To Join or Not to Join: The Role of Information Structures in a Threshold Game*

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Abstract

We analyze a binary-action sequential game with a tipping point in the presence of imperfect information. An information structure summarizes what each agent can observe before making her decision. Focusing on information structures where only "aggregate information" from past history can be observed, we fully characterize information structures that can lead to various (efficient and inefficient) Nash equilibria. When individual decision making can be rationalized using a process of iterative dominance (Moulin (1979)), we derive a necessary and sufficient condition on the information structure under which one obtains a unique and efficient Nash equilibrium outcome. Our results suggest that if sufficient (and not necessarily perfect) information is available, coordination failure can be overcome without centralized intervention.

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JEL Classification: C72, C73, D80.

1 Introduction

A threshold or tipping point is generally defined as a boundary which if crossed leads to an "irrevocable" change of state.[1] Strategic complementarities in games in which the net return from one's actions depends positively on how many others have taken the same action often lead to payoff structures characterized by a threshold. Sequential binary threshold games, models with a tipping point in which agents move in a predetermined sequence and

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[1]The idea seems to have come from the notion of "critical mass" in physics.

each agent has two possible moves, have been found useful in many disciplines.[2] Herding behavior, agglomeration and cascading, evolution of social norms, network formation, formation of clubs and customs unions as well as studies of insurance and of diffusion of technology, represent some examples of problems that can be fruitfully modeled using sequential binary games exhibiting threshold phenomena.

In economics, tipping point models have generally been associated with coordination games. Examples abound of coordination problems with positive externalities: the consumption of a product with a network-externality effect (e.g., fax machines, mobile phones), the adoption of an innovation or standard where compatibility is valuable (e.g., ISO international standards on safety and quality of goods, standardization of screw threads), and market-based externalities where more consumers of a good induce a lower price of the provision of the good or complementary goods (e.g., natural monopolies, insurance markets). Strategic complementarities in these coordination games result in the existence of a "bad" inefficient equilibrium where complete lack of participation leads to inefficiency and a "good" efficient equilibrium with maximal participation.

Cooper and John (1988) have argued that in the presence strategic complementarities, where the marginal return from "increasing" one's strategy is an increasing function of the strategy of others, lies at the heart of the multiplicity of equilibria in coordination games. The classic result in this area is by Milgrom and Roberts (1990) showing that for submodular games the set of serially undominated strategy profiles has a maximal and a minimal elements (pure-strategy Nash equilibria) and are identified as the "bad" and the "good" equilibria referred to above.[3] Milgrom and Roberts argue that since major approaches to non-cooperative games accept that the solution needs to lie in this serially undominated set, these maximal and minimal elements provide upper and lower bounds on the joint behavior of players. For games with strategic complementarities, previous research has explored various approaches that can be used to avoid or reduce the likelihood of coordination failure (i.e., the occurrence of the bad equilibrium). This includes a strand of literature analyzing the effect of financial inducement schemes to overcome coordination failure,[4] and a line of research studying games with strategic complementarities in dynamic environments.[5] Two characteristics distinguish our study from this earlier work. Firstly,

[2]One of the earlier applications in social sciences of a tipping point is Grodzins (1957) in sociology studying white flight from inner cities to the suburbs. This analysis was elaborated and expanded on by Schelling (1969, 1971). This is also a key concept used by Granovetter (1978) to study collective behavior. Evidence of recognition of the tipping point phenomenon can also be found in everyday phrases such as "the straw that broke the camel's back" and "domino effect."

[3]Submodular games are games where the strategy sets have a lattice structure and where there are strategic complementarities. Serially weakly undominated strategies are strategies that remain after serialy crossing out strongly dominated strategies.

[4]See, for example, Dybvig and Spatt (1983) and Park (2004) on the effects of insurance schemes against low adoptions, and Bagnoli and Lipman (1989) on the effects of refund mechanisms.

[5]Examples include dynamic coordination with positive spillovers and irreversible actions (Admati and Perry (1991)), dynamic common interest games with asynchronicity and a finite horizon (Dutta (2012)), and dynamic coordination games with incomplete information (Farrell and Saloner (1985)).

our focus is not on strategic complementarities and positive externalities *per se* but rather on the theoretical implications of a tipping point and with the possibility of a coordination failure. In particular, we allow for negative spillovers ("congestion") in our model by not insisting on strategic complementarities holding everywhere. Secondly, a key aspect of our paper is *the analysis and the modeling of information and its role* in the determination of different kinds of pure-strategy Nash equilibria.

We model information in our sequential game as a signal received by each player before she moves. The information received is anonymous in that it is about how many people rather than who has participated. The message received is (possibly) an imperfect signal about the *aggregate* participation level. We identify key properties of the information structure and their relationship to the occurrence of various types of Nash equilibria, as well as to our game being dominance solvable using a process of iterative elimination of dominated strategies.[6]

We add to the literature in three different ways: Firstly, we fully investigate the relationship between the existence of "intermediate" pure-strategy Nash equilibria (those that lie strictly between the maximal and the minimal equilibrium outcomes referred to above) and the information structure of the game. Secondly, recognizing that for our game, subgame perfection fails to screen out non-credible equilibria we use dominance solvability as the appropriate refinement of Nash equilibrium to explore the relationship between efficiency and the information structure of the game.[7] Thirdly, our results apply to all binary tipping point games in which the agents have a common threshold not just to games with positive spillovers. In particular, our results also apply to games which may only have *local* strategic complementarities and thus games allowing for the interesting possibility of some levels of congestion mixed in with the positive spillovers.[8]

Our main results are as follows. Our first theorem analyzes the relationship between information structures and pure-strategy Nash equilibria. While the existence of the maximal and the minimal Nash equilibrium outcomes, featuring respectively full participation and no participation in equilibrium, is independent of the information structure, the existence of other (possibly efficient and inefficient) Nash equilibria, intermediate between the maximal and the minimal Nash equilibria, is shown to depend critically on the in-

[6] Kohlberg and Mertens (1986) has argued strongly in favor of iterated dominance: "One might argue that, since dominated strategies are never actually chosen and since all players know this, then the deletion of such strategies can have no impact on strategic stability. This would lead to requiring that a strategically stable equilibrium remain so when a dominated strategy is deleted (and hence when the deletion is done iteratively)."

[7] Our dominance solvability is defined using iterative elimination of weakly dominated strategies. It is known that for generic finite extensive form games with perfect information, the solution of subgame perfect equilibrium and the procedure of iterative elimination of weakly dominated strategies are closely related (see Moulin (1986) and Osborne and Rubinstein (1994, Section 6.6)). Notice however that our sequential game is (typically) one with imperfect information.

[8] For games with only local strategic complementarity, consider an adoption game where too many people adopting may lead to increased competition and consequently we may have strategic substitutibility locally at some point rather than global complementarity.

formation structure. We provide a necessary and sufficient condition for the existence of *every* possible type of "intermediate" pure-strategy Nash equilibria lying strictly between no participation and full participation. We find that a defining feature of such an intermediate participation equilibrium is an information structure with the existence of at least a pair of agents who get information signals that are unaffected by each other's act of participation.

Our second result characterizes the relationship between information structures and dominance solvability. We follow Gale (1953), Moulin (1979) and others by iteratively deleting (weakly) dominated strategies in the normal form representation of our game to obtain a set of iteratively undominated (admissible) strategies.[9] We define the game to be dominance solvable when every such iteratively undominated strategy profile leads to the same (unique) outcome. This unique outcome turns out to be the efficient maximal outcome where all agents participate. While the concept of dominance solvability has frequently been used in the literature, our contribution lies in identifying the necessary and sufficient condition on the information structure of the game for it to be dominance solvable.

We define an information chain of length m as a set of m agents who are informationally linked in that the first agent's participation impacts the information signal the second agent receives, and the second agent's participation impacts the information signal the third agent receives, and so on. The tipping point λ represents the degree of participation that is needed for an improvement in the payoffs of the participating agents over the *status quo* to occur. We show that dominance solvability obtains and coordination failure is avoided *if and only if* there exists an information chain of length more than λ. Here, the length of the information chain can be interpreted as the degree of transparency of the information structure, i.e., the degree to which the information about the *aggregate* action of earlier players filters through to agents moving later in the sequence. This result implies that greater the need for coordination (larger the λ) greater the need for informational transparency that is necessary to eliminate coordination failure. Hence, our analysis suggests that a non-coersive decentralized policy tool for preventing coordination failure would be to increase the flow of information between agents and to allow them to act in their rational self interest.

The rest of the paper is organized as follows: Section 2 introduces the model. Section 3 presents our results on the existence of maximal and intermediate pure strategy Nash equilibria, while Section 4 provides a detailed analysis of dominance solvability. All formal proofs of our main results are collected in an appendix.

[9] Also see Gretlein (1982) for a discussion of this procedure in voting games. The procedure of iterative elimination of weakly dominated strategies has also been applied to chess-like games and two-player strictly competitive perfect-information games (Ewerhart (2000,2002)), signaling future actions by burning money (Ben-Porath and Dekel (1992)), dynamic bargaining games with a finite horizon (Tyson (2010)), and auctions (Azrieli and Levin (2011)).

2 The Model

Consider a set of agents $N = \{1, \ldots, n\}$ with $n \geq 3$ who move sequentially. We assume that for all $j \in \{1, \ldots, n-1\}$ agent j moves before agent $(j+1)$ according to the exogenously given order $(1, \ldots, n)$. For $j \in N$, agent j's move $b_j \in \{1, 0\}$ represents the choice between participating ("joining"), $b_j = 1$, and not participating ("not joining"), $b_j = 0$.[10] This gives us an action profile or an outcome $b = (b_j, b_{-j}) = (b_1, \ldots, b_n)$. Following Schelling (1973) we assume that all agents have identical preference orderings on the set of outcomes that depend on the agent's decision (whether or not to participate) and on the *number* of other participants.[11] This is a reasonable assumption when either the individual characteristics of the other agents are unimportant for a problem (such as with equal cost sharing for a public project) or when the information about relevant characteristics of the other agents is just not available (such as with purchasing health insurance when privacy concerns dictate that personal information not be available even though how many other people have joined may be public information). As in Schelling (1973), this allows us to construe the preferences as being represented by a utility function $g : \{1, 0\} \times \{0, \ldots, n-1\} \to \mathbb{R}$. Hence, for $\alpha \in \{0, \ldots, n-1\}$, $g(1, \alpha)$ (resp., $g(0, \alpha)$) represents the value of participating (resp., not participating) when α other individuals participate. Positive and negative spillovers from the decisions of others can then be represented by these functions being increasing and decreasing in α.[12]

In this paper we will analyze a special case of this model that leads to a coordination problem. To do so we will make a simplifying assumption that the payoff from non-participation is constant that can be normalized to zero together with a critical "threshold" assumption on preferences representing the minimum number of participants required for participation to be an improvement over the *status quo.*

2.1 Threshold

Assumption 1 (Tipping Point) *Let f be a function $f : \{1, \ldots, n\} \longrightarrow \mathbb{R}$ given by $f(x) = g(1, x-1)$. There exists $\lambda \in \{2, \ldots, n\}$ such that $f(x) > 0$ for all $x \geq \lambda$ and $f(x) < 0$ for all $x < \lambda$ and the payoff for individual j from an action profile $b = (b_j, b_{-j}) = (b_1, \ldots, b_n)$ is given by*

[10] Possible interpretations of $b_j = 1$ include buying an excludable public good or a club good, buying health insurance sold by the state, adopting a product standard for firms, and adopting a social norm.

[11] This type of anonymous dependence on the number of other participants differs from preferences depending on the *entire* action profile. The latter more general formulation can capture the differential impact on the benefit to participants from different sets of individual participants of the same size. Some problems are such that by their very nature the agents' payoffs depend on information that is non-anonymous. For instance, in the study of white flight from the inner cities to the suburbs, the ethnicity of those who are migrating has an impact on the payoff of the agents. Our model is not suitable for analysis of such problems.

[12] For instance, the value (in terms of personal safety in the case of an accident) of driving a large car ("joining") may positively depend on how many others are driving a large car while the value of driving a small car ("not joining) would depend negatively on how many others drive a large car.

$$u_j(b) = b_j f\left(\sum_{i\in N} b_i\right) \tag{1}$$

The condition $\lambda \geq 2$ in Assumption 1 implies that some minimal (but feasible) degree of coordinated participation is necessary for an improvement over the *status quo*. Geometrically, the assumption implies that the f function has a single crossing property, i.e., f is negative when $x = 1$, is never zero, and crosses the horizontal axis only once on the domain $\{1, ..., n\}$. Note that f is otherwise completely arbitrary permitting both positive and negative slopes which may be used to represent positive and negative spillovers. While these externalities are permitted, it is the implication of having the threshold ("tipping point") and the possibility of a coordination failure, rather than the presence of any externalities, that is of primary interest to us.

Depending on the specific context other additional restrictions may well be appropriate. The standard "convexity" assumption made in economics on technology and preferences, which entails quasi-concavity of utility functions and convexity of cost functions, gives rise to f being quasiconcave and even strictly concave in some representative agent models. Example 1 shows how Assumption 1 and strict concavity of f can arise from economies of scale in production or from positive spillovers in consumption (network externalities) or through some combination of these two effects.

Example 1 (i) (**Network Externalities**) *Suppose the per capita cost of servicing a communication network is a constant* $c > 0$ *and the benefit received by everyone in the network is given by* $v(\sum_N b_i)$ *where* v *is an increasing function. Thus, the net benefit for a typical agent joining the network is* $f(\sum_N b_i) = v(\sum_N b_i) - c$*. Furthermore, if* $v(1) < c$ *and* $v(n) > c$ *then it is easy to check that Assumption 1 and quasiconcavity (resp., strict quasiconcavity) of* f *will (generically) be satisfied if* v *is increasing (resp., strictly increasing) function.*

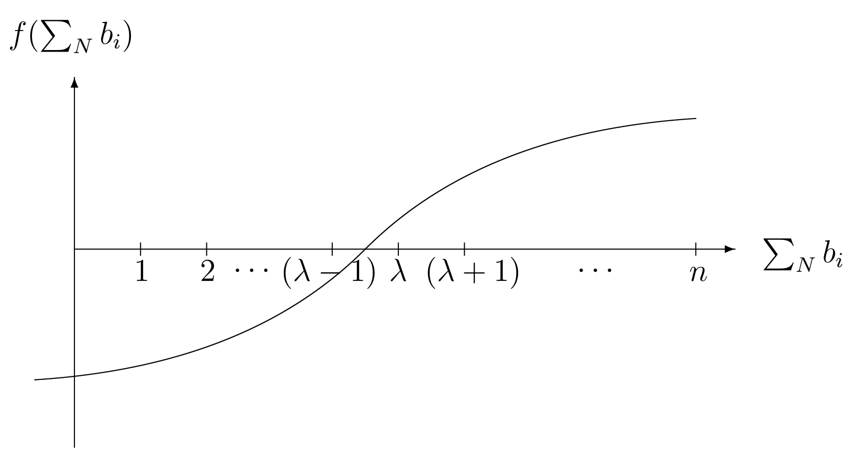

Figure 1. The valuation function $f(\sum_N b_i)$ for Example 1 ((i) and (ii)).

(ii) (**Economies of Scale**) *Consider a health insurance service provided to the agents at average cost where the technology of the industry providing the service is characterized by economies of scale with the average cost,* $C(\sum_N b_i)/(\sum_N b_i)$*, (strictly) declining as more individuals enroll for the service (i.e., it is a "natural monopoly"). If the "gross" value of the service to each agent is a constant* $v > 0$ *and is independent of how many others are getting the service, then the net benefit is* $f(\sum_N b_i) = v - C(\sum_N b_i)/(\sum_N b_i)$ *and the net valuation function* f *will be strictly increasing, capturing the positive spillovers associated with the economies of scale. If further more* $C(1) > v$ *and* $v > C(n)/n$*, then Assumption 1 will (generically) be satisfied and* f *will be strictly quasiconcave.*

In Example 1 (i), if congestion arises when $\sum_N b_i$ is large so as to lead to declining benefits, or in (ii) if after declining for a while, average cost increases as diseconomies of scale set in for large values of $\sum_N b_i$, then the function f will decline monotonically for large values of $\sum_N b_i$. In both these cases, if the effects of such congestion are not too strong, Assumption 1 would still be satisfied in the above examples.

Example 2 illustrates a "bang-bang" type valuation function where positive spillovers occurring immediately around a single value in the domain of f result in a tipping point:

Example 2 (**Voting**) *A group of agents decide whether to join a political party. Agents who join receive benefits from supporting the party if and only if the party wins. If the party loses the supporters of the party are penalized. The party wins if it receives the support of at least* $\lambda \in \{2, \ldots, n\}$ *agents, otherwise it loses. Letting* $b_i = 1$ *represent support for the party, the following function* f *satisfies Assumption 1 and is quasiconcave*:

$$f\left(\sum\nolimits_N b_i\right) = \begin{cases} 1 \; \textit{if} \; \sum_N b_i \geq \lambda, \\ -1, \; \textit{otherwise.} \end{cases}$$

This voting game can be modified so that the valuation function f declines over part of its domain. For instance, if in Example 2 the aggregate constant (unit) benefit of winning that accrues to the supporters of the party is shared among its members, then each individual member's benefit declines as the membership of the party increases beyond λ.[13] Now the function f, which takes value -1 before it reaches the total participation level of λ, reaches a maximum of $1/\lambda$ at the participation level of λ, and is monotonically declining at participation levels greater than λ.

2.2 Information Structure and Information Chain

2.2.1 Information Structure

Before making a move every agent receives a piece of information which depends on the information structure of the model. Let $\kappa(\cdot)$ be an operator that associates with each

[13]In other words, if the aggregate benefit is shared among its members, then $f(\sum_N b_i) = \frac{1}{\sum_N b_i}$ if $\sum_N b_i \geq \lambda$, and $f(\sum_N b_i) = -1$ otherwise.

$j \in N$ a unique number $\kappa(j) \in \{0, \ldots, j-1\}$. The **information structure** $\mathcal{I}$ is defined as an n-vector of non-negative integers $\mathcal{I} = (\kappa(1), \ldots, \kappa(n))$. If $\kappa(j) = 0$ then j receives no information before making her decision. If $\kappa(j) > 0$, then $\kappa(j)$ represents a specific agent moving before j and the report that j receives is an integer from the set $\{0, \ldots, \kappa(j)\}$ indicating exactly how many (though not which) of the agents from the set of agents $\{1, \ldots, \kappa(j)\}$ have chosen to participate.[14] Agent j before making her move only observes the *aggregate* number of participations from the *first* $\kappa(j)$ agents. Whenever $\kappa(j) \geq j'$, any alteration in j''s action, *ceteris paribus*, is reflected in the aggregate report that j receives. So, for agents $j, j' \in N$, if $\kappa(j) \geq j'$ then j's information **covers** j''s action. We will abbreviate this and say that j covers j'.

We will assume that the information structure described above is monotone:

Assumption 2 ($\mathcal{I}$-Monotonicity) *For all $j, j' \in N$, $j \geq j'$ implies that $\kappa(j) \geq \kappa(j')$.*

Assumption 2 implies that agents moving later in the sequence have at least as much *payoff relevant* information as those who move earlier.[15] This assumption ensures that the sequence of numerical reports that agents $1, \ldots, n$ receive is necessarily a (weakly) monotonically increasing set of integers. A useful interpretation of Assumption 2 has to do with the recognition that *aggregate information* about previous moves can be incomplete and that this partial information may become available with a lag. For instance, one can think of a scenario where before taking any action, each agent checks a common website that reports the number of "hits" and the total number of people who have joined. If collecting this information takes time and the information is updated at fixed time intervals, this can result in data collection lags and data publication lags that can be modeled using our information structure. Example 3 illustrates the concept of an information structure and how lags may be modeled for the case of $n = 3$.

Example 3 *Let $N = \{1, 2, 3\}$. There are five possible monotone information structures:*

$$\mathcal{I}_1 = (0,0,0)\,;\, \mathcal{I}_2 = (0,0,1)\,;\, \mathcal{I}_3 = (0,0,2)\,;\, \mathcal{I}_4 = (0,1,1)\,;\, \mathcal{I}_5 = (0,1,2)\,.$$

Information structure $\mathcal{I}_1$ is the extreme case where no one has any information about previous agents' moves, while $\mathcal{I}_5$ represents the polar opposite case where before they move each agent knows the complete aggregate history of previous moves, i.e., each agent knows exactly how many of the previous agents have joined. For instance, if there is no collection lag (the website operators collect and process the information instantaneously) but suppose the website is updated every two periods, then we obtain the information structure $\mathcal{I}_3 =$

[14] Consequently, agent j knows how many agents from $\{1, \ldots, \kappa(j)\}$ have picked the *status quo.* In some cases, for instance if $\kappa(j) = 1$ or if the report received is equal to $\kappa(j)$, j is able to deduce not only how many but also who has joined the group.

[15] If $\kappa(j) = 1$ and $\kappa(j') = 2$, then even though by looking at her signal j knows whether 1 has joined or not joined and j' may not, j' will receive at least as much *payoff relevant* information as j does, since all that matters for payoffs is how many individuals rather than which individuals have joined .

$(0, 0, 2)$. *Here the two-period publication lag implies that agents 1 and 2 get no information before they move while* 3 *has the entire aggregative history before she moves. Suppose that in addition to the publication lag, there is a collection lag and it takes one period to collect and process the data. Then, the report* 3 *receives only includes the information about 1's move, leading to the information structure* $\mathcal{I}_2 = (0, 0, 1)$.

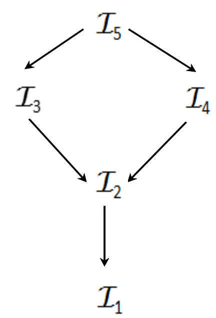

Figure 3. The Information Structures in Example 3.

The information structures $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ and $\mathcal{I}_5$ can be ranked in that there is a greater availability of (payoff relevant) information as we move from $\mathcal{I}_1$ to $\mathcal{I}_5$.[16] However, the information contents of $\mathcal{I}_3$ and $\mathcal{I}_4$ cannot be compared. The lattice showing the quasi-ordering of the information structures in Example 3 is presented in Figure 3, where each arrow represents the relation of "contains more payoff relevant information than."

In general, we can compare information structures by comparing the associated vectors with those which are larger (in the vector sense) representing "smaller" lags and hence a greater availability of information. While some comparisons between information structures may not be possible, the vector comparisons of information structures will nevertheless always be a *pre-order*, a transitive and reflexive (possibly incomplete) binary relation on the set of all information structures with the *maximal* and *minimal* information structures being the n-vectors $\mathcal{I}^{\max} = (0, 1, \ldots, n-1)$ and $\mathcal{I}^{\min} = (0, 0, \ldots, 0)$, respectively. Finally we observe the following:

Proposition 1 *κ is strictly monotone (i.e., if $j' > j$ implies $\kappa(j') > \kappa(j)$) iff $\mathcal{I} = \mathcal{I}^{\max}$.*

2.2.2 Information Chain

Recall that when agent j's information covers agent j', the marginal impact of j''s move on the total number of participations can be observed by j. Thus, j''s knowledge about the aggregate history of the play *and* the impact of j''s action on this aggregate history filter through to agent j. Thus an important property of an information structure is the concept of an *information chain* which provides a measure of the possibility of such transfer of

[16] Agent 3 in the information structure $\mathcal{I}_2$ knows whether 1 has moved but does not in the structiure $\mathcal{I}_3$. Nevertheless agent 3 has more payoff relevant information in $\mathcal{I}_3$ because the payoffs depend on the total number of individuals participating and $\mathcal{I}_3$ provides agent 3 with more of the payoff relevant aggregate participation information.

aggregate information among agents. Intuitively, an information chain measures how *easily* the information flows within the model.

Definition 1 *An ordered set of agents $(i_1, \ldots, i_m) \subseteq N$ forms an* ***information chain*** *of length m if and only if for all $s \in \{2, \ldots, m\}$, agent i_{s-1}'s information covers agent i_s.*

Thus, an information chain of length m in N_1 is a set of m informationally linked agents in N_1 who can be ordered such that i_1's information covers all the others in the chain, i_2's information covers all agents in the chain except i_1, and so on. Clearly, the existence of an information chain of length m implies the existence of an information chain of length m' with $m' \leq m$. From finiteness, it follows that an information chain of maximal length m^* always exists. We interpret m^* as a summary measure of the extent to which available information filters through from agents moving earlier to those moving later in N and as an indicator of the *transparency* of the information structure. Note, however, that m^* is a property of the information structure that does not measure how much information is available in an information structure. In particular, while information structures with strictly more information will have chains of maximal length that are at least as large, information structures with less information may have maximal information chains that are just as large as information structures with more information. In Example 3, the maximal lengths of the information chains for $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_4$ and $\mathcal{I}_5$ in N are respectively $0, 2, 2, 2$ and 3. Though $\mathcal{I}_4 = (0, 1, 1)$ has "more information" than $\mathcal{I}_2 = (0, 0, 1)$, $\mathcal{I}_4$ is *not* any more transparent (as measured by m^*) than $\mathcal{I}_2$.[17]

2.3 Normal Form Game

A pure strategy of agent $j \in N$ is a $(\kappa(j) + 1)$-dimensional binary vector of conditional moves (or actions) $a_j = (a_j(1), \ldots, a_j(\kappa(j) + 1)$. For all $l \in \{1, \ldots, \kappa(j) + 1\}$ we interpret the **conditional move** $a_j(l) \in \{1, 0\}$ as agent j's decision on whether to participate *if* given the information that $(l - 1)$ agents out of the the first $\kappa(j)$ agents are going to participate. The conditional action $a_j(l)$ becomes j's move b_j if and only if j receives the report that exactly $(l - 1)$ out of $\kappa(j)$ previous agents have chosen to participate. In this case we will say that the **pre-requisite** of the l^{th} coordinate of j's strategy has been satisfied. The set of all possible pure strategies of j is denoted by A_j and a pure strategy profile is $a = (a_j, a_{-j}) = (a_1, \ldots, a_n) \in A = \Pi_{j \in N} A_j$.

For any profile a, the path of play is a set of coordinates $a_j(\ell)$ such that for all $j \in \{1, ..., n\}$, the prerequisite of $a_j(\ell)$ is met and hence $a_j(\ell) = b_j(a)$. In this case, we will say that the ℓ^{th}-coordinate of j's strategy is **on the path of play**. For each profile a, one and only one coordinate of each individual is on the path of play and this *unique* path of play yields the action profile $b(a)$. For any other coordinate l whose pre-requisite

[17] An information structure can induce various information chains and possibly multiple information chains with maximal length. We will develop a simple algorithm to find an information chain with maximal length for an information structure in Section 3.2.

is not satisfied, we will say that the l^{th}-coordinate of j's strategy is **off the path of play**. Finally, given a strategy profile a, for all $j \in N$, agent j's payoff (using (1)) is given by $u_j(b(a)) = b_j(a) f\left(\sum_{i \in N} b_i(a)\right)$.

Gathering together the notation developed so far we define the normal form game $\mathcal{G}_0$ as the quadruple $\langle N, A, \mathcal{I}, \{u_j\} \rangle$.

Example 4 illustrates these concepts for a game $\mathcal{G}_0$.

Example 4 *Let* $N = \{1, 2, 3\}$, $f(1) = -3$, $f(2) = f(3) = 4$ *and hence* $\lambda = 2$. *Consider the information structure* $\mathcal{I} = (0, 1, 1)$ *and the following strategy profile* a*:*

Agent j	*1*	*2*	*3*
Strategy a_j	$(\mathbf{0})$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$

Given the strategy profile a*, agents 2 and 3 join if and only if they observe that agent 1 has not joined. The pre-requisites of* $a_2(1)$ *and* $a_3(1)$ *are satisfied and the path of play is represented by the sequence of conditional actions* $a_1(1)$, $a_2(1)$ *and* $a_3(1)$ *(coordinates in boldface), resulting in action profile* $b(a) = (0, 1, 1)$ *and a payoff vector* $(0, 4, 4)$.

3 Information Structure and Nash Equilibria

A pure-strategy Nash equilibrium (PSNE) of the normal form game $\mathcal{G}_0$ is a strategy profile (a_j, a_{-j}) such that for all $j \in N$ and all $a'_j \in A_j$, $u_j(a_j, a_{-j}) \geq u_j(a'_j, a_{-j})$. The action profile $b(a)$ associated with a PSNE a is called a pure-strategy Nash equilibrium outcome (PSNEO). Thus, in Example 4, the strategy profile a is a PSNE and $b(a) = (0, 1, 1)$ is a PSNEO in which two out of the three agents participate.

For a PSNE a, a_j is j's best response to the contingency a_{-j} for all j, and checking whether a strategy profile is a PSNE amounts to verifying that unilateral deviations from the strategy profile are not beneficial for the deviating agent. In addition, we only need to consider those unilateral deviations that change the coordinate (of the deviating agent's strategy) along the path of play. For any strategy profile we will refer to the coordinates of an individual's strategy off the path of play as (payoff) **irrelevant coordinates** for that profile**.** Furthermore, due to Assumption 1, a unilateral change in the coordinate of an agent's strategy along the path of play *necessarily* changes that agent's payoff.[18] Hence, we will refer to such coordinates on the path of play as being (payoff) **relevant coordinates** for that profile.

Assumption 1 implies that the *status quo* (with utility 0) is strictly better than that of participating when the participating group size is less than the tipping point λ. Every agent by playing a strategy in which all the coordinates are zero can guarantee this *status quo* payoff. The *status quo* payoff is hence the reservation payoff and it follows that *all*

[18] The agents payoff either changes from being non zero to being zero or from being zero to being either strictly positive or strictly negative.

agents receive non-negative payoffs in a PSNE and we must either have $\sum_N b_i(a) \geq \lambda$ or $\sum_N b_i(a) = 0$ for a PSNE a. This observation leads us to consider three mutually exclusive (and exhaustive) types of Nash equilibria:

(i) A *maximal PSNE* is a PSNE where all n *agents* participate in equilibrium. A maximal PSNE results in a unique PSNEO with payoffs $(f_1(n), ..., f_n(n))$. This outcome is necessarily Pareto Efficient since any other outcome will reduce the utility of some individual from positive to zero.

(ii) A *minimal PSNE* is a PSNE where *nobody* participates in equilibrium. The associated unique PSNEO with payoffs $(0, \ldots, 0)$ is strongly inefficient in the sense that all achievable benefits from cooperation are lost. In our setting, every other PSNEO weakly Pareto dominates the minimal PSNEO.[19]

(iii) An *intermediate PSNE* is a PSNE where *exactly* $\lambda + \tau$ agents participate in equilibrium for some $\tau \in \{0, \ldots, (n - \lambda - 1)\}$.

While an intermediate PSNEO represents a (weak) Pareto improvement over the *status quo*, an intermediate PSNEO may be either efficient or inefficient depending on the properties of the valuation function f. For instance, if f is non-decreasing, all the intermediate PSNEOs are inefficient since these outcomes are Pareto dominated by the maximal PSNEO. Observe, however, that these intermediate PSNEO may *all* be efficient as well. This will happen for instance when starting from a negative value, f reaches a maximum at the tipping point λ and is strictly decreasing for all participation levels greater than or equal to λ (see the discussion after Example 2). The reason for Pareto efficiency of the intermediate PSNEOs in this case is that when compared to an outcome with a smaller participation level some individual participating in the PSNEO will have to be left out and her payoff will decline from positive to zero; on the other hand, when any intermediate PSNEO (in this case) is compared to an outcome with a larger participation level than λ, since f declines (in Example 2) a movement to an outcome with a higher participation level will make all existing individuals participating in the PSNEO strictly worse off. Indeed, using this argument it is easy to see that the following is true:[20]

Proposition 2 *A PSNEO is efficient if and only if all outcomes with a larger participation level have strictly lower payoffs for the participants.*

While the maximal PSNEO may or may not maximize the utilitarian "sum of utilities" welfare function, it always maximizes the utility of the worst off individual and thus it maximizes the Rawlsian welfarist maximin (or leximin) egalitarian welfare function.

Proposition 3 *A PSNEO maximizes the Rawlsian welfare function* $W = \max_a \min_j \{u_j(a)\}$ *iff it is the payoff from the maximal PSNE. If* f *is weakly increasing the maximal PSNE*

[19] We say weakly Pareto dominates in the sense that some individulas are better off and no individual is worse off.

[20] This condition is clearly satisfied for the PSNE with maximal participation. On the other hand, if f is strictly quasi-concave, then the intermediate PSNEOs can be partitioned such that PSNEOs with participation levels above λ and below "$\arg\max f$" will be inefficient and those with participation levels of "$\arg\max f$" and above will be efficient.

being the only efficient outcome maximizes all Paretian welfare functions and in particular it maximizes the sum of utilities.

We will now discuss how the existence of various PSNEOs of the game $\mathcal{G}_0$ is related to the information structure of the game.

Proposition 4 *The game $\mathcal{G}_0$ always admits maximal and minimal PSNEOs regardless of the information structure $\mathcal{I}$.*

Proposition 4 is entirely driven by Assumption 1. Consider any strategy profile a in which the top (i.e., largest) coordinates of individuals $1, 2, ...,$ n are 1. Any unilateral change in the individual strategies along the path of play would lower the deviating individual's payoffs from positive to zero. Hence a is a PSNE regardless of $\mathcal{I}$. Similarly, since $\lambda \geq 2$, it follows that a strategy profile a where all the coordinates in the strategies of all the individuals are zero is a PSNE regardless of $\mathcal{I}$.

Unlike the case of maximal and minimal PSNEs, whether an intermediate PSNE exists depends on the information structure of $\mathcal{G}_0$. It is easy to check that with $n = 3$ and $\lambda = 2$ and $\mathcal{I}^{\min} = (0, 0, 0)$, no intermediate PSNE can exist since the best response to any contingency when two other agents are participating is to participate. On the other hand, the reader can easily check that in our earlier Example 4 illustrates an intermediate PSNE. Our next result provides a characterization of the information structure $\mathcal{I}$ of $\mathcal{G}_0$ that can give rise to an intermediate PSNE with **exactly** $(\lambda + \tau)$ participants for $\tau \in \{0, \ldots, (n - \lambda - 1)\}$.

Theorem 1 *In the game $\mathcal{G}_0$, let $\tau \in \{0, \ldots, n - \lambda - 1\}$. (i) If there exists $j^* \in \{n - \lambda - \tau + 1, \ldots, n - \tau - 1\}$ such that $\kappa(j^*) \geq n - \lambda - \tau$, $\kappa(j^* + 1) < j^*$, and $n - j^* - \tau - 1 \geq |\{j : \kappa(j) = j^*\}|$ then there exists a PSNE a such that $\sum_N b_i(a) = \sum_{N \setminus \{1, \ldots, n-(\lambda+\tau)\}} = (\lambda + \tau)$. (ii) If there exists a PSNE a such that $\sum_N b_i(a) = (\lambda + \tau)$ then there exists $j^* \in \{n - \lambda - \tau + 1, \ldots, n - \tau - 1\}$ such that $\kappa(j^*) \geq n - \lambda - \tau$, $\kappa(j^* + 1) < j^*$, and $n - j^* - \tau - 1 \geq |\{j : \kappa(j) = j^*\}|$.*

Corollary 1 *In the game $\mathcal{G}_0$, let $\tau \in \{0, \ldots, n - \lambda - 1\}$. Then, there exists a PSNE a such that $\sum_N b_i(a) = (\lambda + \tau)$ if and only if there exists $j^* \in \{n - \lambda - \tau + 1, \ldots, n - \tau - 1\}$ such that $\kappa(j^*) \geq n - \lambda - \tau$, $\kappa(j^* + 1) < j^*$, and $n - j^* - \tau - 1 \geq |\{j : \kappa(j) = j^*|$.*

Corollary 2 *In the game $\mathcal{G}_0$, if either $\mathcal{I} = \mathcal{I}^{\max} = (0, 1, \ldots, n - 1)$ or $\mathcal{I} = \mathcal{I}^{\min} = (0, \ldots, 0)$ then there does not exist an intermediate PSNE.*[21]

Corollary 3 *In the game $\mathcal{G}_0$, there is a PSNE a such that $\sum_N b_i(a) = \lambda$ if and only if there exists $j^* \in \{n - \lambda + 1, \ldots, n - 1\}$ such that $\kappa(j^*) \geq n - \lambda$ and $\kappa(j^* + 1) < j^*$.*

[21] In particular note that in the game $\mathcal{G}_0$, if the information structure is strictly monotone then an intermediate PSNE does not exist.

Corollary 4 *In the game $\mathcal{G}_0$, let $\tau \in \{0, \ldots, n-\lambda-1\}$. If $\mathcal{I} = (\kappa(j))$ where $\kappa(j) = 0$ for all $j \in \{1, ..., n-\lambda-\tau\}$ and $\kappa(j) = n-\lambda-\tau$ for all $j \in \{n-\lambda-\tau+1, ..., n\}$ then $\mathcal{G}_0$ has a PSNE with $\lambda+\tau$ participants.*

For any given participation threshold λ, Theorem 1 and Corollary 1 completely characterize the information structures that can induce intermediate PSNEs and the associated PSNEOs including those PSNEOs where exactly the *first* $(n-\lambda-\tau)$ agents choose the *status quo* and the *last* $(\lambda+\tau)$ agents participate. The rough intuition behind Theorem 1 is that in an intermediate PSNE, some early agents do not want to join because a unilateral deviation from any one of them from not joining to joining triggers a large chain reaction of defections from among the later participants starting with j^*. It is only if the scale of defections is large enough to make the total number of participants fall below the critical number λ will such a deviation be rendered undesirable thus supporting a PSNE. When $\tau \geq 1$, the condition $n - j^* - \tau - 1 \geq |\{j : \kappa(j) = j^*\}|$ of Theorem 1 has bite – it requires that not "too many" agents j moving after j^* have $\kappa(j) = j^*$. In particular, the upper limit on the number of agents j with $\kappa(j) = j^*$ is necessary for this cascading series of defections *to be large enough.*[22] Notice that **none** of the agents j whose information exactly covers j^* (i.e., $\kappa(j) = j^*$) can be a part of this chain of defections since the information report that such an agent j receives remains unchanged when in reaction to the participation of a non-participant j^* defects from being a participant to becoming a nonparticipant. (The use of this type of nonmonotone strategies by the defecting agents may not, *prima facie*, be unreasonable as agents may gang up to prevent others from joining when additional participation reduces the utility of existing participants).[23]

The Corollaries 1 to 4 follow immediately from Theorem 1. Corollary 1 uses (ii) of the theorem together with the fact that when only the last $\lambda+\tau$ individuals participate we do have $\lambda+\tau$ participants to provide a necessary and sufficient condition for the existence of intermediate PSNEs. Corollary 2 concludes that under the maximal and minimal information structures, an intermediate PSNE cannot occur. Corollaries 3 demonstrates that the information structures consistent with intermediate PSNEs always feature a pair of agents, j^* and j^*+1, such that agent (j^*+1)'s information does not cover j^* and both agents are such that their information covers the first $(n-\lambda-\tau)$ agents of the sequence. Corollary 4 shows the existence of information structures such that all feasible intermediate participation levels are possible PSNEOs.

To summarize, three conditions on the information structure are important for the existence of intermediate PSNEs: First, there should be an agent j^*, not covered by the next agent j^*+1, who covers the first $(n-\lambda-\tau)$ agents. Second, there should be enough agents after j^* whose defections are able to bring the total number of participants below λ, i.e., $j^* \leq n-\tau-1$. Thirdly, there should be an upper limit "$n-j^*-\tau-1$" on the number of agents j in $\{j^*+2, ..., n\}$ who have $\kappa(j) = j^*$.

[22] For a relevant deviation to be undesirable for the non-participants, the cascade of defections needs to involve at least τ other agents *after* j^* *and* j^*+1 to defect after the deviation.

[23] See our discussion of credibility of PSNEs in the next section.

4 Information Chains and Dominance Solvability

We first use an example to illustrate that some of the PSNEs identified in Section 3 can be implausible:

Example 5 *Let the game be given by $N = \{1, 2, 3\}$, information structure $\mathcal{I} = (0, 0, 1)$ and payoffs $f(1) = -3$, $f(2) = f(3) = 4$, and hence $\lambda = 2$. Consider the strategy profile*

Agent j	*1*	*2*	*3*
Strategy a_j	$(\mathbf{0})$	$(\mathbf{0})$	$(\mathbf{0}, 0)$

This PSNE is not plausible since it involves agent 3 committing to a conditional move ("threat") off the path of play that can negatively affect her payoff. After observing agent 1 participating agent 3 has a strict incentive to participate. (A similar argument shows that the intermediate PSNE in Example 4 is also not credible.)

How do we eliminate such implausible PSNE equilibria and focus on equilibria that are more compelling? A standard refinement to weed out PSNEs involving non-credible strategies is subgame perfection. However, it is easily verified that the game in Example 5 admits no proper subgames and hence all its PSNEs, *including the non-credible* PSNE in the example, are subgame perfect. To filter out non-credible equilibria finer refinements than subgame perfection (e.g., perfect Bayesian equilibrium and sequential equilibrium) usually entail the use of a cardinal framework and use expected utility maximization together with the possibility of mixed strategies. We take an alternative approach. We use a refinement of PSNE that has been extensively analyzed and used, and one that is easily applicable in our context: dominance solvability. This concept, with an underlying intuition similar to that of subgame perfection, can be traced back to Moulin (1979), relies on *iterated elimination of weakly dominated strategies*, which then results in a set of strategy profiles all of which give the *same* outcome.

We introduce some notation to formalize the process of iterated elimination. In $\mathcal{G}_0 = \langle N, A, \mathcal{I}, \{u_j\}\rangle$, a strategy a'_j is (*weakly*) *dominated* by a strategy a_j if $u_j(a_j, a_{-j}) \geq u_j(a'_j, a_{-j})$ for all a_{-j} and for some a'_{-j}, $u_j(a_j, a'_{-j}) > u_j(a'_j, a'_{-j})$.[24] Let $\mathcal{R}$ be a function which gives us the game $\mathcal{G}_1 = \mathcal{R}(\mathcal{G}_0)$ obtained by eliminating *all dominated strategies of all the agents* in game $\mathcal{G}_0$. Applying the operator $\mathcal{R}$ successively generates a sequence of games $\mathcal{G}_1$, $\mathcal{G}_2$, ..., where $\mathcal{G}_{h+1} = \mathcal{R}(\mathcal{G}_h)$ for $h \geq 0$. Since only strategies are eliminated (and none added), as one goes from one game in the sequence to the next, the strategy profiles in $\mathcal{G}_{h+1}$ is a subset of the set of strategy profiles in games $\mathcal{G}_1, \ldots, \mathcal{G}_h$ with the games in $\{\mathcal{G}_1, \mathcal{G}_2, \ldots\}$ otherwise having the same players and the same information structure as the game $\mathcal{G}_0$. In addition, the payoff functions in $\mathcal{G}_h \in \{\mathcal{G}_1, \mathcal{G}_2, \ldots\}$ are the same as those in $\mathcal{G}_0$ except for the fact that these functions are restricted to a smaller domain in $\mathcal{G}_h$.

[24] We only use weak domination in our arguments and in the absence of ambiguity, hereafter, we will drop the adjective 'weak'.

If $\mathcal{G}_s = \mathcal{R}(\mathcal{G}_s)$ for some $\mathcal{G}_s \in \{\mathcal{G}_1, \mathcal{G}_2, \ldots\}$, then $\mathcal{G}_s$ is said to be *irreducible.* Since $\mathcal{G}_0$ has a finite set of players each with a finite set of strategies, an irreducible game always exists. We will analyze the sequence $\{\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_M\}$ where $\mathcal{G}_M$ is the *first* irreducible game in the sequence. Finally, our concepts of strategy profile, action profile, PSNE, PSNEO, contingency, and the concepts of coordinates of individual strategies being on and off the path of play (i.e., relevant and irrelevant coordinates) defined for $\mathcal{G}_0$ extend naturally and without any ambiguity from $\mathcal{G}_0$ to $\mathcal{G}_h$ for all $h \in \{1, \ldots, M\}$. The game $\mathcal{G}_0 = \langle N, A, \mathcal{I}, \{u_j\}\rangle$ will be said to be *dominance solvable* if $\mathcal{G}_M$ has only one outcome.[25]

To see how this process works, applying the iterated elimination process to Example 5 above gives rise to the following PSNE:

Agent j	1	2	3
Strategy a_j	$(\mathbf{1})$	$(\mathbf{1})$	$(0, \mathbf{1})$

The very first round of elimination removes dominated strategies "$(0, 0)$" and "$(1, 0)$", leaving only two strategies "$(0, 1)$" and "$(1, 1)$" for agent 3. Hence, in $\mathcal{G}_1$ agent 1's strategy "(1)" dominates "(0)" and in $\mathcal{G}_2$ agent 2's strategy "(1)" dominates "(0)", leading to $\mathcal{G}_M = \mathcal{G}_3$ with the unique outcome $b(a) = (1, 1, 1)$. Therefore, dominance solvability here filters out the non-credible equilibrium, resulting in a unique outcome.

However, depending on the information structure, $\mathcal{G}_0$ may not be dominance solvable and the resulting $\mathcal{G}_M$ may have both minimal and intermediate PSNEs.

Example 6 *Consider $\mathcal{G}_0$ with $N = \{1, 2, 3, 4\}$, $\mathcal{I} = (0, 1, 1, 1)$ and payoffs $f(1) = f(2) = -3$, $f(3) = 5$ and $f(4) = 2$, and hence $\lambda = 3$. Here, no strategy in $\mathcal{G}_0$ is dominated (hence $\mathcal{G}_0 = \mathcal{G}_M$) and the intermediate PSNE (a) and minimal PSNE (a') shown below survive the iterated eliminating process:*

Intermediate a :

Agent j	*1*	*2*	*3*	*4*
Strategy a_j	$(\mathbf{0})$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$

Minimal a' :

Agent j	*1*	*2*	*3*	*4*
Strategy a'_j	$(\mathbf{0})$	$(\mathbf{0}, 0)$	$(\mathbf{0}, 0)$	$(\mathbf{0}, 0)$

In general, whether the game $\mathcal{G}_0$ is dominance solvable and whether $\mathcal{G}_M$ necessarily has the maximal PSNE as a unique outcome or whether coordination failure remains a possibility after the iterated elimination of dominated strategies depends on m^*, the maximal length of information chains in $\mathcal{G}_0$ and λ the degree of coordination necessary for improvement over the *status quo*. Recall that m^* is a *property* of the information structure that provides us with a summary measure of information transfer possibilities within the game and represents a measure of the extent to which the information structure allows the information to filter through from players who move earlier to players who move later

[25] This is weaker than the alternative definition that is often used requiring that every individual in $\mathcal{G}_M$ has only one strategy.

in the game allowing the earlier players to predict how a later player would behave if she acted rationally. Whether m^* is greater than or equal to the coordination threshold λ becomes key to determining whether the game $\mathcal{G}_0$ is dominance solvable. Associating rational decentralized decision making with individuals playing strategies which survive iterated dominance and which correspond to some Nash equilibrium, the condition $m^* \geq \lambda$ determines whether decentralized rational decision making leads to the maximal PSNE which is efficient and gives us the Rawlsian maximum welfare.

Theorem 2 *Let $\mathcal{G}_0$ be a game with tipping point λ and let m^* be the maximal length of an information chain in $\mathcal{G}_0$. Then $\mathcal{G}_0$ is dominance solvable if and only if $m^* \geq \lambda$.*

Corollary 5 *(i) If $\mathcal{G}_0$ is dominance solvable, then $\mathcal{G}_M$ has a unique efficient outcome given by the maximal PSNEO. (ii) If $\mathcal{G}_0$ is not dominance solvable, then $\mathcal{G}_M$ has a minimal PSNE in which no individual participates.*

Corollary 6 *(i) If $\lambda = 2$, then $\mathcal{G}_0$ is dominance solvable iff the information structure of $\mathcal{G}_0$ is not minimal. (ii) If $\lambda = n$, then $\mathcal{G}_0$ is dominance solvable iff the information structure of $\mathcal{G}_0$ is maximal.*

Remark 1 Observe that the games in Examples 4 and 5 both have $\lambda = 2$ and $m^* = 2$ and are hence dominance solvable and that the additional information that individual 2 has in Example 4 (as compared to Example 5) is of no importance in predicting the outcome of the game. In contrast, the game in Example 6 has $\lambda = 3 > m^* = 2$, and is hence not dominance solvable.

One implication of Theorem 2 is that policies promoting an improved flow of the information between agents can be used as a policy tool to promote coordination. The intuitive reason as to why the condition "$m^* \geq \lambda$" leads to dominance solvability can be seen from an argument similar to backward induction that we provide in the proof of the sufficiency of the condition for our theorem (and in our discussion of Example 5 above). This should not be a surprise to the reader since dominance solvability has a similar intuitive basis to subgame perfection for games with perfect information. Why "$m^* \geq \lambda$" is necessary for dominance solvability is more opaque and represents a deeper and more important result. It identifies for us the property of the information structure *without* which we *cannot* rule out coordination failure by using dominance solvability. The proof of necessity consists of showing that this backward induction type process will break down and that this will necessarily lead to at least two equilibrium outcomes the maximal PSNEO and the minimal PSNEO surviving in $\mathcal{G}_M$.

An Algorithm for Finding m^*: Since an information structure $\mathcal{I}$ typically admits information chains of various lengths, it is important to identify a maximal information chain of length m^* from any given information structure. We will refer to the *canonical* *information chain* of $\mathcal{G}_0$ as the information chain of maximum length m^* constructed using the following algorithm: Construct an ordered set of agents $(i_1, i_2, \ldots, i_m)$ such that $i_1 = n$ and $i_{s+1} = \kappa(i_s)$ whenever $\kappa(i_s) > 0$ and $\kappa(i_m) = 0$. Since the set of agents is finite and the covering relation is both asymmetric ($i_s \neq \kappa(i_{s+1})$ for all s) and transitive, the sequence is finite and well defined and involves the repeated application of the operator $\kappa(\cdot)$. Thus, $i_2 = \kappa(n)$, $i_3 = \kappa^2(n) = \kappa(\kappa(n))$,$\ldots$, $i_m = \kappa^{m-1}(n)$ and $\kappa(i_m) = \kappa^m(n) = 0$. Figure 3 provides a graphical illustration.

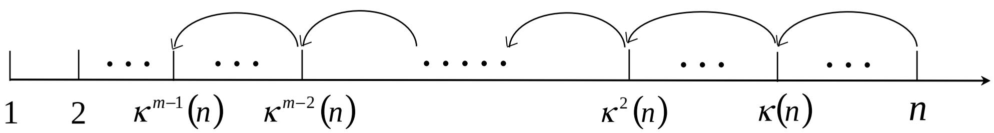

Figure 3. Constructing the Canonical Information Chain of Maximal Length.

We now argue that $m = m^*$ and that this canonical information chain is of maximal length. It suffices to show that if there is an information chain in $\mathcal{G}_0$ given by $(j_1, \ldots, j_t)$ then $m \geq t$. To this end, we establish a one to one function from the set $\{j_1, \ldots, j_t\}$ to a subset $\{i_1, \ldots, i_m\}$ of the canonical sequence. Notice that by the definition of an information chain, "j_1 covers j_2" implies (by $\mathcal{I}$-Monotonicity) "$i_1 = n$ covers j_2" and in particular $\kappa(n) = i_2 \geq j_2$. Similarly, "j_2 covers j_3", using $i_2 \geq j_2$, implies "i_2 covers j_3 and $i_3 = \kappa(i_2) = \kappa^2(n) \geq j_3$." Repeating this argument we have that for each $j_k \in \{j_1, \ldots, j_t\}$ there exists a distinct i_k such that $i_k \geq j_k$. This establishes a one to one function from the set $\{j_1, \ldots, j_t\}$ to $\{i_1, \ldots, i_m\}$, proving that $m \geq t$. Our next proposition summarizes this property of the canonical information chain.

Proposition 5 *For the game $\mathcal{G}_0$ the ordered set of agents given by $(i_1, \ldots, i_m)$, where $i_1 = n$, $i_2 = \kappa(i_1)$,..., and $i_m = \kappa^{m-1}(i_1)$ and $\kappa^m(i_1) = \kappa(i_m) = 0$, is an information chain with maximal length $m = m^*$.*

4.1 Extensions and Robustness

To check the robustness of Theorem 2, we examine the implications for Theorem 2 of relaxing our assumptions on the iterated dominance procedure, on modeling of preferences, and on the information structure of the model.

Order Independence

While the concept of using weak dominance ("admissibility") to eliminate implausible equilibria has strong support in the literature, the process of iterated weak dominance has faced a particular criticism. While our operator $\mathcal{R}$ specifies that in each stage of the iteration all dominated strategies are removed, one can propose alternative operators which remove some but not necessarily all dominated strategies in each stage of the iteration. It has been argued that this change in the order of elimination of dominated strategies can matter and that the different sequences of games generated by alternative operators may lead to different irreducible games. In some of these cases it has been shown that depending on the order of iterative elimination process the irreducible game may not have an unique outcome, creating ambiguity about the outcome of the game. In our case, if the game is dominance solvable, then no matter what alternative operator $\overline{\mathcal{R}} \neq \mathcal{R}$ is used (as long as some dominated strategies are eliminated in each stage of the iteration) the order of elimination will not matter and the irreducible game obtained from alternative operators will have exactly the same unique PSNEO as $\mathcal{G}_M$. The intuition for this being true is that for our particular game, if under our specified operator $\mathcal{R}$, a coordinate of all strategies of some agent becomes 1 at some stage of the reduction process then either this will also be true under any alternative operator or that coordinate will have become irrelevant in the irreducible game. This order independence result is not surprising in our model since our game satisfies Satterthwaite and Sonnenschein's (1981) condition of "non-bossiness" and it has been well understood from the earliest discussion of this subject (Rochet (1980)) that this condition leads to order independence.[26]

Information

The requirement that the information structure be monotone (Assumption 2), i.e., if i_1 covers i_2 and i_2 covers i_3 then i_1 covers i_3, remains critical. In our basic model, we interpreted this assumption as implying that those playing later have just as much information as those playing earlier. Suppose we interpret "how much information an agent gets" by the dimensionality of her strategy space and impose a weaker assumption that later players in the sequence have strategy spaces which are dimensionally no smaller than those of players playing earlier. The following example of a game with sequential structure $1, 2, ..., 7$ shows that our results fail under this weaker assumption. The set of individuals in the cell below each of the players $1, 2, ...7$ shows which individuals are covered by that player (*cf.* 6 covers the individuals in the set $\{4, 3, 1\}$.)

Agent i	1	2	3	4	5	6	7
Set $\varkappa(i)$	$\varnothing$	$\varnothing$	$\{1\}$	$\{1,2,3\}$	$\{4,2,1\}$	$\{4,3,1\}$	$\{4,3,2\}$

Let $\lambda = 4$. One can verify that information chains of length 4 exist (i.e., $(7, 4, 3, 1)$, $(6, 4, 3, 1)$, and $(5, 4, 3, 1)$) and the game is dominance solvable.. However, if we remove agent 7, while a maximal chain still has length 4, the game is no longer dominance solvable. When agent 4 gets a report that two agents out of $\{1, 2, 3\}$ have joined, by joining, agent 4 is not sure that agent 5 or agent 6 will receive a report that three agents have joined since the two participations agent 4 sees may have come from agents 2 and 3 and neither 5 nor 6 covers *both* these agents.

[26] Also see Samuelson (1992), Gilboa, Kalai and Zemel (1993), Mailath, Samuelson and Swinkels (1993), and Marx and Swinkels (1997).

Preferences

In our base model, all we require is that all individuals *have the same ordinal preferences which admit a tipping point.* There are three aspects of this assumption that can easily be relaxed without affecting the theorem, (a) Homogeneity of Preferences (the use of the same valuation function f for all individuals) and (b) Anonymity (dependence of f on the number of participants rather than on the set of individuals participating) and (c) the *status quo* payoff being constant and independent of the number of participants.

We start with a general formulation of the "Tipping Point" Assumption.

For each individual j, partition the set of outcome profiles $\mathcal{B}$ into (i) $\mathcal{B}_j^+(\lambda) = \{b \in \mathcal{B} : b_j = 1, \sum_N b_i \geq \lambda\}$, a set of outcomes where j participates together with at least $(\lambda - 1)$ other individuals, and (ii) $\mathcal{B}_j^-(\lambda) = \{b \in \mathcal{B} : b_j = 1, \sum_N b_i < \lambda\}$, a set of outcomes where j participates and the total number of participants is less than λ; and (iii) $\mathcal{B}_j^0 = \{b \in \mathcal{B} : b_j = 0\}$, outcomes where j does not participate. Endow each agent j with a reflexive and binary weak preference relation $\succsim_j$ on the set of outcome profiles $\mathcal{B}$, with an asymmetric component $\succ_j$ (representing strict preferences) and a symmetric component $\backsim_j$ (representing indifference). In the context of this general definition of preferences, consider the following Common Tipping Point Assumption:

Assumption 3 *(CTP) There exists $\lambda \in \{2, ..., n\}$ such that for all $j \in N$, (if $b \in \mathcal{B}_j^+(\lambda)$ and $b' \in \mathcal{B}_j^0$ then $b \succ_j b'$) and (if $b \in \mathcal{B}_j^-(\lambda)$ and $b' \in \mathcal{B}_j^0$ then $b' \succ_j b$).*

Notice that unlike the preferences in our base model the preferences $\succsim_j$ can be heterogenous, non-anonymous, and intransitive. Since *no* restrictions are imposed on each agent's preference relation *within the sets* $\mathcal{B}_j^+(\lambda)$, $\mathcal{B}_j^-(\lambda)$ and $\mathcal{B}_j^0$, the preferences can be non-anonymous: The dependence of preferences on "who" is participating (rather than just on how many agents are participating) is not necessarily ruled out. Moreover, the condition is consistent with the *status quo* values depending on the set of participants. Also, the lack of any restriction on the binary comparisons using $\succsim_j$ *within* the sets $\mathcal{B}_j^+(\lambda)$, $\mathcal{B}_j^-(\lambda)$ and $\mathcal{B}_j^0$ implies that $\succsim_j$ need *not* even be an ordering on $\mathcal{B}$ (indeed, $\succsim_j$ need neither be transitive nor complete). This model of preferences can thus accommodate a variety of non-traditional preferences, including those not representable by an utility function.[27] The formal definitions of PSNE and Dominance Solvability do not depend on the traditional model of the agents' preferences (as being a weak order) and these concepts can easily be defined in terms of the binary relations $\succsim_j$ and $\succ_j$. As mentioned earlier, for our model, only the changes in strategies along the path of play have an impact on the outcome profile. Thus any such unilateral change along the path of play by any agent j necessarily involves a comparison either between an outcome profile in $\mathcal{B}_j^+(\lambda)$ and an outcome profile in $\mathcal{B}_j^0$ or one between an outcome profile in $\mathcal{B}_j^-(\lambda)$ with one in $\mathcal{B}_j^0$. Since the CTP Assumption 3 is sufficient to inform us on both these types of comparisons and since

[27] For a justification of the use of such a non-traditional ("quasitransitive") preference in games, see Basu and Pattanaik (2014).

these comparisons are *identical* to those implied by the preferences in our base model, this type of extension leaves both our results *and their proofs* intact.

5 Conclusion

We have considered a sequential binary tipping point game and our objective has been to analyze the collective behavior of the agents under various information structures of the game. We have found that the information structure is important in determining the number and type of pure strategy Nash equilibria that occur in such a game. Moreover, while both maximal coordination and coordination failure arise as possible pure strategy Nash equilibrium outcomes in all threshold games irrespective of its information structure, an adequate transmission of and availability of information leads rational agents acting in their self interest to achieve decentralized and efficient coordination. In particular, the greater the degree of coordination necessary to achieve an improvement over the *status quo,* greater the informational transparency required to achieve a maximal pure strategy Nash equilibrium as the unique admissible outcome. Hence, an important implication of our results is that improving the flow of the information among agents acting independently can be used as a policy tool to promote decentralized coordination, avoid inefficiency, and achieve an egalitarian (Rawlsian) maximal welfare in binary threshold models.

Appendix

Proof of Theorem 1. (i) Let $\lambda \in \{2, 3, ..., n-1\}$ and $\tau \in \{0, \ldots, n-\lambda-1\}$. By our hypothesis we have $j^* \geq n-\lambda-\tau+1$ with $\kappa(j^*) \geq n-\lambda-\tau$ and $\kappa(j^*+1) < j^*$ and $n - j^* - \tau - 1 \geq |\{j : \kappa(j) = j^*\}|$. We will construct a PSNE a such that exactly the last $(\lambda + \tau)$ agents participate on the path of play, i.e., $\sum_N b_i(a) == \sum_{N \setminus \{1,2,\ldots n-(\lambda+\tau)\}} = (\lambda + \tau)$.

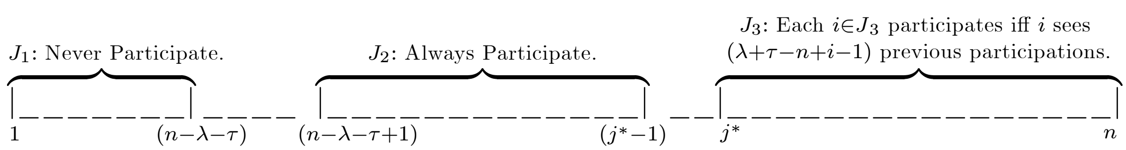

Figure 4. Equilibrium strategy profile a with exactly $(\lambda + \tau)$ participations.

Consider a strategy profile a (see Figure 4) such that:

- For all $j \in J_1 = \{1, \ldots, n-\lambda-\tau\}$, $a_j(l) = 0$ for all $l \in \{1, \ldots, \kappa(j)+1\}$, i.e., all of the first $(n-\lambda-\tau)$ agents never participate regardless of what they observe.

- For all $j \in J_2 = \{n-\lambda-\tau+1, \ldots, j^*-1\}$, $a_j(l) = 1$ for all $l \in \{1, \ldots, \kappa(j)+1\}$, i.e., all agents moving after the agents in J_1 but before agent j^* choose to participate regardless of what they observe.

- For all $j \in J_3 = \{j^*, \ldots, n\}$, $a_j(l) = 1$ iff the l^{th} coordinate of j is on the path of play, i.e., agents moving after agent $(j^* - 1)$ choose to conditionally join *only* in those coordinates that are relevant under a (i.e., these agents have a "1" in the single coordinate of their strategy that is on the path of play and have "0" elsewhere).

By construction, on the path of play only the last $(\lambda + \tau) = |J_2 \cup J_3|$ agents choose to join under a and hence we have $\sum_{i \in N} b_i(a) = \lambda + \tau$. Given the payoff in (1) and Assumption 1, the strategies for agents in J_1, J_2 and J_3 then imply

$$u_j(a) = \begin{cases} 0, \text{ if } j \leq n - \lambda - \tau, \\ f(\lambda + \tau) > 0, \text{ if } j \geq n - \lambda - \tau + 1. \end{cases} \tag{2}$$

We now show that the above profile a is a PSNE. By definition, we only focus on unilateral deviations of strategies that can be on the path of play by some individual j.

If $j \in J_2 \cup J_3$, then $j \geq n - \lambda - \tau + 1$ and using (2), such a unilateral deviation by j decreases j's payoff from $f(\lambda + \tau)$ to 0, which is a reduction in j's utility. Hence j is playing a best response at a.

Next, consider a unilateral deviation of a relevant coordinate by $j \in J_1$ with a resulting profile $a' = (a'_j, -a'_j) = (a'_j, -a_j)$. Given $\kappa(j^* + 1) < j^*$, agents j^* and $(j^* + 1)$ now both observe a signal at a' indicating that one more agent (than at a) has joined. According to a', since $a_{-j} = a'_{-j}$, both j^* and $j^* + 1$ now switch from "join" ($b_{j^*}(a) = b_{j^*+1}(a) = 1$) to "not join" ($b_{j^*}(a') = b_{j^*+1}(a') = 0$) after j's deviation.[28] Now consider agents $j' \geq j^* + 2$ (i.e., $j' \in J_3 \setminus \{j^*, j^* + 1\}$). There are three possible cases:

- If $\kappa(j') < j^*$, j' now (like agents j^* and $(j^* + 1)$) gets a signal under a' indicating that one more agent has joined than under a. Hence, by construction j' will switch from joining to not joining.

- If $\kappa(j') = j^*$, j''s signal will not change and agent j' will, as before, participate.[29]

- If $\kappa(j') > j^*$, then j' gets a signal indicating **at least** one less agent has joined under a' than under a.[30] According to $a_{j'} = a'_{j'}$, j' will switch from joining to not joining.

To summarize, after j's unilateral deviation, the set of agents who choose to join on the path of play is given by $\{j\} \cup J_2 \cup \{j : \kappa(j) = j^*\}$. It follows that

$$\textstyle\sum_{i \in N} b_i(a') = 1 + (j^* - (n - \lambda - \tau + 1)) + |\{j : \kappa(j) = j^*\}| \leq \lambda - 1.$$

[28] To be specific, agents j^* and $(j^* + 1)$ both observe $(j^* - n + \lambda^* + \tau - 1)$ previous agents joining before j's deviation and $(j^* - n + \lambda^* + \tau)$ agents joining after j's deviation.

[29] Since $n - j^* - \tau - 1 \geq |\{j : \kappa(j) = j^*\}|$, there are at most $(n - j^* - \tau - 1)$ such agents.

[30] Since $\kappa(j') \geq j^* + 1$, one is added to j''s information report because j joins, while two is subtracted from j''s report from the defections of j^* and $j^* + 1$.

This implies that j's unilateral deviation is not profitable: j's payoff from her deviation on her relevant coordinate in a is at most $f(\lambda - 1)$ which is less than 0 (Assumption 1).

(ii) Let $\tau \in \{0, \ldots, (n-\lambda-1)\}$. Suppose there is a PSNE a with exactly $(\lambda+\tau)$ agents participating: $\sum_{i\in N} b_i(a) = \lambda + \tau$. Since the set $N_1 = \{(n-\lambda-\tau), (n-\lambda-\tau+1), \ldots, (n-1), n\}$ has $(\lambda+\tau+1)$ agents, $\sum_{i\in N} b_i(a) = \lambda+\tau$ implies that there exists $j \in N_1$ such that $b_j(a) = 0$. Let $\tilde{j}$ be the "last" agent in N_1 such that $b_j(a) = 0$, i.e., let $\tilde{j} = \max\{j : j \in N_1 \text{ and } b_j(a) = 0\}$. By our choice of $\tilde{j}$, we have that

$$
\begin{aligned}
b_j(a) &= 1 \text{ for all } j > \tilde{j}, \text{ and} && (3)\\
\tilde{j} &\geq (n-\lambda-\tau). && (4)
\end{aligned}
$$

Consider a profile $\widetilde{a} = (\widetilde{a}_{\tilde{j}}, a_{-\tilde{j}})$ that differs from a **only** in $\tilde{j}$'s strategy such that the relevant coordinate of $\tilde{j}$ under a is changed from $b_{\tilde{j}}(a) = 0$ to $b_{\tilde{j}}(\widetilde{a}) = 1$. Since a is a PSNE, this unilateral deviation must decrease $\tilde{j}$'s utility to less than zero. In other words at least $(\tau+2)$ agents in $\{(\tilde{j}+1), \ldots, n\}$ who choose to join under a (with $b_j(a) = 1$) must no longer join under $\widetilde{a}$ (with $b_j(\widetilde{a}) = 0$). Define $J = \{j_1, \ldots, j_{\tau+2}\}$ with $j_i < j_{i+1}$ for all $i \leq \tau+1$ be the set of the *first* $(\tau+2)$ agents such that $b_j(a) = 1$ and $b_j(\widetilde{a}) = 0$ for $j \in J$ (i.e., the *first* $(\tau+2)$ agents switching from joining to not joining after $\tilde{j}$'s deviation). This implies that there are at least $\tau+2$ agents moving after $\tilde{j}$ and hence

$$\tilde{j} \leq n - \tau - 2. \tag{5}$$

Inequalities (4) and (5) together imply that

$$(n-\lambda-\tau+1) \leq j_1 \leq (n-\tau-1). \tag{6}$$

Since $\tilde{j}$ has made a unilateral deviation and $j_1 \in J$ is such that $b_{j_1}(a) = 1$ and $b_{j_1}(\widetilde{a}) = 0$, it must be that j_1's information covers $\tilde{j}$, i.e., $\kappa(j_1) \geq \tilde{j}$. By the choice of J, since j_1 and j_2 are the first two agents switching from joining under a to not joining under $\widetilde{a}$, and that all agents j, $j_1 < j < j_2$, have $b_j(a) = b_j(\widetilde{a}) = 1$ by (3), j_2's information does not over j_1, i.e., $\kappa(j_2) < j_1$.[31] Since $j_2 \geq j_1 + 1$, Assumption 2 and $\kappa(j_2) < j_1$ then jointly imply that

$$(j_1+1)\text{'s information does not cover } j_1, \text{ i.e., } \kappa(j_1+1) < j_1. \tag{7}$$

Finally, by our choice of J, since j_1 is the first agent switching after $\tilde{j}$'s deviation, the number of agents who move after agent (j_1+1) and who cover j_1 but not j_2 can be at most $(n-(j_1+1))-\tau = n-j_1-\tau-1$, i.e., $|\{j : \kappa(j) = j_1\}| \leq n-j_1-\tau-1$. The reason for this is that agents in $\{j : \kappa(j) = j_1\}$, when this set is non-empty, receive the same information signal under a and $\widetilde{a}$ and will have the same move under both profiles and cannot belong to J. Since $\lambda \geq 2$, the number of agents in $\{j : \kappa(j) = j_1\}$ cannot be too large for there to

[31] If $\kappa(j_2) \geq j_1$, j_2 would receive the same report under both a and $\widetilde{a}$, contradicting $b_{j_2}(a) \neq b_{j_2}(\widetilde{a})$.

be at least $(\tau + 2)$ agents in J. It is easy to check that if $|\{j : \kappa(j) = j_1\}| > n - j_1 - \tau - 1$ then the number of agents who remain in the set $\{j_1, j_1 + 1, j_1 + 2, \ldots, n\} \backslash \{j : \kappa(j) = j_1\}$ is strictly less than $n - j_1 + 1 - (n - j_1 - \tau - 1) = \tau + 2$. Thus,

$$|\{j : \kappa(j) = j_1\}| \leq n - j_1 - \tau - 1. \quad (8)$$

Using (6), (7), (8) and letting j_1 be the j^* in the statement of the theorem completes the proof. ■

Proof of Theorem 2

We will introduce some notation, define some key concepts, and prove preliminary lemmas that will be used to prove Theorem 2. The first set of lemmas (Lemma 1 to Lemma 4) holds for all $\mathcal{G}_0$ whether or not $\mathcal{G}_0$ is dominance solvable and is unrelated to the particular information structure of $\mathcal{G}_0$. These lemmas provide useful insights into the iterative reduction process of dominated strategies. The second set of lemmas (Lemma 5 and Lemma 6) relates the existence of information chains in $\mathcal{G}_0$ to this reduction process and provides results that represent crucial parts of the proof of Theorem 2.

For convenience, we will abuse notation and use $a_j \in \mathcal{G}_h$ to indicate that a_j is **possible** in the game $\mathcal{G}_h$, i.e., a_j has not been eliminated and is a strategy of j in game $\mathcal{G}_h$. Similarly $a \in \mathcal{G}_h$ will indicate that the profile a is possible in $\mathcal{G}_h$, $a_{-j} \in \mathcal{G}_h$ that the contingency a_{-j} can arise in $\mathcal{G}_h$, and $a_j(l) \in \mathcal{G}_h$ that $a_j\,(l)$ is possible in $\mathcal{G}_h$.

Definition 2 *Let $\mathcal{G}_h \in \{\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_M\}$. The conditional action $a_j\,(l) \in \mathcal{G}_h$ is a **best response conditional action** (BRCA) in $\mathcal{G}_h$ iff there exists $a_{-j} \in \mathcal{G}_h$ such that the l^{th} coordinate of agent j's strategy is on the path of play and a_j is a best response to the contingency a_{-j} in $\mathcal{G}_h$.*

Notice that for the games in $\{\mathcal{G}_1, \ldots, \mathcal{G}_M\}$, as dominated strategies are iteratively eliminated, certain paths of play occurring in earlier games may not appear in later games. As this happens, some coordinates of an agent's strategy become **irrelevant**, i.e., no path of play that is possible in the game passes through that coordinate. Similarly, a coordinate is **relevant** in a game if there is a path of play that goes through that coordinate in the game. It is important to identify which coordinates and which paths persist. The following Lemma 1 shows that an undominated strategy is closely related to BRCA's and these coordinates of individual strategies persist from game to game. The first part of the lemma shows that if some coordinates of a strategy in a game consist of BRCA's then these BRCA's survive in that some undominated strategy in that game has these BRCA's in its coordinates, while the second part shows that for a strategy to be undominated in a game, **all** relevant coordinates of that strategy must be either a BRCA or irrelevant in that game.[32]

[32]Notice that the results in Lemma 1 are stated in the form of a set of coordinates rather than an

Lemma 1 (Persistence) *Let $\mathcal{G}_h \in \{\mathcal{G}_0, \ldots, \mathcal{G}_{M-1}\}$ and $j \in N$. (i) For $a_j \in \mathcal{G}_h$ and $L \subseteq \{1, \ldots, \kappa(j)+1\}$, if for all $l \in L$, $a_j(l)$ is a BRCA then, there exists $a'_j \in \mathcal{G}_{h+1}$ such that $a'_j(l) = a_j(l)$. (ii) If a^*_j is undominated in $\mathcal{G}_h$, then a^*_j is such that for all $l \in \{1, \ldots, \kappa(j)+1\}$, either $a^*_j(l)$ is a BRCA in $\mathcal{G}_h$ or the l^{th} coordinate of j's strategy is irrelevant in $\mathcal{G}_h$.*

Proof. (i) Let $a_j \in \mathcal{G}_h$ be such that $a_j(l)$ is a BRCA in $\mathcal{G}_h$ for all $l \in L$. By the definition of a BRCA, for each such l, there is a strategy profile $a^l = \left(a_j, a^l_{-j}\right) \in \mathcal{G}_h$ such that the l^{th} coordinate of j is on the path of play and a_j is a best response for a^l_{-j} in $\mathcal{G}_h$. If a_j itself is undominated, then $a_j \in \mathcal{G}_{h+1}$ and we are done. If a_j is dominated, then by finiteness of A_j and transitivity of the dominance relation, there is an undominated strategy $a'_j \in \mathcal{G}_h$ that dominates a_j. This implies that for each of the contingencies $a^l_{-j} \in \mathcal{G}_h$, the payoff of j from a'_j is at least as large as that from a_j. But, since a_j is a best response to $a^l_{-j} \in \mathcal{G}_h$, we must have $a'_j(l) = a_j(l)$ for all $l \in L$. Finally, as a'_j is undominated in $\mathcal{G}_h$, $a'_j \in \mathcal{G}_{h+1}$.

(ii) Assume to the contrary that there exists an undominated strategy $a^*_j \in \mathcal{G}_h$ and a non-empty set of coordinates $\widetilde{L}$ of a^*_j where $\widetilde{L} = \{l : l \in \{1, \ldots, \kappa(j)+1\},\ a^*_j(l)$ is relevant and $a^*_j(l)$ is not a BRCA in $\mathcal{G}_h\}$. Replace of all the coordinates of a^*_j in $\widetilde{L}$ with the corresponding BRCA's in $\mathcal{G}_h$ to construct a new strategy a'_j which agrees with a^*_j on all coordinates other than those in $\widetilde{L}$. Thus, by construction, the coordinates of a'_j are either irrelevant or are BRCA's in $\mathcal{G}_h$ and *if* this a'_j exists in $\mathcal{G}_h$ it would dominate a^*_j. Since all contingencies for j in $\mathcal{G}_h$ are also contingencies for j in $\mathcal{G}_{h-1}$ and since every relevant coordinate of a'_j is a BRCA in $\mathcal{G}_h$ it must be the case that these coordinates must also be BRCA's in $\mathcal{G}_{h-1}$. Denoting all these relevant coordinates of a'_j which are BRCA's by L, Lemma 1 (i) (applied to $\mathcal{G}_{h-1}$) tells us that there exists $a''_j \in \mathcal{G}_h$ such that each of a''_j's coordinates either coincides with that of a'_j or the coordinate is irrelevant in $\mathcal{G}_h$. Thus, since a''_j "generates exactly the same outcomes as" a'_j in $\mathcal{G}_h$, it dominates a^*_j contradicting our hypothesis that a^*_j is undominated in $\mathcal{G}_h$. ■

Given our binary setting, another consequence of eliminating dominated strategies is that certain relevant coordinates of an agent's strategies become "fixed" in that for these coordinates all strategies take on the same value. Once this has happened in a game, these coordinates remain fixed in subsequent games. We will be particularly interested in cases where the possibility of non-participation is eliminated by dominance and some coordinate of an agent's strategy is *reduced to 1* in a game and is fixed at 1 in all subsequent games. Moreover, if all relevant coordinates greater than or equal to some coordinate are also fixed at one, we will say that that particular coordinate of the agent's strategy has been *strictly reduced to* 1 and the coordinate is *strictly fixed at 1*.

Definition 3 *Let $\mathcal{G}_h \in \{\mathcal{G}_0, \ldots, \mathcal{G}_{M-1}\}$, $j \in N$ and $l \in \{1, \ldots, \kappa(j)\}$. (i) The conditional action $a_j(l)$ is **fixed at 1** in $\mathcal{G}_{h+1}$ iff for all $a_j \in \mathcal{G}_{h+1}$, $a_j(l) = 1$ and (ii) The conditional*

individual coordinate. Such a somewhat cumbersome formulation is chosen so as to ease our proof for part (ii) of Lemma 1, which currently has no need for arguments in induction.

action $a_j(l)$ is ***strictly fixed at 1*** *in $\mathcal{G}_{h+1}$ iff $a_j(l) = 1$ is* ***fixed at 1*** *in $\mathcal{G}_{h+1}$ and in addition for all $s \in \{1, \ldots, \kappa(j) + 1 - l\}$ either $a_j(l+s) = 1$ or the $(l+s)^{th}$ coordinate of j's strategy is irrelevant in $\mathcal{G}_{h+1}$. If $a_j(l)$ is fixed at 1 in $\mathcal{G}_{h+1}$ and $a_j(l)$ is not fixed at 1 in $\mathcal{G}_h$ we will say that the l^{th} coordinate of j's strategy is* ***reduced to 1*** *in game $\mathcal{G}_h$. If $a_j(l)$ is strictly fixed at 1 in $\mathcal{G}_{h+1}$ and $a_j(l)$ is not strictly fixed at 1 in $\mathcal{G}_h$ we will say that the l^{th} coordinate of j's strategy is* ***strictly reduced to 1*** *in $\mathcal{G}_h$.*

Remark 2 Notice that as per the terminology we have adopted in the above definition that the coordinate is "not fixed" in the game in which it is "reduced" and only becomes fixed in *subsequent* games and $a_j(l) = 1$ being *strictly fixed at 1 in $\mathcal{G}_{h+1}$* implies that $a_j(l) = 1$ is fixed at 1 in $\mathcal{G}_{h+1}$. It is also clear that $a_j(l) = 1$ being *fixed at 1 in $\mathcal{G}_h$* implies that $a_j(l)$ is fixed at 1 in $\mathcal{G}_{h'}$ for all $h' \geq h$. Moreover, since an irrelevant coordinate of a strategy in a game is irrelevant in all subsequent games, it also follows that if $a_j(l)$ is strictly fixed at 1 in $\mathcal{G}_h$ then $a_j(l)$ is strictly fixed at 1 in $\mathcal{G}_{h'}$ for all $h' \geq h$.

We next introduce notation for a particular set of "participating" strategies ($\mathcal{P}$) and a set of "non-participating" strategies ($\mathcal{NP}$). These are simply some book-keeping devices that we shall use to describe the process of the agent's coordinates being "reduced" to 1. Here, $\mathcal{P}$ is the collection of strategy profiles where agents "participate if all the others participate", i.e., all players choose to participate whenever their information signal shows no evidence of previous non-participations. Hence the largest coordinate of each agent's strategy takes the value 1.

$$\mathcal{P} = \{a \in \mathcal{G}_0 : a_j(\kappa(j) + 1) = 1 \text{ for all } j \in N\}.$$

On the other hand, $\mathcal{NP}(r)$ is the set of strategy profiles where agents decide not to participate if they do not observe at least $(r-1)$ participations. Here, *all the coordinates of all the agents' strategies* less than or equal to the r^{th}-coordinate are zero.

$$\mathcal{NP}(r) = \{a \in \mathcal{G}_0 : \text{For all } j \in N,\ a_j(l) = 0 \text{ for all } l \leq \min\{\kappa(j) + 1, r\}, r \in \mathbb{N}\}.$$

Notice that by varying r, we get a nested set of subsets of $\mathcal{NP}(r)$ with $\mathcal{NP}(r-1) \supseteq \mathcal{NP}(r)$. We will use the terminology $\mathcal{P}$ in $\mathcal{G}_h$ (respectively, $\mathcal{NP}(r)$ in $\mathcal{G}_h$) to indicate the subset of the profiles in $\mathcal{P}$ (respectively, the subset of the profiles in $\mathcal{NP}(r)$) that survive the elimination process from $\mathcal{G}_0$ to $\mathcal{G}_h$.[33] We will also use $a_j \in \mathcal{P}$ (respectively, $a_j \in \mathcal{NP}(r)$) to indicate a strategy of j with $a_j(\kappa(j) + 1) = 1$ (respectively, with $a_j(l) = 0$ for all $l \leq \min\{\kappa(j) + 1, r\}$).

Lemma 2 (Unanimous Participation) *Let $\mathcal{G}_h \in \{\mathcal{G}_0, \ldots, \mathcal{G}_M\}$. Then $\mathcal{P} \neq \varnothing$ in $\mathcal{G}_h$ and for all strategy profiles $a \in \mathcal{P}$, a is a PSNE in $\mathcal{G}_h$ with $\sum_N b_j(a) = n$.*

[33] For all games $\mathcal{G}_h \in \{\mathcal{G}_0, \ldots, \mathcal{G}_M\}$, $\mathcal{P} \cap \mathcal{NP}(r) = \varnothing$ for all $r \geq 1$ and both the sets $\mathcal{P}$ and $\mathcal{NP}(r)$ are non-empty in $\mathcal{G}_0$ for all possible values of $r \geq 1$.

Proof. In $\mathcal{G}_0$ since no strategies have been eliminated, $\mathcal{P} \neq \varnothing$. In addition, for all $j \in N$, a_j is a best response to a_{-j} for $(a_j, a_{-j}) \in \mathcal{P}$ in $\mathcal{G}_0$ since any unilateral deviation on the path of play with $a_j(\kappa(j)+1) = 0$ will reduce agent j's payoff from positive to zero. Thus, by the persistence Lemma 1, there is then an undominated strategy a_j with $a_j(\kappa(j)+1) = 1$ in $\mathcal{G}_0$. Since this is true for all j, we have $\mathcal{P} \neq \varnothing$ in $\mathcal{G}_1$. A similar argument establishes the result inductively. ■

The above lemma shows that in all games $\mathcal{G}_h$ there always exists a PSNEO in which each individual receives a payoff of $f(n)$, which is the maximal payoff in our baseline model — hence the efficient PSNEO will never be eliminated in the reduction process.[34] An immediate consequence of the unanimous participation Lemma 2 is Corollary 7, which shows that for dominance solvability it is **necessary** that after the process of iterative dominance, the irreducible game $\mathcal{G}_M$ should satisfy $\mathcal{NP}(1) = \varnothing$.

Corollary 7 *If $\mathcal{NP}(1) \neq \varnothing$ in $\mathcal{G}_M$ then $\mathcal{G}_M$ has at least two outcomes, one in which $\sum_N b_j = n$ and another in which $\sum_N b_j = 0$ and hence $\mathcal{G}_0$ is not dominance solvable.*

The following special type of a prerequisite will play an important role in the sequel:

Definition 4 *The pre-requisite of the l^{th} coordinate of j's strategy is satisfied **exactly** for a contingency a_{-j} in the profile (a_j, a_{-j}) if $\sum_{i=1}^{l-1} b_i(a_j, a_{-j}) = l-1$, where $b(a_j, a_{-j})$ is the action profile induced by (a_j, a_{-j}). The contingency a_{-j} is called an **exact contingency** for the l^{th} coordinate of j's strategy.*

Under an exact contingency a_{-j}, the information that j receives represents a full and complete aggregate report of what *actually* occurs and this report is generated by exactly the *first* $(l-1)$ individuals participating. Our next lemma shows that for all $r \leq \kappa(n)+1$, if $\mathcal{NP}(r) \neq \varnothing$ in $\mathcal{G}_h$, then for each agent, for every (possible) coordinate in the agent's strategy that is no larger than r, there exists an exact contingency in $\mathcal{G}_h$ such that such coordinate of the agent is on the path of play.

Lemma 3 (Exact Contingency) *Let $\mathcal{G}_h \in \{\mathcal{G}_0, \ldots, \mathcal{G}_M\}$ be such that $\mathcal{NP}(r) \neq \varnothing$ in $\mathcal{G}_h$ for some $r \in \{1, \ldots, \kappa(n)+1\}$. Then for $j \in N, l \in \{1, \ldots, \min\{\kappa(j)+1, r\}\}$, there is a strategy profile $a^* \in \mathcal{G}_h$, a^* depending on l, such that the l^{th} coordinate of j's strategy is on the path of play under a^*,and $\sum_{i=1}^{l-1} b_i(a^*) = \sum_N b_i(a^*) = l-1$.*

Proof. The proof is done by construction. First, $\mathcal{P} \neq \varnothing$ in $\mathcal{G}_h$ (Lemma 2) implies that for each $i \in N$, there is a strategy $\hat{a}_i \in \mathcal{G}_h$ with $\hat{a}_i(\kappa(i)+1) = 1$. Since by the hypothesis $\mathcal{NP}(r) \neq \varnothing$ in $\mathcal{G}_h$, there is also a strategy $\check{a}_i \in \mathcal{NP}(r)$ for all $i \in N$ such

[34] A similar argument cannot be applied to the PSNE with $\sum_N b_j(a) = 0$ as under certain information structure, $a_1 = a_1(1) = 0$ may no longer be a BRCA at some stage, while for any j, $a_j(\kappa(j)+1) = 1$ is always a BRCA independently of the information structure in our setting.

that $\check{a}_i\,(1) = \cdots = \check{a}_i\,(\min\{\kappa(i)+1, r\}) = 0$. Let $j \in N$ and $l \in \{1, \ldots, \kappa(j)+1\}$ with $1 \leq l \leq r$ and consider a strategy profile a^* where (see Figure 5)

$$\begin{aligned} \text{for } i &\in \{1, \ldots, l-1\}\,,\, a_i^* = \hat{a}_i, \\ \text{for } i &\in \{l, \ldots, n\}\,,\, a_i^* = \check{a}_i. \end{aligned} \tag{9}$$

$$\overbrace{|\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}|}^{\text{Participate only at the top.}}\overbrace{\text{\_\_\_\_\_\_\_\_\_\_\_\_}|\text{\_\_\_\_\_\_\_\_\_\_\_}|}^{\text{Not participate on the path of play.}}$$

$$1 \qquad\qquad\qquad\qquad\qquad (l-1) \qquad\qquad\qquad j \qquad\qquad\qquad n$$

Figure 5. The Strategy Profile a^* for the l^{th} Coordinate of j's Strategy (An agent $i \leq l-1$ chooses $a_i^* \in \mathcal{P}$, while an agent $i \geq l$ chooses $a_i^* \in \mathcal{NP}\,(r)$).

Hence, under the profile $a^* \in \mathcal{G}_h$, $a_j(l)$ is on the path of play, and $\sum_1^{l-1} b_i(a^*) = \sum_N b(a^*) = l-1$, where $a_i^* \in \mathcal{NP}\,(r)$ for all $i \geq l$, establishing the result. ■

The above construction in (9) yields two key consequences: First, since all strategies are possible in $\mathcal{G}_0$, it follows that $\mathcal{G}_0$ satisfies $\mathcal{NP}\,(\lambda) \neq \varnothing$. Second, for the case where $2 \leq r \leq \lambda - 1$ as one proceeds along the sequence $\mathcal{G}_1, \ldots, \mathcal{G}_{M-1}$, *except* possibly for the first round (from $\mathcal{G}_0$ to $\mathcal{G}_1$), the maximum possible 'reduction' per round of elimination is 'one' in the following sense: If $r \leq \lambda - 1$ and one has $\mathcal{NP}\,(r) \neq \varnothing$ in $\mathcal{G}_h$ (i.e., every agent has a strategy in $\mathcal{G}_h$ with zeros in all coordinates no larger than the r^{th} coordinate), then it holds that $\mathcal{NP}\,(r-1) \neq \varnothing$ in $\mathcal{G}_{h+1}$ (i.e., every agent has a strategy in $\mathcal{G}_{h+1}$ with zeros in all coordinates no larger than the $(r-1)^{\text{th}}$ coordinate). We summarize the above in the next lemma. In particular, the maximal reduction Lemma 4 implies that $\mathcal{G}_1$ satisfies $\mathcal{NP}\,(\lambda - 1) \neq \varnothing$ whether or not the game $\mathcal{G}_0$ is dominance solvable.

Lemma 4 (Maximal Reduction) *(i) $\mathcal{G}_0$ satisfies $\mathcal{NP}\,(\lambda) \neq \varnothing$. (ii) Let $j \in N$ and $\mathcal{G}_h \in \{\mathcal{G}_1, \ldots, \mathcal{G}_{M-1}\}$ be such that $\mathcal{NP}\,(r) \neq \varnothing$ in $\mathcal{G}_h$ for some $r \in \{2, \ldots, \lambda - 1\}$. Then for all $l \in \{1, \ldots, \min\{r-1, \kappa(j)+1\}\}$, $a_j(l) = 0$ is BRCA in $\mathcal{G}_h$ and $\mathcal{NP}\,(r-1) \neq \varnothing$ in $\mathcal{G}_{h+1}$.*

Proof. (i) immediately follows from the fact that all strategies are possible in $\mathcal{G}_0$.

(ii) By the exact contingency Lemma 3, $\mathcal{NP}\,(r) \neq \varnothing$ in $\mathcal{G}_h$ implies the existence of $a^* \in \mathcal{G}_h$ such that the l^{th} coordinate of j's strategy is on the path of play under a^*. As $l \leq r-1 \leq \lambda - 2$, it follows that $a_j\,(l) = 0$ is a BRCA for the contingency a_{-j}^* in (9). By the persistence Lemma 1 (i), j has a strategy in $\mathcal{G}_{h+1}$ with zero in the l^{th} coordinate. As this is true for all j and all $l \leq \lambda - 2$, we have that $\mathcal{NP}\,(r-1) \neq \varnothing$ in $\mathcal{G}_{h+1}$. ■

We now present two lemmas that represent the key steps in the proof of Theorem 2.

Lemma 5 (Sufficiency) *Consider the canonical sequence of agents given by the ordered set $(i_1, i_2, \ldots, i_{m^*})$ where $i_1 = n$ and $i_{s+1} = \kappa(i_s)$ and $\kappa(i_{m^*}) = \kappa^{m^*-1}(n) = 0$. If $m^* \geq \lambda$ then $\mathcal{G}_0$ is dominance solvable.*

Proof. Consider the game $\mathcal{G}_0$. From $m^* \geq \lambda$ and Assumption 2 ($\mathcal{I}$-Monotonicity) we know that i_1 covers at least $(m^* - 1)$ agents, i.e., agents $i_2, \ldots, i_{m^*}$, and hence $\kappa(i_1) = \kappa(n) \geq m^* - 1 \geq \lambda - 1$. Observe that there exists a contingency such that the path of play passes through the λ^{th} coordinate of i_1's strategy in $\mathcal{G}_0$ where all strategies are possible. Moreover, for **every** contingency in $\mathcal{G}_0$ with a path of play passing through the λ^{th} or higher coordinate of i_1, i_1 knows that at least $(\lambda - 1)$ individuals have participated before she moves and hence $a_n(\lambda) = a_{i_1}(\lambda + s) = 1$ is BRCA in $\mathcal{G}_0$.[35] Thus, using the persistence Lemma 1, we can conclude that for $i_1 = n$ the λ^{th} coordinate is reduced to 1 in $\mathcal{G}_1$ and all coordinates $a_{i_1}(\lambda)$ and $a_{i_1}(\lambda + s)$ are strictly fixed at 1 for all $s \in \{1, \ldots, \kappa(n) + 1 - \lambda\}$ in $\mathcal{G}_{1+t}$ for all $t \in \{0, \ldots, M - 1\}$. This implies $\mathcal{G}_1$ satisfies $\mathcal{NP}(\lambda) = \varnothing$. By the maximal reduction Lemma 4, we know that $\mathcal{G}_1$ satisfies $\mathcal{NP}(\lambda - 1) \neq \varnothing$.

Similarly, the existence of a chain of length λ implies that $\kappa(i_2) \geq \lambda - 2$. By the exact contingency Lemma 3, there is an exact contingency in $\mathcal{G}_1$ passing through the $(\lambda - 1)^{\text{th}}$ coordinate of i_2. Since $\kappa(i_1) = i_2$, if $a_{i_2}(\lambda - 1) = 1$, this path of play must pass through the λ^{th} of i_1's strategy. Since $a_{i_1}(\lambda)$ is strictly fixed at 1 in $\mathcal{G}_{1+t}$ for all $t \in \{0, \ldots, M - 1\}$, for **any** contingency in $\mathcal{G}_1$ where the path of play passes through the *$(\lambda - 1)^{\text{th}}$* coordinate or higher of i_2, we have that $a_{i_2}(\lambda - 1) = 1$ is a BRCA. By the persistence Lemma 1, $a_{i_2}(\lambda - 1)$ is strictly fixed at 1 in $\mathcal{G}_{2+t}$ for all $t \in \{0, \ldots, M - 2\}$. In addition, by the maximal reduction Lemma 4, $\mathcal{NP}(\lambda - 2) \neq \varnothing$ in $\mathcal{G}_2$.

Using a similar argument repeatedly for $\mathcal{G}_3, \ldots, \mathcal{G}_{\lambda - 1}$ and $i_3, \ldots, i_\lambda$, we have that $a_{i_\lambda}(1)$ is strictly fixed at 1 in $\mathcal{G}_\lambda$ and that for all $a \in \mathcal{G}_\lambda$, $\sum_N b_i(a) \geq \lambda$. This implies that in $\mathcal{G}_{\lambda+1}$ for every $a \in \mathcal{G}_{\lambda+1}$, for every agent j, and for every relevant coordinate l of j's strategies on the path of play, $a_j(l) = 1$ is a BRCA. Using the persistence Lemma 1, it follows that for all $a \in \mathcal{G}_M$, $\sum_N b_i(a) = n$. Thus, $\mathcal{G}_0$ is dominance solvable. ■

Recall that dominance solvability of $\mathcal{G}_0$ implies $\mathcal{G}_M$ must satisfy $\mathcal{NP}(1) = \varnothing$ (Corollary 7), i.e., in $\mathcal{G}_M$ it should not be possible for every individuals to have some strategy with zero in the first coordinate. Furthermore, with dominance solvability, using the maximal reduction lemma, Lemma 4 (i) and (ii), we know that $\mathcal{G}_1$ satisfies $\mathcal{NP}(\lambda - 1) \neq \varnothing$. Thus, using the maximal reduction Lemma 4 repeatedly we can conclude that as one proceeds along the sequence $\{\mathcal{G}_1, \ldots, \mathcal{G}_M\}$ we will encounter (sequentially) games which satisfy "$\mathcal{NP}(\lambda - 1) = \varnothing$ and $\mathcal{NP}(\lambda - 2) \neq \varnothing$)" followed by games satisfying "$\mathcal{NP}(\lambda - 2) = \varnothing$ and $\mathcal{NP}(\lambda - 3) \neq \varnothing$" and so on until we will get to the set of games satisfying "$\mathcal{NP}(2) = \varnothing$ and $\mathcal{NP}(1) \neq \varnothing$" and then to games satisfying $\mathcal{NP}(1) = \varnothing$, which, as we have noted, is a condition necessary for dominance solvability. To ease exposition, for $\lambda - 1 \geq k \geq 1$, we will adopt the notation of denoting the set of games satisfying "$\mathcal{NP}(\lambda - k + 1) = \varnothing$ and $\mathcal{NP}(\lambda - k) \neq \varnothing$" by the **block** of games $\left\{\mathcal{G}_i^{\lambda-k}\right\}$ with the first and last games in $\left\{\mathcal{G}_i^{\lambda-k}\right\}$ (as ranked in the sequence $\mathcal{G}_1, \ldots, \mathcal{G}_M$) being denoted by $\mathcal{G}_{\min}^{\lambda-k}$ and $\mathcal{G}_{\max}^{\lambda-k}$, respectively.[36] Thus, $\cup_{k=1}^{\lambda-1}\left\{\mathcal{G}_i^{\lambda-k}\right\} = \{\mathcal{G}_1, \ldots, \mathcal{G}_M\}$ and $\{\{\mathcal{G}_i^{\lambda-k}\}\}$ is simply a partition of $\{\mathcal{G}_1, \ldots, \mathcal{G}_M\}$. Note that in all games in the block $\left\{\mathcal{G}_i^{\lambda-k}\right\}$, all agents have strategies in which all the

[35] In $\mathcal{G}_0$, since all strategies are possible, the set of strategies for which this is true is non-empty.

[36] We do not rule out the possibility that $\left\{\mathcal{G}_i^{\lambda-k}\right\}$ can be a singleton and $\mathcal{G}_{\min}^{\lambda-k} = \mathcal{G}_{\max}^{\lambda-k}$.

coordinates less than or equal to the $(\lambda-k)^{\text{th}}$ coordinate is zero, while in the game $\mathcal{G}_{\max}^{\lambda-k}$ the $(\lambda-k)^{\text{th}}$ coordinate reduces to 1 in all strategies for some individual and that this coordinate is fixed at 1 in all subsequent games starting with $\mathcal{G}_{\min}^{\lambda-k-1}$. Using the sets $\left\{\mathcal{G}_i^{\lambda-k}\right\}$ where $1 \le k \le \lambda-1$ we will now establish a property of the canonical sequence of agents $i_1, i_2, \ldots, i_{m^*}$ that will be critically important in demonstrating that the process with successive coordinates becoming fixed at 1 breaks down when λ is strictly less than the length of the longest information chain (m^*).

Our next lemma establishes a key step in the proof of the "necessity" part of Theorem 2. In particular, this lemma shows that in each round of reduction of zeros to ones (in the sense of the maximal reduction Lemma 4) that occurs between the blocks partitioning $\{\mathcal{G}_1, \ldots, \mathcal{G}_M\}$, the reduction of the coordinate always has to start with an agent less than or equal to a distinguished agent from the canonical sequence $i_1, i_2, \ldots, i_{m^*}$ (where the selection of this distinguished agent depends on the block $\left\{\mathcal{G}_i^{\lambda-k}\right\}$ under consideration).

Lemma 6 (Non-Reduction) *Let $s \in \{1, \ldots, \lambda-1\}$. If $\mathcal{G}_h \in \left\{\mathcal{G}_i^{\lambda-s}\right\}$ then for all agents j such that $j > i_{s+1}$ and $\kappa(j) \ge (\lambda-s-1)$, $a_j(\lambda-s) = 0$ is a BRCA in $\mathcal{G}_h$.*

Proof. We will provide a proof by induction. In each step, we explicitly construct a strategy profile to establish the result.

Basis Step. $s = 1$.

We first consider the block $\{\mathcal{G}_i^{\lambda-1}\}$. The analysis here leads to the reduction that takes place between $\mathcal{G}_{\max}^{\lambda-1}$ and $\mathcal{G}_{\min}^{\lambda-2}$, where the $(\lambda-1)^{\text{th}}$ coordinate of some agent becomes fixed at 1. Recall that all games in the block $\{\mathcal{G}_i^{\lambda-1}\}$ satisfy $\mathcal{NP}(\lambda-1) \neq \varnothing$ and $\mathcal{G}_{\min}^{\lambda-1} = \mathcal{G}_1$.

Consider an agent j with $j > i_2$ and $\kappa(j) \ge (\lambda-2)$ in a game $\mathcal{G}_h \in \left\{\mathcal{G}_i^{\lambda-1}\right\}$. Since $\mathcal{NP}(\lambda-1) \neq \varnothing$ in $\mathcal{G}_h$, there exists $a_{j'} \in \mathcal{NP}(\lambda-1)$ in $\mathcal{G}_h$ for all $j' \ge j$. Moreover, since $\kappa(j) \ge (\lambda-2)$ and $\mathcal{NP}(\lambda-1) \neq \varnothing$, the exact contingency Lemma 3 implies that there exists an **exact** contingency a_{-j}^* for the $(\lambda-1)^{\text{th}}$ coordinate of j in $\mathcal{G}_h$.

Construct a contingency $a_{-j} \in \mathcal{G}_h$ as follows (see Figure 6):

- Let all agents $j'' < j$ use their strategies from the exact contingency a_{-j}^*.
- Let all agents $j' > j$ use the strategies $a_{j'} \in \mathcal{NP}(\lambda-1)$.

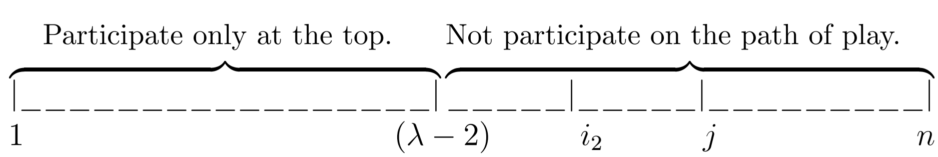

Figure 6. An Illustration of the Contingency a_{-j} for **Basis Step**.

By the construction of a_{-j}, the first $(\lambda-2)$ agents participate on the path of play, while the other agents playing before j choose to not participate. Since, $j > i_2 = \kappa(n) = \kappa(i_1)$, j is not covered by any individual. It follows that all agents moving after j do not participate along the path of play since all such agents after j are using their strategies from $\mathcal{NP}(\lambda-1)$. Hence, irrespective of whether $a_j(\lambda-1)=1$ is possible or not in $\mathcal{G}_h$, $a_j(\lambda-1)=0$ is a BRCA for j in $\mathcal{G}_h$, establishing **Basis Step**.[37]

Applying the above to the game $\mathcal{G}_{\max}^{\lambda-1} \in \left\{\mathcal{G}_i^{\lambda-1}\right\}$ and using the persistence Lemma 1, if $\mathcal{G}_{\min}^{\lambda-2}$, the next game after $\mathcal{G}_{\max}^{\lambda-1}$ in the sequence $\mathcal{G}_1, \ldots, \mathcal{G}_M$, exists, we then have:

$$\text{If } j > i_2 \text{ and } \kappa(j) \geq (\lambda-2), \text{ then there is } a_j \in \mathcal{G}_{\min}^{\lambda-2} \text{ with } a_j(\lambda-1)=0. \quad (10)$$

In other words, a reduction of the $(\lambda-1)^{\text{th}}$ coordinate of any agent moving after i_2 to 1 cannot take place in the block $\left\{\mathcal{G}_i^{\lambda-1}\right\}$ and therefore any such reduction (if any) has to necessarily take place for an agent moving before i_2.

Inductive Step. $s \geq 2$.

Consider the inductive hypothesis: If $\mathcal{G}_h \in \left\{\mathcal{G}_i^{\lambda-s+1}\right\}$ then for all agents j' such that $j' > i_s$ and $\kappa(j') \geq (\lambda-s)$, $a_{j'}(\lambda-s+1)=0$ is a BRCA in $\mathcal{G}_h$.

Using this inductive hypothesis, we need to show that if $\mathcal{G}_h \in \left\{\mathcal{G}_i^{\lambda-s}\right\}$ then for all agents j such that $j > i_{s+1}$ and $\kappa(j) \geq (\lambda-s-1)$, $a_j(\lambda-s)=0$ is a BRCA in $\mathcal{G}_h$.

Analogous to (10), use the inductive hypothesis and Lemma 1 to obtain:

$$\text{If } j' > i_s \text{ and } \kappa(j') \geq (\lambda-s), \text{ then there is } a_{j'} \in \mathcal{G}_{\min}^{\lambda-s} \text{ with } a_{j'}(\lambda-s+1)=0. \quad (11)$$

Consider the first game in the block $\{\mathcal{G}_i^{\lambda-s}\}$: $\mathcal{G}_{\min}^{\lambda-s}$.

Let j be such that $j > i_{s+1}$ and $\kappa(j) \geq (\lambda-s-1)$. We will consider two cases: Case 1. $a_j(\lambda-s)=0$ for all $a_j \in \mathcal{G}_{\min}^{\lambda-s}$. Case 2. There exists $a_j \in \mathcal{G}_{\min}^{\lambda-2}$ with $a_j(\lambda-s)=1$.

Case 1. Since $\mathcal{NP}(\lambda-s) \neq \varnothing$ in $\mathcal{G}_{\min}^{\lambda-s}$, $a_j(\lambda-s)=0$ is possible and the $(\lambda-s)^{\text{th}}$ coordinate of j is relevant (Lemma 3) in $\mathcal{G}_{\min}^{\lambda-s}$. In addition, as $a_j(\lambda-s)=0$ for all $a_j \in \mathcal{G}_{\min}^{\lambda-s}$, $a_j(\lambda-s)=0$ is a BRCA in $\mathcal{G}_h$, completing the proof in this case.

Case 2. In this case we have

$$\text{there exists } a_j \in \mathcal{G}_{\min}^{\lambda-s} \text{ with } a_j(\lambda-s)=1. \quad (12)$$

Let j' be any agent that covers j, i.e., $\kappa(j') \geq j$. Since $\kappa(j) \geq (\lambda-s-1)$ and $\kappa(j') \geq j$, we have $\kappa(j') \geq (\lambda-s)$. In addition, we know that since j' covers j and $j > i_s$ it must be the case that agent j' moves after i_s.[38] Statement (11) shows that for any such j', there is $a_{j'} \in \mathcal{G}_{\min}^{\lambda-s}$ with $a_{j'}(\lambda-s+1)=0$. This allows us to construct the following strategy profile $a \in \mathcal{G}_{\min}^{\lambda-s}$ (and hence a contingency $a_{-j} \in \mathcal{G}_{\min}^{\lambda-s}$) to show that $a_j(\lambda-s)=0$ is a

[37] It is possible for some $j > i_2$ to have $a_j(\lambda-1)=0$ for all $a_j \in \mathcal{G}_h$. For example, consider agent n.

[38] The key difference between **Basis Step** and **Inductive Step** is that an agent j (with $j > i_{s+1}$) may be covered by another agent j' in **Inductive Step** while an agent j (with $j > i_2$) *cannot* be covered by any agent in **Basis Step**. This creates additional complications in constructing the contingency a_{-j} toward the result of $a_j(\lambda-s)=0$ being a BRCA in **Inductive Step.**

BRCA for j in $\mathcal{G}_{\min}^{\lambda-s}$ (see Figure 7):

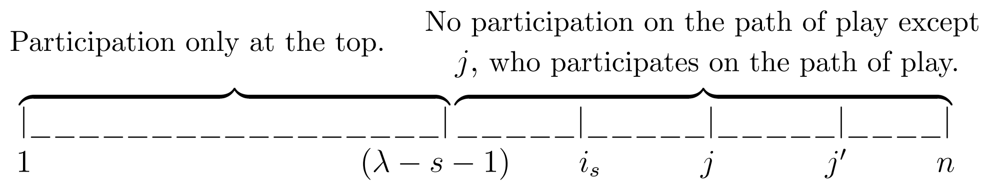

Figure 7. An Illustration of the Strategy Profile a for **Inductive Step**.

- Let agent j use $a_j \in \mathcal{G}_{\min}^{\lambda-s}$ with $a_j(\lambda - s) = 1$ (see (12)).
- Let all agents j'', who do not cover j (i.e., $\kappa(j'') < j$), use the strategies in the corresponding exact contingency a_{-j}^* so that the $(\lambda - s)^{\text{th}}$ coordinate of j's strategy is on the path of play. (This includes all agents moving before j and possibly some agents moving after j. Notice that $\mathcal{NP}(\lambda - s) \neq \varnothing$ in $\mathcal{G}_{\min}^{\lambda-s}$ and the exact contingency Lemma 3 imply that this is possible.)
- Let all agents j' with $j' > j$ and $\kappa(j') \geq j$ use $a_{j'} \in \mathcal{G}_{\min}^{\lambda-s}$ such that $a_{j'}(\lambda - s + 1) = 0$. (Recall that our above arguments for j' imply that this is possible.)

Notice that by construction, there are (exactly) $(\lambda - s)$ agents participating on the path of play (i.e., $\sum_N b_i(a) = \lambda - s$). In particular, only agent j and the first $(\lambda - s - 1)$ agents moving before j participate and no agent moving after j participates. In addition, for each agent j' that covers j, the $(\lambda - s + 1)^{\text{th}}$ coordinate of j' is on the path of play. Since $\sum_N b_i(a) = \lambda - s \leq \lambda - 2$, we have that $a_j(\lambda - s) = 0$ is a BRCA for j in $\mathcal{G}_{\min}^{\lambda-s}$. In addition, under exactly the same strategy profile a, $a_{j'}(\lambda - s + 1) = 0$ is a BRCA for all j' with $j' > j$ and $\kappa(j') \geq j$. If $\mathcal{G}_{\min}^{\lambda-s} = \mathcal{G}_{\max}^{\lambda-s}$, our proof is complete. If not, the persistence Lemma 1 implies that $a_{j'}(\lambda - s + 1) = 0$ persists and is possible in the next game after $\mathcal{G}_{\min}^{\lambda-s}$ which we denote by $\mathcal{G}_{\min+1}^{\lambda-s}$. Hence, if $\mathcal{G}_{\min+1}^{\lambda-s}$ exists, we have the following:

$$\text{if } j' > i_s \text{ and } \kappa(j') \geq (\lambda - s), \text{ then there is } a_{j'} \in \mathcal{G}_{\min+1}^{\lambda-s} \text{ with } a_{j'}(\lambda - s + 1) = 0. \quad (13)$$

Using an argument similar to the above used for $\mathcal{G}_{\min}^{\lambda-s}$, we can show that $a_j(\lambda - s) = 0$ is a BRCA in $\mathcal{G}_{\min+1}^{\lambda-s}$ for any agent $j > i_{s+1}$. The persistence Lemma 1 and the repeated use of this argument establish **Inductive Step** for all games in $\left\{\mathcal{G}_i^{\lambda-s}\right\}$. ■

Proof of Theorem 2. Consider the canonical sequence of agents given by the ordered set $(i_1, i_2, \ldots, i_{m^*})$ where $i_1 = n$, $i_2 = \kappa(i_1) = \kappa^1(n)$, $\cdots$, and $\kappa(i_{m^*}) = \kappa^{m^*-1}(n) = 0$. By Proposition 5, m^* is the maximum length of an information chain in $\mathcal{G}_0$. The sufficiency Lemma 5 already establishes that $m^* \geq \lambda$ implies that $\mathcal{G}_0$ is dominance solvable. Hence we only need to show that if $\mathcal{G}_0$ is dominance solvable then we have $m^* \geq \lambda$.

In $\mathcal{G}_0$, choose any $j \in N$ and any coordinate $l \leq \kappa(j) + 1$ in j's strategy with $l \leq \lambda - 1$. Since all strategies are possible in $\mathcal{G}_0$, we can construct an exact contingency such that the path of play passes through the l^{th} coordinate of j's strategy. Hence, $a_j(l) = 0$ is a BRCA. Using the persistence Lemma 1 (i) it follows that $\mathcal{G}_1$ satisfies $\mathcal{NP}(\lambda - 1) \neq \varnothing$.

By the maximal reduction Lemma 4, as we proceed along the sequence $\mathcal{G}_1, \ldots, \mathcal{G}_M$, we will encounter (sequentially) the $(\lambda - 1)$ non-empty blocks $\left\{\mathcal{G}_i^{\lambda-k}\right\}_{\lambda-1\geq k\geq 1}$ where in the block $\left\{\mathcal{G}_i^{\lambda-k}\right\}$ all agents have strategies in which all the coordinates less than or equal to the $(\lambda - k)^{\text{th}}$ coordinate are zero and in the game $\mathcal{G}_{\max}^{\lambda-k}$ the $(\lambda - k)^{\text{th}}$ coordinate of some individual is reduced to 1 and becomes fixed at 1 in all subsequent games.

Consider the first block $\left\{\mathcal{G}_i^{\lambda-k}\right\}$ with $k = \lambda - 1$.[39] By definition, some agent's $(\lambda - 1)^{\text{th}}$ coordinate is reduced to 1 in game $\mathcal{G}_{\max}^{1}$. There must be some individual j' such that $\kappa(j') \geq \lambda - 1$. Thus, noticing that the hypothesis of Lemma 6 is non-vacuously satisfied, using the persistence Lemma 1, we can conclude that $j' \leq i_2$.

Notice that the existence of these $(\lambda - 2)$ more of such blocks implies that the hypothesis of the non-reduction Lemma 6 is non-vacuously satisfied in each step of the reduction process and that in the game $\mathcal{G}_{\max}^{\lambda-s}$ for $\lambda - 1 \geq s \geq 2$, the $(\lambda - s)^{\text{th}}$ coordinate can be on the path of play and reduces to 1 in $\mathcal{G}_{\max}^{\lambda-h}$ **only** for some individual $j \leq i_s$ where $\kappa(j) \geq \lambda - s - 1$.[40] Hence, for $\mathcal{NP}(1) = \varnothing$ to be true (i.e., the $(\lambda - \lambda + 1)^{\text{th}} = 1^{\text{st}}$ coordinate of some agent's strategy to be reduced to 1), we must have an agent $j \leq i_\lambda$ with $\kappa(j) \geq 0$. Accordingly, we have that $\mathcal{G}_0$ being dominance solvable implies the existence of the canonical sequence $i_1, \ldots, i_{m^*}$ of length at least equal to λ. ■

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[39] Since $\mathcal{G}_1$ belongs to this block, it is nonempty.

[40] Observe that for the l^{th} coordinate of an agent j to be reduced to 1, it must be the case that $\kappa(j) \geq l - 1$. This implies that when we have dominance solvability the hypothesis of the non-reduction lemma is true for each of the blocks $\left\{\mathcal{G}_i^{\lambda-k}\right\}_{\lambda-1\geq k\geq 1}$.

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