Discriminatory Persuasion: How to Convince a Group*

Jimmy Chan† Fei Li‡ and Yun Wang§

December 16, 2015

Abstract

We study a Bayesian persuasion game between a sender and a set of voters. A collective decision is made according to a majority rule. The sender can influence each voter's choice by persuading her individually and privately. Voters are heterogenous in "thresholds of doubt", so some of them are easy to convince while others are hard. The optimal persuasion signal provides voters with discriminatory and correlated information. Furthermore, it has several pivotal realizations, each of which results in a distinctive winning coalition. An individual voter belongs to multiple winning coalitions and she cannot distinguish them given her private signal; and thus information aggregation among voters is inefficient.

Keywords: Bayesian Persuasion, Voting, Selective Communication, Private Persuasion.

JEL Classification Codes: D72, D83

*We thank Ricardo Alonso, Gary Biglaiser, Simon Board, Rahul Deb, John Hey, Maxim Ivanov, Navin Kartik, Anqi Li, Hao Li, Ming Li, Stephen Morris, Peter Norman, George Mailath, Tymofiy Mylovanov, Alessandro Pavan, Xianwen Shi, Joel Sobel, Curtis Taylor, Huseyin Yildirim, Gabor Virag, Zaifu Yang, Jidong Zhou and seminar participants at University of Louisville, University of North Carolina at Chapel Hill, University of Toronto, University of York, 2015 Midwest Theory conference for comments. The usual disclaimer applies.

†School of Economics, Fudan University, School of Economics; jimmy.hing.chan@gmail.com

‡Department of Economics, University of North Carolina Chapel Hill; lifei@email.unc.edu

§Wang Yanan Institute for Studies in Economics, Xiamen University; yunwang@xmu.edu.cn

1 Introduction

An academic department is hiring a junior professor. The search committee ends up with a promising candidate. The hiring decision will be made by the department according to a majority rule. All faculty members have a common interest: they want to hire a good candidate and avoid hiring a bad one. However, making a wise decision requires information. Due to the specialization in the profession, most faculty members lack sufficient information to evaluate the type of the candidate. Suppose that the faculty members are Bayesian decision makers, that is, they are willing to recruit the candidate only if he is good with a sufficiently high probability. Prior to the vote, the search committee can persuade faculty members. Naturally, some people are easy to convince while others are hard to convince. Convincing the latter requires more evidence in favor of the candidate. Suppose the search committee is biased and hopes to hire the candidate regardless of his quality. What is the optimal persuasion strategy? Does the committee want to persuade voters privately or publicly? If privately, whom to persuade?

The above example is far from rare. To implement a new policy, a government communicates with multiple policymakers; an author of a research paper responses to multiple anonymous referees and journal editors in hope to convince them to accept the paper; a political campaign persuades voters through television or online advertising to win their support. In all these examples, the decision is made by a group, and a sender tries to convince a sufficiently large number of group members to support his preferred action. The aim of this paper is to understand the sender's optimal persuasion strategy in such a scenario.

Unlike the previous literature in which information asymmetry is emphasized, we consider a Bayesian persuasion model *à la* Kamenica and Gentzkow (2011) with a novelty that the final decision is made by a group of N voters. The group has to decide between two actions, a and b. Action b is the default choice while action a is chosen only if it receives at least K votes. Voters prefer action a in state A and action b in state B. However, no voter knows the state. All voters are Bayesian, so a voter prefers action a if and only if her belief about the state being A is above a threshold (or cutoff belief). Voters have different thresholds. Voters with high thresholds are hard to convince; voters with low thresholds are easy to convince. The sender wishes to choose a regardless of the state. He can influence voters' beliefs by controlling the information structure, which is a state-dependent *joint distribution* of voters' (realized) individual signals. After observing their own signals, voters update their beliefs accordingly. They then choose an action to vote. Since we assume the sender has perfect commitment power, we can apply the mechanism design approach to derive the sender's optimal persuasion signal. By doing so, we examine the upper bound of the persuasion probability that the sender can achieve.

As a benchmark, we first consider *public persuasion* where voters hold common posterior beliefs after observing realized signals. Because voters make decisions according to their cutoff beliefs, whenever a hard-to-convince voter is willing to vote for a, an easy-to-convince voter is willing to do so as well. Hence, the sender finds it optimal to target the median voter (the Kth most easy-to-convince voter). As long as the median voter is convinced, a winning coalition consisting of K voters forms. However, except for the median voter, members in the winning coalition are over-convinced. That is, their posterior beliefs are strictly above their cutoff ones, which suggests that the sender wastes some information on these voters. Ideally, the sender can save some of the information that would be wasted on these easy-to-convince voters and use it to convince additional hard-to-convince voters without affecting the posterior of the median voter. But two questions immediately arise. First, is this useful? After all, the sender only needs K votes to obtain the desired outcome. Having additional voters on board does not change the final outcome. Second, if it is beneficial, how? Apparently, transferring information across voters leads to heterogenous posterior beliefs among voters. Using public persuasion, the mission is impossible.

We then turn to study *private persuasion*, in which each voter observes only her own realized signal. We demonstrate that the sender finds it beneficial to send private and correlated signals to voters. More interestingly, the sender prefers to persuade not only the K most easy-to-convince voters but also some more hard-to-convince ones. Voters hold heterogenous posterior beliefs. Notice that the result is not obvious because private persuasion does not necessarily lead to heterogenous posterior beliefs among voters when voters can strategically aggregate dispersed information.

The main result of the paper is the characterization of the optimal persuasion signal. To efficiently use information, the sender chooses a signal to create multiple pivotal events for his preferred action. Some voters are pivotal in many winning coalitions. After observing her own realized signal, a voter remains uncertain about the realized winning coalition. As a result, voters cannot efficiently aggregate dispersed information conditional on being pivotal. This gives rise to heterogenous posterior beliefs among voters. Since the sender can control the correlation of the signals that voters receive, he is able to manipulate the way in which voters aggregate information and to convince more voters with less informative signals to achieve a larger persuasion probability.

Although we endow the sender with sufficient information controlling flexibility, we do not allow him to send a non-monotone signal. Specifically, we assume that the signal must satisfy the *monotone likelihood ratio property* (MLRP). This condition is equivalent to monotone Bayesian updating. It intuitively implies that, to convince additional voters to vote for a certain action, the realized signals of voters must lead to a larger aggregate posterior belief about the corresponding state where the aggregated posterior belief is obtained by combining the realizations of all voters. We make this assumption because one can easily implement such a mechanism by using a factor model in practice. Suppose that there are a set of independent simple experiments, that each simple experiment is a state-dependent binary distribution and that its outcome favors either state A or B. The sender controls the probabilities of type-I error and type-II error in each experiment. For each voter, the sender customizes the set of simple experiments to present. In such an environment, it is obvious that the higher the number of voters that are convinced to vote for action x, the (weakly) higher the aggregate posterior belief about the corresponding state is.[1]

Due to strategic information aggregation among voters, there are other constraints on beliefs updating in addition to the standard Bayesian plausibility constraint. Consequently, the conventional "concavification" method (Aumann and Maschler (1995) and Kamenica and Gentzkow (2011)) does not apply. We work directly on the signal space; and thus the number of control variables is remarkably large. As a voter's incentive to follow the recommendation is only driven by the posterior belief conditional on the fact that she is pivotal, we solve an auxiliary problem where the sender controls the probabilities of voters being pivotal and we characterize the optimal solution. Then we show

[1] In principle, the sender can present an identical package of simple experiments to multiple voters. In such a case, having more voters being on board for action x does not increase the aggregate posterior belief.

that the auxiliary problem and the original optimal persuasion problem are equivalent.

Related Literature. Our paper relates to a growing literature on Bayesian persuasion. Brocas and Carrillo (2007), Kamenica and Gentzkow (2011) and Rayo and Segal (2010) study Bayesian persuasion models between one sender and one receiver. In a recent paper, Kolotilin, Li, Mylovanov, and Zapechelnyuk (2015) assume that the receiver has private information about his tastes and the sender is able to provide a targeting signal based on the receiver's reported tastes. They refer to such a model as private persuasion, and they refer to the model where the signal does not depend on the receiver's report as public persuasion. They show that two models are equivalent in terms of the set of implementable payoffs. Unlike their model, multiple receivers exist and interact directly in our model.

As we do in the current paper, Alonso and Camara (2014) and Wang (2012) also consider Bayesian persuasion between one sender and multiple receivers. However, Alonso and Camara (2014) assume that voters have heterogenous preferences on actions given the true state and their paper focuses on public persuasion; while we assume voters have the common preference for each state and study both public and private persuasion. Wang (2012) studies both public and private persuasion in voting games, but in the private case of her model, the signals are assumed to be i.i.d. across voters, which results in the optimality of public persuasion. In our setting where correlated signals are allowed, it is not surprising that private persuasion is strictly better than public persuasion. Our focus is on understanding where the advantage of private persuasion comes from. In a very general framework, Taneva (2014) studies information design in a more abstract setting, and develop a methodology that characterizes the set of equilibria. On the contrary, by focusing on a specific voting game, we are able to derive additional implications for such an environment.

In Caillaud and Tirole (2007), the sender chooses the optimal sequence to communicate with receivers. A receiver learns the state through two channels: (1) directly communicating with the sender, and (2) observing other receivers being convinced. They shed light on the advantage of selective communication to key group members and to engineering persuasion cascades in which receivers who are brought on board sway the opinion of others. By choosing the sequence of persuasion, the sender can manipulate receivers' beliefs through the effects of persuasion cascades. In our model, the sender can send any monotone signal, so he has additional freedom to manipulate information aggregation among voters.

Farrell and Gibbons (1989) and Goltsman and Pavlov (2011) consider cheap talk models with multiple audiences. Unlike our paper, theirs focuses on examining how the communication protocol affects the amount of information *transmitted* from the sender to receivers rather than released (to both parties).

More broadly, our paper relates to the informative voting literature. There is a large body of literature considering information aggregation where voters' (or committee members') private information is exogenously given such as Austen-Smith and Banks (1996), Feddersen and Pesendorfer (1997), Feddersen and Pesendorfer (1998), Li, Rosen, and Suen (2001), and Ekmekci and Lauermann (2015). This literature has been enriched in many dimensions: Gerardi and Yariv (2007) compare various voting rules when voters are allowed to deliberate before casting their votes. Jackson and Tan (2013) allow voters to consult experts before voting and examine how disclosure and voting vary with different voting rules and with the signal precision of experts. Li (2001), Persico (2004), Gerardi and Yariv (2008) and Cai (2009) assume that voters (or committee members) endogenously collect their information individually, which naturally leads to heterogenous private information among voters. The focus of these papers is to design a decision rule that incentivizes voters to acquire information and aggregate their private information efficiently. However, we emphasize the other source of information heterogeneity among voters: the sender's deliberate provision of heterogenous information to different voters.

The rest of the paper is organized as follows. In section 2, we present the model. In section 3, we provide some preliminary analysis of the optimal persuasion problem. In section 4, we study a three-voter case to illustrate the main idea of the paper. In section 5, we consider the general case. Section 6 discusses the robustness of the main result and possible extensions, and section 7 concludes. The Appendix contains omitted proofs.

2 Model

Voters. A group of N voters needs to decide between one of the following two actions:

$$x \in \{a, b\}$$

where b represents the status quo, and a represents the alternative (or risky) action. The collective decision is made according to a K-majority rule where $K < N$: action a is chosen if it receives at least K votes; otherwise, action b is chosen. We assume that the relevant state is binary:

$$\omega \in \{A, B\}.$$

Voters are uncertain about the state and share a common prior belief

$$\mu_0 = \Pr(\omega = A) \in (0, 1).$$

They want to match the state. Specifically, if the final decision is b, each voter obtains 0; otherwise, a voter's payoff is characterized by a von Neumann-Morgerstern utility function

$$\mu_i - (1 - \mu_i) l_i$$

where $\mu_i \in [0, 1]$ is the voter's belief about the state being A and $l_i > 0$: she obtains one unit of benefit if $\omega = A$ while she suffers l_i units of loss if $\omega = B$. The value of l_i measures voter i's "threshold of doubt" - the limit beyond which she will approve the risky action.

While voters hold identical preference over actions conditional on the true state, their preferred action may be different for a given belief μ due to the differences in their "threshold of doubt".[2] In order to convince a voter with higher l_i that the risky action is more promising, one needs to provide more information to increase her belief about the state being A so that $\mu_i - (1 - \mu_i) l_i \geq 0$. We assume that

$$l_1 < l_2 < \ldots\ldots < l_N,$$

so l_1-voter is easiest to convince while l_N-voter is hardest to convince. We assume that

$$\mu_0 - (1 - \mu_0) l_K < 0,$$

so b is the default collective action when the voters receive no further information.

Persuasion Technology. A sender prefers the risky action regardless of the state. He persuades voters to vote for the risky action by controlling their signal structure. Formally,

[2] That is to say, voters have heterogenous preferences on the lottery space.

Definition 1. *A* signal (or information) structure *consists of a set of finite realization spaces* $\{S^i\}_{i=1,2,...N}$ *and a pair of probabilities* $\{\pi(\cdot|\omega)\}_{\omega=A,B} \in \Delta(\times_{i=1,2,..N} S^i)$ *where* S^i *denotes the realization space for voter* i.

Following Kamenica and Gentzkow (2011), we assume the signal structure is observed by voters. A signal realization consists of N individual observations: $s_1, ...s_N$ where $s_i \in S^i, \forall i = 1, 2, ..., N$ and voter i can only observe her own signal realization s_i.

Naturally, a signal realization $s' = (s'_1, s'_2, ...s'_N)$ such that $\{s'_i \in S^i\}_{i=1,2,..N}$ causes posterior beliefs:

$$\mu^i(s'_i) = \frac{\mu_0 \sum_{s:s_i=s'_i} \pi(s|A)}{\mu_0 \sum_{s:s_i=s'_i} \pi(s|A) + (1-\mu_0)\sum_{s:s_i=s'_i} \pi(s|B)} \tag{1}$$

for each voter $i = 1, 2, ...N$. We allow signals to be arbitrarily correlated across voters given the realized state. In general, since $\{s \in \times_i S^i | s_i = s'_i\}$ and $\{s \in \times_i S^i | s_j = s'_j\}$ may not be identical for $i \neq j$, voters do not necessarily hold common posterior beliefs. The sender's goal is to maximize the *persuasion probability*, that is, the probability of the collective decision being a.

As the signal is observed by voters, the sender is able to commit to a persuasion mechanism. Thanks to the revelation principle, one can focus on the *recommendation signals* where $S^i = \{a, b\}, \forall i = 1, 2, ..N$ by applying the revelation principle: a voter only observes a recommendation for her action.[3]

Furthermore, we focus on signals that satisfy the monotone likelihood ratio property. Specifically, we define a (partial) order $\geq$ on the space of recommendation signals as follows. Suppose that $s, \hat{s} \in \{a, b\}^N$. We say $\hat{s} \geq s$ if and only if there exists a $s' \in \{a, b\}^N$ such that $\hat{s} \equiv s \vee s'$. In other words, $\hat{s}_i = a$ unless $s_i = s'_i = b, \forall i = 1, 2, ...N$. We say signal structure $\{\pi(\cdot|\omega)\}_{\omega=A,B}$ satisfy monotone likelihood ratio property (MLRP) if and only, for any $\hat{s} \geq s$,

$$\pi(\hat{s}|A)\pi(s|B) \geq \pi(s|A)\pi(\hat{s}|B) \tag{MLRP}$$

An immediate implication of the MLRP assumption is that, if $\hat{s} \geq s$, for any given prior μ_0, the posterior belief about the state being A caused by $\hat{s}$ is not less than the one caused by s. In other words, $\hat{s}$ *is more favorable than* s *for state* A (Milgrom (1981)). In

[3] See Proposition 2 of Taneva (2014) for the proof in a more general setting.

our setting, MLRP simply suggests that convincing more voters to vote for a requires more good news favoring state A.

Voting Strategies and Equilibrium. Given a recommendation signal $\{\pi(\cdot|\omega)\}_{\omega=A,B}$, one can define a voting game with private information. A pure strategy for voter i is a measurable function from the space of realized signals to the space of actions, i.e., $\sigma_i : \{a,b\} \to \{a,b\}$.

As in most informative voting games, there are trivial equilibria due to coordination failure among voters. Since the information structure is endogenous in our setting, the sender can always send uninformative signals where $s_i = a, \forall i, \omega$ and the persuasion probability is one. To avoid such trivial equilibria, we follow the literature by assuming no voter plays a weakly dominated strategy. In addition, because our purpose is to find the upper bound of the persuasion probability, whenever there exist multiple Bayesian Nash equilibria, we select the one which maximizes the sender's payoff.

Definition 2. *Given a signal structure* $\{\pi(\cdot|\omega)\}_{\omega=A,B}$*, a* voting equilibrium $\{\sigma_i^*\}_{i=1,2,\ldots N}$ *is a sender's optimal Bayesian Nash equilibrium of the voting game where voters do not play a weakly dominated strategy.*

3 Optimal Persuasion Problem

We begin with some notations. Denote $S_x \subset \{a,b\}^N$ as the set of x-winning signals such that

1. $S_a \equiv \{s : |s_i = a| \geq K\}$, and
2. $S_b \equiv \{a,b\}^N \setminus S_a$.

Let $S_x^* \subset S_x$ be the set of pivotal x-winning realized signals where $S_a^* \equiv \{s : |s_i = a| = K\}$, and $S_b^* \equiv \{s : |s_i = a| = K-1\}$. Denote $S_{i,x} \equiv \{s | s_i = x\}$ as the set of realized signals that voter i's realization is $x \in \{a,b\}$. Furthermore, denote

$$S_{i,x}^* \equiv S_{i,x} \cap S_x^*$$

as the set of pivotal x-winning signals for voter i in which voter i is pivotal and her realized recommendation signal is x.

3.1 Incentive Compatiblity

A voter prefers action x only if the realized signal is sufficiently *convincing* to support the corresponding state. In a strategic voting game, voter i can make a change only if she is pivotal. In such a scenario, to convince voter i to vote for action a, the sender needs to influence her pivotal belief, that is, her posterior belief about the state being A conditional on herself being pivotal. Formally, voter i is willing to follow the sender's recommendation to vote for the risky action if and only if

$$\mu^i_{\text{pivotal}}(a) = \frac{\mu_0 \sum_{s\in S^*_{i,a}} \pi(s|A)}{\mu_0 \sum_{s\in S^*_{i,a}} \pi(s|A) + (1-\mu_0)\sum_{s\in S^*_{i,a}} \pi(s|B)} \geq \frac{l_i}{1+l_i} \tag{IC-a}$$

where $\sum_{s\in S^*_{i,a}} \pi(s|\omega)$ represents the probability of voter i being pivotal for action a in state ω. That is to say, to motivate voter i to support action a, her vote must be relevant with a positive probability, and conditional on her being pivotal, the state is A with a sufficiently high probability.

Whenever it is well-defined, the likelihood ratio $\frac{\sum_{s\in S^*_{i,a}} \pi(s|A)}{\sum_{s\in S^*_{i,a}} \pi(s|B)}$ measures the *convincingness* of the signal structure to voter i's vote for action a. The higher the likelihood ratio is, the more convincing the realized individual signal is to voter i. To ensure the collective decision is a, at least K voters' incentive-compatible conditions (IC-a) need to be satisfied. Similarly, if the realized recommendation signal is b, the voter follows the recommendation if and only if

$$\mu^i_{\text{pivotal}}(b) = \frac{\mu_0 \sum_{s\in S^*_{i,b}} \pi(s|A)}{\mu_0 \sum_{s\in S^*_{i,b}} \pi(s|A) + (1-\mu_0)\sum_{s\in S^*_{i,b}} \pi(s|B)} \leq \frac{l_i}{1+l_i} \tag{IC-b}$$

Notice that $\mu^i_{\text{pivotal}}(x)$ may not be well-defined for every i and x. Fix a signal structure. If $\sum_{\omega=A,B}\sum_{s\in S_{i,x}} \pi(s|\omega) = 0$, then voter i is never persuaded to vote for action x, and thus her pivotal belief $\mu^i_{\text{pivotal}}(x)$ is not well-defined.

3.2 Simplifying the Problem

The sender's optimal persuasion problem is

$$Q(\mu_0) = \max_{\pi(\cdot|\omega)} \mu_0 \sum_{s\in S_a} \pi(s|A) + (1-\mu_0)\sum_{s\in S_a} \pi(s|B) \tag{P-0}$$

such that

1. the feasibility constraint: $\pi(\cdot|A), \pi(\cdot|B) \in \Delta(\{a, b\}^N)$,
2. the monotone signal constraint: (MLRP),
3. the incentive-compatible constraints: (IC-a) for voter i if $\sum_{\omega=A,B} \sum_{s \in S_{i,a}} \pi(s|\omega) > 0$ and (IC-b) for voter i if $\sum_{\omega=A,B} \sum_{s \in S_{i,b}} \pi(s|\omega) > 0$.

where the value function $Q(\mu_0)$ denotes the unconditional persuasion probability. We further denote $Q_\omega(\mu_0)$ as the persuasion probability in state $\omega = A, B$. Apparently, $Q(\mu_0) = \mu_0 Q_A(\mu_0) + (1 - \mu_0) Q_B(\mu_0)$.

The following lemmas simplify the analysis by providing some characterizations for solutions of problem (P-0). First, we show that $Q_A(\mu_0) = 1$ for any μ_0.

Lemma 1. *In the optimal persuasion,* $\sum_{s \in S_b} \pi(s|A) = 0$ *so that* $\sum_{s \in S_a} \pi(s|A) = 1$.

Proof. By setting $\pi(s|A) = 0, \forall s \in S_b$, incentive-compatible constraint (IC-b) is trivially satisfied. Also, any signal $s \in S_b$ is "good news" for action b so that the monotonicity condition for signals are still satisfied. As a result, one can set $\sum_{s \in S_a} \pi(s|A) = 1$ to raise the value of the objective function. □

Recall that the sender may recommend his less preferred action b in state B only to convince voters to follow recommendation a. In state A, action b is Pareto inferior, it is unnecessary (and also suboptimal) to be recommended. Notice that an individual voter may still receive a recommendation signal b in state A. In fact, to satisfy a voter's incentive-compatible constraint (IC-a), she has to be pivotal for action a with positive probability. In such an event, $N - K$ other voters receive a recommendation signal for action b.

Thanks to Lemma 1, in any optimal persuasion, the persuasion probability is one in state A, and thus the sender's objective function effectively becomes $\sum_{s \in S_a} \pi(s|B)$. As one can renormalize $l'_i = l_i \frac{1-\mu_0}{\mu_0}$ without affecting the solution, in the remainder of the paper, we assume that $\mu_0 = 0.5$ without loss of any generality.

Assumption 1. *State A and B are ex ante equally likely, i.e.,* $\mu_0 = 0.5$.

The optimal persuasion problem can be further simplified thanks to the MLRP assumption.

Lemma 2. *In an optimal persuasion,* $\pi(s|A) = \pi(s|B) = 0, \forall s \in S_a \setminus S_a^*$.

To understand the intuition, one can consider the likelihood ratio $\frac{\pi(s|A)}{\pi(s|B)}$ as the cost-benefit ratio for a realized signal $s \in S_a$: increasing $\pi(s|A)$ makes signal s better news for state A, so the sender can raise $\pi(s|B)$ to increase the persuasion probability. We refer to $\pi(s|A)$ as the "evidence" that supports $\pi(s|B)$. However, since the total budget of the evidence is constrained by $\sum_{s \in S_a} \pi(s|A) = 1$, one wants to allocate the measure of "evidence" efficiently. Due to the monotonicity constraint (MLRP), for any non-pivotal signal $s' \in S_a \setminus S_a^*$, there exists a pivotal signal $s \in S_a^*$ with a lower cost-benefit ratio. Hence, the sender will be weakly better off by using the pivotal realized signal s.

Lemma 2 further narrows down the set of recommendations used in an optimal persuasion signal: to convince the group to take the collective action a, the sender can focus on pivotal signals in S_a^*.[4] The result is driven by the MLRP assumption. To convince more voters, the signal must be more convincing; and thus, action a is less likely to be recommended in state B, which is obviously undesirable from the sender's perspective. As the sender only needs to convince K voters to choose action a, it is inefficient to use non-pivotal signal $s \in S_a \setminus S_a^*$ and convince more voters.

Furthermore, the results of Lemma 1 and Lemma 2 together imply that only pivotal recommendations $s \in S_a^*$ are effectively adopted, and therefore the MLRP is automatically satisfied in an optimal persuasion signal.

Corollary 1. *In an optimal persuasion signal where* $\pi(s|A) = \pi(s|B) = 0, \forall s \in S_a \setminus S_a^*$*, constraint (MLRP) is not binding.*

Proof. The result immediately comes from Lemma 1 and Lemma 2. □

As a consequence, one can focus on pivotal persuasion, and problem (P-0) becomes

$$Q_B(\mu_0) = \max_{\pi(\cdot|\omega) \in \Delta(S)} \sum_{s \in S_a^*} \pi(s|B) \tag{P-1}$$

$$\sum_{s \in S_{i,a}^*} \pi(s|B) \leq \frac{1}{l_i} \sum_{s \in S_{i,a}^*} \pi(s|A). \tag{IC-a'}$$

[4]Notice that Lemma 2 has no implication on the use of non-pivotal signal for action b.

In problem (P-1), the incentive-compatible condition (IC-a') holds for each voter whether she will be persuaded or not. For a voter whom the sender wants to persuade, either $\sum_{s\in S^*_{i,a}}\pi(s|A) > 0$ or $\sum_{s\in S^*_{i,a}}\pi(s|B) > 0$, so condition (IC-a') is equivalent to (IC-a). For those voters whom the sender does not want to persuade, $\sum_{s\in S^*_{i,a}}\pi(s|B) = \sum_{s\in S^*_{i,a}}\pi(s|A) = 0$, so condition (IC-a') trivially holds.

Notice that the above simplification of the incentive-compatible constraints works because we can focus on pivotal persuasion. In the setting where constraint (MLRP) is not required, Lemma 2 fails, and the sender may find it optimal to use a non-pivotal winning signal: $s \in S_a \setminus S^*_a$. Then condition (IC-a') is insufficient to prevent voters from using weakly dominated strategy to vote for a. For example, one can set an uninformative signal $\pi(s|\omega) = 1$ where $s = (a, a, ...a)$ for $\omega = A, B$. As $\pi(s|\omega) = 0$ for $s \in S^*_{i,a}, \omega = A, B$, condition (IC-a') trivially holds, and the persuasion probability is one. However, none of the N voters is pivotal; thus no voter is persuaded to vote for a. In other words, without (MLRP), programming (P-1) is no longer equivalent to the sender's optimization problem (P-0).

4 A Three-Voter Case

In this section, we study a simple example where $N = 3$ and $K = 2$ to illustrate the intuition of our main results. The sender's problem is

$$\max_{\pi(\cdot|\omega)}\{\pi(aab|B) + \pi(aba|B) + \pi(baa|B)\} \tag{2}$$

$$\begin{aligned}
\text{s.t. } \pi(aab|B)) + \pi(aba|B) &\leq \frac{1}{l_1}[\pi(aab|A) + \pi(aba|A)] &(3)\\
\pi(aab|B) + \pi(baa|B) &\leq \frac{1}{l_2}[\pi(aab|A) + \pi(baa|A)] &(4)\\
\pi(aba|B) + \pi(baa|B) &\leq \frac{1}{l_3}[\pi(aba|A) + \pi(baa|A)] &(5)\\
\pi(\cdot|\omega) \in \Delta(\{a, b\}^3);\ \omega &= A, B &(6)
\end{aligned}$$

First, we consider public persuasion. In this case, voters hold identical posterior beliefs for each realized (public) recommendation. Whenever a hard-to-convince voter is willing to choose a, an easy-to-convince voter is automatically convinced as well. As a consequence, there are only four incentive-compatible recommendation signals:

$$\{bbb, abb, aab, aaa\}.$$

Obviously, it is optimal to persuade only the two easy-to-convince voters. The corresponding revelation mechanism consists of two recommendation signals: aab and bbb. Voter 1 and voter 2 vote for both a and b with positive probabilities; while voter 3 always votes for b. In the optimal public persuasion, voter 2's incentive-compatible constraint (4) is binding so that

$$\pi(aab|B) = \pi(aab|A)/l_2 = 1/l_2.$$

Since $l_1 < l_2$, voter 1's incentive-compatible constraint (3) must be slack. In addition, voter 3's IC constraint (5) is trivially satisfied as she is never pivotal. In the optimal public persuasion, there is only one winning coalition consisting of voter 1 and 2, and there is only one pivotal recommendation signal for the risky action: aab. The risky action is chosen with probability $1/2 + 1/(2l_2)$.

The following lemma suggests that the sender is strictly better off using private persuasion.

Proposition 1. *Public persuasion is strictly suboptimal.*

Proof. We prove the result by finding another signal which strictly increases the probability that the risky action is chosen in state B. In the optimal public persuasion, (3) is slack and (4) is binding. Take $\epsilon = \frac{1}{2}\left(\frac{1}{l_1} - \frac{1}{l_2}\right)$. Let $\hat{\pi}(aab|A) = 1 - \epsilon$, $\hat{\pi}(baa|A) = \epsilon$ and $\hat{\pi}(aba|B) = \delta = \min\{(1-\epsilon)/l_1, \epsilon/l_3\} > 0$. The new signal structure satisfies (3), (4), and (5), and the risky action is chosen with probability $1/l_2 + \delta$. Thus we see that public persuasion is suboptimal. $\square$

The idea is to reallocate "evidence" across voters to manipulate their posterior beliefs conditional on their being pivotal. Recall that to convince voter i, one only needs to ensure that her posterior belief conditional on being pivotal is just above $l_i/(1 + l_i)$. As l_i varies across voters, an efficient persuasion requires that voters hold different posterior beliefs. However, in public persuasion, this is impossible because there is only one winning coalition. Whenever voter 2 is just convinced to choose a, voter is over-convinced, which means the sender "wastes" some "evidence" on voter 1.

Ideally, one would like to reallocate the redundant "evidence" from voter 1 to voter 3 to create additional winning coalitions: one consists of voter 1 and 3 with a recommendation aba, and another one consists of voters 2 and 3 with a recommendation baa. If persuasion is private, such a reallocation of convincingness across voters is possible because multiple pivotal signals for action a can exist. By observing her own signal, a voter is uncertain about the pivotal signal and therefore about the winning coalition of which she is a part. Hence, there can exist a pivotal event where a hard-to-convince voter is willing to choose a, but an easy-to-convince voter is not because the latter does not know who else is in her winning coalition. For example, one can set $\pi(aab|\omega), \pi(aba|\omega), \pi(bbb|\omega) > 0$ for some ω, then there are three pivotal signals for action a, and each voter is involved in two winning coalitions. The signal realization aba is incentive compatible because voter 2 is uncertain if voter 3 is choosing action a. As a result, voters' posterior beliefs are not necessarily identical even if they all observe the same recommendation. As discriminatory persuasion is feasible, the sender can reallocate redundant "evidence" from the easy-to-convince voter to the hard-to-convince voter to strictly increase persuasion probability.

Proposition 2. *When $l_1 > l_3(1/l_2 - 1) + 1$, an optimal persuasion signal satisfies*

$$\begin{aligned} \pi(baa|A) &= \frac{l_3 l_2 - l_3 l_1}{l_1 l_2 + l_2 l_3} \quad ; \quad \pi(baa|B) = 0; \\ \pi(aba|A) &= 0 \quad ; \quad \pi(aba|B) = \frac{l_2 - l_1}{l_1 l_2 + l_2 l_3}; \\ \pi(aab|A) &= \frac{l_1 l_3 + l_1 l_2}{l_1 l_2 + l_2 l_3} \quad ; \quad \pi(aab|B) = \frac{1}{l_2}. \end{aligned}$$

In state B, the persuasion probability is $\frac{l_3+l_2}{l_2(l_1+l_3)}$, which is strictly decreasing in $l_i, \forall i = 1,2,3$, and $\sum_{s \in S_b} \pi(s|B) > 0$. When $l_1 \leq l_3(1/l_2 - 1) + 1$, the persuasion probability is one.

Proposition 2 implies that the persuasion probability can reach one as long as $0 < l_1 \leq l_3(1/l_2 - 1) + 1$. Notice that when l_2 and l_3 are sufficient large, $l_3(1/l_2 - 1) + 1 < 0$, so the persuasion probability is less than one no matter how small l_1 is. Hence, the persuasion probability achieves one only if none of the three voters are sufficiently hard to convince. More importantly, as $l_2 > 1$, the condition requires that $l_1 < 1$. Namely, voter 1 does not need further persuasion to vote for a. In fact, in the optimal persuasion mechanism, the sender will redirect some "evidence" from voter 1 to other voters so that voter 1 is only just convinced by seeing a realization signal a conditional on her being pivotal:

$$\frac{\mu^1_{\text{pivotal}}(a)}{1-\mu^1_{\text{pivotal}}(a)} = \frac{\pi(aab|A)+\pi(aba|A)}{\pi(aab|B)+\pi(aba|B)} = l_1$$

but not convinced by seeing a realized signal b conditional on her being pivotal: $\mu^1(b) = 0$.[5] On the other hand, voters 2 and 3 will obtain more "evidence" so that they are also just convinced by seeing recommendation signal a, respectively.

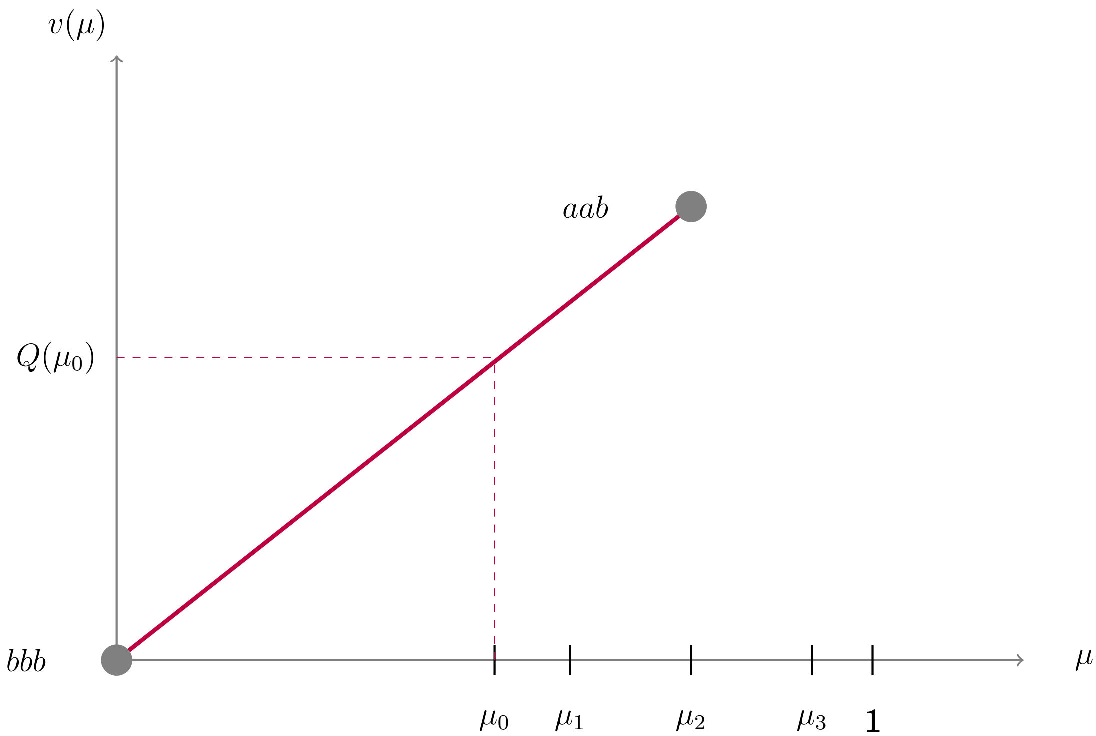
When the persuasion probability is one, there is a continuum of signals maximizing the sender's payoff, so the optimal persuasion rule is not effectively sensitive to parameters. Hence, in the remainder of this section, we focus on the interesting case where the persuasion probability $\frac{l_3+l_2}{l_2(l_1+l_3)} < 1$. In such a case, by using private persuasion rather than public persuasion, the persuasion probability is increased by $\frac{l_2-l_1}{l_1 l_2 + l_2 l_3}$ via sending recommendation aba. Such an advantage is present because, in the optimal private persuasion, voters are uncertain about the winning coalition they are part of given each realized signal. In the three-voter case, there exist multiple possible winning coalitions only if voter 3 is involved, and voter 3 is involved if her voting is informative. Since $\lim_{l_3\to\infty} \pi(baa|A) = (l_2-l_1)/l_2, \lim_{l_3\to\infty} \pi(aab|A) = l_1/l_2$, voter 3 is always involved in the optimal persuasion no matter how hard she is to convince. However, the advantage of private persuasion compared to public persuasion vanishes as (1) $l_3 \to \infty$ or (2) $l_1 \to l_2$. In the former case, when voter 3 is hard to convince, the "evidence" switched from voter 1 can only have very little marginal benefit. In the latter exercise, as voter 1 and 2 become equally hard (or easy) to convince, there is less wasted "evidence" to transfer.

Furthermore, persuasion probability in state B is strictly decreasing in each l_i. This is intuitive. As l_i increases, voter i becomes more cautious so that she needs more "evidence" to support the state being A. As $\sum_{s:s_i=a} \pi(s|A)$ is bounded by one, the sender has to reduce $\sum_{s:s_i=a} \pi(s|B)$, which reduces persuasion probability.

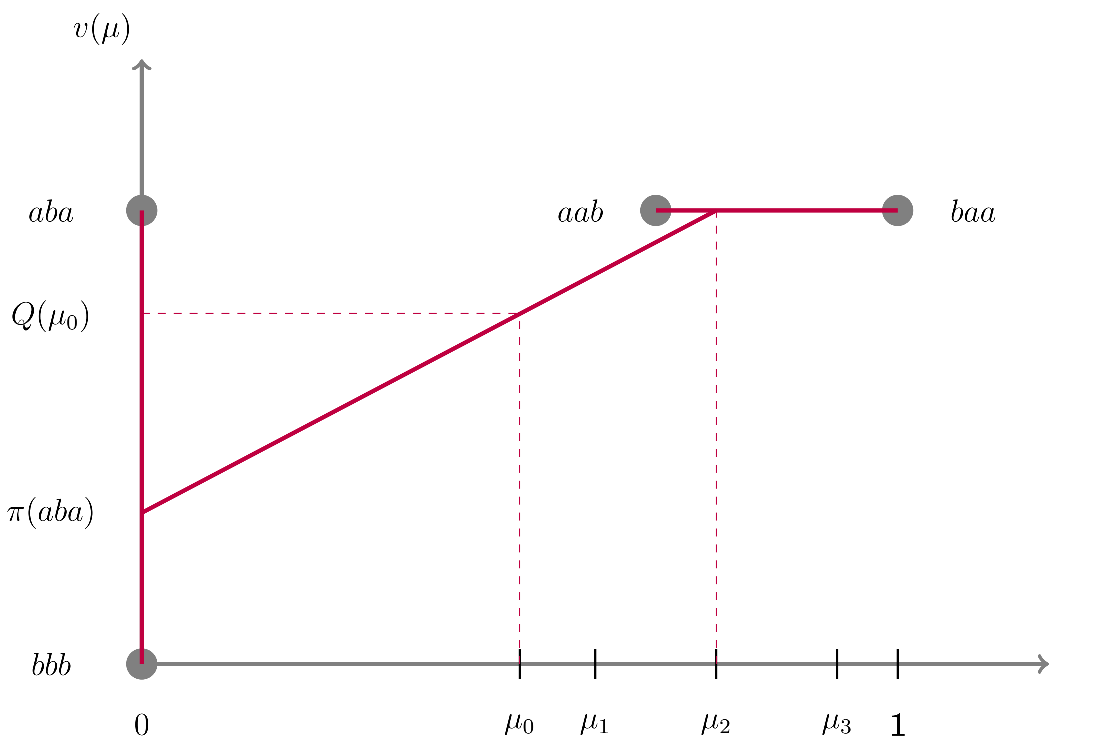
Notice that in the optimal persuasion, all voters' IC constraints are binding because no voter knows which pivotal event she is involved in, and

$$\frac{\pi(aab|A)}{\pi(aab|B)} \in (l_1, l_2); \frac{\pi(aba|A)}{\pi(aba|B)} = 0; \frac{\pi(baa|A)}{\pi(baa|B)} = \infty.$$

[5] Notice that both $\mu^1_{\text{pivotal}}(a)$ and $\mu^1_{\text{pivotal}}(b) < \mu_0$. This is consistent with Bayes's rule because they are posterior beliefs conditional on voter 1 being pivotal. $\sum_s \mu^1_{\text{non-pivotal}}(s) > \mu_0$ so that Bayesian plausibility (Kamenica and Gentzkow (2011)) holds.



(a) Public Persuasion



(b) Private Persuasion

Figure 1: Persuasion Probability

That is to say, if a voter can learn the pivotal event she is involved in, she may become over-convinced or not convinced. For example, by seeing recommendation a, voter 2 cannot tell aab from baa. In the first pivotal event, she is in the winning coalition with voter 1, and because $\frac{\pi(aab|A)}{\pi(aab|B)} < 1$, she is not convinced; while in the second pivotal event, she is in the winning coalition with voter 3, as $\frac{\pi(baa|A)}{\pi(baa|B)} = \infty$, she is over-convinced. Similarly, voter 3 is over-convinced in pivotal event baa but not convinced in event aba. Consequently, although voters are willing to follow the sender's recommendation individually, they will refuse to do so if all recommendation signals are common knowledge; and thus we conclude that in the optimal persuasion, voting does not fully aggregate information in the sense of Feddersen and Pesendorfer (1997). Moreover, the optimal persuasion is not voter-communication-proof. If voters can pairwisely communicate, they can know the realized winning coalition, and some of them may refuse to follow the recommendation. This observation should not cause any surprise. In the extreme case, if all voters can freely communicate their realized signals, private persuasion cannot succeed any better than public persuasion.

5 General Case

In this section, we seek the optimal persuasion mechanism in a general setting with more than three voters. The main result is that the optimal persuasion follows a cutoff rule: the sender only tries to influence the beliefs of voters who are sufficiently easy to convince. Some of them will be convinced with positive probability. Hard to convince voters never vote for action a. Similar to the three-voter case, to create multiple possible winning coalitions, more than K voters will be persuaded with positive probability.

Proposition 3. *In the optimal persuasion, there is a positive integer i^* such that*

1. $\exists s, s' \in S^*_{i,a}$ *such that* $\pi(s|A), \pi(s'|B) > 0$ *for* $i \leq i^*$*,*
2. $\pi(s|A), \pi(s|B) = 0, \forall s \in S^*_{i,a}$ *for* $i > i^*$*, and*
3. $K < i^* \leq N$*,*

In the rest of this section, we prove the above result. The main challenge is that the number of choice variables is too large although it has been significantly reduced by

Lemma 1 and Lemma 2. In principle, given N and K, there are $2C_K^N$ pivotal signals, and the number of variables increases exponentially as N grows.

Notice that in problem (P-1), voter i cannot distinguish different realized signal $s \in S^*_{i,a}$. Her incentive to follow recommendation a relies on $\pi(s|\omega)$ only through the total probability of her being pivotal. A natural idea is to solve the optimal persuasion problem by choosing the probabilities that each voter is pivotal, then pinning down the optimal signal $\pi(s|\omega)$ for $s \in S^*_{i,a}$. Motivated by this idea, consider the following problem:

$$\max_{Q_B,\alpha,\beta} Q_B \tag{P-2}$$

$$\begin{aligned}
\text{s.t. } \alpha_i &\geq l_i(Q_B - \beta_i), \forall i && (7)\\
\sum_{i=1}^{N} \alpha_i &\leq K && (8)\\
\sum_i \beta_i &= (N-K)Q_B && (9)\\
\alpha_i &\in [0,1], \beta_i \in [0, Q_B], \forall i, \text{ and } Q_B \in [0,1] && (10)
\end{aligned}$$

where $\alpha_i = \sum_{s\in S^*_{i,a}} \pi(s|A)$ is the probability that i is pivotal for action a in state A, and β_i represents the probability that a is chosen but i is not pivotal in state B. Because of Lemma 2, in any optimal persuasion, $Q_B - \beta_i = \sum_{s\in S^*_{i,a}} \pi(s|B)$. Hence, (7) is a reformulation of (IC-a'), (8) and (9) hold because each pivotal signal is shared by K voters, and (10) are the feasibility constraints.

Essentially, for each voter, we assemble her pivotal events in problem (P-2). Obviously, for any $\{\pi(s|\omega)\}_{s\in S^*_{i,a}}$ that solves problem (P-1), the corresponding $\{\alpha_i, \beta_i\}_i, Q_B$ automatically solves problem (P-2). The following lemma shows that the opposite direction is also true.

Lemma 3. *1. For any $\{\alpha_i\}_{i=1}^n$ with $\alpha_i \in [0,1], \forall i$ and $\sum_j \alpha_j = K$, there exists a probability distribution $\pi(\cdot|A) \in \Delta(\{a,b\}^N)$ such that $\sum_{s\in S^*_a} \pi(s|A) = 1$, and $\sum_{s\in S^*_{ia}} \pi(s|A) = \alpha_i$.*

2. For any $Q_B \in [0,1]$ and $\{\beta_i\}_{i=1}^N$ with $\beta_i \in [0, Q_B], \forall i$ and $\sum_j \beta_j = (N-K)Q_B$, there exists a probability distribution $\pi(\cdot|B) \in \Delta(\{a,b\}^N)$ such that $\sum_{s^*_b} \pi(s|B) = Q_B$, and $1 - \sum_{s\in S^*_{i,b}} \pi(s|B) = \beta_i$.

As a result of Lemma 3, to solve problem (P-1), we can focus on problem (P-2) without loss of any generality. The following result generalizes the observation of Proposition 1 in the three-voter case. The intuition is similar, so it is removed here.[6]

Lemma 4. *Public persuasion is strictly suboptimal under K-majority rule.*

To solve problem (P-2), we first solve problem (P-3) by fixing $Q_B \in [0,1]$ as a parameter and consider the following problem:

$$U(Q_B) = \min_{\alpha,\beta} \sum_{i=1}^{N} \beta_i \tag{P-3}$$

$$\begin{aligned} \text{s.t. } l_i Q_B &\leq \alpha_i + l_i \beta_i, \forall i && (11)\\ \sum_i \alpha &\leq K, \alpha_i \in [0,1]; \beta_i \in [0, Q_B], \forall i && (12) \end{aligned}$$

Notice that problem (P-3) is not the dual of problem (P-2) unless constraint (9) is also satisfied. In the rest of this section, we first characterize the solution of problem (P-3) for a given Q_B, then we impose constraint (9) to find the solution of problem (P-2).

In a solution of problem (P-3), (11) is binding for each i. Otherwise, one can decrease α_i without raising the value of the objective function; thus one can substitute out β_i so that the sender's problem is to choose $\{\alpha_i\}_{i=1,2,...N}$ to maximize $\sum_{i=1}^{N}(Q_B - \alpha_i/l_i)$ such that constraint (12) holds.

Lemma 5. *For any $Q_B \in [0,1]$, the solution of problem (P-3) satisfies*

$$\hat{\alpha}_i(Q_B) = \begin{cases} 0 & if \quad i > i^* + 1, \\ \min\{1, K - \sum_{j=1}^{i^*} \min\{l_j Q_B, 1\}\} & if \quad i = i^* + 1, \\ \min\{l_i Q_B, 1\} & if \quad i \leq i^*. \end{cases} \tag{13}$$

where

$$i^* = \max\{i \leq N | \sum_{j=1}^{i} \min\{l_j Q_B, q\} < K\}, \tag{14}$$

and

$$\hat{\beta}(Q_B) = Q_B - \frac{\hat{\alpha}(Q_B)}{l_i}. \tag{15}$$

The value function $U(Q_B)$ is strictly increasing in Q_B.

[6] The result remains unless $l_1 = l_2 = ... = l_K$.

Lemma 6. *Suppose that the objective function of problem (P-2) takes value at $Q_B^* \in [0,1]$ at optima. $(\hat{\alpha}(Q_B^*), \hat{\beta}(Q_B^*))$ solves problem (P-2) and $U(Q^*) \leq (N-K)Q_B^*$ where $\hat{\alpha}(\cdot), \hat{\beta}(\cdot)$ are defined in (13) and (15). When $Q_B^* < 1$, $U(Q^*) = (N-K)Q_B^*$, and $(\hat{\alpha}(Q_B^*), \hat{\beta}(Q_B^*))$ is the unique solution of problem (P-2).*

Lemma 5 and Lemma 6 together imply that, in the optimal persuasion, $\beta_i^* = Q_B^*$ as long as $\alpha_i^* = 0$. Hence, if voter i is never pivotal for action a in state A, she is never pivotal in state B. Furthermore, the optimal persuasion follows a cutoff rule. The sender only influences the beliefs of the i^* most easy-to-convince voters. By Lemma 4, we also know that $i^* > K$ so that there exist multiple possible winning coalitions. Hence, we immediately have the result in Proposition 3.

5.1 Comparative Statics

We now analyze how the persuasion probabilities (in state B) are affected when (1) more votes are required for the alternative action, and (2) voters become harder to convince. As in the three-voter case, we are interested in the setting where the persuasion probability is less than one so that both persuasion probability and the optimal signal respond to a slight change in parameters. The following lemma identifies a sufficient condition for such a scenario.

Lemma 7. *If $l_i > 1, \forall i$, $Q_B^* < 1$.*

Lemma 7 says that when each voter needs to be persuaded to vote for a, the persuasion probability is bounded away from one. The condition is intuitive. If each voter needs to be persuaded, a convincing signal will recommend her action b with positive probability. Hence, the persuasion probability cannot be one.

Proposition 4. *When the persuasion probability is less than one, it is strictly decreasing in K and is strictly decreasing in $l_i, \forall i$ s.t. $\alpha_i^* \neq 0$.*

The intuition behind Proposition 4 is simple. As K increases, the sender needs to persuade more voters to vote for a in each winning coalition. As a result, additional hard-to-convince voters must be convinced, which costs a higher probability that b is chosen. As the choice of a non-pivotal voter is irrelevant to the collective decision, the persuasion probability is not affected by the change in the preference of such voters.

On the other hand, when a pivotal voter becomes harder to convince, the sender has to make the signal more convincing. Specifically, he has to reduce the probability that a is recommended in state B, which causes the result.

6 Discussion

Unanimous Rule. In the baseline model, we assume that $K < N$. What happens if the collective decision is made through the unanimous rule ($K = N$)? In such a case, there is a unique feasible pivotal event for action a where all voters vote for it. Hence, the sender is not able to create multiple pivotal events. Although the sender can privately communicate with each voter, the voters will strategically aggregate their private information conditional on being pivotal; and thus voters will hold common posteriors in the unique winning coalition. As a consequence, the advantage of private persuasion no longer exists.

Non-Monotone Signals. Another critical assumption is that signals must satisfy the MLRP. Thanks to this assumption, we can focus on pivotal persuasion (Lemma 2). One may wonder what happens if we open the box of non-pivotal persuasion? It turns out that the problem becomes trivial: the persuasion probability can be arbitrarily close to one regardless of voters' preferences. To understand the result, imagine that the sender commits to the following signal. With probability $1 - \epsilon$, the sender uses a manipulation signal that recommend that voter $i = 1, 2, ..., i^*$ to vote for a regardless of the state where where $i^* \geq K$ and it is determined in (14). With complementary probability, the sender uses a pivotal signal as in Proposition 3 to persuade voter 1,2,..., i^*. Voter i cannot distinguish the manipulation signal from a pivotal signal when her individual recommendation $s_i = a$ for $i = 1, 2, ..., i^*$. In the former case, she is non-pivotal provided that other voters $i' \neq i$ and $i' = 1, 2, ...i^*$ vote for a; while in the latter case, she weakly prefers to vote for a as the signal is sufficiently convincing. Hence, she is willing to follow recommendation a as long as she is pivotal with positive probability. As a result, the persuasion probability is higher than $1 - \epsilon$. By sending ϵ to zero, the persuasion probability converges to one.

7 Conclusion

In numerous organizations and committees, aggregating members' private information is a key to improving the quality of group decisions. Members' private information can arise from two sources: they may actively acquire different information privately, or they may passively receive heterogenous information from another party. The previous literature primarily emphasizes the first channel while our paper rationalizes the second one. In some natural settings, the sender finds it optimal to provide discriminatory information to group members to create uncertainty about the realized pivotal event. By doing so, he can manipulate the information aggregation of voters and successfully persuade them to approve his preferred action with higher probability. It is well known that political campaigns use "big data" to target individual voters and customize the online ads the voters receive.[7] A previous explanation assumes that voters have heterogenous preferences over outcomes.[8] To efficiently persuade them, the sender has to selectively provide information (Hoffmann, Inderst, and Ottaviani (2014)). Our paper provides a new justification for personalized ads that does not rely on voters' heterogenous preferences when information is complete.

As we demonstrated, the sender benefits from the failure of information aggregation among voters. A natural question is whether such a benefit vanishes in large elections. In an informative voting model with exogenous private information, Feddersen and Pesendorfer (1997) show that information fully aggregates in large elections. In our setting, as the population structure changes, the sender modifies the optimal signal accordingly. In a large economy, it seems that the sender has more flexibility to create correlation among the private signals of voters. We will leave this issue for future research.

[7] See West (2013), L. Gordon Crovitz, "How Campaigns Hyper-target Voters Online," Wall Street Journal, Nov 4, 2012; and Tanzina Vega, "Online Data Helping Campaigns Customize Ads," New York Times, Feb. 20, 2012.

[8] Namely, agents have different preference under complete information. For example, some voters want to match the state while other want to mismatch it.

A Appendix

Proof of Lemma 2. In any incentive-compatible signal, $\sum_{s\in S^*_{i,a}}\pi(s|A)\geq l_i\sum_{s\in S^*_{i,a}}\pi(s|B)$ for each $i=1,2,....N$ by constraint (IC-a). Hence, for each i, there exists at least one pivotal signal $s\in S^*_{i,a}$ such that $\pi(s|A)\geq l_i\pi(s|B)$. By the monotonicity constraint (MLRP), for any $s'\in S_a\setminus S^*_a$ and there exists a pivotal $s''\in S^*_{i,a}$, $\pi(s'|A)\pi(s''|B)\geq\pi(s''|A)\pi(s'|B)$. Hence, by increasing $\pi(s''|\omega)$ by $\pi(s'|\omega)$ for $\omega=A,B$, the sender can weakly relax voter i's incentive-compatible constraint where s' appears without affecting other voters'. □

To prove Proposition 2, we start with the assumption that the persuasion probability is less than one in state B and characterize the corresponding solution, then we find the condition under which the persuasion probability is less than one in state B to complete the proof. The following two lemmas characterize the optimal solution of (2) by assuming the persuasion probability is less than one.

Lemma 8. *In the optimal solution, constraints (3-5) are binding.*

Proof. Suppose that (5) is slack, then either $\pi(aba|A)$ or $\pi(baa|A)>0$. In the former case, one can reduce $\pi(aba|A)$ by ϵ, and increase $\pi(aab|A)$ by ϵ so that constraint (3)-(5) are all slack. Then one can increase the persuasion probability by increasing $\pi(aab|B)$. In the latter case, similar logic applies. Hence, in the optimal solution, (5) is binding. One can use the same logic to show that constraint (3) and (4) are both binding. □

Lemma 9. *In the optimal solution,*

1. $\pi(baa|B)\pi(baa|A)=0$*, and*

2. $\pi(aba|B)\pi(aba|A)=0$*.*

Proof. First, suppose not and $\pi(baa|B),\pi(baa|A)>0$. In this case, one can

1. decrease $\pi(baa|B)$ by ϵ and $\pi(baa|A)$ by $l_1\epsilon$,

2. increase $\pi(aab|B)$ by ϵ and $\pi(aab|A)$ by $l_1\epsilon$ where $\epsilon>0$

When ϵ is sufficiently small, it leads to the persuasion probability being unchanged and constraint (5) being slack. By lemma 8, this is suboptimal, so $\pi(baa|B)\pi(baa|A)=0$ is untrue in any optimal persuasion. Similarly, in the optimal persuasion, $\pi(aba|B),\pi(aba|A)>0$ cannot hold.

□

By Proposition 1, public persuasion is strictly suboptimal, so $\pi(baa|B) = \pi(aba|B) = 0$ *is untrue, and* $\pi(baa|A) = \pi(aba|A) = 0$ *is untrue. As a result,*

Corollary 2. *In the optimal solution,*

1. *either* $\pi(baa|B) = \pi(aba|A) = 0$*, or*

2. $\pi(aba|A) = \pi(baa|B) = 0$.

Proof of Proposition 2. By Corollary 2, to search for the optimal persuasion, one can focus on two cases. We start with the case where $\pi(baa|B) = \pi(aba|A) = 0$. Since constraints (3-5) are binding, one can solve the optimal solution of this case by choosing two variables: $\pi(aab|A)$ and $\pi(baa|A)$. The solution is given by

$$\pi(baa|A) = \frac{l_3 l_2 - l_3 l_1}{l_1 l_2 + l_2 l_3} \quad ; \quad \pi(aab|A) = \frac{l_1 l_3 + l_1 l_2}{l_1 l_2 + l_2 l_3};$$
$$\pi(aba|B) = \frac{l_2 - l_1}{l_1 l_2 + l_2 l_3} \quad ; \quad \pi(aab|B) = \frac{1}{l_2}$$

and the persuasion probability in state B is $\frac{l_3+l_2}{l_2(l_1+l_3)}$, which is positive for any $\{l_i\}_i$. In the case where $\pi(aba|A) = \pi(baa|B) = 0$, one can solve the optimal signal as well, but the persuasion probability in state B is 0. Hence, the solution in the first case admits the optimal signal of the original problem. Simple algebra implies that

1. the persuasion probability in state B is less than one if $l_1 > l_3(1/l_2 - 1) + 1$.

2. $\frac{l_3+l_2}{l_2(l_1+l_3)}$ is strictly decreasing in $l_i, i = 1, 2, 3$.

Hence, when $\frac{l_3+l_2}{l_2(l_1+l_3)} \leq 1$, the above signal is the optimal solution. In the case where $\frac{l_3+l_2}{l_2(l_1+l_3)} > 1$, one can find (l'_1, l'_2, l'_3) such that $l_i \leq l_i$ for $i = 1, 2, 3$ and $\frac{l'_3+l'_2}{l'_2(l'_1+l'_3)} = 1$. In the model where voters' preferences are characterized by $\{l'_i\}$, denote the optimal signal $\{\pi(\cdot|\omega)'\}_{\omega=A,B}$ and the persuasion probability is one. Obviously, $\{\pi(\cdot|\omega)'\}_{\omega=A,B}$ is also optimal in the original model with voters' preference $\{l_i\}$.

□

Proof of Lemma 3. We prove part 1. The argument for part 2 is identical. Let $h = C^N_K$ be the number of pivotal signals. It is convenient to state the proposition in matrix form.

Let $s^{[1]}, ..., s^{[h]}$ be an order of the pivotal signals for voting a. Let $\theta = (1-\alpha_1, ..., 1-\alpha_n)$. Let W be a $N \times h$ matrix with

$$W_{ij} = \begin{cases} 1 & if \quad s^{[i]}(j) = b, \\ 0 & if \quad s^{[i]}(j) = a. \end{cases}$$

We need to show that there is an h-th vector $\pi_A = (\pi_{1A}, ..., \pi_{hA})$ such that $\sum_i \pi_{iA} = 1$ and $\pi_{iA} \in [0,1]$, and

$$W\pi_A^T = \theta^T.$$

Suppose by way of contradiction that no such π_A exists. By Farkas' lemma, there exists an n-th vector $\lambda = (\lambda_1, ..., \lambda_N)$ such that

$$W_{ij}^T \lambda^T \geq 0; \quad (16)$$

$$\theta\lambda^T < 0. \quad (17)$$

Note that the row of W_{ij}^T that corresponds to the signal profile where players 1 to $N-K$ observe b begins with $N-K$ ones followed by K zeros. Thus, (16) implies that

$$\sum_{i=1}^{N-K} \lambda_i \geq 0.$$

Since the player ordering is arbitrary, we can assume without loss of generality that λ_i is ascending in i. Hence

$$\begin{aligned} & \min_{x_i} \sum_{i=1}^{N} \lambda_i x_i \text{ s.t. } x_i \in [0,1]\, \forall i, \ \sum_{i=1}^{N} x_i = N-K \ , \\ = & \sum_{i=1}^{N-K} \lambda_i \geq 0. \end{aligned}$$

Since $\alpha_i \in [0,1]$ for all i, and $\sum_{i=1}^{N} x_i = N-K$,

$$\sum_{i=1}^{N} \lambda_i (1-\alpha_i) \geq 0,$$

which contradicts (17). □

Proof of Lemma 4. Suppose not. In the optimal persuasion, $\alpha_i = Q_A = 1, \beta_i = 0, i = 1, 2, ..., K$ and $\alpha_i = 0, \beta_i = Q_B, i > K$, there is only one pivotal event. As $l_1 < l_K$, to satisfy voter K's incentive-compatible constraint, voter 1's incentive-compatible constraint must be slack. Other pivotal voters i' incentive-compatible constraint are satisfied because $l_i \leq l_K, i < K$. For non-pivotal voters, their incentive-compatible constraint are also trivially satisfied. Also, it is obvious that the feasibility constraints for α and β are satisfied:

$$\sum_{i=1}^{K} \alpha_i = K, \sum_{i=K}^{N} \alpha_i = 0, \sum_{i=1}^{K} \beta_i = 0, \sum_{i=K}^{N} \beta_i = (N-K)Q_B$$

and $Q_B < 1$.

Now consider the following procedure.

1. Increase α_1 by ϵ so that voter 1's incentive-compatible constraint is still slack.
2. Decrease α_i by ϵ/K for $i = 2, 3, ...K+1$ so that $\sum_{i=1}^{N} \alpha_i = K$ and voter i's incentive-compatible constraints are slack.
3. One can increase Q_B by fixing $\beta_i, i \leq K+1$ and increasing $\beta_i, i > K+1$ accordingly.

As a result, it is suboptimal to convince $i = 1, 2, ...K$ only.

□

Proof of Lemma 5. The Lagrangian function of (P-3) is

$$\mathcal{L} = \sum_{i=1}^{n} \left(Q_B - \frac{\alpha_i}{l_i} \right) + \rho \left(\sum_{i=1}^{n} \alpha_i - k \right) - \sum_{i=1}^{n} \psi_i \alpha_i + \sum_{i=1}^{n} \xi_i \left(\alpha_i - \min\left(l_i Q_B, 1\right) \right). \tag{18}$$

The Kuhn-Tucker conditions for α_i are that

$$\frac{\partial \mathcal{L}}{\partial \alpha_i} = -\frac{1}{l_i} + \rho - \psi_i + \xi_i = 0 \ \forall i,$$

and ρ, ψ_i, and ξ_i be positive (zero) if the corresponding constraints are binding (non-binding). Hence, for all i,

$$\alpha_i = \begin{cases} 0 & if \quad l_i\rho > 1, \\ [0, \min\left(l_i Q_B, 1\right)] & if \quad l_i\rho = 1, \\ \min\left(l_i Q_B, 1\right) & if \quad l_i\rho < 1. \end{cases} \tag{19}$$

Let $i^* = \max\{i \leq N | \sum_{j=1}^{i} \min\{l_j Q_B, 1\} < K\}$ and set $\rho = \frac{1}{li^*+1}$. Because l_i is increasing,

$$\rho l_i < 1 \quad \text{if} \quad i \leq i^*,$$
$$\rho l_i > 1 \quad \text{if} \quad i > i^* + 1$$

Hence, $\{\alpha_i^*\}_{i=1,2,...N}$ satisfies (13).

Let $U(Q_B)$ denote the solution to P3. Further define $i^{**} = \max\{i \le N | l_i Q_B \le 1\}$, and let $\hat{i} = \min\{i^*, i^{**}\}$. By the envelope theorem, we have

$$\begin{aligned}
\frac{dU}{dQ_B} &= \frac{\partial \mathcal{L}}{\partial Q_B} = N - \sum_{i=1}^{\hat{i}} \left(\frac{1}{l_i} - \rho\right) l_i \\
&= N - \sum_{i=1}^{\hat{i}} \left(1 - \frac{l_i}{l_{\hat{i}}}\right) > N - \hat{i} \ge 0.
\end{aligned}$$

□

Proof of Lemma 6. Recall that $l_K > 1$. When $Q_B = 1/l_K$, we can set

$$\alpha_i = \begin{cases} 0 & if \quad i > K, \\ \frac{l_i}{l_K} & if \quad i \le K, \end{cases}$$

so that the objective function takes the value $(N-K)\frac{1}{l_K}$ and all constraints are satisfied. In particular,

$$\sum_{i=1}^{N} \alpha_i = \sum_{i=1}^{K} \frac{l_i}{l_K} < K$$

and $\alpha_i < \min\{l_i Q_B, 1\}, \forall i > K$. Hence, one can raise $\alpha_i, i > K$ to increase the value of the objective function. Hence, when $Q_B = 1/l_K$,

$$U(Q_B) > (N-K)Q_B$$

Furthermore, when $Q_B > \frac{1}{l_K}$, $\hat{i} < K$, so $\sum_{i=1}^{\hat{i}}(1 - \frac{l_i}{l_K}) < K$

$$\frac{dU}{dQ_B} > N - K$$

so there is at most one solution for $U(Q_B) = (N-K)Q_B$ for $Q_B \in [\frac{1}{l_K}, 1]$. There are two cases: **Case 1.** $\exists Q_B^* \in [\frac{1}{l_K}, 1]$ such that $U(Q_B^*) = (N-K)Q_B^*$. In this case, $U(Q_B) = (N-K)Q_B$ is necessary and sufficient for the feasibility of Q_B in problem (P-2). The necessary part is obvious. For the sufficiency part, note that when $U(Q_B) < (N-K)Q_B$, we can raise β_i so that $\sum_i \beta_i = (N-K)Q_B$ without affecting other constraints. Since dU/Q_B is strictly greater than $(N-K)$ for all $Q_B > 1/l_K$, the optimal Q_B in problem (P-1) is $\min(1, Q_B^*)$. Finally when $Q_B^* < 1$, the constraint $U(Q_B^*) = (N-K)Q_B^*$, and hence the optimal α_i^* must satisfy (13) and the optimal β_i equals $Q_B^* - \alpha_i^*/l_i$. **Case 2.** $U(Q_B) < (N-K)Q_B, \forall Q_B \in [\frac{1}{l_K}, 1]$. Set $Q_B^* = 1$, an optimal solution consists of $\alpha^* = \hat{\alpha}(1)$ according to (13) and $\beta^* = 1 - \alpha_i^*/l_i$.

□

Proof of Lemma 7. By (11) and (10), for any feasible $(Q_B, \{\alpha_i, \beta_i\}_i)$, $NQ - \sum_{i=1,2,..N} \beta_i = KQ \leq \sum_{i=1,2,..N} \frac{\alpha_i}{l_i}$. When $l_i > 1, \forall i$, $\sum_{i=1,2,..N} \frac{\alpha_i}{l_i} < \sum_i \alpha_{i=1,2,..N} \leq K$, so the result immediately follows. □

Proof of Proposition 4. By the proof of Lemma 6, when $Q_B^* \in (0,1)$,

$$U(Q_B^*) = (N-K)Q_B^*. \tag{20}$$

Differentiating it with respect to K yields

$$\frac{\partial Q_B^*}{\partial K} = \frac{Q_B^*}{N - K - U'(Q_B^*)} < 0$$

as $U'(Q_B) > N - K$.

By (18), we have

$$\frac{\partial U(Q)}{\partial l_i} = \frac{\partial \mathcal{L}}{\partial l_i} = \frac{\alpha_i}{l_i^2} + Q_B \xi_i \geq 0$$

As $\alpha_i = 0, \forall i > i^*$, $\xi_i = 0, \forall i > \hat{i}$ and $\hat{i} \leq i^*$, the inequality is strict only if $i \leq i^*$ or $\alpha_i^* > 0$.

Differentiating (20) with respect to l_i yields,

$$\frac{\partial Q}{l_i} = \frac{\partial U(Q_B^*)}{\partial l_i} \frac{1}{N - K - U'(Q_B^*)} < 0$$

if $\alpha_i^* > 0$. □

References

ALONSO, R., AND O. CAMARA *(2014): "Persuading voters,"* Working Paper.

AUMANN, R. J., AND M. MASCHLER *(1995):* Repeated games with incomplete information. *MIT press.*

AUSTEN-SMITH, D., AND J. S. BANKS *(1996): "Information aggregation, rationality, and the Condorcet jury theorem,"* American Political Science Review*, 90(01), 34–45.*

BROCAS, I., AND J. D. CARRILLO *(2007): "Influence through ignorance,"* The RAND Journal of Economics*, 38(4), 931–947.*

CAI, H. *(2009): "Costly participation and heterogeneous preferences in informational committees,"* The RAND Journal of Economics*, 40(1), 173–189.*

CAILLAUD, B., AND J. TIROLE *(2007): "Consensus building: how to persuade a group,"* The American Economic Review*, 97(5), 1877–1900.*

EKMEKCI, M., AND S. LAUERMANN *(2015): "Manipulated electorates and information aggregation,"* Working Paper.

FARRELL, J., AND R. GIBBONS *(1989): "Cheap talk with two audiences,"* The American Economic Review*, pp. 1214–1223.*

FEDDERSEN, T., AND W. PESENDORFER *(1997): "Voting behavior and information aggregation in elections with private information,"* Econometrica*, 65(5), 1029–1058.*

——— *(1998): "Convicting the innocent: The inferiority of unanimous jury verdicts under strategic voting,"* American Political Science Review*, 92(01), 23–35.*

GERARDI, D., AND L. YARIV *(2007): "Deliberative voting,"* Journal of Economic Theory*, 134(1), 317–338.*

——— *(2008): "Information acquisition in committees,"* Games and Economic Behavior*, 62(2), 436–459.*

GOLTSMAN, M., AND G. PAVLOV *(2011): "How to talk to multiple audiences,"* Games and Economic Behavior*, 72(1), 100–122.*

HOFFMANN, F., R. INDERST, AND M. OTTAVIANI *(2014): "Persuasion through selective disclosure: Implications for marketing, campaigning, and privacy regulation,"* Working Paper.

JACKSON, M. O., AND X. TAN *(2013): "Deliberation, disclosure of information, and voting,"* Journal of Economic Theory*, 148(1), 2–30.*

KAMENICA, E., AND M. GENTZKOW *(2011): "Bayesian persuasion,"* American Economic Review*, 101(6), 2590–2615.*

KOLOTILIN, A., M. LI, T. MYLOVANOV, AND A. ZAPECHELNYUK *(2015): "Persuasion of a privately informed receiver,"* Working paper.

LI, H. *(2001): "A theory of conservatism,"* Journal of Political Economy*, 109(3), 617–636.*

LI, H., S. ROSEN, AND W. SUEN *(2001): "Conflicts and common interests in committees,"* The American Economic Review*, 91(5), 1478–1497.*

MILGROM, P. R. *(1981): "Good news and bad news: Representation theorems and applications,"* The Bell Journal of Economics*, pp. 380–391.*

PERSICO, N. *(2004): "Committee design with endogenous information,"* The Review of Economic Studies*, 71(1), 165–191.*

RAYO, L., AND I. SEGAL *(2010): "Optimal information disclosure,"* Journal of Political Economy*, 118(5), 949–987.*

TANEVA, I. *(2014): "Information design,"* Working Paper.

WANG, Y. *(2012): "Bayesian persuasion with multiple receivers,"* Working Paper.

WEST, D. M. *(2013):* Air wars: Television advertising and social media in election campaigns, 1952-2012. *Sage.*