A Foundation of Deterministic Mechanisms[*]

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Abstract

We study a general social choice environment that has multiple agents, a finite set of alternatives, and independent and diffuse information. We show that for any Bayesian incentive compatible mechanism, there exists a deterministic mechanism that i) is Bayesian incentive compatible; ii) delivers the same interim expected utilities/allocation probabilities for all the agents; and iii) delivers the same ex ante expected welfare. Our result holds in settings with a rich class of utility functions, multi-dimensional types, interdependent valuations, and non-transferable utilities. More importantly, the result recovers the optimality of deterministic mechanisms (whether in terms of revenue or efficiency), which sharply contrasts with the existing results in the screening literature. To prove our result, we develop a new methodology of "mutual purification", and establish its link with the literature of mechanism design.

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1 Introduction

Myerson (1981) provides the framework that has become the paradigm for the study of optimal auction design. To recap briefly, he shows that under a "regularity" condition, the optimal auction allocates the good to the bidder with the highest "virtual value", provided that this virtual value is above the seller's opportunity cost. In other words, the optimal auction in Myerson's setting is deterministic.[1] A natural conjecture is that the optimality of deterministic mechanisms generalizes to multidimensional environments; see McAfee and McMillan (1988, Section 4). However, this does not hold when there is a single agent. Several papers have shown that a multiproduct monopolist may find it beneficial to include lotteries as part of the selling mechanism; see for example, Thanassoulis (2004), Manelli and Vincent (2006, 2007), Pycia (2006), Pavlov (2011) and more recently, Hart and Reny (2015) and Rochet and Thanassoulis (2015).[2] In this paper, we recover the optimality of deterministic mechanisms in remarkably general environments with multiple agents.

We study a general social choice environment that has multiple agents, a finite set of alternatives, and independent and diffuse information.[3] Our main result is that for any Bayesian incentive compatible mechanism, there exists a deterministic mechanism that i) is Bayesian incentive compatible; ii) delivers the same interim expected allocation probabilities and the same interim expected utilities for all agents; and iii) delivers the same ex ante expected welfare. Aside from the standard social choice environments with linear utilities and one-dimensional, private types, our result holds in settings with a rich class of utility functions, multi-dimensional types, interdependent valuations, and non-transferable utilities.

Our result implies that any mechanism, including the optimal mechanisms (whether in terms of revenue or efficiency), can be implemented using a deterministic mechanism and nothing can be gained from designing more intricate mechanisms with possibly more complex randomization. As pointed out in Hart and Reny (2015, page 912), Aumann commented that it is surprising that randomization can not increase revenue when there is only one good.

[1] Riley and Zeckhauser (1983) consider a one good monopolist selling to a population of consumers with unit demands and show that lotteries do not help the one good monopolist.

[2] Hart and Reny (2015) study important differences between the one-good and the multiple-good cases. Besides the advantage of randomization in the multiple-good case, they also exhibit the surprising phenomenon that the seller's maximal revenue may well decrease when the buyer's values for the goods increase. Rochet and Thanassoulis (2015) show that the optimality of "stochastic bundling" is a robust phenomenon.

[3] Throughout this paper, we say that an agent has "diffuse information" if her type distribution is atomless.

Indeed, aforementioned papers in the screening literature establish that randomization helps when there are multiple goods. Nevertheless, we show that in general social choice environment with multiple agents, the revenue maximizing mechanism can always be deterministically implemented. This is in sharp contrast with the results in the screening literature.

In order to prove the existence of an equivalent deterministic mechanism, we develop a new methodology of "mutual purification", and establish its link with the literature of mechanism design.[4] The notion of mutual purification is different from the usual purification principle in the literature related to Bayesian games. We shall clarify these two different notions of purification in the next two paragraphs.

It follows from the general purification principle in Dvoretzky, Wald, and Wolfowitz (1950) that any behavioral-strategy Nash equilibrium in a finite-action Bayesian game with independent and diffuse information corresponds to some pure-strategy Bayesian Nash equilibrium with the same payoff.[5] In particular, independent and diffuse information allows the agents to replace their behavioral strategies by some equivalent pure strategies one-by-one.[6] The point is that under the independent information assumption, any agent who has diffuse information could purify her own behavioral strategy regardless whether other agents have diffuse information. Example 2 illustrates this idea of "self purification". Given a behavioral-strategy Nash equilibrium in a 2-agent Bayesian game with independent information, there is an equivalent pure strategy for the agent with diffuse information, while the other agent with an atom in her type space could not purify her behavioral strategy.

In contrast, the purification result of this paper is based on the diffuse information associated with the other agents. Example 3 partially illustrates this idea of "mutual purification". For a given randomized mechanism in a 2-agent setting with independent information, the agent with an atom in her type space can achieve the same interim payoff by some deterministic mechanism, while there does not exist such a deterministic mechanism for the other agent with diffuse information. In other words, our result becomes possible

[4]Some of our technical results extend the corresponding mathematical results in Arkin and Levin (1972); see Footnote 16 for more detailed discussion.

[5]See Radner and Rosenthal (1982), Milgrom and Weber (1985) and Khan, Rath, and Sun (2006). Furthermore, by applying the purification idea to a sequence of Bayesian games, Harsanyi (1973) provided an interpretation of mixed-strategy equilibrium in complete information games; see Govindan, Reny, and Robson (2003) and Morris (2008) for more discussion.

[6]See the proof of Theorem 1 on page 99 of Khan, Rath, and Sun (2006).

because each agent relies on the diffuse information of the other agents rather than her own. This also explains why a similar result does not hold in the one-agent setting since there is no diffuse information from other agents for such a single agent to purify the relevant randomized mechanism. In addition, we emphasize that in the multiple-agent setting, the notion of "mutual purification" requires not only that each agent obtain the same interim payoff under some deterministic mechanism, but also that a *single* deterministic mechanism deliver the same interim payoffs for all the agents *simultaneously*.

As noted in Radner and Rosenthal (1982, page 401), randomization seems to have limited appeal in many practical situations. Thus, it is desirable to work with deterministic mechanisms. Our result implies that for any stochastic mechanism, there exists an equivalent deterministic mechanism which resolves the randomness in a way that respects the underlying incentive constraints. That is, for almost all type profiles, the outcome is deterministic and can thereby be readily implemented without invoking a randomization device. Our paper shares a common motivation with the problem of designing random allocation mechanisms in the matching literature; see for example, Budish, Che, Kojima, and Milgrom (2013). The literature seeks to deterministically implement a random allocation/assignment whereas in our mechanism design setting, for almost all type profiles, the mechanism already specifies a deterministic allocation rule.

This paper also joins a line of literature that studies mechanism equivalence. Though motivations vary, these results show that it is without loss of generality to consider the various subclasses of mechanisms. As in the case of dominant-strategy mechanisms (see Manelli and Vincent (2010) and Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013)) and symmetric auctions (see Deb and Pai (2015)), our findings imply that the requirement of deterministic mechanisms is not restrictive in itself.[7] In this sense, our result also provides a foundation for the use of deterministic allocations in mechanism design settings such as auctions, bilateral trades, and so on.

The rest of the paper is organized as follows. Section 2 introduces the basics. Section 3

[7]Manelli and Vincent (2010) show that for any Bayesian incentive compatible auction, there exists an equivalent dominant-strategy incentive compatible auction that yields the same interim expected utilities for all agents. Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013) extend this equivalence result to social choice environments with linear utilities and independent, one-dimensional, private types; also see Footnote 9 for related discussion. Deb and Pai (2015) show that restricting the seller to a using symmetric auction imposes virtually no restriction on her ability to achieve discriminatory outcomes.

illustrates our equivalence notion and the idea of "mutual purification" through examples. Section 4 presents our equivalence result. Section 5 discusses more literature on the benefit of randomness, in particular more contrasts of our result with the extent literature. We also provide an implementation perspective of our result and discuss the various assumptions of our result. Section 6 concludes. The appendix contains proofs omitted from the main body of the paper.

2 Preliminaries

2.1 Notation

There is a finite set $\mathcal{I} = \{1, 2, \ldots, I\}$ of risk neutral agents with $I \geq 2$ and a finite set $\mathcal{K} = \{1, 2, ..., K\}$ of social alternatives. The set of possible types V_i of agent i is a closed subset of finite dimensional Euclidean space $\mathbb{R}^l$ with generic element v_i. The set of possible type profiles is $V \equiv V_1 \times V_2 \times \cdots \times V_I$ with generic element $v = (v_1, v_2, \ldots, v_I)$. Write v_{-i} for a type profile of agent i's opponents; that is, $v_{-i} \in V_{-i} = \Pi_{j \neq i} V_j$. Denote by λ the common prior distribution on V. For each $i \in \mathcal{I}, \lambda_i$ is the marginal distribution of λ on V_i and is assumed to be atomless. Throughout this paper, types are assumed to be independent.[8] If $(Y, \mathcal{Y})$ is a measurable space, then ΔY is the set of all probability measures on $(Y, \mathcal{Y})$. If Y is a metric space, then we treat it as a measurable space with its Borel σ-algebra.

2.2 Mechanism

The revelation principle applies, and we restrict attention to direct mechanisms characterized by $K + I$ functions, $\{q^k(v)\}_{k \in \mathcal{K}}$ and $\{t_i(v)\}_{i \in \mathcal{I}}$, where v is the profile of reports, $q^k(v) \geq 0$ is the probability that alternative k is implemented with $\sum_{k \in \mathcal{K}} q^k(v) = 1$, and $t_i(v)$ is the monetary transfer that agent i makes to the mechanism designer. Agent i's gross utility in alternative k is $u_i^k(v_i, v_{-i})$.

For simplicity of exposition, we denote

$$Q_i^k(v_i) = \int_{V_{-i}} q^k(v_i, v_{-i}) \lambda_{-i}(\mathrm{d}v_{-i})$$

[8] Note that we do not make any assumption regarding the correlation of the different coordinates of type v_i for any $i \in \mathcal{I}$.

for the interim expected allocation probability (from agent i's perspective) that alternative k is implemented. Also write

$$T_i(v_i) = \int_{V_{-i}} t_i(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i})$$

for the interim expected transfer from agent i to the mechanism designer. Agent i's interim expected utility is

$$\begin{aligned} U_i(v_i) &= \int_{V_{-i}} \left[\sum_{1\le k\le K} u_i^k(v_i, v_{-i})q^k(v_i, v_{-i}) - t_i(v_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}) \\ &= \int_{V_{-i}} \left[\sum_{1\le k\le K} u_i^k(v_i, v_{-i})q^k(v_i, v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}) - T_i(v_i). \end{aligned}$$

A mechanism is Bayesian incentive compatible (BIC) if for each agent $i \in \mathcal{I}$ and each type $v_i \in V_i$,

$$\begin{aligned} &U_i(v_i) \ge 0, \ \text{ and} \\ &U_i(v_i) \ge \int_{V_{-i}} \left[\sum_{1\le k\le K} u_i^k(v_i, v_{-i})q^k(v_i', v_{-i}) - t_i(v_i', v_{-i}) \right] \lambda_{-i}(\mathrm{d}v_{-i}) \end{aligned}$$

for any alternative type $v_i' \in V_i$.

A mechanism (q, t) is said to be "deterministic" if for almost all type profiles, the mechanism implements some alternative k for sure. That is, for λ-almost all $v \in V, q^k(v) = 1$ for some $1 \le k \le K$.

2.3 Mechanism equivalence

We shall employ the following notion of mechanism equivalence.

Definition 1. *Two mechanisms (q, t) and $(\tilde{q}, \tilde{t})$ are equivalent if and only if they deliver the same interim expected allocation probabilities and the same interim expected utilities for all agents, and the same ex ante expected social surplus.*

Remark 1. *Our equivalence is stronger than the prevailing mechanism equivalence notion. For example, Manelli and Vincent (2010) and Gershkov, Goeree, Kushnir, Moldovanu, and*

Shi (2013) define two mechanisms to be equivalent if they deliver the same interim expected utilities for all agents and the same ex ante expected welfare.[9]

Remark 2. *The equivalent deterministic mechanism also guarantees the same ex post monetary transfers, and hence the same expected revenue; see Theorem 1.*

3 Examples

3.1 An illustration of equivalent deterministic mechanism

In the first example, we illustrate our mechanism equivalence notion in a single-unit auction environment.[10] The example is kept deliberately simple and its only purpose is to illustrate what we mean by equivalent deterministic mechanism. our main result is far more general and the proof is much more complex.

Example 1. *There are two bidders, whose valuations are uniformly distributed in* $[0, 1]$*. Consider the following mechanism. Types are divided into intervals of equal probability and types in the same interval are treated equally. If agents' types belong to the same interval, each agent receives the object with probability* $\frac{1}{2}$ *and if agents' types belong to different intervals, the agent whose type belongs to* $[\frac{1}{2}, 1]$ *gets the object. In each cell, the first number is the probability that agent 1 gets the object and the second number is the probability that agent 2 gets the object.*

	$[\frac{1}{2}, 1]$	$[0, \frac{1}{2})$
$[\frac{1}{2}, 1]$	$\frac{1}{2}, \frac{1}{2}$	$1, 0$
$[0, \frac{1}{2})$	$0, 1$	$\frac{1}{2}, \frac{1}{2}$

It is immediate that, the following deterministic mechanism is equivalent in terms of interim expected allocation probabilities. Keeping the transfers unchanged, it is also easy to

[9] Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013, Section 4.1) show that the BIC-DIC equivalence breaks down when requiring the same interim expected allocation probability. They also note that "this notion (of interim expected allocation probabilities) becomes relevant when, for instance, the designer is not utilitarian or when preferences of agents outside the mechanism play a role".

[10] With slight adjustments, this example applies to the irregular case in Myerson's setting where the agents' ironed virtual values are the same in some interval.

see the deterministic mechanism is equivalent in terms of interim expected utilities and ex ante social welfare.

	$[\frac{3}{4},1]$	$[\frac{2}{4},\frac{3}{4})$	$[\frac{1}{4},\frac{2}{4})$	$[0,\frac{1}{4})$
$[\frac{3}{4},1]$	$1,0$	$0,1$	$1,0$	$1,0$
$[\frac{2}{4},\frac{3}{4})$	$0,1$	$1,0$	$1,0$	$1,0$
$[\frac{1}{4},\frac{2}{4})$	$0,1$	$0,1$	$1,0$	$0,1$
$[0,\frac{1}{4})$	$0,1$	$0,1$	$0,1$	$1,0$

In Section 4, we show that for whatever randomized mechanism that the mechanism designer may choose to use, however complicated, there exists an equivalent mechanism that is deterministic. In other words, going from mechanisms that are deterministic to randomized mechanisms in general does not increase the set of obtainable outcomes.

3.2 Self purification and mutual purification

In this section, we provide two examples to demonstrate the conceptual difference between the existing approach of "self purification" and our approach of "mutual purification".

The first example is motivated by the game of matching pennies, while the second example is a single unit auction. Both games have two agents, and share the same information structure as follows.

1. Agent 1's type is uniformly distributed on $(0,1]$ with the total probability $1-\lambda_1(0)$, and has an atom at the point 0 with $\lambda_1(0)>0$.
2. Agent 2's type is uniformly distributed on $[0,1]$.
3. Agents' types are independently distributed.

Example 2 below illustrates the idea of "self purification". The behavioral strategy of agent 2 can be purified since the distribution of agent 2's type is atomless, while the behavioral strategy of agent 1 cannot be purified since agent 1's type has an atom.

Example 2. *Consider an $m\times m$ zero-sum generalized "matching pennies" game with incomplete information, where the positive integer m is sufficiently large such that $\frac{1}{m}<\lambda_1(0)$. The information structure is described in the beginning of this subsection. The action space* *for both agents is* $A_1 = A_2 = \{a_1, a_2, \ldots, a_m\}$. *The payoff matrix for agent 1 is given below. Notice that the payoffs of both agents do not depend on the type profile.*

Agent 2

		a_1	a_2	a_3	$\cdots$	a_m
Agent 1	a_1	1	-1	0	$\cdots$	0
	a_2	0	1	-1	$\cdots$	0
	a_3	0	0	1	$\cdots$	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	a_m	-1	0	$\cdots$	0	1

Suppose that both agents adopt the behavioral strategy $f_1(v) = f_2(v) = \frac{1}{m} \sum_{1 \le s \le m} \delta_{a_s}$, *where* δ_{a_s} *is the Dirac measure at the point* a_s. *It is easy to see that* (f_1, f_2) *is a Bayesian Nash equilibrium and the expected payoffs of both agents are* 0.

Claim 1. *Agent 2 has a pure strategy* f'_2 *such that* (f_1, f'_2) *is still a behavioral-strategy equilibrium and provides both agents the same expected payoffs, while agent 1 does not have such a pure strategy.*

Example 3 below shows how a purification for an agent relies on the diffuse information of the other agent, which partially illustrates the idea of "mutual purification". In particular, for some given randomized mechanism in the 2-agent setting with independent information as specified above, agent 1 who has an atom in her type space can achieve the same interim expected payoff by some deterministic mechanism,[11] while there does not exist such a deterministic mechanism for agent 2 who has diffuse information.

Example 3. *Consider a single unit auction with two agents. The information structure is described as above. The payoff function of agent* i *is* $\epsilon v_i + (1 - v_j)^m$ *for* $i, j = 1, 2$ *and* $i \neq j$, *where* m *is sufficiently large and* ϵ *is sufficiently small such that*

$$\frac{\lambda_1(0)}{2} > \epsilon + \frac{1}{m+1}.$$

The allocation rule q *is defined as follows. Let* $q^i(v)$ *be the probability that agent* i *gets the object, and* $q^1(v_1, v_2) = q^2(v_1, v_2) = \frac{1}{2}$ *for any* (v_1, v_2). *The interim expected payoff of*

[11] For simplicity, we only consider such an equivalence in terms of interim expected payoffs.

agent 1 *with value* v_1 *is*

$$\int_{V_2} (\epsilon v_1 + (1-v_2)^m) q^1(v_1, v_2) \lambda_2(\mathrm{d}v_2) = \frac{\epsilon v_1}{2} + \frac{1}{2(m+1)}.$$

The interim expected payoff of agent 2 *with value* v_2 *is*

$$\int_{V_1} (\epsilon v_2 + (1-v_1)^m) q^2(v_1, v_2) \lambda_1(\mathrm{d}v_1) = \frac{\epsilon v_2}{2} + \frac{\lambda_1(0)}{2} + (1-\lambda_1(0)) \frac{1}{2(m+1)}.$$

Claim 2. *There exists a deterministic mechanism which gives agent 1 the same interim expected payoff; but there does not exist such a deterministic mechanism for agent 2.*

We hasten to emphasize the key difference between our approach and the purification method used in the literature. With the classical purification method, each agent uses her own diffuse information to purify her behavioral strategy, which we call "self purification". In contrast, the purification approach we adopt to achieve our main result is to purify the randomized mechanism via other agents' diffuse information while keeping each agent's interim expected allocation probability and interim expected payoff unchanged *simultaneously*, which we call "mutual purification".

4 Results

4.1 Equivalence

This section establishes the main result of this paper. We consider a general environment in which agents could have nonlinear and interdependent payoffs. In particular, we assume that all agents have "separable payoffs" in the following sense.

Definition 1. *For each* $i \in \mathcal{I}$*, agent* i *is said to have separable payoff if for any outcome* $k \in \mathcal{K}$ *and type profile* $v = (v_1, v_2, \ldots, v_I) \in V$*, her payoff function can be written as follows:*

$$u_i^k(v_1, \ldots, v_I) = \sum_{1 \le m \le M} w_{im}^k(v_i) r_{im}^k(v_{-i}),$$

where M *is a positive integer, and* w_{im}^k *(resp.* r_{im}^k*) is* λ_i*-integrable (resp.* λ_{-i}*-integrable) on* V_i *(resp. on* V_{-i}*) for* $1 \le m \le M$.

That is, the payoff of each agent i is a summation of finite terms, where each term is a product of two components: the first component only depends on agent i's own type, while the second component depends on other agents' types. This setup is sufficiently general to cover most applications. In particular, it includes the interdependent payoff function as in Jehiel and Moldovanu (2001), and obviously covers the widely adopted private value payoffs as a special case.

Theorem 1. *Suppose that for each agent $i \in \mathcal{I}$, his payoff function is separable. Then for any mechanism (q, t), there exists a deterministic allocation rule $\tilde{q}$ such that*

1. *q and $\tilde{q}$ induce the same interim expected allocation probability;*

2. *$(\tilde{q}, t)$ delivers the same interim expected utility with (q, t) for each agent $i \in \mathcal{I}$.*

Thus, if (q, t) is BIC, then $(\tilde{q}, t)$ is also BIC.

Remark 3. *We prove a stronger result. Firstly, it will be clear from the proof of Theorem 1 that the equivalent deterministic mechanism $(\tilde{q}, t)$ also guarantees the same ex post monetary transfers. Therefore, our deterministic mechanism equivalence result does not require transferable utility. Secondly, the equivalence result is immune against coalitions; that is, when there is sharing of information between the coalition members (except for the grand coalition).*[12] *The second point is proved explicitly.*

Let h be a function from V to $\mathbb{R}_{++}^{IKM+1}$ such that $h_0(v) \equiv 1$, and $h_{ikm}(v) = r_{im}^k(v_{-i})$[13] for each $i \in \mathcal{I}$, $1 \le k \le K$ and $1 \le m \le M$.[14] Let $\mathcal{J}$ be the set of all nonempty proper subsets of $\mathcal{I}$, and Υ be the set of all allocation rules. That is, given any $\tilde{q} \in \Upsilon$, $\tilde{q}$ is a measurable function and $\sum_{k \in \mathcal{K}} \tilde{q}^k(v) = 1$ for λ-almost all $v \in V$. For any coalition $J \subseteq \mathcal{I}$, denote $\lambda_J = \bigotimes_{j \in J} \lambda_j$.

Fix a Bayesian incentive compatible mechanism (q, t). We consider the allocation rule $\tilde{q} \in \Upsilon$ such that for any $J \in \mathcal{J}$ and λ_J-almost all $v_J \in V_J$,

$$\mathbb{E}\left(\tilde{q} h_j | v_J\right) = \mathbb{E}\left(q h_j | v_J\right) \tag{1}$$

[12] Jackson and Sonnenschein (2007) also consider the issue of coalitional incentive compatibility. They show that the "linking mechanisms" are immune to manipulations by coalitions.

[13] Throughout this paper, IKM is the product of the integers I, K and M. However, the subscript ikm is not the product of the numbers i, k and m, but refers to the vector (i, k, m) identifying the function r_{im}^k.

[14] Denote $\mathbb{R}_{++}$ as the strictly positive real line. We assume that h is strictly positive without loss of generality. Indeed, we can work with the function h' from V to $\mathbb{R}_{+}^{2IKM+1}$ such that $h'_0(v) \equiv 1$, $h'^1_{ikm}(v) = |r_{im}^k(v_{-i})| + 1$, and $h'^2_{ikm}(v) = r_{im}^k(v_{-i}) + |r_{im}^k(v_{-i})| + 1$ for each $i \in \mathcal{I}$, $1 \le k \le K$ and $1 \le m \le M$. The function h' is strictly positive and suffices for our purpose.

for $j = 0$ or $j = ikm$, $i \in \mathcal{I}$, $1 \leq k \leq K$ and $1 \leq m \leq M$.

Definition 2. *We define the following set* Υ_q*:*

$$\Upsilon_q = \{\tilde{q} \in \Upsilon : \tilde{q} \text{ satisfies Equation (1)}\}.$$

In what follows, we first provide the following characterization result for the set Υ_q: Υ_q is a nonempty, convex and weakly compact set in some Banach space. Therefore, the classical Krein-Milman Theorem (see Royden and Fitzpatrick (2010, p. 296)) implies that Υ_q admits extreme points. We proceed by showing that all extreme points of the set Υ_q are deterministic mechanisms.[15] The existence of a deterministic mechanism that is equivalent in terms of interim expected allocation probabilities immediately follows. The equivalence in terms of interim expected utilities and ex ante expected welfare follows from Equation (4) and the separable payoff assumption. The incentive compatibility of the deterministic mechanism follows from Equation (4) and the assumption that types are independent.

The following lemma characterizes the set Υ_q.

Lemma 1. Υ_q *is a nonempty, convex and weakly compact subset.*

Since Υ_q is a nonempty, convex and weakly compact set, Υ_q has extreme points. The following result shows that all extreme points of Υ_q are deterministic allocations.

Proposition 1. *All extreme points of* Υ_q *are deterministic allocations.*

4.2 The proofs

In this section, we prove Lemma 1, Proposition 1, and finally Theorem 1.

Proof of Lemma 1. Clearly, the set Υ_q is nonempty and convex. We first show that Υ_q is norm closed in $L_1^\lambda(V, \mathbb{R}^K)$, where $L_1^\lambda(V, \mathbb{R}^K)$ is the L_1 space of all measurable mappings from V to $\mathbb{R}^K$ under the probability measure λ.

[15] Manelli and Vincent (2007) use a related technique in the screening literature. Manelli and Vincent (2007) consider revenue maximizing multiproduct monopolist and study the extreme points of the set of feasible mechanisms. They show that, with multiple goods, extreme points could be stochastic mechanisms. In contrast, we work with the mechanism design setting, study a particular set of interest Υ_q and show that all extreme points are deterministic. Apart from this general approach, the technical parts of the proofs are dramatically different.

Suppose that the sequence $\{q_m\} \subseteq \Upsilon_q$ and $q_m \to q_0$ in $L_1^\lambda(V, \mathbb{R}^K)$. Then by the Riesz-Fischer Theorem (see Royden and Fitzpatrick (2010, p. 398)), there exists a subsequence $\{q_{m_s}\}$ of $\{q_m\}$, which converges to q_0 λ-almost everywhere. Since $\sum_{k\in\mathcal{K}} q_{m_s}^k(v) = 1$ for λ-almost all v, $\sum_{k\in\mathcal{K}} q_0^k(v) = 1$ for λ-almost all v. As a result, $q_0 \in \Upsilon$.

For any $k \in \mathcal{K}$, $J \in \mathcal{J}$, and $\mathcal{B}(V_J) \otimes \left(\bigotimes_{1\le j\le I, j\notin J}\{V_j, \emptyset\}\right)$-measurable bounded mapping $p\colon V \to \mathbb{R}^K$,

$$\int_V (q_0^k h_j) p\lambda(\mathrm{d}v) = \lim_{s\to\infty} \int_V (q_{m_s}^k h_j) p\lambda(\mathrm{d}v) = \int_V (q^k h_j) p\lambda(\mathrm{d}v)$$

for $j = 0$ or $j = ikm$. The first equality is due to the dominated convergence theorem, and the second equality holds since $\{q_{m_s}\} \subseteq \Upsilon_q$. Thus, $q_0 \in \Upsilon_q$, which implies that Υ_q is norm closed in $L_1^\lambda(V, \mathbb{R}^K)$.

Since Υ_q is convex, Υ_q is also weakly closed in $L_1^\lambda(V, \mathbb{R}^K)$ by Mazur's Theorem (see Royden and Fitzpatrick (2010, p. 292)). As Υ is weakly compact in $L_1^\lambda(V, \mathbb{R}^K)$, we have that Υ_q is weakly compact in $L_1^\lambda(V, \mathbb{R}^K)$, and hence has extreme points. □

Below, we give the proof of Proposition 1.

Proof of Proposition 1. Pick an allocation rule $\tilde{q} \in \Upsilon_q$ which is not deterministic, we shall show that $\tilde{q}$ is not an extreme point of Υ_q.

Since $\tilde{q}$ is not deterministic, there is a positive number $0 < \delta < 1$, a Borel measurable set $D \subseteq V$ such that $\lambda(D) > 0$, and indices j_1, j_2 such that $\delta \le \tilde{q}^{j_1}(v), \tilde{q}^{j_2}(v) \le 1 - \delta$ for any $v \in D$. For any $J \in \mathcal{J}$, let D_J be the projection of D on $\prod_{j\in J} V_j$. For any $v_J \in D_J$, let $D_{-J}(v_J) = \{v_{-J}\colon (v_J, v_{-J}) \in D\}$ (abbreviated as D_{v_J}).

Consider the following problem on $\alpha \in L_\infty^\lambda(D, \mathbb{R})$: for any $J \in \mathcal{J}$ and $v_J \in D_J$,

$$\int_{D_{-J}(v_J)} \alpha(v_J, v_{-J}) h(v_J, v_{-J}) \lambda_{-J}(\mathrm{d}v_{-J}) = 0. \tag{2}$$

Recall that h is a function taking values in $\mathbb{R}^{IKM+1}$. For simplicity, denote $l_0 = IKM+1$. Define the set $\mathscr{E}$ as

$$\mathscr{E} = \{h(v) \cdot \sum_{J\in\mathcal{J}} \psi_J(v_J)\colon \psi_J \in L_\infty^\lambda(D_J, \mathbb{R}^{l_0}), \forall J \in \mathcal{J}\}.$$

Then a bounded measurable function α in $L_\infty^\lambda(D, \mathbb{R})$ is a solution to Problem (2) if and only if $\int_D \alpha\varphi d\lambda = 0$ for any $\varphi \in \mathscr{E}$. Lemma 4 in the Appendix shows that $\mathscr{E}$ is not dense in

$L_1^\lambda(D, \mathbb{R})$.[16] By Corollary 5.108 in Aliprantis and Border (2006), Problem (2) has a nontrivial bounded solution α.

Without loss of generality, we assume that $|\alpha| \leq \delta$. We extend the domain of α to V by letting $\alpha(v) = 0$ when $v \notin D$. For every $v \in V$, define

$$\hat{q}(v) = \tilde{q}(v) + \alpha(v) \left(e_{j_1} - e_{j_2} \right);$$

$$\overline{q}(v) = \tilde{q}(v) + \alpha(v) \left(e_{j_2} - e_{j_1} \right).$$

Then $\sum_{k \in \mathcal{K}} \hat{q}^k(v) = \sum_{k \in \mathcal{K}} \overline{q}^k(v) = \sum_{k \in \mathcal{K}} \tilde{q}^k(v) = 1$. If $v \in D$, then $0 \leq \hat{q}^{j_1}(v), \overline{q}^{j_2}(v) \leq 1$ as $\delta \leq \tilde{q}^{j_1}(v), \tilde{q}^{j_2}(v) \leq 1 - \delta$, and $\hat{q}^j(v) = \overline{q}^j(v) = \tilde{q}^j(v)$ for $j \neq j_1, j_2$. If $v \notin D$, then $\hat{q}(v) = \overline{q}(v) = \tilde{q}(v)$ as $\alpha(v) = 0$. Thus, $\hat{q}$, $\overline{q} \in \Upsilon$.

For any $J \in \mathcal{J}$ and $\mathcal{B}(V_J) \otimes \left(\bigotimes_{1 \leq j \leq I, j \notin J} \{V_j, \emptyset\} \right)$-bounded measurable mapping $p \in L_\infty^\lambda(V, \mathbb{R}^K)$,

$$\int_V (\hat{q} \cdot p) h \lambda(\mathrm{d}v) = \int_V (\tilde{q} \cdot p) h \lambda(\mathrm{d}v) + \int_V \alpha(v) \left((e_{j_1} - e_{j_2}) \cdot p(v) \right) h(v) \lambda(\mathrm{d}v).$$

Since

$$\begin{aligned}
&\int_V \alpha(v) \left((e_{j_1} - e_{j_2}) \cdot p(v) \right) h(v) \lambda(\mathrm{d}v) \\
&= \int_{V_J} \int_{V_{-J}} \alpha(v) \left((e_{j_1} - e_{j_2}) \cdot p(v) \right) h(v) \lambda_{-J}(\mathrm{d}v_{-J}) \lambda_J(\mathrm{d}v_J) \\
&= \int_{V_J} (e_{j_1} - e_{j_2}) \cdot p(v) \int_{V_{-J}} \alpha(v) h(v) \lambda_{-J}(\mathrm{d}v_{-J}) \lambda_J(\mathrm{d}v_J) \\
&= 0,
\end{aligned}$$

we have that

$$\int_V (\hat{q} \cdot p) h \lambda(\mathrm{d}v) = \int_V (\tilde{q} \cdot p) h \lambda(\mathrm{d}v),$$

[16] Lemma 4 as well as its preparatory lemmas and their proofs are given in the Appendix. The result in Lemma 4 provides the key for proving Proposition 1. Our technical lemmas in the Appendix extend the corresponding mathematical results in Arkin and Levin (1972) from the special case with $I = 2$ and λ the uniform distribution on $[0, 1] \times [0, 1]$ to the general setting in this paper. Those mathematical results in Arkin and Levin (1972) were then used to show the following result (see Theorem 2.3 therein): "Suppose that $f_1 \in L_1^\eta(X \times Y, \mathbb{R}^{l_1})$, $f_2 \in L_1^\eta(X \times Y, \mathbb{R}^{l_2})$ and $f_3 \in L_1^\eta(X \times Y, \mathbb{R}^{l_3})$, where $X = Y = [0, 1]$ and η is the uniform distribution on $[0, 1] \times [0, 1]$. Let A be the simplex $\{a = (a_1, \ldots, a_K) \colon \sum_{1 \leq k \leq K} a_k = 1, a_k \geq 0\}$. Given any measurable function α from $X \times Y$ to A, there exists another measurable function $\overline{\alpha}$ from $X \times Y$ to the vertices of the simplex A such that $\int_{[0,1]} f_1(x, y) \alpha(x, y) \, \mathrm{d}y = \int_{[0,1]} f_1(x, y) \overline{\alpha}(x, y) \, \mathrm{d}y$, $\int_{[0,1]} f_2(x, y) \alpha(x, y) \, \mathrm{d}x = \int_{[0,1]} f_2(x, y) \overline{\alpha}(x, y) \, \mathrm{d}x$ and $\int_{[0,1]} \int_{[0,1]} f_3(x, y) \alpha(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_{[0,1]} \int_{[0,1]} f_3(x, y)$."

which implies that $\hat{q} \in \Upsilon_q$. Similarly, one can show that $\overline{q} \in \Upsilon_q$. Since $\hat{q}$ and $\overline{q}$ are distinct and $\tilde{q} = \frac{1}{2}(\hat{q} + \overline{q})$, $\tilde{q}$ is not an extreme point of Υ_q. □

Now we are ready to prove our main result.

Proof of Theorem 1. Fix a mechanism (q, t). The proof is then divided into two steps. In the first step, we obtain a deterministic allocation rule $\tilde{q}$ which has the same interim expected allocation probability with q. In the second step, we verify that $(\tilde{q}, t)$ and (q, t) deliver the same interim expected utility for each agent.

By Proposition 1, every extreme point of Υ_q is a deterministic allocation rule. Therefore, we can fix a measurable allocation rule $\tilde{q}$ such that

1. $\tilde{q}^k(v) = 0$ or 1 for λ-almost all $v \in V$ and $1 \leq k \leq K$;

2. for any agent i and λ_i-almost all $v_i \in V_i$,

$$\int_{V_{-i}} \tilde{q}(v_i, v_{-i}) \lambda_{-i}(\mathrm{d}v_{-i}) = \int_{V_{-i}} q(v_i, v_{-i}) \lambda_{-i}(\mathrm{d}v_{-i}), \tag{3}$$

and

$$\int_{V_{-i}} \tilde{q}(v_i, v_{-i}) h_{jkm}(v_i, v_{-i}) \lambda_{-i}(\mathrm{d}v_{-i}) = \int_{V_{-i}} q(v_i, v_{-i}) h_{jkm}(v_i, v_{-i}) \lambda_{-i}(\mathrm{d}v_{-i}) \tag{4}$$

for any $j \in \mathcal{I}$, $1 \leq k \leq K$ and $1 \leq m \leq M$.

Let D_i be the subset of V_i such that Equation (3) or (4) does not hold. Then $\lambda_i(D_i) = 0$. Define a new allocation rule $\hat{q}$ such that

$$\hat{q}(v) = \begin{cases} q(v), & \text{if } v_i \in D_i \text{ for some } i \in \mathcal{I}; \\ \tilde{q}(v), & \text{otherwise.} \end{cases}$$

Then $\hat{q}^k(v) = 0$ or 1 for λ-almost all $v \in V$ and $1 \leq k \leq K$.

Fix agent i and $v_i \in V_i$. If $v_i \in D_i$, then $\hat{q}(v_i, v_{-i}) = q(v_i, v_{-i})$ and $\int_{V_{-i}} \hat{q}(v_i, v_{-i}) \lambda_{-i}(\mathrm{d}v_{-i}) = \int_{V_{-i}} q(v_i, v_{-i}) \lambda_{-i}(\mathrm{d}v_{-i})$. If $v_i \notin D_i$, then

$$\begin{aligned} \int_{V_{-i}} \hat{q}(v_i, v_{-i}) \lambda_{-i}(\mathrm{d}v_{-i}) &= \int_{D_{-i}} \hat{q}(v_i, v_{-i}) \lambda_{-i}(\mathrm{d}v_{-i}) + \int_{V_{-i} \setminus D_{-i}} \hat{q}(v_i, v_{-i}) \lambda_{-i}(\mathrm{d}v_{-i}) \\ &= \int_{D_{-i}} q(v_i, v_{-i}) \lambda_{-i}(\mathrm{d}v_{-i}) + \int_{V_{-i} \setminus D_{-i}} \tilde{q}(v_i, v_{-i}) \lambda_{-i}(\mathrm{d}v_{-i}) \end{aligned}$$

$$= 0 + \int_{V_{-i}} \tilde{q}(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i})$$
$$= \int_{V_{-i}} q(v_i, v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}),$$

where $D_{-i} = \cup_{j\in\mathcal{I},j\neq i}\left(D_j \times \prod_{s\in\mathcal{I},s\neq i,j} V_s\right)$. The first equality holds by dividing V_{-i} as D_{-i} and $V_{-i}\setminus D_{-i}$. The second equality is due to the definition of $\hat{q}$. The third equality holds since $\lambda_{-i}(D_{-i}) = 0$. The last equality is due to the condition that $v_i \notin D_i$. As a result, Equation (3) holds for $\hat{q}$ and every $v_i \in V_i$. Similarly, one can check that Equation (4) also holds for $\hat{q}$ and every $v_i \in V_i$.

Suppose that the mechanism $(\hat{q}, t)$ is adopted. By Equation (3), the allocation rules q and $\hat{q}$ induce the same interim expected allocation. We need to check that they induce the same interim expected utility. If agent i observes the state v_i but reports v_i', then his payoff is

$$\int_{V_{-i}} \left[\sum_{1\le k\le K} u_i^k(v_i, v_{-i})\hat{q}^k(v_i', v_{-i}) - t_i(v_i', v_{-i})\right]\lambda_{-i}(\mathrm{d}v_{-i})$$
$$= \sum_{1\le k\le K}\sum_{1\le m\le M}\int_{V_{-i}} w_{im}^k(v_i)r_{im}^k(v_{-i})\hat{q}^k(v_i', v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) - T_i(v_i')$$
$$= \sum_{1\le k\le K}\sum_{1\le m\le M} w_{im}^k(v_i)\int_{V_{-i}} r_{im}^k(v_{-i})\hat{q}^k(v_i', v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) - T_i(v_i')$$
$$= \sum_{1\le k\le K}\sum_{1\le m\le M} w_{im}^k(v_i)\int_{V_{-i}} h_{ikm}(v)\hat{q}^k(v_i', v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) - T_i(v_i')$$
$$= \sum_{1\le k\le K}\sum_{1\le m\le M} w_{im}^k(v_i)\int_{V_{-i}} h_{ikm}(v)q^k(v_i', v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) - T_i(v_i')$$
$$= \sum_{1\le k\le K}\sum_{1\le m\le M} w_{im}^k(v_i)\int_{V_{-i}} r_{im}^k(v_{-i})q^k(v_i', v_{-i})\lambda_{-i}(\mathrm{d}v_{-i}) - T_i(v_i')$$
$$= \int_{V_{-i}} \left[\sum_{1\le k\le K} u_i^k(v_i, v_{-i})q^k(v_i', v_{-i}) - t_i(v_i', v_{-i})\right]\lambda_{-i}(\mathrm{d}v_{-i}).$$

The first and second equalities follow from the separable payoff assumption. The fourth equality follows from Equation (4) and also the assumption that types are independent. All other equalities are simple algebras. Thus, these two mechanisms (q, t) and $(\hat{q}, t)$ deliver the same interim expected utility for every agent. If (q, t) is Bayesian incentive compatible, then $(\hat{q}, t)$ is clearly Bayesian incentive compatible. This completes the proof. $\square$

5 Discussions

5.1 Benefit of randomness revisited

As is well known, it is not clear what optimal mechanisms for selling two goods look like, even in the *single agent* case. The two-dimensional problem is notoriously difficult, and the mechanisms that amount to solving one-dimensional problems - such as separate selling and bundling - do not maximize the revenue in general. This motivates the following question: how good are simple mechanisms for selling two goods? Hart and Nisan (2013, 2014) and Briest, Chawla, Kleinberg, and Weinberg (2015) show the surprising result that in the general case where the valuations may be correlated, simple mechanisms (separate selling, bundling, and deterministic mechanisms) cannot guarantee any positive fraction of the optimal revenue.

Hart and Nisan (2013, 2014) then ask the same questions for more general auction settings involving *multiple agents*. They observe that their result also applies to multiple-agent settings, simply by considering a single "significant" agent together with multiple "negligible" (in the extreme, 0-value for all items) agents. However, this result does not hold with a small perturbation of the prior. The next paragraph elaborates this point.

We shall perturb the prior in Hart and Nisan (2013, 2014) a little bit such that the new prior is atomless. In their paper, the prior λ_1 of the significant buyer has countable support, say $\{a_1, a_2, \ldots\}$ (see Hart and Nisan's prior constructed for their Theorem A). For sufficiently small $\epsilon > 0$, we can construct a new prior G_ϵ of two buyers such that i) its marginal on the significant buyer has support $\cup_{k\geq 1}[a_k - \frac{\epsilon}{2^k}, a_k + \frac{\epsilon}{2^k}]$, and the probability $\lambda_1(a_k)$ is uniformly distributed on $[a_k - \frac{\epsilon}{2^k}, a_k + \frac{\epsilon}{2^k}]$ for each k;[17] ii) its marginal on the negligible buyer is uniform on $[0, \epsilon]$; and (iii) the prior of these two buyers are independent. Then, the prior G_ϵ converges to Hart and Nisan's prior at least weakly and yet the prior satisfies our assumption of being independent and atomless. Thus, our result implies that the multiple agents version of Hart and Nisan's result is a knife-edge case, since with a small perturbation, deterministic and stochastic mechanisms generate the same revenue.

It is worth mentioning that while our result requires independence across agents, we do not make any assumption regarding the correlation of the different coordinates of type v_i for any agent $i \in \mathcal{I}$. In other words, while the correlation across valuations plays an important

[17] If some intervals have intersections, then the probability of the intersection is the corresponding summation.

role in determining the benefit of randomness in the screening context, this is no longer the case with multiple agents. For arbitrary correlation among different dimensions of an agent's type, as long as we maintain the independence of types across agents, there exists an equivalent deterministic mechanism for any stochastic mechanism.

Chawla, Malec, and Sivan (2015) consider *multi-agent* setting and focus on the case where the agents' values are independent both across different agents' types and different coordinates of an agent's type. In particular, Chawla, Malec, and Sivan (2015, Theorem13) establish a constant factor upper bound for the benefit of randomness when the agents' values are independent. In the special case of multi-unit multi-item auctions, they show that the revenue of any Bayesian incentive compatible, individually rational randomized mechanism is at most 33.75 times the revenue of the optimal deterministic mechanism. In this paper, we push this result to the extreme and show that the revenue maximizing auction can be deterministically implemented.[18]

5.2 An implementation perspective

We have motivated our result broadly, in terms of revenue, social surplus, interim expected allocation probabilities, interim expected utilities and even ex post payments. Alternatively, we may take an implementation perspective to formulate our result. Beyond the equivalence notion discussed throughout the paper, the deterministic allocation rule can also be required to pick some allocation in the support of the randomized allocation in the stochastic mechanism for each type profile v. Therefore, when a stochastic mechanism implements some social goal (i.e., at every type profile v, every realized allocation is consistent with the social goal), our equivalent deterministic mechanism also has the same property. We shall explain this point in the following paragraph.

Suppose that q is a random allocation rule. Given the K alternatives, the set of all nonempty subsets of $\{1, \ldots, K\}$ can have at most $2^K - 1$ elements $\{C_j\}_{1 \leq j \leq 2^K - 1}$. As a result, the set of type profiles V can be divided into $2^K - 1$ disjoint subsets $\{D_j\}_{1 \leq j \leq 2^K - 1}$ such that

1. the support of $q(v)$ is C_j for all $v \in D_j$;

2. $\lambda(\cup_{1 \leq j \leq 2^K - 1} D_j) = 1$.

[18] Chawla, Malec, and Sivan (2015, page 316) remarked that "our bounds on the benefit of randomness are in some cases quite large and we believe they can be improved".

We define $2^K - 1$ functions $\{\beta_j\}_{1 \leq j \leq 2^K - 1}$ such that $\beta_j = 1 + \mathbf{1}_{D_j}$ for each j; that is, β_j is the summation of 1 and the indicator function of the set D_j. Instead of working with the function h, we can work with the new function $h' = (h, \beta_1, \ldots, \beta_{2^K - 1})$. Lemma 1 and Proposition 1 stills hold and we can obtain a deterministic mechanism $\tilde{q}$ such that

$$\int_V q\beta_j \, \mathrm{d}\lambda = \int_V \tilde{q}\beta_j \, \mathrm{d}\lambda$$

for each j, and

$$\int_V q \, \mathrm{d}\lambda = \int_V \tilde{q} \, \mathrm{d}\lambda.$$

That is, $\int_{D_j} q \, \mathrm{d}\lambda = \int_{D_j} \tilde{q} \, \mathrm{d}\lambda$ for each j. Since $\sum_{k \in C_j} q^k(v) = 1$ for λ-almost all $v \in D_j$, $\int_{D_j} \sum_{k \in C_j} q^k(v)\lambda(\mathrm{d}v) = \lambda(D_j)$, which implies that $\int_{D_j} \sum_{k \in C_j} \tilde{q}^k(v)\lambda(\mathrm{d}v) = \lambda(D_j)$. As a result, for λ-almost all $v \in D_j$, $\tilde{q}^k = 1$ for some $k \in C_j$. This proves our claim that the deterministic allocation rule lies in the support of the random allocation rule.

5.3 Assumptions

Our result relies on several assumptions. In this subsection, we briefly discuss these assumptions. The requirement of multiple agents needs no further explanation. Atomless distribution is an indispensable requirement for almost all purification results. See Example 3 where we cannot purify the allocation for agent 2 while keeping her interim expected utility unchanged because agent 1's type distribution has an atom, let alone the stronger requirement that the deterministic mechanism requires such purification for all agents simultaneously. While our result requires independence, it is worth mentioning that we only require independence across agents and we do not make any assumption regarding the correlation of the different coordinates of type v_i for any agent $i \in \mathcal{I}$; see Section 5.1 for a fuller discussion. Though separable payoff is a restriction, this setup is sufficiently general to cover most applications. In particular, it includes the interdependent payoff function as in Jehiel and Moldovanu (2001), and obviously covers the widely adopted private value payoffs as a special case; see Section 4.1 for details.

6 Conclusion

In this paper, we prove that in a general social choice environment with multiple agents, for any stochastic mechanism, there exists an equivalent deterministic mechanism. Deterministic mechanisms have important advantages. For example, in the case of allocating indivisible objects, it is clearly desirable that the mechanism allocates the objects in a deterministic way. Our result implies that the requirement of deterministic mechanisms is not restrictive in itself. That is, even if one is constrained to employ only deterministic mechanisms, there is no loss of revenue or social welfare. Therefore, our result provides a foundation for the use of deterministic mechanisms in mechanism design settings, such as auctions, bilateral trades, etc.

The contrast with the results in the screening literature is particularly interesting for us. It is by now well known that, in the case of one-dimensional, private types, the optimal mechanism is deterministic, whether there is a single agent or multiple agents. When types are multidimensional, while the optimal mechanism might be stochastic in the single agent context, we recover the optimality of deterministic mechanisms when there are multiple agents. This reflection highlights the difference between settings with one agent and multiple agents, and leaves us wondering whether there are other interesting distinctions. In this sense, this paper takes one step in understanding such differences.

A Appendix

A.1 Proof of Claims 1 and 2

Proof of Claim 1. It is easy to see that the following pure strategy f_2' gives agent 2 the same expected payoff and (f_1, f_2') is still a Bayesian Nash equilibrium, where

$$f_2'(v) = \begin{cases} a_s, & v \in [\frac{s-1}{m}, \frac{s}{m}), 1 \leq s \leq m-1; \\ a_m, & v \in [\frac{m-1}{m}, 1]. \end{cases}$$

We next show that there does not exist a pure strategy g_1 of agent 1 such that g_1 is a component of a Bayesian Nash equilibrium with each agent's expected payoff being 0. Suppose that (g_1, g_2) is a Bayesian Nash equilibrium such that g_1 is a pure strategy of agent 1. Let $D_s = \{v_1 \in V_1 \colon g_1(v_1) = a_s\}$ for $1 \leq s \leq m$. Without loss of generality, we assume that $0 \in D_1$. Let $S = \arg\max_{1 \leq s \leq m} \lambda_1(D_s)$. Since $\lambda_1(D_s) \geq \lambda_1(D_1) \geq \lambda_1(0) > \frac{1}{m}$ for each $s \in S$, S must be a strict subset of $\{1, \ldots, m\}$. Without loss of generality, we assume that $s^* \in S$ and $s^* + 1 \notin S$. Given agent 1's strategy g_1, agent 2 can adopt the pure strategy $g_2'(v_2) = a_{s^*+1}$ for any $v_2 \in V_2$. Then the expected payoff of agent 2 is $\lambda_1(D_{s^*}) - \lambda_1(D_{s^*+1}) > 0$ with the strategy profile (g_1, g_2'). Since (g_1, g_2) is a Bayesian Nash equilibrium, the expected payoff of agent 2 must be at least $\lambda_1(D_{s^*}) - \lambda_1(D_{s^*+1})$ with the strategy profile (g_1, g_2), which is strictly positive. This is a contradiction. $\square$

Proof of Claim 2. We first construct a deterministic mechanism which gives agent 1 the same interim expected payoff. Define a function G on $V_1 \times V_2 = [0,1]^2$ by letting

$$G(v_1, v_2) = \int_0^{v_2} \left[\epsilon v_1 + (1 - v_2')^m\right] \lambda_2(\mathrm{d}v_2') - \left[\frac{\epsilon v_1}{2} + \frac{1}{2(m+1)}\right],$$

for any $(v_1, v_2) \in V_1 \times V_2$. It is clear that for any $v_1 \in [0,1]$, $G(v_1, 0) < 0 < G(v_1, 1) = \frac{\epsilon v_1}{2} + \frac{1}{2(m+1)}$. One can also check that $\frac{\partial G}{\partial v_2} = \epsilon v_1 + (1 - v_2)^m > 0$ for any $v_1 \in [0,1]$ and $v_2 \in [0,1)$. Hence, for each $v_1 \in [0,1]$, there exists a unique number $g(v_1) \in (0,1)$ such that $G(v_1, g(v_1)) = 0$. By the usual implicit function theorem, g must be differentiable, and hence measurable. Let $\hat{q}^1(v_1, v_2) = 1$ if $0 \le v_2 \le g(v_1)$ and 0 otherwise, and $\hat{q}^2(v_1, v_2) = 1 - \hat{q}^1(v_1, v_2)$. Then the mechanism $\hat{q}$ gives agent 1 the same interim expected payoff.

We next show that there does not exist any deterministic mechanism that gives agent 2 the same interim expected payoff. Suppose that there exists a deterministic mechanism $\tilde{q}$ that gives agent 2 the same interim expected payoff. Fix value $v_2 \in V_2 = [0,1]$.

Suppose that $\tilde{q}^2(0, v_2) = 1$. Then the interim expected payoff of agent 2 with value v_2 is

$$\int_{V_1} (\epsilon v_2 + (1 - v_1)^m) \tilde{q}^2(v_1, v_2) \lambda_1(\mathrm{d}v_1) \ge (\epsilon v_2 + 1)\lambda_1(0).$$

Recall that $\frac{\lambda_1(0)}{2} > \epsilon + \frac{1}{m+1}$. Hence we have

$$(\epsilon v_2 + 1)\lambda_1(0) \ge \lambda_1(0) > \frac{\lambda_1(0)}{2} + \epsilon + \frac{1}{m+1} > \frac{\epsilon v_2}{2} + \frac{\lambda_1(0)}{2} + (1 - \lambda_1(0))\frac{1}{2(m+1)}.$$

Thus, the interim expected payoff of agent 2 under the mechanism $\tilde{q}$ is strictly greater than the interim expected payoff of agent 2 under the mechanism q. This is a contradiction. Therefore, we must have $\tilde{q}^2(0, v_2) = 0$ since $\tilde{q}$ is a deterministic mechanism.

Next, since $\tilde{q}^2(0, v_2) = 0$, the interim expected payoff of agent 2 is

$$\begin{aligned}
&\int_{V_1} (\epsilon v_2 + (1 - v_1)^m) \tilde{q}^2(v_1, v_2) \lambda_1(\mathrm{d}v_1) = \int_{(0,1]} (\epsilon v_2 + (1 - v_1)^m) \tilde{q}^2(v_1, v_2) \lambda_1(\mathrm{d}v_1) \\
&\le (1 - \lambda_1(0)) \int_0^1 (\epsilon v_2 + (1 - v_1)^m)\, \mathrm{d}v_1 = (1 - \lambda_1(0))\epsilon v_2 + \frac{1 - \lambda_1(0)}{m+1} \\
&< \epsilon + \frac{1}{m+1} < \frac{\lambda_1(0)}{2}
\end{aligned}$$

$$< \frac{\epsilon v_2}{2} + \frac{\lambda_1(0)}{2} + (1-\lambda_1(0))\frac{1}{2(m+1)}.$$

That is, the interim expected payoff of agent 2 under the mechanism $\tilde{q}$ is strictly less than the interim expected payoff of agent 2 under the mechanism q. This is also a contradiction. Therefore, there does not exist any deterministic mechanism that gives agent 2 the same interim expected payoff. □

A.2 Technical lemmas

In the following, we present several lemmas as the technical preparations of Proposition 1.

If $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are measurable spaces, then a measurable rectangle is a subset $A \times B$ of $X \times Y$, where $A \in \mathcal{X}$ and $B \in \mathcal{Y}$ are measurable subsets of X and Y, respectively. The "sides" A, B of the measurable rectangle $A \times B$ can be arbitrary measurable sets; they are not required to be intervals. A discrete rectangle is a measurable rectangle such that each of its sides is a finite set.

Lemma 2. *Let D be a Borel measurable subset of V, and $F \subseteq V$ a measurable rectangle with sides $Y_i \subseteq V_i$ of measure l_i, $i \in \mathcal{I}$. Assume that $\lambda(D \cap F) \geq (1-\epsilon)\lambda(F)$ for some $0 < \epsilon < 1$. Then for any i,*

$$\lambda_i\{v_i \in V_i \colon \lambda_{-i}(D_{v_i} \cap F_{v_i}) > (1-\sqrt{\epsilon})\lambda_{-i}(F_{v_i})\} \geq (1-\sqrt{\epsilon})l_i.$$

Proof. Denote

$$\Gamma_i = \{v_i \in V_i \colon \lambda_{-i}(D_{v_i} \cap F_{v_i}) > (1-\sqrt{\epsilon})\lambda_{-i}(F_{v_i})\}.$$

Let Γ_i^C be the complement of Γ_i in V_i. Then

$$\begin{aligned}
(1-\epsilon)\Pi_{1\leq j\leq I} l_j &= (1-\epsilon)\lambda(F) \\
&\leq \lambda(D \cap F) \\
&= \Big(\int_{\Gamma_i} + \int_{\Gamma_i^C}\Big)\lambda_{-i}(D_{v_i} \cap F_{v_i})\lambda_i(\mathrm{d}v_i) \\
&= \int_{\Gamma_i} \lambda_{-i}(D_{v_i} \cap F_{v_i})\lambda_i(\mathrm{d}v_i) + \int_{\Gamma_i^C} \lambda_{-i}(D_{v_i} \cap F_{v_i})\lambda_i(\mathrm{d}v_i) \\
&\leq \int_{\Gamma_i} \lambda_{-i}(F_{v_i})\lambda_i(\mathrm{d}v_i) + (1-\sqrt{\epsilon})\int_{\Gamma_i^C} \lambda_{-i}(F_{v_i})\lambda_i(\mathrm{d}v_i) \\
&= \sqrt{\epsilon}\int_{\Gamma_i} \lambda_{-i}(F_{v_i})\lambda_i(\mathrm{d}v_i) + (1-\sqrt{\epsilon})\int_{Y_i} \lambda_{-i}(F_{v_i})\lambda_i(\mathrm{d}v_i)
\end{aligned}$$

$$= \sqrt{\epsilon}\lambda_i(\Gamma_i) \cdot \Pi_{j \neq i} l_j + (1 - \sqrt{\epsilon})\Pi_{1 \leq j \leq I} l_j.$$

The first inequality holds due to the condition that $\lambda(D \cap F) \geq (1-\epsilon)\lambda(F)$. The second inequality is true since $\lambda_{-i}(D_{v_i} \cap F_{v_i}) \leq (1-\sqrt{\epsilon})\lambda_{-i}(F_{v_i})$ for $v_i \in \Gamma_i^C$. All the equalities are just simple algebras. Rearranging the terms, we have

$$\lambda_i(\Gamma_i) \geq (1 - \sqrt{\epsilon}) l_i.$$

This completes the proof. $\square$

Lemma 3. *Let D be a Borel measurable subset of V with $\lambda(D) > 0$, $\tilde{i}_1, \ldots, \tilde{i}_I$ be positive natural numbers, and $0 < \epsilon < 1$ be sufficiently small such that $\epsilon' = \Pi_{1 \leq j \leq I} \tilde{i}_j \cdot \epsilon < 1$ and $\Pi_{1 \leq j \leq I} \tilde{i}_j \cdot \epsilon'^{\frac{1}{2^I}} < 1$.*

Consider the system of measurable rectangles $F^{i_1,\ldots,i_I} = \prod_{1 \leq j \leq I} Y_j^{i_j}$, where $1 \leq i_j \leq \tilde{i}_j$ and $Y_j^1, \ldots, Y_j^{\tilde{i}_j}$ are pairwise disjoint subsets on V_j for $1 \leq j \leq I$ such that $\lambda(F^{i_1,\ldots,i_I} \cap D) \geq (1-\epsilon)\lambda(F^{i_1,\ldots,i_I})$. Then there exists a discrete rectangle $\{v_1^{i_1}, \ldots, v_I^{i_I}\}_{\{1 \leq i_j \leq \tilde{i}_j, 1 \leq j \leq I\}}$ such that

1. *$(v_1^{i_1}, \ldots, v_I^{i_I}) \in F^{i_1,\ldots,i_I} \cap D$ for $1 \leq i_j \leq \tilde{i}_j$ and $1 \leq j \leq I$;*
2. *for each $1 \leq j \leq I$, $\{v_j^{i_j}\}$ are different points for $1 \leq i_j \leq \tilde{i}_j$.*

Proof. First, we consider the set

$$\Gamma_1^{i_1,\ldots,i_I} = \{v_1 \in Y_1^{i_1} \colon \lambda_{-1}(D_{v_1} \cap F_{v_1}^{i_1,\ldots,i_I}) > (1 - \sqrt{\epsilon'})\lambda_{-1}(F_{v_1}^{i_1,\ldots,i_I})\}.$$

Denote $\Gamma_1^{i_1} = \cap_{1 \leq i_k \leq \tilde{i}_k, 2 \leq k \leq I} \Gamma_1^{i_1,\ldots,i_I}$. We have

$$\begin{aligned}
\lambda_1(\Gamma_1^{i_1}) &= \lambda_1(Y_1^{i_1}) - \lambda_1\left(\cup_{1 \leq i_k \leq \tilde{i}_k, 2 \leq k \leq I}(Y_1^{i_1} \setminus \Gamma_1^{i_1,\ldots,i_I})\right) \\
&\geq \lambda_1(Y_1^{i_1}) - \sum_{1 \leq i_k \leq \tilde{i}_k, 2 \leq k \leq I} \left(\lambda_1(Y_1^{i_1}) - \lambda_1(\Gamma_1^{i_1,\ldots,i_I})\right) \\
&\geq \lambda_1(Y_1^{i_1}) - \sum_{1 \leq i_k \leq \tilde{i}_k, 2 \leq k \leq I} \left(\lambda_1(Y_1^{i_1}) - (1 - \sqrt{\epsilon'})\lambda_1(Y_1^{i_1})\right) \\
&= \left(1 - \Pi_{2 \leq k \leq I} \tilde{i}_k \cdot \sqrt{\epsilon'}\right) \lambda_1(Y_1^{i_1}) \\
&> 0.
\end{aligned}$$

The second inequality holds due to Lemma 2. We fix points $y_1^{i_1} \in \Gamma_1^{i_1}$ arbitrarily, as long as they are all distinct.

Second, let

$$\Gamma_2^{i_1,\ldots,i_I} = \{v_2 \in Y_2^{i_2} \colon (\bigotimes_{3\leq k\leq I} \lambda_k)(D_{(y_1^{i_1},v_2)} \cap F^{i_1,\ldots,i_I}_{(y_1^{i_1},v_2)}) > (1-\epsilon'^{\frac{1}{4}})(\bigotimes_{3\leq k\leq I} \lambda_k)(F^{i_1,\ldots,i_I}_{(y_1^{i_1},v_2)})\}.$$

Since $y_1^{i_1} \in \Gamma_1^{i_1}$ for any i_1, we have $y_1^{i_1} \in \Gamma_1^{i_1,\ldots,i_I}$ and

$$(\bigotimes_{2\leq k\leq I} \lambda_k)(D_{y_1^{i_1}} \cap F^{i_1,\ldots,i_I}_{y_1^{i_1}}) > (1-\sqrt{\epsilon'})(\bigotimes_{2\leq k\leq I} \lambda_k)(F^{i_1,\ldots,i_I}_{y_1^{i_1}}).$$

By Lemma 2, we have

$$\lambda_2(\Gamma_2^{i_1,\ldots,i_I}) \geq (1-\epsilon'^{\frac{1}{4}})\lambda_2(Y_2^{i_2}).$$

Denote $\Gamma_2^{i_2} = \cap_{1\leq i_j\leq \tilde{i}_j, j\neq 2}\Gamma_2^{i_1,\ldots,i_I}$. We have

$$\begin{aligned}
\lambda_2(\Gamma_2^{i_2}) &= \lambda_2(Y_2^{i_2}) - \lambda_2\left(\cup_{1\leq i_k\leq \tilde{i}_k, k\neq 2}(Y_2^{i_2} \setminus \Gamma_2^{i_1,\ldots,i_I})\right) \\
&\geq \lambda_2(Y_2^{i_2}) - \sum_{1\leq i_k\leq \tilde{i}_k, k\neq 2} \left(\lambda_2(Y_2^{i_2}) - \lambda_2(\Gamma_2^{i_1,\ldots,i_I})\right) \\
&\geq \lambda_2(Y_2^{i_2}) - \sum_{1\leq i_k\leq \tilde{i}_k, k\neq 2} \left(\lambda_2(Y_2^{i_2}) - (1-\epsilon'^{\frac{1}{4}})\lambda_2(Y_2^{i_2})\right) \\
&= \left(1 - \Pi_{1\leq k\leq I, k\neq 2}\tilde{i}_k \cdot \epsilon'^{\frac{1}{4}}\right)\lambda_2(Y_2^{i_2}) \\
&> 0.
\end{aligned}$$

We fix points $y_2^{i_2} \in \Gamma_2^{i_2}$ arbitrarily, as long as they are all distinct, and are also different from $\{y_1^{i_1}\}$.

Repeating this procedure until $I-1$, we can find $y_k^{i_k} \in \Gamma_k^{i_k}$ for $1 \leq i_k \leq \tilde{i}_k$ and $1 \leq k \leq I-1$, where $\Gamma_k^{i_k} = \cap_{1\leq i_j\leq \tilde{i}_j, j\neq k}\Gamma_k^{i_1,\ldots,i_I}$ and $\lambda_k(\Gamma_k^{i_k}) > 0$. In particular,

$$\begin{aligned}
\Gamma_{I-1}^{i_1,\ldots,i_I} = &\Big\{v_{I-1} \in Y_{I-1}^{i_{I-1}} \colon \lambda_I(D_{(y_1^{i_1},\ldots,y_{I-2}^{i_{I-2}},v_{I-1})} \cap F^{i_1,\ldots,i_I}_{(y_1^{i_1},\ldots,y_{I-2}^{i_{I-2}},v_{I-1})}) \\
&> (1-\epsilon'^{\frac{1}{2^{I-1}}})\lambda_I(F^{i_1,\ldots,i_I}_{(y_1^{i_1},\ldots,y_{I-2}^{i_{I-2}},v_{I-1})})\Big\}.
\end{aligned}$$

Finally, consider the set

$$E^{i_I} = \cap_{1\leq i_k\leq \tilde{i}_k, 1\leq k\leq I-1}\left(D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} \cap Y_I^{i_I}\right).$$

Notice that $F^{i_1,\cdots,i_I}_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} = Y_I^{i_I}$ for any $i_1,\ldots,i_I$. Then

$$\lambda_I(E^{i_I}) = \lambda_I\left(\cap_{1\leq i_k\leq \tilde{i}_k, 1\leq k\leq I-1}(D_{(y_1^{i_1},\ldots,y_{I-1}^{i_{I-1}})} \cap Y_I^{i_I})\right)$$

$$
\begin{aligned}
&= \lambda_I(Y_I^{i_I}) - \lambda_I\left(\cup_{1\le i_k\le \tilde{i}_k, 1\le k\le I-1}(Y_I^{i_I}\setminus D_{(y_1^{i_1},\dots,y_{I-1}^{i_{I-1}})})\right)\\
&\ge \lambda_I(Y_I^{i_I}) - \sum_{1\le i_k\le \tilde{i}_k, 1\le k\le I-1}\left(\lambda_I(Y_I^{i_I}) - \lambda_I(D_{(y_1^{i_1},\dots,y_{I-1}^{i_{I-1}})}\cap Y_I^{i_I})\right)\\
&= \lambda_I(Y_I^{i_I}) - \sum_{1\le i_k\le \tilde{i}_k, 1\le k\le I-1}\left(\lambda_I(Y_I^{i_I}) - \lambda_I(D_{(y_1^{i_1},\dots,y_{I-1}^{i_{I-1}})}\cap F^{i_1,\cdots,i_I}_{(y_1^{i_1},\dots,y_{I-1}^{i_{I-1}})})\right)\\
&> \lambda_I(Y_I^{i_I}) - \sum_{1\le i_k\le \tilde{i}_k, 1\le k\le I-1}\left(\lambda_I(Y_I^{i_I}) - (1-\epsilon'^{\frac{1}{2^{I-1}}})\lambda_I(F^{i_1,\cdots,i_I}_{(y_1^{i_1},\dots,y_{I-1}^{i_{I-1}})})\right)\\
&= \lambda_I(Y_I^{i_I}) - \sum_{1\le i_k\le \tilde{i}_k, 1\le k\le I-1}\left(\lambda_I(Y_I^{i_I}) - (1-\epsilon'^{\frac{1}{2^{I-1}}})\lambda_I(Y_I^{i_I})\right)\\
&= \left(1 - \Pi_{1\le k\le I-1}\tilde{i}_k\cdot \epsilon'^{\frac{1}{2^{I-1}}}\right)\lambda_I(Y_I^{i_I})\\
&> 0.
\end{aligned}
$$

The second inequality holds since $y_{I-1}^{i_{I-1}} \in \Gamma_{I-1}^{i_{I-1}} \subseteq \Gamma_{I-1}^{i_1,\dots,i_I}$, and hence

$$
\lambda_I(D_{(y_1^{i_1},\dots,y_{I-1}^{i_{I-1}})}\cap F^{i_1,\dots,i_I}_{(y_1^{i_1},\dots,y_{I-1}^{i_{I-1}})}) > (1-\epsilon'^{\frac{1}{2^{I-1}}})\lambda_I(F^{i_1,\dots,i_I}_{(y_1^{i_1},\dots,y_{I-1}^{i_{I-1}})}).
$$

Fix points $y_I^{i_I} \in E^{i_I}$ arbitrarily, as long as they are all different, and are different from $\{y_j^{i_j}\}_{1\le j\le I-1, 1\le i_j\le \tilde{i}_j}$. By the choice of E^{i_I}, $(y_1^{i_1},\dots,y_I^{i_I}) \in F^{i_1,\dots,i_I}\cap D$ for any $1\le i_j\le \tilde{i}_j$ and $1\le j\le I$. This completes the proof. □

Now we prove the last lemma.

Lemma 4. *There is a measurable function $d(v)$ with a finite set of values, which cannot be approximated in measure on $(D,\mathcal{B}(D),\lambda)$ by functions in $\mathscr{E}$ (recall that $\mathscr{E}$ is defined in the proof of Proposition 1).*

Proof. Let $g = \mathbf{1}_D$ be the indicator function of the set D, and $g_\delta(v) = \frac{1}{\lambda(B(v,\delta))}\int_{B(v,\delta)} g\,\mathrm{d}\lambda$. By Lemma 4.1.2 in Ledrappier and Young (1985), $g_\delta \to g$ for λ-almost all $v\in\mathbb{R}^{lI}$ as $\delta\to 0$. Without loss of generality, we assume that this convergence result holds for each point of D and the function h is continuous on D.

Fix natural numbers $\tilde{i}_j$ satisfying the condition that $l\cdot\sum_{J\in\mathcal{J}}(\Pi_{j\in J}\tilde{i}_j) < \Pi_{1\le j\le I}\tilde{i}_j$. For any discrete rectangle $L = \{(v_1^{i_1},\dots,v_I^{i_I})\in D\colon 1\le i_j\le\tilde{i}_j, 1\le j\le I\}$, we associate a linear mapping T_L from $\mathbb{R}^{\Pi_{1\le j\le I}\tilde{i}_j}$ to $\mathbb{R}^{l_0\cdot\sum_{J\in\mathcal{J}}(\Pi_{j\in J}\tilde{i}_j)}$:

$$
T_L(w) = \{\sum_{j\notin J, 1\le i_j\le\tilde{i}_j} h(v_1^{i_1},\dots,v_I^{i_I})\cdot w^{i_1,\dots,i_I}\}_{1\le i_j\le\tilde{i}_j, j\in J, J\in\mathcal{J}},
$$

where $l_0 = IKM + 1$, w is a vector with dimensions $\tilde{i}_1, \ldots, \tilde{i}_I$ and $w^{i_1,\ldots,i_I}$ is the corresponding component.

Fix a discrete rectangle $\overline{L} \subseteq D$ such that

- $\overline{L} = \{(\overline{v}_1^{i_1}, \ldots, \overline{v}_I^{i_I}) \in D \colon 1 \leq i_j \leq \tilde{i}_j, 1 \leq j \leq I\}$;
- the rank of the mapping $T_{\overline{L}}$ is maximal, say r.

Consider the system of $\sum_{J \in \mathcal{J}} (\Pi_{j \in J} \tilde{i}_j)$ homogeneous linear equations with $\Pi_{1 \leq j \leq I} \tilde{i}_j$ unknowns:

$$T_{\overline{L}}(w) = 0.$$

We take r equations and r unknowns for which the corresponding determinant is nonzero. Without loss of generality, we focus on this $r \times r$ matrix and denote it as $\overline{L}_s$, then $\det(\overline{L}_s) \neq 0$. For any discrete rectangle L, denote L_s as the restriction of the vector generated by the operator T_L onto the same matrix. Since h is continuous, $\det(L_s) \neq 0$ for any discrete rectangle L in a small open neighborhood of $\overline{L}$.

Let $w_{\overline{L}}$ be a nontrivial solution of the system corresponding to the discrete rectangle $\overline{L}$ in the sense that $T_{\overline{L}}(w_{\overline{L}}) = 0$. For any discrete rectangle $L \subseteq D$ such that $\det(L_s) \neq 0$, we provide a solution w_L below such that $T_L(w_L) = 0$.

- Since $\det(L_s) \neq 0$, the rank of the system corresponding to the operator T_L is at least r. Due to the choice of $\overline{L}$, the rank of the system corresponding to the operator T_L is at most r, and hence is r. As a result, the equations that do not occur in the determinant $\det(L_s)$ are linear combinations of the r equations that do.
- We focus on the r equations that occur in the determinant $\det(L_s)$, and let $w_L^{i_1,\ldots,i_I} = w_{\overline{L}}^{i_1,\ldots,i_I}$ if the column corresponding to the unknown $w_L^{i_1,\ldots,i_I}$ does not occur in the determinant $\det(L_s)$.
- The remaining r unknowns of $w_L^{i_1,\ldots,i_I}$, corresponding to the columns that occur in the determinant $\det(L_s)$, can be obtained by Cramer's rule.

By the above construction, it is obvious that w_L depends continuously on the r nodes of the discrete rectangle L corresponding to the columns of $\det(L_s)$.

Pick numbers $d^{i_1,\ldots,i_I}$ subject to $\sum_{1\le i_j\le \tilde{i}_j, 1\le j\le I} d^{i_1,\ldots,i_I}\cdot w_L^{i_1,\ldots,i_I} = 1$. Consider the measurable rectangles

$$G^{i_1,\ldots,i_I} = \{v = (v_1,\ldots,v_I)\in \mathbb{R}^{lI} \colon |v_j - \overline{v}_j^{i_j}| \le \delta,\ 1\le j\le I\},$$

and

$$F^{i_1,\ldots,i_I} = \{v = (v_1,\ldots,v_I)\in V \colon |v_j - \overline{v}_j^{i_j}| \le \delta,\ 1\le j\le I\}.$$

Then for sufficiently small δ, $\{G^{i_1,\ldots,i_I}\}$ are pairwise disjoint, and $\{F^{i_1,\ldots,i_I}\}$ are also pairwise disjoint.

By the first paragraph of this proof, $\frac{1}{\lambda(B(v,\delta))}\int_{B(v,\delta)} \mathbf{1}_D \,\mathrm{d}\lambda \to \mathbf{1}_D(v)$ for each $v\in D$. Since $(\overline{v}_1^{i_1},\ldots,\overline{v}_I^{i_I})\in D$, $\lambda(G^{i_1,\ldots,i_I}\cap D)\ge (1-\epsilon)\lambda(G^{i_1,\ldots,i_I})$ for sufficiently small δ, where ϵ is given in the proof of Lemma 3. Since D is a subset of V, we have

$$\lambda(F^{i_1,\ldots,i_I}\cap D) = \lambda(G^{i_1,\ldots,i_I}\cap D) \ge (1-\epsilon)\lambda(G^{i_1,\ldots,i_I}) \ge (1-\epsilon)\lambda(F^{i_1,\ldots,i_I}).$$

In addition, since $\sum_{1\le i_j\le \tilde{i}_j, 1\le j\le I} d^{i_1,\ldots,i_I}\cdot w_L^{i_1,\ldots,i_I}$ is continuous in the discrete rectangle, for sufficiently small δ, $\sum_{1\le i_j\le \tilde{i}_j, 1\le j\le I} d^{i_1,\ldots,i_I}\cdot w_L^{i_1,\ldots,i_I} \ge \frac{1}{2}$ for

$$L = \{(v_1^{i_1},\ldots,v_I^{i_I})\in F^{i_1,\ldots,i_I}\cap D \colon 1\le i_j\le \tilde{i}_j, 1\le j\le I\}.$$

To summarize, we pick $\delta > 0$ sufficiently small such that

1. $\lambda(F^{i_1,\ldots,i_I}\cap D)\ge (1-\epsilon)\lambda(F^{i_1,\ldots,i_I})$; and

2. $\sum_{1\le i_j\le \tilde{i}_j, 1\le j\le I} d^{i_1,\ldots,i_I}\cdot w_L^{i_1,\ldots,i_I} \ge \frac{1}{2}$ for any discrete rectangle

$$L = \{(v_1^{i_1},\ldots,v_I^{i_I})\in F^{i_1,\ldots,i_I}\cap D \colon 1\le i_j\le \tilde{i}_j, 1\le j\le I\}.$$

Let

$$d(v) = \begin{cases} d^{i_1,\ldots,i_I}, & \text{if } v\in F^{i_1,\ldots,i_I}\cap D;\\ 0, & \text{otherwise.}\end{cases}$$

If it could be approximated by functions in $\mathscr{E}$ on $(D,\mathcal{B}(D),\lambda)$ in measure, then there is a sequence $d_n(v) = h(v)\cdot\sum_{J\in\mathcal{J}}\psi_J^n(v_J)$ which converges to d on some Borel measurable subset C such that $\lambda(C)=\lambda(D)$.

By condition (1) above and Lemma 3, there exists a discrete rectangle $L = \{(v_1^{i_1}, \ldots, v_I^{i_I})\}_{\{1 \le i_j \le \tilde{i}_j, 1 \le j \le I\}}$ such that $(v_1^{i_1}, \ldots, v_I^{i_I}) \in F^{i_1,\ldots,i_I} \cap C$ for $1 \le i_j \le \tilde{i}_j$ and $1 \le j \le I$. Since $\sum_{1 \le i_j \le \tilde{i}_j, j \notin J} w_L^{i_1,\ldots,i_I} h(v_1^{i_1}, \ldots, v_I^{i_I}) = 0$ for any $J \in \mathcal{J}$, we have

$$
\begin{aligned}
& \sum_{1 \le i_j \le \tilde{i}_j, 1 \le j \le I} d^{i_1,\ldots,i_I} \cdot w_L^{i_1,\ldots,i_I} \\
&= \lim_{n \to \infty} \sum_{1 \le i_j \le \tilde{i}_j, 1 \le j \le I} d_n(v_1^{i_1}, \ldots, v_I^{i_I}) \cdot w_L^{i_1,\ldots,i_I} \\
&= \lim_{n \to \infty} \sum_{1 \le i_j \le \tilde{i}_j, 1 \le j \le I} \left(h(v_1^{i_1}, \ldots, v_I^{i_I}) \cdot \sum_{J \in \mathcal{J}} \psi_J^n(v_J^{i_J}) \right) w_L^{i_1,\ldots,i_I} \\
&= \lim_{n \to \infty} \sum_{1 \le i_j \le \tilde{i}_j, 1 \le j \le I} \left\{ \left(w_L^{i_1,\ldots,i_I} h(v_1^{i_1}, \ldots, v_I^{i_I}) \right) \cdot \sum_{J \in \mathcal{J}} \psi_J^n(v_J^{i_J}) \right\} \\
&= \lim_{n \to \infty} \sum_{J \in \mathcal{J}} \sum_{1 \le i_j \le \tilde{i}_j, 1 \le j \le I} \left\{ \left(w_L^{i_1,\ldots,i_I} h(v_1^{i_1}, \ldots, v_I^{i_I}) \right) \cdot \psi_J^n(v_J^{i_J}) \right\} \\
&= \lim_{n \to \infty} \sum_{J \in \mathcal{J}} \sum_{1 \le i_j \le \tilde{i}_j, j \in J} \left\{ \sum_{1 \le i_j \le \tilde{i}_j, j \notin J} w_L^{i_1,\ldots,i_I} h(v_1^{i_1}, \ldots, v_I^{i_I}) \right\} \cdot \psi_J^n(v_J^{i_J}) \\
&= 0,
\end{aligned}
$$

where $v_J^{i_J}$ denotes the vector $(v_j^{i_j})_{j \in J}$. However, $\sum_{1 \le i_j \le \tilde{i}_j, 1 \le j \le I} d^{i_1,\ldots,i_I} \cdot w_L^{i_1,\ldots,i_I} \ge \frac{1}{2}$ by condition (2) above, which is a contradiction. As a result, the function d cannot be approximated by functions in $\mathscr{E}$ on $(D, \mathcal{B}(D), \lambda)$ in measure. This completes the proof. □

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