Universal Interactive Preferences*

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Abstract

We prove that a universal preference type space exists under more general conditions than those postulated by Epstein and Wang (1996). To wit, it suffices that preferences can be encoded monotonically in rich enough ways by collections of continuous, monotone real-valued functionals over acts, which determine—even in discontinuous fashion—the preferences over limit acts. The proof relies on a generalization of the method developed by Heifetz and Samet (1998a).

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1 INTRODUCTION

Classical game theory has largely been developed under the assumption that players have Savage (1954) preferences, and can hence be modeled as maximizing subjective expected utilities. In single-person decision problems, in contrast, a voluminous literature axiomatizes and analyzes many additional classes of preference relations, which are obviously relevant in strategic interactions as well. How should games of incomplete information be modeled and handled with such more general preferences?

With Savage (1954) preference relations, games with incomplete information are modeled by probabilistic type spaces (Harsanyi (1967-68)). Each type of each player is

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associated with a probabilistic belief over the space of states of 'nature'—the players' von Neumann and Morgenstern (1944) utility indices from their action profiles, the external signals they get, etc.—and the other players' types. A strategy of each player is a measurable mapping from her types to her actions. Thus, from each player's perspective, her own actions coupled with the strategy profiles of the other players constitute acts from their types and nature into the space of everybody's action profiles; integration with respect to the probabilistic belief of each of the player's types of the payoffs associated by nature to each action profile defines a Savage (1954) preference relation over these acts. Moreover, by considering the type's marginal belief over nature, over nature and the other players' marginal beliefs on nature, etc., we see how each type's belief encapsulates an infinite hierarchy of beliefs of all orders.

Type spaces can be readily extended to more general classes of preferences by endowing each type directly with a preference relation over acts which are measurable functions from nature and the others' types into everybody's action profiles. The type's marginal preference over constant acts, over acts which are measurable with respect to the other players' marginal preferences over constant acts, etc., form a hierarchy of preferences. In the particular case in which the preference relations satisfy Savage (1954) axioms and, for each player, states of nature associate real-valued von Neumann and Morgenstern (1944) payoffs with the players' action profiles, each of these preference relations can be represented by a probability measure over nature and the other players' types, as in Harsanyi's formulation.

Given a class of preference relations over acts, does the corresponding class of type spaces contain a universal space, i.e. one which 'embeds' all others in the sense of containing all preference hierarchies which appear in some type space? This is a pertinent question since, in applications, 'small' type spaces are tailored to the problem at hand, and it is important to know whether any generality is lost by this restriction or rather the same analysis could, in principle, be carried out in a universal space and deliver the same result. Furthermore, robustness results are most relevant if they obtain in a universal space, which allows for all possible perturbations, rather than within any particular, restricted type space.

For the case of preferences based on probabilistic beliefs, Mertens and Zamir (1985), followed by Brandenburger and Dekel (1993), Heifetz (1993), and Mertens et al. (1994) showed that under suitable topological or regularity assumptions, the set of all hierarchies of probabilistic beliefs constitutes a type space, which is hence universal.[1] In the absence of regularity, however, Heifetz and Samet (1999) showed that

[1] Other developments under regularity assumptions include Battigalli and Siniscalchi (1999) for

there exist hierarchies of beliefs which are not types in any type space. Nevertheless, Heifetz and Samet (1998a) showed that even in the absence of regularity, the set of all profiles of belief hierarchies appearing in type spaces is itself a type space, which is universal.[2]

What happens with more general classes of preferences? Epstein and Wang (1996) showed, under topological assumptions, that when preferences are regular in the appropriate sense, the set of all preference hierarchies forms a type space and Chen (2010) proved that it is universal. Alternatively, if one restricts attention to algebras of events then Di Tillio (2008) showed that a universal space exists under very mild conditions. However, what happens in the absence of regularity and when the pertinent class of events forms a σ-algebra?

In this paper, we show that a universal space exists under milder and more general conditions on preferences than those postulated by Epstein and Wang (1996). To wit, it suffices that preferences can be encoded monotonically in rich enough ways by finite or countably infinite collections of continuous real-valued functionals over acts, which determine—even in discontinuous fashion—the preferences over limit acts. For example, a lexicographic preference represented by a (finite or countably infinite) sequence of $|L|$ continuous functionals is not itself continuous, since an act may be superior to all acts in some increasing sequence, but inferior to their limit. Nevertheless, there does exist a universal space in the category of type spaces where each type is associated with a lexicographic preference representable by a collection of $|L|$ continuous functionals over acts. Furthermore, this existence result applies to both well-founded[3] and non-well-founded lexicographic preferences.[4]

To prove this result we proceed in two steps. First, in Section 3, generalizing

conditional beliefs in dynamic games, Mariotti et al. (2005) for compact possibility models, Ahn (2007) for compact sets of probabilistic beliefs, Gul and Pesendorfer (2010) to study interdependent preferences that accommodate reciprocity, Bergemann et al. (2011) to study strategic distinguishability of types, Heifetz et al. (2012) to study unawareness, and Heifetz and Kets (2012) to study bounded reasoning. Lee (2013) constructs the universal type space for lexicographic preferences under topological assumptions.

[2] Meier (2008), Pinter and Udvari (2011), Heinsalu (2012), Kets (2012), and Pinter (2012) provide recent developments of more general type spaces using the Heifetz and Samet (1998a) approach, while Moss and Viglizzo (2004) formulate type spaces as coalgebras and show the existence of a final coalgebra which provides the universal type space.

[3] Well-founded lexicographic beliefs are sequences of beliefs such that the "most important" belief in any subsequence is always well-defined. Rényi (1956) explored notions closely related to non-well-founded lexicographic beliefs.

[4] The epistemic characterization of iterated admissibility in Lee (2015b) uses hierarchies of lexicographic beliefs that can be extended to *non-well-founded* lexicographic beliefs but not extended to well-founded ones. This suggests that the analysis in Lee (2015b) could be carried out in type spaces, which are simpler objects than hierarchy spaces.

Heifetz and Samet (1998a), we collect all hierarchies of preference representations that appear in type spaces in the category, and show that the resulting collection is a universal type space. A crucial point of the argument made in Proposition 3 is that even if the ever-extended preference representation is not itself continuous, the fact that it is encoded by continuous functionals is sufficient to imply that the limit preference representation is uniquely defined. One must then furthermore show that this limit preference representation varies in a measurable way with the hierarchy. This follows from a functional monotone class theorem employed in Lemma 1.

Second, in Section 4 we partition the universal type space into equivalence classes of types whose preference representations express the same preference relation over acts that depend on nature, on nature and others' preference relations over acts that depend on nature, etc. We show that this quotient space is universal in the category of preference type spaces, i.e. type spaces partitioned into members consisting of types with the same preferences on acts over nature and the other players' partition members. Heifetz and Samet (1998a) bypassed this second step by working in the first place with the standard representation of each Savage preference relation, namely the functional which attaches the value 0 to the constant act 0 and the value 1 to the constant act 1. This standard representation of the Savage preference relation constitutes integration with respect to a probability measure, interpreted as the type's "belief". However, for more general classes of preferences, such as lexicographic preferences, there does not necessarily exist within the cone of preference representations a "standard" representation, which is preserved under type morphisms, and hence the need to work explicitly with equivalence classes of preference representations.

The paper is organized as follows. Section 2 introduces preliminary notation and definitions. Section 3 contains the definitions and statement of the results for the existence of the universal type space, which is the main result of the section. Section 4 introduces and develops the notion of a preference type space and contains the main result of the paper: the existence of the universal preference type space. Section 5 concludes by discussing the main conceptual ideas and challenges of the paper. All proofs are collected in the Appendix A. A supplementary appendix provides examples of preferences for which the present paper's results apply: Appendix B.1 considers lexicographic expected utility and shows (in Section B.1.1) that the representation of lexicographic expected utility preferences satisfy a key monotone determination property (Definition 7). Appendix B also provides additional examples of preferences to which the present results apply.

Let L be a countable index set and $\trianglerighteq$ be a partial order on $\mathbb{R}^L$ for which the upper and lower contour sets for all $r = (r_\ell)_{\ell \in L} \in \mathbb{R}^L$

$$\{r' \in \mathbb{R}^L : r' \trianglerighteq r\} \quad \text{and} \quad \{r' \in \mathbb{R}^L : r \trianglerighteq r'\} \tag{1}$$

are Borel. For any measurable space Y with an associated σ-algebra Σ_Y, we denote by $\mathcal{F}(Y)$ the set of all real-valued bounded acts, i.e., bounded Σ_Y-measurable functions from Y to the set of outcomes $\mathbb{R}$. We say that a reflexive and transitive binary relation $\gtrsim$ over $\mathcal{F}(Y)$, henceforth termed a preference relation, admits a **monotone continuous** $(L, \trianglerighteq)$**-representation** if there exists a function

$$U \colon \mathcal{F}(Y) \to \mathbb{R}^L$$

that satisfies the following three conditions.

1. Representation: For $f, g \in \mathcal{F}(Y)$

$$g \gtrsim f \iff U(g) \trianglerighteq U(f). \tag{2}$$

2. Representation continuity: For $\{g_n\}_{n \geqslant 1} \subseteq \mathcal{F}(Y)$ and $g \in \mathcal{F}(Y)$,

$$(\forall y \in Y \quad g_n(y) \to g(y)) \implies U_\ell(g_n) \to U_\ell(g) \tag{3}$$

for all $\ell \in L$ with U_ℓ denoting the ℓ-th coordinate of U.[5]

3. Representation monotonicity: For $f, g \in \mathcal{F}(Y)$,

$$f \geqslant g \implies U_\ell(f) \geqslant U_\ell(g) \tag{4}$$

for all $\ell \in L$.

Definition 1. *$\mathcal{R}$ is called a representation class if, for any measurable space Y, $\mathcal{R}(Y)$ is some set of monotone continuous $(L, \trianglerighteq)$-representations equipped with the σ-algebra*

[5] This is not continuity of the preference relation. It is only the continuity of the representation functions $U = (U_\ell)_{\ell \in L}$. For example, this condition is satisfied by lexicographic expected utility preferences, which are not continuous.

generated by sets of the following form.

$$[f \sqsupseteq_\ell g] = \{U \in \mathcal{R}(Y) : U_\ell(f) \geqslant U_\ell(g)\} \qquad f, g \in \mathcal{F}(Y), \qquad \ell \in L$$

For measurable spaces Y and Z and a measurable function $\phi\colon Y \to Z$, define the map $\breve{\phi}\colon \mathcal{R}(Y) \to \mathcal{R}(Z)$ as follows.

$$\forall U \in \mathcal{R}(Y) \quad \forall f \in \mathcal{F}(Z) \quad \breve{\phi}(U)(f) = U(f \circ \phi) \tag{5}$$

The map $\breve{\phi}$, if well-defined (see the next definition) is measurable because, for every $f, g \in \mathcal{F}(Z)$ and $\ell \in L$,

$$\breve{\phi}^{-1}([f \sqsupseteq_\ell g]) = \{U \in \mathcal{R}(Y) : U_\ell(f \circ \phi) \geqslant U_\ell(g \circ \phi)\} \in \Sigma_{\mathcal{R}(Y)}.$$

Definition 2. *Representation class $\mathcal{R}$ is image-regular if, for all measurable spaces Y and Z and every measurable function $\phi\colon Y \to Z$, the map $\breve{\phi}$ is well-defined.*

Given that $\phi\colon Y \to Z$ can be viewed as a projection map from $\{(y, z) \in Y \times Z \mid z = \phi(y)\}$ on its second coordinate Z, image-regularity is simply the requirement that, for each $r \in \mathcal{R}(Y)$, the corresponding "marginal representation" of preferences on $\mathcal{F}(Z)$ is well-defined.

For the remainder, fix a representation class $\mathcal{R}$ and assume that it is image-regular. Image-regularity is sufficient for the existence of a universal type space, as defined in Definition 5 in Section 3.1 below.

For all $f, g \in \mathcal{F}(Y)$, let

$$[f \sqsupseteq g] = \{U \in \mathcal{R}(Y) : U(f) \trianglerighteq U(g)\}.$$

Then $[f \sqsupseteq g] \in \Sigma_{\mathcal{R}(Y)}$ since the sets in (1) are Borel.[6]

The set of players is I and $I_0 = I \cup \{0\}$ denotes the set of players and "nature" (player 0). As usual, for any collection $\{Y_i\}_{i \in I_0}$, $Y_{-i} = \bigtimes_{i' \in I_0 \setminus \{i\}} Y_{i'}$. We consider the

[6] Since L is countable, this follows from noting that (i) if $B \subseteq \mathbb{R}^L$ is Borel then

$$\{U \in \mathcal{R}(Y) \mid U(f) \notin B\} = \mathcal{R}(Y) \backslash \{U \in \mathcal{R}(Y) \mid U(f) \in B\}$$

and (ii) if $B_1, B_2, \dots \subseteq \mathbb{R}^L$ are Borel then

$$\{U \in \mathcal{R}(Y) \mid U(f) \in \bigcup_{n=1}^{\infty} B_n\} = \bigcup_{n=1}^{\infty} \{U \in \mathcal{R}(Y) \mid U(f) \in B_n\}$$

product, finite or infinite, of measurable spaces as a measurable space with the product σ-algebra.

3 TYPE SPACES

Let S be a measurable space of states of nature. Fix the countable index set L and partial order $\trianglerighteq$ satisfying (1).

Definition 3. *An $(L, \trianglerighteq)$ type space on S is a tuple $\langle (T_i)_{i\in I_0}, (m_i)_{i\in I}\rangle = \langle T, m\rangle$, such that*

1. *$T_0 = S$; and*
2. *for each $i \in I_0$, T_i is a measurable space; and*
3. *for each $i \in I$, $m_i\colon T_i \to \mathcal{R}(T_{-i})$ is measurable.*

Given a type space $\langle T, m\rangle$ and $i \in I$, elements of T are called states of the world and an element of T_i is called an i-type. For any $f, g \in \mathcal{F}(T_{-i})$, let

$$[f \sqsupseteq_\ell g]^i = \{t_i \in T_i : m_i(t_i)_\ell(f) \geqslant m_i(t_i)_\ell(g)\}$$

and the belief operator $[f \sqsupseteq g]^i$ is defined as

$$[f \sqsupseteq g]^i = \{t_i \in T_i : m_i(t_i)(f) \trianglerighteq m_i(t_i)(g)\}.$$

Then recalling that $[f \sqsupseteq g] = \{U \in \mathcal{R}(T_{-i}) : U(f) \trianglerighteq U(g)\}$, we have that

$$[f \sqsupseteq g]^i = m_i^{-1}([f \sqsupseteq g]).$$

Let $\langle T, m\rangle$ and $\langle T', m'\rangle$ be $(L, \trianglerighteq)$ type spaces on S. Type morphisms, defined next, are mappings that preserve the representation structures as given by m and m'.

Definition 4. *A type morphism from $\langle T, m\rangle$ to $\langle T', m'\rangle$ is a function $\phi = (\phi_i)_{i\in I_0}\colon T \to T'$ such that*

1. *$\phi_0\colon T_0 \to T_0'$ is the identity on S; and*
2. *for each $i \in I_0$, $\phi_i\colon T_i \to T_i'$ is measurable; and*
3. *for each $i \in I$ and $t_i \in T_i$, $m_i'(\phi_i(t_i)) = \check{\phi}_i(m_i(t_i))$, i.e., for every $f \in \mathcal{F}(T_{-i}')$,*

$$m_i'(\phi_i(t_i))(f) = m_i(t_i)(f \circ \phi). \tag{6}$$

Then, it can be verified that a type morphism ϕ preserves belief operators, i.e., for each $i \in I$, $f \in \mathcal{F}(T'_{-i})$,

$$\phi_i^{-1}([f \sqsupseteq g]^i) = [f \circ \phi_{-i} \sqsupseteq g \circ \phi_{-i}]^i. \tag{7}$$

3.1 THE UNIVERSAL TYPE SPACE

Definition 5. *An $(L, \trianglerighteq)$ type space $\langle T^*, m^* \rangle$ on S is universal if, for every $(L, \trianglerighteq)$ type space $\langle T, m \rangle$ on S, there exists a unique type morphism from $\langle T, m \rangle$ to $\langle T^*, m^* \rangle$.*

The existence of a universal type space is the main result of this section and is established in the remainder.

3.2 MAIN MEASURE-THEORETIC LEMMA

The main measure-theoretic lemma needed for the construction of the universal type space is the following.

Lemma 1. *Let (Y, Σ_Y) be a measurable space. Let $\mathcal{F}_0 \subseteq \mathcal{F}(Y)$ be such that the σ-algebra Σ_Y is generated by*

$$\mathbf{A}_{\mathcal{F}_0} = \{f^{-1}(E) : f \in \mathcal{F}_0, \quad E \subseteq \mathbb{R} \textit{ is Borel}\}$$

and such that $\mathcal{F}_0$ satisfies the following properties.

1. *The constant function $1 \in \mathcal{F}_0$*
2. *For any $f, f' \in \mathcal{F}_0$ and $\alpha, \alpha' \in \mathbb{R}$, $\alpha f + \alpha' f' \in \mathcal{F}_0$.*
3. *For any $f, f' \in \mathcal{F}_0$, $\min\{f, f'\} \in \mathcal{F}_0$.*

Let $\Sigma_{\mathcal{F}_0}$ be the σ-algebra on $\mathcal{R}(Y)$ generated by sets of the form

$$[f \sqsupseteq_\ell g] \quad \textit{for } \ell \in L \textit{ and } f, g \in \mathcal{F}_0.$$

Then $\Sigma_{\mathcal{R}(Y)} = \Sigma_{\mathcal{F}_0}$.

3.3 HIERARCHIES OF REPRESENTATIONS

We now define spaces of hierarchies of preference representations H_i^k for each $k \geqslant 0$ and $i \in I_0$. For every $k \geqslant 0$, $H_0^k = S$ and for every $i \in I$, H_i^0 is a singleton. As usual

$H^k = \bigtimes_{i \in I_0} H_i^k$. We define inductively

$$H_i^{k+1} = H_i^k \times \mathcal{R}(H_{-i}^k) = H_i^0 \times \left(\bigtimes_{k=0}^{k'} \mathcal{R}(H_{-i}^{k'}) \right). \tag{8}$$

The space of i-hierarchies for player $i \in I$ is

$$H_i = H_i^0 \times \left(\bigtimes_{k=0}^{\infty} \mathcal{R}(H_{-i}^{k'}) \right) \tag{9}$$

and the projection from H_i to H_i^k is denoted ϖ_i^k.

Given an $(L, \trianglerighteq)$ type space T, we can define an i-description map $h_i \colon T_i \to H_i$ for each $i \in I_0$ as follows. For all $k \geqslant 0$, let h_0^k be the identity on S. For $i \in I$, $h_i^0 \colon T_i \to H_i^0$ is uniquely defined since H_i^0 is a singleton. Inductively, define $h_i^{k+1} \colon T_i \to H_i^{k+1}$ for $k \geqslant 0$ by

$$h_i^{k+1}(t_i) = \left(h_i^k(t_i), \check{h}_{-i}^k(m_i(t_i)) \right) = \left(h_i^0(t_i), \check{h}_{-i}^0(m_i(t_i)), \ldots, \check{h}_{-i}^k(m_i(t_i)) \right) \tag{10}$$

where $\check{h}_{-i}^k \colon \mathcal{R}(T_{-i}) \to \mathcal{R}(H_{-i}^k)$ is the mapping between the sets of representations as defined by (5) in Section 2. Now define $h_i \colon T_i \to H_i$, $i \in I$ as the unique function that satisfies for all $k \geqslant 0$, $h_i^k = \varpi_i^k(h_i)$, i.e.,

$$h_i(t_i) = \left(h_i^0(t_i), \check{h}_{-i}^0(m_i(t_i)), \ldots, \check{h}_{-i}^k(m_i(t_i)), \ldots \right) \tag{11}$$

and define h_0 to be the identity on S. The first result is as follows.

Proposition 1. *Type morphisms preserve i-descriptions.*

We can now define the universal type space by setting $T_0^* = S$ and T_i^* to be the set of all i-descriptions in H_i, i.e., all hierarchies $t_i^* \in H_i$ for which $t_i^* = h_i(t_i)$ for some $t_i \in T_i$ in some type space $\langle T, m \rangle$ over S. The σ-algebra of T_i^* is the one inherited from H_i. We define $m_i^* \colon T_i^* \to \mathcal{R}(T_{-i}^*)$ by

$$m_i^*(t_i) = \check{h}_{-i}(m_i(t_i)). \tag{12}$$

The next result establishes that $\langle T^*, m^* \rangle$ thus defined is a $(L, \trianglerighteq)$ type space.

Proposition 2. *$\langle (T_i^*)_{i \in I_0}, (m_i^*)_{i \in I} \rangle$ is a $(L, \trianglerighteq)$ type space on S.*

Proposition 3. *For every $(L, \trianglerighteq)$ type space $\langle T, m \rangle$, the description map $h \colon T \to T^*$ is a type morphism.*

Lemma 2. *The hierarchy description maps $h_i\colon T_i^* \to T_i^*$ are the identity maps.*

We now state and prove the result on the existence of the universal $(L, \trianglerighteq)$ type space.

Theorem 1. *$\langle (T_i^*)_{i\in I_0}, (m_i^*)_{i\in I}\rangle$ is the universal $(L, \trianglerighteq)$ type space on S.*

4 PREFERENCE TYPE SPACES AND THE UNIVERSAL PREFERENCE TYPE SPACE

Having shown the existence of the universal type space for representations, we now establish the main result of the paper: the existence of the universal *preference* type space, as defined in Definition 14 in Section 4.3 below. This is essential since, in general, many representations may stand for the same preference, whereas it is the preference relation itself that is of economic relevance. We first introduce some terminology and notation.

4.1 PRELIMINARIES

Definition 6. *Let Y be a measurable space. A filtration of Σ_Y is a sequence $(\Sigma_Y^k)_{k\geqslant 0}$ of sub-σ-algebras on Y such that*

1. *$\Sigma_Y^k \subseteq \Sigma_Y^{k+1}$ for all $k \geqslant 0$; and*
2. *Σ_Y is generated by $\bigcup_{k\geqslant 0} \Sigma_Y^k$.*

Definition 7. *Representation class $\mathcal{R}$ is preference monotone determined if, for every measurable space Y, the two statements below are equivalent for all $U, V \in \mathcal{R}(Y)$ and for every filtration $(\Sigma_Y^k)_{k\geqslant 0}$ of Σ_Y.*

1. *For all $k \geqslant 0$, the preferences represented by U and V coincide on Σ_Y^k-measurable acts.*

$$\forall k \geqslant 0 \quad \forall f, g \in \mathcal{F}(\Sigma_Y^k), \quad U(f) \trianglerighteq U(g) \iff V(f) \trianglerighteq V(g)$$

2. *The preferences represented by U and V coincide on all Σ_Y-measurable acts.*

$$\forall f, g \in \mathcal{F}(\Sigma_Y) \quad U(f) \trianglerighteq U(g) \iff V(f) \trianglerighteq V(g)$$

Monotone determination, in the familiar setting of expected utility preferences, is the property that when and if an extension of a projective system of probability measures (i.e., a sequence of probability measures on an inverse system of measurable sets) exists, then it is unique. This property is used by Heifetz and Samet (1998a) to obtain their universal type space result. In the more general settings allowed here, monotone determination says that if an inverse system of preferences (i.e., a sequence of preferences over acts defined on a projective system of measurable sets) has an an extension to a preference over acts defined on the projective limit of the system, then it is unique. In the appendices, we prove that monotone determination is satisfied by several well-known classes of preferences that are of interest.

Definition 8. *We say that $\mathcal{R}$ is regular if it is both image-regular and preference monotone determined.*

For the remainder, assume that $\mathcal{R}$ is regular. Regularity of $\mathcal{R}$ is sufficient for the existence of a universal preference type space. Given a regular representation class $\mathcal{R}$, we define the corresponding preference class $\mathcal{P}$ as follows.

Definition 9. *For any measurable space Y, given a regular representation class $\mathcal{R}(Y)$ let $\mathcal{P}(Y)$ denote the set of preference relations $\succsim$ on $\mathcal{F}(Y)$ that admit a representation in $\mathcal{R}(Y)$. Each $\succsim \in \mathcal{P}(Y)$ is identified with the equivalence class of all $U \in \mathcal{R}(Y)$ that represent $\succsim$.*

$\mathcal{P}(Y)$ is equipped with the σ-algebra generated by sets of the following form.

$$[[f \sqsupseteq g]] = \{\succsim \in \mathcal{P}(Y) : f \succsim g\} \quad f, g \in \mathcal{F}(Y)$$

Definition 10. *For each $U \in \mathcal{R}(Y)$, let $p_Y(U)$ denote the equivalence class in $\mathcal{P}(Y)$ to which U belongs. The map $p_Y : \mathcal{R}(Y) \to \mathcal{P}(Y)$ is measurable because*

$$p_Y^{-1}([[f \sqsupseteq g]]) = [f \sqsupseteq g] \quad \textit{for every } f, g \in \mathcal{F}(Y).$$

Definition 11. *For measurable spaces Y and Z and a measurable function $\phi : Y \to Z$, define the map $\widehat{\phi} : \mathcal{P}(Y) \to \mathcal{P}(Z)$ as follows.*

$$\forall \succsim \in \mathcal{P}(Y) \quad \widehat{\phi}(\succsim) = \{\check{\phi}(U) \mid U \in \succsim\}$$

The map $\widehat{\phi}$ is measurable because, for every $f, g \in \mathcal{F}(Z)$,

$$\widehat{\phi}^{-1}([[f \sqsupseteq g]]) = \{\succsim \in \mathcal{P}(Y) : f \circ \phi \succsim g \circ \phi\} \in \Sigma_{\mathcal{P}(Y)}.$$

We now introduce the notion of a preference type space and a preference type morphism between preference type spaces.

Definition 12. *A $(L, \trianglerighteq)$ preference type space on S is a tuple $\langle (\Pi_i)_{i \in I_0}, (\mu_i)_{i \in I_0} \rangle = \langle \Pi, m \rangle$ such that for some $(L, \trianglerighteq)$ type space $\langle T, m \rangle$ on S, for every $i \in I_0$, Π_i is a measurable space which is a partition of T_i, i.e., Π_i consists of the partition members $\{\Pi_i(t_i) \mid t_i \in T_i\}$, where the following are satisfied.*

1. *Π_0 is the partition of S to singletons, i.e., $\Pi_0(s) = \{s\}$ for all $s \in S$, endowed with the σ-algebra inherited from S, i.e., $E \subseteq S$ is measurable in S iff*

$$\{\Pi_0(s) : s \in E\} = \{\{s\} : s \in E\} \quad \text{is measurable in } \Pi_i.$$

2. *For all $i \in I$, $\mu_i \colon \Pi_i \to \mathcal{P}(\Pi_{-i})$ is a measurable map defined by*

$$\mu_i(\Pi_i(t_i)) = p_{\Pi_{-i}}(\check{\Pi}_{-i}(m_i(t_i))) \quad \forall t_i \in T_i.$$

3. *For all $i \in I_0$, the map $t_i \mapsto \Pi_i(t_i)$ is measurable.*

We say that $\langle \Pi, \mu \rangle$ is based on $\langle T, m \rangle$.

In particular, the condition in Definition 12.2 implies that $\mu_i(\Pi_i(t_i)) = \mu_i(\Pi_i(t_i'))$ whenever $\Pi_i(t_i) = \Pi_i(t_i')$.

In general, there may be many preference-type spaces based on the same $\langle T, m \rangle$. In particular, $\langle T, m \rangle$ itself can be viewed as a preference type space based on itself where, for all $i \in I$, Π_i is the partition of T_i into singletons and the measurable structure of Π_i is inherited from that of T_i and $\mu_i = p_{\Pi_{-i}} \circ \check{\Pi}_{-i} \circ m_i$.[7]

Definition 13. *Let $\langle \Pi, \mu \rangle$ and $\langle \Pi', \mu' \rangle$ be $(L, \trianglerighteq)$ preference-type spaces on S. A preference-type morphism from $\langle \Pi, \mu \rangle$ to $\langle \Pi', \mu' \rangle$ is a function $\phi = (\phi_i)_{i \in I_0} \colon \Pi \to \Pi'$ such that*

1. *$\phi_0 \colon \Pi_0 \to \Pi_0'$ is the identity on S,*
2. *for each $i \in I_0$, $\phi_i \colon \Pi_i \to \Pi_i'$ is a measurable function, and*
3. *for each $i \in I$ and $\pi_i \in \Pi_i$, $\mu_i'(\phi_i(\pi_i)) = \widehat{\phi}_{-i}(\mu_i(\pi_i))$.*

[7] That is, $\Pi_i(t_i) = \{t_i\}$ for all $t_i \in T_i$ and $E \subseteq T_i$ is measurable in T_i if and only if

$$\{\Pi_i(t_i) : t_i \in E\} = \{\{t_i\} : t_i \in E\}$$

is measurable in Π_i.

Given that, as explained above, every type space $\langle T, m\rangle$ can be viewed also as a preference type space where Π_i is the partition of T_i to singletons for all $i \in I$, and given that it is the preferences of types $t_i \in T_i$ that is of economic relevance, more so than the particular representation $m_i(t_i)$ of these preferences, it is of interest to know if a universal preference type space as defined next exists.

Definition 14. *A $(L, \trianglerighteq)$ preference-type space $\langle \Pi^*, \mu^*\rangle$ on S is universal if, for every preference-type space $\langle \Pi, \mu\rangle$ on S, there exists a unique preference type morphism from $\langle \Pi, \mu\rangle$ to $\langle \Pi^*, \mu^*\rangle$.*

The positive answer to this existence question, which is the main result of this paper, is given by the following theorem.

Theorem 2. *There exists a universal preference-type space on S.*

We prove this result by constructing the putative universal $(L, \trianglerighteq)$ preference type space $\langle \Pi^*, \mu^*\rangle$ as a preference type space based on $\langle T^*, m^*\rangle$ and establishing that there is a unique preference type morphism to $\langle \Pi^*, \mu^*\rangle$ from any preference type space $\langle \Pi, \mu\rangle$.

$\langle \Pi^*, \mu^*\rangle$ is constructed as follows. For each $k \geqslant 0$, let $\Pi_0^{*,k}$ be the partition of $T_0^* = S$ into singletons. Let $\Sigma_{\Pi_0^{*,k}}$ be σ-algebra inherited from $\Sigma_{T_0^*} = \Sigma_S$. For each $i \in I$ and $k = 0$, let $\Pi_i^{*,0} = \{T_i^*\}$. For each $i \in I$ and $k \geqslant 0$, define $\Pi_i^{*,k+1}$ inductively.

Suppose that we have already defined the partitions $\Pi_i^{*,0}, \ldots, \Pi_i^{*,k}$ for each $i \in I$. Let $\Pi_i^{*,k+1}$ be the partition of T_i^* into equivalence classes of types that induce the same preferences on $\mathcal{F}(\Pi_{-i}^{*,k})$, i.e.,

$$\forall t_i^* \in T_i^* \quad \Pi_i^{*,k+1}(t_i^*) = \{y_i^* \in T_i^* : p_{\Pi_{-i}^{*,k}}(\check{\Pi}_{-i}^{*,k}(m_i^*(t_i^*))) = p_{\Pi_{-i}^{*,k}}(\check{\Pi}_{-i}^{*,k}(m_i^*(y_i^*)))\}$$

Let $\Sigma_{\Pi_i^{*,k+1}}$ be generated by the following family of sets.

$$\{[f \sqsupseteq g]^i : f, g \in \mathcal{F}(\Pi_{-i}^{*,k})\}$$

Let Π_i^* denote the join (coarsest common refinement) of the weakly refining sequence of partitions $(\Pi_i^{*,k})_{k\geqslant 0} = (\Pi_i^{*,0}, \Pi_i^{*,1}, \Pi_i^{*,2}, \ldots)$, i.e.,

$$\Pi_i^* = \bigvee_{k\geqslant 0} \Pi_i^{*,k}$$

Let $\Sigma_{\Pi_i^*}$ denote the σ-algebra generated by the union of the weakly refining sequence of σ-algebras $(\Sigma_{\Pi_i^{*,k}})_{k\geqslant 0} = (\Sigma_{\Pi_i^{*,0}}, \Sigma_{\Pi_i^{*,1}}, \Sigma_{\Pi_i^{*,2}}, \dots)$.

Finally, define $\mu_i^*\colon \Pi_i^* \to \mathcal{P}(\Pi_{-i}^*)$ by the following.

$$\forall t_i^* \in T_i^* \quad \mu_i^*(\Pi_i^*(t_i^*)) = p_{\Pi_{-i}^*}\left(\check{\Pi}_{-i}^*(m_i^*(t_i^*))\right)$$

Then μ_i^* is well-defined, i.e., $\mu_i^*(\Pi_i^*(t_i^*)) = \mu_i^*(\Pi_i^*(t_i^{*\prime})$, whenever $\Pi_i^*(t_i^*) = \Pi_i^*(t_i^{*\prime})$ due to the monotone determination property (Definition 7). The next result establishes that $\langle \Pi^*, \mu^* \rangle$ is indeed a preference type space based on $\langle T^*, m^* \rangle$.

Proposition 4. *For all $i \in I$, μ_i^* is measurable.*

Let $\langle \Pi, \mu \rangle$ be a preference-type space based on the type space $\langle T, m \rangle$. Let η_0 be the identity on S. For each $i \in I$, let $\eta_i\colon \Pi_i \to \Pi_i^*$ be defined by the following.

$$\forall t_i \in T_i \quad \eta_i(\Pi_i(t_i)) = \Pi_i^*(h_i(t_i))$$

where h_i is the representation i-description map associated with $\langle T, m \rangle$.

Proposition 5. *η is a preference-type morphism.*

To establish that η is in fact the unique preference type morphism from any preference type space to $\langle \Pi^*, \mu^* \rangle$, we first establish a property of any preference type morphism. Let $\langle \Pi, \mu \rangle$ be a preference-type space based on the type space $\langle T, m \rangle$. Let ψ be a preference-type morphism from $\langle \Pi, \mu \rangle$ to $\langle \Pi^*, \mu^* \rangle$. Then $\mu_i^*(\psi_i(\pi_i)) = \widehat{\psi}_{-i}(\mu_i(\pi_i))$ for each $i \in I$ by Definition 13.3. Furthermore, we can inductively define $\psi^k\colon \Pi \to \Pi^{*,k}$ for each $k \geqslant 0$ as follows. For each $k \geqslant 0$, let ψ_0^k be the identity on S. For each $i \in I$, $\psi_i^0\colon \Pi_i \to \Pi_i^{*,0}$ is uniquely defined because $\Pi_i^{*,0}$ is a singleton. Let $\psi_i^{k+1} = \widehat{\psi}_{-i}^k \circ \mu_i$.

Lemma 3. *For all $i \in I_0$ and $k \geqslant 0$, $\psi_i^k = \Pi_i^{*,k} \circ \psi_i$.*

The next result establishes the uniqueness of η, which completes the proof of Theorem 2.

Proposition 6. *If ψ is a preference-type morphism from $\langle \Pi, \mu \rangle$ to $\langle \Pi^*, \mu^* \rangle$, then $\psi = \eta$.*

5 CONCLUDING REMARKS

The universal type space and universal preference type space construction above can be applied to wide-ranging classes of preferences. These include lexicographic expected utility preferences (Appendix B.1), continuous preferences under risk and ambiguity, and some instances of preferences over menus (Appendix B).

In the construction of a universal preference type space, our starting point was the main conceptual idea of Heifetz and Samet (1998a), namely to collect from all the type spaces in the category all the hierarchies that describe the state of mind of each player about nature, about nature and the others' state of mind about nature, and so forth. The collection of these description hierarchies would constitute a universal space if each such hierarchy determines uniquely the limit state of mind of the player in a measurable way. When such a continuity property does not hold (like in the case of dichotomous knowledge, see Heifetz and Samet (1998b)), a universal space fails to exist. While implementing this idea when a "state of mind" is a preference relation over acts rather than a belief, we had to address several challenges.

First, whereas for the case of probabilistic beliefs, (Heifetz and Samet, 1998a, Lemma 4.5) used a monotone class theorem to prove the measurability of the limit state of mind, for the case of preferences we had to to employ a stronger tool, namely a double application of a functional monotone class theorem (Lemma 1). Second, and more importantly, we are able to cater to classes of preferences of economic relevance, like lexicographic preferences, that are themselves discontinuous but nevertheless satisfy a weaker, limit determination property: each description hierarchy of states of mind uniquely determines a limit state of mind, even if in a discontinuous way. We showed how this weaker property is sufficient for a universal space to exist. Last, in the classes of preferences of interest, the same preference relation may be represented by numerous functional tuples, and from within the positive cone of representations of the same preference relation there might emerge no natural candidate as the "standard" representation, a standardization which would be preserved under type morphisms. Thus, unlike in the case of Savage (1954) preferences over acts, which have a standard representation by the integral of these acts with respect to a probability measure, for lexicographic expected utility preferences of 3 or more levels, for example, such an anchor representation does not exist. This challenge led us to proceed in two steps.

First (Section 3), we constructed the universal type space in the category of type spaces actually used by game theorists, namely type spaces in which each type is associated in a measurable way with a particular representation of the preference relation of that type. In this universal space, there are numerous types which, albeit with different representations, entertain the same preference relation on acts over nature and the other players' types. Therefore in the second step (Section 4), we partitioned the universal type space into equivalence classes of types whose preference representations express the same preference relation over acts that depend on nature, on nature and others' preference relations over acts that depend on nature, and so on (Definition 14). We showed that this quotient space is universal in the category of preference type spaces, i.e. type spaces partitioned into members consisting of types with the same preferences on acts over nature and the other players' partition members (Definition 12).

The class of type spaces forms a sub-class of preference type spaces, because each type space can be viewed, in particular, as a preference type space by partitioning each player's types to singletons, and associating each such singleton with the preference relation represented by the functional-tuple of that type. It is this preference relation (within its measurable structure) rather than the particular preference representation (within its stronger, coordinate-wise measurable structure of functional-tuples) that is of economic relevance, and hence the significance of the existence of a universal preference type space (Theorem 2), into which every type space can be embedded in a unique way by a preference type morphism. The universal preference type space might be guaranteed to be a type space itself (selecting for each type a functional-tuple representing its preference relation in a measurable way) only under very specific topological assumptions (see for example Lee (2015a)). Here, in contrast, like in Heifetz and Samet (1998a), we adopted a purely measurable approach, and hence the strength and wide potential applicability of the results.

REFERENCES

D. Ahn. Hierarchies of ambiguous beliefs. *Journal of Economic Theory*, 136:286–301, 2007.

P. Battigalli and M. Siniscalchi. Hierarchies of conditional beliefs and interactive epistomology for dynamic games. *Journal of Economic Theory*, 88:188–230, 1999.

D. Bergemann, S. Morris, and S. Takahashi. Interdependent preferences and strategic distinguishability. mimeo, Princeton University Economic Theory Center Research Paper No. 008-2011, 2011.

L. Blume, A. Brandenburger, and E. Dekel. Lexicographic probabilities and choice under uncertainty. *Econometrica*, 59(1):pp. 61–79, 1991. ISSN 00129682.

A. Brandenburger and E. Dekel. Hierarchies of belief and common knowledge. *Journal of Economic Theory*, 59:189–198, 1993.

S. Cerria-Vioglio, M. Marinacci, F. Maccheroni, and L. Montrucchio. Uncertainty-averse preferences. *Journal of Economic Theory*, 146:1275–1330, 2011.

Y.-C. Chen. Universality of the epstein-wang type structure. *Games and Economic Behavior*, 68(1):389 – 402, 2010. ISSN 0899-8256.

C. Dellacherie and P.-A. Meyer. *Probabilities and Potential A*. North Holland Mathematics Studies 29. North Holland, New York, 1978.

A. Di Tillio. Subjective expected utility in games. *Theoretical Economics*, 3:287–323, 2008.

L. Epstein. An axiomatic model of non-bayesian updating. *Review of Economic Studies*, 73:413–436, 2006.

L. Epstein and T. Wang. 'Beliefs about beliefs' without probabilities. *Econometrica*, 64:1343–1373, 1996.

F. Gul and W. Pesendorfer. Temptation and self-control. *Econometrica*, 69:1403–1435, 2001.

F. Gul and W. Pesendorfer. Interdependent preference models as a theory of intentions. mimeo, Princeton University, 2010.

J. C. Harsanyi. Games with incomplete information played by 'bayesian' players: Parts i-iii. *Management Science*, 14:159–182, 320–334, and 486–502, 1967-68.

A. Heifetz. The bayesian formulation of incomplete information–the non-compact case. *International Journal of Game Theory*, 21:329–338, 1993.

A. Heifetz and W. Kets. All types naive and canny. mimeo, Northwestern University, 2012.

A. Heifetz and D. Samet. Topology-free typology of beliefs. *Journal of Economic Theory*, 82:324–341, 1998a.

A. Heifetz and D. Samet. Knowledge spaces with arbitrarily high rank. *Games and Economic Behavior*, 22:260–273, 1998b.

A. Heifetz and D. Samet. Coherent beliefs are not always types. *Journal of Mathematical Economics*, 32:475–488, 1999.

A. Heifetz, M. Meier, and B. Schipper. Unawareness, beliefs, and speculative trade. mimeo, UC Davis, 2012.

S. Heinsalu. Universal type space with unawareness. mimeo, Yale University, 2012.

B. Hill. An additively separable representation in the savage framework. *Journal of Economic Theory*, 145:2044–2054, 2010.

E. Karni, D. Schmeidler, and K. Vind. On state dependent preferences and subjective probabilities. *Econometrica*, 51:1021–1031, 1983.

W. Kets. Bounded reasoning and higher-order uncertainty. mimeo, Northwestern University, 2012.

B. S. Lee. Conditional beliefs and higher-order preferences. mimeo, University of Toronto, 2013.

B. S. Lee. Generalizing type spaces. mimeo, 2015a.

B. S. Lee. Admissibility and assumption. mimeo, 2015b.

T. Mariotti, M. Meier, and M. Piccione. Hierarchies of beliefs for compact possibility models. *Journal of Mathematical Economics*, 21:303–324, 2005.

M. Meier. Universal knowledge-belief structures. *Games and Economic Behavior*, 62: 53–66, 2008.

J.-F. Mertens and S. Zamir. Formulation of bayesian analysis for games with incomplete information. *International Journal of Game Theory*, 14:1–29, 1985.

J.-F. Mertens, S. Sorin, and S. Zamir. Repeated games: Part a: Background material. CORE Discussion Paper 9420, Universite Catholique de Louvain., 1994.

L. S. Moss and I. D. Viglizzo. Harsanyi type spaces and final coalgebras constructed from satisfied theories. *Electronic Notes in Theoretical Computer Science*, 106: 279–295, 2004. ISSN 1571-0661.

M. Pinter. Type spaces with non-additive beliefs. mimeo, Corvinus University of Budapest, 2012.

M. Pinter and Z. Udvari. Generalized type spaces. mimeo, MPRA Paper No. 34107, 2011.

A. Rényi. On conditional probability spaces generated by a dimensionally ordered set of measures. *Theory of Probability & Its Applications*, 1(1):55–64, 1956.

L. J. Savage. *The foundations of statistics.* John Wiley and Sons, 1954.

D. Schmeidler. Subjective probability and expected utility without additivity. *Econometrica*, 57:571–587, 1989.

M. Siniscalchi. Vector expected utility and attitudes toward variation. *Econometrica*, 77:801–855, 2009.

J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1944.

P. Wakker and A. Tversky. An axiomatisation of cumulative prospect theory. *Journal of Risk and Uncertainty*, 7:147–176, 1993.

P. Wakker and H. Zank. State dependent utility for savage's state space. *Mathematics of Operations Research*, 24:8–34, 1999.

A PROOFS

Proof of Lemma 1. Clearly $\Sigma_{\mathcal{R}(Y)} \supseteq \Sigma_{\mathcal{F}_0}$, since $\Sigma_{\mathcal{R}(Y)}$ is generated by

$$\{[f \sqsupseteq_\ell g] \mid f, g \in \mathcal{F}(Y), \ell \in L\}$$

which is a subset of the family of sets

$$\{[f \sqsupseteq_\ell g] \mid f, g \in \mathcal{F}_0, \ell \in L\}$$

that generates $\Sigma_{\mathcal{F}_0}$. We now establish in two steps that $\Sigma_{\mathcal{R}(Y)} \subseteq \Sigma_{\mathcal{F}_0}$. First, let $\mathcal{F}' \subseteq \mathcal{F}(Y)$ be the collection of acts f such that $[f \sqsupseteq g]_\ell \in \Sigma_{\mathcal{F}_0}$ for all $g \in \mathcal{F}_0$ and $\ell \in L$. We prove that $\mathcal{F}' \supseteq \mathcal{F}(Y)$ by employing the functional monotone class theorem (Dellacherie and Meyer, 1978, theorem 22.3, p.15-1).[8] Given assumptions (i)-(iii) on $\mathcal{F}_0$, and the fact that $\mathcal{F}(Y)$ is the set of Σ_Y-measurable acts while Σ_Y is generated by $\mathbf{A}_{\mathcal{F}_0}$, it remains to show that $\mathcal{F}'$ is closed under bounded monotone convergence. Indeed, let $\{f_n\}_{n=1}^{\infty}$ be a bounded monotonically increasing sequence of functions in $\mathcal{F}'$ converging to $f \in \mathcal{F}(Y)$.[9] Then for all $g \in \mathcal{F}_0$ and $\ell \in L$, by representation continuity

[8] The corresponding notation there has $\mathcal{H} = \mathcal{F}'$ and $\mathcal{C} = \mathcal{F}_0$.

[9] That is, $\{f_n\}_{n=1}^{\infty}$ is a sequence for which (i) there exists $M < \infty$ such that $0 \leqslant f_n(y) \leqslant M$ for all $y \in Y$ and $n = 1, 2, \ldots$ and (ii) $f_n(y)$ is increasing in n for all $y \in Y$.

(3) and monotonicity (4),

$$[f \sqsupseteq_\ell g] = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n \geqslant m} \left[f_n \sqsupseteq_\ell g - \frac{1}{k} \right] \in \Sigma_{\mathcal{F}_0}$$

and hence $f \in \mathcal{F}'$, as required.

Second, let $\mathcal{F}'' \subseteq \mathcal{F}(Y)$ be the collection of acts g such that $[f \sqsupseteq_\ell g] \in \Sigma_{\mathcal{F}_0}$ for all $f \in \mathcal{F}' = \mathcal{F}(Y)$ and $\ell \in L$. Here too we prove that $\mathcal{F}'' \supseteq \mathcal{F}(Y)$ by employing the functional monotone class theorem, and the crucial step is to show that $\mathcal{F}''$ is closed under bounded monotone convergence. Indeed, let $\{g_n\}_{n=1}^{\infty}$ be a bounded monotonically increasing sequence of functions in $\mathcal{F}''$ converging to $g \in \mathcal{F}(Y)$. Then for all $f \in \mathcal{F}(Y)$

$$[f \sqsupseteq_\ell g] = \bigcap_{n \geqslant 1} [f \sqsupseteq_\ell g_n] \in \Sigma_{\mathcal{F}_0}$$

and hence $g \in \mathcal{F}''$, as required. From the two steps together we conclude that $\mathcal{F}(Y) \times \mathcal{F}(Y) = \mathcal{F}' \times \mathcal{F}''$ is the collection of act-pairs (f, g) for which $[f \sqsupseteq_\ell g] \in \Sigma_{\mathcal{F}_0}$ for all $\ell \in L$. Hence $\Sigma_{\mathcal{F}_0}$ contains all the generators of $\Sigma_{\mathcal{R}(Y)}$, and therefore $\Sigma_{\mathcal{R}(Y)} \subseteq \Sigma_{\mathcal{F}_0}$. □

Proof of Proposition 1. Let $\phi \colon T \to T'$ be a type morphism. We have to show that $h'_i(\phi_i(t_i)) = h_i(t_i)$ for all $t_i \in T_i$ and $i \in I_0$. For $i = 0$, this follows immediately since $\phi_0, h_0^k, h_0, h_0'^k, h_0'$ are all the identity map on S. For $i \in I$, $h_i^0(t_i) = h_k'^0(\phi_i(t_i))$ since H_i^0 is a singleton. Suppose, inductively, that we have already proved that $h_i^k(t_i) = h_i'^k(\phi_i(t_i))$ for every $t_i \in T_i$ and every $i \in I_0$. In the following sequence of equalities, the second equality stems from the fact that type morphisms preserve preference representations (6) and the induction hypothesis is used in the third equality. For any $f \in \mathcal{F}(H_{-i}^k)$

$$\begin{aligned} \check{h}_{-i}'^k(m_i'(\phi_i(t_i)))(f) &= m_i'(\phi_i(t_i))(f \circ h_{-i}'^k) \\ &= m_i(t_i)(f \circ h_{-i}'^k \circ \phi_{-i}) = m_i(t_i)(f \circ h_{-i}^k) = \check{h}_{-i}^k(m_i(t_i))(f) \end{aligned}$$

It then follows that

$$\begin{aligned} h_i'^{k+1}(\phi_i(t_i)) &= \Big(h_i'^k(\phi_i(t_i)), \check{h}_{-i}'^k(m_i'(\phi_i(t_i))) \Big) \\ &= \Big(h_i^k(t_i), \check{h}_{-i}^k(m_i(t_i)) \Big) = h_i^{k+1}(t_i) \end{aligned}$$

as needed. □

Proof of Proposition 2. To show that $\langle T^*, m^* \rangle$ is a type space on S, we have to show that m_i^* is a measurable mapping for each $i \in I$. For t_i^*, let t_i be the i-type used to define $m_i^*(t_i^*) \in \mathcal{R}(T_{-i}^*) \subseteq \mathcal{R}(H_{-i})$, i.e., $m_i^*(t_i^*) = \check{h}_{-i}(m_i(t_i))$. Consider the preference relation on $\mathcal{F}(H_{-i}^k)$ induced by $m^*(t_i^*)$, i.e., $\breve{\varpi}_{-i}^k(m_i^*(t_i^*))$ where $\breve{\varpi}_{-i}^k\colon \mathcal{R}(H_{-i}) \to \mathcal{R}(H_{-i}^k)$ is the representation mapping defined in (5) corresponding to the projection $\varpi_i^k\colon H_i \to H_i^k$. Then,

$$
\begin{aligned}
\breve{\varpi}_{-i}^k(m_i^*(t_i^*)) &= \breve{\varpi}_{-i}^k(\check{h}_{-i}(m_i(t_i))) && (13)\\
&= \check{h}_{-i}^k(m_i(t_i)) \quad \text{since } h_i^k(t_i) = \varpi_i^k(h_i(t_i)) && (14)\\
&= (k+1)^{th} \text{ coordinate of } h_i(t_i) && (15)\\
&= (k+1)^{th} \text{ coordinate of the hierarchy } t_i^* && (16)\\
&\equiv (t_i^*)^{k+1} && (17)
\end{aligned}
$$

Let $\mathcal{G}_k \subseteq \mathcal{F}(H_{-i})$ be the set of acts that are measurable with respect to H_{-i}^k, i.e., $\mathcal{G}_k$ is the set of acts f_k such that for every Borel measurable $E \subseteq \mathbb{R}$ there exists some measurable $E_k \subseteq H_{-i}^k$ for which $f_k^{-1}(E) = (\varpi_{-i}^k)^{-1}(E_k)$. Let

$$
\mathcal{G} = \bigcup_{k=0}^{\infty} \mathcal{G}_k
$$

Then $\mathbf{A}_{\mathcal{G}} = \{f^{-1}(E) : f \in \mathcal{G}, E \subseteq \mathbb{R} \text{ Borel}\}$ is the collection of all cylinders with finite-dimensional bases, which generates the σ-algebra on H_{-i}. Moreover, (i) the constant act 1 is in $\mathcal{G}_0$ and hence in $\mathcal{G}$. Furthermore, if $f, f' \in \mathcal{G}$ then $f \in \mathcal{G}_k$ and $f' \in \mathcal{G}_{k'}$ for some k, k', and if, without loss of generality $k \geqslant k'$ then $f' \in \mathcal{G}_k$. It thus follows that (ii) $\alpha f + \alpha' f' \in \mathcal{G}_k \subset \mathcal{G}$ for every $\alpha, \alpha' \in \mathbb{R}$, and (iii) $\min\{f, f'\} \in \mathcal{G}_k \subset \mathcal{G}$. Lemma 1 then implies that $\Sigma_{\mathcal{R}(H_{-i})} = \Sigma_{\mathcal{G}}$, i.e., that $\Sigma_{\mathcal{R}(H_{-i})}$ is generated by the sets of the form

$$
\{[f \sqsupseteq_\ell g] \mid f, g \in \mathcal{G}, \ell \in L\} = \bigcup_{k=0}^{\infty} \{[f_k \sqsupseteq_\ell g_k] \mid f_k, g_k \in \mathcal{G}_k, \ell \in L\}
$$

But if $f_k, g_k \in \mathcal{G}_k, \ell \in L$, then denoting by $f^k, g^k \in \mathcal{F}(H_{-i}^k)$ the acts on H_{-i}^k for which $f_k = f^k \circ \varpi_{-i}^k$, $g_k = g^k \circ \varpi_{-i}^k$, from (13) we get that

$$
\begin{aligned}
(m_i^*)^{-1}([f_k \sqsupseteq_\ell g_k]) &= \{t_i^* \mid (m_i^*(t_i^*))_\ell(f_k) \geqslant (m_i^*(t_i^*))_\ell(g_k)\} && (18)\\
&= \{t_i^* \mid ((t_i^*)^{k+1})_\ell(f^k) \geqslant ((t_i^*)^{k+1})_\ell(g^k)\} && (19)
\end{aligned}
$$

which are hence measurable subsets in H_i. This proves that m_i^* is a measurable mapping as required. $\square$

Proof of Proposition 3. The functions $(h_i)_{i\in I}$, are measurable and h_0 is the identity. Since the range of h_i is T_i^*, it is also measurable as a function to T_i^*. Also, from (13), it follows that for acts f_k in $\mathcal{F}(H_{-i})$ that are measurable with respect to the σ-algebra on H_{-i}^k, $(m_i^*(t_i^*))_\ell(f_k)$ does not depend on the specific type t_i chosen to define $m_i^*(t_i^*)$, since there exists $f^k \in \mathcal{F}(H_{-i}^k)$ such that $f_k = f^k \circ \varpi_{-i}^k$ and so

$$(m_i^*(t_i^*))_\ell(f_k) = ((t_i^*)^{k+1})_\ell(f^k) = (\check{h}_{-i}(m_i(t_i)))_\ell(f_k) = (m_i(t_i))_\ell(f_k \circ h_{-i}) \tag{20}$$

for any t_i such that $h_i(t_i) = t_i^*$ and every $\ell \in L$. Now, every measurable act $f \in \mathcal{F}(H_{-i})$ is a pointwise limit of a sequence of functions $f_k \in \mathcal{F}(H_{-i})$ which are, respectively, measurable with respect to the σ-algebra on H_{-i}^k. The continuity of $(m_i^*(t_i^*))_\ell$ and $(m_i(t_i))_\ell$ in (3) then implies that

$$(m_i^*(t_i^*))_\ell(f) = \lim_{k\to\infty}(m_i^*(t_i^*))_\ell(f_k) = \lim_{k\to\infty}(m_i(t_i))_\ell(f_k \circ h_{-i}) = (m_i(t_i))_\ell(f \circ h_{-i}) \tag{21}$$

for every $\ell \in L$ and $i \in I$, which proves that h is a type morphism. $\square$

Proof of Lemma 2. It suffices to show that for each k and $i \in I$, the function h_i^k on T^* is the projection on H_i^k. We show this by induction on k. It is clearly true for $k = 0$. Suppose that $h^k = \varpi^k$. By definition, $(h_i(t^*))^{k+1} = \check{h}_{-i}^k(m_i^*(t_i^*))$. Using the induction hypothesis we get $\check{h}_{-i}^k(m_i^*(t_i^*)) = \check{\varpi}_{-i}^k(m_i^*(t_i^*))$, implying that $(h_i(t^*))^{k+1} = \check{\varpi}_{-i}^k(m_i^*(t_i^*)) = (t_i^*)^{k+1}$. $\square$

Proof of Theorem 1. For any type space $\langle T, m\rangle$, the description map $h\colon T \to T^*$ is a type morphism by Proposition 3. We need to show that it is unique. Suppose $\phi\colon T \to T^*$ is a type morphism. Then for each $i \in I$ and $t_i \in T_i$, $h_i(t_i) = h_i(\phi_i(t_i))$ by Proposition 1. However, from Lemma 2, we get $h_i(\phi_i(t_i)) = \phi_i(t_i)$. Hence, $\phi_i = h_i$ and the result follows. $\square$

Proof of Proposition 4. Measurability of $\mu_i^\star$ follows from the measurability of $p_{\Pi_{-i}^*}$, $\check{\Pi}_{-i}^*$, and $m_i^\star$. $\square$

Proof of Proposition 5. For all $i \in I$, $\Sigma_{\Pi_{-i}^*}$ is generated by $(\Sigma_{\Pi_{-i}^{*,k}})_{k\geqslant 0}$. Due to monotone determination, it therefore suffices to show the following for all $\Sigma_{\Pi_{-i}^{*,k}}$-measurable $f, g \in \mathcal{F}(\Pi_{-i}^*)$.

$$\eta_i^{-1}([[f \sqsupseteq g]]^i) = \mu_i^{-1}([[f \circ \eta_{-i} \sqsupseteq g \circ \eta_{-i}]])$$

Note that $\eta_i^{-1}([[f \sqsupseteq g]]^i)$ is equal to the following by definition.

$$
\begin{aligned}
&= \{\pi_i \in \Pi_i : \eta_i(\pi_i) \in [[f \sqsupseteq g]]^i\} \\
&= \{\Pi_i(t_i) : \eta_i \circ \Pi_i(t_i) \in [[f \sqsupseteq g]]^i \wedge t_i \in T_i\} \quad (\text{replacing } \pi_i \text{ with } \Pi_i(t_i) \text{ where } t_i \in T_i) \\
&= \{\Pi_i(t_i) : \Pi_i^* \circ h_i(t_i) \in [[f \sqsupseteq g]]^i \wedge t_i \in T_i\} \quad (\text{using } \eta_i \circ \Pi_i = \Pi_i^* \circ h_i) \\
&= \{\Pi_i(t_i) : \Pi_i^* \circ h_i(t_i) \in \{\pi_i^* \in \Pi_i^* : \mu_i^*(\pi_i^*) \in [[f \sqsupseteq g]]\} \wedge t_i \in T_i\} \quad (\text{expanding } [[f \sqsupseteq g]]^i) \\
&= \{\Pi_i(t_i) : \mu_i^*(\Pi_i^* \circ h_i(t_i)) \in [[f \sqsupseteq g]] \wedge t_i \in T_i\} \quad (\text{replacing } \pi_i^* \text{ with } \Pi_i^* \circ h_i(t_i))
\end{aligned}
$$

This is in turn equal to the following.

$$
\begin{aligned}
&= \{\Pi_i(t_i) : \mu_i^* \circ \Pi_i^*(h_i(t_i)) \in [[f \sqsupseteq g]] \wedge t_i \in T_i\} \\
&= \{\Pi_i(t_i) : p_{\Pi_{-i}^*} \circ \check{\Pi}_{-i}^*(m_i^*(h_i(t_i))) \in [[f \sqsupseteq g]] \wedge t_i \in T_i\} \quad (\text{by definition of } \mu_i^*) \\
&= \{\Pi_i(t_i) : \check{\Pi}_{-i}^*(m_i^*(h_i(t_i))) \in [f \sqsupseteq g] \wedge t_i \in T_i\} \quad (\text{by definition of } p_{\Pi_{-i}^*}) \\
&= \{\Pi_i(t_i) : m_i^*(h_i(t_i)) \in [f \circ \Pi_{-i}^* \sqsupseteq g \circ \Pi_{-i}^*] \wedge t_i \in T_i\} \quad (\text{by definition of } \check{\Pi}_{-i}^*) \\
&= \{\Pi_i(t_i) : \check{h}_{-i} \circ m_i(t_i) \in [f \circ \Pi_{-i}^* \sqsupseteq g \circ \Pi_{-i}^*] \wedge t_i \in T_i\} \quad (\text{since } h\colon T \to T^* \text{ is a type morphism}) \\
&= \{\Pi_i(t_i) : m_i(t_i) \in [f \circ \Pi_{-i}^* \circ h_{-i} \sqsupseteq g \circ \Pi_{-i}^* \circ h_{-i}] \wedge t_i \in T_i\} \quad (\text{by definition of } \check{h}_{-i}) \\
&= \{\Pi_i(t_i) : m_i(t_i) \in [f \circ \eta_{-i} \circ \Pi_{-i} \sqsupseteq g \circ \eta_{-i} \circ \Pi_{-i}] \wedge t_i \in T_i\} \quad (\text{using } \eta_i \circ \Pi_i = \Pi_i^* \circ h_i) \\
&= \{\Pi_i(t_i) : \check{\Pi}_{-i} \circ m_i(t_i) \in [f \circ \eta_{-i} \sqsupseteq g \circ \eta_{-i}] \wedge t_i \in T_i\} \quad (\text{by definition of } \check{\Pi}_{-i}) \\
&= \{\Pi_i(t_i) : p_{\Pi_{-i}} \circ \check{\Pi}_{-i} \circ m_i(t_i) \in [[f \circ \eta_{-i} \sqsupseteq g \circ \eta_{-i}]] \wedge t_i \in T_i\} \quad (\text{by definition of } p_{\Pi_{-i}}) \\
&= \{\Pi_i(t_i) : \mu_i(\Pi_i(t_i)) \in [[f \circ \eta_{-i} \sqsupseteq g \circ \eta_{-i}]] \wedge t_i \in T_i\} \quad (\text{because } \langle \Pi, \mu \rangle \text{ is based on } \langle T, m \rangle) \\
&= \{\pi_i \in \Pi_i : \mu_i(\pi_i) \in [[f \circ \eta_{-i} \sqsupseteq g \circ \eta_{-i}]]\} \quad (\text{replacing } \Pi_i(t_i), \text{ ranging over } t_i \in T_i, \text{ with } \pi_i \in \Pi_i) \\
&= \mu_i^{-1}([[f \circ \eta_{-i} \sqsupseteq g \circ \eta_{-i}]])
\end{aligned}
$$

It remains to be shown that $\eta_i \colon \Pi_i \to \Pi_i^*$ is measurable for all $i \in I_0$. The map $\eta_0 \colon \Pi_0 \to \Pi_0^*$ is measurable because it is the identity on S. For $i \in I$, we proceed by induction. For $k = 0$, $\eta_i \colon \Pi_i \to \Pi_i^*$ is $(\Sigma_{\Pi_i}, \Sigma_{\Pi_i^*,0})$-measurable by definition.

Suppose $\eta_i \colon \Pi_i \to \Pi_i^*$ is $(\Sigma_{\Pi_i}, \Sigma_{\Pi_i^*,k})$-measurable for $k \geqslant 0$. We want to show that $\eta_i \colon \Pi_i \to \Pi_i^*$ is $(\Sigma_{\Pi_i}, \Sigma_{\Pi_i^*,k+1})$-measurable. The σ-algebra $\Sigma_{\Pi_i^*,k+1}$ is generated by sets of the form $[[f \sqsupseteq g]]^i$ for $\Sigma_{\Pi_{-i}^*,k}$-measurable $f, g \in \mathcal{F}(\Pi_{-i}^*)$. We therefore need to show that $\eta_i^{-1}([[f \sqsupseteq g]]^i) \subseteq \Pi_i$ is measurable for all $\Sigma_{\Pi_{-i}^*,k}$-measurable $f, g \in \mathcal{F}(\Pi_{-i}^*)$. Let $f, g \in \mathcal{F}(\Pi_{-i}^*)$ be $\Sigma_{\Pi_{-i}^*,k}$-measurable. Then $f \circ \eta_{-i}$ and $g \circ \eta_{-i}$ are measurable by the inductive hypothesis. It then follows that $\eta_i^{-1}([[f \sqsupseteq g]]^i)$ is measurable because $\eta_i^{-1}([[f \sqsupseteq g]]^i) = \mu_i^{-1}([[f \circ \eta_{-i} \sqsupseteq g \circ \eta_{-i}]])$ and μ_i is measurable. $\square$

Proof of Lemma 3. It is trivial that $\psi_i^0 = \Pi_i^{*,0} \circ \psi_i$. Assume the inductive hypothesis that $\psi_i^k = \Pi_i^{*,k} \circ \psi_i$. Recall that $\psi_i^{k+1} = \widehat{\psi}_{-i}^k \circ \mu_i$.

$$\begin{aligned}\psi_i^{k+1} = \widehat{\psi}_{-i}^k \circ \mu_i = (\widehat{\Pi}_{-i}^{*,k} \circ \widehat{\psi}_{-i}) \circ \mu_i &= \widehat{\Pi}_{-i}^{*,k} \circ (\widehat{\psi}_{-i} \circ \mu_i)\\ &= \widehat{\Pi}_{-i}^{*,k} \circ (\mu_i^* \circ \psi_i)\\ &= (\widehat{\Pi}_{-i}^{*,k} \circ \mu_i^*) \circ \psi_i\\ &= \Pi_i^{*,k+1} \circ \psi_i\end{aligned}$$

□

Proof of Proposition 6. The inductive definition of $(\psi^k)_{k\geqslant 0}$ depends only on ψ^0. The inductive definition of $(\eta^k)_{k\geqslant 0}$ depends only on η^0. However, if ψ and η are type morphisms, then $\psi^0 = \eta^0$. It follows that $(\psi^k)_{k\geqslant 0} = (\eta^k)_{k\geqslant 0}$, which implies the following.

$$\begin{aligned}\forall i \in I \quad \psi_i(\pi_i) &= \bigcap_{k\geqslant 0} (\Pi_i^{*,k+1} \circ \psi_i)(\pi_i) = \bigcap_{k\geqslant 0} \psi_i^{k+1}(\pi_i) = \bigcap_{k\geqslant 0} (\widehat{\psi}_{-i}^k \circ \mu_i)(\pi_i)\\ &= \bigcap_{k\geqslant 0} (\widehat{\eta}_{-i}^k \circ \mu_i)(\pi_i) = \bigcap_{k\geqslant 0} \eta_i^{k+1}(\pi_i) = \bigcap_{k\geqslant 0} (\Pi_i^{*,k+1} \circ \eta_i)(\pi_i) = \eta_i(\pi_i)\end{aligned}$$

Furthermore, $\psi_0 = \eta_0$ because ψ and η are preference-type morphisms. Therefore, $\psi = \eta$. □

Proof of Theorem 2. $\langle \Pi^*, \mu^* \rangle$ is a preference type space by Proposition 4. For any preference type space $\langle \Pi, \mu \rangle$, η is a preference type morphism to $\langle \Pi^*, \mu^* \rangle$ by Proposition 5 and it is unique by Proposition 6. The result follows. □

B APPLICABLE CLASSES OF PREFERENCE RELATIONS

B.1 LEXICOGRAPHIC EXPECTED UTILITY PREFERENCES

When $L \subseteq \mathbb{N}$ and $\trianglerighteq$ is the lexicographic order on $\mathbb{R}^L$, then $\succsim$ admits a *lexicographic expected utility* representation if each U_ℓ is a continuous linear functional. The following results in Section B.1.1 show that this representation class satisfies the monotone determination property (Definition 7) and hence the results of Section 4 apply to provide the existence of the universal preference type space.

B.1.1 Lexicographic probability systems: preference monotone determination

Suppose $L = \mathbb{N}$ and $\mathcal{R}(Y)$ is the set of all monotone continuous $(L, \trianglerighteq)$-representations of lexicographic expected utility (LEU) preferences on $\mathcal{F}(Y)$ for any measurable space Y. We show that $\mathcal{R}$ satisfies preference monotone determination (Definition 7) in Corollary 1 below, which follows from results that we prove next.

While we assume that L is countably infinite, the proofs below establish monotone determination for LEU preferences represented by both finite and countably infinite LPSs. LEU preferences that can be represented by finite LPSs are those that have "minimal" representations with a "repeating tail" in a way that will become clear in Definition 16.

Let $\mathcal{U}(Y)$ denote the set of all bounded linear functionals on $\mathcal{F}(Y)$ so that $\mathcal{R}(Y) = \prod_{\ell \in L} \mathcal{U}(Y)$ and let $\geqslant^{\mathsf{lex}}$ denote the lexicographic order.

Definition 15. *Let $\ell' \in L$ and let $(U_\ell)_{\ell \leqslant \ell'} \in \prod_{\ell \leqslant \ell'} \mathcal{U}(Y)$. $(U_\ell)_{\ell \leqslant \ell'}$ is* ***non-minimal*** *if there is some $\ell'' < \ell'$ such that, for all $f, g \in \mathcal{F}(Y)$:*

$$(U_\ell(f))_{\ell \leqslant \ell''} \geqslant^{\mathsf{lex}} (U_\ell(g))_{\ell \leqslant \ell''} \iff (U_\ell(f))_{\ell \leqslant \ell''+1} \geqslant^{\mathsf{lex}} (U_\ell(f))_{\ell \leqslant \ell''+1}$$

$(U_\ell)_{\ell \leqslant \ell'}$ is ***minimal*** *if it is not non-minimal.*

Definition 16. *Let $U = (U_\ell)_{\ell \in L} \in \mathcal{R}(Y)$. U is* ***minimal*** *if, for all $\ell' \in L$, it must be the case that either $(U_\ell)_{\ell \leqslant \ell'+1}$ is minimal or $U_{\ell'} = U_{\ell'+1}$. That is, a minimal U will either have no non-minimal initial segment or have a minimal initial segment and a repeating tail.*

$$(\overbrace{U_1, U_2, \ldots, U_{\ell'}}^{\text{min. init. segment}}, \underbrace{U_{\ell'}, U_{\ell'}, \ldots}_{\text{repeating tail}})$$

The arguments showing the following remarks can be seen in Blume et al. (1991) (Theorem 3.1 and the subsequent discussion on p.66), which axiomatizes lexicographic expected utility preferences.

Remark 1. *Let $\ell' \in L$ and let $(U_\ell)_{\ell \leqslant \ell'}, (V_\ell)_{\ell \leqslant \ell'} \in \prod_{\ell \leqslant \ell'} \mathcal{U}(Y)$ be minimal. Furthermore, let the following hold for all $f, g \in \mathcal{F}(Y)$:*

$$(U_\ell(f))_{\ell \leqslant \ell'} \geqslant^{\mathsf{lex}} (U_\ell(g))_{\ell \leqslant \ell'} \iff (V_\ell(f))_{\ell \leqslant \ell'} \geqslant^{\mathsf{lex}} (V_\ell(f))_{\ell \leqslant \ell'}$$

This is equivalent to the following: There is some $((\alpha_j^\ell)_{j \leqslant \ell})_{\ell \leqslant \ell'} \in \prod_{\ell \leqslant \ell'} \prod_{j \leqslant \ell} \mathbb{R}$ such *that* $(\alpha_\ell^\ell)_{\ell\leqslant\ell'} > (0)_{\ell\leqslant\ell'}$ *and*

$$(V_\ell)_{\ell\leqslant\ell'} = \left(\sum_{j\leqslant\ell}\alpha_j^\ell U_j\right)_{\ell\leqslant\ell'}.$$

Remark 2. *Let* $\ell' \in L$ *and let* $(U_\ell)_{\ell\leqslant\ell'+1} \in \prod_{\ell\leqslant\ell'}\mathcal{U}(Y)$ *be non-minimal. If* $(U_\ell)_{\ell\leqslant\ell'}$ *is minimal then there is some* $(\alpha_j^{\ell'})_{j\leqslant\ell'} \in \prod_{j\leqslant\ell'}\mathbb{R}$ *such that*

$$U_{\ell'+1} = \sum_{j\leqslant\ell'}\alpha_j^{\ell'}U_j$$

Lemma 4. *Let* $(\Sigma_Y^n)_{n\geqslant 0}$ *be a filtration of* Σ_Y*. Let* $U \in \mathcal{R}(Y)$ *and* $\ell' \in L$ *such that* $(U_\ell)_{\ell\leqslant\ell'}$ *is minimal and denote by* U^n *the restriction of* U *to acts in* $\mathcal{F}(Y)$ *that are* Σ_Y^n*-measurable. Then there is some* $N_{\ell'}$ *such that, for all* $n \geqslant N_{\ell'}$*,* $(U_\ell^n)_{\ell\leqslant\ell'}$ *is minimal.*

Proof. The proof is by induction as follows.

Base case. The case when $\ell' = 1$ is trivial.

Inductive hypothesis. There is some $N_{\ell'}$ such that, for all $n \geqslant N_{\ell'}$, $(U_\ell^n)_{\ell\leqslant\ell'}$ is minimal.

Inductive Step. Suppose that $(U_\ell)_{\ell\leqslant\ell'+1}$ is minimal. We need to show that there is some $N_{\ell'+1}$ such that, for all $n \geqslant N_{\ell'+1}$, $(U_\ell^n)_{\ell\leqslant\ell'+1}$ is minimal. Suppose by way of contradiction that, for all $N_{\ell'+1}$, there is some $n \geqslant N_{\ell'+1}$ such that $(U_\ell^n)_{\ell\leqslant\ell'+1}$ is non-minimal. Without loss of generality, let $N_{\ell'+1} \geqslant N_{\ell'}$. Since $(U_\ell^n)_{\ell\leqslant\ell'}$ must be minimal by the inductive hypothesis, there is exactly one $(\alpha_j^{\ell'})_{j\leqslant\ell'}$ such that

$$U_{\ell'+1}^n = \sum_{j\leqslant\ell'}\alpha_j^{\ell'}U_j^n \tag{22}$$

due to the required linear independence. Note that if $U_{\ell'+1}^{n+1} = \sum_{j\leqslant\ell'}\alpha_j^{\ell'}U_j^{n+1}$, then it must be the case that $U_{\ell'+1}^n = \sum_{j\leqslant\ell'}\alpha_j^{\ell'}U_j^n$. This is due to the fact that U^n and U^{n+1} coincide on $\mathcal{F}(\Sigma_Y^n)$ by definition since $\Sigma_Y^n \subseteq \Sigma_Y^n$. It follows that (22) must hold for all $n \geqslant 1$ for some fixed $(\alpha_j^{\ell'})_{j\leqslant\ell'}$. From this and Remark 2 it follows that

$$U_{\ell'+1} = \sum_{j\leqslant\ell'}\alpha_j^{\ell'}U_j$$

which contradicts the fact that $(U_\ell)_{\ell\leqslant\ell'+1}$ is minimal. □

Proposition 7. *Let $(\Sigma_Y^n)_{n\geqslant 0}$ be a filtration of Σ_Y. Let $U, V \in \mathcal{R}(Y)$ so that*

$$\forall n \quad \forall f, g \in \mathcal{F}(Y) \text{ that are } \Sigma_Y^n\text{-measurable}, \quad U(f) \trianglerighteq U(g) \iff V(f) \trianglerighteq V(g). \tag{23}$$

Then $p_Y(U) = p_Y(V)$, which is equivalent to

$$\forall f, g \in \mathcal{F}(Y) \quad U(f) \trianglerighteq U(g) \iff V(f) \trianglerighteq V(g).$$

Proof. Let $\mathcal{F}(Y, \Sigma_Y^n) \subseteq \mathcal{F}(Y)$ denote the Σ_Y^n-measurable acts and denote by U^n, V^n the respective restrictions of U, V to $\mathcal{F}(Y, \Sigma_Y^n)$. Without loss of generality, assume that U and V are minimal. The proof is by induction on ℓ'. What we want to prove is that the following holds for all $f, g \in \mathcal{F}(Y)$.

$$\forall \ell' \in L \quad (U_\ell(f))_{\ell\leqslant\ell'} \geqslant^{\mathsf{lex}} (U_\ell(g))_{\ell\leqslant\ell'} \iff (V_\ell(f))_{\ell\leqslant\ell'} \geqslant^{\mathsf{lex}} (V_\ell(g))_{\ell\leqslant\ell'}$$

Base Case. We can rewrite (23) more succinctly as

$$p_{(Y,\Sigma_Y^n)}(U|_{\mathcal{F}(Y,\Sigma_Y^n)}) = p_{(Y,\Sigma_Y^n)}(V|_{\mathcal{F}(Y,\Sigma_Y^n)}),$$

which makes it obvious that $U_1^n = V_1^n$ for all n. Furthermore, if $U_1^n = V_1^n$ for all n, then $U_1 = V_1$. Therefore, for all $f, g \in \mathcal{F}(Y)$:

$$(U_\ell(f))_{\ell\leqslant 1} \geqslant^{\mathsf{lex}} (U_\ell(g))_{\ell\leqslant 1} \iff (V_\ell(f))_{\ell\leqslant 1} \geqslant^{\mathsf{lex}} (V_\ell(g))_{\ell\leqslant 1}$$

Inductive Hypothesis (ℓ'). For all $f, g \in \mathcal{F}(Y)$:

$$(U_\ell(f))_{\ell\leqslant\ell'} \geqslant^{\mathsf{lex}} (U_\ell(g))_{\ell\leqslant\ell'} \iff (V_\ell(f))_{\ell\leqslant\ell'} \geqslant^{\mathsf{lex}} (V_\ell(g))_{\ell\leqslant\ell'}$$

Inductive Step. We want to show that, for all $f, g \in \mathcal{F}(Y)$,

$$(U_\ell(f))_{\ell\leqslant\ell'+1} \geqslant^{\mathsf{lex}} (U_\ell(g))_{\ell\leqslant\ell'+1} \iff (V_\ell(f))_{\ell\leqslant\ell'+1} \geqslant^{\mathsf{lex}} (V_\ell(g))_{\ell\leqslant\ell'+1}$$

If $(U_\ell)_{\ell\leqslant\ell'}$ is the longest minimal initial segment of U, then no further work is needed. Therefore, now consider the case when $(U_\ell)_{\ell\leqslant\ell'+1}$ is minimal.

By Lemma 4 and the minimality of U and V, there is some $N_{\ell'+1}$ such that, for all $n \geqslant N_{\ell'+1}$, $(U_\ell^n)_{\ell\leqslant\ell'+1}$ and $(V_\ell^n)_{\ell\leqslant\ell'+1}$ are both minimal.[10]

[10] Apply Lemma 4 to U and V separately to find the corresponding $N_{\ell'+1}$ for each and take the maximum of the two numbers.

By (23) and Remark 2, there is some $(\alpha_j^{\ell'+1})_{j\leqslant \ell'+1}$ such that $\alpha_{\ell'+1}^{\ell'+1} > 0$ and

$$V_{\ell'+1}^n = \sum_{j\leqslant \ell'+1} \alpha_j^{\ell'+1} U_j^n.$$

By the minimality of $(U_\ell^n)_{\ell\leqslant \ell'+1}$, there is exactly one such $(\alpha_j^{\ell'+1})_{j\leqslant \ell'+1}$. Therefore, the analogous property carries over to the limit extensions, i.e.,

$$V_{\ell'+1} = \sum_{j\leqslant \ell'+1} \alpha_j^{\ell'+1} U_j,$$

which, when combined with the Inductive Hypothesis, shows the Inductive Step by Remark 2. □

Corollary 1. *$\mathcal{R}$ satisfies monotone determination.*

Proof. This is immediate because Proposition 7 holds for all nonempty measurable Y. □

B.2 CONTINUOUS PREFERENCES UNDER RISK AND AMBIGUITY

If $|L| = 1$ and $\trianglerighteq$ is the usual order $\geqslant$ on $\mathbb{R}$, $\succsim$ admits a continuous representation for preferences under risk and ambiguity. These preferences include uncertainty-averse preferences (Cerria-Vioglio et al., 2011) and vector expected utility preferences (Siniscalchi, 2009) and cumulative prospect theory preferences (Wakker and Tversky, 1993). For these classes of preferences, as we describe below, there exist 'standard' representations of beliefs, as separate from tastes, such as the representation of monotone continuous Savage (1954) preferences by a countably additive probability measure. The standard representation is unique and the result of Section 3 provides the existence of a universal type space in the category of type spaces where each type is associated in a measurable way with the standard representation of the preferences.

However, it is also possible to consider representations where beliefs and tastes are not separated and a standard representation may not exist. In particular, for many preferences, the standard representations capture the 'belief' part of preferences and not the 'tastes' part. The latter would be, for example, the so-called Bernoulli or von Neumann-Morgenstern utility function u which captures the risk attitude in the case of Savage (1954) preferences. The Bernoulli utility function is typically only identified up to positive affine transformations. That is, u and $u' = \alpha u + \beta, \alpha >$

$0, \beta \in \mathbb{R}$ both represent the same risk attitude. Representations which do not separate beliefs and tastes have been extensively studied under the label of 'state-dependent' representations, See, for example, Karni et al. (1983) and references therein. In this case, there typically does not exist a 'standard' or unique representation that also incorporates differences in tastes. However, the approach of Section 4, by working with the equivalence classes of representations, provides the existence of the universal preference type space where the representations do not distinguish between tastes and beliefs.

Below, we first provide a list of standard representations of the belief part of preferences, for many preferences under risk and ambiguity to which the results of this paper are applicable.[11] Uniqueness of the standard representation implies that $\mathcal{R}$ is image regular (Definition 2) and results of Section 3 provide the existence of the universal type space where preferences are uniquely represented. We then discuss the case of state-dependent representations for which there may not be a standard or unique representation in the relevant class of representations and where the results of Section 4 apply to provide the existence of the universal preference type space.

B.2.1 Standard representations of beliefs[12]

For uncertainty averse preferences, let $\mathcal{R}(Y)$ be encoded by the set of all lower semi-continuous and linearly continuous functions $G\colon \mathbb{R} \times \Delta^{\sigma}(Y) \to (-\infty, \infty]$, where $\Delta^{\sigma}(Y)$ is the set of all countably additive probabilities over (Y, Σ_Y). The results of Cerria-Vioglio et al. (2011) (Theorem 7, Lemma 48, Lemma 57) imply that each element of $\mathcal{R}(Y)$ uniquely represents a continuous functional U over $\mathcal{F}(Y)$ that represents preference $\gtrsim$ over $\mathcal{F}(Y)$, where

$$U(f) = \min_{p \in \Delta} G\left(\int f \,\mathrm{d}p, p\right)$$

and

$$G(t, p) = \sup\left\{U(f) \;\middle|\; f \in \mathcal{F}(Y), \int f \,\mathrm{d}p \leqslant t\right\}$$

[11] The list is certainly not comprehensive and is only meant to indicate the scope of the present paper's approach.

[12] The results of Heifetz and Samet (1998a) already cover the case of preferences where beliefs are represented by continuous non-additive measures or capacities, (see Schmeidler (1989) for an axiomatization of such preferences).

Examples of uncertainty averse preferences, such as multiple prior preferences, variational preferences, smooth ambiguity preferences and the specific functional form of $G(\cdot)$ for each, are provided in Cerria-Vioglio et al. (2011). Restricting $\mathcal{R}$ to the set of all G corresponding to any of the above subclasses also satisfies image regularity.[13]

For vector expected utility preferences, axiomatized in Siniscalchi (2009), let $\mathcal{R}(Y)$ be encoded by a collection of tuples $(p, n, (\zeta_i)_{0\leqslant i\leqslant n}, A)$ where $p \in \Delta^\sigma(Y)$, $n \in \mathbb{N} \cup \{\infty\}$, $(\zeta_i)_{0\leqslant i\leqslant n} \in \mathcal{F}^n(Y)$, $A\colon \mathbb{R}^n \to \mathbb{R}$ satisfy the following conditions.[14]

$$\text{For every } 0 \leqslant i \leqslant n, \quad \mathbb{E}_p[\zeta_i] = 0$$
$$A((0)_{0\leqslant i\leqslant n}) = 0 \quad \text{and}$$
$$A((r_i)_{0\leqslant i\leqslant n}) = \begin{cases} A((-r_i)_{0\leqslant i\leqslant n}) & \text{if } (r_i)_{0\leqslant i\leqslant n} = (\mathbb{E}_p[\zeta_i \cdot f])_{0\leqslant i\leqslant n} \text{ for some } f \in \mathcal{F}(Y) \\ 0 & \text{otherwise.} \end{cases}$$

The result of Siniscalchi (2009) (Theorem 1) implies each element of $\mathcal{R}(Y)$ is a continuous functional U over $\mathcal{F}(Y)$, which represents vector expected utility preference $\succsim$ over $\mathcal{F}(Y)$ where

$$U(f) = \mathbb{E}_p[f] + A\left((\mathbb{E}_p[\zeta_i \cdot f])_{0\leqslant i\leqslant n}\right).$$

For cumulative prospect theory preferences, axiomatized in Wakker and Tversky (1993), let $\mathcal{R}(Y)$ be encoded by a collection of continuous capacity pairs $(\nu_1, \nu_2) \in bv_1^{m\sigma}(Y) \times bv_1^{m\sigma}(Y)$, where $bv_1^{m\sigma}(Y)$ is the collection of set functions on $\nu\colon \Sigma_Y \to [0, \infty]$ satisfying (i) $\nu(\varnothing) = 0 = 1 - \nu(Y)$, (ii) (monotonicity) $\nu(A) \leqslant \nu(B)$ if $A \subseteq B$, (iii) (continuity) for each A, $\lim_{n\to\infty} \nu(A_n) = \nu(A)$ whenever $(A_n) \uparrow A$ or $(A_n) \downarrow A$. The result of Wakker and Tversky (1993) (Theorem 6.3) implies that each element of $\mathcal{R}(Y)$ is a continuous functional U over $\mathcal{F}(Y)$, which represents preference $\succsim \in \mathcal{F}(Y)$, where $f^+ = \max\{0, f\}$, $f^- = -\min\{0, f\}$, and $\bar{\nu}_2(A) = 1 - \nu_2(A^c)$ for all $A \in \Sigma_Y$ and

$$U(f) = \int f^+ \, \mathrm{d}\nu_1 + \int f^- \, \mathrm{d}\bar{\nu}_2$$

[13] For example, variational preferences are represented by G satisfying $G(t, p) = t + c(p)$, where $c\colon \Delta^\sigma(Y) \to [0, \infty]$ is a lower semicontinuous convex function, with $\min_{p\in\Delta^\sigma(Y)} c(p) = 0$. Suppose $\mathcal{R}(Y)$ is encoded by the set of all such G. Then $\mathcal{R}$ is image-regular and players preferences are of the variational preferences class.

[14] In what follows, $(\mathbb{E}_p[\zeta_i \cdot f])_{0\leqslant i\leqslant n} = 0$ if $n = 0$.

B.2.2 Non-separation of beliefs and tastes

Suppose $\mathcal{R}(Y)$ is encoded by the set of *all* continuous, additively separable, monotonic functionals over $\mathcal{F}(Y)$, where for each $U \in \mathcal{R}(Y)$, there exists $\mu \in \Delta^\sigma(Y)$ and strictly increasing continuous $V_y : \mathbb{R} \to \mathbb{R}$, $y \in Y$, such that, for all $f \in \mathcal{F}(Y)$

$$U(f) = \int V_y(f(y))\, \mathrm{d}\mu.$$

If $(\mu^*, (V_y^*)_{y\in Y})$ represents the same preference as $(\mu, (V_y)_{y\in Y})$ then (i) $\mu(A) = 0 \iff \mu^*(A) = 0$ and (ii) $V_y^* = \alpha(y) + \sigma\beta(y)V_y$ where $\sigma > 0$ is a constant, $\alpha : Y \to \mathbb{R}$ is measurable, and β is the Radon-Nikodym density of μ with respect to μ^*. Then $U \in \mathcal{R}(Y)$ represents preferences $\gtrsim$ over $\mathcal{F}(Y)$ that are complete, transitive, continuous, and satisfy the sure-thing principle as required in Wakker and Zank (1999) (Theorem 12), but do not separately represent beliefs and tastes in a unique way.[15]

However, $\mathcal{R}(Y)$ is image-regular and is preference monotone determined (Lemma 5 below) and $U \in \mathcal{R}(Y)$ satisfy (2)–(4). Section 4 provides the existence of the universal preference type space for such preferences.

Lemma 5. *$\mathcal{R}$ satisfies monotone determination.*

Proof. Let $U, U^* \in \mathcal{R}(Y)$ and let $(\Sigma_Y^n)_{n\geqslant 0}$ be a filtration of Σ_Y. Denote by $\mathcal{F}(Y, \Sigma_Y^n) \subseteq \mathcal{F}(Y)$ the set of Σ_Y^n-measurable acts and by $U^n = (\mu^n, (V_y^n)_{y\in Y}), U^{*n} = (\mu^{*n}, (V_y^{*n})_{y\in Y})$ the respective restrictions of U, V to $\mathcal{F}(Y, \Sigma_Y^n)$. Suppose

$$\forall n \quad \forall f, g \in \mathcal{F}(Y, \Sigma_Y^n) \quad U(f) \geqslant U(g) \iff U^*(f) \geqslant U^*(g). \tag{24}$$

That is, $\forall n$, U^n, U^{*n} represent the same preference over $\mathcal{F}(Y, \Sigma_Y^n)$. Then, $V_y^{*n} = \alpha^n(y) + \sigma^n\beta^n(y)V_y^n$ where $\sigma^n > 0$ is a constant, $\alpha^n : Y \to \mathbb{R}$ is measurable, and β^n is the Radon-Nikodym density of μ^n with respect to μ^{*n}. So, for any $f, g \in \mathcal{F}(Y, \Sigma_Y^n)$

$$U^{*n}(f) = \int_Y \alpha^n\, \mathrm{d}\mu^* + \sigma^n U^n(f).$$

Let $f \in \mathcal{F}(Y)$ (resp. $g \in \mathcal{F}(Y)$) and $(f^n)_{n\geqslant 1}$ be a sequence converging pointwise to f, $f^n \in \mathcal{F}(Y, \Sigma_Y^n)$ (resp. $(g^n)_{n\geqslant 1}$ be a sequence converging pointwise to g, $g^n \in \mathcal{F}(Y, \Sigma_Y^n)$).

[15] The results of this paper would not apply to the additively separable representations in the more recent work of Hill (2010) that need not satisfy monotonicity.

Then,

$$\begin{aligned}
&U^*(f) - U^*(g) \geqslant 0 \\
&\iff \lim_{n\to\infty} (U^*(f^n) - U^*(g^n)) \geqslant 0 \\
&\iff \lim_{n\to\infty} (U^{*n}(f^n) - U^{*n}(g^n)) \geqslant 0 \\
&\iff \lim_{n\to\infty} \sigma^n (U^n(f^n) - U^n(g^n)) \geqslant 0 \\
&\iff \lim_{n\to\infty} \sigma^n (U(f^n) - U(g^n)) \geqslant 0 \\
&\iff \lim_{n\to\infty} (U(f^n) - U(g^n)) \geqslant 0,
\end{aligned}$$

since

$$\forall n \quad \sigma^n = \frac{U^{*n}(1) - U^{*n}(0)}{U^n(1) - U^n(0)} = \frac{U^*(1) - U^*(0)}{U(1) - U(0)} > 0.$$

This yields the desired result since $\lim_{n\to\infty}(U(f^n) - U(g^n)) = U(f) - U(g)$.[16] □

B.3 PREFERENCES OVER MENUS

$U \in \mathcal{R}(Y)$ can be used to characterize some instances of preferences over countable menus of acts in $\mathcal{F}(Y)$ that feature behavior such as self-control and temptation. For instance, self-control preferences over menus (Gul and Pesendorfer (2001), Epstein (2006)) are defined in terms of a commitment utility C and temptation utility T for each act.

Suppose C and T are continuous linear functionals on $\mathcal{F}(Y)$ and let $\mathcal{R}(Y)$ be encoded by $\Delta^\sigma(Y) \times \Delta^\sigma(Y)$, the set of countably additive measures over (Y, Σ_Y). Then every C is uniquely determined by some $p \in \Delta^\sigma(Y)$ and T by some $q \in \Delta^\sigma(Y)$. Self-control preferences over countable menus along the lines of Epstein (2006) (Theorem 1) can be characterized via a commitment utility $C\colon \mathcal{F}(Y) \to \mathbb{R}$ and a temptation utility $T\colon \mathcal{F}(Y) \to \mathbb{R}$, which combine in the representation $V\colon \mathcal{F}^{\mathbb{N}}(Y) \to \mathbb{R}$ defined by

$$V(f) = \sup_{n=1,2,\ldots} \{C(f_n) + T(f_n)\} - \sup_{n=1,2\ldots} T(f_n) \tag{25}$$

[16] The choice of acts 1 and 0 to evaluate σ^n is not special. Any constant acts a, b such that $a \neq b$ will suffice. This reflects a restriction on how constant acts are compared under the preference representation of Wakker and Zank (1999) (Theorem 12).

Here $L = \mathbb{N} \times \mathbb{N}$, where for every $(f_1, \ldots, f_n, \ldots) \in \mathcal{F}^{\mathbb{N}}(Y)$,

$$U((f_1, \ldots, f_n, \ldots)) = ((C(f_1), \ldots, C(f_n), \ldots), (T(f_1), \ldots, T(f_n), \ldots))$$

with the order $\trianglerighteq$ on $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ defined by

$$\begin{aligned} &((c_n)_{n=1}^{\infty}, (t_n)_{n=1}^{\infty}) \trianglerighteq ((c'_n)_{n=1}^{\infty}, (t'_n)_{n=1}^{\infty}) \\ &\qquad\qquad \iff \sup_{n=1,2,\ldots} \{c_n + t_n\} - \sup_{n=1,2,\ldots} t_n \geqslant \sup_{n=1,2,\ldots} \{c'_n + t'_n\} - \sup_{n=1,2,\ldots} t'_n \end{aligned}$$

This order satisfies condition (1) since coordinate-projections in $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$, sums of measurable functions and suprema of measurable functions are themselves measurable functions. In this case, Section 3 provides the existence of the universal type space in the category of type spaces where each type is measurably associated with a standard representation. On the other hand, for the Gul and Pesendorfer (2001) style representation, (C, T) need not be identified with probability measures, and may be represented by a additively separable representation, along the lines of Wakker and Zank (1999) (Theorem 12). In this case, the preference type space approach of Section 4 provides the existence of the universal preference type space.