Mechanism Design by an Informed Seller

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Abstract

This paper studies the design of a selling mechanism by a privately informed seller of an indivisible object, whose private information directly affects the the buyers' valuations on the object. The analysis focuses on *safe mechanisms* (Myerson, 1983) which are incentive compatible and individually rational for all the players, regardless of the buyers' beliefs about the seller's type. Among this class of mechanisms, we characterize the *Rothschild-Stiglitz-Wilson (RSW) mechanism* (Maskin and Tirole, 1992) which maximizes the revenue of each type of the seller. The RSW mechanism only differs in reserve prices from the revenue-maximizing mechanism where the seller's information is public. Specifically, the lowest type of the seller has the same reverve prices in the two mehcanisms, while all other types of the seller set higher reserve prices in the RSW mechansim. This result indicates that for an informed seller, the reverse price is the least costly device of signaling. The paper also shows that the RSW mechanism can be supported as the the seller's equilibrium strategy in the seller-optimal separating equilibrium of the mechanism-selection game where the inscrutability principle (Myerson, 1983) does not apply.

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1 Introduction

The standard literature of mechanism design usually assumes that a mechanism designer has no private information relevant to the design of a mechanism. This assumption does not perfectly fit some situations involving the design of selling mechanisms. For example, when designing an artwork auction, the seller may have private information about some characteristics of the object, such as the evaluation of the object by a connoisseur and the easiness of reselling the objects; in keyword auctions, the auctioneers are usually better informed than the buyers about the frequencies that the keywords are searched; in a spectrum auction, the government's agenda of regulating the telecommunication industry, which is important for the bidders to set their bidding strategies, may be unknown to them at the time of bidding.

This paper departs from the standard literature, and studies the mechanism design problem facing a seller of an indivisible object, who learns a private signal regarding the quality of the object prior to choosing the selling mechanism. The valuations of the seller and buyers on the object are all increasing in this signal. It is obvious that the privacy of seller information induces a signaling consideration: the choice of mechanism by the seller may (partially) reveal her information to the buyers. This paper is devoted to studying how this signaling issue will change the design of the selling mechanism from the case where the seller has no private information.

This paper is mostly related to the seminal work of Myerson (1983) and Maskin and Tirole (1990, 1992) on mechanism design by an informed principal. Myerson (1983) lays down the foundation of analyzing this problem by developing several equilibrium concepts with different strengths. Maskin and Tirole (1990,1992) focus their analysis on the one-principal/one-agent situation. Maskin and Tirole (1992) consider the case that principal's private information *directly* affects the agent's utility, which is the same as the case considered in the current paper.

Mylovanov and Tröger have several papers on the informed-principal problem. Their work focuses on the private-value case in which the principal's information does not directly enter the agents' utility functions; it affects the payoffs of the agents through its effects on the principal's equilibrium behavior.

This paper is also closely related to the signaling literature. In signaling games, the privately informed sender takes a costly action which may partially or fully reveal her information in equilibrium. The structure of the informed-principal mechanism selection game is the same as that of the signaling game: the choice of mechanism may reveal the principal's information. The main difference is that the "action" in the signaling model is changed to "mechanism" in the informed-principal problem.

The signaling games developed in Jullien and Mariotti (2006) and Cai, Riley and Ye (2007) can be taken as semi-auction-selection games. In those games, the auction format is fixed, and the privately informed seller has freedom in setting the reserve price of the auction. Cai, Riley and Ye (2007) find that in this game, there is a unique separating equilibrium in which the lowest type of the seller sets the reverse price that same as that in the case where her information is publicly known, and other types set higher reserve prices compared with the public information case.

Our model setup is similar to that of Cai, Riley and Ye (2007), except that we endow the seller with the freedom of choosing every element of a mechanism, instead of only the reserve price. Our analysis focuses on the safe mechanisms (Myerson, 1983) which are mediated mechanisms that are incentive compatible and individually rational for all the players, regardless of the buyers' beliefs about the seller's type. I give a full characterization of the set of safe mechanisms, and derive the Rothschild-Stiglitz-Wilson (RSW) mechanism (Maskin and Tirole, 1992) which is the safe mechanism maximizing the revenue of each type of the seller among all the safe mechanisms. The main finding of the paper is that the RSW mechanism only differs in reserve prices from the revenue-maximizing mechanism where the seller's information is public. Specifically, the lowest type of the seller has the same reserve prices in the two mehcanisms, while all other types of the sellers set higher reserve prices in the RSW mechanism. This result indicates that for an informed seller, the reverse price is the least costly device of signaling.

This result is in fact intuitive. Given that buyers' valuations on the object are increasing in the seller's type, a low type seller would like to have the buyers believe that she is of some higher type, so that she has a chance to sell the object at a higher price. There are two ways to disincentivize a lower type seller to mimic a higher type: (1) increasing the reserve price in the mechanism adopted by the high-type seller, thus decreasing the probability of selling the object, and (2) decreasing the payment of the buyers in the mechanism chosen by the high type seller, thus reducing the revenue of the lower type seller from mimicking the higher type. Both channels can work independently to eliminate the "cheating benefits" of a lower type seller, and separate a higher type from lower types. However, the first approach is better from the perspective of a high type seller, because it increases the probability that the seller to keeps the object, which makes the higher type seller lose less than the lower type, as the higher type values the object more.[1] The second approach induces the same loss to both types of the seller, as it is equivalent to letting the higher type seller to give up the same amount of revenue as the low type.

The paper also shows that the RSW mechanism can be supported as the seller's equilibrium strategy in the seller-optimal separating equilibrium of the mechanism-selection game where the inscrutability principle (Myerson, 1983) does not apply. Combined with the characterization of the RSW mechanism, we will see that this result provides a support for the literature on reserve price signaling (Jullien and Mariotti, 2006; Cai, Riley and Ye, 2007): the seller only uses the reserve price to signal herself, even if she has the freedom to vary the whole mechanism.

This paper is organized as follows: section 2 sets up the models; section 3 discusses mechanism design with a mediator, and focuses on showing how the full-information optimal mechanism fails

[1] When the reserve prices are increased, the payments of the buyers become higher in the case that there is only one buyer having a type higher than his reserve price. However, compared with the effect of reserve prices on the probability of trading, this is second order.

to be incentive compatible and characterizing the RSW mechanism; section 4 analyzes mechanism design when the principle of instrutability fails, and tries to connect equilibria of this game with the set of safe mechanisms.

2 Model Setup

The setup of the model is standard. There is an indivisible object for sale. The owner of the object would like to design a revenue-maximizing auction mechanism and sell the object to n potential buyers.

The seller *privately* observes a signal s which determines her valuation $v_0(s)$ of the object. We assume that $v_0(s)$ is increasing in s and twice continuously differentiable with bounded derivatives. It is common knowledge for all the players that s is drawn from the distribution $f_0 : [\underline{s}, \bar{s}] \mapsto R_{++}$. For any buyer $i = 1, 2, \ldots, n$, his valuation $v_i(s, t_i)$ for the object depends on the signal s of the seller and the private signal t_i of his own, and is increasing and twice continuously differentiable with bounded derivatives in both signals. t_i is drawn from distribution $f_i : [\underline{t}_i, \bar{t}_i] \mapsto R_{++}$, which is common knowledge. Signals $s, t_1, t_2, \ldots, t_n$ are all independent. To simplify notations, we define $t \equiv (t_1, t_2, \ldots, t_n)$ and use S, T_i, and T to denote $[\underline{s}, \bar{s}]$, $[\underline{t}_i, \bar{t}_i]$, and $\times_{i=1}^{n} [\underline{t}_i, \bar{t}_i]$, respectively.

The seller chooses an auction mechanism after she learns her private signal. The most of our analysis will focus on incentive feasible direct revelation mechanism, according to the revelation principle. A direct revelation auction mechanism M consists of an allocation function $x : S \times T \to [0,1]^n$ and a payment function $p : S \times T \to R^n$. Specifically,

$$\begin{aligned} x(s,t) &= (x_1(s,t), x_2(s,t), \ldots, x_n(s,t)), \\ p(s,t) &= (p_1(s,t), p_2(s,t), \ldots, p_n(s,t)), \end{aligned}$$

where $x_i(s,t)$ and $p_i(s,t)$ are respectively buyer i's probability of getting the object and payment under (s,t). $x(s,t)$ should satisfy the feasibility constraint

$$\textstyle\sum_{i=1}^{n} x_i(s,t) \leq 1 \text{ and } x_i(s,t) \geq 0, \text{ for } \forall s, t. \tag{1}$$

We use $x_0(s,t)$ in this paper to denote the probability that the seller keeps the object, i.e., $x_0(s,t) = 1 - \sum_{i=1}^{n} x_i(s,t)$ for $\forall s, t$. By abusing notations a little bit, we define $x_i(s,t_i) = \int_{T_{-i}} x_i(s, t_i, t_{-i}) f_{-i}(t_{-i}) dt_{-i}$, $p_i(s,t_i) = \int_{T_{-i}} p_i(s, t_i, t_{-i}) f_{-i}(t_{-i}) dt_{-i}$, and $x_0(s) = \int_T x_0(s,t) f(t) dt$, where $t_{-i} = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$, $T_{-i} = \times_{j \neq i}^{n} T_j$ and $f_{-i}(t_{-i}) = \prod_{j \neq i}^{n} f_j(t_j)$.

In the rest of our analysis, we say that a direct mechanism is *incentive feasible* if and only if the feasibility condition (1) and all the incentive compatibility (IC) and individual rationality (IR) constraints of the players are satisfied. The specification of IC and IR constraints of a direct mechanism depends on whether the mechanism design game is mediated or not. In the following section, we first consider the mediated case.

3 Mechanism Design with a Mediator

In a mediated mechanism design game, the informed seller first selects a mechanism which will be executed by an interest-neutral, non-strategic mediator. Under the selected mechanism, all players including the seller take strategies in their strategy sets specified by the mechanism. The allocation of the object and the payment of each player are realized based on the strategy profile of the players. Since a mechanism is selected after the seller learns about her type, the choice of mechanism may partially or completely reveal her information. Thus, on observing the selected mechanism, the buyers may update their belief about the seller's type.

For any selected mechanism and an associated (common) posterior of the buyers about the seller's type, according to the revelation principle, there exists a direct revelation mechanism which is incentive feasible under the same posterior and yields the same payoffs for all the players. Suppose that a direct mechanism M is selected, and the posterior of the buyers about the seller's type is $f_0(\cdot|M)$. If M is incentive feasible, it should satisfy condition (1), and the seller with any type $s \in S$ should have incentive to tell the truth, i.e.,

$$U_0(M|s) \geq U_0(M, s'|s), \text{ for } \forall s, s' \in S, \text{ where} \tag{2}$$

$$\begin{aligned} U_0(M|s) &= \int_T \left\{ v_0(s)\, x_0(s,t) + \sum_{i=1}^{n} p_i(s,t) \right\} f(t)\, dt, \text{ and} \\ U_0(M, s'|s) &= \int_T \left\{ v_0(s)\, x_0(s',t) + \sum_{i=1}^{n} p_i(s',t) \right\} f(t)\, dt, \ f(t) = \prod_{i=1}^{n} f_i(t_i). \end{aligned} \tag{3}$$

$U_0(M|s)$ is the expected truth-telling payoff of any type-s seller, $U_0(M, s'|s)$ is the expected payoff of misreporting her type as s'.

For buyer i, incentive feasibility of M requires that given that all other players report their types truthfully, he would like to participate and report his type truthfully, i.e.,

$$\begin{aligned} U_i(t_i|M) &\geq U_i(t_i', t_i|M), \text{ and} \\ U_i(t_i|M) &\geq 0 \qquad \text{for } \forall t_i, t_i', \text{ where} \end{aligned} \tag{4}$$

$$\begin{aligned} U_i(t_i|M) &= \int_S [v_i(s,t_i)\, x_i(s,t_i) - p_i(s,t_i)]\, f_0(s|M)\, ds, \text{ and} \\ U_i(t_i', t_i|M) &= \int_S [v_i(s,t_i)\, x_i(s,t_i') - p_i(s,t_i')]\, f_0(s|M)\, ds. \end{aligned} \tag{5}$$

$U_i(t_i|M)$ is buyer $i's$ expected payoff from telling the truth, $U_i(t_i', t_i|M)$ is that from misreporting his type as t_i'.

According to Milgrom and Segal (2002), we can rewrite the constraints (2) and (4). The results are summarized in the following lemma. The proof of this lemma is put in Appendix A, as the method used in the proof is standard in the literature of mechanism design.

Lemma 1 *For any direct mechanism M and the associated posterior $f_0(\cdot|M)$ of the buyers, we can derive that*

1. *IC constraint* (2) *for the seller holds if and only if*

$$x_0(s) \geq x_0(s') \quad \text{if } s \geq s', \forall s, s' \in S \tag{6}$$

$$U_0(M|s) = \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0(\tilde{s})\, d\tilde{s} + U_0(M|\underline{s}),\ \forall s \in S. \tag{7}$$

2. *Buyers' IC and IR constraints in* (4) *hold if and only if*

$$\int_{t_i'}^{t_i} \left\{ \int_S v_{i2}'\left(s, \tilde{t}_i\right) \left[x_i\left(s, \tilde{t}_i\right) - x_i\left(s, t_i'\right)\right] f_0(s|M)\, ds \right\} d\tilde{t}_i \geq 0,\ \forall t_i, t_i' \in T_i, \tag{8}$$

$$U_i(t_i|M) = \int_{\underline{t}_i}^{t_i} \left[\int_S v_{i2}'\left(s, \tilde{t}_i\right) x_i\left(s, \tilde{t}_i\right) f_0(s|M)\, ds \right] d\tilde{t} + U_i(\underline{t}_i|M),\ \forall t_i \in T_i, \tag{9}$$

$$U_i(\underline{t}_i|M) \geq 0\ . \tag{10}$$

As one can see, different from the classical mechanism design problems, to make a mechanism with an informed principal incentive feasible, it requires that the probability of the seller keeping this object be nondecreasing with her type (condition (6)) and expect payoff of the seller be in a certain structure (condition (7)).

This lemma can also be applied to the interdependent value case in which every player's valuation for the object depends on all other buyers' private information in the form that $v_0(s,t) = u_0(s) + \varphi_0(t)$ and $v_i(s,t) = u_i(s,t_i) + \varphi_i(t_{-i})$. If $u_i(s,t_i)$ is in additive form, i.e., $u_i(s,t_i) = g_i(s) + h_i(t_i)$, then condition (8) can be simplified to

$$\int_S x_i(s,t_i)\, f_0(s|M)\, ds \geq \int_S x_i(s,t_i')\, f_0(s|M)\, ds \text{ for } t_i \geq t_i'.$$

That is, the expected probability of buyer i getting this object is increasing in t_i.

3.1 Inscrutability Principle

The game of mechanism design with an informed seller has the feature of signaling games that the choice of mechanism may signal to the buyers about the seller type. This signaling feature potentially complexes our analysis dramatically. However, thanks to the *principle of inscrutability* introduced in Myerson (1983), in analyzing seller's revenue-maximizing mechanisms, we can with loss of generality assume that all types of the seller in equilibrium propose the same mechanism, thus the choice of mechanism reveals no private information of the seller in the mediated mechanism design game. This principle can be formally stated in the following lemma.

Lemma 2 *(Inscrutability Principle, Myerson (1983)) For any equilibrium of the mechanism design game with an informed seller, there exists an equilibrium in which all types of the seller choose the same incentive feasible mechanism and get the same payoffs as in the original equilibrium.*

In this subsection, we prove this principle for the case in which every type of seller plays a pure strategy in the mechanism selection stage. The proof for the case of mixed strategy can be found in Appendix B.

Proof. Consider a partition $\left\{S^l\right\}_{l\in I}$ of S, where I is a set of index that can be finite or infinite. $\left\{S^l\right\}_{l\in I}$ is defined in a way that different types of the seller in the same element S^l choose the same mechanism M^l, but the mechanisms across elements are different, and the seller in S^l has no incentive to change his mechanism to one corresponding to $S^{l'}$, $l' \neq l$.[2] Thus, by choosing mechanism M^l, the seller signals the buyers that her type belongs to S^l. According to the revelation principle, we can assume that M^l is a direct incentive feasible mechanism without loss of generality. That is, M^l satisfies (1), (2), and (4). The self-signaling property of the partition and the incentive compatibility of M^l imply that

$$U_0\left(M^l|s\right) \geq U_0(M^{l'}|s), \text{ for } s \in S^l, l' \neq l. \tag{11}$$

We can construct an inscrutable mechanism M in the following way,

$$\begin{cases} x\left(s,t\right) = x^l\left(s,t\right) \\ p\left(s,t\right) = p^l\left(s,t\right) \end{cases}, \text{ if } s \in S^l, l \in I \tag{12}$$

where $x^l\left(s,t\right)$ and $p^l\left(s,t\right)$ are the allocation function and payment function of mechanism M^l, respectively. It is clear that if every type of the seller chooses the mechanism M, IC constraints for the buyers still hold, because

$$\begin{aligned} U_i(t_i|M) &= \int_{l\in I}\left[U_i(t_i|M^l)\cdot\int_{S^l} f_0\left(s\right)ds\right]dl \\ &\geq \int_{l\in I}\left[U_i\left(t_i', t_i|M^l\right)\cdot\int_{S^l} f_0\left(s\right)ds\right]dl = U_i\left(t_i', t_i|M\right), \end{aligned}$$

[2] If the seller with her type in one element of the partition would like to deviate to the mechanism corresponding to another element of the partion, then we can redefine the partition.

where the first and last equalities are from the definition of M, and the inequality is derived using the incentive compatibility of M^l.[3] The IR constraints can be verified similarly.

The construction of M immediately implies that any type of the seller has no incentive to misreport her signal due to the incentive compatibility of M^l, $l \in I$, and (11). By the truthfully reporting her type, the seller gets the same payoff as in the original strategy $\{M^l\}_{l \in I}$. ■

Given this principle, in the rest of our analysis of the mediated game, we can focus on the case in which all types of the seller choose the same mechanism. This means that on the equilibrium path, the posterior of the buyers after observing the selected mechanism can always be assumed to be the same as their prior.

3.2 Full-information Optimal Mechanism

As we move from the full-information case in which the seller's signal is public knowledge to the case of informed seller, a natural question for one to ask is whether the full-information optimal mechanism is incentive feasible in the informed-seller case. We address this question in this subsection and study how the privacy of the seller's information changes the seller's problem.

In the full-information case, the seller is a mechanism designer, but not a player in a mechanism. Each type $s \in S$ of the seller chooses the mechanism solving the following problem,

$$\max_{x(s,t)\ ,p(s,t)} \int_T \left\{ v_0(s)\, x_0(s,t) + \sum_{i=1}^{n} p_i(s,t) \right\} f(t)\, dt$$

$$\text{s.t.} \qquad U_i(t_i|s,M) \geq U_i(t_i', t_i|s,M), \text{ and } U_i(t_i|s,M) \geq 0, \text{ for } \forall t_i, t_i' ,$$

$$\text{Feasibility Condition (1)},$$

where $U_i(t_i|s,M) = v_i(s,t_i)\, x_i(s,t_i) - p_i(s,t_i)$ and $U_i(t_i', t_i|s,M) = v_i(s,t_i)\, x_i(s,t_i') - p_i(s,t_i')$. According to Milgrom and Segal (2002), the IC and IR constraints can be replaced by the following conditions

$$x_i(s,t_i) \geq x_i(s,t_i'), \text{ if } t_i \geq t_i' \ , \tag{13}$$

$$U_i(t_i|s,M) = \int_{\underline{t}_i}^{t_i} v_{i2}'\left(s,\tilde{t}_i\right) x_i\left(s,\tilde{t}_i\right) d\tilde{t} + U_i\left(\underline{t}_i|s,M\right) \ , \tag{14}$$

$$U_i\left(\underline{t}_i|s,M\right) \geq 0 \ . \tag{15}$$

Solving the $p_i(s,t_i)$ from (14), substituting it into the objective function, and using integration

[3] This argument for the buyers shows that moving from the informative strategy $\{M^l\}_{l \in I}$ to the inscrutable strategy M, the interim expected payoffs of the buyers may be changed. But their *ex post* payoffs are unchanged.

by parts, we can obtain

$$\max_{x(s,t)} v_0(s) + \int_T \sum_{i=1}^{n} \left[J_i(s,t_i) - v_0(s)\right] x_i(s,t) f(t)\, dt - \sum_{i=1}^{n} U_i(\underline{t}_i | s, M)$$
$$\text{s.t.} \quad (13), \ (15), \ \text{and} \ (1).$$

$J_i(s,t_i)$ is the virtual valuation of buyer i with signal t_i and has expression

$$J_i(s,t_i) = v_i(s,t_i) - \frac{1-F_i(t_i)}{f_i(t_i)} v'_{i2}(s,t_i), \tag{16}$$

where $v'_{i2}(s,t_i)$ is the first order derivative of $v_i(s,t_i)$ with respect to t_i. To simplify our analysis, throughout this paper, we put the following assumption which is satisfied for some commonly used functional forms of v_i when the hazard rate $f_i(t_i)/[1-F_i(t_i)]$ is nondecreasing in t_i, for example, the linear form $v_i(s,t_i) = \alpha s + \beta t_i, \alpha, \beta > 0$ and the multiplicative form $v_i(s,t_i) = u(s) \cdot t_i$.

Assumption 1 $J_i(s,t_i) = v_i(s,t_i) - \frac{1-F_i(t_i)}{f_i(t_i)} v'_{i2}(s,t_i)$ *is increasing in* t_i, *and* $J_i(s,\bar{t}_i) > v_0(s)$, *for* $\forall i, \forall s \in S$.

Under this assumption, the optimal allocation rule $x^F : S \times T \to [0,1]^n$ can be characterized as

$$\begin{cases} x_i^F(s,t) > 0, \text{only if } J_i(s,t_i) \geq \max\left\{\max_{k \neq i}\{J_k(s,t_k)\}, v_0(s)\right\}, \\ \sum_{i=1}^{n} x_k^F(s,t) = 1, \text{if } \max_k \{J_k(s,t_k)\} \geq v_0(s). \end{cases} \tag{17}$$

The superscript F is to indicate that it is of the full-information optimal mechanism. One can find that x^F automatically satisfies the monotonicity constraint (13). The expected payoff of each buyer with the lowest type in optimality is 0, i.e., $\sum_{i=1}^{n} U_i(\underline{t}_i|s,M) = 0$, which can be immediately seen from the seller's maximization problem above. The optimal payment rule $p^F : S \times T \to R^n$, which is not unique, can be solved from (14). In the rest of the paper, we use M^F to denote the full-information optimal mechanism composed of $\left(x^F, p^F\right)$.

To proceed, let $U_0^F(s)$ be the optimal expected payoff of type-s seller in the full-information case, i.e.,

$$\begin{aligned} U_0^F(s) &= v_0(s) + \int_T \sum_{i=1}^{n} \left[J_i(s,t_i) - v_0(s)\right] x_i^F(s,t) f(t)\, dt \\ &= v_0(s) x_0^F(s) + \int_T \sum_{i=1}^{n} J_i(s,t_i) x_i^F(s,t) f(t)\, dt. \end{aligned} \tag{18}$$

We define $g(s,x) \equiv v_0(s) x_0 + \int_T \sum_{i=1}^{n} J_i(s,t_i) x_i(t) f(t)\, dt$ in which $x_0 = 1 - \int_T \sum_{i=1}^{n} x_i(t) f(t)\, dt$, and let $g'_1(s,x)$ and $J'_{i1}(s,t_i)$ be the partial derivatives of g and J_i with respect to s, respectively. It is easy to verify that $g(s,x)$ is absolutely continuous and differentiable with respect to s for any feasible allocation rule x, and since the derivatives of v_0 and J_i are all bounded, there exists a large number b such that

$$\sup_{x(t)} |g_1'(s,x)| = \sup_{x(t)} \left| v_0'(s)\, x_0 + \int_T \sum_{i=1}^{n} J_{i1}'(s,t_i)\, x_i(t)\, f(t)\, dt \right| \leq b \text{ for } \forall s.$$

According to Theorem 1 and Theorem 2 of Milgrom and Segal (2002), we can derive that

$$U_0^F(s) = \int_{\underline{s}}^{s} g_1'\left(\tilde{s}, x^F(\tilde{s})\right) d\tilde{s} + U_0^F(\underline{s}), \tag{19}$$

in which $x^F(s,\cdot) : T \rightarrow [0,1]^n$ represents the full-information optimal allocation rule for the type-s seller, and

$$g_1'\left(s, x^F(s,\cdot)\right) = v_0'(s)\, x_0^F(s) + \int_T \sum_{i=1}^{n} J_{i1}'(s,t_i)\, x_i^F(s,t)\, f(t)\, dt. \tag{20}$$

If the seller's signal becomes private, is the full-information optimal mechanism M^F incentive feasible in the mediated game? To answer this question, we check the IC and IR constraints for all the players. For each buyer, one can see that (13), (14), and (15) are sufficient for (8), (9), and (10), regardless of the posterior of the buyers. Thus, if all other players report truthfully, it is incentive compatible and individually rational for a buyer to report his type truthfully. But this may not be the case for the seller. If M^F is executed by a mediator, the constraint (6) *might be* satisfied, and the expected revenue for type-s seller to tell the truth is $U_0\left(M^F|s\right) = U_0^F(s)$, i.e., equal to her full-information optimal expected revenue. Since $U_0^F(s)$ satisfies (19), there is

$$U_0\left(M^F|s\right) = \int_{\underline{s}}^{s} g_1'\left(\tilde{s}, x^F(\tilde{s})\right) d\tilde{s} + U_0\left(M^F|\underline{s}\right).$$

But this is not consistent with the constraint (7) which requires that

$$U_0\left(M^F|s\right) = \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^F(\tilde{s})\, d\tilde{s} + U_0\left(M^F|\underline{s}\right)$$

for M^F to be incentive compatible for the seller, because

$$\int_{\underline{s}}^{s} g_1'\left(\tilde{s}, x^F(\tilde{s})\right) d\tilde{s} - \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^F(\tilde{s})\, d\tilde{s} = \int_{\underline{s}}^{s} \int_T \sum_{i=1}^{n} J_{i1}'(\tilde{s},t_i)\, x_i^F(\tilde{s},t)\, f(t)\, dt d\tilde{s}. \tag{21}$$

Therefore, as long as the difference in (21) is not zero, M^F is not incentive feasible due to the failure of the seller's IC constraint.[4]

[4] This result indicates that if the seller's signal does not enter the buyers' valuation for the object, i.e., the model is a private-value model, then $J_{i1}'(s,t_i) = 0$ for $\forall s, t_i$, and the difference in (21) is equal to 0. Therefore, M^F is incentive feasible. This result implies that in the private-value case, the expected revenue of the seller with private

When the seller's signal directly affects the valuation of the buyers, i.e., in the interdependent value case, it is hard to have the difference in (21) be zero for any $s \in S$. If $J_i(s, t_i)$ changes with s in an arbitrary way, it becomes impossible for one to track the deviations of M^F from the incentive feasible mechanisms for different $s \in S$. In order to get rid of this problem, in the rest of the paper we are going to impose the following important assumption:

Assumption 2 *$J_i(s, t_i)$ is increasing in s for $\forall t_i \in T_i, \forall i,$.*

This assumption is also satisfied for some commonly adopted functional forms of v_i, for example, the linear form $v_i(s, t_i) = \alpha s + \beta t_i$, $\alpha, \beta > 0$, and the multiplicative form $v_i(s, t_i) = u(s) \cdot t_i$ when $t_i - (1 - F_i(t_i)) / f_i(t_i) > 0$ for $\forall t_i$. Under this assumption, the term in (21) is always positive. This indicates that M^F gives too much revenue to the seller of type $s > \underline{s}$ than that required by the IC constraint. The next lemma points out the source that M^F fails to be incentive compatible for the seller.

Lemma 3 *Given Assumption 2, a lower type seller would have incentive to misreport her type (locally) upwards when M^F is executed by the mediator and all buyers truthfully report their types.*

Proof. When M^F is executed by the mediator and all buyers report sincerely, a type-s seller, $s \in [\underline{s}, \bar{s})$, would get expected revenue $U_0\left(M^F|s\right) = U_0^F(s)$ from reporting truthfully, where $U_0^F(s)$ is given by (18). If the seller misreports her type as $\hat{s}$ instead, she can get expected revenue

$$\begin{aligned} U_0\left(M^F, \hat{s}|s\right) &= v_0(s)\, x_0^F(\hat{s}) + \int_T \sum_{i=1}^n J_i(\hat{s}, t_i)\, x_i^F(\hat{s}, t)\, f(t)\, dt \\ &= U_0\left(M^F|\hat{s}\right) - \left[v_0(\hat{s}) - v_0(s)\right] x_0^F(\hat{s}). \end{aligned} \tag{22}$$

To examine the incentive of misreporting, we take the difference of these two expected revenues and obtain

$$\begin{aligned} U_0\left(M^F, \hat{s}|s\right) - U_0\left(M^F|s\right) &= U_0\left(M^F|\hat{s}\right) - U_0\left(M^F|s\right) - \left[v_0(\hat{s}) - v_0(s)\right] x_0^F(\hat{s}) \\ &= \int_s^{\hat{s}} g_1'\left(\tilde{s}, x^F(\tilde{s})\right) d\tilde{s} - \int_s^{\hat{s}} v_0'(\tilde{s})\, x_0^F(\hat{s})\, d\tilde{s} \\ &= \int_s^{\hat{s}} \left\{ v_0'(\tilde{s}) \left[x_0^F(\tilde{s}) - x_0^F(\hat{s})\right] + \int_T \sum_{i=1}^n J_{i1}'(\tilde{s}, t_i)\, x_i^F(\tilde{s}, t)\, f(t)\, dt \right\} d\tilde{s}, \end{aligned}$$

where the first equality is obtained using (22), the second equality is due to (19), and the third equality is derived according to the definition of $g_1'\left(s, x^F(s)\right)$ in (20).

Let $\hat{s} = s + \Delta$. When Δ is an arbitrarily small positive number, due to the continuity of x_0^F,

information is no less than that in the full-information case. This is consistent with the finding of Mylovanov and Tröger (2012).

$v_0'(\tilde{s})\left[x_0^F(\tilde{s}) - x_0^F(\hat{s})\right]$ is arbitrarily small for $\tilde{s} \in [s, \hat{s})$.[5] However, the second term in the large bracket is bounded from zero. Thus, in this case there is $U_0\left(M^F, \hat{s}|s\right) - U_0\left(M^F|s\right) > 0$. That is, the type-$s$ seller gets better off from misreporting (locally) upwards. ■

This lemma is not as redundant as it seems. Pinning down the reason why M^F fails the IC constraint of the seller is not only helpful for us to learn about the effect of private information on the design of auction mechanism, but also, and more importantly, gives us an idea on how to adjust the M^F so that the modified the mechanism is incentive feasible. This idea is proved to be crucial for our analysis in the following subsection on safe mechanisms.

3.3 Safe Mechanisms

Subsection 3.1 shows that in the mediated mechanism design game, without loss of generality, one can assume that all types of the seller propose the same mechanism in equilibrium, and the posterior of the buyers about the seller's type is the same as their prior after observing the selected mechanism. This inscrutability principle frees us from belief updating in equilibrium analysis, but does not completely free us from concerns on beliefs: the prior of the buyers still plays an important in determining the incentive feasibility of a mechanism.

This subsection is denoted to studying the set of mechanisms whose incentive feasibility is completely belief-free. We call this set of mechanisms as safe mechanisms, which is firstly defined and studied by Myerson (1983). Below is the formal definition of the safe mechanisms.

Definition 1 *A mechanism is a safe if it is incentive feasible and satisfies the IC and IR constraints of the buyers when the seller's signal is publicly known.*

The concept of safe mechanism is one that lies between interim incentive feasible mechanism and *ex post* incentive feasible mechanism. The set of interim incentive feasible mechanisms is characterized by lemma 1, it only requires that a mechanism is incentive compatible when the players only know their own types. An *ex post* mechanism requires that no one have incentive to misreport his/her type when all other players types are publicly known, thus is more demanding than a safe mechanism.

According to this definition, one can see that if mechanism M is safe, then it satisfies constraints (6), (7) in lemma 1, and (13), (14), and (15). The incentive feasibility of safe mechanisms is belief-free, because regardless of $f(\cdot|M)$, (13), (14), and (15) imply (8), (9), and (10) in lemma 1.

The next lemma characterizes the set of safe mechanisms:

[5] The continuity of x_0^F can be easily proved. Let $r_i(s)$ be the "reserve price" set by the seller of type s for buyer i. That is, if buyer i has type $t_i < r_i(s)$, he has no chance to get this object regardless of the types of other buyers. The value of $r_i(s)$ is determined by $J_i(s, r_i(s)) - v_0(s) = 0$. The differentiability of J_i with respect to its two arguments implies that $r_i(s)$ is differentiable in s. The definition of x_0^F gives that $x_0^F(s) = \prod_{i=1}^n F_i(r_i(s))$. F_i are continutous, thus $x_0^F(s)$ is continuous.

Lemma 4 *A mechanism M is safe if and only if it satisfies the following constraints, for $\forall s \in S$, $t \in T$,*

$$v_0(s) + \int_T \sum_{i=1}^{n} [J_i(s,t_i) - v_0(s)]\, x_i(s,t) f(t)\, dt - \sum_{i=1}^{n} U_i(\underline{t}_i | s, M) = \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0(\tilde{s})\, d\tilde{s} + U_0(M|\underline{s}), \tag{23}$$

$$x_0(s) \geq x_0(s'),\ x_i(s,t_i) \geq x_i(s,t_i'),\ \text{if } s \geq s' \in S,\ t_i \geq t_i' \in T_i,$$
$$\textstyle\sum_{i=1}^{n} x_i(s,t) \leq 1,\ x_i(s,t) \geq 0,$$
$$U_i(\underline{t}_i | s, M) \geq 0,\ \text{and}$$
$$p_i(s,t_i) = v_i(s,t_i)\, x_i(s,t_i) - \int_{\underline{t}_i}^{t_i} v_{i2}'\left(s,\tilde{t}_i\right) x_i\left(s,\tilde{t}_i\right) d\tilde{t}_i - U_i(\underline{t}_i | s, M).$$

Condition (23) is derived by rewriting $U_0(M|s)$ into a function of allocation rule and $U_i(\underline{t}_i|s,M)$ as we did in the full-information case in rewriting the seller's objective function. Specifically, given the definition of $U_i(t_i|s,M)$, we can solve out $p_i(s,t_i)$ from (14). Substituting the expression of $p_i(s,t_i)$ into $U_0(M|s)$ and using integration by parts, we can transform (7) to (23). All other conditions in the lemma are directly from (1), (6), (13), (14), and (15). We list all of them here explicitly for the convenience of readers.

It is easy to see that the set of safe mechanisms is nonempty. The mechanism which has the seller always keep the object, i.e., the mechanism with $x_i(s,t) = 0$ and $p_i(s,t) = 0$ for $i = 1, 2, \ldots, n$, $s \in S, t \in T$, satisfies all the constraints in lemma 4, thus is safe.

Moreover, the set is not a singleton and is convex. Based on the insight provided by lemma 3, we can construct a safe mechanism $\hat{M}$ by modifying M^F.[6] Specifically, let the allocation rule $\hat{x}$ of $\hat{M}$ be defined as

$$\begin{cases} \hat{x}_0(\underline{s}) = x_0^F(\underline{s}),\ \hat{x}_0(s) = \sup_{\tilde{s} \in [\underline{s}, s]} \left\{ x_0^F(\tilde{s}) \right\} \text{ for } s > \underline{s}, \text{ and} \\ \hat{x}(s,\cdot) \in \arg\max_{x(t)} \left\{ \int_T \sum_{i=1}^{n} [J_i(s,t_i) - v_0(s)]\, x_i(t) f(t)\, dt, \text{ s.t. } x_0 = \hat{x}_0(s) \right\}, \end{cases} \tag{24}$$

in which $\hat{x}(s,\cdot) : T \to [0,1]^n$ is the allocation rule of $\hat{x}$ given s. $U_i\left(\underline{t}_i | s, \hat{M}\right) \geq 0$, $i = 1, 2, \ldots, n$, be constructed such that

$$\sum_{i=1}^{n} U_i\left(\underline{t}_i | s, \hat{M}\right) = v_0(s) + \int_T \sum_{i=1}^{n} [J_i(s,t_i) - v_0(s)]\, \hat{x}_i(s,t) f(t)\, dt - \int_{\underline{s}}^{s} v_0'(\tilde{s})\, \hat{x}_0(\tilde{s})\, d\tilde{s} - U_0\left(M^F | \underline{s}\right).$$

Due to assumption 2, $\sum_{i=1}^{n} U_i\left(\underline{t}_i | s, \hat{M}\right)$ is positive for $s > \underline{s}$. The proof can be found in the Appendix C. The payment rule $\hat{p}(s,t)$ of $\hat{M}$ can be defined based on $\hat{x}_i(s,t)$ and $U_i\left(\underline{t}_i | s, \hat{M}\right)$

[6] This is not the simplest way of constructing a safe mechanism. Choosing to contruct this mechanism, instead of other simpler ones, would facilitate our discussion in the rest of the paper.

using the last constraint in lemma 4. One can verify that $\hat{M}$ satisfies all the constraints in lemma 4. The convexity of the set of safe mechanisms is straightforward, so we omit its proof here.

In the case of mechanism design by an informed seller, we usually cannot find an incentive feasible mechanism that maximizes the revenue of *each type* of the seller. However, if we reduce our choice set to the set of safe mechanisms, we can find such a mechanism. That is, there is a safe mechanism that yields each type of the seller a payoff no less than does any other safe mechanism. Maskin and Tirole (1992) call such a mechanism as Rothschild-Stiglitz-Wilson (RSW) mechanism. Their analysis of the one-principal/one-agent case is centered on this mechanism.

The next proposition characterizes the RSW mechanism in our auction setting. One can find that a RSW mechanism can be derived by properly raising the reserve prices of M^F only. The lowest type of the seller gets the same revenue as in M^F, but all other types of the seller get worst off than in M^F.

Proposition 1 *Under Assumption 1 and Assumption 2, a safe mechanism M^* is a RSW mechanism if and only if it satisfies the following three conditions,*

1. $x^*(s,\cdot) \in \arg\max_{x(t)} \left\{ \int_T \sum_{i=1}^n \left[J_i(s,t_i) - v_0(s)\right] x_i(t) f(t)\, dt,\ s.t.\ x_0 = x_0^*(s) \right\}$, *and* $x_0^*(s) \geq x_0^F(s)$, $\forall s \in S$;

2. $U_i\left(\underline{t}_i | s, M^*\right) = 0,\ \forall s,\ i = 1, 2, \ldots n$;

3. *The lowest type of the seller gets her optimal full-information payoff, i.e.,*

$$U_0\left(M^* | \underline{s}\right) = U_0\left(M^F | \underline{s}\right).$$

Proof. (1) The sufficiency of the three conditions:

Suppose that M^* is a safe mechanism satisfying all the three conditions, but there exists a safe mechanism M such that for some $s \in S$, $U_0(M|s) > U_0(M^*|s)$. If this is true, then there should be

$$\int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0(\tilde{s})\, d\tilde{s} + U_0(M|\underline{s}) > \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + U_0\left(M^F|\underline{s}\right),$$

according to (23). Since $U_0(M|\underline{s}) \leq U_0\left(M^F|\underline{s}\right)$, we have

$$\int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0(\tilde{s})\, d\tilde{s} > \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s}.$$

This inequality holds only if there is a set $S^M \subset [\underline{s}, s]$ such that $x_0(s) > x_0^*(s)$ for $s \in S^M$. Let $s^{\sup}$ be the supremum of S^M. If $s^{\sup} \notin S^M$, then we can find a $s^{\varepsilon} \in S^M$ that is arbitrarily close to $s^{\sup}$. Define

$$s^M = \begin{cases} s^{\sup}, \text{ if } s^{\sup} \in S^M; \\ s^{\varepsilon}, \text{ if } s^{\sup} \notin S^M. \end{cases}$$

So $s^M \in S^M$, and

$$x_0\left(s^M\right) > x_0^*\left(s^M\right). \tag{25}$$

Then we have

$$\int_{\underline{s}}^{s^M} v_0'\left(\tilde{s}\right) x_0\left(\tilde{s}\right) d\tilde{s} + U_0\left(M|\underline{s}\right) > \int_{\underline{s}}^{s^M} v_0'\left(\tilde{s}\right) x_0^*\left(\tilde{s}\right) d\tilde{s} + U_0\left(M^F|\underline{s}\right),$$

because the set $\left[s^M, s^{\sup}\right]$ is arbitrarily small or empty and for $\forall s' \in \left(s^{\sup}, s\right]$, $x_0\left(s'\right) \leq x_0^*\left(s'\right)$. According to equation (23) and condition 2 above which implies that $\sum_{i=1}^n U_i\left(\underline{t}_i|s^M, M^*\right) = 0$, we obtain

$$\begin{aligned} & \int_T \sum_{i=1}^n \left[J_i\left(s^M, t_i\right) - v_0\left(s^M\right)\right] x_i\left(s^M, t\right) f\left(t\right) dt - \sum_{i=1}^n U_i\left(\underline{t}_i|s^M, M\right) \\ > \; & \int_T \sum_{i=1}^n \left[J_i\left(s^M, t_i\right) - v_0\left(s^M\right)\right] x_i^*\left(s^M, t\right) f\left(t\right) dt. \end{aligned}$$

$\sum_{i=1}^n U_i\left(\underline{t}_i|s^M, M\right)$ is nonnegative, so

$$\int_T \sum_{i=1}^n \left[J_i\left(s^M, t_i\right) - v_0\left(s^M\right)\right] x_i\left(s^M, t\right) f\left(t\right) dt > \int_T \sum_{i=1}^n \left[J_i\left(s^M, t_i\right) - v_0\left(s^M\right)\right] x_i^*\left(s^M, t\right) f\left(t\right) dt,$$

Since M^* satisfies condition 1 of the proposition, this inequality contradicts (25). Thus, M^* is a RSW mechanism.

(2) The necessity of the three conditions:

Suppose that M^* is a RSW mechanism. It is easy to show that the third condition is satisfied by M^*. Equation (23) indicates that $U_0\left(M^F|\underline{s}\right) \geq U_0\left(M^*|\underline{s}\right)$. Since no other safe mechanisms could yield any type of the seller a higher payoff than does M^*, there is $U_0\left(M^*|\underline{s}\right) \geq U_0\left(\hat{M}|\underline{s}\right) = U_0\left(M^F|\underline{s}\right)$, in which $\hat{M}$ is the safe mechanism constructed below lemma 4. So $U_0\left(M^*|\underline{s}\right) = U_0\left(M^F|\underline{s}\right)$.

We prove the first and second conditions together. Suppose that the first condition is not satisfied by M^*, then it must be the case that for some s, either

$$x^*\left(s, \cdot\right) \notin \arg\max_{x(t)} \left\{ \int_T \sum_{i=1}^n \left[J_i\left(s, t_i\right) - v_0\left(s\right)\right] x_i\left(t\right) f\left(t\right) dt, \text{ s.t. } x_0 = x_0^*\left(s\right) \right\}$$

or $x_0^*\left(s\right) < x_0^F\left(s\right)$ or both. If the first case happens, then we can always construct a new RSW mechanism $\bar{M}^*$ by adjusting the allocation rule of M^* such that

$$\bar{x}^*\left(s, \cdot\right) \in \arg\max_{x(t)} \left\{ \int_T \sum_{i=1}^n \left[J_i\left(s, t_i\right) - v_0\left(s\right)\right] x_i\left(t\right) f\left(t\right) dt, \text{ s.t. } x_0 = x_0^*\left(s\right) \right\},$$

so $\bar{x}_0^*(s) = x_0^*(s)$. The expected payoff to the lowest type of each buyer under $\bar{M}^*$ is defined in the following way so that the equation (23) holds,

$$\begin{aligned}\sum_{i=1}^{n} U_i\left(\underline{t}_i|s, \bar{M}^*\right) &= \sum_{i=1}^{n} U_i\left(\underline{t}_i|s, M^*\right) + \int_T \sum_{i=1}^{n} J_i(s, t_i)\, \bar{x}_i^*(s,t)\, f(t)\, dt - \int_T \sum_{i=1}^{n} J_i(s,t_i)\, x_i^*(s,t)\, f(t)\, dt \\ &> \sum_{i=1}^{n} U_i\left(\underline{t}_i|s, M^*\right) \geq 0. \end{aligned} \tag{26}$$

The first inequality is from the definition of $\bar{x}^*$. The second inequality is based on the supposition that M^* is a RSW mechanism. (26) implies that the failure of the first part of condition 1 is equivalent to the failure of conditions 2. Thus, in the rest of our proof, we assume that the M^* satisfies the first part of condition 1, and show that $x_0^*(s) \geq x_0^F(s)$ and condition 2 must hold.

Suppose for some s_1, $x_0^*(s_1) < x_0^F(s_1)$. According to the continuity of $x_0^F(s)$, there exists $s_2 < s_1$ (s_2 close to s) such that $x_0^*(s_1) < x_0^F(s_2)$. We construct a safe mechanism M which has

$$x_i^M(s,t) = \begin{cases} x_i^*(s,t), \text{ for } s < s_2, \\ x_i^F(s_2,t), \text{ for } s \geq s_2 \end{cases} \tag{27}$$

and $U_i\left(\underline{t}_i|s, M\right)$ satisfies

$$\begin{aligned} &\sum_{i=1}^{n} U_i\left(\underline{t}_i|s, M\right) \\ = &\begin{cases} \sum_{i=1}^{n} U_i\left(\underline{t}_i|s, M^*\right), \text{ if } s < s_2, \\ v_0(s)\, x_0^M(s) + \int_T \sum_{i=1}^{n} J_i(s,t_i)\, x_i^M(s,t)\, f(t)\, dt - \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^M(\tilde{s})\, d\tilde{s} - U_0\left(M^F|\underline{s}\right), \text{ if otherwise.} \end{cases} \end{aligned} \tag{28}$$

Proof for the safety of M can be found in the Appendix D. In mechanism M, the seller with a signal $s \in [s_2, s_1]$ gets higher expected revenue than under M^*, because for $s \in [s_2, s_1]$,

$$\begin{aligned} \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^M(\tilde{s})\, d\tilde{s} + U_0\left(M^F|\underline{s}\right) &= \int_{\underline{s}}^{s_2} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \int_{\underline{s_2}}^{s} v_0'(\tilde{s})\, x_0^F(s_2)\, d\tilde{s} + U_0\left(M^F|\underline{s}\right) \\ &> \int_{\underline{s}}^{s_2} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + \int_{\underline{s_2}}^{s} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + U_0\left(M^F|\underline{s}\right) \\ &= \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + U_0\left(M^F|\underline{s}\right), \end{aligned}$$

where the first equality is from the definition of x^M, the inequality is based on that $x_0^*(s_1) < x_0^F(s_2)$. This is a contradiction to that M^* is a RSW mechanism.

Let us turn to condition 2. Suppose that for some $\hat{s} > \underline{s}$, $\sum_{i=1}^{n} U_i\left(\underline{t}_i|\hat{s}, M^*\right) > 0$. Then we can show that there exist some $\underline{\delta} > 0$ such that over the interval $[\hat{s} - \underline{\delta}, \hat{s}]$,

$$\inf_{s \in [\hat{s} - \underline{\delta}, \hat{s}]} \sum_{i=1}^{n} U_i \left(\underline{t}_i | s, M^* \right) > 0. \tag{29}$$

The proof of it is put in the Appendix E. Given this result, we can construct a safe mechanism M^ε making all types of the seller in the set $[\hat{s} - \underline{\delta}, \hat{s}]$ strictly better off. Specifically, the allocation rule of M^ε is defined as

$$\begin{aligned}
x^\varepsilon (s, t) &= x^* (s, t), \text{ for } s < \hat{s} - \underline{\delta}, \\
x^\varepsilon (s, \cdot) &= \arg\max_{x(t)} \left\{ \int_T \textstyle\sum_{i=1}^n \left[J_i (s, t_i) - v_0 (s) \right] x_i (t) f (t) \, dt, \text{ s.t. } x_0 = x_0^* (s) + \varepsilon \right\}, \text{ for } s \in [\hat{s} - \underline{\delta}, \hat{s}], \\
x^\varepsilon (s, t) &= x^\varepsilon (\hat{s}, t), \text{ for } s > \hat{s}.
\end{aligned}$$

The $\sum_{i=1}^n U_i \left(\underline{t}_i | s, M^* \right)$ of M^ε is constructed in the following way,

$$\begin{aligned}
& \sum_{i=1}^{n} U_i \left(\underline{t}_i | s, M^\varepsilon \right) \\
= & \begin{cases}
\sum_{i=1}^n U_i \left(\underline{t}_i | s, M^* \right), \text{ if } s < \hat{s} - \underline{\delta} \\
\sum_{i=1}^n U_i \left(\underline{t}_i | s, M^* \right) - \left\{ \begin{array}{c} \int_T \sum_{i=1}^n J_i (s, t_i) x_i^* (t) f (t) \, dt \\ - \int_T \sum_{i=1}^n J_i (s, t_i) x_i^\varepsilon (t) f (t) \, dt - \varepsilon v_0 (\hat{s} - \underline{\delta}) \end{array} \right\}, \text{ if } s \in [\hat{s} - \underline{\delta}, \hat{s}] \\
v_0 (s) x_0^\varepsilon (s) + \int_T \sum_{i=1}^n J_i (s, t_i) x_i^\varepsilon (s, t) f (t) \, dt - \int_{\underline{s}}^{s} v_0' (\tilde{s}) x_0^\varepsilon (\tilde{s}) \, d\tilde{s} - U_0 \left(M^F | \underline{s} \right), \text{ if } s > \hat{s}
\end{cases}.
\end{aligned}$$

When ε is small enough, M^ε is safe. The proof is put in the Appendix F. Here we show that the seller with $s \in [\hat{s} - \underline{\delta}, \hat{s}]$ gets better off. For $s \in [\hat{s} - \underline{\delta}, \hat{s}]$, the difference of the seller's payoffs under M^ε and M^* is

$$\int_{\underline{s}}^{s} v_0' (\tilde{s}) x_0^\varepsilon (\tilde{s}) \, d\tilde{s} - \int_{\underline{s}}^{s} v_0' (\tilde{s}) x_0^* (\tilde{s}) \, d\tilde{s} = \int_{\hat{s} - \underline{\delta}}^{s} v_0' (\tilde{s}) \varepsilon d\tilde{s} > 0.$$

This again contradicts that M^* is a RSW mechanism. Thus, condition 2 must be satisfied for M^*. This finally completes our proof. ■

Lemma 3 points out that it is the seller's incentive to misreport upwards that prevents M^F to be incentive compatible. Intuitively, this is because an upward deviation allows them to sell the object at a relatively higher price, even though the probability of trading might be reduced. This intuition suggests two ways to disincentivize the lower types of the seller to misreport, thus make a mechanism incentive compatible, based on the fact that the payoff of the seller is totally determined by x and $\sum_{i=1}^n U_i \left(\underline{t}_i | s, M \right)$. These two ways are, (1) to increase the reserve prices of mechanisms adopted by a high-type seller, thus decrease the probability of selling the object, and (2) to increase the expected payoffs of the lowest types of the buyers (this is equivalent to uniformly decreasing the payment of the buyers).

Proposition 1 tells us that from the seller's perspective, the first approach outperforms the second one. This is because the first one is less costly to the higher type seller whom the lower type is trying to mimic. Both of these two approaches eliminate the incentive of the lower type seller to misreport upward by reducing her "cheating benefits" by certain amount. To achieve this reduction, the second approach is equivalent to letting the higher type seller to give up the same amount of revenue. The first way increases the probability of the seller to receive no payments and keeping the object, which makes the higher type seller lose less than the lower type, as the higher type values the object more.[7]

There are two ways to prove the existence of RSW mechanisms. The first way is like what Maskin and Tirole (1992) did for the one-principal/one-agent case. The other way is to reduce the problem of proving the existence of the mechanism to the problem of proving the existence of a function $x_0 : S \to [0, 1]$ satisfying conditions (23) and the first condition in Proposition 1.

Proposition 2 *There exists a RSW mechanism.*

Proof. See Appendix G. ■

Corollary 1 *In a RSW mechanism,* $x_0 : S \to [0, 1]$*, the probability that the seller keeps the object, is continuous and strictly increasing in* s *if* $x_0 (s) < 1$*.*

Proof. See Appendix G. ■

4 Mechanism Design without Inscrutability

In the section above, we considered a mechanism selection game in which the seller is inscrutable in equilibrium. In this section, we analyze a game in which the principle of inscrutability fails. That is, we can no long restrict our attention to the cases where all possible types of the seller propose the same mechanism.

In the game of this section, a mechanism selected by a seller is not a function of a selller's report. Taking direct revelation mechanisms for example. A direct revelation auction mechanism M consists of an allocation function $x : T \to [0, 1]^n$ and a payment function $p : T \to R^n$, with

$$\begin{aligned} x(t) &= (x_1(t), x_2(t), \ldots, x_n(t)), \\ p(t) &= (p_1(t), p_2(t), \ldots, p_n(t)), \end{aligned}$$

[7] When the reserve prices are increased, the payments of the buyers become higher in the case that there is only one buyer having a type higher than his reserve price. However, compared with the effect of reserve prices on the probability of trading, this is second order.

where $x_i(t)$ and $p_i(t)$ are respectively buyer i's probability of getting the object and payment under t. $x(t)$ should satisfy the feasibility constraint

$$\sum_{i=1}^{n} x_i(t) \leq 1 \text{ and } x_i(t) \geq 0, \text{ for } \forall t.$$

Based on the insights derived in the above section, we have the following proposition.

Proposition 3 *Suppose that the strategy space of the seller is the set of mechanisms which have buyer-equilibria regardless of buyers' beliefs above the seller's type. The RSW mechanism in the mediated mechanism selection game can be supported as the equilibrium strategy of the seller in the seller-optimal separating equilibrium.*

Proof. The trickiest part of this proof is to find a system of belief to support RSW mechanism as the seller's equilibrium strategy. We put this proof in Appendix H. ■

This proposition, combining with the characterization of the RSW mechanism, provides a support for the literature on reserve price signaling (Jullien and Mariotti, 2006; Cai, Riley and Ye, 2007): the seller only uses the reserve price to signal herself, even if she has the freedom to vary the whole mechanism.

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Appendix

A. Proof for Lemma 1

To be added.

B. Proof for Lemma 3.1

To be added.

C. $\hat{M}$ is a Safe Mechanism

It is clear that $\hat{M}$ in subsection 3.3 immediately satisfies all the constraints for a safe mechanism, except that $\sum_{i=1}^{n} U_i\left(\underline{t}_i|s,\hat{M}\right) \geq 0$. Now we prove that this is also true. The definition of $\hat{x}$ implies that for $s_1 < s_2 \in S$, either there exists a $s' \in (s_1, s_2]$ such that $x_0^F(s') = \hat{x}_0(s_2)$ or $\hat{x}_0(s_1) = \hat{x}_0(s_2)$. In the first case,

$$
\begin{aligned}
& \sum_{i=1}^{n} U_i\left(\underline{t}_i|s_2,\hat{M}\right) - \sum_{i=1}^{n} U_i\left(\underline{t}_i|s_1,\hat{M}\right) \\
\geq\ & v_0(s_2)\, x_0^F(s') + \int_T \sum_{i=1}^{n} J_i(s_2,t_i)\, x_i^F(s',t)\, f(t)\, dt - v_0(s_1)\,\hat{x}_0(s_1) - \int_T \sum_{i=1}^{n} J_i(s_1,t_i)\,\hat{x}_i(s_1,t)\, f(t)\, dt \\
& -\int_{s_1}^{s_2} v_0'(\tilde{s})\,\hat{x}_0(\tilde{s})\, d\tilde{s} \\
\geq\ & v_0(s_2)\, x_0^F(s') + \int_T \sum_{i=1}^{n} J_i(s',t_i)\, x_i^F(s',t)\, f(t)\, dt - v_0(s_1)\,\hat{x}_0(s_1) - \int_T \sum_{i=1}^{n} J_i(s_1,t_i)\,\hat{x}_i(s_1,t)\, f(t)\, dt \\
& -\int_{s_1}^{s_2} v_0'(\tilde{s})\,\hat{x}_0(\tilde{s})\, d\tilde{s} \\
\geq\ & U_0\left(M^F|s'\right) - U_0\left(M^F|s_1\right) + \left[v_0(s_2) - v_0(s')\right] x_0^F(s') - \int_{s_1}^{s_2} v_0'(\tilde{s})\,\hat{x}_0(\tilde{s})\, d\tilde{s} \\
=\ & \int_{s_1}^{s'} g_1'\left(\tilde{s}, x^F(\tilde{s})\right) d\tilde{s} - \int_{s_1}^{s'} v_0'(\tilde{s})\,\hat{x}_0(\tilde{s})\, d\tilde{s} \\
=\ & \int_{s_1}^{s'} \left\{ v_0'(\tilde{s})\left[x_0^F(\tilde{s}) - \hat{x}_0(\hat{s})\right] + \int_T \sum_{i=1}^{n} J_{i1}'(\tilde{s},t_i)\, x_i^F(\tilde{s},t)\, f(t)\, dt \right\} d\tilde{s}.
\end{aligned}
$$

The first inequality uses the definition of $\hat{x}$ in (24) and the fact that $x_0^F(s') = \hat{x}_0(s_2)$. The second inequality is derived using the monotonicity of J_i and nonnegativity of $x_i^F(s',t)$. The third inequality is based on the definition of $U_0\left(M^F|s'\right)$ and the optimality of x^F. The first equality uses the result of (19) and the fact that $\hat{x}_0(s) = x_0^F(s')$ for $s \in [s', s_2]$. The last equality is obtained by substituting the expression of $g_1'\left(\tilde{s}, x^F(\tilde{s})\right)$ into the first equality. It is clear that if s_1 and s_2 are arbitrarily close to each other, then this difference is positive.

If $\hat{x}_0(s_1) = \hat{x}_0(s_2)$, it is obviously true that $\sum_{i=1}^n U_i\left(\underline{t}_i|s_2, \hat{M}\right) - \sum_{i=1}^n U_i\left(\underline{t}_i|s_1, \hat{M}\right) \geq 0$. Thus, no matter which case happens, $\sum_{i=1}^n U_i\left(\underline{t}_i|s, \hat{M}\right)$ is nondecreasing in s. Since $\sum_{i=1}^n U_i\left(\underline{t}_i|\underline{s}, \hat{M}\right) = 0$, $\sum_{i=1}^n U_i\left(\underline{t}_i|s, \hat{M}\right)$ is never negative.

D. M **defined by** (27) **and** (28) **is Safe**

Since M^* is safe, the construction of M immediately implies that it satisfies equation (23) and the feasibility condition. According to (27), $x_i^M(s, t_i)$ and $x_0^M(s)$ both satisfy the monotonicity conditions. $\sum_{i=1}^n U_i(\underline{t}_i|\underline{s}, M)$ is always nonnegative, because for $s < s_2$, $\sum_{i=1}^n U_i(\underline{t}_i|s, M) = \sum_{i=1}^n U_i(\underline{t}_i|s, M^*) \geq 0$, and for $s = s_2$,

$$\begin{aligned}
\sum_{i=1}^n U_i(\underline{t}_i|s_2, M) &= v_0(s)\,x_0^F(s_2) + \int_T \sum_{i=1}^n J_i(s_2, t_i)\,x_i^F(s_2, t)\,f(t)\,dt - \int_{\underline{s}}^{s_2} v_0'(\tilde{s})\,x_0^*(\tilde{s})\,d\tilde{s} - U_0\left(M^F|\underline{s}\right) \\
&\geq v_0(s)\,x_0^*(s_2) + \int_T \sum_{i=1}^n J_i(s_2, t_i)\,x_i^*(s_2, t)\,f(t)\,dt - \int_{\underline{s}}^{s_2} v_0'(\tilde{s})\,x_0^*(\tilde{s})\,d\tilde{s} - U_0\left(M^F|\underline{s}\right) \\
&= \sum_{i=1}^n U_i(\underline{t}_i|s_2, M^*) \geq 0.
\end{aligned}$$

The weak inequality is due to the optimality of $x^F(s_2, \cdot)$ at s_2.

For $s > s_2$,

$$\begin{aligned}
\sum_{i=1}^n U_i(\underline{t}_i|s, M) &= v_0(s)\,x_0^M(s) + \int_T \sum_{i=1}^n J_i(s, t_i)\,x_i^M(s, t)\,f(t)\,dt - \int_{\underline{s}}^{s} v_0'(\tilde{s})\,x_0^M(\tilde{s})\,d\tilde{s} - U_0\left(M^F|\underline{s}\right) \\
&= v_0(s)\,x_0^F(s_2) + \int_T \sum_{i=1}^n J_i(s, t_i)\,x_i^F(s_2, t)\,f(t)\,dt \\
&\quad - \int_{\underline{s}}^{s_2} v_0'(\tilde{s})\,x_0^*(\tilde{s})\,d\tilde{s} - \int_{s_2}^{s} v_0'(\tilde{s})\,x_0^F(s_2)\,d\tilde{s} - U_0\left(M^F|\underline{s}\right) \\
&= v_0(s_2)\,x_0^F(s_2) + \int_T \sum_{i=1}^n J_i(s, t_i)\,x_i^F(s_2, t)\,f(t)\,dt - \int_{\underline{s}}^{s_2} v_0'(\tilde{s})\,x_0^*(\tilde{s})\,d\tilde{s} - U_0\left(M^F|\underline{s}\right) \\
&> v_0(s_2)\,x_0^F(s_2) + \int_T \sum_{i=1}^n J_i(s_2, t_i)\,x_i^F(s_2, t)\,f(t)\,dt - \int_{\underline{s}}^{s_2} v_0'(\tilde{s})\,x_0^*(\tilde{s})\,d\tilde{s} - U_0\left(M^F|\underline{s}\right) \\
&= \sum_{i=1}^n U_i(\underline{t}_i|s_2, M) \geq 0.
\end{aligned}$$

Therefore, M is a safe mechanism.

E. Proof for (29)

Suppose that for some $\hat{s}$, $\sum_{i=1}^n U_i(\underline{t}_i|\hat{s}, M^*) > 0$, then there must exist $\delta > 0$, such that for $\forall s \in [\hat{s} - \delta, \hat{s}]$, $\sum_{i=1}^n U_i(\underline{t}_i|s, M^*) > 0$. Suppose this is not true, i.e., for $\forall \delta > 0$, $\exists s \in [\hat{s} - \delta, \hat{s}]$

such that $\sum_{i=1}^{n} U_i\left(\underline{t}_i|s, M^*\right) = 0$, then we can find a sequence $\{s_n\}_{n=1}^{\infty}$ converging to $\hat{s}$ with $\sum_{i=1}^{n} U_i\left(\underline{t}_i|s_n, M^*\right) = 0$ for $\forall n$. According to equation (23), we have

$$v_0\left(s_n\right) + \int_T \sum_{i=1}^{n} \left[J_i\left(s_n, t_i\right) - v_0\left(s_n\right)\right] x_i^*\left(s_n, t\right) f\left(t\right) dt = \int_{\underline{s}}^{s_n} v_0'\left(\tilde{s}\right) x_0^*\left(\tilde{s}\right) d\tilde{s} + U_0\left(M^*|\underline{s}\right).$$

By continuity,

$$\lim_{n\to\infty} v_0\left(s_n\right) + \int_T \sum_{i=1}^{n} \left[J_i\left(s_n, t_i\right) - v_0\left(s_n\right)\right] x_i^*\left(s_n, t\right) f\left(t\right) dt = \int_{\underline{s}}^{\hat{s}} v_0'\left(\tilde{s}\right) x_0^*\left(\tilde{s}\right) d\tilde{s} + U_0\left(M^*|\underline{s}\right). \quad (30)$$

However, since $x_0^*\left(s_n\right) \leq x_0^*\left(\hat{s}\right)$ and condition 1 of Proposition 1 is satisfied, there is

$$v_0\left(s_n\right)+\int_T \sum_{i=1}^{n} \left[J_i\left(s_n, t_i\right) - v_0\left(s_n\right)\right] x_i^*\left(s_n, t\right) f\left(t\right) dt \geq v_0\left(s_n\right)+\int_T \sum_{i=1}^{n} \left[J_i\left(s_n, t_i\right) - v_0\left(s_n\right)\right] x_i^*\left(\hat{s}, t\right) f\left(t\right) dt.$$

The expressions on both sides of the inequality are continuous. By taking limit of them, we can derive

$$\int_{\underline{s}}^{\hat{s}} v_0'\left(\tilde{s}\right) x_0^*\left(\tilde{s}\right) d\tilde{s} + U_0\left(M^*|\underline{s}\right) \geq v_0\left(\hat{s}\right) + \int_T \sum_{i=1}^{n} \left[J_i\left(\hat{s}, t_i\right) - v_0\left(\hat{s}\right)\right] x_i^*\left(\hat{s}, t\right) f\left(t\right) dt.$$

This contradicts equation (23), given $\sum_{i=1}^{n} U_i\left(\underline{t}_i|\hat{s}, M^*\right) > 0$. There, we have shown that $\exists \delta > 0$, such that for $\forall s \in [\hat{s} - \delta, \hat{s}]$, $\sum_{i=1}^{n} U_i\left(\underline{t}_i|s, M^*\right) > 0$.

Now we further show that $\exists \underline{\delta} \in (0, \delta)$ such that $\inf_{s\in[\hat{s}-\underline{\delta},\hat{s}]} \sum_{i=1}^{n} U_i\left(\underline{t}_i|s, M^*\right) > 0$. We again prove this by contradiction. Suppose this is not the case, then it means that for $\forall \underline{\delta} \in (0, \delta)$, $\inf_{s\in[\hat{s}-\underline{\delta},\hat{s}]} \sum_{i=1}^{n} U_i\left(\underline{t}_i|s, M^*\right) = 0$. If so, for decreasing positive sequences $\{\delta_m\}_{m=1}^{\infty}$ and $\{\varepsilon_m\}_{m=1}^{\infty}$ with $\delta_1 < \delta$, $\lim_{m\to\infty} \delta_m = 0$ and $\lim_{m\to\infty} \varepsilon_m = 0$, we can construct a sequence $\{s_m\}_{m=1}^{\infty}$ such that

$$s_m \in [\hat{s} - \delta_m, \hat{s}] \text{ and } \textstyle\sum_{i=1}^{n} U_i\left(\underline{t}_i|s_m, M^*\right) < \varepsilon_m.$$

This implies that

$$\lim_{m\to\infty} s_m = \hat{s} \text{ and } \lim_{m\to\infty} \textstyle\sum_{i=1}^{n} U_i\left(\underline{t}_i|s_m, M^*\right) = 0.$$

According to equation (23),

$$\begin{aligned} & \lim_{m\to\infty} v_0\left(s_m\right) + \int_T \sum_{i=1}^{n} \left[J_i\left(s_m, t_i\right) - v_0\left(s_m\right)\right] x_i^*\left(s_m, t\right) f\left(t\right) dt \\ = & \lim_{m\to\infty} \int_{\underline{s}}^{s_m} v_0'\left(\tilde{s}\right) x_0^*\left(\tilde{s}\right) d\tilde{s} + \lim_{m\to\infty} \sum_{i=1}^{n} U_i\left(\underline{t}_i|s_m, M^*\right) + U_0\left(M^*|\underline{s}\right) \\ = & \int_{\underline{s}}^{\hat{s}} v_0'\left(\tilde{s}\right) x_0^*\left(\tilde{s}\right) d\tilde{s} + U_0\left(M^*|\underline{s}\right). \end{aligned} \quad (31)$$

However, due to that $x_0^*(s_m) \leq x_0^*(\hat{s})$ and condition 1,

$$v_0(s_m)+\int_T \sum_{i=1}^{n} [J_i(s_m,t_i) - v_0(s_m)]\, x_i^*(s_m,t)\, f(t)\, dt \geq v_0(s_m)+\int_T \sum_{i=1}^{n} [J_i(s_m,t_i) - v_0(s_m)]\, x_i^*(\hat{s},t)\, f(t)\, dt.$$

Taking limits on both sides, we derive

$$\int_{\underline{s}}^{\hat{s}} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + U_0(M^*|\underline{s}) \geq v_0(\hat{s}) + \int_T \sum_{i=1}^{n} [J_i(\hat{s},t_i) - v_0(\hat{s})]\, x_i^*(\hat{s},t)\, f(t)\, dt,$$

according to (31). This is a contradiction to equation (23), given $\sum_{i=1}^{n} U_i(\underline{t}_i|\hat{s}, M^*) > 0$. Therefore, we have proved that $\exists \underline{\delta} \in (0,\delta)$ such that $\inf_{s\in[\hat{s}-\underline{\delta},\hat{s}]} \sum_{i=1}^{n} U_i(\underline{t}_i|s, M^*) > 0$.

F. M^ε is a Safe Mechanism

Verifying that M^ε satisfies the equation (23), monotonicity condition, feasibility condition is straightforward, so we only need to show that $\sum_{i=1}^{n} U_i(\underline{t}_i|s, M^\varepsilon) \geq 0$, for $\forall s$. For $s < \hat{s} - \underline{\delta}$, $\sum_{i=1}^{n} U_i(\underline{t}_i|s, M^\varepsilon) = \sum_{i=1}^{n} U_i(\underline{t}_i|s, M^*) \geq 0$. For $s \in [\hat{s} - \underline{\delta}, \hat{s}]$,

$$\sum_{i=1}^{n} U_i(\underline{t}_i|s, M^\varepsilon) \geq \inf_{s\in[\hat{s}-\underline{\delta},\hat{s}]} \sum_{i=1}^{n} U_i(\underline{t}_i|s, M^*) - \left\{ \begin{array}{c} \int_T \sum_{i=1}^{n} J_i(s,t_i)\, x_i^*(t)\, f(t)\, dt \\ -\int_T \sum_{i=1}^{n} J_i(s,t_i)\, x_i^\varepsilon(t)\, f(t)\, dt - \varepsilon v_0(\hat{s}-\underline{\delta}) \end{array} \right\},$$

which is positive when ε is small enough. For $s > \hat{s}$,

$$\begin{aligned}
\sum_{i=1}^{n} U_i(\underline{t}_i|s, M^\varepsilon) &= v_0(s)\, x_0^\varepsilon(\hat{s}) + \int_T \sum_{i=1}^{n} J_i(s,t_i)\, x_i^\varepsilon(\hat{s},t)\, f(t)\, dt - \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^\varepsilon(\tilde{s})\, d\tilde{s} - U_0\left(M^F|\underline{s}\right) \\
&= v_0(\hat{s})\, x_0^\varepsilon(\hat{s}) + \int_T \sum_{i=1}^{n} J_i(s,t_i)\, x_i^\varepsilon(\hat{s},t)\, f(t)\, dt - \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^\varepsilon(\tilde{s})\, d\tilde{s} - U_0\left(M^F|\underline{s}\right) \\
&\geq v_0(\hat{s})\, x_0^\varepsilon(\hat{s}) + \int_T \sum_{i=1}^{n} J_i(\hat{s},t_i)\, x_i^\varepsilon(\hat{s},t)\, f(t)\, dt - \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^\varepsilon(\tilde{s})\, d\tilde{s} - U_0\left(M^F|\underline{s}\right) \\
&= \sum_{i=1}^{n} U_i(\underline{t}_i|\hat{s}, M^\varepsilon) \geq 0,
\end{aligned}$$

in which the first equality and second equality are from the definition of $x^\varepsilon(s,\cdot)$ for $s > \hat{s}$, the inequality is due to that $J_i(s,t_i)$ is increasing in s, the last equality holds because at $s = \hat{s}$, M^ε satisfies equation (23).

G. Existence of RSW Mechanisms

According to Proposition 1 and Lemma 4, a mechanism $M^* \equiv (x^*, p^*)$ is a RSW mechanism if and only if

1. The allocation rule x^* satisfies

$$v_0(s) + \int_T \sum_{i=1}^{n} [J_i(s,t_i) - v_0(s)]\, x_i^*(s,t)\, f(t)\, dt = \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + U_0\left(M^F|\underline{s}\right),$$

and $x^*(s,\cdot)$ is a solution of

$$\max_{x(t)}\left\{\int_T \sum_{i=1}^n \left[J_i(s,t_i)-v_0(s)\right]x_i(t)\,f(t)\,dt,\text{ s.t. } x_0=x_0^*(s)\right\}, \tag{32}$$

with $x_0^*(s)\geq x_0^F(s)$, and $x_0^*(s)$ and $x_i^*(s,t_i)$ are respectively weakly increasing in s and t_i;

2. The payment rule satisfies

$$p_i^*(s,t_i)=v_i(s,t_i)\,x_i^*(s,t_i)-\int_{\underline{t}_i}^{t_i} v_{i2}'\left(s,\tilde{t}_i\right)x_i^*\left(s,\tilde{t}_i\right)d\tilde{t}_i.$$

These conditions imply that to prove the existence of a RSW mechanism, the key is to prove the existence of a function $x_0^*(s)$ that is greater than $x_0^F(s)$ and weakly increasing in s, and satisfies

$$v_0(s)+\max_{x(t)}\left\{\int_T \sum_{i=1}^n \left[J_i(s,t_i)-v_0(s)\right]x_i(t)\,f(t)\,dt,\text{ s.t. } x_0=x_0^*(s)\right\}=\int_{\underline{s}}^{s} v_0'(\tilde{s})\,x_0^*(\tilde{s})\,d\tilde{s}+U_0\left(M^F|\underline{s}\right). \tag{33}$$

This is because once there exists such a $x_0^*(s)$, we can easily derive x^* and p^* satisfying the above conditions, thus derive a RSW mechanism. Intuitively, given $x_0^*(s)$, $x^*(s,\cdot)$ solving the maximization problem (32) would allocate the object to a buyer with the highest virtual surplus, and if the maximum virtual surplus under profile t' is higher than that under t, then $\sum_{i=1}^n x_i^*(s,t')\geq\sum_{i=1}^n x_i^*(s,t)$. Thus, characterizing $x^*(s,\cdot)$ is equivalent to finding a value $\underline{J}$ of virtual valuation such that the seller keeps the object if and only if the maximum virtual valuation under a profile t is lower than $\underline{J}$. $\underline{J}$ should satisfy

$$\max_k\left\{J_k\left(s,\underline{t}_k\right):k=1,\ldots,n\right\}\equiv J^{\min}(s)\leq\underline{J}\leq J^{\max}(s)\equiv\max_k\left\{J_k\left(s,\bar{t}_k\right):k=1,\ldots,n\right\}. \tag{34}$$

In line with this intuition, we define $r_i(s,\underline{J})$ as the minimum type of buyer i that has a chance to get the object given s and $\underline{J}$, thus if $J_i(s,\underline{t}_i)\leq\underline{J}\leq J_i(s,\bar{t}_i)$, $r_i(s,\underline{J})$ satisfies $J_i(s,r_i(s,\underline{J}))=\underline{J}$; if $J_i(s,\bar{t}_i)<\underline{J}$, $r_i(s,\underline{J})=\bar{t}_i$.[8] The continuity and monotonicity of $J_i(s,t_i)$ in s and t_i imply that $r_i(s,\underline{J})$ is continuously decreasing in s and continuously increasing in $\underline{J}$. The probability of the seller keeping the object given $\underline{J}$ is thus $\prod_{i=i}^n F_i(r_i(s,\underline{J}))$, because the object is left unsold if and only if the virtual valuations of the buyers are all smaller than $\underline{J}$. Condition (33) can be rewritten as

$$v_0(s)+\sum_{i=1}^n\int_{r_i(s,\underline{J}^*(s))}^{\bar{t}_i}\int_{T_{-i}(s,t_i)}\left[J_i(s,t_i)-v_0(s)\right]f(t)\,dt=\int_{\underline{s}}^{s} v_0'(\tilde{s})\,x_0^*(\tilde{s})\,d\tilde{s}+U_0\left(M^F|\underline{s}\right), \tag{35}$$

[8] According to condition (34), one can varify that $J_i(s,\underline{t}_i)>\underline{J}$ will not happen.

where $\underline{J}^*(s)$ and $T_{-i}(s,t_i)$ are defined as

$$\prod_{i=i}^{n} F_i\left(r_i\left(s, \underline{J}^*(s)\right)\right) = x_0^*(s),$$

$$T_{-i}(s,t_i) \equiv \left\{t_{-i} \in T_{-i} : J_i(s,t_i) \geq \max_k \left\{J_k(s,t_k) : k < i\right\},\ J_i(s,t_i) > \max_k \left\{J_k(s,t_k) : k > i\right\}\right\}.$$

So for any $t_{-i} \in T_{-i}(s,t_i)$, buyer i is the highest indexed agent with the maximum virtual valuation. This indicates that equation (35) corresponds to an allocation rule which satisfies (32) and allocates the object to the highest indexed buyer when there is tie in the maximium virtual valuation.

Proving the existence of a RSW mechanism is reduced to proving the existence of a function $x_0^* : S \to [0,1]$ that is increasing in s and bounded by x_0^F and 1, i.e., $x_0^F(s) \leq x_0^*(s) \leq 1$, $\forall s$, and satisfies equation (35) for any s. To proceed, we define function $D(\underline{J}, s, x_0)$ and mapping Γ,

$$D(\underline{J}, s, x_0) = v_0(s) + \sum_{i=1}^n \int_{r_i(s,\underline{J})}^{\bar{t}_i} \int_{T_{-i}(s,t_i)} \left[J_i(s,t_i) - v_0(s)\right] f(t)\, dt - \left[\int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0(\tilde{s})\, d\tilde{s} + U_0\left(M^F | \underline{s}\right)\right]$$

$$\Gamma x_0(s) = \left\{\prod_{i=1}^{n} F_i\left(r_i\left(s, \underline{J}(s)\right)\right) : \underline{J}(s) = \arg \min_{\max\{J^{\min}(s), v_0(s)\} \leq \underline{J} \leq J^{\max}(s)} \left|D(\underline{J}, s, x_0)\right|\right\}.$$

Once we prove that Γ has a fixed point x_0^* that is increasing in s and bounded by x_0^F and 1, and makes $D(\underline{J}^*(s), s, x_0^*) = 0$, $\forall s$, then the existence of a RSW mechanism is proved.

We use Schauder's fixed point theorem to complete this proof. First of all, we specify the domain of Γ. Let $C(S)$ denote the set of bounded continuous functions $h : S \to R$ endowed with sup norm, $\|h\| = \sup_{s \in S} |h(s)|$. Thus, $C(S)$ is a Banach space. We use $X_0(S)$ to represent the set of continuous functions $x_0 : S \to R$ which are increasing and bounded by x_0^F and 1, so $X_0(S) \subset C(S)$. $X_0(S)$ is nonempty, convex, and compact. Nonemptiness and convexity are obvious. Here we show its compactness. $X_0(S)$ is bounded, as each of its element is bounded. Now we prove that it is closed. Let $\{x_0^n\}_{n \in \mathbb{N}}$ be a sequence in $X_0(S)$ converging to x_0, so

$$\lim_{n \to \infty} x_0^n(s) = x_0(s), \forall s \in S. \tag{36}$$

Since $x_0^n(s), \forall n$,belong to closed interval $\left[x_0^F(s), 1\right]$, $x_0(s) \in \left[x_0^F(s), 1\right]$. Thus, x_0 is bounded by x_0^F and 1. x_0 is increasing in s. If not, there would exist $s < s'$ such that $x_0(s) > x_0(s')$. Due to (36), for any $\varepsilon/2 > 0$, $\exists N$, $\forall n > N$,

$$\left|x_0^n(s) - x_0(s)\right| < \varepsilon/2,$$

or equivalently,

$$x_0(s) - \varepsilon/2 < x_0^n(s) < x_0(s) + \varepsilon/2,$$

and $\exists N'$, $\forall n > N'$,

$$\left|x_0^n(s') - x_0(s')\right| < \varepsilon/2,$$

or equivalently,

$$x_0(s') - \varepsilon/2 < x_0^n(s') < x_0(s') + \varepsilon/2.$$

For $n > \max\{N, N'\}$ and $\varepsilon < x_0(s) - x_0(s')$, we would have

$$x_0^n(s) - x_0^n(s') > x_0(s) - x_0(s') - \varepsilon > 0,$$

which contradicts the assumption that $x_0^n \in X_0(S)$. The continuity of x_0 can also be proved easily. Since x_0^n are continuous, $\forall \varepsilon > 0$, $\exists \delta > 0$, if $|s - s'| < \delta$, then

$$\left|x_0^n(s) - x_0^n(s')\right| < \varepsilon/3, \forall n.$$

Also, due to the convergence of $\{x_0^n\}_{n\in\mathbb{N}}$, we have that $\exists N$,

$$\begin{aligned}\left|x_0^n(s) - x_0(s)\right| &< \varepsilon/3, \\ \left|x_0^n(s') - x_0(s')\right| &< \varepsilon/3.\end{aligned}$$

Hence, $\forall \varepsilon > 0$, $\exists \delta > 0$, if $|s - s'| < \delta$,

$$\left|x_0(s) - x_0(s')\right| \le \left|x_0^n(s) - x_0(s)\right| + \left|x_0^n(s') - x_0(s')\right| + \left|x_0^n(s) - x_0^n(s')\right| < \varepsilon.$$

This completes the proof of the continuity of x_0. Therefore, $x_0 \in X_0(S)$, and $X_0(S)$ is compact.

Γ maps $X_0(S)$ into a subset $\hat{C}(S)$ of $C(S)$ which includes continuous function bounded by x_0^F and 1. To prove this, we show that for any $x_0 \in X_0(S)$, Γx_0 is bounded by x_0^F and 1 and continuous. Given that $\max\left\{J^{\min}(s), v_0(s)\right\} \le \underline{J} \le J^{\max}(s)$, it is easy to derive

$$x_0^F(s) = \prod_{i=1}^{n} F_i\left(r_i\left(s, \max\left\{J^{\min}(s), v_0(s)\right\}\right)\right) \le \Gamma x_0(s) \le \prod_{i=1}^{n} F_i\left(r_i\left(s, J^{\max}(s)\right)\right) = 1, \forall s.$$

Thus, Γx_0 is bounded by x_0^F and 1. Since $D\left(\underline{J}, s, x_0\right)$ is continuous in $\underline{J}$ and s given the continuity of $r_i\left(s, \underline{J}\right)$ and $\int_{T_{-i}(s,t_i)} f_{-i}(t_{-i})\, dt_{-i}$, and the interval $\left[\max\left\{J^{\min}(s), v_0(s)\right\}, J^{\max}(s)\right]$ is compact and continuous in s, $\underline{J}(s)$ is continuous in s according to the Theorem of the Maximum. This consequently implies that $\Gamma x_0(s) = \prod_{i=i}^{n} F_i\left(r_i\left(s, \underline{J}(s)\right)\right)$ is continuous in s.

Γ is a continuous mapping. Consider a converging sequence $\{x_0^n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} x_0^n = x_0$ in sup norm. That is, $\forall \varepsilon > 0$, $\exists N > 0$, $\forall n > N$, there is

$$\left\|x_0^n - x_0\right\| < \varepsilon.$$

Since $\|\cdot\|$ is sup norm, we have

$$\left|\int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0^n(\tilde{s})\, d\tilde{s} - \int_{\underline{s}}^{s} v_0'(\tilde{s})\, x_0(\tilde{s})\, d\tilde{s}\right| = \left|\int_{\underline{s}}^{s} v_0'(\tilde{s}) \left[x_0^n(\tilde{s}) - x_0(\tilde{s})\right] d\tilde{s}\right| < \varepsilon \left[v_0(\bar{s}) - v_0(\underline{s})\right], \quad (37)$$

and $\forall s \in S$,

$$\left|D\left(\underline{J}, s, x_0^n\right) - D\left(\underline{J}, s, x_0\right)\right| < \varepsilon \left[v_0(\bar{s}) - v_0(\underline{s})\right]. \quad (38)$$

Define $\underline{J}^n(s)$ as

$$\underline{J}^n(s) = \arg \min_{\max\{J^{\min}(s), v_0(s)\} \leq \underline{J} \leq J^{\max}(s)} \left|D\left(\underline{J}, s, x_0^n\right)\right|.$$

Now we show that $\underline{J}^n \to \underline{J}$ in sup norm. Let

$$A = \left\{(s, \underline{J}) : s \in S, \text{ and } \underline{J} \in \left[\max\left\{J^{\min}(s), v_0(s)\right\}, J^{\max}(s)\right]\right\}.$$

It is obvious that A is compact. We define a subset A_ε of A by

$$A_\varepsilon = \left\{(s, \underline{J}) \in A : \left|\underline{J} - \underline{J}(s)\right| \geq \varepsilon\right\}. \quad (39)$$

A_ε is compact, and for ε small enough, it is nonempty. The result is trivial when A_ε is empty. For any ε, let

$$\delta = \min_{(s, \underline{J}) \in A_\varepsilon} \left|\left|D\left(\underline{J}, s, x_0\right)\right| - \left|D\left(\underline{J}(s), s, x_0\right)\right|\right|. \quad (40)$$

The continuities of D in $\underline{J}, s$ and of $\underline{J}(s)$ in s guarantee the existence of δ. According to (38), $\forall \delta > 0$, $\exists N_\delta$, $\forall n > N_\delta$,

$$\left|D\left(\underline{J}, s, x_0^n\right) - D\left(\underline{J}, s, x_0\right)\right| < \frac{\delta}{2}, \quad (41)$$

so

$$\begin{aligned}
& \left|\left|D\left(\underline{J}^n(s), s, x_0\right)\right| - \left|D\left(\underline{J}(s), s, x_0\right)\right|\right| \\
= \; & \left|D\left(\underline{J}^n(s), s, x_0\right)\right| - \left|D\left(\underline{J}(s), s, x_0\right)\right| \\
\leq \; & \left|D\left(\underline{J}^n(s), s, x_0\right)\right| - \left|D\left(\underline{J}^n(s), s, x_0^n\right)\right| + \left|D\left(\underline{J}(s), s, x_0^n\right)\right| - \left|D\left(\underline{J}(s), s, x_0\right)\right| \\
\leq \; & \left|D\left(\underline{J}^n(s), s, x_0\right) - D\left(\underline{J}^n(s), s, x_0^n\right)\right| + \left|D\left(\underline{J}(s), s, x_0^n\right) - D\left(\underline{J}(s), s, x_0\right)\right| \\
< \; & \delta.
\end{aligned}$$

The equality and first inequality are based on the definitions of $\underline{J}^n(s)$ and $\underline{J}(s)$, the second inequality is using the triangle inequality of absolute values. The last inequality is from (41). Thus, from (39) and (40), for $n > N_\delta$,

$$\left|\underline{J}^n(s) - \underline{J}(s)\right| < \varepsilon, \forall s \in S.$$

This is equivalent to that $\underline{J}^n \to \underline{J}$ in sup norm. To proceed, we define function

$$H\left(s,\underline{J}\right)=\prod_{i=1}^{n}F_i\left(r_i\left(s,\underline{J}\right)\right)-\prod_{i=1}^{n}F_i\left(r_i\left(s,\underline{J}\left(s\right)\right)\right)$$

and $\underline{J}_{\varepsilon}^{u}\left(s\right)$ and $\underline{J}_{\varepsilon}^{d}\left(s\right)$ for $\varepsilon>0$ by

$$\begin{aligned}\underline{J}_{\varepsilon}^{u}\left(s\right) &= \begin{cases}\min\left\{\underline{J}:H\left(s,\underline{J}\right)\geq\varepsilon\right\}, \text{ if } \left\{\underline{J}:H\left(s,\underline{J}\right)\geq\varepsilon\right\} \text{ is nonempty;}\\ J^{\max}\left(s\right), \text{ if } \left\{\underline{J}:H\left(s,\underline{J}\right)\geq\varepsilon\right\} \text{ is empty.}\end{cases}\\ \underline{J}_{\varepsilon}^{d}\left(s\right) &= \begin{cases}\max\left\{\underline{J}:H\left(s,\underline{J}\right)\leq-\varepsilon\right\}, \text{ if } \left\{\underline{J}:H\left(s,\underline{J}\right)\leq-\varepsilon\right\} \text{ is nonempty;}\\ \max\left\{J^{\min}\left(s\right),v_0\left(s\right)\right\}, \text{ if } \left\{\underline{J}:H\left(s,\underline{J}\right)\leq-\varepsilon\right\} \text{ is empty.}\end{cases}\end{aligned}$$

Let

$$\hat{\delta}_{\varepsilon}=\inf_{s\in S}\left\{\max\left\{\underline{J}_{\varepsilon}^{u}\left(s\right)-\underline{J}\left(s\right),\underline{J}\left(s\right)-\underline{J}_{\varepsilon}^{d}\left(s\right)\right\}\right\}.$$

It is clear that $\hat{\delta}_{\varepsilon}>0$ for $\varepsilon>0$. Employing the convergence of $\underline{J}^n$, we have for $\hat{\delta}_{\varepsilon}>0$, $\exists N_{\hat{\delta}_{\varepsilon}}$, $\forall n>N_{\hat{\delta}_{\varepsilon}}$,

$$\left|\underline{J}^n\left(s\right)-\underline{J}\left(s\right)\right|<\hat{\delta}_{\varepsilon},\ \forall s\in S,$$

so $\underline{J}^n\left(s\right)\in\left(\underline{J}_{\varepsilon}^{d}\left(s\right),\underline{J}_{\varepsilon}^{u}\left(s\right)\right)$, and

$$\sup_{s\in S}\left|\prod_{i=1}^{n}F_i\left(r_i\left(s,\underline{J}^n\left(s\right)\right)\right)-\prod_{i=1}^{n}F_i\left(r_i\left(s,\underline{J}\left(s\right)\right)\right)\right|=\left\|\Gamma x_0^n-\Gamma x_0\right\|<\varepsilon.$$

Therefore, $\lim_{n\to\infty}\Gamma x_0^n=\Gamma x_0$ in sup norm. Γ is continuous.

Γ does not map $X_0\left(S\right)$ into itself, as we can guarantee that Γx_0 is increasing in s. Here we define another mapping Ψ over $\hat{C}\left(S\right)$, with $\Psi h\left(s\right)=\sup_{\hat{s}\in[\underline{s},s]}h\left(\hat{s}\right)$, for $h\in\hat{C}\left(S\right)$. It is obvious that $\Psi h\left(s\right)$ is increasing in s. So Ψ maps $\hat{C}\left(S\right)$ into $X_0\left(S\right)$, and the compound mapping $\Psi\circ\Gamma$ maps $X_0\left(S\right)$ into itself.

Ψ is continuous. Consider a converging sequence $\left\{h^n\right\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}h^n=h$ in sup norm, then $\forall\varepsilon>0$, $\exists N_{\varepsilon}$, $\forall n>N^{\varepsilon}$,

$$\left\|h^n-h\right\|<\varepsilon,$$

which is equivalent to

$$\left|h^n\left(s\right)-h\left(s\right)\right|<\varepsilon,\forall s\in S. \tag{42}$$

According to the definition of Ψ, for $s\in S$

$$\begin{aligned}\left|\Psi h^n\left(s\right)-\Psi h\left(s\right)\right| &= \left|\sup_{\hat{s}\in[\underline{s},s]}h^n\left(\hat{s}\right)-\sup_{\hat{s}\in[\underline{s},s]}h\left(\hat{s}\right)\right|\\ &= \left|h^n\left(s'\right)-h\left(s''\right)\right|,\end{aligned}$$

where s' and s'' are values on $[\underline{s}, s]$ that maximize h^n and h respectively. If $h^n(s') - h(s'') > 0$, then

$$|h^n(s') - h(s'')| = h^n(s') - h(s'') \leq h^n(s') - h(s');$$

if $h^n(s') - h(s'') \leq 0$, then

$$|h^n(s') - h(s'')| = h(s'') - h^n(s') \leq h(s'') - h^n(s'').$$

Thus, given (42), for $n > N_\varepsilon$, we have

$$|\Psi h^n(s) - \Psi h(s)| < \varepsilon, \forall s \in S.$$

That is,

$$\|\Psi h^n - \Psi h\| < \varepsilon.$$

This completes the proof that Ψ is continuous.

Given all the results above, one can see that the compound mapping $\Psi \circ \Gamma$ is continuous and maps from $X_0(S)$, which is non-empty, convex, and compact, into itself. According to Schauder's fixed point theorem, $\Psi \circ \Gamma$ has a fixed point on $X_0(S)$, that is, there exists a $x_0^* \in X_0(S)$ satisfying

$$x_0^* = \Psi \circ \Gamma x_0^*.$$

It is obvious that $x_0^*(\underline{s}) = x_0^F(\underline{s})$.

x_0^* is also a fixed point of Γ. To prove this, we first show that if for some $s \in S$, $x_0^*(s) = 1$, then $x_0^*(s') = \Gamma x_0^*(s') = 1$ for $s' \geq s$. Because if $x_0^*(s) = 1$, then there exists $\hat{s} \leq s$, $\Gamma x_0^*(\hat{s}) = 1$ which implies that

$$D(J^{\max}(\hat{s}), \hat{s}, x_0^*) = v_0(\hat{s}) - \left[\int_{\underline{s}}^{\hat{s}} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + U_0\left(M^F|\underline{s}\right)\right] \geq 0,$$

so for any $\hat{s}' \in (\hat{s}, \bar{s}]$,

$$\begin{aligned} D(J^{\max}(\hat{s}'), \hat{s}', x_0^*) &= v_0(\hat{s}') - \left[\int_{\underline{s}}^{\hat{s}'} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + U_0\left(M^F|\underline{s}\right)\right] \\ &= D(J^{\max}(\hat{s}), \hat{s}, x_0^*) + \int_{\hat{s}}^{\hat{s}'} v_0'(\tilde{s})\left[1 - x^*(\tilde{s})\right] d\tilde{s} \geq 0, \end{aligned}$$

Hence, $\Gamma x_0^*(\hat{s}') = 1 = x_0^*(\hat{s}')$.

Now we show that if $x_0^*(s) < 1$, $x_0^*(s)$ is strictly increasing in s. We prove this by contradiction. Suppose that for some $s' < s''$, $x_0^*(s') = x_0^*(s'') < 1$. Let $\check{s} = \inf\{\hat{s} \in S : x_0^*(\hat{s}) = x_0^*(s'')\}$. The continuity and monotonicity of x_0^* guarantee the existence of $\check{s}$, and $x_0^*(s) < x_0^*(\check{s})$ for $s < \check{s}$,

$x_0^*(s) = x_0^*(s'')$ for any $s \in [\check{s}, s'']$. Moreover, $x_0^*(\check{s}) = \Gamma x_0^*(\check{s}) \geq x_0^F(\check{s})$, as $x_0^*(s) < x_0^*(\check{s})$ for $s < \check{s}$. It is not possible to have $D(\underline{J}(\check{s}), \check{s}, x_0^*) = 0$, because if so, for $s \in (\check{s}, s'']$,

$$\begin{aligned}
& v_0(s) x_0^*(s) + \int_T \sum_{i=1}^n J_i(s, t_i) x_i^*(s, t) f(t) dt - \int_{\underline{s}}^{s} v_0'(\tilde{s}) x_0^*(\tilde{s}) d\tilde{s} + U_0\left(M^F | \underline{s}\right) \\
\geq \; & v_0(s) x_0^*(\check{s}) + \int_T \sum_{i=1}^n J_i(s, t_i) x_i^*(\check{s}, t) f(t) dt - \int_{\underline{s}}^{s} v_0'(\tilde{s}) x_0^*(\tilde{s}) d\tilde{s} + U_0\left(M^F | \underline{s}\right) \\
> \; & v_0(s) x_0^*(\check{s}) + \int_T \sum_{i=1}^n J_i(\check{s}, t_i) x_i^*(\check{s}, t) f(t) dt - \int_{\underline{s}}^{s} v_0'(\tilde{s}) x_0^*(\tilde{s}) d\tilde{s} + U_0\left(M^F | \underline{s}\right) \\
= \; & D(\underline{J}(\check{s}), \check{s}, x_0^*) + [v_0(s) - v_0(\check{s})] x_0^*(\check{s}) - \int_{\check{s}}^{s} v_0'(\tilde{s}) x_0^*(\tilde{s}) d\tilde{s} \\
= \; & 0.
\end{aligned}$$

In the first line, $x^*(s, \cdot)$ denote the allocation rule maximizing the virtual surplus given $x_0^*(s)$. The first inequality is due to the optimality of $x^*(s, \cdot)$. The second inequality is because that $J_i(s, t_i)$ is strictly increasing in s. The first equality is derived from the definition of $D(\underline{J}(\check{s}), \check{s}, x_0^*)$, and the last equality is resulted from $D(\underline{J}(\check{s}), \check{s}, x_0^*) = 0$ and $x_0^*(s) = x_0^*(s'')$ for any $s \in [\check{s}, s'']$. This sequence of inequalities implies that $\Gamma x_0^*(s) > \Gamma x_0^*(s)$. This contradicts the definition of $x_0^*(s)$. Thus, it is only possible to have $D(\underline{J}(\check{s}), \check{s}, x_0^*) < 0$, which indicates that $\Gamma x_0^*(\check{s}) = x_0^F(\check{s}) = \Psi x_0^F(\check{s})$. Define

$$D(s, x_0^*) = v_0(s) x_0^*(s) + \int_T \sum_{i=1}^n J_i(s, t_i) x_i^*(s, t) f(t) dt - \int_{\underline{s}}^{s} v_0'(\tilde{s}) x_0^*(\tilde{s}) d\tilde{s} + U_0\left(M^F | \underline{s}\right),$$

$$\underline{\check{s}} = \sup\{s \in [\underline{s}, \check{s}] : D(s, x_0^*) = 0\}.$$

The set $\{s \in [\underline{s}, \check{s}] : D(s, x_0^*) = 0\}$ is nonempty, as it includes $\underline{s}$, so $\underline{\check{s}}$ exists. The continuity of $x_0^*(s)$ implies that $\underline{\check{s}} < \check{s}$ and $D(\underline{\check{s}}, x_0^*) = 0$, $D(s, x_0^*) > 0$, for $s \in (\underline{\check{s}}, \check{s}]$. Suppose that $x_0^*(s') > \Psi x_0^F(s')$ for some $s' \in (\underline{\check{s}}, \check{s}]$. Given that $x_0^*(s') = \Psi \circ \Gamma x_0^*(s')$, there exists $s'' \leq s'$, $x_0^*(s') = \Gamma x_0^*(s'')$, thus $\Gamma x_0^*(s'') > \Psi x_0^F(s') \geq x_0^F(s'')$, which implies that $D(s'', x_0^*) = 0$, $s'' \leq \underline{\check{s}}$, and $x_0^*(s) = x_0^*(s'')$ for any $s \in (s'', s']$. The argument above already points out that this is not possible. Therefore, $x_0^*(s') = \Psi x_0^F(s')$, $\forall s' \in (\underline{\check{s}}, \check{s}]$. Continuity of $x_0^*(s), \Psi x_0^F(s)$ implies that $x_0^*(\underline{\check{s}}) = \Psi x_0^F(\underline{\check{s}})$. Then we can obtain

$$\begin{aligned}
D(\check{s}, x_0^*) - D(\underline{\check{s}}, x_0^*) \; = \; & v_0(\check{s}) x_0^*(\check{s}) + \int_T \sum_{i=1}^n J_i(\check{s}, t_i) x_i^*(\check{s}, t) f(t) dt \\
& - v_0(\underline{\check{s}}) x_0^*(\underline{\check{s}}) + \int_T \sum_{i=1}^n J_i(\underline{\check{s}}, t_i) x_i^*(\underline{\check{s}}, t) f(t) dt - \int_{\underline{\check{s}}}^{\check{s}} v_0'(\tilde{s}) x_0^*(\tilde{s}) d\tilde{s} \\
\geq \; & 0,
\end{aligned}$$

according to (24) and Appendix C. This contradicts the supposition that $D(\check{s}, x_0^*) = D(\underline{J}(\check{s}), \check{s}, x_0^*) < 0$. This finally completes the proof that $x_0^*(s)$ is strictly increasing in s if $x_0^*(s) < 1$. Hence, $x_0^*(s) =$

$\Gamma x_0^*(s)$ for $x_0^*(s) < 1$. Combining with the result above that $x_0^*(s) = \Gamma x_0^*(s)$ for $x_0^*(s) = 1$, x_0^* is a fixed point of Γ.

In the rest of the proof, we show that for x_0^*,

$$D\left(\underline{J}(s), s, x_0^*\right) = 0, \forall s \in S.$$

We still prove this by contradiction. First, suppose that $\exists s'$, $D\left(\underline{J}(s'), s', x_0^*\right) > 0$, which implies that $x_0^*(s') = 1$. Continuity of x_0^* and $D\left(\underline{J}(s), s, x_0^*\right)$ indicates that $\exists \hat{s} < s'$, such that $x_0^*(\hat{s}) = 1$ and $D\left(\underline{J}(\hat{s}), \hat{s}, x_0^*\right) = 0$. The monotonicity of x_0^* gives us

$$\begin{aligned}
D\left(\underline{J}(s'), s', x_0^*\right) &= v_0(s') - \left[\int_{\underline{s}}^{s'} v_0'(\tilde{s})\, x_0^*(\tilde{s})\, d\tilde{s} + U_0\left(M^F|\underline{s}\right)\right] \\
&= D\left(\underline{J}(\hat{s}), \hat{s}, x_0^*\right) + \int_{\hat{s}}^{s'} v_0'(\tilde{s})\left[1 - x_0^*(\tilde{s})\right] d\tilde{s} \\
&= 0,
\end{aligned}$$

which violates the supposition that $D\left(\underline{J}(s'), s', x_0^*\right) > 0$. Second, suppose that $\exists s''$, $D\left(\underline{J}(s''), s'', x_0^*\right) < 0$, which implies that $x_0^*(s'') = x_0^F(s'')$. Still, employing the continuity of x_0^* and $D\left(\underline{J}(s), s, x_0^*\right)$ yields that $\exists \hat{s} < s''$, such that $x_0^*(s) = x_0^F(s)$ for $s \in [\hat{s}, \bar{s}]$, $D\left(\underline{J}(\hat{s}), \hat{s}, x_0^*\right) = 0$.

$$\begin{aligned}
D\left(\underline{J}(s''), s'', x_0^*\right) - D\left(\underline{J}(\hat{s}), \hat{s}, x_0^*\right) &= v_0(s'')\, x_0^F(s'') + \int_T \textstyle\sum_{i=1}^n J_i(s'', t_i)\, x_i^F(s'', t)\, f(t)\, dt \\
&\quad - v_0(\hat{s})\, x_0^F(\hat{s}) + \int_T \textstyle\sum_{i=1}^n J_i(\hat{s}, t_i)\, x_i^F(\hat{s}, t)\, f(t)\, dt \\
&\quad - \int_{\check{\underline{s}}}^{\check{s}} v_0'(\tilde{s})\, x_0^F(\tilde{s})\, d\tilde{s} \\
&> 0.
\end{aligned}$$

The inequality is due to that fact that full-information optimal mechanism is not incentive compatible. (See equation (21)). Therefore, $D\left(\underline{J}(s), s, x_0^*\right) = 0$, $\forall s \in S$.

H.