Strategy-proof Pareto-improvement*

Samson Alva
University of Texas at San Antonio
samson.alva@utsa.edu

Vikram Manjunath
Texas A&M University
vikram@dosamobile.com

April 15, 2016

Abstract

We consider a general model of allocating discrete resources when agents have unit demand. Our main result is that every individually rational and non-wasteful rule is (weakly) Pareto-dominated by at most one strategy-proof rule. An immediate implication is that if a strategy-proof rule is individually rational and non-wasteful, then it is strategy-proofness-constrained Pareto-efficient.

By specializing our model, we show that it applies to a broad class of economic problems from auctions to object allocation to matching with contracts.

Keywords: strategy-proofness, object allocation, non-wastefulness, stability, market design, school choice, matching with contracts
JEL Codes: C78; D47; D71

1 Introduction

We study a model that encompasses most of the commonly studied models with discrete goods where agents have unit demand. Notable examples are object allocation, priority augmented object allocation, matching with contracts, and auctions. To be specific, our model consists of a set of agents, a set of objects, and the terms under which an agent may consume an object. Each object comes with a collection of ways in which it can be feasibly allocated: sets of agent-term pairs. These constitute within-object constraints. Our model also allows the specification of across-object constraints. Feasibility requires that an allocation satisfy both within and across-object constraints. Agents have weak preferences over object-term pairs and being unmatched.

*We thank Lars Ehlers, Patrick Harless, Sean Horan, Thayer Morrill, Tayfun Sönmez, William Thomson, Guoqiang Tian, Utku Ünver, Rodrigo Velez, Bumin Yenmez, and participants at the Texas Theory Camp at UT Austin and the Workshop on Matching and Choice Theory at ITAM for helpful comments and discussions.

The main question that we answer is this: Suppose that a mechanism designer is tasked with selecting a strategy-proof rule, but is constrained to select a rule that every agent finds to be at least as good as some fixed benchmark—perhaps a status quo rule or a rule that reflects some normative considerations. Is there more than one solution to his problem?

Our main result (Theorem 1) is that *an individually rational and non-wasteful*[1] *benchmark rule can be (weakly) Pareto-dominated by at most one strategy-proof rule*. An immediate implication is that if the benchmark is strategy-proof itself, then no other strategy-proof rule Pareto-dominates it. Thus, every individually rational and non-wasteful strategy-proof rule is strategy-proofness-constrained (or "second best") Pareto-efficient.

We shed more light on the structure of the set of strategy-proof rules with regards to the Pareto-relation. If one strategy-proof rule Pareto-dominates another strategy-proof rule, then the dominating rule is "less wasteful" than the other (Theorem 2). In other words, the Pareto-improvement cannot come from better allocation of the same discrete resources, but from allocation of *more* of these discrete resources. This is because a wasteful strategy-proof rule is not Pareto-dominated by an equally wasteful strategy-proof rule (Lemma 3). Thus, by allocating more and more of the available resources, it is possible to achieve strategy-proof Pareto-improvements. However, once the resources are used maximally, even if not Pareto-efficiently, Theorem 1 implies that no further strategy-proof Pareto-improvements are possible. That is, individual rationality and non-wastefulness are sufficient conditions for a strategy-proof rule to be strategy-proofness-constrained Pareto-efficient. However, we show that these are not necessary conditions: there may be strategy-proof, but wasteful, rules that cannot be Pareto-improved upon in a strategy-proof way (Proposition 3).

Key to proving our results is what we call the Structure Lemma: for every profile of preferences, the set of individually rational and non-wasteful allocations can be partitioned so that *a*) for each pair of allocations in the same component, the set of agents who are assigned an object is the same and *b*) two allocations in separate components cannot be Pareto-compared.

In Section 5, we specialize our model in two ways to demonstrate the implications of our results for "market design." First, we assume that agents have strict preferences. Second, we augment the model by prioritizing the feasible sets of each object through choice correspondences.[2] Given a set of agents and terms under which they wish to consume a particular object, the object's choice correspondence says which of the feasible subsets of these agent-term pairs have the highest priority. For such matching with contracts (Hatfield and Milgrom, 2005) settings, *stability* has been the main normative consideration. We show that under certain conditions—*size monotonicity*[3] and a mild consistency

[1] We extend the definition of non-wastefulness from the object allocation problem (Balinski and Sönmez, 1999) to our more general setting.

[2] To better accommodate ties in priorities of feasible sets (as in the school choice model), we have chosen to model them as choice *correspondences* rather than choice functions. To our knowledge Erdil and Kumano (2014) is the first and only other analysis in the literature on matching that considers choice correspondences as a model primitive.

[3] We use the naming convention of Alkan and Gale (2003). Alkan (2002) calls this "cardinal monotonicity" and Hatfield and Milgrom (2005) refers to it as the "law of aggregate demand."

property that we call *idempotence*[4]—every stable allocation is non-wasteful (Proposition 4). Thus, Theorem 1 says that a stable rule is Pareto-dominated by at most one strategy-proof rule. So if a stable rule is strategy-proof, it is strategy-proofness-constrained Pareto-efficient.[5]

Without further conditions on choice correspondences, the existence of the *agent-optimal stable rule* is not guaranteed.[6] However, when such a rule does exist, our main result implies that it is the unique stable and strategy-proof rule. Moreover, the Structure Lemma implies that, in such cases, every stable allocation is in the same component of the above mentioned partition of the individually rational and non-wasteful allocations. Consequently, we have an equivalence of the existence of the agent-optimal stable rule on one hand and the "Rural Hospitals Theorem" of Roth (1986) alongside the existence of a stable and strategy-proof rule on the other (Propositions 5 and 6).

The *priority augmented object allocation* model[7] involves a particular kind of choice correspondence: Each object is associated with a "capacity" and a priority order over the agents that may consume it. If the agents in one set have higher priority than the agents in another, then the former set is prioritized over the latter. Respecting priorities is an important normative consideration as a form of fairness (Balinski and Sönmez, 1999): an agent has the *right* to protest an allocation if he would rather have an object that is assigned to someone of lower priority. However, individual rationality, non-wastefulness and respect for priorities are together equivalent to stability, which often implies involves efficiency losses. What if, instead of demanding that priorities be respected, we are willing to consider allocations that are not necessarily stable themselves but Pareto-dominate some stable allocation? After all, if an agent were to protest the chosen allocation on the grounds that it is not stable, we can point to the stable allocation that it Pareto-dominates as justification: no agent would desire a move to that stable allocation. Thus, we consider "stable-dominating" allocations. While stability is at odds with Pareto-efficiency (except for particular kinds of priorities (Ergin, 2002; Ehlers and Erdil, 2010)), the requirement that an allocation be stable-dominating is not. If the priorities are strict, the agent-optimal stable rule is well defined and our results say that this is the unique stable-dominating strategy-proof rule. On the other hand, under weak priorities there may not be an agent-optimal stable rule. In such cases, our results still say that no stable and strategy-proof rule can be Pareto-improved upon in a strategy-proof way. While there may be many stable-dominating strategy-proof rules, we show that none of them are group strategy-proof. This answers, in the negative, the question of whether any of the many group strategy-proof rules for this model (variants of "Top Trading Cycles" rules, such as those defined by Pycia and Ünver (2015)) are stable-dominating.

[4]This is a weaker consistency condition than "irrelevance of rejected contracts" (Aygün and Sönmez, 2013).

[5]Versions of this result have appeared previously in the literature, the broadest of these being due to Hirata and Kasuya (2015), which we discuss further, along with Erdil (2014), in Section 1.1. Among earlier such results are those by Abdulkadiroğlu et al. (2009) and Kesten and Kurino (2015) who show that "deferred acceptance" is not Pareto-dominated by any strategy-proof rule.

[6]See Hatfield and Kojima (2010) for sufficient conditions.

[7]This is also known as the "school choice" model (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003).

The remainder of the paper is organized as follows. In Section 1.1 we discuss the work most closely related to ours. We introduce our model and give key definitions in Section 2. In Section 3 we state and prove the Structure Lemma. Our main results are in Section 4. We demonstrate the implications of these results for matching with contracts by specializing our model in Section 5 and augmenting it with choice correspondences. We further specialize choice correspondences to model priority augmented object allocation in Section 6. Finally, in Section 7 we discuss the implications of our results for auctions, reallocation of objects from an endowment, and market design.

1.1 Related Literature

We defer to Section 7 the discussion of what our results imply for many of the specific models subsumed by ours. In this section, we focus on two related recent works.

The first is Erdil (2014). He specifically considers the object allocation problem with strict preferences where each object has a certain capacity and analyzes stochastic allocation. He shows a version of the Structure Lemma for his model. Further, he shows what is an implication of Theorem 1 in our setting: non-wastefulness and individual rationality are sufficient conditions for a strategy-proof rule to be strategy-proofness-constrained Pareto-efficient. His last result that is related to our results is an analog of Theorem 2, which says that if a strategy-proof rule dominates another, then the dominating rule allocates more resources.

On one hand, our model is significantly more general than his as it allows complex within-object and across-object feasibility constraints, indifferences, and multiple terms under which an object may be consumed. On the other hand, his analysis is of stochastic allocation. Consequently, our results do not imply one another's.

The second is Hirata and Kasuya (2015). They study the matching with contracts model with choice *functions* that satisfy the "irrelevance of rejected contracts" condition of Aygün and Sönmez (2013). Since their analysis is in the matching with contracts framework, their results are related to results in Sections 5 and 6 where we demonstrate the implications of our results for this framework. In this context, they show that there is at most one strategy-proof and stable rule. In Section 5, we show that this result does not hold under the assumptions that we make on choice correspondences, even if these correspondences are single-valued. They also show that whenever the agent-optimal stable rule is well defined, it is the only candidate for a stable and strategy-proof rule. Their proof does not rely on the Rural Hospitals Theorem, which fails in their setting even if the doctor-optimal stable rule is well defined. In contrast, the existence of a doctor-optimal stable rule and the Rural Hospitals Theorem are intimately tied together under our conditions. Finally, they show that individual rationality and non-wastefulness are sufficient for a strategy-proof rule to be strategy-proofness-constrained Pareto-efficient, which is for our model an implication of Theorem 1, as discussed above. Their assumption of the irrelevance of rejected contracts condition, while quite mild, is technically stronger than our assumption of idempotence. However, we require size-monotonicity of choice correspondence to show that stability implies non-wastefulness. So our results in Sections 5 and 6 do not imply theirs, nor do their results imply ours.

2 The Model

Let $\boldsymbol{N}$ be a finite and nonempty set of agents and let $\boldsymbol{O}$ be a finite and nonempty set of objects. Let $\boldsymbol{T}$ be the nonempty set of all possible terms under which an agent may be assigned an object. A triple $(i,o,t) \in N \times O \times T$ represents the assignment of o to i under the terms t. Let $\boldsymbol{X \subseteq N \times O \times T}$ denote the set of permissible triples of this sort.

Given $x \in X$, we denote the agent associated with x by $\boldsymbol{x_N}$. Similarly, $\boldsymbol{x_O}$ is the object associated with x and $\boldsymbol{x_T}$ are the terms associated with it. For each $Y \subseteq X$, the elements of Y associated with $i \in N$ are $\boldsymbol{Y(i)}$, and elements of Y associated with $o \in O$ are $\boldsymbol{Y(o)}$. For each $Y \subseteq X$, the set of agents involved in elements of Y are $\mathbf{Y_N} \equiv \{y_N : y \in Y\}$. When T is a singleton, each $x \in X$ is fully identified by the involved agent and object. In such cases, for each $i \in N$, each element of $X(i)$ is identified by an element of O while for each $o \in O$, each element of $X(o)$ is identified by an element of N.

Agents have unit demand, so each $i \in N$ is either assigned a singleton consisting of an element of $X(i)$ or assigned nothing, $\emptyset$. His preferences are a complete, reflexive, and transitive binary relation on singletons and the empty set. We denote it by R_i. We write, for each $x,y \in X(i)$, $\boldsymbol{x\ R_i\ y}$ to mean that i finds $\{x\}$ to be at least as good as $\{y\}$. Similarly, if he finds $\{x\}$ to be at least as good as the empty set, we write $x\ R_i\ \emptyset$ and vice versa. We use $\boldsymbol{P_i}$ to denote strict preference and $\boldsymbol{I_i}$ to denote indifference, the asymmetric and symmetric components of R_i, respectively. Let $\boldsymbol{\mathcal{R}_i}$ be a set of preference relations for i that satisfy the following two conditions. For each $R_i \in \mathcal{R}_i$:

- No indifference with $\emptyset$: there is no $x \in X(i)$ such that $x\ I_i\ \emptyset$.[8]
- Richness: for each distinct pair $x,y \in X(i)$, if $x\ P_i\ y\ P_i\ \emptyset$, then there is $R_i' \in \mathcal{R}_i$ such that *a*) $x\ R_i'\ \emptyset\ R_i'\ y$, and *b*) for each $z \in X(i)$, if $z\ P_i'\ \emptyset$ then $z\ P_i\ y$.

The *preference domain* is $\boldsymbol{\mathcal{R}} \equiv \times_{i\in N}\mathcal{R}_i$.

Let $\boldsymbol{\mathcal{P}_i} \subseteq \mathcal{R}_i$ be the subset of strict (antisymmetric) preferences. For each $R_i \in \mathcal{P}_i$, the symmetric component I_i is trivial, so the asymmetric component P_i completely identifies R_i. When working in this strict preference setting (Sections 5 and 6), we refer to $P_i \in \mathcal{P}_i$. The *strict subdomain* of $\mathcal{R}$ is $\boldsymbol{\mathcal{P}} \equiv \times_{i\in N}\mathcal{P}_i$.

Within-object Constraints Each object can only be allocated in certain ways. For each $o \in O$, the **feasible sets** for o are a collection of subsets of $X(o)$. We denote them by $\boldsymbol{F_o}$. We assume that it is always possible to leave an object unallocated. That is, $\emptyset \in F_o$.

As an example, consider the school choice model of Abdulkadiroğlu and Sönmez (2003), where each object is a school with capacity $q \in \mathbb{Z}_+$. In this model, there is only one term under which a student can be assigned to a school. For each $o \in O$, F_o consists of all subsets of $X(o)$ of no more than q elements.[9]

The feasible sets, however, may be more complex than that. Suppose o is a flight and the terms T consist of "assigned aisle seat" (α), "assigned window seat" (ω), or "unassigned seat" (υ). Physically, suppose that a flight has two seats: an aisle seat and a window

[8] See Sönmez (1999) and Erdil and Ergin (2015) for other instances where this assumption is made.

[9] In these environments T is a singleton, so, as previously discussed, feasible sets for each $o \in O$ are identified by subsets of N while preferences are identified by orderings of $O \cup \{\emptyset\}$.

seat. Financially, suppose that a flight is canceled unless both seats are filled. Given three agents i_1, i_2, and i_3, which subsets of $\{(i_1, o, v), (i_2, o, v), (i_3, o, \omega), (i_3, o, \alpha)\}$ can be allocated? If the flight is canceled, then we can allocate $\emptyset$. The constraint of having only one aisle and one window seat, together with the financially-motivated lower bound, requires that the feasible set for o comprise the following subsets: $\{(i_1, o, v), (i_2, o, v)\}$, $\{(i_1, o, v), (i_3, o, \omega)\}$, $\{(i_1, o, v), (i_3, o, \alpha)\}$, $\{(i_2, o, v), (i_3, o, \omega)\}$, $\{(i_2, o, v), (i_3, o, \alpha)\}$, and $\{(i_3, o, \omega), (i_3, o, \alpha)\}$. Note that feasibility for an object need not imply feasibility for an agent—i_3 can occupy just one seat so, while $\{(i_3, o, \omega), (i_3, o, \alpha)\}$ maybe feasible for this object, it is not feasible from the agent's perspective.[10]

Feasibility and Allocations An **allocation** μ is a *feasible* subset of X. The feasible sets for each object allow us to model within-object constraints on how it may be allocated. So $\mu \subseteq X$ can only be feasible if it contains a feasible set for each object. Since agents have unit demand, for μ to be feasible it must contain no more than one element for each agent. Since we allow across-object constraints as well, not all such μ are feasible. The set of allocations is $\mathcal{F} \subseteq 2^X$ such that for each $\mu \in \mathcal{F}$, *a*) for each $i \in N$, $|\mu(i)| \leq 1$, and *b*) for each $o \in O$, $\mu(o) \in F_o$. We say that $\mathcal{F}$ is **Cartesian** if there are no across-object constraints: if $\mu \subseteq X$ satisfies Condition (*a*) and Condition (*b*) above, then $\mu \in \mathcal{F}$.[11]

As an example, the across-object constraints allow us to model problems like the distribution of a social endowment of objects and money (Alkan et al., 1991). Suppose that there are Ω units of money to be distributed among the agents along with O. This could, for instance, be compensation for the tasks that the objects represents. Then, we set $T \equiv \mathbb{R}$ so that the terms under which an agent is assigned an object is the amount of money that he gets along with his object. For each $o \in O$, we set $F_o \equiv \{\{(i, o, t)\} : (i, t) \in N \times T\}$ so that each object can be assigned with any amount of money. Since any amount of money is permissible, $X \equiv N \times O \times T$. Finally, the across-object constraints ensure that no more than Ω units of money are allocated: given $\mu \subseteq X$, $\mu \in \mathcal{F}$ if and only if *a*) for each $i \in N$, $|\mu(i)| \leq 1$, so that each agent consumes at most one object, *b*) for each $o \in O$, $|\mu(o)| \leq 1$, so that each object is consumed by at most one agent, and *c*) $\sum_{x \in \mu} x_T \leq \Omega$, so that no more than Ω units of money are distributed.

Our analysis is for fixed N, O, X, and $\mathcal{F}$. Thus, an economy is entirely described by $R \in \mathcal{R}$. A **rule**, $\varphi : \mathcal{R} \to \mathcal{F}$, associates each economy with an allocation. For each $R \in \mathcal{R}$ and each $i \in N$, i's assignment at R is $\varphi_i(R)$. We denote by $\varphi_N(R)$ the set of agents who are assigned an object at R. That is, $\varphi_N(R) = \{i \in N : \varphi_i(R) \in X(i)\}$.

[10]This flexibility in defining F_o turns out to be useful when we consider the implication of our results for the matching with contracts setting, in particular when we consider Hatfield and Kominers (2014b) in Section 7.3.2.

[11]Since we permit an object to be consumed under different terms, instead of having multiple objects we could have instead had a single object and treated the *actual* objects as terms under which this single object is consumed. However, we find it convenient to separate the within-object and across-object feasibility constraints when relating our results to more structured models in the later part of this paper.

2.1 Properties of Allocations and Rules

Individual rationality An allocation is **individually rational** if every agent finds his assignment to be at least as good as being assigned nothing. That is, for each $R \in \mathcal{R}$ and each $\mu \in \mathcal{F}$, we say that μ is individually rational at R if for each $i \in N$, $\mu(i) \; R_i \; \varnothing$.

A rule φ is *individually rational* if, for each $R \in \mathcal{R}$, $\varphi(R)$ is individually rational at R.

Non-wastefulness In environments where each object is associated with a "capacity" (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003), an object can typically be allocated to any set of agents no bigger than its capacity. A natural requirement is that an agent ought not to prefer an object that has remaining capacity to his assignment. If he were to, we could allow him to consume this available resource at no expense to the other agents.

These environments are described by feasible sets where for each $o \in O$, there is $q_o \in \mathbb{Z}_+$ such that for each $M \subseteq N, M \in F_o$ if and only if M contains no more than q_o agents. We say that such F is "capacity-based." We say that $\mathcal{F}$ is capacity-based if it is Cartesian and each F_o is capacity-based.

For capacity-based $\mathcal{F}$ and strict preferences, Balinski and Sönmez (1999) say that $\mu \in \mathcal{F}$ is non-wasteful at $P \in \mathcal{P}$ if there is no object with remaining capacity that some agent finds preferable to his assignment at μ. That is, μ is **Balinski-Sönmez-non-wasteful** at P if there is no $o \in O$ such that $|\mu(o)| < q_o$ and $i \in N$ such that $o \; P_i \; \mu(i)$.

Before we extend this concept beyond just capacity-based settings with strict preferences, we present three examples that illustrate some challenges.

Example 1. *Without capacity-based feasible sets, there is no fixed capacity.*

Let $O \equiv \{o\}$, $N \equiv \{i_1, i_2, i_3\}$ and $F_o \equiv \{\varnothing, \{i_1\}, \{i_2\}, \{i_3\}, \{i_2, i_3\}\}$. Consider $R \in \mathcal{R}$ as follows:

R_{i_1}	R_{i_2}	R_{i_3}
o	o	o
$\varnothing$	$\varnothing$	$\varnothing$

What is the "capacity" of o? The largest set of agents that may consume o contains two elements while the smallest non-trivial set contains only one. If we naïvely extend Balinski-Sönmez-non-wastefulness by setting the capacity of o to be two, then allocating it to i_1 would be wasteful, even though this is the only allocation where i_1 receives his top choice. On the other hand if we set the capacity of o to be one, then allocating it to i_2 alone would *not* be wasteful even though o could be assigned to i_3 as well. Neither of these is sensible. This demonstrates the difficulty in extending non-wastefulness in a way that specifies a fixed capacity for each object.

Example 2. *Ignoring indifference can be wasteful.*

Let $O \equiv \{o_1, o_2\}$, $N \equiv \{i_1, i_2\}$, $F_{o_1} \equiv \{\varnothing, \{i_1\}, \{i_2\}\}$, $F_{o_2} \equiv \{\varnothing, \{i_1\}, \{i_2\}\}$, and $\mathcal{F}$ be Cartesian. Consider $R \in \mathcal{R}$ as follows:

R_{i_1}	R_{i_2}
o_1, o_2	o_1
$\varnothing$	$\varnothing$

There are two allocations of interest here. The first is where i_1 is assigned o_1 and i_2 is assigned $\varnothing$. The second is where i_1 is assigned o_2 and i_2 is assigned $\varnothing$. The second allocation is clearly wasteful, since i_2 could be assigned o_1. Is the first allocation wasteful? Since i_1 is indifferent between o_1 and o_2, -non-wastefulness would not say that it is wasteful to assign him o_2. So, a naïve extension of Balinski-Sönmez-non-wastefulness to our setting would not perceive waste in the first allocation though it would in the second. However, the first allocation is welfare-equivalent to the unambiguously wasteful second allocation. A sensible version of non-wastefulness for our general setting should rule out the first allocation as well.

Example 3. *Complementarities in feasibility.*

Let $O \equiv \{o_1, o_2\}$, $N \equiv \{i_1, i_2\}$, $F_{o_1} \equiv \{\varnothing, \{i_1, i_2\}\}$, $F_{o_2} \equiv \{\varnothing, \{i_1\}, \{i_2\}\}$, and $\mathcal{F}$ be Cartesian. Consider $R \in \mathcal{R}$ as follows:

R_{i_1}	R_{i_2}
o_1	o_2
o_2	o_1
$\varnothing$	$\varnothing$

There are two allocations of interest for this economy. The first is where both agents are assigned o_1. This is the only allocation where i_1 receives his top choice. The second is where i_1 is assigned $\varnothing$ and i_2 is assigned o_2. This is the only allocation where i_2 receives his top choice. At either of these allocations there is an agent who prefers the unallocated object to what he receives. However, the only way he can be assigned this unallocated object is by making the other agent worse off. A sensible version of non-wastefulness in our general setting should not rule out either of these allocations.

Though there is no fixed notion of capacity, as demonstrated by Example 1, and because there can be welfare-equivalent allocations, as demonstrated by Example 2, non-wastefulness should seek to ensure that each object is utilized to the greatest extent possible. Yet, as demonstrated by Example 3, it should take care to ensure that increasing the utilization of an object by allocating it to agents who prefer it does not harm other agents.

Summarizing the discussion above, given $R \in \mathcal{R}$, we say that $\mu \in \mathcal{F}$ is *wasteful* if there are $o \in O$, $i \in N$, and $\nu \in \mathcal{F}$, such that *a*) $|\nu(o)| > |\mu(o)|$, so that ν allocates o more than μ does, *b*) $\nu(i)\ P_i\ \mu(i)$, so that i prefers his assignment at ν to that at μ, and *c*) for each $j \in N \setminus \{i\}$, $\nu(j)\ R_i\ \mu(j)$, so that no agent is made worse off. If it is not wasteful, then μ **non-wasteful**.

We show that for the domain of problems where BS-non-wastefulness is defined, non-wastefulness is an equivalent to it. The proof is in Appendix A.1.

Proposition 1. *Suppose that T is a singleton, $\mathcal{F}$ is capacity-based, and preferences are strict. Then μ is Balinski-Sönmez-non-wasteful if and only if it is non-wasteful.*

Our results do not rely on the full strength of non-wastefulness. Non-wastefulness of an allocation rules out the existence of allocations that beneficially increase utilization of *some* resources. We define a weaker version that only rules out the existence of allocations for which we can beneficially increase utilization of the aggregate resources, or from another point of view, allocations for which we can assign more agents to objects.

Given $R \in \mathcal{R}$, we say that $\mu \in \mathcal{F}$ is *strongly wasteful* if there exists an *unassigned* agent who prefers to be assigned some object under some terms and there is $\nu \in \mathcal{F}$ that achieves this without harming any other agent. That is, there are $i \in N$, $x \in X(i)$, and $\nu \in \mathcal{F}$ such that *a*) $x = \nu(i)\ P_i\ \mu(i) = \varnothing$, and *b*) for each $j \in N$, $\nu(j)\ R_j\ \mu(j)$. If it is not strongly wasteful, then μ **weakly non-wasteful.**[12]

A rule φ is *non-wasteful* if, for each $R \in \mathcal{R}$, $\varphi(R)$ is non-wasteful. We define weak non-wastefulness of φ similarly.

Pareto-domination One allocation (weakly) **Pareto-dominates** another if each agent finds the first to be at least as desirable as the second. That is, for each $R \in \mathcal{R}$, and each pair $\mu, \nu \in \mathcal{F}$, we say that μ Pareto-dominates ν at R if for each $i \in N$, $\mu(i)\ R_i\ \nu(i)$.[13] The Pareto-domination relation is reflexive and transitive but not complete. We say that a pair of allocations is **Pareto-comparable** if they can be compared according to the Pareto-relation.

A pair of allocations is **Pareto-connected** (within the individually rational and non-wasteful set) if there is a sequence of individually rational and weakly non-wasteful allocations starting at one and ending at the other such that successive allocations are Pareto-comparable. That is, for each pair $\mu, \mu' \in \mathcal{F}$, μ and μ' are Pareto-connected if there exists a sequence of individually rational and weakly non-wasteful allocations, $(\mu_k)_{k=0}^{K}$, with $\mu_0 \equiv \mu$ and $\mu_K \equiv \mu'$, such that for every $k \in \{1, \ldots, K\}$, μ_k and μ_{k-1} are Pareto-comparable.

For each $R \in \mathcal{R}$ and each pair $\nu, \mu \in \mathcal{F}$ we say that μ **strictly Pareto-dominates** ν at R if it Pareto-dominates ν and there is $i \in N$ such that $\mu(i)\ P_i\ \nu(i)$. If $\mu \in \mathcal{F}$ is such that there is no allocation that strictly Pareto-dominates it at R, we say that μ is **Pareto-efficient** at R.

For each pair of rules φ and φ', φ (weakly) *Pareto-dominates* φ' if, for each $R \in \mathcal{R}$, $\varphi(R)$ Pareto-dominates $\varphi'(R)$ at R. If φ' Pareto-dominates φ and for some $R \in \mathcal{R}$, $\varphi(R)$ strictly Pareto-dominates $\varphi(R)$ at R, then φ strictly Pareto-dominates φ. They are *Pareto-connected* if for each $R \in \mathcal{R}$, $\varphi(R)$ and $\varphi'(R)$ are Pareto-connected at R. We say that φ is *Pareto-efficient* if for each $R \in \mathcal{R}$, $\varphi(R)$ is Pareto-efficient at R. A pair of rules φ and φ' are **welfare-equivalent** if, for each $R \in \mathcal{R}$ and for each $i \in N$, $\varphi_i(R)\ I_i\ \varphi'_i(R)$. If they are not welfare-equivalent, we say that they are **welfare-distinct**.

Strategy-proofness A rule is **strategy-proof** if no agent can benefit by misreporting his preferences, no matter what other agents do. That is, φ is strategy-proof if for each $R \in \mathcal{R}$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, $\varphi_i(R)\ R_i\ \varphi_i(R'_i, R_{-i})$.

Extending this concept to groups of agents, a rule is **group strategy-proof** if no group of agents can benefit by misreporting their preferences. That is, φ is group strategy-proof if for each $R \in \mathcal{R}$ and each $S \subseteq N$, there is no $R'_S \in \times_{i \in S} \mathcal{R}_i$, such that for each $i \in S$,

[12] Just as non-wastefulness reduces to Balinski-Sönmez-non-wastefulness for strict preferences and capacity based feasibility, weak non-wastefulness reduces to the corresponding notion defined by Ehlers and Klaus (2014).

[13] In this case, Some authors say that μ *weakly* Pareto-dominates ν. However, since this is the main form of Pareto-dominance that we consider, we drop the qualifier.

$\varphi_i(R'_S, R_{-S})\ R_i\ \varphi_i(R)$ and for some $i \in S$, $\varphi_i(R'_S, R_{-S})\ P_i\ \varphi_i(R)$. If we only rule out R'_S such that for each $i \in S$, $\varphi_i(R'_S, R_{-S})\ P_i\ \varphi_i(R)$, then φ is **weakly group strategy-proof**.

3 The Individually Rational and Non-wasteful Set

The set of individually rational and weakly non-wasteful allocations turns out to have a nice structure. Note that the Pareto-connectedness relation is reflexive and symmetric. Moreover, restricted to the set of individually rational and weakly non-wasteful allocations, it is an equivalence relation as it is also transitive. Therefore, the Pareto-connectedness relation partitions the set of individually rational and weakly non-wasteful allocations, where each component of the partition consists of allocations that are Pareto-connected to one another.

Our main result about allocations of a given economy is the following Structure Lemma, which states that the set of agents who are assigned an object is the same for every allocation in a given component of this partition.

Lemma 1 (Structure Lemma). *For each $R \in \mathcal{R}$ and each pair $\mu, \nu \in \mathcal{F}$ that are individually rational, weakly non-wasteful, and Pareto-connected, the set of assigned agents is the same at each of μ and ν. That is, $\nu_N = \mu_N$.*

Proof. We begin with the following claim.

Claim: For each pair $\mu, \nu \in \mathcal{F}$, if μ is individually rational and weakly non-wasteful and ν Pareto dominates μ, then the conclusion of the Lemma holds and ν is individually rational and weakly non-wasteful itself.

Proof of claim: Since ν Pareto-dominates μ, which is individually rational, for each $i \in N$, $\nu(i)\ R_i\ \mu(i)\ R_i\ \varnothing$. Thus ν is individually rational.

For each $i \in N$, if $\mu(i) \neq \varnothing$, then by the "no indifference with $\varnothing$" assumption on $\mathcal{R}_i$, $\mu(i)\ P_i\ \varnothing$. Since $\nu(i)\ R_i\ \mu(i)$, $\nu(i) \neq \varnothing$. Thus, $\mu_N \subseteq \nu_N$.

If $\mu_N \subsetneq \nu_N$ then there is $i \in \nu_N \setminus \mu_N$. So $\mu(i) = \varnothing$ and $\nu(i) \neq \varnothing$. Since $\nu(i)\ R_i\ \mu(i) = \varnothing$, by the "no indifference with $\varnothing$" assumption on $\mathcal{R}_i$, $\nu(i)\ P_i\ \mu(i) = \varnothing$. This contradicts the assumption that μ is weakly non-wasteful. Therefore, $\nu_N = \mu_N$.

Next, we show that ν is weakly non-wasteful. If ν is strongly wasteful, then there exists $i \in N$ and $\gamma \in \mathcal{F}$ such that $\nu(i) = \varnothing$, $\gamma(i)\ P_i\ \nu(i)$, $|\gamma| > |\nu|$, and for each $j \in N$, $\gamma(j)\ R_j\ \nu(j)$. Since $\mu_N = \nu_N$, $\mu(i) = \varnothing$ and since ν Pareto-dominates μ, $\gamma(i)\ P_i\ \mu(i)$ and for every $j \in N$, $\gamma(j)\ R_j\ \mu(j)$. Also, since $|\mu| = |\nu|$, $|\gamma| > |\mu|$. This contradicts the weak non-wastefulness of μ. Thus, ν is weakly non-wasteful. This completes the proof of the claim.

To complete the proof of the Lemma, suppose $\mu, \nu \in \mathcal{F}$ are individually rational, weakly non-wasteful, and Pareto-connected allocations. Then, there exists a sequence of individually rational and weakly non-wasteful allocations, $\left(\mu^k\right)_{k=0}^K$, with $\mu^0 = \mu$ and $\mu^K = \nu$, such that for every $k \in \{1, \ldots, K\}$, μ^k and μ^{k-1} are Pareto-comparable. That is, either μ^k Pareto-dominates μ^{k-1} or vice versa. In either case, the claim establishes that $\mu^k_N = \mu^{k-1}_N$. Thus, the Lemma is proved. □

If we restrict attention to non-wasteful allocations, we can say more about their relation. The following lemma strengthens the hypothesis of the claim in the proof of the Structure Lemma from weak non-wastefulness to non-wastefulness and draws the stronger conclusion that the Pareto-dominating allocation assigns each object to the same number of agents. The proof is in Appendix A.2.

Lemma 2. *For each $R \in \mathcal{R}$ and each pair $\mu, \nu \in \mathcal{F}$ if μ is individually rational and non-wasteful and ν Pareto-dominates μ, then the same set of agents is assigned an object at ν as at μ and each object is assigned to the same number of agents. That is, $\nu_N = \mu_N$ and for each $o \in O$, $|\nu(o)| = |\mu(o)|$.*

Lemma 2 is reminiscent of the Rural Hospitals Theorem, which concludes that at each *stable* allocation the same agents are matched and each object is matched to the same number of agents. In environments where the feasible sets are augmented with choice functions such that an "agent-optimal stable allocation" is guaranteed to exist, Lemma 2 implies the Rural Hospitals Theorem. We discuss this connection in Section 5, where we consider stability.

4 Strategy-proof Pareto-improvement

We now present our main result. Given an individually rational and weakly non-wasteful benchmark rule, we are interested in strategy-proof rules that Pareto-improve upon it. When a mechanism designer is constrained this way in his choice of a strategy-proof rule, Theorem 1 says that his problem has a unique solution (in terms of welfare) if it has one at all.

Theorem 1. *For each individually rational and weakly non-wasteful benchmark rule $\underline{\varphi}$, there is at most one strategy-proof rule, in welfare terms, that Pareto-dominates it.*

In order to prove Theorem 1, we prove two more results that are of independent interest. These results, along with some additional results that we show later in this section shed light on the structure of the set of strategy-proof rules.

Lemma 3 states that two welfare-distinct individually rational rules for which, at every preference profile, the set of assigned agents coincide cannot both be strategy-proof.

Lemma 3. *Let φ and φ' be individually rational rules for which there is $i \in N$ such that they are not welfare-equivalent for i. Suppose, further, that for each $R \in \mathcal{R}$, $\varphi_i(R) = \varnothing$ if and only if $\varphi'_i(R) = \varnothing$. At most one of φ and φ' is strategy-proof.*

Proof. Since φ and φ' are not welfare-equivalent for i, there is $R \in \mathcal{R}$ such that $\neg(\varphi_i(R)\ I_i\ \varphi'_i(R))$. Without loss of generality, suppose that $\varphi_i(R)\ P_i\ \varphi'_i(R)$. We prove the lemma by assuming that φ is strategy-proof and concluding from this that φ' is not strategy-proof.

Let $x \equiv \varphi_i(R)$ and $y \equiv \varphi'_i(R)$. By individual rationality of φ', $y\ R_i\ \varnothing$. Since $x\ P_i\ y$, we have $x\ P_i\ \varnothing$. Thus $x \neq \varnothing$. By the hypothesis of the Lemma, $y \neq \varnothing$ so by the "no indifference with $\varnothing$" assumption, $y\ P_i\ \varnothing$.

Since $x\ P_i\ y$, by richness of $\mathcal{R}_i$, there is $R'_i \in \mathcal{R}_i$ such that *a*) $x\ R'_i\ \varnothing\ R'_i\ y$, and *b*) for each $z \in X(i)$, if $z\ P'_i\ \varnothing$, then $z\ P_i\ y$.

Let $z \equiv \varphi_i(R'_i, R_{-i})$. Since φ is strategy-proof, $z\ R'_i\ x$. By definition of R'_i, $x\ R'_i\ \emptyset$. By the "no indifference with $\emptyset$" assumption, $x\ P'_i\ \emptyset$, so $z \neq \emptyset$.

Let $z' \equiv \varphi'_i(R'_i, R_{-i})$. By individual rationality of φ', $z'\ R'_i\ \emptyset$. By definition of R'_i, we have $\emptyset\ R'_i\ y$. By the "no indifference with $\emptyset$" assumption, $\emptyset\ P'_i\ y$. Thus, $z'\ P'_i\ y$. By the hypothesis of the lemma, since $z \neq \emptyset$, we have $z' \neq \emptyset$. Thus, by definition of R'_i, $z'\ P_i\ y$. Then $\varphi'_i(R'_i, R_{-i}) = z'\ P_i\ y = \varphi'_i(R)$, so φ' is not strategy-proof. □

Theorem 1 follows from the next result, which states that among a set of Pareto-connected, individually rational, and weakly non-wasteful rules, at most one of them can be strategy-proof. It is a consequence of the Structure Lemma and Lemma 3.

Proposition 2. *If a pair of distinct strategy-proof, weakly non-wasteful, and individually rational rules are Pareto-connected, then they are welfare-equivalent.*

Proof. Consider a pair φ and φ' of weakly non-wasteful and individually rational rules. If they are Pareto-connected, then by the Structure Lemma, for each $R \in \mathcal{R}$ and each $i \in N$, $\varphi_i(R) = \emptyset$ if and only if $\varphi'_i(R) = \emptyset$. If they are distinct, then by Lemma 3, either they are welfare-equivalent or at most one of them is strategy-proof. Since both are strategy-proof, they must be welfare-equivalent. □

An immediate implication of Proposition 2 is that no strategy-proof, individually rational, and weakly non-wasteful rule is strictly Pareto-dominated by a strategy-proof rule.

4.1 Pareto-improvements over a wasteful rule

It is clear that Theorem 1 does not generally hold for strongly wasteful benchmark rules. If, for instance, the constant rule that assigns $\emptyset$ to every agent is the benchmark, then every individually rational rule dominates it. There may be, however, many strategy-proof and individually rational rules: every serial dictatorship meets these requirements.[14]

We can still say something about strategy-proof rules that Pareto-dominate one another. An implication of Lemma 3 is that among individually rational and strategy-proof rules, a strict Pareto-improvement involves leaving fewer agents unassigned. Theorem 1 says that it is not possible to strictly Pareto-improve, in a strategy-proof way, on a rule that is allocating an adequate amount of the available resources, even if that is done in an Pareto-inefficient way. On the other hand, the following result says that if a strategy-proof rule is ever to be strictly Pareto-improved upon by another strategy-proof rule, the Pareto-improvement comes from allocating resources to *more* agents and not (by Lemma 3) from better allocation to the *same* agents.

Theorem 2. *Let φ and φ' be a pair of strategy-proof rules. If φ is individually rational and φ' strictly Pareto-dominates φ, then at each $R \in \mathcal{R}$, $\varphi(R)_N \subseteq \varphi'(R)_N$, and at some $\tilde{R} \in \mathcal{R}$, $\varphi(\tilde{R})_N \subsetneq \varphi'(\tilde{R})_N$.*

[14] In settings where there are complementarities in feasibility, however, a serial dictatorship rule may not be individually rational.

Proof. Since φ' Pareto-dominates φ, which is individually rational, φ' is also individually rational. Furthermore, for each $R \in \mathcal{R}$ and each $i \in N$, if $\varphi_i(R) \neq \emptyset$, then $\varphi'_i(R)\ R_i\ \varphi_i(R)\ P_i\ \emptyset$ so $\varphi'_i(R) \neq \emptyset$. Thus, $\varphi'(R)_N \supseteq \varphi(R)_N$.

If for each $R \in \mathcal{R}$, $\varphi'(R)_N = \varphi(R)_N$, then, by Lemma 3, this contradicts the assumption that both φ and φ' are strategy-proof. Thus, there is some $\tilde{R} \in \mathcal{R}$ such that $\varphi'(R)_N \supsetneq \varphi(R)_N$. □

Restricting attention to strategy-proof rules, the strategy-proofness-constrained Pareto-frontier consists of strategy-proof rules that are not Pareto-dominated by another strategy-proof rule. Theorem 1 tells us that every individually rational and weakly non-wasteful strategy-proof rule is on this frontier. Theorem 2 says that every rule below this frontier leaves more agents unassigned than any point towards the frontier above it. We might ask whether every point on the frontier is weakly non-wasteful. Proposition 3 says that this is not the case. That is, the strategy-proofness-constrained Pareto-frontier may contain strongly wasteful rules. While a full description of such a rule is not straightforward, we prove that one exists.

Proposition 3. *There exists a strategy-proof and strongly wasteful rule that is not Pareto-dominated by any welfare-distinct strategy-proof rule.*

In proving prove Proposition 3 (Appendix A.3) we actually show more: there exists a strongly wasteful and *group* strategy-proof rule that is not Pareto-dominated by a strategy-proof rule. Thus, even among group strategy-proof rules, weak non-wastefulness is not a necessary condition for a rule to be strategy-proofness-constrained Pareto-efficient.

5 Choice and Stability

The purpose of this section is to demonstrate the implications of our results for market design applications based on the matching with contracts model. In such applications, there is more information available about each object than just the feasible sets. The constraints imposed by this information is what, in many applications, keeps the benchmark rule below the Pareto-frontier. These might be priorities over agents as in the school choice model of Abdulkadiroğlu and Sönmez (2003), objectives of the army as in Sönmez and Switzer (2013), and so on.

Since the goal of this section (and of Section 6) is demonstrative, we assume that each agent has strict preferences.[15] That is, we restrict attention to economies in $\mathcal{P}$.

We model the extra information about how feasible sets are prioritized by associating each $o \in O$ with a **choice correspondence**, $\boldsymbol{C_o : 2^{X(o)} \rightrightarrows 2^{X(o)}}$, such that *a*) for each $Y \subseteq X(o), C_o(Y) \subseteq 2^Y$, and *b*) range$(C_o) = F_o$. Condition (*a*) says that from any set C_o must select only subsets of it, while Condition (*b*) says that the feasible sets are exactly those that are chosen from *some* set. To satisfy Condition (*b*), it would suffice, for instance, to select every feasible set from itself.[16]

[15] See Bogomolnaia et al. (2005) and Erdil and Ergin (2015) for more on the problems that arise when modeling indifference in such settings.

[16] An alternative approach is to start with C_o as the primitive and define F_o to be its range.

Our model is very close to the matching with contracts model (Hatfield and Milgrom, 2005) except that we associate each object with a choice correspondence rather than a choice function. We are back in the matching with contracts framework if we require that, for each $o \in O$, C_o is single-valued, or is a *choice function*. Since applications like school choice (Abdulkadiroğlu et al., 2009) which involve "weak priorities" are better modeled with choice correspondences, we adopt this more general approach.[17]

We place two restrictions on choice correspondences. The first says that the choices from every set should be at least as large as every choice from a subset. We say that C is **size monotonic** if, for each $o \in O$ and each pair of finite $Y, Y' \subseteq X(o)$ such that $Y \subseteq Y'$, for each $Z \in C_o(Y)$ and each $Z' \in C_o(Y')$, $|Z| \leq |Z'|$.[18] [19]

The second restriction is that if a set is among those chosen from a larger set, it ought to be among what is chosen from itself. We say that C is **idempotent** if, for each $o \in O$, and each $Y \in \text{range}(C_o)$, $Y \in C_o(Y)$.

Unlike the rest of the literature on matching with contracts, we do not assume that objects' choice satisfy the *irrelevance of rejected contracts* (IRC) condition of Aygün and Sönmez (2013), which is equivalent to the *weak axiom of revealed preference* as observed by Alva (2016). Idempotence is weaker than IRC.

Stability An allocation is stable if no set of agents can drop their assignments in favor of being assigned to a new object under some terms that the object would "choose." That is $\mu \in \mathcal{F}$ is *stable* if it is individually rational[20] and there are no $o \in O$ and $Y \subseteq X(o) \setminus \mu(o)$ such that a) for each $i \in N, |Y(i)| \leq 1$, b) for each $y \in Y, y \; P_{y_N} \; \mu(y_N)$, c) $\mu(o) \notin C_o(\mu(o) \cup Y)$, and d) there is $Z \in C_o(\mu(o) \cup Y)$ such that $Y \subseteq Z$. Condition (a) says that the set Y contains at most one element associated with each agent. Condition (b) says that the agent involved in each element of Y finds it preferable to his assignment at μ. These are familiar conditions from the definition of stability for choice functions. Since we are concerned with choice correspondences, the remainder of the definition needs to be broken into two parts. The first, Condition (c), says that $\mu(o)$ is not among what is chosen when Y is available. The second, Condition (d), says that there is some chosen set that contains Y.[21] The standard definition of stability typically does not include Condition (c) since it is implied by Condition (d) when choice correspondences satisfy IRC.

Stability is relevant if the choice correspondences represent more than feasibility constraints: they may represent the rights of agents with regards to the objects or particular design goals of the policy maker. A stable rule is thus a natural candidate for a benchmark rule that the mechanism designer may need to Pareto-improve upon. Since Theorem 1 only applies to weakly non-wasteful benchmarks, such a rule would have to be weakly non-wasteful to invoke it. In many applications, like school choice where

[17]To our knowledge, the analysis of the school choice model by Erdil and Kumano (2014) is the first instance where choice correspondences have been used in a matching framework.

[18]The only role that size monotonicity plays in our analysis is in showing that every stable allocation is also non-wasteful.

[19]Setting $Y = Y'$, size monotonicity implies that for each pair $Z, Z' \in C_o(Y)$, $|Z| = |Z'|$.

[20]Individual rationality as defined in Section 2.1 accounts for agents' preferences while feasibility as defined in Section 2 accounts for objects' choice correspondences.

[21]If, for each $o \in O$, C_o is single-valued, this definition is equivalent to the standard definition of stability.

there is more structure on the choice correspondences, it is obvious that stability implies non-wastefulness, which in turn implies weak non-wastefulness. However, without any restrictions on choice correspondences, this may not be the case.[22] Our assumption that choice correspondences are size monotonic and idempotent ensure that this implication holds. The complexity of proving this implication is due to the fact that we have not assumed IRC.

Before showing that stability implies non-wastefulness, we start with a definition and a lemma. For each $P \in \mathcal{P}$, each $\mu \in \mathcal{F}$ and each $o \in O$, let $Y_o^\mu(P)$ be the elements of $X(o)$ such that the involved agent prefers it to what he is assigned at μ. That is, $Y_o^\mu(P) \equiv \{x \in X(o) : x \; P_{x_N} \; \mu(x_N)\}$.

Lemma 4. *Suppose that C is size monotonic and idempotent. For each $P \in \mathcal{P}$, each $\mu \in \mathcal{F}$ such that μ is stable at P, each $o \in O$, each finite $Y \subseteq Y_o^\mu(P)$, and each $Z \in C_o(\mu(o) \cup Y)$, $|Z| = |\mu(o)|$.*

Proof. We proceed by induction over subsets of $Y_o^\mu(P)$. Let $Y \subseteq Y_o^\mu(P)$.

For the base case, where $Y = \varnothing$, since $\mu(o) \in \text{range}(C_o)$, by idempotence of C, $\mu(o) \in C_0(\mu(o) \cup Y)$ and by size monotonicity of C, for each $Z \in C_o(\mu(o))$, $|Z| = |\mu(o)|$.

As an induction hypothesis, assume that for each $Y' \subsetneq Y$ and each $Z \in C_o(\mu(o) \cup Y')$, $|Z| = |\mu(o)|$. Equivalently, for each $T \subseteq \mu(o) \cup Y$ such that $Y \not\subseteq T$, for each $Z \in C_o(\mu(o) \cup T)$, $|Z| = |\mu(o)|$.

The induction step is to show that for each $Z \in C_o(\mu(o) \cup Y)$, $|Z| = |\mu(o)|$. Let $Z \in C_o(\mu(o) \cup Y)$. *1)* By idempotence of C, $Z \in C_o(Z)$. *2)* Since $Z \subseteq \mu(o) \cup Z$, by size monotonicity of C and item (1) above, for each $Z' \in C_o(\mu(o) \cup Z)$, $|Z| \leq |Z'|$. *3)* By stability of μ, either $Y \not\subseteq Z$ or $\mu(o) \in C_o(\mu(o) \cup Y)$. If $\mu(o) \in C_o(\mu(o) \cup Y)$, then by size monotonicity of C, for each $Z \in C_o(\mu(o) \cup Y)$, $|Z| = |\mu(o)|$, concluding the proof. Thus, we consider the case where $Y \not\subseteq Z$. *4)* By the induction hypothesis and item (3) above, for each $Z' \in C_o(\mu(o) \cup Z)$, $|Z'| = |\mu(o)|$. *5)* By items (2) and (4) above, $|Z| \leq |\mu(o)|$. *6)* Since $\mu \in \mathcal{F}$, by idempotence of C, $\mu(o) \in C_o(\mu(o))$. *7)* By size monotonicity of C and item (6) above, since $Z \in C_o(\mu(o) \cup Y)$, $|\mu(o)| \leq |Z|$. By items (5) and (7) above, $|Z| = |\mu(o)|$. □

We now show that every stable allocation is non-wasteful.

Proposition 4. *Suppose that C is size monotonic and idempotent. For each $P \in \mathcal{P}$ and each $\mu \in \mathcal{F}$, if μ is stable at P, then μ is non-wasteful at P.*

Proof. Suppose that μ is wasteful. Then there are $o \in O$ and $\nu \in \mathcal{F}$ such that $|\nu(o)| > |\mu(o)|$ and for each $y \in \nu(o) \setminus \mu(o)$, $y \; P_{y_N} \; \mu(y_N)$. Let $Y \equiv \nu(o) \setminus \mu(o)$. Since $Y \subseteq Y_o^\mu(P)$ and Y is finite, by Lemma 4, for each $Z' \in C_o(\mu(o) \cup Y)$, $|Z'| = |\mu(o)|$. However, $\nu(o) \subseteq \mu(o) \cup Y$. So, by size monotonicity of C, for each $Z \in C_o(\nu(o))$ and each $Z' \in C_o(\mu(o) \cup Y)$, $|Z| \leq |Z'| = |\mu(o)|$. Since $\nu \in \mathcal{F}$, $\nu \in F_o = \text{range}(C_o)$. So by idempotence of C, $\nu(o) \in C_o(\nu(o))$. Thus, $|\nu(o)| \leq |\mu(o)|$. This contradicts the definition of ν. □

[22] Let $N \equiv \{i_1, i_2\}$, $T \equiv \{t_1, t_2\}$, and $O \equiv \{o\}$. Since O is a singleton, we suppress it in the tuples that make up $X(o) \equiv \{(i_1, t_1), (i_1, t_2), (i_2, t_1)\}$. Let C_o be such that for each $Y \subseteq X(o)$, if $(i_1, t_1) \in Y$ then $C_o(Y) = \{(i_1, t_1)\}$ and otherwise $C_o(Y) = \{Y\}$. Clearly, C_o is not size monotonic. Let $P \in \mathcal{P}$ be such that $(i_1, t_2) \; P_{i_1} \; (i_1, t_1) \; P_{i_1} \; \varnothing$ while $(i_2, t_1) \; P_{i_2} \; \varnothing$. At these preferences, $\mu \in \mathcal{F}$ such that $\mu_{i_1} = (i_1, t_1)$ and $\mu(i_2) = \varnothing$ is stable. However, it is strongly wasteful as there is $\nu \in \mathcal{F}$ such that $\nu(i_1) = (i_1, t_2)$ and $\nu(i_2) = (i_2, t_1)$, which makes both agents better off and $|\nu| > |\mu|$.

Now that we have established that stability implies non-wastefulness, we have the following corollary which follows from Theorem 1 and Proposition 4.

Corollary 1. *If C is size monotonic and idempotent, then a stable rule is Pareto-dominated by at most one strategy-proof rule.*

An implication of Corollary 1 is that if a rule is stable and itself strategy-proof, then it is not Pareto-dominated by any other stable and strategy-proof rule. Hirata and Kasuya (2015) show that this result holds for single valued choice functions that satisfy the IRC condition. In fact, they show that for such choice functions, there is at most one strategy-proof and stable rule. This latter result, unfortunately does not hold in our setting as demonstrated by the following example.

Example 4. *There may be more than one strategy-proof and stable rule.*

Hatfield and Kominers (2014b) show that in a "slot-specific priorities" setting, if the order of precedence in which slots are filled depends on the set of agents that are being considered, then the choice function may violate the IRC condition. Here, we provide an example where the ranking according to which agents are chosen depends upon the agents being compared.

Consider a situation where there are two positions for teachers at one school o. There are four candidates $N \equiv \{m_1, m_2, p_1, p_2\}$. There is only one term each teacher can be hired under, so T is a singleton. Let C_o be a single-valued choice correspondence described by the following process.

Two of the teachers, m_1 and m_2, specialize in math and the other two, p_1 and p_2, specialize in physics. The math teachers are able to teach physics but not as well as the physics teachers, and vice versa. As overall teachers, m_2 is the best, followed by p_1, m_1, and p_2 in that order.

If more math specialists are being considered than physics specialists, then the math faculty are more likely to weigh in, so the positions are filled according to how good a math teacher the candidates are. Vice versa if there are more physics specialists. If there are equal numbers of math and physics specialists, the candidates are compared based on their overall teaching ability.

Below, the boxed elements show the choices from each set of candidates.

$$\{m_1, \boxed{m_2, p_1}, p_2\}$$

$$\{\boxed{m_1, m_2}, p_1\} \quad \{\boxed{m_1, m_2}, p_2\} \quad \{m_1, \boxed{p_1, p_2}\} \quad \{m_2, \boxed{p_1, p_2}\}$$

$$\{\boxed{m_1, m_2}\} \quad \{\boxed{m_1, p_1}\} \quad \{\boxed{m_2, p_1}\} \quad \{\boxed{m_1, p_2}\} \quad \{\boxed{m_2, p_2}\} \quad \{\boxed{p_1, p_2}\}$$

$$\{\boxed{m_1}\} \quad \{\boxed{m_2}\} \quad \{\boxed{p_1}\} \quad \{\boxed{p_2}\}$$

For each pair $\pi \subseteq N$ such that $|\pi| \leq 2$, let $\mu^\pi \in \mathcal{F}$ be such that it assigns agents in π to o and leaves the others unassigned. That is, $\mu^\pi(o) = \pi$ and for each $i \in N \setminus \pi, \mu^\pi(i) = \varnothing$.

For each $P \in \mathcal{P}$, let $G(P) \equiv \{i \in N : o \; P_i \; \varnothing\}$.

Consider the rule φ defined by setting, for each $P \in \mathcal{P}$,

$$\varphi(P) \equiv \begin{cases} \mu^{\{m_1,m_2\}} & \text{if } \{m_1,m_2\} \subseteq G(P), \\ \mu^{\{p_1,p_2\}} & \text{if } \{p_1,p_2\} \subseteq G(P) \text{ and } \{m_1,m_2\} \not\subseteq G(P), \\ \mu^{G(P)} & \text{otherwise.} \end{cases}$$

We show, in Appendix A.4 that φ is stable and strategy-proof.

Now, consider the rule φ' defined by setting, for each $P \in \mathcal{P}$,

$$\varphi'(P) \equiv \begin{cases} \mu^{\{p_1,p_2\}} & \text{if } \{p_1,p_2\} \subseteq G(P), \\ \mu^{\{m_1,m_2\}} & \text{if } \{m_1,m_2\} \subseteq G(P) \text{ and } \{p_1,p_2\} \not\subseteq G(P), \\ \mu^{G(P)} & \text{otherwise.} \end{cases}$$

Since it is symmetric to φ, φ' is also stable and strategy-proof. In fact, both of these rules are group strategy-proof.

Notice that C satisfies our assumptions of size monotonicity and idempotence, but violates IRC. Furthermore, neither of the above rules can be generated by a "cumulative offer" algorithm as the cumulative offer algorithm, regardless of the order, outputs the unstable allocation $\mu^{\{m_2,p_1\}}$ at this preference profile.

Since the choice of "tie breaker" is endogenous, C violates the IRC condition. While this may appear strange, for the school choice model with "weak priorities", Ehlers and Erdil (2010) provide an example where fixed tie breaking implies a loss of efficiency. Thus, it may be worthwhile to consider such choice functions.

5.1 The Agent-Optimal Stable Rule and the Rural Hospitals Theorem

Since we do not make assumptions about choice correspondences beyond idempotence and size monotonicity, the existence of a stable allocation is not guaranteed, let alone the lattice structure of the stable set. Nonetheless, suppose that $C = (C_o)_{o\in O}$ is such that for every $P \in \mathcal{P}$, there exists a stable allocation that Pareto-dominates every other stable allocation. That is, C is such that the agent-optimal stable allocation always exists. Then the *agent-optimal stable rule*, φ^{AOS}, is well defined.

Another property of the set of stable allocations that our assumptions do not guarantee is the so-called "Rural Hospitals Theorem." It states that the conclusion of Lemma 2 holds for every pair of stable allocations. By Lemma 2, if φ^{AOS} is well defined, then at every $P \in \mathcal{P}$, the entire stable set is Pareto-comparable to φ^{AOS}.

It turns out that *a*) φ^{AOS} being well defined, *b*) the Rural Hospitals Theorem, and *c*) the existence of a stable and strategy-proof rule are intimately connected. Hatfield and Kojima (2010) provide conditions on C that guarantee each of the above when, for each $o \in O$, C_o is single-valued.[23] We show below that the first statement is equivalent to the combination of the second and third. The proofs of the following propositions are in Appendix A.4.

[23]They show, given IRC, that "unilateral substitutes" is sufficient for the first and, with the addition of size monotonicity, is sufficient for the second and third.

Proposition 5. *Suppose that C is size monotonic and idempotent. If φ^{AOS} is well defined, then a) φ^{AOS} is the unique stable and strategy-proof rule, and b) the Rural Hospitals Theorem holds.*

Our results shed more light on what drives the Rural Hospitals Theorem. The first is non-wastefulness. The Lemma 2 says, regardless of stability, that at any pair of Pareto-connected allocations the conclusion of the Rural Hospital Theorem holds. The only additional thing required is that the stable set be Pareto-connected, which the existence of the agent-optimal stable matching guarantees.

Proposition 6. *Suppose that C is size monotonic and idempotent. If a) the Rural Hospitals Theorem holds and b) there exists a stable and strategy-proof rule φ, then $\varphi = \varphi^{AOS}$ and it is the unique stable and strategy-proof rule.*

When C is such that, for each $o \in O$, C_o is single valued, Hatfield and Kojima (2009) use the Rural Hospitals Theorem to establish that φ^{AOS} is weakly group strategy-proof. An implication of Proposition 5 is that whenever φ^{AOS} is well defined, and our conditions on choice correspondences hold, it is weakly group strategy-proof. Since the proof of Proposition 7 is a straightforward adaptation of the proof in Hatfield and Kojima (2009), we omit it.

Proposition 7. *Suppose that C is size monotonic and idempotent. If φ^{AOS} is well-defined, then it is weakly group strategy- proof.*

A consequence of weak group strategy-proofness is that φ^{AOS} is weakly Pareto-efficient over its range.[24] Since it has full range, this means that it is weakly Pareto-efficient.

Corollary 2. *Suppose that C is size monotonic and idempotent. If φ^{AOS} is well-defined, then for each $P \in \mathcal{P}$, there is no allocation that every finds better than $\varphi^{AOS}(P)$.*

6 Priority-augmented Allocation

In this section, we specialize our model to study the "priority-augmented" object allocation problem, which is identical to the school choice model of Abdulkadiroğlu and Sönmez (2003). In this specialized model, T is a singleton and for each $o \in O$, F_o is capacity based. Let q_o be the capacity for o. Additionally, o is associated with a "priority" order over N, denoted by $\succsim_o$, which is complete, transitive, and reflexive.[25] The **priority structure** is $\succsim \equiv (\succsim_o)_{o \in O}$.

The priority structure specifies certain "rights" that agents have with regards to the objects. Suppose that an agent prefers a particular object o to the one that he is assigned. If o is assigned to someone else who has *strictly* lower priority, then he has the right to protest this allocation. For each $\mu \in \mathcal{M}$ we say that μ **respects priorities** if no agent can protest on such grounds. That is, there is no pair $i, j \in N$ and $o \in O$ such that $\mu(i) = o$, $o \; P_j \; \mu(j)$, and $j \succ_o i$.

[24] We say that $\mu \in \mathcal{F}$ is weakly Pareto-efficient if there is no $\nu \in \mathcal{F}$ such that for each $i \in N$, $\nu(i) \; P_i \; \mu(i)$.

[25] That is, $\succsim_o$ is a weak order.

Respecting priorities and stability Interpreting respect for the priorities as a "fairness" constraint (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu et al., 2009), we are interested in rules that are individually rational, fair and non-wasteful. Here we define a choice correspondence for every object and show that respect for priorities alongside individual rationality and non-wastefulness is equivalent to stability with respect to these correspondences.

Given a priority structure, $\succsim$, for each $o \in O$, we define a choice correspondence, C_o as follows. For each $Y \subseteq N$,

$$C_o(Y) \equiv \begin{cases} \{Y\} & \text{if } Y \in F_o, \\ \{Z \subseteq Y : |Z| = q_o \text{ and for each } i \in Z \text{ and each } j \in Y \setminus Z, i \succsim_o j\} & \text{otherwise.} \end{cases}$$

That is, for each subset of agents, if it is feasible (that is, it contains no more than q_o elements), then the entire set is the only one that is chosen. If it is not, then all subsets that contain exactly q_o elements are chosen, except for ones that include agents of *strictly* lower priority than an excluded agent. Obviously, C is size monotonic and idempotent.

Proposition 8. *For each $P \in \mathcal{P}$, each $\mu \in \mathcal{F}$ is stable with respect to C if and only if it is individually rational, non-wasteful and respects the priorities $\succsim$.*

We prove this Proposition in Appendix A.5. It says that, among individually rational and non-wasteful rules, the requirement that a rule or allocation respect priorities is equivalent to the requirement that it be stable. Since they are equivalent, we speak of an allocation being stable rather than saying that it being individually rational, non-wasteful and respecting priorities.

When priorities are strict (that is, they contain no ties between agents), the set of stable allocations forms a lattice and the agent-optimal stable rule is well defined. However, when priorities contain ties, there may not exist a single stable allocation that Pareto-dominates every other stable allocation. Consequently, there may be several allocations that are not Pareto-dominated by any other stable allocation. Since φ^{AOS} is not well defined, a common approach to handling weak priorities in the literature on school choice is to use a "tie breaker" to form strict priorities from weak ones. Let $\tau \equiv (\tau_o)_{o \in O}$ be a profile of linear orders over N, one for each object. For each such τ, let $\succsim^{\tau}$ be the priorities **tie broken by τ**. That is, for each pair $i, j \in N$ such that $i \neq j$, $i \succ_o^{\tau} j$ if either *a*) $i \succ_o j$ or *b*) $i \sim_o j$ and $i \; \tau_o \; j$. Let $\mathcal{T}$ be the set of all such profiles of tie breakers.

The following proposition says that the set of stable allocations for a given profile of priorities is the union of the stable sets for every tie broken version of the priorities. Thus, the following proposition gives us a third way of identifying the same set, alongside *a*) non-wastefulness and respect of priorities and *b*) stability with regards to the priorities. The proof is in Appendix A.5.

Proposition 9. *For each $P \in \mathcal{P}$, and each $\mu \in \mathcal{F}$, μ is stable with respect to $\succsim$ if and only if there is $\tau \in \mathcal{T}$ such that μ is stable with respect to $\succsim^{\tau}$.*

Proposition 9 shows that the stable set with weak priorities is the union of lattices: the stable sets for the tie broken priorities, which are strict. The rural hospitals theorem applies to each of these lattices. If two of them intersect, then, by Lemma 2, it applies to their union.

Since the agent-optimal stable rule is well defined for strict priorities, given $\succsim$ and $\tau \in \mathcal{T}$, we define the agent-optimal stable rule for the priorities tie broken by τ as $\varphi^{AOS\tau}$. These are the rules studied by Abdulkadiroğlu et al. (2009).[26]

Ergin (2002) shows that, for strict priorities, unless they satisfy a restrictive condition that he calls "acyclicity", stability and efficiency are at odds. That is, unless priorities are acyclic, the agent-optimal stable rule is not Pareto-efficient. When priorities are weak, it is therefore clear that arbitrarily breaking ties could cause Pareto-inefficiency. In fact, Erdil and Ergin (2008) show that, no matter how ties are broken, the agent-optimal stable rule with tie broken priorities may not always select an allocation that is undominated by a stable allocation. Ehlers and Erdil (2010) further provide an example where there exists a stable, group strategy-proof, and Pareto-efficient rule while for no $\tau \in \mathcal{T}$ is $\varphi^{AOS\tau}$ Pareto-efficient.

6.1 Pareto-improving over a stable benchmark

Since $\varphi^{AOS\tau}$ may not be Pareto-efficient, Abdulkadiroğlu et al. (2009) considered the following question: For each $\tau \in \mathcal{T}$, is it possible to find a strategy-proof rule that Pareto-dominates $\varphi^{AOS\tau}$? They showed that this is impossible.[27] However, even if the answer were positive, the allocations chosen by the Pareto-dominating rule would not be stable themselves. The reason such a rule would pass muster in a setting with priorities is that it Pareto-dominates a stable rule: $\varphi^{AOS\tau}$. As we have explained above, there is nothing special about these rules, other than strategy-proofness, when priorities are weak. In fact, they may even select allocations that are Pareto-dominated by other stable allocations. The exercise is then to justify the choice of a rule on the grounds that it Pareto-dominates *some* stable allocation at every profile of preferences. If an agent were to protest the violation of his priority at some object, we may point to the Pareto-dominated stable allocation where this protest would be moot: every agent, including the protesting one, is at least as well off as at that stable allocation. If a rule Pareto-dominates some stable rule, we call it a **stable-dominating** rule.

Since φ^{AOS} is well defined for strict priorities, Theorem 1 implies the following, which is stronger than the known result that φ^{AOS} is the only stable and strategy-proof rule (Alcalde and Barberà, 1994).

Corollary 3. *If $\succsim$ consists of strict priorities, then φ^{AOS} is the unique strategy-proof stable-dominating rule.*

On the other hand, for weak priorities, it says less about stable-dominating rules. It only implies the following result.

[26] Ehlers and Klaus (forthcoming) show, for this specialized model, that even when there are no priorities if a rule is strategy-proof, satisfies some additional basic properties, and has desirable comparative statics—in the form of either "population monotonicity" or "resource monotonicity"—it would have to be φ^{AOS} for *some* priorities or choice functions.

[27] While Abdulkadiroğlu et al. (2009) show this result in a setting similar to ours where agents may rank $\varnothing$ anywhere in their preference relation, Kesten and Kurino (2015) have shown that this holds even when we restrict attention to the subdomain of preferences where $\varnothing$ is worse than every object.

Corollary 4. *No stable and strategy-proof rule, including for each $\tau \in \mathcal{T}$, $\varphi^{AOS\tau}$, is Pareto-dominated by any other strategy-proof rule. Further no pair of stable and strategy-proof rules is Pareto-comparable.*

Abdulkadiroğlu et al. (2009) have previously shown that for each $\tau \in \mathcal{T}$, $\varphi^{AOS\tau}$ is strategy-proofness-constrained Pareto-efficient. As Ehlers and Erdil (2010) have shown, $\varphi^{AOS\tau}$s are not the only stable and strategy-proof rules. Corollary 4 extends the result to all of these rules.

In what follows we say more about stable-dominating rules. When priorities are so weak as to be degenerate (in the sense of every agent having equal priority at each object), stability reduces to non-wastefulness, thus every Pareto-efficient allocation is also stable. On the other hand, if priorities are strict and every object has the same priority over agents, then there is a unique stable allocation and it is itself Pareto-efficient. The tension between stability and efficiency is thus dependent on the priority structure.

Ehlers and Erdil (2010) define a property of the priority structure that guarantees that every constrained efficient stable allocation is actually Pareto-efficient, where constrained efficiency of a stable allocation means that no other stable allocation Pareto-dominates it.[28] They say that $\succsim$ contains a *weak cycle* if is a distinct triple $i, j, k \in N$ and a distinct pair $x, y \in O$ such that *a*) (loop) $i \succsim_x j \succ_x k$ and $k \succsim_y i$, and *b*) (scarcity) there exist disjoint $N_x \subseteq N \setminus \{i, j, k\}$ and $N_y \subseteq N \setminus \{i, j, k\}$ such that for each $l \in N_x$, $l \succsim_x j$ and for each $l \in N_y$, $l \succsim_y i$, $|N_x| = q_x - 1$, and $|N_y| = q_y - 1$. They say that $\succsim$ is *strongly acyclic* if it does not contain any *weak cycles*.

While Ehlers and Erdil (2010) show that this condition makes guarantees about the set of constrained efficient stable allocations, under a slightly stronger condition, we can say more about all stable allocations. We define a *weak* cycle* exactly as a weak cycle except that we only require $N_y \subseteq N \setminus \{i, k\}$ rather than $N_y \subseteq N \setminus \{i, j, k\}$ in the scarcity condition. We say that $\succsim$ is *strongly* acyclic* if it does not contain an *weak* cycle*.

Proposition 10. *If $\succsim$ is strongly* acyclic and $\mu \in \mathcal{F}$ is stable, then every $\mu' \in \mathcal{F}$ that Pareto-dominates μ is also stable.*

In Appendix A.5 where we prove Proposition 10, we provide an example that shows that it does not hold under the slightly weaker condition of Ehlers and Erdil (2010).

An implication of Proposition 10 is that, if priorities are strongly* acyclic, then a rule that is stable-dominating is itself stable.

Corollary 5. *If $\succsim$ is strongly* acyclic then φ is stable-dominating if and only if φ is stable.*

While we cannot pin down all stable-dominating rules for general priority structures, we are able to say more if they are also group strategy-proof. For the following result we assume a stronger richness condition on the domain of preferences than the one in Section 2. We say that $\mathcal{P}$ is **rich*** if for each $i \in N$ and each $o \in o$, there is $P_i \in \mathcal{P}_i$ such that for each $o' \in O \setminus \{o\}$, $o \; P_i \; \varnothing \; P_i \; o'$.

[28] Ehlers and Westkamp (2011) provide conditions on priorities that guarantee the existence of a strategy-proof rule that selects a constrained efficient stable allocation for every profile of preferences.

Proposition 11. *If $\mathcal{P}$ is rich* and φ is group strategy-proof and stable-dominating, then φ is itself stable.*

We prove Proposition 11 in Appendix A.5.

The broad class of group strategy-proof and Pareto-efficient rules defined by Pycia and Ünver (2015) for the subdomain of $\mathcal{P}$ where agents rank $\varnothing$ below every $o \in O$ are readily extended to $\mathcal{P}$.[29] An implication of Proposition 11 is the following: Unless the priority structure is such that there exists a stable, group strategy-proof, and Pareto-efficient rule,[30] none of these rules (including the "top trading cycles" rules of Abdulkadiroğlu and Sönmez (2003)) can be justified on the basis being stable-dominating.

7 Applications

7.1 Auctions

To demonstrate the implications of our results in the auctions setting, we specialize our model. We let $T \subseteq \mathbb{R}_+$ be the possible payments by an agent so that $(i,o,t) \in X$ represents i receiving o while making a payment of t. Thus, agents have preferences over $O \times T$ and are monotone decreasing in the payment. For each $o \in O$, $F_o \equiv \{\{(i,o,t)\} : i \in N, t \in T\}$ so that each object may be consumed by at most one agent and any payment is possible. Thus, a feasible allocation is one where each agent consumes either $o \in O$ while paying $t \in T$ or $\varnothing$. Like Milgrom and Segal (2015) we consider situations where an agent who is not assigned an object does not make (or receive) a payment.[31]

As a simple example, we start with the case of a single object. The first price auction, interpreted as a direct mechanism, is a non-wasteful rule: the agent with the highest valuation receives o and pays his valuation while other agents receive $\varnothing$. It is well known to fail strategy-proofness. The second price auction, where the agent with the highest valuation receives o while paying the second highest valuation, is strategy-proof and Pareto-dominates the first price auction. Theorem 1 says that *the second price auction is the unique strategy-proof rule that Pareto-dominates the first price auction.*

For an arbitrary number of objects, the first price auction generalizes to the *maximum price Walrasian* rule.[32] Similarly, the second price auction generalizes to the *minimum price Walrasian* rule.[33] Since the minimum price Walrasian rule Pareto-dominates the maximum price Walrasian rule, which is in turn non-wasteful, we have the following corollary to Theorem 1.

---

[29]See Svensson (1999), Pápai (2000), Pycia and Ünver (2014) and Bade (2014) for more on group strategy-proof rules for such economies.

[30] Han (2015) provides necessary and sufficient conditions on priorities for this.

[31]Morimoto and Serizawa (2015) impose this as a requirement on rules instead and call it "no subsidy for losers."

[32] Demange and Gale (1985) define Walrasian equilibria for such economies and show that there is a "maximum" price Walrasian equilibrium.

[33]For quasi-linear settings, this is equivalent to the Vickery-Clarke-Groves (Vickrey, 1961; Clarke, 1971; Groves, 1973) mechanism (Leonard, 1983), which we know to be strategy-proof.

Corollary 6. *The minimum price Walrasian rule is the only strategy-proof rule to Pareto-dominate the maximum price Walrasian rule.*

Leonard (1983) shows that the minimum price Walrasian rule is the only strategy-proof Walrasian rule. Corollary 6 says more: it is the only strategy-proof rule that Pareto-dominates the maximum price Walrasian rule.

7.2 Reallocating objects from an endowment

Consider the model of Shapley and Scarf (1974). Their model relates to our general model in the following way. For each $o \in O$, F_o is capacity based with unit capacity, $|O| = |N|$, and T is a singleton. There is a reference allocation $\omega \in \mathcal{F}$ where, for each $i \in N$, ω_i is "i's endowment." Finally, each agent finds $\varnothing$ to be their worst possible assignment. That is, every $o \in O$ is ranked above $\varnothing$. Denote by $\underline{\mathcal{P}} \subseteq \mathcal{P}$ the subdomain of such preference profiles.

Due to the restriction that $\varnothing$ is ranked at the bottom of each preference relation and since $|O| = |N|$, it is reasonable to only consider allocations where no agent is ever assigned $\varnothing$. "Individual rationality" in the literature following Shapley and Scarf (1974) is defined to mean that each agent receives an object that he finds at least as desirable as his endowment. That is, the endowment places a lower bound on each agent's welfare. Translating this to the language that we use, this requires a rule to Pareto-dominate a benchmark rule that selects ω at every profile of preferences.

We consider the broader preference domain where each agent may rank $\varnothing$ anywhere in his preference relation. Consequently, the benchmark of each agent consuming his endowment may not be individually rational in our sense, since there may be an agent who finds his endowment to be worse than $\varnothing$. A better benchmark, then would allow such an agent to discard his endowment if he finds it to be worse than $\varnothing$. Of course, this could be wasteful: i's trash may be j's treasure. Often, in such situations, there are institutional rules that determine how these discarded resources are distributed among other agents. For instance, fantasy sports leagues have "waivers systems"[34] where a waiver order over the "fantasy owners" is fixed beforehand and any players that the owners discard (or "waive") can be picked up by other owners in this order. Another such example is the allocation of offices: if an employee empties his office, most workplaces allow other employees to take the option to take it in order of seniority.

Suppose, then, that we have such an order, $\succ$, along with an endowment, ω. The process described above is formalized as Algorithm 1 in Appendix B. This procedure takes the endowment, preference profile, and order as arguments and returns a non-wasteful and individually rational allocation the Pareto-dominates the endowment. If the status quo is to use such a procedure, then any change to a new rule would have to Pareto-dominate it. Thus, we have a benchmark rule $\underline{\varphi}^{\omega,\succ}$.

It turns out that there is, in fact, a strategy-proof rule that Pareto-dominates this benchmark. We define it as follows. For each $o \in O$, let $\tilde{\succ}_o$ be an ordering of N that agrees with $\succ$ on all agents but $\omega(o)$ and ranks $\omega(o)$ above all other agents. For each

[34] See, for instance, https://fantasybowl.com/rtfm/?topic=waiver-wire.

$P \in \mathcal{P}^N$, the pair $(P, \tilde{\succ})$ forms a priority-augmented object allocation problem. Let $\varphi(P)$ be the agent-optimal stable allocation at $(P, \tilde{\succ})$. It is easy to see that φ Pareto-dominates $\underline{\varphi}^{\omega,\succ}$. Since $\underline{\varphi}^{\omega,\succ}$ is individually rational and non-wasteful, Theorem 1 says that φ is actually the *only* strategy-proof rule that Pareto-dominates $\underline{\varphi}^{\omega,\succ}$.

Notice that $\tilde{\succ}$ contains "Ergin-cycles" (Ergin, 2002), so $\varphi(R)$ is not Pareto-efficient. From this and Theorem 1, we have the following corollary.

Corollary 7. *There is no Pareto-efficient and strategy-proof rule that Pareto-dominates $\underline{\varphi}^{\omega,\succ}$.*

On the subdomain $\underline{\mathcal{P}}$, for each ω, there is a unique core allocation (Roth and Postlewaite, 1977). A very well known result is that the rule that selects the core allocation at every preference profile in $\underline{\mathcal{P}}$ is actually the only strategy-proof and Pareto-efficient rule that Pareto-dominates ω (Ma, 1994). This is in stark contrast with Corollary 7. For each $P \in \underline{\mathcal{P}}$, $\underline{\varphi}^{\omega,\succ}(P) = \omega$. So, on the subdomain $\underline{\mathcal{P}}$, the requirement that a rule Pareto-dominate $\underline{\varphi}^{\omega,\succ}$ reduces to the requirement that it Pareto-dominate ω.

7.3 Market design

In this subsection, we explore the implications of our results for the several branches of the literature on market design: 1. school choice with diversity constraints; 2. matching with slot-specific priorities; and 3. matching with distributional goals.

7.3.1 School choice with diversity constraints

Public school districts are often concerned not only about parental preferences but also about the composition of the student body at each school when seeking an assignment of students to school seats, usually in terms of the diversity of the student body along dimensions such as race or socioeconomic background. This problem of *controlled school choice* (Abdulkadiroğlu and Sönmez, 2003) has been studied in a matching framework, where diversity objectives are modeled in the choice functions of schools by imposing type-specific quotas, reserves, or floors.

Weighing quotas versus reserves with two types of students, Hafalir et al. (2013) advocate that minority students should be prioritized over majority students (minority reserves or soft bounds) rather than setting an upper bound on the number of majority students (majority quotas or hard bounds), showing that the student-proposing deferred acceptance rule with minority reserves Pareto-dominates the student-proposing deferred acceptance rule with majority quotas. In effect, majority quotas can be wasteful if a school has empty seats but has reached its quota of majority students. Their result does not contradict our main result, however. If these quotas are treated as a feasibility requirement for an allocation, then our result applies, stating that there is no strategy-proof rule that Pareto-improves upon the student-proposing deferred acceptance rule with quotas and respects these quotas. However, interpreting the distributional goal as reserve requirements removes the "artificial" constraints of quotas, making the student-proposing deferred acceptance rule with quotas wasteful and allowing strategy-proof Pareto-improvements. Pareto-improvement and individual rationality of the student-proposing deferred acceptance rule with quotas implies that every student that is matched under the student-proposing deferred acceptance rule with quotas is matched under deferred acceptance with minority reserves. By Theorem 2, we can also conclude that at some preference profiles strictly more students must be matched.

Ehlers et al. (2014) studies controlled school choice with more than two types, and with lower-bound, as well as upper-bound, constraints. They define a notion of fairness and non-wastefulness that accounts for the diversity constraints as soft bounds, construct school choice functions that use lower and upper bounds on types as soft bounds, and show that these choice functions are "substitutable" and size monotonic.[35] Then the student-proposing deferred acceptance rule defines the agent-optimal stable rule, where stability is defined with respect to the constructed choice functions and is equivalent to fairness and non-wastefulness. Our results in Section 5 immediately apply, when feasible sets for the objects are defined as the range of the choice functions.[36]

An assignment that is stable under these choice functions with soft bounds can violate the diversity constraints when viewed as hard bounds even when there exists another stable assignment that satisfies the constraints. In fact, as both Ehlers et al. (2014) and Bo (forthcoming) point out, the student-proposing deferred acceptance rule with these constructed choice functions may pick a stable matching that violates constraints even when there is a stable matching that does not. They show that the college-proposing deferred acceptance rule with these constructed choice functions always picks the stable matching that is closest to satisfying the constraints amongst all stable matchings. However, it is not strategy-proof. One might then seek an answer to the following question: Is there a strategy-proof allocation rule that provides at least the same level of welfare to students as the college-proposing deferred acceptance rule, where the rule can violate stability with the aim of getting fewer violations of the diversity constraints? By Theorem 1, the answer is no, because the only candidate for such a rule is the student-proposing deferred acceptance rule since it is strategy-proof and dominates the school-proposing deferred acceptance rule. The conclusion, then, is that an attempt to maintain the welfare of students while reducing diversity constraint violations requires a redesign of the choice functions themselves. However, Ehlers et al. (2014) also show that there is no other "acceptant" choice function that is closer to the diversity constraints than their constructed choice function. Therefore, one can argue that the student-proposing deferred acceptance rule is the only strategy-proof rule that balances student welfare with diversity constraints.

7.3.2 Slot-specific priorities

In some real-world allocation problems, there are multiple policy objectives that have to be reconciled. For example, the US Army has one policy objective of prioritizing the assignment of military cadets to military specialities (called branches) on the basis of an

[35] They also study what can be achieved if the constraints are interpreted as hard bounds, and include lower-bound constraints at each object. Such hard constraints can be accommodated in our model, by appropriate definition of the feasible set for each object.

[36] Our results require the existence of an "outside option" for students, which Ehlers et al. (2014) do not assume. Nevertheless, their results for the soft-bounds would still apply if an outside option exists for each student.

"order-of-merit" list, which accounts for academic, physical, and military performance. Singular objectives such as these are straightforwardly accommodated in the model of Balinski and Sönmez (1999). However, the US Army also has an objective to increase retention of its military cadets past their initial term of service. Towards this objective, the US Army seeks to prioritize cadets willing to serve longer terms. These two objectives have to be traded off in some manner. The slot-specific priorities framework is one way to handle these multi-objective matching problems.

Sönmez and Switzer (2013) and Sönmez (2013) study this problem of matching of military cadets to US Army branches, using the matching with contracts model based on Hatfield and Milgrom (2005) and Hatfield and Kojima (2010). Each cadet is matched to a branch with a term specifying years of service. The rule that each of these papers proposes is based on the "cadet-proposing cumulative offer process." The twin objectives of prioritizing on the basis of the order-of-merit list and on the basis of terms of service is achieved by dividing the capacity of a branch into two, the first portion of which goes towards the order-of-merit list objective, and the second goes towards the retention objective. They show that this approach to handling two priority rankings allows a strategy-proof allocation rule, the cadet-optimal stable rule, that satisfies a desirable fairness requirement with respect to the two priority rankings (stability) and always picks the best such fair allocation in terms of welfare of the cadets (cadet-optimal). Because the designed choice functions satisfy unilateral substitutes and size monotonicity, by Hatfield and Kojima (2010) the cumulative offer allocation rule is the cadet-optimal stable rule. Our results in Section 5 produce the corollary that the cadet-optimal stable rule is the only strategy-proof rule that Pareto-improves upon a stable rule in a strategy-proof manner. Thus, strategy-proof Pareto-improvements in this setting can only be achieved by modifying the choice functions of branches themselves.

Corollary 8. *The cadet-optimal stable rule is the only strategy-proof stable-dominating rule.*

Kominers and Sönmez (2015) go further by allowing for any number of distinct object-specific priority rankings. Their matching model with general slot- specific priorities nests the cadet-branch matching model and the two-type controlled school choice problem of Hafalir et al. (2013). The capacity of an object is divided into individual slots, and each slot has its own priority ranking over contracts involving that object. The choice function of an object is defined using these slot-specific priorities together with a parameter, the precedence order of slots. However, slot-generated choice functions need not satisfy size monotonicity or unilateral substitutes, though they do satisfy IRC and "bilateral substitutes." Kominers and Sönmez study the cumulative offer process of Hatfield and Milgrom (2005) and show that it defines a strategy-proof allocation rule in their slot-specific priorities model.

Without size monotonicity, Corollary 1 that stable and strategy-proof rules admit no strategy-proof and strict Pareto-improvements does not immediately apply, because non-wastefulness of stable allocations cannot be guaranteed. However, the proof technique in Kominers and Sönmez (2015) can be adapted to obtain the desired result.

Kominers and Sönmez associate every slot-specific model with a representative matching-with-contracts model satisfying substitutability and size monotonicity. They show that each stable allocation in the representation is associated to a unique stable allocation in the original model, and so the agent-optimal stable rule in the representation defines a stable rule in the original model. Moreover, they show that the strategy-proofness of the agent-optimal stable rule in the representation implies strategy-proofness of the associated stable rule in the original model. Finally, they show that this strategy-proof and stable rule is actually equivalent to the rule defined by using the cumulative offer process.

The representation retains the same set of feasible allocations as the original model in the sense that every feasible allocation in the original model has a unique representative feasible allocation. As a consequence, the cumulative offer rule is non-wasteful, since it is represented by the agent-optimal stable rule, which is non-wasteful. By Corollary 2 and the strategy-proofness of the cumulative offer rule, it is not Pareto-dominated by any other strategy-proof rule. Example 4 of Kominers and Sönmez (2015) shows it is possible that a stable allocation Pareto-dominates the cumulative offer allocation. Our result for this slot-specific model states that no such improvements can be made without giving up strategy-proofness. We summarize this result in the following corollary.

Corollary 9. *The cumulative offer rule of Kominers and Sönmez (2015) is a non-wasteful rule and is not Pareto-dominated by another strategy-proof rule.*

Unlike in models where the agent-optimal stable rule is well-defined, a slot-specific model might have stable allocations that match different numbers of agents for some preference profiles, since the choice functions may not be size monotonic.[37] However, there is a particular subset of stable allocations, the set that is associated to stable allocations in the representation, for which the structure results of Section 3 apply. Thus, this subset of stable allocations is non-wasteful, and the only strategy-proof rule that always dominates some stable allocation from this subset is the cumulative offer rule.

Hatfield and Kominers (2014b) propose the notion of "substitutably completable" choice functions, and show that the agent-proposing cumulative offer rule is strategy-proof if choice functions are completable to substitutable and size monotonic choice functions. As an application, they show that the slot-specific priorities model defines choice functions that are substitutably completable.

A choice function can be "completed" by altering the choice function only at sets where some agent is associated with more than one element, and to do so only by adding elements involving agents who are already associated with an element in the chosen set. Thus, a completion of a choice function always agrees with the original choice function on sets that have no more than one element per agent.

The proof technique of Hatfield and Kominers is similar in spirit to that of Kominers and Sönmez (2015). They define a representation of the original model with a many-to-many matching with contracts model[38] that satisfies substitutes and size monotonicity. If the choice functions can be completed so that substitutability and size monotonicity is satisfied, then such a representation exists, and the agent-proposing deferred acceptance in the representative model is strategy-proof and stable in the original model. Any allocation that is feasible in the original economy is feasible in the associated economy, so anything that is non-wasteful in the associated economy is non-wasteful in the original. Thus,

[37] See Example 2 in Kominers and Sönmez (2015).

[38] See Hatfield and Kominers (2014a), and also Klaus and Walzl (2009).

the agent-proposing cumulative offer rule in Hatfield and Kominers (2014b) is on the strategy-proofness-constrained Pareto-frontier.

7.3.3 Distributional goals and dynamic reserves

Policy makers sometimes seek to balance the welfare of agents while also achieving a particular distribution of resources. School choice with "diversity constraints" is an example of this type of real-world problem, and one modeling approach is to have ceilings, floors, or reserves for each type of agent, while using an object-specific master priority list. An alternative approach is to model these types of policy concerns through the targeting of a specified distribution, while specifying how to reallocate unused reserves of capacity when available.

Westkamp (2013) proposes a model of matching with "complex constraints" that allows capacity to be redistributed between different priorities at an object, with an initial target distribution specified exogenously. At each object there is a finite sequence of priority rankings over agents, as in the model of Kominers and Sönmez (2015), and an associated sequence of capacity redistribution functions, which dictate how many slots are available at a particular priority ranking as a function of the vector of unused capacity at earlier priority rankings.

This model is a generalization of the two-type controlled school choice model in Abdulkadiroğlu and Sönmez (2003), and also captures the model of Hafalir et al. (2013).[39] In one way, the complex constraints model is more general than Kominers and Sönmez (2015), because it allows different priority rankings at each slot but also allows the capacity of the slot to be affected by the choice at slots earlier in the precedence order. On the other hand, Kominers and Sönmez (2015) allow many possible contractual terms between an agent and object.

Westkamp (2013) shows that, with some conditions on how capacity is redistributed, the object choice functions satisfy substitutability and size monotonicity. Therefore, the rule he proposes is the agent-optimal stable allocation rule, which is strategy-proof. Our results in Section 5 then imply that this rule is the unique strategy-proof stable-dominating rule.

Aygün and Turhan (2016) propose a model of matching with dynamic reserves, which combines the slot-specific model of Kominers and Sönmez (2015) with the complex constraints model of Westkamp (2013), by allowing capacity transfers across slots while allowing multiple contractual terms per agent-object pair. When each slot has a target capacity and has "responsive" priorities over sets of contracts, and capacity redistribution satisfies conditions similar to those of Westkamp (2013), Aygün and Turhan (2016) show that the cumulative offer process defines a strategy-proof and stable rule. They show that the choice functions of objects in their setting have substitutable completions that satisfy size monotonicity, and so strategy-proofness of the cumulative offer rule follows from the results of Hatfield and Kominers (2014b). Consequently, by our previous discussion of Hatfield and Kominers (2014b), the cumulative offer rule in the model of dynamic reserves is non-wasteful and on the strategy-proofness-constrained Pareto-frontier.

[39] When there are more than two types of agents, neither the controlled school choice model with quotas nor the complex constraints model generalize each other.

References

Abdulkadiroğlu, Atila, Parag A Pathak, and Alvin E Roth (2009) "Strategy-Proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match," *American Economic Review*, Vol. 99, No. 5, pp. 1954–1978. [3], [14], [19], [20], [21]

Abdulkadiroğlu, Atila and Tayfun Sönmez (2003) "School Choice: A Mechanism Design Approach," *American Economic Review*, Vol. 93, No. 3, pp. 729–747, June. [3], [5], [7], [13], [18], [19], [22], [24], [28]

Alcalde, José and Salvador Barberà (1994) "Top Dominance and the Possibility of Strategy-Proof Stable Solutions to Matching Problems," *Economic Theory*, Vol. 4, No. 3, pp. 417–35, May. [20]

Alkan, Ahmet (2002) "A class of multipartner matching markets with a strong lattice structure," *Economic Theory*, Vol. 19, No. 4, pp. 737–746. [2]

Alkan, Ahmet and David Gale (2003) "Stable Schedule Matching Under Revealed Preference," *Journal of Economic Theory*, Vol. 112, No. 2, pp. 289–306. [2]

Alkan, Ahmet, Gabrielle Demange, and David Gale (1991) "Fair Allocation of Indivisible Goods and Criteria of Justice," *Econometrica*, Vol. 59, No. 4, pp. 1023–1039. [6]

Alva, Samson (2016) "WARP and Combinatorial Choice in Matching," working paper, University of Texas at San Antonio. [14]

Aygün, Orhan and Tayfun Sönmez (2013) "Matching with Contracts: Comment," *American Economic Review*, Vol. 103, No. 5, pp. 2050–51. [3], [4], [14]

Aygün, Orhan and Bertan Turhan (2016) "Dynamic Reserves in Matching Markets: Theory and Applications," working paper, ITAM. [28]

Bade, Sophie (2014) "Pareto Optimal Strategy-proof and Non-bossy Matching Mechanisms," working paper, Royal Holloway College, University of London. [22]

Balinski, Michel and Tayfun Sönmez (1999) "A Tale of Two Mechanisms: Student Placement," *Journal of Economic Theory*, Vol. 84, No. 1, pp. 73–94, January. [2], [3], [7], [19], [26]

Bo, Inacio (forthcoming) "Fair Implementation of Diversity in School Choice," *Games and Economic Behavior*. [25]

Bogomolnaia, Anna, Rajat Deb, and Lars Ehlers (2005) "Strategy-proof assignment on the full preference domain," *Journal of Economic Theory*, Vol. 123, No. 2, pp. 161–186, Aug. [13]

Clarke, Edward H. (1971) "Multipart pricing of public goods," *Public Choice*, Vol. 11, No. 1, pp. 17–33. [22]

Demange, Gabrielle and David Gale (1985) "The Strategy Structure of Two-sided Matching Markets," *Econometrica*, Vol. 53, No. 4, pp. 873–88, July. [22]

Ehlers, Lars and Aytek Erdil (2010) "Efficient assignment respecting priorities," *Journal of Economic Theory*, Vol. 145, No. 3, pp. 1269–1282, May. [3], [17], [20], [21], [36]

Ehlers, Lars and Bettina Klaus (2014) "Strategy-Proofness Makes the Difference: Deferred-Acceptance with Responsive Priorities," *Mathematics of Operations Research*, Vol. 39, No. 4, pp. 949–966, November. [9]

——— (forthcoming) "Object Allocation via Deferred-Acceptance: Strategy-Proofness and Comparative Statics," *Games and Economic Behavior*. [20]

Ehlers, Lars and Alexander Westkamp (2011) "Strategy-Proof Tie-Breaking," working paper, CIREQ. [21]

Ehlers, Lars, Isa E. Hafalir, M. Bumin Yenmez, and Muhammed A. Yildirim (2014) "School choice with controlled choice constraints: Hard bounds versus soft bounds," *Journal of Economic Theory*, Vol. 153, pp. 648 – 683. [25]

Erdil, Aytek (2014) "Strategy-proof stochastic assignment," *Journal of Economic Theory*, Vol. 151, pp. 146 – 162. [3], [4]

Erdil, Aytek and Haluk Ergin (2008) "What's the Matter with Tie-Breaking? Improving Efficiency in School Choice," *American Economic Review*, Vol. 98, No. 3, pp. 669–89, June. [20], [36]

——— (2015) "Two-Sided Matching with Indifferences," working paper, University of California, Berkeley. [5], [13]

Erdil, Aytek and Taro Kumano (2014) "Prioritizing Diversity in School Choice," working paper, University of Cambridge. [2], [14]

Ergin, Haluk I. (2002) "Efficient Resource Allocation on the Basis of Priorities," *Econometrica*, Vol. 70, No. 6, pp. 2489–2497. [3], [20], [24]

Groves, Theodore (1973) "Incentives in Teams," *Econometrica*, Vol. 41, No. 4, pp. 617–631. [22]

Hafalir, Isa E., M. Bumin Yenmez, and Muhammed A. Yildirim (2013) "Effective Affirmative Action in School Choice," *Theoretical Economics*, Vol. 8, No. 2, pp. 325–363. [24], [26], [28]

Han, Xiang (2015) "Priority-augmented House Allocation," working paper, Southern Methodist University. [22]

Hatfield, John William and Fuhito Kojima (2009) "Group incentive compatibility for matching with contracts," *Games and Economic Behavior*, Vol. 67, No. 2, pp. 745–749. [18]

——— (2010) "Substitutes and stability for matching with contracts," *Journal of Economic Theory*, Vol. 145, No. 5, pp. 1704–1723, September. [3], [17], [26]

Hatfield, John William and Scott Duke Kominers (2014a) "Contract Design and Stability in Many-to-Many Matching," working paper, University of Texas at Austin. [27]

——— (2014b) "Hidden Substitutes," working paper, University of Texas at Austin. [6], [16], [27], [28]

Hatfield, John William and Paul Milgrom (2005) "Matching with Contracts," *American Economic Review*, Vol. 95, No. 4, pp. 913–935, September. [2], [14], [26], [35]

Hirata, Daisuke and Yusuke Kasuya (2015) "On Stable and Strategy-Proof Rules in Matching Markets with Contracts," working paper, Hitotsubashi University. [3], [4], [16]

Kesten, Onur and Morimitsu Kurino (2015) "Do outside options matter in matching? A new perspective on the trade-offs in school choice," working paper, Carnegie Mellon University. [3], [20]

Klaus, Bettina and Markus Walzl (2009) "Stable many-to-many matching with contracts," *Journal of Mathematical Economics*, Vol. 45, pp. 422–434. [27]

Kominers, Scott Duke and Tayfun Sönmez (2015) "Matching with Slot-Specific Priorities: Theory," working paper, Boston College. [26], [27], [28]

Leonard, Herman B (1983) "Elicitation of Honest Preferences for the Assignment of Individuals to Positions," *Journal of Political Economy*, Vol. 91, No. 3, pp. 461–79, June. [22], [23]

Ma, Jinpeng (1994) "Strategy-Proofness and the Strict Core in a Market with Indivisibilities," *International Journal of Game Theory*, Vol. 23, No. 1, pp. 75–83. [24]

Milgrom, Paul and Ilya Segal (2015) "Deferred-Acceptance Auctions and Radio Spectrum Reallocation," working paper, Stanford University. [22]

Morimoto, Shuhei and Shigehiro Serizawa (2015) "Strategy-proofness and efficiency with non-quasi-linear preferences: a characterization of minimum price Walrasian rule," *Theoretical Economics*, Vol. 10, No. 2, May. [22]

Pápai, Szilvia (2000) "Strategyproof Assignment by Hierarchical Exchange," *Econometrica*, Vol. 68, No. 6, pp. 1403–1434, Nov. [22]

Pycia, Marek and M. Utku Ünver (2014) "Outside Options in Neutral Discrete Resource Allocation," working paper, Boston College. [22]

——— (2015) "Incentive Compatible Allocation and Exchange of Discrete Resources," working paper, Boston College. [3], [22] Roth, Alvin E. (1986) "On the Allocation of Residents to Rural Hospitals: A General Property of Two-Sided Matching Markets," *Econometrica*, Vol. 54, No. 2, pp. 425–427. [3]

Roth, Alvin E and Andrew Postlewaite (1977) "Weak versus strong domination in a market with indivisible goods," *Journal of Mathematical Economics*, Vol. 4, No. 2, pp. 131–137, Aug. [24]

Shapley, Lloyd and Herbert Scarf (1974) "On cores and indivisibility," *Journal of Mathematical Economics*, Vol. 1, pp. 23–37. [23]

Sönmez, Tayfun (1999) "Strategy-Proofness and Essentially Single-Valued Cores," *Econometrica*, Vol. 67, No. 3, pp. 677–690, May. [5]

Sönmez, Tayfun (2013) "Bidding for Army Career Specialties: Improving the ROTC Branching Mechanism," *Journal of Political Economy*, Vol. 121, No. 1, pp. 186–219, Mar. [26]

Sönmez, Tayfun and Tobias B. Switzer (2013) "Matching With (Branch-of-Choice) Contracts at the United States Military Academy," *Econometrica*, Vol. 81, No. 2, pp. 451–488. [13], [26]

Svensson, Lars-Gunnar (1999) "Strategy-proof allocation of indivisible goods," *Social Choice and Welfare*, Vol. 16, pp. 557–567. [22]

Vickrey, William (1961) "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance*, Vol. 16, No. 1, pp. 8–37, March. [22]

Westkamp, Alexander (2013) "An analysis of the German university admissions system," *Economic Theory*, Vol. 53, No. 3, pp. 561–589. [28]

Appendices

A Proofs omitted from the body

A.1 Equivalence of non-wastefulness definitions

Proof of Proposition 1. Suppose that μ is Balinski-Sönmez-wasteful. Then there are $o \in O$ and $i \in N$ such that $o \; P_i \; \mu(i)$ and $|\mu(i)| < q_o$. Let $\nu \equiv (\mu \cup \{(i,o)\}) \setminus \mu(i)$. Then $|\nu(o)| = |\mu(o)| + 1 \leq q_o$ and for each $o' \in O \setminus \{o\}, |\nu(o')| \in \{|\mu(o')| - 1, |\mu(o')|\} \leq q_{o'}$. Thus, $\nu \in \mathcal{F}$ and $|\nu(o)| > |\mu(o)|$. Furthermore, $\nu(i) \; P_i \; \mu(i)$ while for each $j \in N \setminus \{i\}, \nu(i) = \mu(i)$. Thus, μ is wasteful.

Suppose that μ is Balinski-Sönmez-non-wasteful. Consider $\nu \in \mathcal{F}$ such that for each $i \in N$, $\nu(i) \; R_i \; \mu(i)$ and for some $i \in N$, $\nu(i) \; P_i \; \mu(i)$. Let $o \in O$. If $|\mu(o)| < q_o$, by Balinski-Sönmez-non-wastefulness, there is no $i \in N$ such that $o \; P_i \; \mu(i)$. So $|\nu(o)| \leq |\mu(o)|$. If $|\mu(o)| = q_o$, by feasibility of ν, $|\nu(o)| \leq q_o = |\mu(o)|$. Thus, μ is non-wasteful. □

A.2 Pareto-dominating an individually rational and non-wasteful allocation

Proof of Lemma 2. Since μ is non-wasteful, it is weakly non-wasteful as well. So by the Structure Lemma, $\mu_N = \nu_N$, so $|\mu| = |\nu|$.

Since μ is non-wasteful, for each $o \in O$, $|\mu(o)| \geq |\nu(o)|$. Since for each pair $o, o' \in O$, $\mu(o)$ and $\mu(o')$ are disjoint, $\sum_{o \in O} |\mu(o)| = |\mu|$. Similarly, $\sum_{o \in O} |\nu(o)| = |\nu|$. Thus, $\sum_{o \in O} |\mu(o)| = \sum_{o \in O} |\nu(o)|$. So for each $o \in O$, $|\mu(o)| \geq |\nu(o)|$. □

A.3 Existence of strategy-proof, constrained efficient and wasteful rule

Proof of Proposition 3. Let $\{i_1, i_2, i_3, \dots\}$ be a labeling of the N and $\{a, b, c, \dots\}$ be a labeling of O. Suppose that T is a singleton, $|N| \geq 3$, $|O| \geq 3$, for each $o \in O$, $F_o = \{\{i\} : i \in N\} \cup \{\emptyset\}$, $\mathcal{F}$ is Cartesian, and for each $i \in N$, the preference domain $\mathcal{R}_i$ is the set of all strict preferences over $O \cup \{\emptyset\}$.[40] Then $\mathcal{R} = \mathcal{P}$, and satisfies our assumptions from Section 2.

Consider the benchmark rule, $\underline{\varphi}$, defined by setting, for each $P \in \mathcal{P}$,

$$\underline{\varphi}_{i_1}(P) = P_{i_1}\text{-}\max(O \setminus \{a\}).^{41}$$

$$\underline{\varphi}_{i_2}(P) = \begin{cases} P_{i_2}\text{-}\max(O \setminus \underline{\varphi}_{i_1}(P)) & \text{if } \underline{\varphi}_{i_1}(P) \neq c \text{ or} \\ P_{i_2}\text{-}\max(O \setminus (\underline{\varphi}_{i_1}(P) \cup \underline{\varphi}_{i_3}(P)) & \text{otherwise} \end{cases}$$

$$\underline{\varphi}_{i_3}(P) = \begin{cases} P_{i_3}\text{-}\max(O \setminus (\underline{\varphi}_{i_1}(P) \cup \underline{\varphi}_{i_2}(P)) & \text{if } \underline{\varphi}_{i_1}(P) = c \text{ or} \\ P_{i_3}\text{-}\max(O \setminus \underline{\varphi}_{i_1}(P)) & \text{otherwise} \end{cases}$$

and for $k > 3$,

$$\underline{\varphi}_{i_k}(P) = P_{i_k}\text{-}\max(O \setminus (\underline{\varphi}_{i_1}(P) \cup \cdots \cup \underline{\varphi}_{i_{k-1}}(P)))$$

In words, this rule assigns to i_1 his most preferred object except for a. The remaining objects are distributed among the remaining agents sequentially in the order i_2, i_3, $i_4, \dots$ if i_1 is not assigned c. The places of i_2 and i_3 are swapped if i_1 is assigned c. Since i_1 is barred from receiving a, this rule is strongly wasteful: at each $P \in \mathcal{P}$ such that for each $i \in N \setminus \{i_1\}, \emptyset \; P_i \; a$ and for each $o \in O \setminus \{a\}$, $a \; P_{i_1} \; \emptyset \; P_{i_1} \; o$, $\underline{\varphi}_{i_1}(P) = \emptyset$ and a is not assigned, rendering $\underline{\varphi}$ strongly wasteful.

While it may be possible to find a strategy-proof rule that Pareto-dominates $\underline{\varphi}$, we show that every such rule is, itself, strongly wasteful. Thus, there *exists* a strongly wasteful strategy-proof rule that cannot be Pareto-dominated by another strategy-proof rule.

To prove this claim, suppose that φ is weakly non-wasteful and Pareto-dominates $\underline{\varphi}$,

[40] The requirement of at least as many objects as agents is not needed. An example with three agents and two objects is available upon request.

[41] Given $P_i \in \mathcal{P}_i$ and $A \subseteq O$, we denote the best element of A according to P_i by P_i-max(A).

and consider $P \in \mathcal{P}$ as follows:

$$\begin{array}{ccccc} P_{i_1} & P_{i_2} & P_{i_3} & \text{and for } k > 3, & P_{i_k} \\ \hline a & b & b & & \varnothing \\ \varnothing & a & \varnothing & & \\ & \vdots & & & \end{array}$$

By definition of $\underline{\varphi}$, we have $\underline{\varphi}_{i_1}(P) = \varnothing$, $\underline{\varphi}_{i_2}(P) = b$, and for each $i \in N \setminus \{i_1, i_2\}$, $\varphi_i(P) = \varnothing$. Since φ Pareto-dominates $\underline{\varphi}$ and φ is weakly non-wasteful, $\varphi_{i_1}(P) = a$, $\varphi_{i_2} = b$ and for each $i \in N \setminus \{i_1, i_2\}$, $\varphi_i(P) = \varnothing$.

Now consider $P_{i_1} \in \mathcal{P}_{i_1}$ as follows:

$$\begin{array}{c} P'_{i_1} \\ \hline a \\ c \\ \vdots \end{array}$$

By definition of $\underline{\varphi}$, we have $\underline{\varphi}_{i_1}(P'_{i_1}, P_{-i_1}) = c$, $\underline{\varphi}_{i_2}(P'_{i_1}, P_{-i_1}) = a$, $\underline{\varphi}_{i_3}(P'_{i_1}, P_{-i_1}) = b$, and for each $i \in N \setminus \{i_1, i_2, i_3\}$, $\underline{\varphi}_i(P'_{i_1}, P_{-i_1})) = \varnothing$. Since this allocation is Pareto-efficient at (P'_{i_1}, P_{-i_1}) and φ Pareto-dominates $\underline{\varphi}$, $\varphi(P'_{i_1}, P_{-i_1}) = \underline{\varphi}(P'_{i_1}, P_{-i_1})$. But then

$$\varphi(P_{i_1}, P_{-i_1}) = a \; P'_{i_1} \; c = \varphi(P'_{i_1}, P_{-i_1}),$$

so φ is not strategy-proof.

We conclude that every strategy-proof rule that Pareto-dominates $\underline{\varphi}$ is, itself, strongly wasteful. □

Notice that $\underline{\varphi}$ is actually *group strategy-proof*. Since no weakly non-wasteful strategy-proof rule Pareto-dominates it, it follows then, that there exists a strongly wasteful and *group* strategy-proof rule that is not Pareto-dominated by a strategy-proof rule.

A.4 Proofs of Propositions in Section 5

Proof that φ in Example 4 is stable and strategy-proof. We first establish that φ is stable by considering four cases.

Case 1: $m_1, m_2 \in G(P)$. Then $\varphi(P) = \mu^{\{m_1, m_2\}}$. Regardless of whether $p_1, p_2 \in G(P)$, there is no $Y \subseteq G(P) \setminus \{m_1, m_2\}$ such that $Y \subseteq C(\mu^{\{m_1, m_2\}} \cup Y)$. Thus, $\varphi(P)$ is stable.

Case 2: $m_1 \notin G(P)$ but $m_2 \in G(P)$. If $p_1, p_2 \in G(P)$, then $\varphi(P) = \mu^{\{p_1, p_2\}}$. Since $C(\{m_2, p_1, p_2\}) = \{p_1, p_2\}$, $\varphi(P)$ is stable. Otherwise, $\varphi(P) = \mu^{G(P)}$ and each agent receives his top choice. Thus $\varphi(P)$ is stable.

Case 3: $m_1 \in G(P)$ but $m_2 \notin G(P)$. This is symmetric to Case 2.

Case 4: $m_1, m_2 \notin G(P)$. Since, $\varphi(P) = \mu^{G(P)}$ and each agent receives his top choice, $\varphi(P)$ is stable.

To show that φ is strategy-proof, we again consider the same four cases.

Case 1: $m_1, m_2 \in G(P)$. Then $\varphi(P) = \mu^{\{m_1,m_2\}}$ and neither m_1 nor m_2 can benefit by misreporting his preferences. Regardless $P_{\{p_1,p_2\}}$, φ selects $\mu^{\{m_1,m_2\}}$ so neither of p_1 or p_2 can benefit by misreporting his preference either.
Case 2: $m_1 \notin G(P)$ but $m_2 \in G(P)$. If $p_1, p_2 \in G(P)$, then $\varphi(P) = \mu^{\{p_1,p_2\}}$, so neither p_1 nor p_2 benefits by misreporting and since φ selects $\mu^{\{p_1,p_2\}}$ regardless of m_2's preference, he has no incentive to misreport either. Otherwise, $\varphi(P) = \mu^{G(P)}$ and no agent can benefit by misreporting since he receives his top choice.
Case 3: $m_1 \in G(P)$ but $m_2 \notin G(P)$. This is symmetric to Case 2.
Case 4: $m_1, m_2 \notin G(P)$. Since $\varphi(P) = \mu^{G(P)}$, no agent can benefit by misreporting since he receives his top choice.

□

Proof of Proposition 5. By definition, for each $P \in \mathcal{P}$, $\varphi^{AOS}(P)$ Pareto-dominates every stable allocation at P. By Lemma 2 the Rural Hospitals Theorem holds.

The proofs of Theorems 10 and 11 of Hatfield and Milgrom (2005) use only the conclusion of the Rural Hospitals Theorem to show that φ^{AOS} is strategy-proof, so their proof applies unchanged in our setting.

Now we show that it is the unique stable and strategy-proof rule. If any other rule φ is stable, then it is Pareto-dominated by φ^{AOS}. By Theorem 1, if $\varphi \neq \varphi^{AOS}$, then it is not strategy-proof. Thus φ^{AOS} is the only stable and strategy-proof rule. □

Proof of Proposition 6. If φ is not the agent-optimal stable rule, then there are $P \in \mathcal{P}$ and $\nu \in \mathcal{F}$ such that ν is stable at P and $\varphi(P)$ does not Pareto-dominate ν. So there is $i \in N$ such that $\nu(i)\ P_i\ \varphi_i(P)$.

Let $x \equiv \nu(i)$ and $y \equiv \varphi_i(P)$. Since φ is stable, it is individually rational. Thus, $x\ P_i\ y\ R_i\ \emptyset$. Since $x \neq \emptyset$, by the Rural Hospitals Theorem, $y \neq \emptyset$.

Let $P'_i \in \mathcal{P}_i$ be such that *a*) for each $z \in X(i)$ if $z\ P'_i\ x$ then $z\ P_i\ x$ and if $z\ P'_i\ y$ then $z\ P_i\ y$ and *b*) $x\ P'_i\ \emptyset\ P'_i\ y$. Let $P' \equiv (P'_i, P_{-i})$.

We first show that ν is stable at P'. If not, there are $o \in O$ and $Y \subseteq X(o)$ such that for each $z \in Y$, $z\ P'_{z_N}\ \nu(z_N)$, $\nu(o) \notin C_o(\nu(o) \cup Y)$ and there is $Z \in C_o(\nu(o) \cup Y)$ such that $Y \subseteq Z$. Since, for each $j \in N \setminus \{i\}$, $P'_j = P_j$ and ν is stable at P, there is $z \in Y$ such that $z_N = i$. However, since $z\ P'_i\ \nu(i) = x$ we have $z\ P_i\ x$. This contradicts the stability of ν at P. Thus ν is stable at P'.

By the Rural Hospitals Theorem, since $\nu(i) \neq \emptyset$ and since both ν and $\varphi(P')$ are stable at P', $\varphi_i(P') \neq \emptyset$. Since φ is individually rational, $\varphi'_i(P')\ P'_i\ \emptyset$. By definition of P'_i, $\varphi'_i(P')\ P'_i\ y$. Thus, we have $\varphi'_i(P')\ P_i\ y$. However, this contradicts the strategy-proofness of φ since $x = \varphi(P'_i, P_{-i})\ P_i\ \varphi_i(P_i, P_{-i}) = y$. □

A.5 Proofs of Propositions in Section 6

Proof of Proposition 8. Let μ be stable with respect to C. By definition of stability it is individually rational. Since C is size monotonic and idempotent, by Proposition 4, μ is non-wasteful. If it violates priorities, there are a pair $i, j \in N$ and $o \in O$ such that $\mu(i) = o$, $o\ P_j\ \mu(j)$, and $j \succ_o i$. Since μ is non-wasteful, $|\mu(o)| = q_o$. Since $j \succ_o i$, $C_o(\mu(o) \cup \{j\}) = \{(\mu(o) \setminus \{i\}) \cup \{j\}\}$. This contradicts the stability of μ.

Suppose that μ is individually rational, non-wasteful and respects priorities. If it is not stable, then there are $o \in O$ and $Y \subseteq N \setminus \mu(o)$ such that for each $i \in Y$, $o\ P_i\ \mu(i)$ and *a*) $Y \subseteq Z$ for some $Z \in C_o(\mu(o) \cup Y)$ and *b*) $\mu(o) \notin C_o(\mu(o) \cup Y)$. Since, for each $i \in Y$, $o\ P_i\ \mu(i)$, and μ is non-wasteful, $|\mu(o)| = q_o$. Thus, for each $Z \in C_o(\mu(o) \cup Y)$, $|Z| = |\mu(o)|$. Since $\mu(o) \notin C_o(\mu(o) \cup Y)$, there are $i \in Y$ and $j \in \mu(o)$ such that $i \succ_o j$. This contradicts the assumption that μ respects priorities. $\square$

Proof of Proposition 9. Let μ be stable with respect to $\succsim$. Let $\tau \in \mathcal{T}$ be such that for each $o \in O$ and each pair $i, j \in N$ such that $i \in \mu(o)$ but $j \notin \mu(o)$, $i\ \tau_o\ j$.

If μ is not stable with respect to $\succsim^\tau$ there are a pair $i, j \in N$ and $o \in O$ such that $\mu(j) = o$, $i \succ^\tau j$, and $o\ P_i\ \mu(i)$. Since μ is stable, $j \succsim_o i$. Since $i \succ^\tau j$, $i \succsim_o j$. Thus, $i \sim_o j$ so $i\ \tau_o\ j$. However, $i \notin \mu(o)$ while $j \in \mu(o)$. This contradicts the definition of τ.

Let μ be stable with respect to $\succsim^\tau$. Then, for each $j \in N$ such that $o\ P_j\ \mu(j)$ and each $i \in \mu(o)$, $i \succ_o^\tau j$. Thus, $i \succsim_o j$. Thus, μ is stable with respect to $\succsim$. $\square$

Proof of Proposition 10. By Lemma 2, every allocation that Pareto-improves on μ reallocates objects among agents in (possibly several) cycles so that each agent obtains the object assigned by μ to the next agent in the cycle. If the same object appears twice in the same cycle, we can divide the cycle into two separate cycles. Thus it suffices to show that for every cycle $C \equiv \{i_1, \ldots, i_k\}$ such that for each pair $i, j \in C$, $o_i \neq o_j$, $\mu(i_1)\ P_{i_k}\ \mu(i_k)$, and for each $l \in \{1, \ldots, k-1\}$, $\mu(i_{l+1})\ P_{i_l}\ \mu(i_l)$, the allocation μ' defined below is stable. For each $i \in N$,

$$\mu'(i) = \begin{cases} \mu(i) & \text{if } i \notin C, \\ \mu(i_{l+1}) & \text{if } i = i_l \text{ where } l \in \{1, \ldots, k-1\}, \\ \mu(i_1) & \text{if } i = i_k. \end{cases}$$

For each $l \in \{1, \ldots, k\}$, let $o_l \equiv \mu(i_l)$. Since μ is stable and $o_l\ P_{i_{l-1}}\ \mu(i_{l-1})$, for each $l \in \{2, \ldots, k\}$, $i_l \succsim_{o_l} i_{l-1}$ and $|\mu(o_l)| = q_{o_l}$. Let $N_{o_l} \equiv \mu(o_l) \setminus \{i_l\}$. Since $|\mu(o_l)| = q_{o_l}$, $|N_{o_l}| = q_{o_l} - 1$. Since $\mu(i_{l-1}) \neq o_l$, $i_{l-1} \notin N_{o_l}$. Since μ is stable, for each $k \in N_{o_l}$, $k \succsim_{o_l} i_{l-1}$. Similarly, $i_1 \succsim_{o_1} i_k$ and letting $N_{o_1} \equiv \mu(o_1) \setminus \{i_1\}$, $|N_{o_1}| = q_{o_l} - 1$ and $i_k \notin N_{o_1}$. Thus, for each $l \in \{2, \ldots, k\}$, $i_l, i_{l-1} \notin N_{o_l}$ and $i_1, i_k \notin N_{o_1}$, and for each $k \in N_{o_1}$, $k \succsim_{o_1} i_k$.

Suppose that μ' violates priorities. Without loss of generality, there is j such that $j \succ_{o_2} i_1$ and $o_2\ P_j\ \mu'(j)$. However, since $\mu'(j)\ P_j\ \mu(j)$ and μ is stable, $i_2 \succsim_{o_2} j$. Thus, $i_2 \succsim_{o_2} j \succ_{o_2} i_1$. Since μ is stable, for each $k \in N_{o_2}$, $k \succsim_{o_2} j$. Since $\succsim$ is strongly* acyclic, either $i_2 \succ_{o_3} i_1$ or $i_1 \in N_{o_3}$. If $i \in N_{o_3}$ then $k = 2$, and since $i_1, i_2 \notin N_{o_1}$ and $i_1, j, i_2 \notin N_{o_2}$, and $\succsim$ is strongly* acyclic, $i_2 \succ_{o_1} i_1$, contradicting $i_1 \succsim_{o_1} i_2$. Thus $i_2 \succ_{o_3} i_1$ so that $i_3 \succsim_{o_3} i_2 \succ_{o_3} i_1$. Again, since $\succsim$ is strongly* acyclic, either $i_3 \succ_{o_4} i_1$ or $i_1 \in N_{o_4}$. If $i \in N_{o_4}$ then $k = 3$, and since $i_1, i_3 \notin N_{o_1}$ and $i_1, i_2, i_3 \notin N_{o_3}$, and $\succsim$ is strongly* acyclic, $i_3 \succ_{o_1} i_1$, contradicting $i_1 \succsim_{o_1} i_3$. Thus $i_3 \succ_{o_4} i_1$ so that $i_4 \succsim_{o_4} i_3 \succ_{o_4} i_1$. Repeating the argument, we have $i_k \succsim_{o_k} i_{k-1} \succ_{o_k} i_1$. However, since $i_1 \succsim_{o_1} i_k$, $i_1, i_k, i_{k-1} \notin N_{o_k}$ and $i_1, i_k \notin N_{o_1}$, this contradicts the assumption that $\succsim$ is strongly* acyclic. Thus, μ' is stable. $\square$

The following example shows that Proposition 10 does not hold under the slightly weaker condition of Ehlers and Erdil (2010). In the proof of the proposition, we see that every "improving cycle" is a "stable improving cycle" (Erdil and Ergin, 2008) under the stronger condition. Under the weaker condition, one can only show that whenever there is an improving cycle, there is at least one stable improving cycle.

Example 5. *Proposition 10 does not hold if $\succsim$ is only strongly acyclic.*

Let $O \equiv \{o_1, o_2\}$ and $N \equiv \{i_1, i_2, i_3\}$. Let $q_{o_1} = 2$ and $q_{o_1} = 1$. Define $\succsim$ as follows:

$\succsim_{o_1}$	$\succsim_{o_2}$
i_1, i_2, i_3	i_2, i_3
	i_1

Since there are only three agents, the scarcity condition is never met. Thus, $\succsim$ is strongly acyclic despite the loop condition being satisfied.

Consider $P \in \mathcal{P}$ as follows:

P_{i_1}	P_{i_2}	P_{i_3}
o_2	o_1	o_2
o_1	o_2	o_1

Let $\mu \in \mathcal{F}$ be such that $\mu(i_1) = o_1$, $\mu(i_2) = o_2$, and $\mu(i_3) = o_1$. Though μ is stable, it is not Pareto-efficient. There are two Pareto-improving cycles: i_1 and i_2 trade their assignments or i_2 and i_3 trade their assignments. The former leads to an unstable allocation.

Proof of Proposition 11. If φ is not stable, there is $P \in \mathcal{P}$ such that $\varphi(P)$ is not stable. Let $\mu \equiv \varphi(P)$. Since φ is stable-dominating, there exists $\underline{\mu} \in \mathcal{F}$ such that $\underline{\mu}$ is stable and μ Pareto-dominates it. By Proposition 4, $\underline{\mu}$ is non-wasteful. So by the Structure Lemma, μ is non-wasteful. Since μ is not stable, it does not respect priorities. So there are a pair $i, j \in N$ and $o \in O$ such that $\mu(i) = o$, $j \succ_o i$, and $o \; P_j \; \mu(j)$ and $|\mu(o)| = q_o$. Since $j \succ_i i$, $|\{k \in N \setminus \{j\} : \mu(k) = o \text{ and } k \succsim_o j\}| \leq q_o - 1$.

Let $S \equiv N \setminus \{j\}$ and consider $\overline{P}_S \in \times_{k \in S} \mathcal{P}_k$ such that for each $k \in S$ and each $\underline{o} \in O \setminus \{\mu(k)\}$, $\mu(k) \; \overline{P}_k \; \varnothing \; \overline{P}_k \; \underline{o}$, which is admissible given the rich* assumption on the preference domain. Then $|\{k \in N \setminus \{j\} : o \; \overline{P}_k \; \varnothing \text{ and } k \succsim_o j\}| \leq q_o - 1$. So for each $\nu \in \mathcal{F}$ such that ν is stable at $(\overline{P}_S, P_j)$, $\nu(j) \; R_j \; o$.

Let $\overline{\mu} = \varphi(\overline{P}_S, P_j)$. By definition of $\overline{P}_S$, for each $k \in S$, $\mu(k) \; \overline{R}_k \; \overline{\mu}(k)$. If there is $k \in S$ such that $\overline{\mu}(k) \neq \mu(k)$, then $\mu(k) \; \overline{P}_i \; \overline{\mu}(k)$. This contradicts the group strategy-proofness of φ as S may beneficially report P_S when the true preferences are $\overline{P}_S$. Thus, for each $k \in S$, $\overline{\mu}(k) = \mu(k)$. Since $\overline{\mu}$ Pareto-dominates a stable allocation, $\overline{\mu}(j) \; R_j \; o \; P_j \; \mu(j)$. Thus, $\overline{\mu}$ Pareto-dominates μ. This contradicts the group strategy-proofness of φ since N may beneficially report $(\overline{P}_S, P_j)$ when the true preferences are P. □

B Waiver Algorithm

The procedure "WaiverOrder" takes as input an initial allocation, ω, a preference profile, P, and a waiver order over the agents, $\succ$. The output is a non-wasteful allocation where agents drop and pick up available objects in the order of $\succ$, starting from the initial allocation ω.

Algorithm 1 Procedure to waive and pick up objects on waivers

```
procedure WaiverOrder(ω, P, ≻)
    y = ∅
    for i ∈ N do
        y_i = { ω_i   if ω_i P_i ∅
              { ∅     otherwise
    A = {o ∈ O : y(o) = ∅}                          ▷ Set of unassigned objects.
    while x is wasteful at P do
        for i = first in ≻ to i = last in ≻ do
            for a ∈ A do
                if a P_i y_i then
                    A = (A \ a) ∪ y_i               ▷ Remove a from and add y_i to A.
                    y_i = a                         ▷ Assign a to i.
                    go to 7
    return y
```