dynamic equation:

$$\dot{p}_1\left(t\right) = \underbrace{-\lambda\left(t\right)p_1\left(t\right)\left(1-p_1\left(t\right)\right)}_{\text{updating beliefs about initial type}} + \underbrace{\overline{\gamma}_1\left(t\right)p_2\left(t\right)}_{\text{probability that type 2 became 1}} \qquad (2)$$

First, players update their beliefs about their neighbor's types based on the fact they do not see her stopping. Second, the types of neighbors may evolve dynamically since even if the neighbor initially had two neighbors (probability $p_2\left(t\right)$), her other neighbor might have stopped in the time interval (probability $\gamma(t)$), thus changing her type into a type 1.

The two effects go in opposite direction. The first effect makes you less confident that the neighbor started off as a type 1 but the second makes it more likely that she became one over time. Overall, we show that these two effects perfectly balance each other and we find that the beliefs about the neighbor's type do not evolve as presented in the following result:

Proposition 2 *If* $\overline{\gamma}_1 > \overline{\gamma}_2$ *then:*

- *Type 1 players mix at constant rate* $\lambda_1 = \frac{\overline{\gamma}_1}{p_1(0)}$
- *The belief that a random neighbor is of type 1 remains constant, equal to* $p_1(0)$ *throughout the game*

In this case, the players play an infinite war of attrition as if they were facing a single player mixing at rate $\overline{\gamma}_1$. Their beliefs about the neighbor's type remain fixed. Only the players of type 1 situated at the extremities of the line mix at any point in time. Overall, the network shrinks in size over time, hence the terminology. The pattern is therefore very different than in the benchmark case where there was no network structure.

To understand more in depth why the two effects perfectly balance each other, consider what happens in a small period of time dt. Suppose among possible neighbors at date t, there are N_1 of type 1 and N_2 of type 2 (so that $p_1(t) = \frac{N_1}{N_1+N_2}$). In the period dt, a proportion $\lambda_1 N_1$ will stop. At the same time, a proportion $\overline{\gamma}_1 N_2$ will be transformed in types 1. Overall, at the end of the period there are $N_1 - \lambda_1 N_1 + \overline{\gamma}_1 N_2$ neighbors of type 1 and $N_1 - \lambda_1 N_1 + N_2$ total number of players. Given that $\lambda_1 N_1 = \overline{\gamma}_1 N_2 + \overline{\gamma}_1 N_1$ (i.e $\lambda_1 p_1 = \overline{\gamma}_1$), we find that the initial proportions are unchanged.

These results lead to some interesting comparative statics on the speed of entry in the network.

Proposition 3 *The average time before an average member of the network stops is given by*

$$E[T] \quad = \quad (q_1 + q_2)\frac{1}{2\overline{\gamma}_1}$$

It is

1. *Increasing in* B_0*, decreasing in* B_1 *and independent of* B_2
2. *Increasing in* $q_1 + q_2$*.*

It is natural that decreasing the incentives of type 1 to stop (by increasing B_0 or decreasing B_1) delays entry. Interestingly, the rate of stopping is independent of B_2. Given the shape of the equilibrium, this is straightforward, types 1 will be the unique players to mix throughout the game and their incentives are independent of B_2.

3.4 Fragmenting networks

We now consider the case $\overline{\gamma}_2 > \overline{\gamma}_1$. We show that in this case, types 2 have the highest incentives to stop first.

As in the previous case the evolution of beliefs about the neighbor's type are the result of two effects: updating based on the fact that the neighbor did not stop and dynamic evolution of beliefs. However the major difference is that in this case both effects go in the same direction and as time passes it becomes increasingly likely that the neighbor is of type 1.

Overall, we show in the proof of Proposition 4 that the evolution of beliefs will be characterized by:

$$\begin{aligned}
\dot{p}_2(t) &= \underbrace{-\lambda(t)\,p_2(t)\,(1-p_2(t))}_{\text{updating beliefs about initial type}} - \underbrace{\gamma_2(t)\,p_2(t)}_{\text{probability that type 2 became 1}} \\
&= -\gamma_2(t) \qquad (3)
\end{aligned}$$

As in the benchmark case of section 1, at some date t_2 players are sure that their neighbor is not of type 2, i.e $p_2(t_2) = 0$. At that date, types 1 mix exactly as in the benchmark case.

The rate of stopping by types 2 does not however follow the same dynamics as in the benchmark case. If he decides to stop, he gets B_2 as in the benchmark case. When he waits, it is in the hope that one of his two neighbors stops in the meantime, at which point he will become a type 1 with value $V_1(t)$ that varies over time, while it was constant in the benchmark. Thus the stopping rate of a random neighbor will be given by:

$$\gamma_2(t) = \frac{rB_2}{2\,(V_1(t) - B_2)},$$

where the value $V_1(t)$ is defined by the following Bellman equation:

$$V_1(t) = \gamma_2(t)\,B_0 dt + (1 - \gamma_2 dt)\,(1 - r dt)\left(V_1(t) + \dot{V}_1(t)\,dt\right)$$

Indeed, the payoff of a player of type $k = 1$ at a date t where only types $k = 2$ are mixing is composed of the expected payoff in the period dt, which is B_0 if the neighbor stops (probability $\gamma_2(t)$), plus the continuation value. As long as players types $k = 1$ strictly prefer to wait, we have $V_1(t) > B_1$, but $V_1(t)$ is strictly decreasing in time. We will see that there is a moment t_2 at which $V_1(t)$ hits B_1, and from then on types $k = 1$ start mixing.

Proposition 4 *If $\overline{\gamma}_2 > \overline{\gamma}_1$ then there exits a date t_2 such that:*

- *For $t < t_2$ only types $k = 2$ are mixing and the expected rate of stopping of a random neighbor is $\gamma_2(t) = \frac{rB_2}{2(V_1(t)-B_2)}$, where $V_1(t)$ is the value function of type $k = 1$. We have $B_0 > V_1(t) > B_1$ and*

$$\dot{V}_1(t) = -\frac{rB_2(B_0 - V_1(t))}{2(V_1(t) - B_2)} + rV_1(t) < 0. \tag{4}$$

- *At time $t = t_2$, we have $V_1(t_2) = B_1$ and $p_2(t_2) = 0$*
- *For $t > t_2$ players of type $k = 1$ mix at a constant hazard rate $\overline{\gamma}_1$*
- *If $p_2(0) < \frac{1}{2}$, then the compared to the benchmark case, $t_2 > t_b^2$*

Compared to the benchmark model, there are two main forces that affect the time t where the players are sure the other player is not of type 2 (i.e t_2 in the case under consideration and t_b^2 in the benchmark model). First, types 2 mix at a lower rate for two reasons: they have two neighbors, so the chance of at least one stopping is higher than in the benchmark model. Furthermore, the value obtained if one neighbor stops, V_1, is higher than in the benchmark, B_1. Both these effects imply that there are more incentives to wait and the stopping rate will be lower. At the same time, as time passes, some neighbors of type 2 become type 1 which give less incentives to wait. If the proportion of types 2 is initially small as indicated in the last result of Proposition 4, the first effect will dominate.

The dynamic evolution is very different than in section 3.3. Only types 2, situated at the heart of the network as opposed to its extremities, initially mix. At some point one of them randomly stops. The initial network is then fragmented in two smaller networks and the same process repeats itself. We explore in section 6 the consequences of this fragmentation process in terms of spatial distribution of players at the end of the game.

4 Subsidies for adoption

In this section, we examine the welfare impact of introducing subsidies aimed at solving the coordination problems characterizing the waiting game. This is particularly relevant for our leading application to technology adoption. Many countries have in place large scale subsidy programs to support adoption of technologies. This includes subsidies for agricultural techniques (such as fertilizers) in developing countries, health saving technologies, or environmentally friendly technologies in developed countries.

Of course, different motivations drive public intervention in these different areas. The main justification for subsidies in the case of environmentally friendly technologies, and to some extent health related products, is the internalization of an externality. For agricultural techniques, as reported in Dufflo et al. (2011), there is much less consensus on the source of market failure justifying state intervention. Some cite informational problems while others invoke behavioral biases. In this paper we highlight another source linked to coordination failures.

In this context we examine the welfare effect of different types of subsidy programs that are observed in practice. In particular we compare the effect of a one time policy, a subsidy for adoption that applies only to early adopters (i.e in our model those who adopt at date zero) to that of a permanent subsidy. We find that the welfare impact of the policies depends on the expected size of the network, ie on the relative number of types 2.

Typically there is a deadweight loss of funds raised to finance such subsidy programs. To calculate overall welfare we thus assume that, for a given financial cost c of subsidies, the welfare cost is given by $(1+\alpha)c$. Furthermore, we present the results in the case of shrinking networks. In principle, if we return to the model of section 2.1 where we give micro foundations for the application to adoption, both cases are possible, depending on the sequence of spillover factors (σ_i). However, unless σ_2 is very large compared to σ_1, we will be in the shrinking network case. This also seems to be the most relevant case empirically: speed of technology diffusion is often described using measures of distance covered by year (see survey by Geroski).

4.1 Temporary subsidy

We first examine the effect of a temporary subsidy. Without a subsidy, in the case of shrinking networks, players of type 1 mix at the start of the game. A subsidy, encourages a mass of players to immediately adopt. If the subsidy is large, all types 1 immediately adopt. If the subsidy is smaller, some type 1 players have an incentive to wait, in the hope that their neighbor will be one of the early adopters. In fact, for small values of s we have that the proportion $\pi(s)$ of those who adopt at time zero is such that a type 1, given that other types 1 randomize at rate $\pi(s)$, is indifferent between adopting immediately (and get B_1+s) and waiting to either get B_0 or play the waiting game with payoff B_1:

$$B_1 + s = p_1\pi(s)B_0 + [p_1(1-\pi(s)) + p_2]\,B_1$$

i.e

$$\pi(s) \quad = \quad \frac{s}{p_1\left(B_0 - B_1\right)} \tag{5}$$

So players of type 1 will mix at rate $\pi(s)$ as long as $s \leq p_1\left(B_0 - B_1\right)$.

This policy affects the payoff of all types. Type 0, who would have adopted anyway, gets in addition the subsidy. Types 1 get $B_1 + s$ in equilibrium since they are indifferent between adopting now or waiting. Finally, types 2 get a higher expected payoff than without subsidies as they might benefit from the fact that one or two of their neighbors adopts early. However, note that if none of their neighbors end up adopting early and they remain a type 2 at time 0, then the expected payoff is the same as in the baseline case with no subsidy. Indeed the policy affects the probability of facing a type 1 at the start of the game ($p_1(0)$), but does not affect the adoption rate of a neighbor given by $\overline{\gamma}_1$. If more types 1 adopted at date zero because of the policy, this increases $p_1(0)$ and the remaining types will just mix at a lower rate (since $\overline{\gamma}_1 = \lambda_1 p_1(0)$), leaving the expected adoption rate of a neighbor unaffected.

Using the notation G^{te} for the expected gain of the temporary policy te, C^{te} for the expected cost and W^{te} for the total welfare, we have:

$$G^{te}\left(s\right) = q_0\left(B_0 + s\right) + q_1\left(B_1 + s\right) + q_2 V_2'$$

where $V_2' > V_2$ (V_2 is the expected payoff of a type 2 absent subsidies) is presented in the appendix.

The financial cost of the policy is given by:

$$C^{te}(s) \quad = \quad s\left(q_0 + q_1\pi(s)\right)$$

Overall, we find the following result that characterizes in particular total welfare $W^{te}(s) = G^{te}(s) - (1+\alpha)C^{te}(s)$

Proposition 5 *A temporary subsidy $s \leq p_1\left(B_0 - B_1\right)$:*

- *Makes a proportion $\pi(s)$ of types 1 adopt at time zero, where $\pi(s)$ is characterized* *by:*

$$\pi(s) \;\; = \;\; \frac{s}{p_1\left(B_0 - B_1\right)}$$

- *There exists q_0^*, decreasing in q_2 and increasing in q_1 such that:*
  - *if $q_0 > q_0^*$, it is optimal not to implement a temporary subsidy*
  - *if $q_0 \leq q_0^*$, there is a unique optimal subsidy s^**

We find that introducing a temporary subsidy is welfare increasing, conditional on having initially sufficiently few types 0 in the population. Indeed, the benefit of the policy comes from the fact that it partially solves the coordination problem due to the waiting game, by pushing a proportion $\pi(s)$ of types 1 to immediately adopt and speeds up the adoption of types 2. There is however a cost attached to this policy which is that types 0, who would have adopted in any case now receive a subsidy which is costly to finance. The policy is thus welfare enhancing if and only if there is a sufficiently few types 0.

The second result is that the smaller the expected size of the lines (i.e the larger q_1 and the smaller q_2), the more attractive the subsidy program becomes (q_0^* is decreasing in q_2 and increasing in q_1). In terms of costs the only first order cost is the subsidy paid to type 0. The subsidy paid to types 1 is of second order since the probability that a type 1 takes the subsidy is proportional to s. Furthermore the marginal benefit of the policy is higher for types 1 who directly benefit from it than for types 2 who indirectly benefit through the adoption decision of types 1.

4.2 Permanent subsidy

We now examine an alternative solution which is to propose a subsidy that does not expire. In this case, there is no initial mass of adoption, types 1 initially mix between adopting and waiting (at a different rate than without subsidy), while types 2 wait. The expected payoff of types 0 and types 1 is the same as under the temporary subsidy. Types 2 get a different payoff: no one enters at date 0, but the entry rate of types 1 is now faster. We denote the policy *pe* (for permanent subsidy). We have that:

$$G^{pe}\left(s\right) = q_0\left(B_0 + s\right) + q_1\left(B_1 + s\right) + q_2 \frac{2\left(B_1 + s\right)^2}{B_0 + B_1 + 2s}$$

We compare these two policies for small level of subsidies, in other words, we compare the respective marginal welfare gains when $s = 0$. We find that the temporary subsidy brings smaller gains but at a smaller cost and will be preferred when the lines are small.

Proposition 6

- *The marginal benefit and marginal cost of the temporary subsidy are smaller than that of a permanent subsidy when* $s = 0$
- *The temporary subsidy is preferred if* q_2 *is small*

The expected benefit of both policies is the same for types 0 and 1: they both get $B_i + s$. So if there were no types 2, the difference would just depend on expected costs. These costs are higher under the permanent subsidy where all players end up obtaining the subsidy. The temporary subsidy is thus preferred when q_2 is small. The permanent subsidy policy becomes more attractive for networks of larger expected size (i.e where q_2 is high), since types 2 benefit from the policy because it speeds up the entry rate of all types 1.

We conclude the welfare analysis by pointing out that policies targeted at certain types would be preferable to both policies considered up till now. For instance, paying the subsidy to players with no neighbors is in the context of our model a pure welfare loss. Of course such targeted policies seem extremely hard to put in place.

5 Large networks

This section is incomplete

We now consider larger network where the type k can take values in $\{0, .., N\}$. As in the previous cases, only one type k can be mixing at any point in time. Indeed, if type k is mixing on the interval $\tau \in [0, T]$, it has to be the case that:

$$(1 - F(\tau))^k = \exp\left(-\frac{rB_k}{(V_{k-1} - B_k)}\tau\right)$$

Given that the probability that a random neighbor stops is described by $F(t)$ for all types, for two types k and k' to both mixing at date t, it has to be the case that:

$$r\frac{B_k}{k\left(V_{k-1}(t) - B_k\right)} = r\frac{B_{k'}}{k'\left(V_{k'-1}(t) - B_{k'}\right)}$$

which is unlikely to hold. Thus it is natural to expect that a single type will be mixing at a given instant.

It will turn out to be useful to introduce the following hazard rates:

$$\overline{\gamma}_k \equiv r \frac{B_k}{k\left(B_{k-1} - B_k\right)}$$

Following the logic of the line case, we will examine two cases in turn:

- Case where $\overline{\gamma}_k$ is a increasing sequence
- Case where $\overline{\gamma}_k$ is an decreasing sequence

5.1 Shrinking networks

We first examine the case where $\overline{\gamma}_k$ is a decreasing sequence, which corresponds to the case of shrinking networks. At the start of the game, types 1 are mixing and while these types are still present in the game, they are the only ones mixing. Two things can occur: either types 1 never disappear and the proportions of different types converge to particular limit. Alternatively, types 1 disappear in finite time and a second phase starts, where types 2 are mixing. Following the same logic, types 2 can either never disappear or disappear in finite time.

We start by focusing on subgames where only types k and above are left. In such a subgame, because $\overline{\gamma}_k$ is a increasing sequence, only types k are mixing, and thus the only neighbors who stop are those of type k. However they might stop for two distinct reasons: either because of their own mixing, or because one of their own neighbors stopped, transforming them in types $k-1$ who immediately stop. Thus the entry rate of a neighbor is given by:

$$\gamma = p_k \lambda_k + (k-1) p_k \gamma$$

So that

$$\gamma = \frac{p_k \lambda_k}{1 - (k-1) p_k}$$

which imposes the constraint

$$p_k \leq \frac{1}{k-1} \tag{6}$$

If types 1 and 2 do not disappear, the highest type that ever randomizes is type 2, and hence then condition (6) will not be violated. Proposition 7 characterizes conditions under which types 1 and 2 don't disappear in finite time. The conditions depend on one statistic summarizing the shape of the network, i.e the expected number of neighbors of a neighbor, that we denote $L(t)$

$$L(t) = \sum_k (k-1)p_k(t)$$

Proposition 7 *If $L(0) \leq \frac{5}{2}$, then the unique symmetric equilibrium is such that as t goes to infinity, $p(t)$ converges to some limit vector p^* such that $p_1^* + p_2^* = 1$ and $p_3^* = ... = p_N^* = 0$.. Furthermore:*

- *If $L(0) < 1$ then types 1 never disappear, $p_1^* > 0$*
- *If $L(0) \in (1, \frac{5}{2})$ then there exists a date $\hat{t}$ such that $p_1(\hat{t}) = 0$*

A key property we use is that in the first phase of the game, when types 1 are mixing, $L(t)$ is constant. Thus if $L(0) < 1$, types 1 never disappear. We also show that $L(0)$ can determine whether we are in a subgame where all types 1 disappear but types 2 remain until the end of the game and in fact their proportion converges to 1.

The part of this section that describes what happens when types 1 and 2 disappear is to be added here!

5.2 Fragmenting networks

We now consider the case where γ_k is a decreasing sequence. We show that the dynamics are very similar to the dynamics observed in the fragmenting case on the line. At the start of the game, players with the highest incentive to enter are the most connected players (with N neighbors). As they mix, beliefs are updated in such a way that if a neighbor has not entered yet, it becomes increasingly unlikely that she was of type N. As in section 3.4, the two forces affecting beliefs, i.e the updating and the dynamic evolution of beliefs go in the same direction, as shown below:

$$\dot{p}_N = -(N-1)\gamma p_N - \gamma(1-p_N) = -\gamma(1+(N-2)p_N)$$

At some finite date t_N, all types N will have disappeared, while all other types will still be present. Thus, at $t = t_N$, we start a similar subgame with the types $N-1$ mixing and the process unfolds in the same way. This evolution of the network, that gradually fragments in smaller networks, reaches a date t_2 where only isolated pairs of types 1 are left and they then play an infinite war of attrition.

Proposition 8 *There exists an decreasing sequence of dates* $\{t_2, .., t_N, t_{N+1}\}$*, with* $t_{N+1} = 0$*, such that:*

- *For dates* $t_{k+1} < t < t_k$*, with* $k \in \{2, N+1\}$ *types* k *are mixing and the expected stopping rate of a neighbor is* $\gamma(t) = \frac{rB_k}{k(V_{k-1}(t)-B_k)}$*. At date* $t = t_k$*,* $p_l = 0$ *for all* $l \geq k$*, i.e all types higher than* k *have disappeared.*
- *For dates* $t > t_2$ *players of type* $k = 1$ *mix at a constant hazard rate* $\gamma_1 = \frac{rB_1}{B_0 - B_1}$

We thus see that the case of fragmenting networks for larger network size follows the same pattern as in the case of the line. The network gradually fragments into smaller network size until a date where only isolated pairs are left.

6 Spatial inefficiency

In this last section, we consider in more detail another application of our model, the exit decision by firms. The network corresponds to a particular spatial distribution of firms. These firms are currently making zero profits. If they exit the market, they get a payoff B_i (they can sell the machinery for instance) and if they are the last firm standing, they make a profit $B_0 > B_i\ \forall i > 0$. This fits exactly the framework of our model. Note that we could alternatively have chosen a model, closer to the classic war of attrition, where only the last firm gets benefit B_0 and all firms have to pay a flow cost c while staying in. We consider this alternative model in Appendix B3 and show it leads to equivalent results.

As in the classical war of attrition, without the network structure, we found a timing inefficiency in Proposition 4. In the case of the fragmenting network, we uncover a second potential source of inefficiency of a spatial nature due to the network structure, linked in particular to the final spatial distribution of firms.

In this application, it is natural to think that the shape of the final distribution of firms can then be of great significance. For instance, suppose that customers at a distance of more than one link from a firm cannot be profitably served by that firm given their transport cost. In that case, the socially optimal distribution of firms would be equally spaced firm, as represented in Figure 1. However, in our exit model, there is no reason that the final spatial distribution will be equally spaced. Consider the dynamic evolution of the network represented in Figure 1. The first firm to exit is firm 3 leaving two pairs whose members will eventually play a war of attrition. This war of

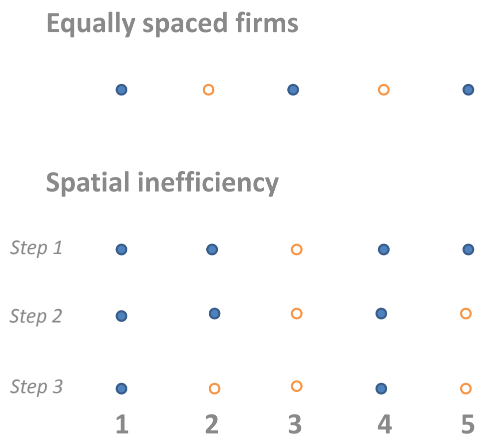

Figure 1: Spatial inefficiency

attrition might result in players 2 and 5 exiting, leaving the customers located between the locations of 2 and 3 stranded. We describe this as a spatial inefficiency. We will now characterize how likely this is to occur on an infinite line.

We start with the limit case such that initially the line is fully connected so that $p_2(0) = 1$. As described in Proposition 4, the equilibrium is such that initially only types 2 mix until a date t_2 is reached where only isolated pairs of players are left. To characterize the spatial inefficiency, we are interested in two elements. First, the proportion of firms remaining at the end of the game, proportion we denote p_e (for exit). If firms were equally spaced, this proportion would be exactly $1/2$. Second, we would like to describe the random variable measuring the gap between two consecutive firms at the end of the game, random variable that we denote l_g. If firms were equally spaced, this random variable would be degenerate at value 1. We first observe that the random variable can in fact take values only in $\{1, 2, 3\}$. At date t_2, the gap between two firms is at most 1, since players of type 1 do not enter in the first phase. The maximum value of 3 for l_g will be achieved in the case where at the end of the first phase, two pairs are seperated by a gap and the second phase sees the players close to the gap exit first.

Let us first compute the fraction of the firms that exit in the first phase of the game, i.e. before time t_2. It is equivalent to the probability with which i will exit before one of her neighbors do. If firms were all mixing at a constant rate, this probability would be exactly 1/3. However, as time passes, the neighbors start mixing at a lower rate on average since they might have transformed into a type 1. Overall, we find that this probability is in fact close to 1/2.

At the end of the first phase, three types of nodes exist:

- x nodes that exited
- $2y$ nodes in pairs
- z nodes singletons

The ratio of firms that exit in the first phase is thus $\frac{x}{x+2y+z}$. The ratio of firms that stay in at the end is $\frac{z+y}{x+2y+z}$. We have the additional constraint that to the left of each node that exited, there is either a pair or a singleton, thus $x = y + z$. Overall, this implies that these two proportions are equal and this provides a way to calculate p_e as presented in the first result of Proposition 9.

At the end of the first phase, i.e at date t_2, a gap is surrounded either by pairs or singletons. In fact, the probability of having a pair to the right of the gap is independent of having a pair to the left. This provides a direct way of calculating the final distribution of variable l_g, as expressed in the following distribution.

Proposition 9 *At the end of the game, the spatial distribution of firms is such that:*

- *The proportion of remaining firms is* $p_e = \frac{1}{2}\left(1 - e^{-2}\right)$
- *The probability distribution of the* l_g*, the gap between two consecutive firms is:*

$$P\left[l_g = 3\right] = p^2\frac{1}{4} \backsimeq 0.02$$
$$P\left[l_g = 2\right] = p^2\frac{1}{2} + 2p(1-p)\frac{1}{2} \backsimeq 0.26$$
$$P\left[l_g = 1\right] = p^2\frac{1}{4} + 2p(1-p)\frac{1}{2} + (1-p)^2 \backsimeq 0.72$$

where $p = 2\frac{1}{1+e^2}$

Proposition 9 describes precisely how the final distribution of firms differs from an equally spaced distribution. Overall, approximatively 43 percent of firms remain at the end of the game. This implies that at least some firms are separated by a gap of more than 2. In fact we find that 28 percent of firms are in this situation, while gaps of 3 are rather rare.

Note that if the final spatial distribution were equally spaced with exactly one inactive firm between each two active firms, the fraction of players that stay would be $1/2$. Hence, what we refer to as spatial inefficiency is the finding that in equilibrium this fraction is significantly lower than this: $\omega \approx 0.432 < 1/2$.

7 Conclusion

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8 Appendix A

Proposition 1:

We prove the result in the case $\mu_1 > \mu_2$. The other case is perfectly symmetric.

Denote $F(t)$ the probability that the other player stops before time t. The expected payoff for type j of the strategy "stop at time τ if the other player has not yet stopped"is given by:

$$W_j(\tau) = \left[\int_0^\tau e^{-rt}\left(B_{j-1}\right)f(t)dt + (1-F(\tau))e^{-r\tau}(B_j)\right].$$

For a player of type j to be ready to mix in an interval $[t,t']$, it has to be the case he is indifferent between stopping at any date $\tau \in [t,t']$. We must therefore have:

$$\begin{aligned}
e^{-r\tau}B_{j-1}f(\tau) - f(\tau)e^{-r\tau}B_j - r(1-F(\tau))e^{-r\tau}B_j &= 0 \\
(B_{j-1}-B_j) - r\frac{(1-F(\tau))}{f(\tau)}B_j &= 0. \\
\frac{f(\tau)}{(1-F(\tau))} = r\frac{B_j}{B_{j-1}-B_j} &\equiv \mu_j
\end{aligned}$$

Therefore, if one type is mixing, the other one won't be. Since $\mu_1 > \mu_2$, initially only types 1 are mixing. Furthermore, while players of type 1 are mixing, the rate of entry of the other player is a fixed rate μ_1, where $\mu_1 = p_1(t)\lambda_1(t)$ combines the probability that the other player is of type 1 and the rate of entry of a type 1.

The updated belief that the other player is of type 1 is then given by Baye's rule:

$$p_1(t+dt) = \frac{p_1(t)(1-\lambda_1(t)dt)}{p_1(t)(1-\lambda dt)+(1-p_1(t))} \tag{7}$$

So that

$$\frac{p_1(t+dt)-p_1(t)}{dt} = \frac{1}{dt}\frac{p_1(t)(1-\lambda_1(t)dt) - p_1(t)(p_1(t)(1-\lambda_1(t)dt)+(1-p_1(t)))}{p_1(t)(1-\lambda_1(t)dt)+(1-p_1(t))} \tag{8}$$

Taking limits we have:

$$\dot{p}_1(t) = -\lambda_1(t)(t)\,p_1(t)\,(1-p_1(t)) \tag{9}$$

Since $\mu_1 = p_1(t)\lambda_1(t)$, we have:

$$\dot{p}_1(t) = -\gamma_1(t)(1 - p_1(t)) \tag{10}$$

The solution of this differential equation is:

$$1 - p_1(t) = (1 - p_1(0))e^{-\gamma_1(t)t} \tag{11}$$

$p_1(t)$ is strictly decreasing over time. Thus there exists a time t_1^b such that $p_1(t_1^b) = 0$. It is defined by:

$$t_1^b = -\frac{ln(1 - p_1(0))}{\gamma_1}$$

After that date only players of type 2 are left and they mix at constant rate γ_2 as in classical waiting games.

Proposition 2:

We first establish the result that $p_1(t)$ remains constant throughout the game. We define two events:

- NE the event that no entry takes place in the interval $[t, t+\epsilon]$
- CS (change state) the event that the neighbor changes state during the interval $[t, t+\epsilon]$, which can only mean that his other neighbor stopped, i.e he moved from being a type 2 to a type 1.

Using these notations, we have:

$$\begin{aligned} p_1(t+\epsilon) &= \frac{P[k=1 \cap NE \cap CS]}{P[NE]} + \frac{P[k=1 \cap NE \cap CS^C]}{P[NE]} \\ &= \frac{p_2(t)(1-\lambda_1(t)\epsilon)\overline{\gamma}_1\epsilon}{P[NE]} + \frac{P[NE|k=1 \cap SC^C]P[k=1 \cap SC^C]}{P[NE|k=1]p_1(t) + (1-p_1(t))} \end{aligned}$$

We now examine:

$$\begin{aligned}
\frac{p_1(t+\epsilon)-p_1(\epsilon)}{\epsilon} &= \frac{p_2(t)(1-\lambda_1(t)\epsilon)\gamma_1}{P[NE]}+\frac{1}{\epsilon}\frac{p_1(t)(1-\lambda_1(t)\epsilon)-p_1(t)(p_1(t)(1-\lambda_1(t)\epsilon)+(1-p_1(t)))}{P[NE]} \\
&= \frac{p_2(t)(1-\lambda\epsilon)\gamma_1}{P[NE]}+\frac{1}{\epsilon}\frac{p_1(t)(1-p_1(t))\lambda\epsilon}{P[NE]} \\
&= \frac{p_2(t)(1-\lambda_1(t)\epsilon)\gamma_1}{P[NE]}+\frac{p_1(t)(1-p_1(t))\lambda_1(t)}{P[NE]}
\end{aligned}$$

Taking the limit when ϵ goes to zero, we have $P[NE]$ converges to one so that

$$\dot{p}_1(t)=\gamma_1(1-p_1(t))-\lambda p_1(t)(1-p_1(t))$$

Finally, by definition, $\gamma_1=\lambda_1 p_1$, so that

$$\dot{p}_1=0$$

This establishes the first part of the proposition.

Finally given that $p_1(t)$ and $\gamma_1=\overline{\gamma}_1$ do not depend on time, the rate of mixing of types 1 $\lambda_1(t)$ also remains constant and is equal to $\lambda_1=\frac{\overline{\gamma}_1}{p_1(0)}$ as indicated in the first result of the proposition. ■

Proof Proposition 3:

We first derive the average time before stopping of a random member of the network. If the player is of type 0 (probability q_0), he enters immediately. If he is of type 1 (probability q_1), his stopping rate is $\lambda_1+\overline{\gamma}_1$, since he stops either because of his own mixing or because a neighbor stops. Finally, if he is of type 2, he fist needs to transition to being a type 1, which occurs at a rate $2\overline{\gamma}_1$, then follows the same dynamic as a type 1. Overall we have that the expected waiting time, that we denote T is given by;

$$\begin{aligned}
E[T] &= q_0 0+q_1\frac{1}{\lambda_1+\overline{\gamma}_1}+q_2\left[\frac{1}{2\overline{\gamma}_1}+\frac{1}{\lambda_1+\overline{\gamma}_1}\right] \\
&= q_2\frac{1}{2\overline{\gamma}_1}+(q_1+q_2)\frac{1}{\lambda_1+\overline{\gamma}_1}
\end{aligned}$$

In proposition 2, we established that $\lambda_1=\frac{\overline{\gamma}_1}{p_1}$

Furthermore, the following relationship holds generally between q_k and p_k (see e.g.

Jackson, 2008):

$$p_k = \frac{kq_k}{\sum\limits_{k'=0}^{\infty}(k' q_{k'})}$$

in other words:

$$p_1 = \frac{q_1}{q_1 + 2q_2}$$

Replacing we have:

$$E[T] \;=\; (q_1 + q_2)\frac{1}{2\overline{\gamma}_1}$$

$E[T]$ is thus like $\overline{\gamma}_1$ increasing in B_0, decreasing in B_1 and independent of B_2. Furthermore, since $\overline{\gamma}_1$ is independent of q_1 and q_2, $E[T]$ is overall increasing in $q_1 + q_2$.

Proof Proposition 4:

Following the same arguments as in Proposition 2, we can establish that while players of type 2 are mixing, the evolution of beliefs evolves according to:

$$\dot{p}_2(t) = -\gamma_2(t)p_2(t) - \lambda_2(t)p_2(t)(1 - p_2(t))$$

Given that $\gamma_2(t) = \lambda_2(t)p_2(t)$, we obtain that

$$\dot{p}_2(t) = -\gamma_2(t)$$

i.e

$$p_2(t) = p_2(0) - \int_0^t \gamma_2(s)\, ds$$

$p_2(t)$ is a strictly decreasing function of t so that we can find a date t_2 such that $p_2(t_2) = 0$.

Furthermore we have

$$\gamma_2(t) = \frac{rB_2}{2(V_1(t) - B_2)},$$

where the value $V_1(t)$ is defined by the following Bellman equation:

$$V_1(t) = \gamma_2(t)\, B_0 dt + (1 - \gamma_2(t)dt)(1 - rdt)\left(V_1(t) + \dot{V}_1(t)\, dt\right)$$

Using the value of γ_2, we obtain:

$$\dot{V}_1(t) = -\frac{rB_2(B_0 - V_1(t))}{2(V_1(t) - B_2)} + rV_1(t) < 0. \tag{12}$$

This establishes the first result of the proposition

To establish the last result, we compare the values of $\dot{p}_2(t)$ in the two cases.

Here we have:

$$\dot{p}_2(t) = -\frac{rB_2}{2(V_1(t) - B_2)}$$

In the benchmark case we had:

$$\dot{p}_2(t) = -(1 - p_2(t))\frac{rB_2}{(B_1(t) - B_2)}$$

Given that $V_1(t) \geq B_1$ and $p_2(t) \leq p_2(0) < \frac{1}{2}$, it is the case that the posterior probability decreases faster in the benchmark case, so that $t_2 > t_b^2$.

Proposition 5

The expected welfare gain (not including costs) due to the temporary subsidy was derived in the main text:

$$G^{te}(s) = q_0(B_0 + s) + q_1(B_1 + s) + q_2\widetilde{V}_2$$

where

$$\widetilde{V}_2 = V_2 + 2p_1(1 - p_1)\pi(s)(B_1 - V_2) + p_1^2\left(2\pi(1 - \pi)(B_1 - V_2) + \pi^2(B_0 - V_2)\right) \tag{13}$$

and V_2 is the expected payoff of a type 2 absent a subsidy

$$V_2 = \frac{2\overline{\gamma}_1}{2\overline{\gamma}_1 + r}B_1$$

Replacing by the value of $\overline{\gamma}_1$, we have:

$$V_2 = \frac{2(B_1)^2}{B_0 + B_1}$$

Rewriting equation (13), we obtain:

$$\widetilde{V_2} = V_2 + \left(2p_1\pi - 2p_1^2\pi^2\right)(B_1 - V_2) + p_1^2\pi^2(B_0 - V_2)$$

Using the fact that $\pi(s) = \frac{s}{p_1(B_0 - B_1)}$, we have

$$\begin{aligned}\widetilde{V_2} &= V_2 + \left(2p_1\pi - 2p_1^2\pi^2\right)(B_1 - V_2) + p_1^2\pi^2(B_0 - V_2) \\ &= V_2 + 2p_1\pi(B_1 - V_2) + p_1^2\pi^2(B_0 - 2B_1 + V_2)\end{aligned}$$

We simplify the previous expression using the fact that $V_2 = \frac{2(B_1)^2}{B_0+B_1}$

$$\begin{aligned} &= V_2 + 2s\frac{B_1 - V_2}{B_0 - B_1} + s^2\frac{B_0}{B_0^2 - B_1^2} \\ &= V_2 + 2s\frac{B_1}{B_0 + B_1} + s^2\frac{B_0}{B_0^2 - B_1^2}\end{aligned}$$

The financial cost of the policy is given by:

$$\begin{aligned}C^{te}(s) &= s\left(q_0 + q_1\pi(s)\right) \\ &= s\left(q_0 + q_1\frac{s}{p_1\left(B_0 - B_1\right)}\right) \\ &= s\left(q_0 + (q_1 + 2q_2)\frac{s}{(B_0 - B_1)}\right)\end{aligned}$$

So taking the derivative with respect to the level of subsidy of the total welfare function, we have:

$$(W^{te})'(s) = -q_0\alpha + q_1 + 2q_2\frac{B_1}{B_0 + B_1} + 2s\left(q_2\frac{B_0}{B_0^2 - B_1^2} - (1+\alpha)(q_1 + 2q_2)\frac{1}{(B_0 - B_1)}\right)$$

and

$$(W^{te})''(s) = 2\left(q_2\frac{B_0}{B_0^2 - B_1^2} - (1+\alpha)(q_1 + 2q_2)\frac{1}{(B_0 - B_1)}\right) < 0$$

The second derivative is negative since

$$\frac{B_0}{B_0^2 - B_1^2} < \frac{2}{(B_0 - B_1)}$$

Finally we have

$$\begin{aligned}(W^{te})'(0) &= -q_0\alpha + q_1 + 2q_2\frac{B_1}{B_0+B_1} \\ &= -q_0(1+\alpha) - q_2\frac{B_0-B_1}{B_0+B_1} + 1\end{aligned}$$

So that $(W^{te})'(0) \geq 0$ if and only if:

$$q_0 \leq \frac{1}{1+\alpha}\left(1 - q_2\frac{B_0-B_1}{B_0+B_1}\right) \equiv q_0^*$$

As established in the proposition q_0^* is decreasing in q_2.

Given that W^{te} is concave, if $q_0 \geq q_0^*$, temporary subsidies should not be used. If $q_0 < q_0^*$, the optimal subsidy s^* solves $(W^{te})'(s^*) = 0$, i.e

$$\begin{aligned}s^* &= \frac{-q_0\alpha + q_1 + 2q_2\frac{B_1}{B_0+B_1}}{-q_2\frac{B_0}{B_0^2-B_1^2} + (1+\alpha)(q_1+2q_2)\frac{1}{(B_0-B_1)}} \\ &= \frac{-\alpha + q_1(1+\alpha) + 2q_2(\frac{B_1}{B_0+B_1} + \frac{\alpha}{2})}{-q_2\frac{B_0}{B_0^2-B_1^2} + (1+\alpha)(q_1+2q_2)\frac{1}{(B_0-B_1)}}\end{aligned}$$

Proposition 6

We first derive the benefits and costs of the temporary subsidy.

We first focus on the expected benefit. A player with no neighbors will stop immediately and will get payoff $B_0 + s$. A player with one neighbor will start randomizing, and since she is indifferent between stopping and waiting her payoff is $B_1 + s$. A player with two neighbors will wait and her payoff is

$$\begin{aligned}V_2(s) &= \int_{t=0}^{\infty} 2\overline{\gamma}_1(s)\, e^{-2\overline{\gamma}_1(s)t} e^{-rt}(B_1+s)\, dt \\ &= \frac{2\overline{\gamma}_1(s)(B_1+s)}{2\overline{\gamma}_1(s)+r}\end{aligned}$$

Using the fact that $\overline{\gamma}_1 = r\frac{B_1+s}{B_0-B_1}$, we have:

$$V_2(s) = \frac{2(B_1+s)^2}{B_0+B_1+2s}.$$

Therefore we have:

$$G^{pe}(s) = q_0 (B_0 + s) + q_1 (B_1 + s) + q_2 \frac{2 (B_1 + s)^2}{B_0 + B_1 + 2s}$$

$$(G^{pe})'(s) = q_0 + q_1 + \frac{4q_2 (B_1 + s)(B_0 + s)}{(B_0 + B_1 + 2s)^2},$$

$$(G^{pe})'(0) = q_0 + q_1 + \frac{4q_2 B_0 B_1}{(B_0 + B_1)^2}.$$

We now calculate the expected cost. Consider this separately for each type k. For type $k = 0$, the payment is made immediately, so the cost is simply s. For type $k = 1$, payment accrues at time τ that is exponential with parameter $\lambda_1(s) + \overline{\gamma}_1(s)$, so discounted cost is

$$\mathbb{E}\left(e^{-r\tau} s\right) = s \int_0^\infty \left(\lambda_1(s) + \overline{\gamma}_1(s)\right) e^{-(\lambda_1(s)+\overline{\gamma}_1(s))t} e^{-rt} dt = \frac{\lambda_1(s) + \overline{\gamma}_1(s)}{\lambda_1(s) + \overline{\gamma}_1(s) + r} s.$$

Finally, type $k = 2$ becomes type $k = 1$ at time τ_1 that is exponential with parameter $2\overline{\gamma}_1(s)$, and then will wait another time interval τ to stop. The expected payment is therefore

$$\mathbb{E}\left(e^{-r(\tau_1+\tau)} s\right) = \mathbb{E}\left(e^{-r\tau_1}\right) \mathbb{E}\left(e^{-r\tau}\right) s = \frac{2\overline{\gamma}_1(s)\left(\lambda_1(s) + \overline{\gamma}_1(s)\right)}{\left(2\overline{\gamma}_1(s) + r\right)\left(\lambda_1(s) + \overline{\gamma}_1(s) + r\right)} s.$$

We have then

$$C^{pe}(s) = q_0 s + q_1 \frac{\lambda_1(s) + \overline{\gamma}_1(s)}{\lambda_1(s) + \overline{\gamma}_1(s) + r} s + q_2 \frac{2\overline{\gamma}_1(s)\left(\lambda_1(s) + \overline{\gamma}_1(s)\right)}{\left(2\overline{\gamma}_1(s) + r\right)\left(\lambda_1(s) + \overline{\gamma}_1(s) + r\right)} s.$$

Substituting in $\overline{\gamma}_1(s)$ and $\lambda_1(s)$ from above, and using $q_0 + q_1 + q_2 = 1$, we get:

$$\begin{aligned} C^{pe}(s) &= s \frac{\left[q_0 B_0 + B_1 \left(q_0 + 2(q_1 + q_2)\right) + 2(q_0 + q_1 + q_2) s\right]}{B_0 + B_1 + 2s} \\ &= s \frac{\left[q_0 B_0 + (2 - q_0) B_1 + 2s\right]}{B_0 + B_1 + 2s} = s \frac{q_0 (B_0 - B_1) + 2(B_1 + s)}{B_0 + B_1 + 2s} \end{aligned}$$

and

$$(C^{pe})'(s) = \frac{q_0 \left(B_0^2 - B_1^2\right) + 2B_1 (B_0 + B_1) + 4(B_0 + B_1 + s) s}{(B_0 + B_1 + 2s)^2},$$

$$(C^{pe})'(0) = \frac{q_0 \left(B_0^2 - B_1^2\right) + 2B_1 (B_0 + B_1)}{(B_0 + B_1)^2}.$$

We can now compare permanent and temporary subsidies. We first compare benefits. We have:

$$\begin{aligned}(G^{te})'(0) - (G^{pe})'(0) &= q_2\left[2\frac{B_1}{B_0+B_1} - 4\frac{B_1B_0}{(B_0+B_1)^2}\right] \\ &= q_2 2\frac{B_1(B_1-B_0)}{(B_0+B_1)^2} < 0\end{aligned}$$

We can now compare costs

$$\begin{aligned}(C^{te})'(0) - (C^{pe})'(0) &= q_0 - \frac{q_0\left(B_0^2 - B_1^2\right) + 2B_1\left(B_0+B_1\right)}{\left(B_0+B_1\right)^2} \\ &= (q_0-1)\frac{2B_1}{B_0+B_1} < 0\end{aligned}$$

So if q_2 is small, prefer the temporary subsidy at $s=0$.

Proposition 7

Step 1: There exists no symmetric equilibrium where for some date t types 1 and 2 have disappeared.

Suppose we reach such a subgame. Then two cases can arise:

- Either types $k \in \{3, .., N-1\}$ disappear in finite time
- Or there exists a type k such that a date t' exists where for all $t > t'$ $p_k(t) > 0$ and $p_{k'}(t) = 0$ for $k' < k$

In the first case we reach a subgame where only types $N-1$ and N are left. In this case the dynamics are given by:

$$\begin{aligned}\dot{p}_{N-1} &= (N-2)\gamma p_N \\ \dot{p}_N &= -(N-2)\gamma p_N.\end{aligned}$$

So that p_N converges to 0 and p_{N-1} converges to 1. We have shown that for a symmetric equilibrium to exist, it has to be the case that $p_k \leq \frac{1}{k-1}$. Thus in this case no symmetric equilibrium exists.

Consider the second case and place ourselves at the start of the subgame where all types below k have disappeared. The system is then described by:

$$\begin{aligned}
\dot{p}_k &= k\gamma p_{k+1} - \gamma(1-p_k) \\
\dot{p}_{k+1} &= -(k-1)\gamma p_{k+1} + (k+1)\gamma p_{k+2} \\
&\ldots \\
\dot{p}_{N-1} &= -(N-3)\gamma p_{N-1} + (N-1)\gamma p_N \\
\dot{p}_N &= -(N-2)\gamma p_N.
\end{aligned}$$

In the limit, if types k don't disappear, p_k converges to 1 and we reach a contradiction as p_k crosses the $\frac{1}{k-1}$ thresshold.

Step 2: If $p_1(t) > 0$ then $L(t)$ is constant and if $p_1(t) = 0$ then $L(t)$ is strictly deceasing.

Shown above for $p_1(t) > 0$. For $p_1(t) = 0$, we have

$$\begin{aligned}
\sum_{k=1}^{N} (k-1)\,\dot{p}_k &= -(k-1)\gamma(1-p_k) \\
&< 0.
\end{aligned}$$

Step 3: If $L(0) < 1$ then types 1 never disappear $p_1^* > 0$.

Suppose on the contrary there exists a date t such that they disappear. At that date, since according to step 2, $L(t)$ is constant, we have $L(t) = L(0) < 1$, which is impossible if $p_1(t) = 0$.

Step 4: Suppose $L(0) \in (1, \frac{5}{2})$ then there exists a date $\hat{t}$ such that $p_1(\hat{t}) = 0$ and the unique symmetric equilibrium is such that as t goes to infinity, $p\,(t)$ converges to some limit vector p^* such that $p_1^* + p_2^* = 1$ and $p_3^* = ... = p_N^* = 0.$.

Suppose that there was no date $\hat{t}$ such that $p_1(\hat{t}) = 0$, i.e types 1 did not disappear. If that were not the case, then in the limit only types 1 and 2 would be left, implying that $L(0)$ would be smaller than 1 using Step 2. We reach a contradiction.

Consider the subgame starting at date $\hat{t}$ where types 2 start mixing. The evolution of beliefs is such that

$$\begin{aligned}
\dot{p}_2 &= 2\gamma p_3 - \gamma(1 - p_2) \\
\dot{p}_3 &= -\gamma p_3 + 3\gamma p_4 \\
&\ldots \\
\dot{p}_{N-1} &= -(N-3)\gamma p_{N-1} + (N-1)\gamma p_N \\
\dot{p}_N &= -(N-2)\gamma p_N.
\end{aligned}$$

Suppose that there exists a date $\tilde{t} > \hat{t}$, such that $p_2(\tilde{t}) = 0$. Then the evolution of beliefs would imply

$$\dot{p}_2 = 2\gamma p_3 - \gamma$$

This is only compatible with $p_2(\tilde{t}) = 0$ if $p_3 \leq \frac{1}{2}$. At date $\tilde{t}$, we have:

$$L(\tilde{t}) = 2p_3(\tilde{t}) + \sum_{k>3}(k-1)p_k(\tilde{t})$$

The minimum value of $L(\tilde{t})$ compatible with condition $p_3 \leq \frac{1}{2}$ will then be achieved if $p_3(\tilde{t}) = p_4(\tilde{t}) = \frac{1}{2}$ for which $L(\tilde{t}) = \frac{5}{2}$. We thus conclude that

$$L(\tilde{t}) \geq \frac{5}{2}$$

According to step 2, this is a contradiction since $L(t)$ is weakly decreasing in t

Proposition 8

$$
\begin{aligned}
\dot{p}_k &= -(k-1)\gamma p_k - \gamma(1-p_k) = -\gamma(1+(k-2)p_3) \\
\dot{p}_{k-1} &= (k-1)\gamma p_k - (k-2)\gamma p_{k-1} + \gamma p_{k-1} \\
&\ldots \\
\dot{p}_2 &= 2\gamma p_3 - \gamma p_2 + \gamma p_2 = 2\gamma p_3 \\
\dot{p}_1 &= \gamma p_2 + \gamma p_1 = \gamma(1-p_3)
\end{aligned}
$$

Two results that are true for all k:

- none of the types $k' < k$ can disappear in finite time while beliefs are governed by these dynamics. Indeed for any of those k', if you take $p_{k'} = 0$ then $\dot{p}_{k'} > 0$
- type k disappears in finite time since $\dot{p}_k < -\gamma$

Proposition 9

As described in the main text, the first step to calculate p_e is to determine the probability with which i exits before one of her neighbors does, a probability we denote ω.

For $t < t_2$ firm i exits with a hazard rate $\lambda(t)$ as long as none of her two neighbors have exited. Denote by $f(t)$ the probability density function for i's planned exit time (i.e. time to exit if none of her neighbors have yet stopped):

$$f(t) = \lambda(t) \cdot e^{-\int_0^t \lambda(s)ds}.$$

The hazard rate with which a neighbor of i exits is $\gamma(s)$ so that the probability that none of i's neighbors have exited at time t (given that i has not) is

$$e^{-\int_0^t 2\gamma(s)ds}.$$

Using this, we can write the probability that i exits before one of her neighbors as:

$$\omega = \int_0^{t_2} f(t) \cdot e^{-\int_0^t 2\gamma(s)ds} dt = \int_0^{t_2} \lambda(t) \cdot e^{-\int_0^t \lambda(s)ds} \cdot e^{-\int_0^t 2\gamma(s)ds} dt. \tag{14}$$

To evaluate this expression, we have to utilize the connection between $\lambda(s)$ and $\gamma(s)$.

Recall that $p_2(t)$ evolves according to

$$\dot{p}_2(t) = -\lambda(t) \cdot p_2(t),$$

so that with boundary condition $p_2(0) = 1$ we have

$$p_2(t) = e^{-\int_0^t \lambda(s)ds}.$$

Moreover, the relationship between $\lambda(t)$ and $\gamma(t)$ is

$$\lambda(t) = \frac{\gamma(t)}{p_2(t)},$$

so that

$$f(t) = \lambda(t) \cdot e^{-\int_0^t \lambda(s)ds} = \gamma(t).$$

Using this, (14) reduces to the following

$$\omega = \int_0^{t_2} \gamma(t) \cdot e^{-\int_0^t 2\gamma(s)ds} dt = \frac{1}{2}\left(1 - e^{-2}\right) \approx 0.432,$$

where we have utilized

$$\begin{aligned} \frac{d}{dt}\int_0^t \gamma(s)\,ds &= \gamma(t), \text{ and} \\ \int_0^{t'} \gamma(s)\,ds &= 1. \end{aligned}$$

As explained in the main text, we have $p_e = \omega$, and this establishes the first result.

We now determine the distribution of random variable l_g. First point we establish is that conditional on being at a node in x, the probability that there is a pair on the right is independent of the type of nodes on the left. Indeed conditional on the node being in x, the two direct neighbors do not exit. The behavior of the firms positioned two nodes away is then only determined by their other neighbor and so what happens on the right is independent of what occurs on the left.

This probability can in fact be derived directly: $p = \frac{y}{y+z} = \frac{y}{x}$ Using the fact that

$$\begin{aligned}
\omega &= \frac{x}{x+2y+z} \\
&= \frac{x}{2x+y} \\
&= \frac{1}{2+p}
\end{aligned}$$

yields

$$\begin{aligned}
p &= \frac{1}{\omega} - 2\omega \\
&= 2\frac{1}{1+e^2}
\end{aligned}$$

We are now in a position to the probability distribution of l_g. Consider a gap at date t_2. A gap $l_g = 3$ can only occur at the end of the game if to the right and to the left of the initial gap (probability p^2), there was a pair, and the firms closer to the gap exited (probability $\frac{1}{4}$). For a gap of size two to appear, you need at least one pair. The distribution is thus given as in the main text:

$$P\left[l_g = 3\right] = p^2\frac{1}{4}$$

$$P\left[l_g = 2\right] = p^2\frac{1}{2} + 2p(1-p)\frac{1}{2}$$

$$P\left[l_g = 1\right] = p^2\frac{1}{4} + 2p(1-p)\frac{1}{2} + (1-p)^2$$

9 Appendix B

B1: Informational spillovers

Suppose that the process of adoption has different costs depending on the choices made. We place ourselves in the case of the line where $N = 2/$ There are two choices to be made in adopting, for instance different organizational dimensions, $a_1 \in \{L, R\}$ and $a_2 \in \{L, R\}$. The state of nature is described by $\theta = \{\theta_1, \theta_2\}$ determines which adoption technique is less costly. The cost of adoption is $c = c_1 + c_2$ where $c_i = c_L 1_{a_i = \theta_i} + c_H 1_{a_i \neq \theta_i}$. When you observe a neighbor, with probability 1/2 you learn perfectly about dimension 1 and with probability 1/2 about dimension 2 (regardless of the choice that neighbor actually made). Note that this ensures that there is no inference made on the information the neighbor's neighbor held.

In this case

$$
\begin{aligned}
B_2 &= B - 2\frac{1}{2}(c_L + c_H) = B - (c_L + c_H) \\
B_1 &= B - c_L - \frac{1}{2}(c_L + c_H) = B - (\frac{3}{2}c_L + \frac{1}{2}c_H) \\
B_0 &= B - c_L - \frac{1}{2}(c_L) - \frac{1}{4}(c_L + c_H) = B - (\frac{7}{4}c_L + \frac{1}{4}c_H)
\end{aligned}
$$

So that

$$
\begin{aligned}
B_0 - B_1 &= \frac{1}{4}(c_H - c_L) \\
B_1 - B_2 &= \frac{1}{2}(c_H - c_L)
\end{aligned}
$$

In this case you have $\gamma_1 > \gamma_2$, so that this setup will naturally correspond to the shrinking network setup.

B2: generalization with two state variables

In the application to the adoption of technologies, a more general model should keep track of two state variables:

- a the number of active neighbors
- i the number of inactive neighbors

The inactive neighbors are the neighbors who stopped in the past thus providing payoff spillovers. The number of inactive neighbors therefore determines the payoff when stopping. The number of inactive neighbors will impact the incentives to wait.

Types are thus described by (a, i). Keeping with the example of the line, we have $a \in \{0, 1, 2\}$ and $i \in \{0, 1, 2\}$. A random member of the network can be of types $(2, 0)$, $(1, 0)$, $(1, 1)$, $(0, 2)$, $(0, 1)$ or $(0, 0)$. As in the main text, we assume that the type distribution has full support at date 0, in other words there could be some stopping at date zero for exogenous reasons.

In the model used in the core of the paper and in particular in section 3, we restrict ourselves to one state variable. The implicit assumption we make is that $a + i = 2$, i.e everyone starts with the same number of neighbors, some active and some inactive. Thus in the main part of the paper there were only three possible types $(2, 0)$, $(1, 1)$, and $(0, 2)$. We now show that the general pattern is preserved with a slight complication due to the existence of types $(1, 0)$.[5]

We use the same notation for payoffs. B_2 is the payoff for 0 inactive neighbors, B_1 for one inactive and B_0 for two inactive. The payoff is increasing in the number of inactive neighbors since each neighbors that adopts increases the payoff from stopping, so that $B_2 < B_1 < B_0$.

results

As in the main model we introduce some important measures.

$$\overline{\gamma}_{(1,0)} := \frac{rB_2}{B_1 - B_2}$$
$$\overline{\gamma}_{(1,1)} := \frac{rB_1}{B_0 - B_1}$$
$$\overline{\gamma}_{(2,0)} := \frac{rB_2}{2\,(B_1 - B_2)}$$

Consistently with the equivalence between types here and in the model of section 3, we see that $\overline{\gamma}_{(1,1)} = \overline{\gamma}_1$ and $\overline{\gamma}_{(2,0)} = \overline{\gamma}_2$. We consider two cases: $\overline{\gamma}_{(1,0)} > \overline{\gamma}_{(1,1)}$ and $\overline{\gamma}_{(1,0)} < \overline{\gamma}_{(1,1)}$.

Case 1: $\overline{\gamma}_{(1,0)} > \overline{\gamma}_{(1,1)}$

In this case types $(1, 0)$ have the highest incentives to stop. Indeed these types always

[5] types $(0, 2)$, $(0, 1)$ or $(0, 0)$ do not have any active neighbors and therefore stop immediately regardless whether they have 0, 1 or 2 inactive neighbors.

have a higher incentive to stop then types $(2,0)$, since they get the same benefit from stopping B_2, but they get lower benefit of waiting $\mu(B_1 - B_2)$, whereas types $(2,0)$ get benefit $(2\mu(V_1 - B_2)$ with $V_1 > B_1)$. We now describe the evolution of beliefs.

$$\dot{p}_{(1,0)}(t) = -\lambda(t)\, p_{(1,0)}(t)\left(1 - p_{(1,0)}(t)\right) < 0$$

As time passes, players become less confident that their neighbor is of type $(1,0)$. Whereas in section 3 there were two countervailing forces affecting beliefs, here the second force is not present since types $(2,0)$, if their other neighbor happens to stop, will turn into a type $(1,1)$, not a type $(1,0)$.

Thus at some date $t_{(1,0)}$ all types $(1,0)$ will have stopped. We are then back to the case studied in section 3 with only types $(1,1)$ and $(2,0)$. Depending on the relative size of $\overline{\gamma}_{(1,1)} := \frac{rB_1}{B_0 - B_1}$ and $\overline{\gamma}_{(2,0)} := \frac{rB_2}{2(B_1 - B_2)}$, we will be either in the case of shrinking or fragmenting networks.

Case 2: $\overline{\gamma}_{(1,1)} > \overline{\gamma}_{(1,0)}$

Types $(1,1)$ initially mix. The evolution of beliefs is given by:

$$\begin{aligned}\dot{p}_{(1,1)}(t) &= -\lambda(t)\, p_{(1,1)}(t)\left(1 - p_{(1,1)}(t)\right) + \overline{\gamma}_{(1,1)}(t)\, p_{(2,0)}(t) \\ &= -\overline{\gamma}_{(1,1)}\left(1 - p_{(1,1)}(t) - p_{(2,0)}(t)\right) < 0\end{aligned}$$

In this case, as in the case studied in section 3, there are two forces affecting the belief $p_{(1,1)}(t)$. However, the dominating effect is the evolution of beliefs and as time passes, active members of the network become less confident that their neighbor is of type $(1,1)$. At some date $t_{(1,1)}$, among active members of the networks, only types $(1,0)$ and $(2,0)$ remain. The networks are therefore formed of lines of random sizes with types $(1,0)$ at the extremities. Types $(1,0)$ then have a strictly higher incentive to adopt. As soon as a type $(1,0)$ adopts, the neighbor, if he is of type $(2,0)$, transforms into a type $(1,1)$ and thus immediately adopts. Thus entry by a type $(1,0)$ creates an immediate cascade that immediately covers the entire line. It is therefore as if types $(1,0)$ were playing a waiting game with no type uncertainty. They therefore mix at rate $\overline{\gamma}_{(1,0)}$ and as soon as one adopts, so does the entire line.

B3: War of attrition

We present here a more classical version of the war of attrition, adding as in the rest of the paper the network structure. Firms decide when to exit, where exit is irreversible. Staying in costs $c > 0$ per unit of time, but there is no discounting.

Once both neighbors of a firm exit, the remaining isolated firm gets prize B. As in the rest of the paper, each player only observes whether her neighbors are active or not, but cannot see the status of any other player in the network.

We show there exists a symmetric equilibrium, characterized by a date $t' > 0$ such that within $(0, t')$ all those players who have two active neighbors mix, and within (t', ∞) there are only players with one active neighbor left (i.e. isolated pairs of players) who play a standard war of attrition with each other.

Denote by $V(t)$ the value of a player, who has one active neighbor left (so that one of her two neighbors have exited). We have $V(t) > 0$ for $t \in (0, t')$ and $V(t') = 0$.

Let us denote by $\gamma(t)$ the hazard rate with which an arbitrary neighbor exits at time t, where $t \in (0, t')$. For a randomizing player to be indifferent, the benefit of delaying exit by dt must equate the cost of doing so, i.e. $2\gamma(t)\, dtV(t) = cdt$, so that

$$2\gamma(t) V(t) = c, \tag{15}$$

or

$$\gamma(t) = \frac{c}{2V(t)}. \tag{16}$$

The Bellman equation for the player who has only one neighbor left can be written:

$$V(t) = \gamma(t)\, dtB + (1 - \gamma(t)\, dt) \left(V(t) + \dot{V}(t)\, dt \right) - cdt, \tag{17}$$

which gives

$$\dot{V}(t) = -\gamma(t)(B - V(t)) + c. \tag{18}$$

Plugging (16) in (18) gives us a differential equation for $V(t)$:

$$\dot{V}(t) = -\frac{cB}{2V(t)} + \frac{3}{2}c. \tag{19}$$

Starting with any initial value $V(0)$ such that $0 < V(0) < \frac{B}{3}$ this has a solution $V(t)$ that is decreasing and hits zero at some time point t'.