Two-sided investments and matching with multi-dimensional cost types and attributes

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Abstract

I study settings in which heterogeneous buyers and sellers, characterized by cost types, must invest in attributes before they compete for partners in a frictionless, continuum assignment market. I define Cole, Mailath and Postlewaite's (2001a) notion of ex-post contracting equilibrium in a general assignment game framework. Ex-ante efficient investment and matching can always be supported in equilibrium. The main part of the paper sheds light on what enables and what precludes coordination failures resulting in mismatch of agents (from an ex-ante perspective) and/or pairwise inefficient investments. A kind of technological multiplicity is the key source of potential inefficiencies. Absence of technological multiplicity rules out pairwise inefficient investments, and it heavily constrains mismatch in multi-dimensional environments with differentiated agents. An example with simultaneous under- and over-investment shows that even extreme exogenous heterogeneity may not suffice to rule out inefficient equilibria in environments with technological multiplicity.

Keywords: Matching, pre-match investments, coordination failure, mismatch, multi-dimensional attributes, assignment game, optimal transport.

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1 Introduction

In many markets, participants make investments before they enter and compete for partners with whom they then trade or form some kind of productive relationship. Probably the most salient example is the labor market, where individuals invest in human capital before they try to find a job while employers invest in technology or create firms before hiring new workers. Premarital investments, made by men and women before entering the marriage market, constitute another important case that has attracted a lot of attention by economists (e.g. Peters and Siow 2002; Iyigun and Walsh 2007; Chiappori, Iyigun and Weiss 2009).

A sizeable literature has examined, in the context of one-to-one matching models with a pre-match investment stage, how various frictions in the (post-investment) matching market, such as search frictions (e.g. Acemoglu 1996; Acemoglu and Shimer 1999) or asymmetric information (Mailath, Postlewaite and Samuelson 2012, 2013) lead to inefficient investment incentives. Hold-up problems (e.g. Williamson 1985) arising from bargaining power in frictionless markets with a small (finite) number of agents (Cole, Mailath and Postlewaite 2001b; Felli and Roberts 2001) and consequences of non-transferable utility (e.g. Peters and Siow 2002) have been studied as well.[1]

In a seminal article, Cole, Mailath and Postlewaite (2001a, henceforth (CMP)) showed that investments are not necessarily efficient even if utility is transferable (TU) and the matching market is frictionless and competitive, in the sense that agents take as fixed the utilities that must be provided to potential matching partners: while there always is an efficient *ex-post contracting equilibrium*, there may be inefficient equilibria as well. The notion of ex-post contracting equilibrium (see below) captures a market incompleteness associated with situations where agents' sunk investments endogenously determine which markets are open at the matching stage. Inefficient equilibria may be interpreted as arising from coordination failures in investment choices.

The main goal of this paper is to shed light on what enables and what constrains, or even precludes, investment coordination failures in ex-post contracting equilibrium. The model and analysis build on (CMP), but I allow for more general investment choices and match surplus functions, and for more general forms of ex-ante heterogeneity of agents. In particular, investment choices may be multi-dimensional, reflecting the fact that investments affect several relevant quality and/or skill dimensions (and may lead to endogenous specialization that cannot be represented in a one-dimensional model) in many interesting environments. Accordingly, there may be multi-dimensional heterogeneity of characteristics representing "costs" or "abilities"

[1] I discuss the related literature on "investment and matching" in more detail in Section 2.

before investment. Two different kinds of inefficiency can occur. First, there may be *inefficiency of joint investments*, meaning that the equilibrium investments do not maximize net surplus for some matches that form. Secondly, there may be a *mismatch* of agents, meaning that some agents match with "wrong" partners from the perspective of ex-ante efficiency. This latter inefficiency is impossible in (CMP)'s framework of one-dimensional heterogeneity and positive assortative matching.

The main contributions of the paper are as follows. First, I develop a new sufficient condition, *absence of technological multiplicity*, for ruling out inefficiency of joint investments. Secondly, I provide an analysis of mismatch in a particular model featuring multi-dimensional heterogeneity. In particular, I show how powerful results from the mathematical theory on optimal transport problems can be used to study mismatch, and to develop sufficient conditions for ruling out mismatch, in such settings. Thirdly, I provide new insights about the role of ex-ante heterogeneity for eliminating inefficiencies in environments plagued with technological multiplicity. In an important subsequent contribution, Nöldeke and Samuelson (2014) have further advanced the study of ex-post contracting equilibria, in particular by extending the analysis to environments with imperfectly transferable utility (Legros and Newman 2007).

To be more precise about the model, consider a continuum of buyers and sellers. All agents must first, simultaneously and non-cooperatively, invest in costly *attributes*. In the second stage (ex-post), they compete for partners in a frictionless one-to-one matching market, pair off and divide a surplus that depends on both parties' investments. Agents are ex-ante heterogeneous, differing in their costs for making the various possible investments. Utility is transferable, and the technology is deterministic in the sense that investments determine attributes and attributes determine (gross) surplus. Like (CMP), I label agents "sellers" and "buyers," but one could equally call them "workers" and "firms" (or "women" and "men").

Competitive equilibria of the ex-post *continuum transferable utility assignment game* (Gretzky, Ostroy and Zame 1992, 1999) feature an efficient, surplus-maximizing matching of buyers and sellers on the basis of their sunk investments. The appropriate equilibrium notion for the two-stage model is less obvious.[2] In an ex-post contracting equilibrium, every attribute choice has to respond optimally, given the agent's costs, to the correctly anticipated trading possibilities and payoffs in the equilibrium market. An agent who deviates by choosing an otherwise non-existent attribute can match with any marketed attribute from the other side, leave (slightly more than) the equilibrium payoff to the partner and keep the remaining surplus. This pins down payoffs outside of the equilibrium support. Alternatively, one could require that individual invest-

[2] Compare the discussion in Mailath, Postlewaite and Samuelson (2013).

ments must be optimal with respect to a market-clearing payoff/price system for *all ex-ante possible* attributes. In this case, coordination failures are ruled out. However, by assuming that investments are directed by a price system that attaches market prices to attributes from the other side that nobody chooses, one essentially ignores that sunk investments determine actual trading possibilities. Mailath, Postlewaite and Samuelson (2013, pg. 537) remark: "On the one hand, we find the existence of such prices counterintuitive. On the other hand and more importantly, like Makowski and Ostroy (1995), we expect coordination failures to be endemic when people must decide what goods to market, and hence we think it important to work with a model that does not preclude them."

In (CMP), buyers and sellers choose between *one-dimensional* investment levels, agents can be *completely ordered* in terms of their *marginal cost of investment*, and gross surplus is a strictly *supermodular* function of investment levels.[3] Combining the *Spence-Mirrlees single crossing conditions* of this "1-d supermodular framework" with the absence of hold-up problems,[4] (CMP) proved that there always is an equilibrium in which all agents invest and match efficiently. They also gave examples of inefficient equilibria in which parts of both populations under-invest (over-invest) because suitable complementary investments are missing. Moreover, (CMP) showed that these coordination failures are ruled out if agents are very heterogeneous ex-ante.

I use the Kantorovich duality theorem from the mathematical theory of optimal transport (Villani 2009) to define ex-post contracting equilibrium in more general environments. The duality theorem characterizes all *stable and feasible bargaining outcomes*, i.e. pairs of efficient matching and core payoff function (equivalently, competitive equilibria, see Section 2), of any continuous assignment game (Shapley and Shubik 1971; Gretzky, Ostroy and Zame 1992, 1999). Consequently, the model allows for the representation of quite general preference relations both before and after investment.[5]

I verify first that virtually any stable and feasible bargaining outcome of the benchmark assignment game in which buyers and sellers can bargain and write complete contracts before they invest, so that partners choose *jointly optimal* attributes, can be supported by an ex-post contracting equilibrium ((CMP)'s nice explicit proof heavily

[3]Such assumptions à la Becker (1973) have been made by a majority of papers that study two-sided matching with quasi-linear utility.

[4]By assumption, no single individual from the continuum economy can affect the market payoffs of others. See Cole, Mailath and Postlewaite (2001b), (CMP) and the discussion of Makowski (2004) in Section 2. TU and frictionless matching eliminate other potential sources of hold-up.

[5]Moreover, the approach resolves some technical issues, concerning feasibility in a continuum model, that have been discussed at some length in (CMP).

used the conditions of the 1-d supermodular framework).[6]

The important open task is to shed light on the complex interplay of technology (surplus and cost functions), competition and exogenous heterogeneity that determines whether inefficient equilibria are possible and, if so, what kind of coordination failures may occur. In the 1-d supermodular framework, matching has to be positively assortative in *any* equilibrium, both ex-post, in investment levels, and from the ex-ante view, in cost types. While it may sometimes happen that investments are not jointly optimal, there can never be any *mismatch* of buyers and sellers. By contrast, beyond the 1-d supermodular framework it is a priori unclear which matching patterns can be part of an equilibrium.

A preliminary observation is very useful. Due to competition ex-post ("no hold-up"), every agent's investment has to maximize net surplus conditional on the attribute of his partner: for all equilibrium pairs, attribute choices form a Nash equilibrium of a *hypothetical* "full appropriation" (FA) game. In this game, both agents internalize their investment's full net effect on joint surplus. Starting from this simple property of equilibria, I gain significant insights into the main sources, forms and limitations of coordination failure in a sequence of different environments.

Jointly optimal attributes are always a Nash equilibrium of the FA game between a buyer and a seller. If FA games have multiple equilibria for some pairs (and not all Nash profiles maximize actual net surplus), this *technological multiplicity* is a key source of potential inefficiency. In the 1-d supermodular framework, technological multiplicity is *necessary* for inefficiency. Beyond that framework, mismatch is sometimes possible even if equilibria of FA games are unique (Section 5.2.1). On the other hand, I find that the absence of technological multiplicity also restricts mismatch (Sections 5.2.2 and 5.2.3). In such cases, jointly optimal attributes (and hence equilibrium investments) depend continuously on the types of partners. Moreover, any attribute choice shows the agent's specialization for the intended match, but it also strongly reflects his own cost type. Therefore, marketed attributes are potentially attractive targets for deviations by types that are not too different from the agent's partner, especially if these types would be more suitable matches for the agent ex-ante. Profitable deviations must be ruled out by sufficiently high equilibrium payoffs. These requirements constrain mismatch and may even completely preclude it, in particular if agents are ex-ante differentiated (not necessarily very heterogeneous) and/or attributes very strongly depend on own cost types. In some environments, it is possible to check for coordination failures by going through all possible cases, or by using a lot of a priori

[6] Often, though not always, there is a unique ex-ante stable and feasible bargaining outcome. Compare Section 2 and footnote 34.

knowledge about the structure of equilibria (Sections 5.2.1 and 5.2.2). In more complex settings, one has to use the characterization of ex-ante efficient matchings that is implied by the Kantorovich duality theorem to evaluate whether (local and/or global) mismatch is possible. I provide an example of such an analysis in Section 5.2.3.

(CMP)'s examples show that severe coordination failures can happen in environments with technological multiplicity. These problems are exacerbated outside of the 1-d supermodular framework (Section 5.3). In particular, mismatch becomes a common feature of inefficient equilibria. Accordingly, the effect that a high level of exogenous heterogeneity may rule out coordination failures (by *ensuring* that market segments needed for ex-ante efficiency are open in equilibrium) is much weaker.

Finally, I show that even extreme exogenous diversity may be insufficient to eliminate inefficient equilibria. This holds also in the 1-d supermodular framework (Section 5.4). The finding highlights the importance of the precise form of technological multiplicity, and it complements the picture of the most interesting inefficiencies in (CMP)'s original model.

2 Related literature

(CMP)'s analysis motivated this paper. In a recent comprehensive study that builds on the results of (CMP) and of the present paper, Nöldeke and Samuelson (2014) have extended the investigation of ex-post contracting equilibria to environments with imperfectly transferable utility (ITU). In particular, they generalized the idea of absence of technological multiplicity to ITU environments and provided corresponding sufficient conditions based on quasi-concavity of utility functions. Moreover, they proposed sufficient conditions to rule out mismatch in ITU settings with one-dimensional heterogeneity (related to the *generalized increasing differences* conditions of Legros and Newman 2007) and pointed out some consequences of the separability assumption made in most of the literature (including the present paper) that agents' preferences at the matching stage do no depend on initial types.

In a companion paper for (CMP), Cole, Mailath and Postlewaite (2001b) examined the case of finitely many buyers and sellers. An efficient equilibrium that does not hinge on special off-equilibrium bargaining outcomes exists whenever a non-generic "double-overlap" condition is satisfied. Generically, full efficiency is achievable only if off-equilibrium outcomes punish deviations, which requires unreasonable sensitivity to whether the deviating agent is a buyer or a seller. A particular and limited form of mismatch is sometimes possible due to the externality that a single agent can exert on others by "shooting for a better partner" through an aggressive investment. This type of coordination failure was first identified by Felli and Roberts (2001).[7] Makowski (2004) analyzed a continuum model in which single agents are, and expect to be, pivotal for aggregate market outcomes whenever the endogenous market has a non-singleton core. He showed that results similar to those of Cole, Mailath and Postlewaite (2001b) hold in this case. In particular, hold-up and inefficiencies à la Felli-Roberts are possible. As the focus of the present paper is on whether trading options in endogenous markets necessarily limit or rule out coordination failures, I follow (CMP) and assume that a single agent is not, and does not expect to be, pivotal for aggregate market outcomes in a very large economy.[8]

In a model with *non-transferable utility* (NTU) and continuum populations, Peters and Siow (2002) showed that there is an equilibrium that is Pareto efficient.[9] Under their assumptions (fifty-fifty sharing of an additive match surplus, ordered cost types) the equilibrium also maximizes aggregate surplus.[10] Acemoglu (1996) formalized hold-up problems associated with search frictions in the second-stage matching market. He demonstrated how such frictions result in a "pecuniary" (Acemoglu 1996, pg. 779) externality that can explain social increasing returns in human (and physical) capital accumulation, in a model without technological externalities. Mailath, Postlewaite and Samuelson (2012, 2013) introduced another friction, namely that sellers cannot observe buyers' attributes and are (therefore) restricted to uniform pricing. They studied the impact that agents' *premuneration values* have for the (in)efficiency of investments in this case. Premuneration values add up to joint surplus and describe individual values or benefits from a particular match in the absence of payments.[11]

The seminal paper on the transferable utility assignment game is Shapley and Shubik (1971). For the case of finitely many buyers and sellers, they proved that the core of the assignment game is equivalent to the set of Walrasian equilibria, and to the solutions of a linear program. More precisely, solutions to the linear program of max-

[7]They studied the interplay of hold-up and coordination failure when double-overlap does not hold, and when buyers bid for sellers in a particular non-cooperative game.

[8]Note that such assumptions are in principle consistent with introducing small amounts of uncertainty, e.g. in the form of a small "probability of death" between investment and market participation. See also Gall (2013), Gall, Legros, and Newman (2013) and Bhaskar and Hopkins (2013) for models with a stochastic investment technology.

[9]Bhaskar and Hopkins (2013) pointed to a limitation of this result by showing that, for deterministic investments and NTU, the set of equilibria can be very large.

[10]Gall, Legros and Newman (2013) examined investment distortions when equilibrium matching under NTU is not surplus-maximizing, focusing on inefficiencies due to excessive segregation in higher education markets. They identified re-match policies, in particular affirmative action policies, that can increase aggregate welfare. Gall (2013) further analyzed the surplus efficiency of investments in a one-sided market with non-transferabilities described by general Pareto frontiers (as in Legros and Newman, 2007).

[11]For example, buyers get the good in case of trade, firms own the output produced by a worker who has to bear the cost of effort, and so on.

imizing aggregate surplus are Walrasian allocations, and dual solutions are elements of the core and correspond to Walrasian equilibrium payoffs: no matter how surplus would be divided in the absence of payments, there are Walrasian, "personalized" prices (Mailath, Postlewaite and Samuelson 2012, 2013) that correct these divisions such that payoffs correspond to the core element. Furthermore, in the language of the two-sided matching literature, core payoffs are stable and feasible surplus shares. Gretzky, Ostroy and Zame (1992) extended these equivalences to continuum models, in which the heterogeneous populations of buyers and sellers are described by non-negative Borel measures on the spaces of possible attributes. Gretzky, Ostroy and Zame (1999) identified several equivalent conditions for perfect competitiveness of an assignment economy with continuous surplus function, in the sense that individuals (in the continuum model, infinitesimal individuals) are unable to manipulate prices in their favor. Among these conditions are that the core is a singleton and that all agents appropriate their full marginal product. Gretzky, Ostroy and Zame showed that perfect competitiveness is a generic property for continuum assignment games with continuous match surplus,[12] and that most large finite assignment games are "approximately perfectly competitive."

The linear program associated with the assignment game, the optimal transport problem, is the object of study of an extensive mathematical literature. Villani (2009) is an excellent reference that surveys a multitude of results, including the fundamental duality theorem that I use in this paper. More advanced topics comprise sufficient conditions for uniqueness of optimal transports/ Walrasian allocations (Gretzky, Ostroy and Zame (1999) were concerned with uniqueness of payoffs) and for *purity* of these assignments (that is, each type of agent is matched to exactly one type of agent from the other side), a delicate regularity theory, and many other things.

Equilibria which require that investments, in the two-stage model, are optimal with respect to a market-clearing payoff system for all ex-ante possible attributes are equivalent to hedonic equilibria in a hedonic pricing model (Rosen 1974) with quasi-linear utility (Ekeland 2005, 2010; Chiappori, McCann and Nesheim 2010). In this hedonic pricing model, any seller chooses a bundle of a seller and a buyer attribute at production cost equal to her cost for the seller attribute, and a buyer's utility from purchasing a bundle equals the gross surplus net of his cost for the buyer attribute. In hedonic equilibrium, sellers' production and buyers' consumption decisions must be optimal with respect to a market-clearing price system for all possible bundles. Using a convex programming approach, Ekeland proved existence and efficiency of hedonic equilibria.

[12]Intuitively, the existence of a long side and a short side of the market as well as "overlaps" of matched agent types are generic and pin down unique core utilities.

Chiappori, McCann and Nesheim simplified this equivalence between hedonic pricing and ex-ante efficient matching by relating both to the (linear) optimal transport problem. Like Ekeland, they also established sufficient conditions for uniqueness and purity of an optimal matching (which combine generalized single-crossing conditions for the surplus function with mild conditions on the distributions of types), as well as a weaker condition that suffices for uniqueness.

Some less closely related papers analyzed how heterogeneous agents compete for partners through costly *signals* in the assortative framework. In Hoppe, Moldovanu and Sela (2009), investments are wasteful and may be used to signal private information about characteristics that determine match surplus (which is shared fifty-fifty). They studied how the heterogeneity of (finite or infinite) populations affects the amount of wasteful signalling. Hopkins (2012) examined a model in which investments signal private information about productive characteristics *and* affect surplus (investments are partially wasteful). His main results nicely identify comparative statics effects due to changes in the populations, both under NTU and under TU.

The plan of the paper is as follows. In Section 3, I introduce the model and define ex-post contracting equilibrium. Section 4 contains the result on existence of ex-ante efficient equilibria. In the main part of the paper, Section 5, I study sources, forms and limitations of potential coordination failures. All proofs are in the Appendix.

3 Model

3.1 Agent populations, costs, and match surplus

There is a continuum of buyers and sellers. All agents have quasi-linear utility functions, and utility is transferable. At time $t = 0$, all agents simultaneously and non-cooperatively invest in costly *attributes*. If a buyer of *type* $b \in B$ chooses an attribute $x \in X$, he incurs a cost $c_B(x, b)$. Similarly, a seller of type $s \in S$ can invest into attribute $y \in Y$ at cost $c_S(y, s)$. B, S, X and Y are compact metric spaces,[13] and $c_B : X \times B \to \mathbb{R}_+$ and $c_S : Y \times S \to \mathbb{R}_+$ are continuous functions. If a buyer with attribute x and a seller with attribute y form a match at time $t = 1$ (ex-post), they generate a *gross surplus* $v(x, y)$. The function v describes the gains from trade (or, in a "worker-firm" context, the surplus from joint production) for any pair of attributes. I assume that $v : X \times Y \to \mathbb{R}_+$ is continuous and that unmatched agents obtain a surplus

[13] I suppress metrics and induced topologies in the notation.

of zero.[14] From an ex-ante perspective, the *maximal net surplus* that a buyer-seller pair (b, s) can generate is

$$w(b, s) = \max_{x \in X, y \in Y} h(x, y|b, s), \tag{1}$$

where

$$h(x, y|b, s) := v(x, y) - c_B(x, b) - c_S(y, s).$$

For all $(b, s) \in B \times S$, *jointly optimal* attributes $(x^*(b, s), y^*(b, s))$ maximizing $h(\cdot, \cdot|b, s)$ exist because X and Y are compact, and v, c_B and c_S are continuous (the pair $(x^*(b, s), y^*(b, s))$ need not be unique). By the Maximum Theorem, w is continuous.

The heterogeneous ex-ante populations of buyers and sellers are described by Borel probability measures μ_B on B and μ_S on S.[15] The "generic" case with a long side and a short side of the market (more buyers than sellers, or vice versa) is easily included by adding, topologically isolated, "dummy" types on the short side. Dummy types $b_\varnothing \in B$ and $s_\varnothing \in S$ always choose dummy attributes $x_\varnothing \in X$ and $y_\varnothing \in Y$ at a cost of zero. $x_\varnothing$ and $y_\varnothing$ are prohibitively costly for all $b \neq b_\varnothing$, $s \neq s_\varnothing$, so that no real agent ever chooses them. The assumption that unmatched agents create no surplus yields $v(x_\varnothing, \cdot) \equiv 0$ and $v(\cdot, y_\varnothing) \equiv 0$. All statements about functional forms and properties of v, c_B, c_S and w consistently refer to non-dummy types and attributes.

3.2 Transferable utility assignment games

At $t = 1$, buyers and sellers compete for partners in a frictionless market with complete information and transferable utility (TU). This *continuum transferable utility assignment game* is characterized by the gross surplus function v[16] and by measures μ_X on X and μ_Y on Y that describe the heterogeneous post-investment populations. A *stable and feasible bargaining outcome* of such an assignment game consists of i) a surplus-maximizing (efficient) *matching* of μ_X and μ_Y and ii) core payoffs, for all attributes from the supports of μ_X and μ_Y (the existing attributes).[17] These payoffs yield a feasible division of surplus in all matched pairs, and they make the underlying matching stable. Surplus-efficiency of the matching is necessary for stability because

[14] The latter assumption is made for simplicity only. A model in which it may be valuable that some agents stay unmatched even though potential partners are still available is ultimately equivalent.

[15] I use normalized measures, which is common in optimal transport. Gretzky, Ostroy and Zame (1992, 1999) worked with non-negative Borel measures. This is useful for analyzing the "social gains function" that plays a key role in their work.

[16] As v is continuous, the framework is equivalent to those of Gretzky, Ostroy and Zame (1999) and Gretzky, Ostroy and Zame (1992, Section 3.5).

[17] In ex-post contracting equilibrium, all agents choose attributes from these supports. See Section 3.3, where I also define the payoffs after a deviation by a single agent.

of TU. Furthermore, stable and feasible bargaining outcomes are equivalent to competitive equilibria (see Section 2 for a detailed explanation). The cooperative formulation has the advantage that one can focus directly on payoffs as divisions of joint surplus, without specifying individual utility functions.

As in (CMP), there is another relevant assignment game. It corresponds to the benchmark case in which buyers and sellers can bargain and write complete contracts before they invest, so that partners choose jointly optimal attributes. This assignment game is described by w, μ_B and μ_S, and its stable and feasible bargaining outcomes are ex-ante efficient.

In Theorem 1 below, I summarize results of Theorem 5.10 from Villani (2009) that characterize *all* stable and feasible bargaining outcomes of *any* given assignment game with continuous surplus. I use this characterization to define ex-post contracting equilibrium, which requires that individual investments are "best-replies" to the trading possibilities and bargaining outcome of the endogenous second-stage market, and to verify the existence of efficient equilibria. Moreover, I need the information that Theorem 1 (applied to (μ_B, μ_S, w)) provides about the structure of ex-ante efficient matchings for the study of potential coordination failures (in Section 5.2.3).

To avoid additional notation, I state the general results of this section for assignment games (μ_X, μ_Y, v). The exposition is concise and collects only what is needed for the subsequent analysis of ex-post contracting equilibria. Readers who are interested in proofs and further details should consult Chapters 4 and 5 of Villani (2009) and Gretzky, Ostroy and Zame (1992, 1999).

The possible *matchings* of μ_X and μ_Y are the measures π on $X \times Y$ with marginals μ_X and μ_Y.[18] Let $\Pi(\mu_X, \mu_Y)$ denote the set of all these matchings. The linear program for an efficient matching is to find a $\pi^* \in \Pi(\mu_X, \mu_Y)$ that attains

$$\sup_{\pi \in \Pi(\mu_X, \mu_Y)} \int v \, d\pi.$$

The dual program is to find payoff functions $\psi_X^* : \mathrm{Supp}(\mu_X) \to \mathbb{R}$ and $\psi_Y^* : \mathrm{Supp}(\mu_Y) \to \mathbb{R}$ (the measure supports $\mathrm{Supp}(\mu_X)$ and $\mathrm{Supp}(\mu_Y)$ describe the sets of existing attributes[19]) with the following property: ψ_X^* and ψ_Y^* minimize aggregate payoffs among all $\psi_X \in L^1(\mu_X)$ and $\psi_Y \in L^1(\mu_Y)$ that are stable in the sense that

[18]Matchings are called *couplings* in the optimal transport literature. As surplus is non-negative and unmatched agents create zero surplus, there is no need to explicitly consider the possibility that agents stay single. Agents who get a dummy partner are of course de facto unmatched.

[19]More precisely, for any $x \in \mathrm{Supp}(\mu_X)$, every neighborhood of attributes containing x has strictly positive mass.

$\psi_Y(y) + \psi_X(x) \geq v(x,y)$ for all $(x,y) \in \text{Supp}(\mu_X) \times \text{Supp}(\mu_Y)$.[20] That is, ψ_X^* and ψ_Y^* must attain

$$\inf_{\{(\psi_X,\psi_Y)|\ \psi_Y(y)+\psi_X(x)\geq v(x,y) \text{ for all } (x,y)\in \text{Supp}(\mu_X)\times\text{Supp}(\mu_Y)\}} \left(\int \psi_Y\, d\mu_Y + \int \psi_X\, d\mu_X \right).$$

To find solutions ψ_X^* and ψ_Y^*, one may restrict attention to functions ψ_X that are *v-convex* with respect to the sets $\text{Supp}(\mu_X)$ and $\text{Supp}(\mu_Y)$ and set $\psi_Y := \psi_X^v$, the so-called *v-transform* of ψ_X.

Definition 1. *A function $\psi_X : \text{Supp}(\mu_X) \to \mathbb{R}$ is called* v-convex, *w.r.t. the sets* $\text{Supp}(\mu_X)$ *and* $\text{Supp}(\mu_Y)$*, if there is a function ζ: $\text{Supp}(\mu_Y) \to \mathbb{R} \cup \{+\infty\}$ such that*

$$\psi_X(x) = \sup_{y\in\text{Supp}(\mu_Y)} (v(x,y) - \zeta(y)) =: \zeta^v(x), \quad \textit{for all } x \in \text{Supp}(\mu_X).$$

The function $\psi_X^v(y) := \sup_{x\in\text{Supp}(\mu_X)} (v(x,y) - \psi_X(x))$, defined on $\text{Supp}(\mu_Y)$*, is called the* v-transform *of ψ_X. The* v-subdifferential *of ψ_X, $\partial_v\psi_X$ is defined as*

$$\partial_v\psi_X := \{(x,y) \in \text{Supp}(\mu_X) \times \text{Supp}(\mu_Y)|\ \psi_X^v(y) + \psi_X(x) = v(x,y)\}.$$

Remark 1. *i) $\psi_X : \text{Supp}(\mu_X) \to \mathbb{R}$ is v-convex if and only if $\psi_X = (\psi_X^v)^v$ (see Proposition 5.8 in Villani (2009)).*[21]

ii) The relation $\psi_X(x) = \sup_{y\in\text{Supp}(\mu_Y)} (v(x,y) - \psi_Y(y))$ reflects buyers' price-taking behavior with respect to market payoffs ψ_Y for existing seller attributes. In any relationship, a buyer with attribute x can claim the gross surplus net of the seller's payoff, and he may optimize over all $y \in \text{Supp}(\mu_Y)$ (and vice versa for sellers).

iii) As v is continuous and $\text{Supp}(\mu_X)$ and $\text{Supp}(\mu_Y)$ are compact, any v-convex function is continuous, and so is its v-transform.[22] *In particular, v-subdifferentials are closed.*

iv) The v-subdifferential $\partial_v\psi_X$ is the set of those (x,y) for which the "stable" payoffs $\psi_X(x)$ and $\psi_Y(y) = \psi_X^v(y)$ are feasible*, so that they might actually be generated by the given pair.*

Theorem 1 has two parts. The first part states that efficient matchings and dual solutions exist ("sup" turns into "max," and "inf" turns into "min" in the statement

[20] These pointwise inequalities must hold for a pair of representatives from the L^1-equivalence classes of ψ_X and ψ_Y.

[21] This is a generalization of the usual Legendre duality for convex functions. In that case, $X = Y = \mathbb{R}^n$ and $v(x,y) = x \cdot y$ is the standard inner product in Euclidean space.

[22] The proofs of these claims are straightforward. They also follow immediately from the proof of Theorem 6 in Gretzky, Ostroy and Zame (1992), who "v-convexify" a given dual solution to extract a continuous representative in the same L^1-equivalence class.

of Theorem 1) and that the optimal values of both programs coincide. The second part shows that any v-convex dual solution ψ^*_X defines payoffs $(\psi^*_X, (\psi^*_X)^v)$ that can be interpreted as feasible surplus shares in all (pointwise, not just almost surely) pairs that are part of any efficient matching: the support of any π^* is contained in the v-subdifferential of any ψ^*_X. This leads to a satisfactory formal definition of the stable and feasible bargaining outcomes of any given assignment game (Definition 3).[23] Moreover, Theorem 1 clarifies the structure of efficient matchings. Any matching $\pi \in \Pi(\mu_X, \mu_Y)$ that is concentrated on a *v-cyclically monotone* set is efficient (it is easy to see that the v-subdifferential $\partial_v \psi_X$ of a v-convex function ψ_X is a v-cyclically monotone set).

Definition 2. *A set $A \subset X \times Y$ is called* v-cyclically monotone *if for all $K \in \mathbb{N}$, $(x_1, y_1), ..., (x_K, y_K) \in A$ and $y_{K+1} = y_1$, the following inequality is satisfied.*

$$\sum_{i=1}^{K} v(x_i, y_i) \geq \sum_{i=1}^{K} v(x_i, y_{i+1}).$$

Theorem 1. *The following identity holds:*

$$\max_{\pi \in \Pi(\mu_X, \mu_Y)} \int v \, d\pi = \min_{\{\psi_X | \psi_X \text{ is } v-\text{convex w.r.t. } \mathrm{Supp}(\mu_X) \text{ and } \mathrm{Supp}(\mu_Y)\}} \left(\int \psi^v_X \, d\mu_Y + \int \psi_X \, d\mu_X \right).$$

If $\pi \in \Pi(\mu_X, \mu_Y)$ is concentrated on a v-cyclically monotone set then it is optimal. Moreover, there is a closed set $\Gamma \subset \mathrm{Supp}(\mu_X) \times \mathrm{Supp}(\mu_Y)$ such that

$$\begin{cases} \pi \text{ is optimal in the primal problem if and only if } \mathrm{Supp}(\pi) \subset \Gamma, \\ a\ v\text{-convex } \psi_X \text{ is optimal in the dual problem if and only if } \Gamma \subset \partial_v \psi_X. \end{cases}$$

Definition 3. *A* stable and feasible bargaining outcome *for the assignment game (μ_X, μ_Y, v) is a pair (π^*, ψ^*_X), such that $\pi^* \in \Pi(\mu_X, \mu_Y)$ is an optimal solution for the primal linear program, and the v-convex function ψ^*_X is an optimal solution for the dual linear program.*

3.3 Ex-post contracting equilibria

I use subscripts to distinguish between matchings of cost types $\pi_0 \in \Pi(\mu_B, \mu_S)$ and matchings of attributes $\pi_1 \in \Pi(\mu_X, \mu_Y)$. In ex-post contracting equilibrium, individual investments must be "best-replies" (given the agent's cost type) to the trading possibilities and equilibrium outcome (π^*_1, ψ^*_X) of the market (μ_X, μ_Y, v) that results from

[23](CMP) had to invest some effort to define feasibility appropriately in cases where $\mathrm{Supp}(\mu_X)$ and $\mathrm{Supp}(\mu_Y)$ are not connected, even in the special assortative framework.

others' sunk investments.[24] However, attribute choices do not have to be optimal with respect to a market-clearing payoff system for all ex-ante possible attributes. Payoffs for non-marketed seller attributes, say, do not influence buyers' investment decisions. Hence, investment coordination failures are possible (compare the explanations in Sections 1 and 2). A buyer with attribute $x \in X$ can match with any seller, leave the market payoff $\psi_Y^*(y) = (\psi_X^*)^v(y)$ to her and keep the remaining surplus. That is, the *gross payoff* for x is

$$r_X(x) = \sup_{y \in \mathrm{Supp}(\mu_Y)} \left(v(x,y) - \psi_Y^*(y) \right).$$

r_X pins down payoffs after unilateral deviations to attributes outside of the equilibrium support ($x \in X \setminus \mathrm{Supp}(\mu_X)$), and it coincides with ψ_X^* on $\mathrm{Supp}(\mu_X)$. Similarly, a seller choosing attribute $y \in Y$ gets gross payoff

$$r_Y(y) = \sup_{x \in \mathrm{Supp}(\mu_X)} \left(v(x,y) - \psi_X^*(x) \right),$$

which coincides with ψ_Y^* on $\mathrm{Supp}(\mu_Y)$. r_X and r_Y are continuous (by the Maximum Theorem and continuity of ψ_Y^* and ψ_X^*).

(CMP) depicted investment behavior that *might* be part of an equilibrium by functions $\beta : B \to X$ and $\sigma : S \to Y$.[25] I describe candidates for equilibrium *investment profiles* by measurable functions $\beta : B \times S \to X$ and $\sigma : B \times S \to Y$, together with a "pre-assignment" π_0 of buyers and sellers.[26] The motivation for this choice will become apparent from Corollary 1 below.

In particular, different agents with the same cost type may choose distinct attributes. This must often happen if agents with the same type have to match with different types of partners, e.g. because distributions have atoms or type spaces are discrete.[27] An innocuous technical condition (Definition 4) ensures that post-investment populations are adequately described by the image measures of π_0 under β and σ, $\mu_X := \beta_\# \pi_0$ and $\mu_Y := \sigma_\# \pi_0$.[28]

[24] Like (CMP), I assume that a single agent cannot affect aggregate market outcomes, see Section 2.

[25] More precisely, in their model, $B = S = [0,1]$, $\mu_B = \mu_S = U[0,1]$, $X = Y = \mathbb{R}_+$, and β and σ are "well-behaved," i.e. strictly increasing with finitely many discontinuities, Lipschitz on intervals of continuity points, and without isolated values.

[26] The technical Lemma 8 in the Appendix shows that, for any π_0, $\mathrm{Supp}(\pi_0)$ is an equivalent description of the sets of existing buyer and seller types, $\mathrm{Supp}(\mu_B)$ and $\mathrm{Supp}(\mu_S)$.

[27] On the other hand, the optimal transport literature has found conditions that ensure "pure" optimal matchings in much more general situations than those considered in (CMP), see Section 2.

[28] That is, for all Borel sets $\mathcal{X} \subset X$ and $\mathcal{Y} \subset Y$, $\mu_X(\mathcal{X}) = \pi_0(\beta^{-1}(\mathcal{X}))$ and $\mu_Y(\mathcal{Y}) = \pi_0(\sigma^{-1}(\mathcal{Y}))$.

Definition 4. *An investment profile* (β, σ, π_0) *is said to be* regular *if* $\beta(b,s) \in \mathrm{Supp}(\mu_X)$ *and* $\sigma(b,s) \in \mathrm{Supp}(\mu_Y)$ *for all* $(b,s) \in \mathrm{Supp}(\pi_0)$.

Definition 4 corresponds to a "no isolated values" condition for β and σ in (CMP). In a regular investment profile, all agents choose attributes that do not get lost in the description (μ_X, μ_Y, v) of the attribute assignment game. Moreover, at $t = 1$, there are (almost) equivalent alternatives for each agent's attribute.[29]

Definition 5. *An* ex-post contracting equilibrium *is a tuple* $((\beta, \sigma, \pi_0), (\pi_1^*, \psi_X^*))$, *in which* (β, σ, π_0) *is a regular investment profile and* (π_1^*, ψ_X^*) *is a stable and feasible bargaining outcome for* (μ_X, μ_Y, v), *such that for all* $(b,s) \in \mathrm{Supp}(\pi_0)$ *it holds*

$$\psi_X^*(\beta(b,s)) - c_B(\beta(b,s), b) = \sup_{x \in X} (r_X(x) - c_B(x,b)) =: r_B(b),$$

and

$$\psi_Y^*(\sigma(b,s)) - c_S(\sigma(b,s), s) = \sup_{y \in Y} (r_Y(y) - c_S(y,s)) =: r_S(s).$$

In the remainder of the paper, the functions r_B and r_S *always denote net payoffs in ex-post contracting equilibrium.* r_B and r_S are continuous (by the Maximum Theorem).

3.3.1 Two basic properties of equilibria

Full appropriation games

Due to the absence of hold-up problems, every agent's equilibrium investment must maximize net match surplus *contingent on* the attribute of her equilibrium partner.

Lemma 1. *Let* $((\beta, \sigma, \pi_0), (\pi_1^*, \psi_X^*))$ *be an ex-post contracting equilibrium. For any* $(b, s') \in \mathrm{Supp}(\pi_0)$ *and* $(\beta(b,s'), y) \in \mathrm{Supp}(\pi_1^*)$, $\beta(b,s')$ *satisfies* $\beta(b,s') \in \mathrm{argmax}_{x \in X} (v(x,y) - c_B(x,b))$. *Similarly, for any* $(b', s) \in \mathrm{Supp}(\pi_0)$ *and any* $(x, \sigma(b',s)) \in \mathrm{Supp}(\pi_1^*)$, $\sigma(b',s)$ *satisfies* $\sigma(b',s) \in \mathrm{argmax}_{y \in Y} (v(x,y) - c_S(y,s))$.

In particular, the investments of a buyer of type b and a seller of type s who are matched must be a Nash equilibrium (NE) of a *hypothetical* complete information game with strategy spaces X and Y and payoffs $v(x,y) - c_B(x,b)$ and $v(x,y) - c_S(y,s)$. I refer to this game as a "full appropriation" (FA) game between b and s. Jointly optimal

[29] In the Appendix, I check that the sets $\beta(\mathrm{Supp}(\pi_0))$ and $\sigma(\mathrm{Supp}(\pi_0))$ are contained and dense in $\mathrm{Supp}(\mu_X)$ and $\mathrm{Supp}(\mu_Y)$ (Lemma 9). As a consequence, the fact that $\beta(\mathrm{Supp}(\pi_0))$ and $\sigma(\mathrm{Supp}(\pi_0))$ are not necessarily closed or even merely measurable does not cause problems, and one may use the stable and feasible bargaining outcomes for (μ_X, μ_Y, v) as defined in Section 3.2 to formulate agents' investment problems. Compare also Lemma 10 in the Appendix.

attributes $(x^*(b,s), y^*(b,s))$ are always a Nash equilibrium of the FA game between b and s, but there may be other pure strategy NE.[30]

Corollary 1. *Let* $((\beta, \sigma, \pi_0), (\pi_1^*, \psi_X^*))$ *be an ex-post contracting equilibrium. Then, if* $(\beta(b,s'), \sigma(b',s)) \in \text{Supp}(\pi_1^*)$ *for some* $(b,s'), (b',s) \in \text{Supp}(\pi_0)$*,* $(\beta(b,s'), \sigma(b',s))$ *is a Nash equilibrium of the FA game between* b *and* s*.*

Corollary 1 motivates the use of a pre-assignment to describe potential equilibrium investments. One can restrict attention to (regular) investment profiles (β, σ, π_0) for which $(\beta(b,s), \sigma(b,s))$ is a NE of the FA game for all (b,s), tentatively set $\pi_1 = (\beta, \sigma)_{\#}\pi_0$ and check whether this investment behavior and matching can indeed occur in an equilibrium. In these cases, π_0 actually describes a matching of buyer and seller types that is compatible with the matching of attributes.

(CMP)'s "constrained efficiency" property

(CMP) noted an indirect but useful constrained efficiency property of ex-post contracting equilibria. Attributes of the following kind cannot exist in the equilibrium market: the attribute is part of a pair of attributes that some buyer and some seller could use for "blocking" the equilibrium outcome in a world of ex-ante contracting (joint net surplus exceeds the sum of net equilibrium payoffs). Lemma 2 states the result in the notation of this paper. A very simple proof may be found in the Appendix.

Lemma 2 (Lemma 2 of (CMP))**.** *Let* $((\beta, \sigma, \pi_0), (\pi_1^*, \psi_X^*))$ *be an ex-post contracting equilibrium. Suppose that there are* $b \in \text{Supp}(\mu_B)$*,* $s \in \text{Supp}(\mu_S)$ *and* $(x,y) \in X \times Y$ *such that* $h(x,y|b,s) > r_B(b) + r_S(s)$*. Then,* $x \notin \text{Supp}(\mu_X)$ *and* $y \notin \text{Supp}(\mu_Y)$*.*

4 Efficient ex-post contracting equilibria

The stable and feasible bargaining outcomes (π_0^*, ψ_B^*) of (μ_B, μ_S, w) describe how buyers and sellers would match and divide net surplus if they could bargain and write complete contracts before they invest (so that partners choose jointly optimal attributes $(x^*(b,s), y^*(b,s))$). Theorem 1 applied to (μ_B, μ_S, w) characterizes these ex-ante efficient outcomes and ensures existence. Theorem 2 below shows that any (π_0^*, ψ_B^*) that satisfies the following very mild technical condition can be achieved in ex-post contracting equilibrium.

[30] In accordance with the deterministic functions β and σ, I consider only pure strategy NE without further mentioning it.

Condition 1. *There is a selection* (β^*, σ^*) *from the solution correspondence for* (1) *such that* $(\beta^*, \sigma^*, \pi_0^*)$ *is a regular investment profile.*[31]

Theorem 2. *Let* (π_0^*, ψ_B^*) *be a stable and feasible bargaining outcome for* (μ_B, μ_S, w) *that satisfies Condition 1, and let* (β^*, σ^*) *be the corresponding selection. Then the regular investment profile* $(\beta^*, \sigma^*, \pi_0^*)$ *is part of an ex-post contracting equilibrium* $((\beta^*, \sigma^*, \pi_0^*), (\pi_1^*, \psi_X^*))$ *with* $\pi_1^* = (\beta^*, \sigma^*)_\# \pi_0^*$.[32]

Theorem 2 shows that (CMP)'s main existence and efficiency result does not hinge on supermodular surplus and cost functions, which imply ordered preferences and assortative matching. This is remarkable because single-crossing conditions took center stage in their proof. One may ask for which (μ_B, μ_S, w) stable and feasible bargaining outcomes are unique, but this (difficult) question is not of central importance for the present paper and has been studied elsewhere.[33]

5 Inefficient equilibria

In this section, I aim to clarify sources, forms and limitations of potential investment coordination failures. The core of the analysis consists of a sequence of in-depth studies of five examples in Sections 5.2 - 5.4. It is useful, and important for the economic interpretation of inefficiencies, to distinguish two different manifestations of coordination failure. These are not mutually exclusive. First, agents might choose inefficient specializations and match with partners that they should not match with from the ex-ante view. Secondly, attributes in equilibrium partnerships could differ from jointly optimal ones.

Thanks to Corollary 1, one can restrict attention to equilibria of the following form: $(\beta(b, s), \sigma(b, s))$ is a NE of the FA game for all (b, s), and π_1^* satisfies $\pi_1^* = (\beta, \sigma)_\# \pi_0$. In some cases, matchings of μ_B and μ_S other than π_0 are also compatible with the equilibrium.[34] I say that an equilibrium exhibits *mismatch* if it is not compatible with any ex-ante optimal matching. With regard to a compatible matching of μ_B and μ_S

[31] By the Maximum Theorem, the solution correspondence for the problem (1) is upper-hemicontinuous, so that a measurable selection always exists.

[32] If π_0^* is pure, the equilibrium investments could also be represented as functions that depend only on an agent's own type (as in (CMP)). Even then, π_0^* still serves to describe π_1^*, the efficient matching of the attribute economy that supports the ex-ante efficient matching of buyer and seller types.

[33] For example, Gretzky, Ostroy and Zame (1999) have shown that for a given continuous surplus function w and *generic* population measures μ_B and μ_S, core payoffs are unique. On the other hand, much research on optimal transport has been devoted to establishing sufficient conditions for unique, or more often unique and pure, optimal matchings.

[34] This happens if buyers (say) with different types choose the same attribute and if this kind of attribute is matched with investments that stem from different seller types.

there is *inefficiency of joint investments* if a *strictly positive mass of agents* is matched with attributes that are not jointly optimal. In particular, an equilibrium is ex-ante efficient if and only if it has the following two properties: it does not exhibit mismatch, and there is a compatible ex-ante optimal matching for which there is no inefficiency of joint investments. Corollary 1 implies a simple sufficient condition for ruling out inefficiency of joint investments.

Corollary 2. *Assume that for all* $b \in \text{Supp}(\mu_B)$ *and* $s \in \text{Supp}(\mu_S)$*, the FA game between b and s has a unique NE (which then coincides with the unique pair of jointly optimal attributes). Then, ex-post contracting equilibria cannot feature inefficiency of joint investments.*

I say that an environment displays *technological multiplicity* if FA games have more than one pure strategy NE for some $(b, s) \in \text{Supp}(\mu_B) \times \text{Supp}(\mu_S)$.

Consider first the following slight generalization of (CMP)'s model.[35]

Condition 2 (The 1-d supermodular framework)**.** *Let* $X \setminus \{x_\varnothing\}, Y \setminus \{y_\varnothing\}, B \setminus \{b_\varnothing\}, S \setminus \{s_\varnothing\} \subset \mathbb{R}_+$*, and assume that* v *is strictly supermodular in* (x, y)*,* c_B *is strictly submodular in* (x, b)*, and* c_S *is strictly submodular in* (y, s)*.*[36]

Under Condition 2, mismatch is impossible. Every equilibrium has to be compatible with the positively assortative matching of buyer and seller types (Corollary 4 in the Appendix). In particular, technological multiplicity is necessary for the existence of inefficient equilibria (Lemma 13 in the Appendix).

Beyond the 1-d supermodular framework, mismatch is sometimes possible even if all FA games have unique NE (see Example 1 below). On the other hand, intuition suggests that the absence of technological multiplicity also restricts mismatch. In such cases, jointly optimal investments are continuous on $\text{Supp}(\mu_B) \times \text{Supp}(\mu_S)$, and $(\beta(b, s), \sigma(b, s)) = (x^*(b, s), y^*(b, s))$ must hold for all equilibrium pairs. In this sense, all pairs must invest according to the same, efficient, "technological regime." Moreover, any attribute choice displays the specialization for the intended match, but it also *strongly reflects the agent's own type* (see also footnote 40). Therefore, every seller s' who is not too different from s must receive a "high" payoff just to ensure that investing optimally for the marketed attribute $x^*(b, s)$ (which is not very different from $x^*(b, s')$) is not a profitable deviation for s'. These payoff bounds are particularly demanding for sellers s' who would be more suitable partners (than s) for b ex-ante. Similar

[35]No smoothness is assumed, cost functions need not be convex in attribute choice, and types do not have to be uniformly distributed on intervals.

[36]See e.g. Milgrom and Roberts (1990) or Topkis (1998) for formal definitions of these very well-known concepts.

observations apply for $y^*(b, s)$ and buyers b', and for all equilibrium pairs. Ruling out each and any profitable deviation should be impossible if a lot of net surplus is lost through mismatch, especially if ex-ante populations are differentiated (and/or attributes very strongly reflect own cost types).

Examples 2 and 3 serve to study this intuition. They show more rigorously how ex-ante differentiation tends to preclude mismatch in a particular model with two-dimensional types and attributes. It should be noted that similar forces shape equilibrium matching patterns whenever agents invest according to NE of FA games that vary continuously in (b, s), even if this is inefficient (due to technological multiplicity). Examples 1 and 2 are still close to the 1-d supermodular framework. The analysis of Example 3 is substantially more involved and uses results from optimal transport, in particular Theorem 1.

In Sections 5.3 and 5.4, I turn to environments with technological multiplicity. (CMP)'s examples show that severe coordination failures can happen. In the environment that (CMP) consider, jointly optimal investments are increasing in the types of (assortatively matched) buyers and sellers. They jump upward once, at an "indifference pair" that can generate the same net surplus in each of two distinct NE of the FA game. In the under-investment equilibrium, the pairs with costs below those of the indifference pair make "low regime" investments, which still is a NE of the FA game between the partners. This inefficiency is ruled out, however, if (and only if) ex-ante populations are so heterogeneous that making low investments is not a NE of the FA game for the pairs with the lowest costs. In this case, the top pairs *must* make high regime investments. A possible jump from low to high attributes must then happen at the indifference pair (as net equilibrium payoffs have to be continuous in type).

Section 5.3 illustrates how the problems caused by technological multiplicity are aggravated outside of the 1-d supermodular framework. In particular, mismatch can easily occur, potentially without any inefficiency of joint investments. The positive effects of exogenous heterogeneity (for *ensuring* that market segments needed for ex-ante efficiency are open in equilibrium) still exist, but they can be much weaker than in (CMP).

I also demonstrate that, in contrast to what might be suggested by (CMP)'s examples, inefficient equilibria can exist even if agents are extremely heterogeneous (Section 5.4). The example, in which under- and over-investment occur simultaneously, also adds to a more comprehensive picture of the most interesting inefficiencies in the 1-d supermodular framework.

As before, all proofs are in the Appendix.

5.1 The basic module for examples

(CMP) generated an analytically convenient environment with technological multiplicity by defining gross surplus in a piecewise manner from two simpler surplus functions. This approach can fruitfully be pushed further. I systematically use a basic module that satisfies Condition 2 to construct examples, both with and without technological multiplicity. Apart from those in Section 5.4, the environments themselves do not satisfy Condition 2.

Basic module. *Let* $0 < \alpha < 2$, $\gamma > 0$, $f(z) = \gamma z^\alpha$ *for* $z \in \mathbb{R}_+$, $X \setminus \{x_\varnothing\} = Y \setminus \{y_\varnothing\} = \mathbb{R}_+$ *and* $v(x, y) := f(xy)$.[37] *Furthermore, let* $c_B(x, b) = x^4/b^2$ *and* $c_S(y, s) = y^4/s^2$ *for* $b, s \in \mathbb{R}_+ \setminus \{0\}$.

Symmetry just keeps the analysis reasonably tractable. None of the effects that I illustrate in the following sections depends on symmetry assumptions. Observe that for all $b \neq b_\varnothing$ and $s \neq s_\varnothing$, there is a trivial NE of the FA game between b and s, namely $(x, y) = (0, 0)$. This stationary point, which is not even a *local* maximizer of the net surplus $h(\cdot, \cdot|b, s)$, should be viewed as the only unpleasant feature of an otherwise very convenient example. Throughout Section 5, I focus on *non-trivial* equilibria, in which agents who prepare for matching with a non-dummy partner do not make zero investments. In other words, equilibria that arise only because of the pathological stationary point of the basic module are ignored![38]

Lemma 3. *In the basic module, the FA game between* $b \neq b_\varnothing$ *and* $s \neq s_\varnothing$ *has a unique non-trivial NE given by the jointly optimal attributes*[39]

$$(x^*(b, s), y^*(b, s)) = \left(\left(\frac{\gamma\alpha}{4}\right)^{\frac{1}{4-2\alpha}} b^{\frac{4-\alpha}{8-4\alpha}} s^{\frac{\alpha}{8-4\alpha}}, \left(\frac{\gamma\alpha}{4}\right)^{\frac{1}{4-2\alpha}} s^{\frac{4-\alpha}{8-4\alpha}} b^{\frac{\alpha}{8-4\alpha}}\right). \tag{2}$$

The maximal net surplus is

$$w(b, s) = \kappa(\alpha, \gamma)(bs)^{\frac{\alpha}{2-\alpha}}, \tag{3}$$

where

$$\kappa(\alpha, \gamma) = \gamma^{\frac{2}{2-\alpha}} \left(\frac{\alpha}{4}\right)^{\frac{\alpha}{2-\alpha}} \left(1 - \frac{\alpha}{2}\right). \tag{4}$$

[37] For any given μ_B and μ_S, one could replace attribute choice sets by sufficiently large compact intervals $[0, \bar{x}]$ and $[0, \bar{y}]$ without affecting any of the subsequent analysis.

[38] Eliminating the trivial NE explicitly by modifying the functions is possible but not worth the effort. It would make the following study a lot messier.

[39] Observe that $x^*(b, s)$ strongly reflects b. In fact, the geometric weight on b is always larger than the one on s (as $\alpha < 2$).

Moreover, the following identity is satisfied for all b, s, s':

$$\max_{y\in Y}\left(v(x^*(b,s'),y)-c_S(y,s)\right) \;=\; b^{\frac{\alpha}{2-\alpha}}s^{\frac{2\alpha}{4-\alpha}}(s')^{\frac{\alpha^2}{(4-\alpha)(2-\alpha)}}\gamma^{\frac{2}{2-\alpha}}\left(\frac{\alpha}{4}\right)^{\frac{\alpha}{2-\alpha}}\left(1-\frac{\alpha}{4}\right). \quad (5)$$

In particular, if $x^*(b,s')$ is an equilibrium investment of buyer type b (who prepares for matching with s'), then the net payoff for a seller s who optimally prepares for and matches (deviates to a match) with $x^*(b,s')$ is

$$b^{\frac{\alpha}{2-\alpha}}s^{\frac{2\alpha}{4-\alpha}}(s')^{\frac{\alpha^2}{(4-\alpha)(2-\alpha)}}\gamma^{\frac{2}{2-\alpha}}\left(\frac{\alpha}{4}\right)^{\frac{\alpha}{2-\alpha}}\left(1-\frac{\alpha}{4}\right)-c_B(x^*(b,s'),b)-r_B(b). \quad (6)$$

Here, $r_B(b)+c_B(x^*(b,s'),b)$ is the equilibrium gross payoff that must be left to buyer b with attribute $x^*(b,s')$. Analogous formulae apply for buyers.

5.2 Mismatch and its constraints in a multi-dimensional model without technological multiplicity

Consider the following "bilinear model."

The bilinear model. *Let* $\mathrm{Supp}(\mu_B)\subset\mathbb{R}^2_+\setminus\{0\}\cup\{b_\varnothing\}$, $\mathrm{Supp}(\mu_S)\subset\mathbb{R}^2_+\setminus\{0\}\cup\{s_\varnothing\}$ *and* $X\setminus\{x_\varnothing\}=Y\setminus\{y_\varnothing\}=\mathbb{R}^2_+$. *For non-dummy types and attributes* $b=(b_1,b_2)$, $s=(s_1,s_2)$, $x=(x_1,x_2)$ *and* $y=(y_1,y_2)$, *surplus and costs are given by* $v(x,y)=x\cdot y=x_1y_1+x_2y_2$, $c_B(x,b)=\frac{x_1^4}{b_1^2}+\frac{x_2^4}{b_2^2}$ *and* $c_S(y,s)=\frac{y_1^4}{s_1^2}+\frac{y_2^4}{s_2^2}$. *In particular,* $w(b,s)=\frac{1}{8}(b_1s_1+b_2s_2)$.

The functions v, c_B and c_S are additively separable and correspond to setting $\gamma=\alpha=1$ (in the basic module) for each of two relevant dimensions. FA games have unique non-trivial Nash equilibria, so that (non-trivial) inefficiency of joint investments is impossible. The functions v and w are bilinear (the formula for w follows from (3) and (4)). Bilinear surplus/valuation functions have been used in much classical work on screening and mechanism design. In the optimal transport problem, bilinear surplus corresponds to the classical case of quadratic transportation cost.

From (5), it follows that $\max_{y\in Y}\left(v(x^*(b,s'),y)-c_S(y,s)\right)=\sum_{i=1}^2\frac{3}{16}s_i^{\frac{2}{3}}(s_i')^{\frac{1}{3}}b_i$. Similarly, for buyers, $\max_{x\in X}\left(v(x,y^*(b',s))-c_B(x,b)\right)=\sum_{i=1}^2\frac{3}{16}b_i^{\frac{2}{3}}(b_i')^{\frac{1}{3}}s_i$. Moreover, $c_B(x^*(b,s),b)=c_S(y^*(b,s),s)=\frac{1}{16}(b_1s_1+b_2s_2)$. I allow that one coordinate of a type is equal to zero, meaning that any strictly positive investment in the corresponding dimension is infinitely costly (so that the agent makes zero investments in that dimension). This assumption is not fully compatible with the model of Section 3.1, but it serves to avoid unnecessary ε-arguments in Examples 1 and 2.

5.2.1 An example of mismatch

Example 1. *Let* $\mu_S = a_H\delta_{(s_H,s_H)} + (1-a_H)\delta_{(s_L,s_L)}$*, where* $0 < s_L < s_H$ *and* $0 < a_H < 1$*. Moreover,* $\mu_B = a_1\delta_{(b_1,0)} + a_2\delta_{(0,b_2)} + (1 - a_1 - a_2)\delta_{b_\varnothing}$*, where* $0 < a_1, a_2, b_1, b_2$ *and* $a_1 + a_2 < 1$*. Finally, let* $b_1 > b_2$ *and* $a_H < a_1 + a_2$*.*

In Example 1, sellers (workers) are generalists who can invest in both dimensions. There are only two possible types, and these are completely ordered. As $w((b'_1, b'_2), (s_1, s_1)) = \frac{1}{8}(b'_1 + b'_2)s_1$, net surplus is strictly supermodular in s_1 and $b'_1 + b'_2$. Hence, ex-ante optimal matchings are positively assortative with respect to these sufficient statistics (even for arbitrary distributions of buyer types). Sellers are on the long side of the market and face two types/sectors of specialized buyers (employers). There is a slight abuse of notation in Example 1 because b_1 and b_2 refer to different buyers, not to a generic buyer type (b_1, b_2). To make the problem interesting, sector 1 is more productive ex-ante ($b_1 > b_2$), and not all buyers can get high seller types ($a_H < a_1 + a_2$). The ex-ante efficient equilibrium of Theorem 2 always exists. What about other, inefficient equilibria?

Claim 1. *Consider the environment of Example 1. If* $a_H > a_2$*, then there is exactly one additional non-trivial, mismatch inefficient equilibrium if and only if*

$$\frac{b_2}{b_1} \geq 1 - \frac{s_L}{s_H} \quad \textit{and} \quad \frac{3}{2}\frac{b_2}{b_1} \geq \frac{1 - \frac{s_L}{s_H}}{1 - (\frac{s_L}{s_H})^{\frac{2}{3}}}.$$

Otherwise, only the ex-ante efficient equilibrium exists. If $a_H < a_2$*, then there is exactly one additional non-trivial, mismatch inefficient equilibrium if and only if*

$$\frac{2}{3}\frac{b_2}{b_1} \geq \frac{\left(\frac{s_H}{s_L}\right)^{\frac{2}{3}} - 1}{\frac{s_H}{s_L} - 1}.$$

Otherwise, only the ex-ante efficient equilibrium exists.

In the inefficient equilibrium for $a_H > a_2$, all $(0, b_2)$-buyers match with (s_H, s_H)-sellers, and sector 1 attracts both high and low seller types. Both conditions impose lower bounds on the ratio $\frac{b_2}{b_1}$ in terms of how different seller types are. The first one is a participation constraint for $(0, b_2)$-buyers. Given the payoff they have to leave to (s_H, s_H)-sellers, it must be weakly profitable for them to invest and enter the market. The second condition ensures that low-type sellers do not want to deviate and match with $x^*((0, b_2), (s_H, s_H))$-attributes, given the payoff that must be left to buyers from sector 2. The first condition is more stringent for small values of $\frac{s_L}{s_H}$ (in which case

(s_H, s_H)-types must receive a high payoff), while it is the other way round for $\frac{s_L}{s_H}$ close to 1.

In the inefficient equilibrium for $a_H < a_2$, all (s_H, s_H)-sellers are "depleted" by $(0, b_2)$-buyers. The remaining $(0, b_2)$-buyers and $(b_1, 0)$-buyers match with (s_L, s_L)-sellers. The lower bound on $\frac{b_2}{b_1}$ ensures that (s_H, s_H)-sellers do not want to deviate and match with $x^*((b_1, 0), (s_L, s_L))$-attributes. It is most stringent if $\frac{s_H}{s_L}$ is close to 1, in which case the investments made by the more productive sector of buyers are very suitable also for (s_H, s_H)-sellers.

5.2.2 The limits of mismatch: a simple example

The example of this section is a variation of the previous one. Sellers are generalists, and they are on the long side of the market (in particular, there are sufficiently many sellers such that non-zero investments must be made in both sectors in any non-trivial equilibrium). However, their population is more differentiated than in Example 1. Buyers belong to one of two specialized sectors again, but they can be heterogeneous and it need not be the case that one sector is uniformly more productive than the other one.

Example 2. Supp$(\mu_S) = \{(s_1, s_1)|s_L \leq s_1 \leq s_H\}$*, for some* $s_L < s_H$*.* μ_S *admits a bounded density, uniformly bounded away from zero, with respect to Lebesgue measure on* $[s_L, s_H]$*.* μ_B *is compactly supported in the union of* $(\mathbb{R}_+ \setminus \{0\}) \times \{0\}$*,* $\{0\} \times (\mathbb{R}_+ \setminus \{0\})$ *and* $\{b_\varnothing\}$*. The restrictions of* μ_B *to* $(\mathbb{R}_+ \setminus \{0\}) \times \{0\}$ *and* $\{0\} \times (\mathbb{R}_+ \setminus \{0\})$ *have interval support and admit bounded densities, uniformly bounded away from zero, with respect to Lebesgue measure on these intervals.*

Claim 2. *Consider the environment of Example 2. The only non-trivial ex-post contracting equilibrium is the ex-ante optimal one.*

The diversity of seller types and sellers' ability to deviate to marketed attributes from the other sector suffice to rule out any mismatch. The proof is indirect but constructive. Given an arbitrary candidate for a mismatch inefficient equilibrium, I identify some seller types who must have a profitable deviation.

5.2.3 The limits of mismatch continued: a fully multi-dimensional case

Checking for coordination failures by going through all possible cases, as in Example 1, is not viable in complex environments. Moreover, one usually does not know ex-ante optimal matchings explicitly[40] and has little, if any, a priori knowledge of structural

[40] Closed form solutions exist only in very few, exceptional cases.

constraints for equilibrium matchings (by contrast, in Example 2, π_0^* is assortative in sufficient statistics, and any equilibrium matching must be assortative within each of the two sectors). In such cases, optimal transport results may be helpful, especially the characterization that a matching is optimal if and only if it is concentrated on a w-cyclically monotone set. One can derive necessary properties of matchings from equilibrium conditions and try to evaluate whether these preclude mismatch, both "locally" (for subpopulations, if these should be matched to each other) and "globally."

I give an example of this kind of analysis in the present section, for an environment that features truly multi-dimensional heterogeneity. The environment is chosen such that the regularity theory for optimal transport (for bilinear surplus, which corresponds to quadratic transportation cost) ensures that the ex-ante optimal matching is unique, pure (given by a bijection between buyer and seller types) and smooth.

Condition 3. $\mathrm{Supp}(\mu_B), \mathrm{Supp}(\mu_S) \subset (\mathbb{R}_+ \setminus \{0\})^2$ *are closures of bounded, open and uniformly convex sets with smooth boundaries. Moreover,* μ_B *and* μ_S *are absolutely continuous with respect to Lebesgue measure, with smooth densities bounded from above and below on* $\mathrm{Supp}(\mu_B)$ *and* $\mathrm{Supp}(\mu_S)$.

Theorem 12.50 and Theorem 10.28 in Villani (2009) yield:

Theorem. *Under Condition 3, the stable and feasible bargaining outcomes of* (μ_B, μ_S, w) *satisfy:* ψ_B^* *is unique up to an additive constant and* smooth. *Moreover, the unique ex-ante optimal matching* π_0^* *is given by a smooth bijection* $T^* : \mathrm{Supp}(\mu_B) \to \mathrm{Supp}(\mu_S)$ *satisfying* $\frac{1}{8}T^*(b) = \nabla\psi_B^*(b)$.

I show that under very mild additional assumptions on the supports, *any* smooth and pure matching of buyers and sellers that is compatible with an ex-post contracting equilibrium must be ex-ante optimal.

Example 3. μ_B *and* μ_S *satisfy Condition 3. Furthermore,* $\left(\frac{s_1}{b_1}\frac{b_2}{s_2} + \frac{s_2}{b_2}\frac{b_1}{s_1}\right) < 32$ *for all* $b \in \mathrm{Supp}(\mu_B)$, $s \in \mathrm{Supp}(\mu_S)$.

Claim 3. *Consider the environment of Example 3. If* $T : \mathrm{Supp}(\mu_B) \to \mathrm{Supp}(\mu_S)$ *is a smooth pure matching of buyer and seller types that is compatible with an ex-post contracting equilibrium, then* T *is ex-ante efficient.*

The idea of proof is as follows. I note first that whenever an equilibrium matching is locally given by a smooth map T, then T corresponds to the gradient of the buyer net payoff function, $\nabla r_B(b) = \frac{1}{8}T(b)$ (Lemma 4 and Corollary 3). Then, I use a local version of the fact that both buyers and sellers must not have incentives to change investments and match with other marketed attributes from the other side. This yields bounds on $DT(b) = 8\,\mathrm{Hess}\, r_B(b)$ (the Hessian) and on the inverse of this matrix (Lemma 5). Taken together, these bounds force $\mathrm{Hess}\, r_B$ to be positive semi-definite under the mild additional assumptions on supports (Lemma 6). It then follows that r_B is convex, so that the matching associated with T is concentrated on the subdifferential of a convex function. This is a w-cyclically monotone set, and hence T is ex-ante optimal by Theorem 1.

The rest of this section contains the above-mentioned sequence of preliminary results. Throughout, η denotes a *direction* ($\eta \in \mathbb{R}^2$ and $|\eta| = 1$) and $\cdot$ is the standard inner product on $\mathbb{R}^2$. Most importantly, T always stands for a one-to-one onto matching of buyer and seller types that is compatible with an ex-post contracting equilibrium.

Lemma 4. *Let T be smooth in a neighborhood of $b \in \mathrm{Supp}(\mu_B)$. Then it holds for all admissible directions η:*

$$\frac{1}{8}T(b) \cdot \eta = \lim_{t \to 0} \frac{r_B(b + t\eta) - r_B(b)}{t}.$$

Corollary 3. *Let T be smooth on an open set $U \subset \mathrm{Supp}(\mu_B)$. Then r_B is smooth on U and satisfies $\nabla r_B(b) = \frac{1}{8}T(b)$ for all $b \in U$.*

Lemma 5. *Let T be smooth on an open set $U \subset \mathrm{Supp}(\mu_B)$ and consider $b \in U$. Then, $T(b)$ and the symmetric, non-singular linear map $DT(b) = 8\,\mathrm{Hess}\, r_B(b)$ satisfy: both $3DT(b) + \begin{pmatrix} \frac{T(b)_1}{b_1} & 0 \\ 0 & \frac{T(b)_2}{b_2} \end{pmatrix}$ and $3DT(b)^{-1} + \begin{pmatrix} \frac{b_1}{T(b)_1} & 0 \\ 0 & \frac{b_2}{T(b)_2} \end{pmatrix}$ are positive semi-definite.*

Lemma 6. *Let T be smooth on an open set $U \subset \mathrm{Supp}(\mu_B)$. For $b \in U$, if $\left(\frac{T(b)_1}{b_1}\frac{b_2}{T(b)_2} + \frac{T(b)_2}{b_2}\frac{b_1}{T(b)_1}\right) < 32$, then $DT(b) = 8\,\mathrm{Hess}\, r_B(b)$ is positive semi-definite.*

5.3 Technological multiplicity and severe coordination failures

The example of this section combines the one of Section 5.2.2 with an under-investment example à la (CMP). Population measures (and cost functions) are as in Example 2, with support $\{(s_1, s_1) | s_L \leq s_1 \leq s_H\}$ ($s_L < s_H$) for μ_S, $\{(0, b_2) | b_{2,L} \leq b_2 \leq b_{2,H}\}$ ($b_{2,L} < b_{2,H}$) for the sector 2 population of buyers, and $\{(b_1, 0) | b_{1,L} \leq b_1 \leq b_{1,H}\}$ ($b_{1,L} < b_{1,H}$) for the sector 1 population of buyers. The technology for sector 1 is as in Example 2, but match surplus in sector 2 has an additional regime of increased complementarity for high attribute choices: $v(x,y) = x_1 y_1 + \max(f_1, f_2)(x_2 y_2)$, where $f_1(z) = z$ and $f_2(z) = \frac{1}{2} z^{\frac{3}{2}}$.

Lemma 7 (for $K = 2$) shows that the surplus for sector 2 is strictly supermodular (I use Lemma 7 again, for a case with $K = 3$, in Section 5.4).[41]

Lemma 7. *Let $K \in \mathbb{N}$, $0 < \alpha_1 < ... < \alpha_K < 2$, $\gamma_1, ..., \gamma_K > 0$ and $f_i(z) = \gamma_i z^{\alpha_i}$ for $i = 1, ..., K$. For $i < j$, there is a unique $z_{ij} \in \mathbb{R}_+ \setminus \{0\}$ in which f_j crosses f_i (from below). z_{ij} is given by*

$$z_{ij} = \left(\frac{\gamma_i}{\gamma_j}\right)^{\frac{1}{\alpha_j - \alpha_i}}.$$

Consider parameter constellations for which $z_{12} < z_{23} < ... < z_{(K-1)K}$. In this case, $(\max_{i=1,...,K} f_i)(xy)$ defines a strictly supermodular function in $(x,y) \in \mathbb{R}^2_+$.

If the surplus for sector 2 were globally given by f_1, the unique non-trivial NE of the FA game between $(0, b_2)$ and (s_1, s_1) would be $(x,y) = \left(\left(0, \frac{1}{2} b_2^{\frac{3}{4}} s_1^{\frac{1}{4}}\right), \left(0, \frac{1}{2} b_2^{\frac{1}{4}} s_1^{\frac{3}{4}}\right)\right)$, yielding net surplus $\frac{1}{8} b_2 s_1$. The corresponding expressions for f_2 are $(x,y) = \left(\left(0, \frac{3}{16} b_2^{\frac{5}{4}} s_1^{\frac{3}{4}}\right), \left(0, \frac{3}{16} b_2^{\frac{3}{4}} s_1^{\frac{5}{4}}\right)\right)$ and $\kappa\left(\frac{3}{2}, \frac{1}{2}\right)(b_2 s_1)^3 = \frac{3^3}{2^{15}}(b_2 s_1)^3$. Hence, pairs with $b_2 s_1 < \frac{2^6}{3^{\frac{3}{2}}} =: \tau$ are better off with the f_1-technology, and pairs with $b_2 s_1 > \tau$ are better off with the f_2-technology. The true technology is defined via f_1 for $x_2 y_2 < z_{12} = 4$ and via f_2 for $x_2 y_2 > 4$. Still, the identified attributes are the jointly optimal choices for all b_2 and s_1, as $x_2 y_2 = \frac{1}{4} b_2 s_1$ and $x_2 y_2 = \frac{3^2}{2^8}(b_2 s_1)^2$ evaluated at the indifference pairs $b_2 s_1 = \tau$ are equal to $\frac{2^4}{3^{\frac{3}{2}}} < 4$ and $\frac{2^4}{3} > 4$ respectively. However, as in (CMP), the "low regime" investments still yield NE of the FA game for some range of b_2 and s_1 with $b_2 s_1 > \tau$ (and "high regime" investments still yield NE of the FA game for some range of b_2 and s_1 with $b_2 s_1 < \tau$).

Consider now a situation in which ex-ante efficiency requires that high cost investments are made in sector 2. This is the case if and only if $(0, b_{2,H})$ is matched to a (s_1^*, s_1^*) satisfying $b_{2,H} s_1^* > \tau$ in the ex-ante efficient equilibrium. In contrast to Example 2, w is not globally supermodular with regard to 1-d sufficient statistics, so that the problem of finding the ex-ante optimal matching is actually non-local and difficult, with potentially very complicated solutions.[42] However, for the present purposes, it is not necessary to solve the ex-ante assignment problem explicitly.

If all sector 2 pairs invest according to the low cost regime (which is inefficient by assumption), then Claim 2 implies that $(0, b_{2,H})$ is matched to the seller type $(s_{1,q}, s_{1,q})$ who satisfies $\mu_S(\{(s_1, s_1) | s_1 \geq s_{1,q}\}) = q$, for $q = \mu_B(\{b | b_1 + b_2 \geq b_{2,H}\})$. In contrast

[41] As (CMP) noted, the piecewise construction matters only for analytical convenience. One could smooth out kinks without affecting any results.

[42] I briefly illustrate this difficulty by spelling out the 2-cycle condition in the Appendix.

to Example 2, this means a mismatch in the present case! This inefficient situation (in which efficient investment opportunities in sector 2 are missed, and some high type sellers invest for sector 1 while they should invest for sector 2) is ruled out if and only if the low regime investments are in fact not a NE of the FA game between $(0, b_{2,H})$ and $(s_{1,q}, s_{1,q})$ (and would trigger an upward deviation by at least one of the two parties). Whether this is true depends crucially on q, and hence on sector 1 of the buyer population. In particular, whether the coordination failure is precluded or not depends on the full ex-ante populations, not just on supports (as in (CMP)).

Note finally that if the inefficient equilibrium exists, it exhibits inefficiency of joint investments if $b_{2,H}s_{1,q} > \tau$, while all agents make jointly optimal investments if $b_{2,H}s_{1,q} \leq \tau$.

5.4 Simultaneous under- and over-investment: the case of missing middle sectors

In the 1-d supermodular example of this section, "lower middle" types under-invest and bunch with low types who invest efficiently, while "upper middle" types over-invest and bunch with high types who invest efficiently. In particular, *the attribute market lacks an efficient middle sector*. As in (CMP), there is no bunching in a literal sense: attribute choices are strictly increasing in type. It should rather be understood as bunching in the same connected component of the attribute market.

I use the construction of Lemma 7 and the notation introduced there. v has three different regimes of complementarity, i.e. $K = 3$, $z_{12} < z_{23}$ and $v(x, y) = (\max_{i=1,\dots,3} f_i)(xy)$. Population measures are absolutely continuous with respect to Lebesgue measure and have interval support. For simplicity, they are symmetric, i.e. $\mu_B = \mu_S$. I denote the interval support by $I \subset \mathbb{R}_+ \setminus \{0\}$. By Corollary 4 in the Appendix, b is matched to $s = b$ in any ex-post contracting equilibrium.

If surplus were globally given by $f_i(xy)$ ($i \in \{1, 2, 3\}$) rather than v, then the non-trivial NE of the FA game for (b, b) would be unique and given by

$$x_i^*(b, b) = y_i^*(b, b) = \left(\frac{\gamma_i \alpha_i}{4}\right)^{\frac{1}{4-2\alpha_i}} b^{\frac{1}{2-\alpha_i}}. \tag{7}$$

The net surplus that the pair (b, b) would generate according to f_i is $w_i(b) = \kappa_i b^{\frac{2\alpha_i}{2-\alpha_i}}$, where $\kappa_i = \kappa(\alpha_i, \gamma_i)$. For $i < j$, w_j crosses w_i exactly once in $\mathbb{R}_+ \setminus \{0\}$ and this crossing is from below (as for the functions f_i, f_j). The type at which this crossing occurs is

$$b_{ij} = \left(\frac{\kappa_i}{\kappa_j}\right)^{\frac{1}{2\alpha_j/(2-\alpha_j)-2\alpha_i/(2-\alpha_i)}} = \left(\frac{\kappa_i}{\kappa_j}\right)^{\frac{(2-\alpha_i)(2-\alpha_j)}{4(\alpha_j-\alpha_i)}}.$$

I write x_{iij} for the attributes that the indifference types b_{ij} would use under f_i, and x_{jij} for those attributes they would use under f_j. These are given by

$$x_{iij} = z_{ij}^{\frac{1}{2}} \left(\frac{\alpha_i}{4}\right)^{\frac{1}{4-2\alpha_i}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{2-\alpha_j}{4(\alpha_j-\alpha_i)}},$$

and

$$x_{jij} = z_{ij}^{\frac{1}{2}} \left(\frac{\alpha_j}{4}\right)^{\frac{1}{4-2\alpha_j}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{2-\alpha_i}{4(\alpha_j-\alpha_i)}}.$$

Thus, x_{iij} and x_{jij} depend on γ_i, γ_j only through γ_i/γ_j, and moreover, x_{jij}/x_{iij} depends only on α_i and α_j. It follows that

$$\frac{x_{jij}}{x_{iij}} = \left(\frac{\alpha_j}{4}\right)^{\frac{1}{4-2\alpha_j}} \left(\frac{\alpha_i}{4}\right)^{-\frac{1}{4-2\alpha_i}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}}(2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}}(2-\alpha_j)}\right)^{\frac{1}{4}} = \left(\frac{\alpha_j(2-\alpha_i)}{\alpha_i(2-\alpha_j)}\right)^{\frac{1}{4}}.$$

This ratio is greater than 1 (as $0 < \alpha_i < \alpha_j < 2$), so that there is an upward jump in attribute choice where types would like to switch from the f_i to the f_j surplus function.

If parameters are such that $b_{12} < b_{23}$, then f_1 would be the best surplus function for $b < b_{12}$, f_2 would be best for $b_{12} < b < b_{23}$, and f_3 would be best for $b_{23} < b$.

However, the true gross surplus function is v with its three different regimes. The above comparison of net surplus from optimal choices for globally valid f_1, f_2 and f_3 is sufficient to find the ex-ante efficient equilibrium if and only if the "jump attributes" actually lie in the valid regimes. Formally, this requires

$$x_{112}^2 < z_{12} < x_{212}^2 < x_{223}^2 < z_{23} < x_{323}^2. \tag{8}$$

If $b_{12} < b_{13} < b_{23}$, it is clear from (7) that $x_{112} < x_{113}$, $x_{212} < x_{223}$ and $x_{313} < x_{323}$. I show next that the following two conditions may simultaneously be satisfied:

i) (8) holds,

ii) the jump from x_{113} to x_{313} (which is not part of the efficient equilibrium!), is also between valid regimes, that is $x_{113}^2 < z_{12}$ and $z_{23} < x_{313}^2$.

Indeed, let $\alpha_1 = 0.1$, $\alpha_2 = 0.6$, $\alpha_3 = 1.6$, $\gamma_1 = 1$, $\gamma_2 = 1.5$ and $\gamma_3 = 1$. Then

$z_{12} = 4/9$, $z_{23} = 3/2$, $b_{12} \approx 1.5823$, $b_{13} \approx 1.8908$, $b_{23} \approx 1.9266$ and jump attributes for all three possible jumps lie in the valid regimes: $x^2_{112} \approx 0.2326$, $x^2_{113} \approx 0.2806$, $x^2_{212} \approx 0.6637$, $x^2_{223} \approx 0.8793$, $x^2_{313} \approx 2.4459$ and $x^2_{323} \approx 2.6863$.

In the Appendix, I show that for the these parameters and any symmetric populations with interval support I and $b_{13} \in I$, the inefficient outcome in which types $b < b_{13}$ make investments $\beta(b) = \sigma(b) = \left(\frac{\gamma_1 \alpha_1 b^2}{4}\right)^{\frac{1}{4-2\alpha_1}}$ and types $b > b_{13}$ make investments $\beta(b) = \sigma(b) = \left(\frac{\gamma_3 \alpha_3 b^2}{4}\right)^{\frac{1}{4-2\alpha_3}}$ can be supported by an ex-post contracting equilibrium with symmetric payoffs $\psi_X(x) = \psi_Y(x) = v(x,x)/2$ on $\mathrm{cl}(\beta(I)) = \mathrm{cl}(\sigma(I))$ ($\mathrm{cl}(\cdot)$ denotes the closure of a set). Investments in this inefficient equilibrium are depicted in Figure 2. Figure 1 shows investments in the efficient equilibrium. The dotted lines indicate that investing according to the respective regime remains a NE of the FA game for a range of pairs (b, b) beyond the indifference pairs (b_{ij}, b_{ij}).

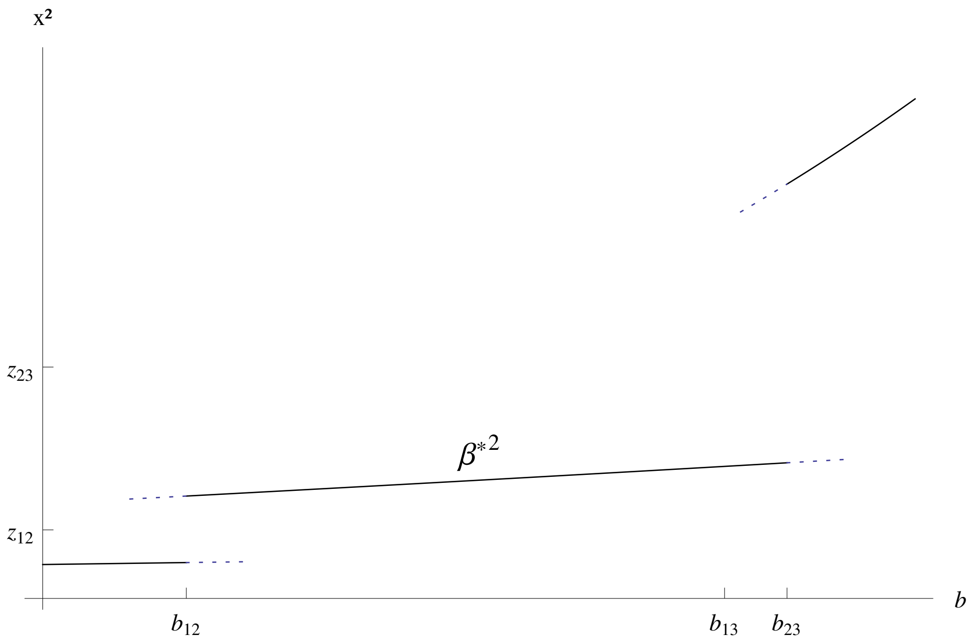

Figure 1: Investments in the efficient equilibrium

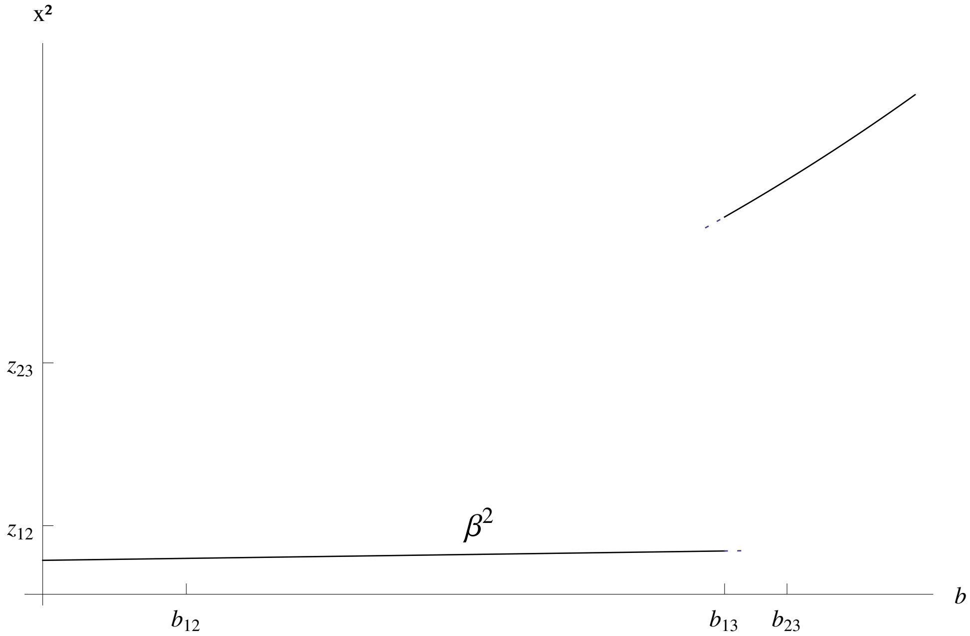

Figure 2: Investments in the inefficient equilibrium

Appendix

Proofs for Sections 3.3 and 4

The next lemma shows that the sets of existing buyer and seller types, $\mathrm{Supp}(\mu_B)$ and $\mathrm{Supp}(\mu_S)$, may equivalently be described by $\mathrm{Supp}(\pi_0)$ for any $\pi_0 \in \Pi(\mu_B, \mu_S)$.

Lemma 8. *Consider the projections $P_B(b,s) = b$ and $P_S(b,s) = s$. For any $\pi_0 \in \Pi(\mu_B, \mu_S)$, the identities $\mathrm{Supp}(\mu_B) = \mathrm{P}_B(\mathrm{Supp}(\pi_0))$ and $\mathrm{Supp}(\mu_S) = \mathrm{P}_S(\mathrm{Supp}(\pi_0))$ hold.*

Proof of Lemma 8. I prove the claim for μ_B only and show $\mathrm{P}_B(\mathrm{Supp}(\pi_0)) \subset \mathrm{Supp}(\mu_B)$ first. Consider any $(b,s) \in \mathrm{Supp}(\pi_0)$. Then, for any open neighborhood U of b, $\pi_0(U \times S) > 0$ and hence $\mu_B(U) > 0$. Thus, $b \in \mathrm{Supp}(\mu_B)$.

I next prove the slightly less trivial inclusion $\mathrm{Supp}(\mu_B) \subset \mathrm{P}_B(\mathrm{Supp}(\pi_0))$. Assume to the contrary that there is some $b \in \mathrm{Supp}(\mu_B)$ that is not contained in $\mathrm{P}_B(\mathrm{Supp}(\pi_0))$. The latter assumption implies that for all $s \in S$ there are open neighborhoods $U_s \subset B$ of b and $V_s \subset S$ of s such that $\pi_0(U_s \times V_s) = 0$. As S is compact, the open cover $\{V_s\}_{s \in S}$ of S contains a finite subcover $\{V_{s_1}, ..., V_{s_k}\}$. Moreover, $U := \bigcap_{i=1}^k U_{s_i}$ is an open neighborhood of b and $U \times S \subset \bigcup_{i=1}^k U_{s_i} \times V_{s_i}$. This leads to the contradiction $0 < \mu_B(U) = \pi_0(U \times S) \leq \pi_0(\bigcup_{i=1}^k U_{s_i} \times V_{s_i}) = 0$. $\square$

The following two technical lemmas merely serve to verify that for a regular investment profile (β, σ, π_0), the assignment game $(\beta_\# \pi_0, \sigma_\# \pi_0, v)$ and its stable and feasible bargaining outcomes, as defined in Section 3.2, adequately describe existing attributes and payoffs in the second-stage market.

Lemma 9. *Let* (β, σ, π_0) *be a regular investment profile. Then* $\beta(\mathrm{Supp}(\pi_0))$ *is dense in* $\mathrm{Supp}(\beta_\# \pi_0)$*,* $\sigma(\mathrm{Supp}(\pi_0))$ *is dense in* $\mathrm{Supp}(\sigma_\# \pi_0)$*, and* $(\beta, \sigma)(\mathrm{Supp}(\pi_0))$ *is dense in* $\mathrm{Supp}((\beta, \sigma)_\# \pi_0)$*.*

Proof of Lemma 9. I prove the claim for $\beta(\mathrm{Supp}(\pi_0))$. Assume to the contrary that there is some $x \in \mathrm{Supp}(\beta_\# \pi_0)$ and an open neighborhood U of x such that $U \cap \beta(\mathrm{Supp}(\pi_0)) = \emptyset$. Then $\beta_\# \pi_0(U) > 0$ (by definition of the support) and on the other hand $\beta_\# \pi_0(X \setminus U) \geq \pi_0(\mathrm{Supp}(\pi_0)) = 1$. Contradiction. □

Lemma 10. *Let* (β, σ, π_0) *be a regular investment profile. Let* $\psi_X : \beta(\mathrm{Supp}(\pi_0)) \to \mathbb{R}$ *be v-convex with respect to the (not necessarily closed) sets* $\beta(\mathrm{Supp}(\pi_0))$ *and* $\sigma(\mathrm{Supp}(\pi_0))$*, let* ψ_X^v *be its v-transform, and let* $\pi_1 \in \Pi(\beta_\# \pi_0, \sigma_\# \pi_0)$ *be such that* $\psi_X^v(y) + \psi_X(x) = v(x, y)$ *on a dense subset of* $\mathrm{Supp}(\pi_1)$*. Then there is a unique extension of* (ψ_X, ψ_X^v) *to a v-dual pair with respect to the compact metric spaces* $\mathrm{Supp}(\beta_\# \pi_0)$ *and* $\mathrm{Supp}(\sigma_\# \pi_0)$*, and with this extension* (π_1, ψ_X) *becomes a stable and feasible bargaining outcome in the sense of Definition 3.*

Proof of Lemma 10. First, define for all $y \in \mathrm{Supp}(\sigma_\# \pi_0)$,

$$\psi_{Y0}(y) := \sup_{x \in \beta(\mathrm{Supp}(\pi_0))} (v(x, y) - \psi_X(x)).$$

By definition, ψ_{Y0} coincides with ψ_X^v on the set $\sigma(\mathrm{Supp}(\pi_0)) \subset \mathrm{Supp}(\sigma_\# \pi_0)$, which is a dense subset by Lemma 9. Next, set for all $x \in \mathrm{Supp}(\beta_\# \pi_0)$,

$$\psi_{X1}(x) := \sup_{y \in \mathrm{Supp}(\sigma_\# \pi_0)} (v(x, y) - \psi_{Y0}(y)),$$

and finally for all $y \in \mathrm{Supp}(\sigma_\# \pi_0)$,

$$\psi_{Y1}(y) := \sup_{x \in \mathrm{Supp}(\beta_\# \pi_0)} (v(x, y) - \psi_{X1}(x)).$$

By definition, ψ_{X1} is a v-convex function with respect to the compact metric spaces $\mathrm{Supp}(\beta_\# \pi_0)$ and $\mathrm{Supp}(\sigma_\# \pi_0)$, and ψ_{Y1} is its v-transform. ψ_{X1} coincides with ψ_X on $\beta(\mathrm{Supp}(\pi_0))$, and ψ_{Y1} equals ψ_X^v on $\sigma(\mathrm{Supp}(\pi_0))$. Indeed, for any $x = \beta(b, s)$ with $(b, s) \in \mathrm{Supp}(\pi_0)$, the set of real numbers used to define the supremum $\psi_X(x)$

is contained in the one used to define $\psi_{X1}(x)$. Assume then for the sake of deriving a contradiction that $\psi_{X1}(\beta(b,s)) > \psi_X(\beta(b,s))$. Then, there must be some $y \in \text{Supp}(\sigma_{\#}\pi_0)$, such that $v(\beta(b,s),y) > \psi_X(\beta(b,s)) + \psi_{Y0}(y)$ and hence in particular $v(\beta(b,s),y) > \psi_X(\beta(b,s)) + v(\beta(b,s),y) - \psi_X(\beta(b,s))$, which yields a contradiction. A completely analogous argument shows that $\psi_{Y1}(\sigma(b,s)) = \psi_X^v(\sigma(b,s))$ for all $(b,s) \in \text{Supp}(\pi_0)$. Thus, $\psi_X(x) := \psi_{X1}(x)$ and $\psi_X^v(y) := \psi_{Y1}(y)$ are well-defined (and unique) extensions to a v-dual pair with respect to $\text{Supp}(\beta_{\#}\pi_0)$ and $\text{Supp}(\sigma_{\#}\pi_0)$.

As $\partial_v\psi_X$ is closed for the extended ψ_X, it follows that $\text{Supp}(\pi_1) \subset \partial_v\psi_X$. Hence (π_1, ψ_X) is a stable and feasible bargaining outcome in the sense of Definition 3. □

Proof of Lemma 1. Assume to the contrary that there is some x such that $v(x,y) - c_B(x,b) > v(\beta(b,s'),y) - c_B(\beta(b,s'),b)$. $(\beta(b,s'),y) \in \text{Supp}(\pi_1^*)$ implies $\psi_X^*(\beta(b,s')) = v(\beta(b,s'),y) - \psi_Y^*(y)$. Hence,

$$\begin{aligned}\psi_X^*(\beta(b,s')) - c_B(\beta(b,s'),b) &= v(\beta(b,s'),y) - \psi_Y^*(y) - c_B(\beta(b,s'),b) \\ &< v(x,y) - \psi_Y^*(y) - c_B(x,b) \leq r_B(b),\end{aligned}$$

which contradicts the assumption that $\beta(b,s')$ is an equilibrium choice of buyer b. The proof for sellers is analogous. □

Proof of Lemma 2. Assume to the contrary that $x \in \text{Supp}(\mu_X)$. Then,

$$r_S(s)+\psi_X^*(x)-c_B(x,b) \geq v(x,y)-\psi_X^*(x)-c_S(y,s)+\psi_X^*(x)-c_B(x,b) > r_B(b)+r_S(s).$$

The first inequality follows from the definition of r_S, and the second holds by assumption. It follows that $\psi_X^*(x) - c_B(x,b) > r_B(b)$, a contradiction (formally, $\psi_X^*(x) = v(x,y') - \psi_Y^*(y')$ for some $y' \in \text{Supp}(\mu_Y)$ matched with x under π_1^* and this leads to a contradiction to the definition of r_B). The proof for $y \notin \text{Supp}(\mu_Y)$ is analogous. □

Proof of Theorem 2. Let $\psi_S^* := (\psi_B^*)^w$ (the w-transform of ψ_B^* with respect to $\text{Supp}(\mu_B)$ and $\text{Supp}(\mu_S)$) denote the payoffs for seller types in the ex-ante stable and feasible bargaining outcome. Remember the following implications of Theorem 1.

$$\begin{gathered}\psi_S^*(s) + \psi_B^*(b) = w(b,s) \quad \text{for all } (b,s) \in \text{Supp}(\pi_0^*) \\ \psi_S^*(s) + \psi_B^*(b) \geq w(b,s) \quad \text{for all } b \in \text{Supp}(\mu_B), s \in \text{Supp}(\mu_S).\end{gathered} \tag{9}$$

By assumption, it holds for all $b \in B, s \in S$ that

$$v(\beta^*(b,s), \sigma^*(b,s)) - c_B(\beta^*(b,s), b) - c_S(\sigma^*(b,s), s) = w(b,s),$$

and moreover that $(\beta^*, \sigma^*, \pi_0^*)$ is a regular investment profile. It is intuitively quite clear that the matching of attributes $\pi_1^* = (\beta^*, \sigma^*)_\# \pi_0^* \in \Pi(\beta^*_\# \pi_0^*, \sigma^*_\# \pi_0^*)$, must be optimal. Indeed, from a social planner's point of view, and modulo technical details, the problem of finding an ex-ante optimal matching of buyers and sellers with corresponding mutually optimal investments is equivalent to a two-stage optimization problem for which the planner must first decide on investments for all agents and then match the two resulting populations optimally.

I next define the v-convex payoff function for buyer attributes ψ_X^* that is the other part of the equilibrium stable and feasible bargaining outcome for $(\beta^*_\# \pi_0^*, \sigma^*_\# \pi_0^*, v)$. Optimality of π_1^* will be (formally) shown along the way.[43] For any x for which there is some $(b,s) \in \mathrm{Supp}(\pi_0^*)$ such that $x = \beta^*(b,s)$, set

$$\psi_X^*(x) := \psi_B^*(b) + c_B(x,b).$$

This is well-defined. Indeed, take any other $(b', s') \in \mathrm{Supp}(\pi_0^*)$ with $x = \beta^*(b', s')$. Since ψ_B^* is a w-convex dual solution, it holds that $\mathrm{Supp}(\pi_0^*) \subset \partial_w \psi_B^*$ (by Theorem 1). Thus, $v(x, \sigma^*(b,s)) - c_B(x,b) - c_S(\sigma^*(b,s), s) = w(b,s) = \psi_S^*(s) + \psi_B^*(b)$. Moreover, $v(x, \sigma^*(b,s)) - c_B(x,b') - c_S(\sigma^*(b,s), s) \leq w(b', s) \leq \psi_S^*(s) + \psi_B^*(b')$, where the first inequality follows from the definition of w, and the second one follows from (9). This implies $c_B(x,b) - c_B(x,b') \leq \psi_B^*(b') - \psi_B^*(b)$, and hence $\psi_B^*(b) + c_B(x,b) \leq \psi_B^*(b') + c_B(x,b')$. Reversing roles in the above argument shows that $\psi_X^*(x)$ is well-defined.

Similarly, for any y for which there is some $(b,s) \in \mathrm{Supp}(\pi_0^*)$ such that $y = \sigma^*(b,s)$,

$$\psi_Y^*(y) := \psi_S^*(s) + c_S(y,s)$$

is well-defined. $\psi_X^*(x)$ and $\psi_Y^*(y)$ are the gross payoffs that agents get in their ex-ante efficient matches if the net payoffs are ψ_B^* and ψ_S^*. From the equality in (9) and from the definitions of ψ_X^* and ψ_Y^*, it follows that for all $(b,s) \in \mathrm{Supp}(\pi_0^*)$,

$$\begin{aligned} v(\beta^*(b,s), \sigma^*(b,s)) &= w(b,s) + c_B(\beta^*(b,s), b) + c_S(\sigma^*(b,s), s) \\ &= \psi_B^*(b) + \psi_S^*(s) + c_B(\beta^*(b,s), b) + c_S(\sigma^*(b,s), s) \\ &= \psi_X^*(\beta^*(b,s)) + \psi_Y^*(\sigma^*(b,s)). \end{aligned} \tag{10}$$

[43]It should be kept in mind that there may be other stable and feasible bargaining outcomes for $(\beta^*_\# \pi_0^*, \sigma^*_\# \pi_0^*, v)$. These are incompatible with (two-stage) ex-post contracting equilibrium however.

Moreover, the inequality in (9) implies for any $x = \beta^*(b, s)$ and $y = \sigma^*(b', s')$ with $(b, s), (b', s') \in \text{Supp}(\pi_0^*)$,

$$\begin{aligned} \psi_X^*(x) + \psi_Y^*(y) &= \psi_B^*(b) + c_B(x, b) + \psi_S^*(s') + c_S(y, s') \\ &\geq w(b, s') + c_B(x, b) + c_S(y, s') \geq v(x, y). \end{aligned} \tag{11}$$

(10) and (11) imply that with respect to the sets $\beta^*(\text{Supp}(\pi_0^*))$ and $\sigma^*(\text{Supp}(\pi_0^*))$, ψ_X^* is a v-convex function, and ψ_Y^* is its v-transform. Furthermore (by (10)), the set $(\beta^*, \sigma^*)(\text{Supp}(\pi_0^*))$, which by Lemma 9 is dense in $\text{Supp}(\pi_1^*)$, is contained in the v-subdifferential of ψ_X^*. Completing ψ_X^* as in Lemma 10 yields the stable and feasible bargaining outcome (π_1^*, ψ_X^*) for $(\beta_\#^*\pi_0^*, \sigma_\#^*\pi_0^*, v)$.

It remains to be shown that no agent has an incentive to deviate. So assume that there is a buyer of type $b \in \text{Supp}(\mu_B)$ for whom it is profitable to deviate. Then, there must be some $x \in X$ such that

$$\sup_{y \in \text{Supp}(\sigma_\#^*\pi_0^*)} (v(x, y) - \psi_Y^*(y)) - c_B(x, b) > \psi_B^*(b).$$

Hence, there is some $y \in \text{Supp}(\sigma_\#^*\pi_0^*)$ for which

$$v(x, y) - \psi_Y^*(y) - c_B(x, b) > \psi_B^*(b).$$

As $\sigma^*(\text{Supp}(\pi_0^*))$ is dense in $\text{Supp}(\sigma_\#^*\pi_0^*)$ and by continuity of v and ψ_Y^*, it follows that there is some $(b', s') \in \text{Supp}(\pi_0^*)$ such that

$$v(x, \sigma^*(b', s')) - \psi_S^*(s') - c_S(\sigma^*(b', s'), s') - c_B(x, b) > \psi_B^*(b).$$

Hence in particular $w(b, s') > \psi_S^*(s') + \psi_B^*(b)$, which contradicts (9). The argument for sellers is analogous. □

Proofs for Section 5

Some basic facts about the 1-d supermodular framework

As is well known, strict supermodularity of v forces optimal matchings to be positively assortative for any attribute assignment game. The Kantorovich duality theorem can be used for a very short proof.

Lemma 11. *Let Condition 2 hold. Then, for any (μ_X, μ_Y, v), the unique optimal matching is the positively assortative one.*

Proof of Lemma 11. By Kantorovich duality, the support of any optimal matching π_1^* is a v-cyclically monotone set. In particular, for any $(x, y), (x', y') \in \mathrm{Supp}(\pi_1^*)$ with $x > x'$, $v(x, y)+v(x', y') \geq v(x, y')+v(x', y)$ and hence $v(x, y)-v(x', y) \geq v(x, y')-v(x', y')$. As v has strictly increasing differences, it follows that $y \geq y'$. □

Lemma 12. *Let Condition 2 hold. Then, in any ex-post contracting equilibrium, attribute choices are non-decreasing with respect to agents' own type.*

Proof of Lemma 12. From Definition 5, $\beta(b, s) \in \mathrm{argmax}_{x \in X}(r_X(x) - c_B(x, b))$. The objective is strictly supermodular in (x, b). By Theorem 2.8.4 from Topkis (1998), all selections from the solution correspondence are non-decreasing in b. The argument for sellers is analogous. □

Corollary 4. *Let Condition 2 hold. Then every ex-post contracting equilibrium is compatible with the positively assortative matching of buyer and seller types.*

The positively assortative matching may assign buyers of the same type to different seller types, and vice versa, whenever μ_B or μ_S have atoms, but this does not affect the result.

Lemma 13. *Let Condition 2 hold, and assume that for all $b \in \mathrm{Supp}(\mu_B)$ and $s \in \mathrm{Supp}(\mu_S)$, the FA game between b and s has a unique NE . Then every ex-post contracting equilibrium is ex-ante efficient.*

Proof of Lemma 13. By Corollary 4, every equilibrium is compatible with the positively assortative matching of buyer and seller types. In particular, this is true for the ex-ante efficient equilibrium that was constructed in Theorem 2 (by (upper hemi-) continuity of the solution correspondence for (1), Condition 1 is automatically satisfied if $(x^*(b, s), y^*(b, s))$ is unique for all (b, s)). By Corollary 2, inefficiency of joint investments is impossible. This proves the claim. □

Proofs for Section 5.1

Proof of Lemma 3. Any NE of the FA game for (b, s) must be a stationary point of $h(x, y|b, s) = \gamma(xy)^\alpha - \frac{x^4}{b^2} - \frac{y^4}{s^2}$. By behavior of this function on the main diagonal $x = y$ for small x, as well as by the asymptotic behavior as $x \to \infty$ or $y \to \infty$, there is an interior global maximum. Necessary first order conditions are

$$\begin{cases} \gamma\alpha x^{\alpha-1}y^\alpha &= \frac{4}{b^2}x^3 \\ \gamma\alpha x^\alpha y^{\alpha-1} &= \frac{4}{s^2}y^3 \end{cases} \Rightarrow \begin{cases} y &= (\frac{4}{\gamma\alpha b^2})^{1/\alpha}x^{(4-\alpha)/\alpha} \\ x &= (\frac{4}{\gamma\alpha s^2})^{1/\alpha}y^{(4-\alpha)/\alpha}. \end{cases}$$

Plugging in yields a unique stationary point apart from $(0,0)$, given by

$$\begin{cases} x^{(4-\alpha)^2/\alpha^2-1} = (\frac{\gamma\alpha s^2}{4})^{1/\alpha}(\frac{\gamma\alpha b^2}{4})^{(4-\alpha)/\alpha^2} \\ y^{(4-\alpha)^2/\alpha^2-1} = (\frac{\gamma\alpha b^2}{4})^{1/\alpha}(\frac{\gamma\alpha s^2}{4})^{(4-\alpha)/\alpha^2} \end{cases} \Rightarrow \begin{cases} x = (\frac{\gamma\alpha}{4})^{1/(4-2\alpha)} b^{(4-\alpha)/(8-4\alpha)} s^{\alpha/(8-4\alpha)} \\ y = (\frac{\gamma\alpha}{4})^{1/(4-2\alpha)} s^{(4-\alpha)/(8-4\alpha)} b^{\alpha/(8-4\alpha)}. \end{cases}$$

This proves (2). Net match surplus is

$$\begin{aligned} w(b,s) &= \gamma(x^*(b,s)y^*(b,s))^\alpha - \frac{x^*(b,s)^4}{b^2} - \frac{y^*(b,s)^4}{s^2} \\ &= \gamma\left(\frac{\gamma\alpha}{4}\right)^{\frac{\alpha}{2-\alpha}}(bs)^{\frac{\alpha}{2-\alpha}} - \frac{1}{b^2}\left(\frac{\gamma\alpha}{4}\right)^{\frac{2}{2-\alpha}} b^{\frac{4-\alpha}{2-\alpha}} s^{\frac{\alpha}{2-\alpha}} - \frac{1}{s^2}\left(\frac{\gamma\alpha}{4}\right)^{\frac{2}{2-\alpha}} s^{\frac{4-\alpha}{2-\alpha}} b^{\frac{\alpha}{2-\alpha}} \\ &= \kappa(\alpha,\gamma)(bs)^{\frac{\alpha}{2-\alpha}}, \end{aligned}$$

where

$$\kappa(\alpha,\gamma) = \gamma^{\frac{2}{2-\alpha}}\left(\left(\frac{\alpha}{4}\right)^{\frac{\alpha}{2-\alpha}} - 2\left(\frac{\alpha}{4}\right)^{\frac{2}{2-\alpha}}\right) = \gamma^{\frac{2}{2-\alpha}}\left(\frac{\alpha}{4}\right)^{\frac{\alpha}{2-\alpha}}\left(1-\frac{\alpha}{2}\right).$$

This proves (3) and (4).

Now, let $x = x^*(b,s')$. From the first order condition for the seller of type s, it follows that $y = \left(\frac{\gamma\alpha s^2}{4}\right)^{\frac{1}{4-\alpha}} x^{\frac{\alpha}{4-\alpha}}$. Hence,

$$\begin{aligned} \gamma(xy)^\alpha - \frac{y^4}{s^2} &= \gamma x^{\frac{4\alpha}{4-\alpha}}\left(\frac{\gamma\alpha}{4}\right)^{\frac{\alpha}{4-\alpha}} s^{\frac{2\alpha}{4-\alpha}} - \frac{1}{s^2}\left(\frac{\gamma\alpha s^2}{4}\right)^{\frac{4}{4-\alpha}} x^{\frac{4\alpha}{4-\alpha}} \\ &= s^{\frac{2\alpha}{4-\alpha}}\left(\left(\frac{\gamma\alpha}{4}\right)^{\frac{1}{4-2\alpha}} b^{\frac{4-\alpha}{8-4\alpha}}(s')^{\frac{\alpha}{8-4\alpha}}\right)^{\frac{4\alpha}{4-\alpha}}\left(\gamma\left(\frac{\gamma\alpha}{4}\right)^{\frac{\alpha}{4-\alpha}} - \left(\frac{\gamma\alpha}{4}\right)^{\frac{4}{4-\alpha}}\right) \\ &= b^{\frac{\alpha}{2-\alpha}} s^{\frac{2\alpha}{4-\alpha}}(s')^{\frac{\alpha^2}{(4-\alpha)(2-\alpha)}}\gamma^{\frac{2}{2-\alpha}}\left(\frac{\alpha}{4}\right)^{\frac{\alpha}{2-\alpha}}\left(1-\frac{\alpha}{4}\right). \end{aligned}$$

This proves (5). □

Proofs for Section 5.2.1

Proof of Claim 1. It is impossible that (s_L, s_L)-sellers are matched while some (s_H, s_H)-sellers remain unmatched. This follows immediately from (5) and (6) (the net return from making zero investments and matching with a dummy is zero for both types).

Case $a_H > a_2$: So, some (s_H, s_H)-sellers must match with $(b_1, 0)$-buyers. In particular, as equilibrium partners must have jointly optimal attributes (by the uniqueness of non-trivial NE of FA games), $r_B(b_1, 0) + r_S(s_H, s_H) = \frac{1}{8} b_1 s_H$. An equilibrium is not compatible with the ex-ante optimal matching if and only if some $((0, b_2), (s_H, s_H))$-

pairs *and* $((b_1,0),(s_L,s_L))$-pairs exist as well, in any compatible matching. In such an equilibrium, $r_B(0,b_2)+r_S(s_H,s_H)=\frac{1}{8}b_2s_H$ and $r_B(b_1,0)+r_S(s_L,s_L)=\frac{1}{8}b_1s_L$. Thus, by strict supermodularity, $r_B(0,b_2)+r_S(s_L,s_L)+r_B(b_1,0)+r_S(s_H,s_H)=\frac{1}{8}b_2s_H+\frac{1}{8}b_1s_L<\frac{1}{8}b_2s_L+\frac{1}{8}b_1s_H=\frac{1}{8}b_2s_L+r_B(b_1,0)+r_S(s_H,s_H)$. Hence, $((0,b_2),(s_L,s_L))$-pairs cannot be part of the equilibrium.

So, the only candidate for an inefficient ex-post contracting equilibrium is the one in which only $((0,b_2),(s_H,s_H))$-, $((b_1,0),(s_H,s_H))$- and $((b_1,0),(s_L,s_L))$-pairs exist. As sellers are on the long side, some (s_L,s_L)-types remain unmatched and make zero investments, so that $r_S(s_L,s_L)=0$. Thus, $r_B(b_1,0)=\frac{1}{8}b_1s_L$, $r_S(s_H,s_H)=\frac{1}{8}b_1(s_H-s_L)$ and $r_B(0,b_2)=\frac{1}{8}b_2s_H-r_S(s_H,s_H)$. In particular, neither $(b_1,0)$ nor (s_H,s_H) have profitable deviations. The remaining equilibrium conditions are that $(0,b_2)$-types do not want to deviate to zero investments, i.e. $r_B(0,b_2)\geq 0$ (there is only one suitable attribute to match with for them in the candidate equilibrium, the one chosen by (s_H,s_H)-types for sector 2), and that (s_L,s_L)-types cannot get a strictly positive net payoff from investing to match with $x^*((0,b_2),(s_H,s_H))$. According to (5) and (6), the latter condition is equivalent to

$$\frac{3}{16}b_2s_L^{\frac{2}{3}}s_H^{\frac{1}{3}}-\frac{1}{16}b_2s_H-r_B(0,b_2)\ \leq\ 0.$$

Plugging in $r_B(0,b_2)$ and rearranging terms yields

$$\frac{3}{2}\frac{b_2}{b_1}\ \geq\ \frac{1-\frac{s_L}{s_H}}{1-(\frac{s_L}{s_H})^{\frac{2}{3}}}.$$

Finally, $r_B(0,b_2)\geq 0$ may be rewritten as $\frac{b_2}{b_1}\geq 1-\frac{s_L}{s_H}$.

Case $a_H<a_2$: As before, inefficiency requires the existence of both $((0,b_2),(s_H,s_H))$- and $((b_1,0),(s_L,s_L))$-pairs. Some $((0,b_2),(s_L,s_L))$-pairs necessarily exist as well. As in the previous case, the additional existence of $((b_1,0),(s_H,s_H))$-pairs would lead to an immediate contradiction. So, the only possibility is that all (s_H,s_H)-sellers are depleted by sector 2. It follows that $r_S(s_L,s_L)=0$, $r_B(0,b_2)=\frac{1}{8}b_2s_L$, $r_S(s_H,s_H)=\frac{1}{8}b_2(s_H-s_L)$ and $r_B(b_1,0)=\frac{1}{8}b_1s_L$. Buyers and (s_L,s_L)-sellers have no profitable deviations. The remaining equilibrium condition for (s_H,s_H) is

$$\frac{1}{8}b_2\,(s_H-s_L)\ \geq\ \frac{3}{16}b_1\,s_H^{\frac{2}{3}}s_L^{\frac{1}{3}}-\frac{3}{16}b_1\,s_L,$$

which may be rewritten as

$$\frac{2}{3}\frac{b_2}{b_1} \geq \frac{\left(\frac{s_H}{s_L}\right)^{\frac{2}{3}} - 1}{\frac{s_H}{s_L} - 1}.$$

□

Proofs for Section 5.2.2

Proof of Claim 2. Assume that there is an equilibrium that is not ex-ante efficient. Then, in any matching of μ_B and μ_S compatible with the equilibrium, there exist $(s_1', s_1'), (s_1'', s_1'')$ and b', b'' with $s_1' < s_1''$ and $|b'| > |b''|$, such that (b', s') and (b'', s'') are matched (with jointly optimal investments). As equilibrium matching is positively assortative within each sector (according to Corollary 4), b' and b'' must be from different sectors. W.l.o.g. $b' = (b_1', 0), b'' = (0, b_2'')$. Define open right-neighborhoods $R_\varepsilon(s_1) := \{t_1 | s_1 < t_1 < s_1 + \varepsilon\}$, and

$$\hat{s}_1 := \inf\{s_1 \geq s_1' | \text{for all } \varepsilon > 0 \text{ there are } t_1 \in R_\varepsilon(s_1) \text{ with investments } y^*((0, \cdot), (t_1, t_1))\}.$$

The set used to define the infimum is non-empty as a seller of type (s_1'', s_1'') makes investment $y^*((0, b_2''), (s_1'', s_1''))$, μ_S is absolutely continuous w.r.t. Lebesgue measure and investment profiles are regular. Hence, $\hat{s}_1$ exists and satisfies $s_1' \leq \hat{s}_1 \leq s_1''$. If $\hat{s}_1 > s_1'$, then every left-neighborhood of $\hat{s}_1$ contains sellers investing for sector 1. If $\hat{s}_1 = s_1'$, then $(\hat{s}_1, \hat{s}_1)$ invests for sector 1 by assumption. In either case, regularity (and completion) implies that there are suitable attributes for $(\hat{s}_1, \hat{s}_1)$ in both sectors: there are $(\hat{b}_1, 0)$, $\hat{b}_1 \geq b_1'$ and $(0, \hat{b}_2)$, $\hat{b}_2 \leq b_2''$ (in particular $\hat{b}_2 < \hat{b}_1$) such that $x^*((0, \hat{b}_2), (\hat{s}_1, \hat{s}_1)), x^*((\hat{b}_1, 0), (\hat{s}_1, \hat{s}_1)) \in \text{Supp}(\mu_X)$. $(\hat{s}_1, \hat{s}_1)$ must be indifferent between the two corresponding equilibrium matches. This implies $r_S(\hat{s}_1, \hat{s}_1) = \frac{1}{8}\hat{b}_1\hat{s}_1 - r_B(\hat{b}_1, 0) = \frac{1}{8}\hat{b}_2\hat{s}_1 - r_B(0, \hat{b}_2)$. By construction, there are buyers from sector 2 just above $\hat{b}_2$ who invest for seller types just above $\hat{s}_1$ and vice versa. I show next that these seller types can profitably deviate to match with $x^*((\hat{b}_1, 0), (\hat{s}_1, \hat{s}_1))$. This yields the desired contradiction. Indeed, on the one hand, r_S must be right-differentiable at $\hat{s}_1$ with derivative $\frac{1}{8}\hat{b}_2$. However, if $s_1 > \hat{s}_1$ invests for and matches with $x^*((\hat{b}_1, 0), (\hat{s}_1, \hat{s}_1))$, this type gets a payoff of

$$\frac{3}{16}\hat{b}_1 s_1^{\frac{2}{3}}\hat{s}_1^{\frac{1}{3}} - \frac{1}{16}\hat{b}_1\hat{s}_1 - r_B(\hat{b}_1, 0) = \frac{3}{16}\hat{b}_1 s_1^{\frac{2}{3}}\hat{s}_1^{\frac{1}{3}} - \frac{3}{16}\hat{b}_1\hat{s}_1 + r_S(\hat{s}_1, \hat{s}_1).$$

The leading order term in the expansion of the first two terms on the right hand side (around $\hat{s}_1$) is $\frac{1}{8}\hat{b}_1(s_1 - \hat{s}_1)$. This contradicts the conclusion about the derivative of r_S obtained from sector 2 (as $\hat{b}_2 < \hat{b}_1$). $\square$

Proofs for Section 5.2.3

Proof of Lemma 4. I show $\limsup_{t\to 0, t>0} \frac{r_B(b+t\eta)-r_B(b)}{t} \leq \frac{1}{8}T(b)\cdot\eta$ and $\liminf_{t\to 0,t>0} \frac{r_B(b+t\eta)-r_B(b)}{t} \geq \frac{1}{8}T(b)\cdot\eta$. Assume to the contrary that $\limsup_{t\to 0,t>0} \frac{r_B(b+t\eta)-r_B(b)}{t} > \frac{1}{8}T(b)\cdot\eta$. Then there is an $a > \frac{1}{8}T(b)\cdot\eta$ and a monotone decreasing sequence (t_n) with $\lim_{n\to\infty} t_n = 0$ such that $r_B(b+t_n\eta) \geq r_B(b) + t_n a$. Consider the sellers $T(b+t_n\eta)$. Net payoffs must satisfy

$$\begin{aligned} r_S(T(b+t_n\eta)) &= \frac{1}{8}(b+t_n\eta)\cdot T(b+t_n\eta) - r_B(b+t_n\eta) \\ &\leq \frac{1}{8}(b+t_n\eta)\cdot T(b+t_n\eta) - r_B(b) - t_n a \\ &= r_s(T(b)) + t_n(\frac{1}{8}b\cdot DT(b)\eta + \frac{1}{8}T(b)\cdot\eta - a) + o(t_n). \end{aligned}$$

On the other hand, if seller $T(b+t_n\eta)$ invests optimally to match with $x^*(b,T(b))$ she gets:

$$\begin{aligned} &\sum_{i=1}^{2} \frac{3}{16} T(b+t_n\eta)_i^{\frac{2}{3}} T(b)_i^{\frac{1}{3}} b_i - \frac{1}{16} b\cdot T(b) - r_B(b) \\ =& \sum_{i=1}^{2}\left(\frac{3}{16}\left(T(b)_i^{\frac{2}{3}} + \frac{2}{3}T(b)_i^{-\frac{1}{3}} t_n (DT(b)\eta)_i\right) T(b)_i^{\frac{1}{3}} b_i - \frac{3}{16} b_i T(b)_i\right) + r_S(T(b)) + o(t_n) \\ =& r_S(T(b)) + \frac{1}{8} t_n b\cdot DT(b)\eta + o(t_n). \end{aligned}$$

It follows that for small t_n, $T(b+t_n\eta)$ has a profitable deviation. This contradicts equilibrium. Thus, $\limsup_{t\to 0,t>0} \frac{r_B(b+t\eta)-r_B(b)}{t} \leq \frac{1}{8}T(b)\cdot\eta$. $\liminf_{t\to 0,t>0} \frac{r_B(b+t\eta)-r_B(b)}{t} \geq \frac{1}{8}T(b)\cdot\eta$ may be shown by an analogous argument, using deviations by buyers. $\square$

Proof of Corollary 3. At $b \in U$, derivatives in all directions η exist and are given by $\frac{1}{8}T(b)\cdot\eta$ (Lemma 4). These are smooth on U since T is smooth. Hence r_B is smooth on U and satisfies $\nabla r_B = \frac{1}{8}T$. $\square$

Proof of Lemma 5. Given $b \in U$, an arbitrary direction η and $t > 0$, consider buyers $b+t\eta$ and b. Ex-post contracting equilibrium requires in particular that $b+t\eta$ does not want to deviate from his match with $T(b+t\eta)$ and invest (optimally) for $y^*(b,T(b))$

instead. Moreover, $T(b+t\eta)$ must not want to deviate and match with $x^*(b,T(b))$. The two resulting conditions are:

$$\sum_{i=1}^{2}\frac{3}{16}(b_i+t\eta_i)^{\frac{2}{3}}b_i^{\frac{1}{3}}T(b)_i \;\leq\; r_B(b+t\eta)+\frac{1}{2}r_B(b)+\frac{3}{2}r_S(T(b)), \tag{12}$$

and

$$\sum_{i=1}^{2}\frac{3}{16}T(b+t\eta)_i^{\frac{2}{3}}T(b)_i^{\frac{1}{3}}b_i \;\leq\; r_S(T(b+t\eta))+\frac{1}{2}r_S(T(b))+\frac{3}{2}r_B(b). \tag{13}$$

I next derive the second order approximations of the left and right hand side of (12), using $\nabla r_B(b)=\frac{1}{8}T(b)$, $\operatorname{Hess} r_B(b)=\frac{1}{8}DT(b)$, and the following identity:

$$(b_i+t\eta_i)^{\frac{2}{3}}=b_i^{\frac{2}{3}}+\frac{2}{3}b_i^{-\frac{1}{3}}t\eta_i-\frac{1}{9}b_i^{-\frac{4}{3}}t^2\eta_i^2+o(t^2).$$

$$\begin{aligned}\sum_{i=1}^{2}\frac{3}{16}(b_i+t\eta_i)^{\frac{2}{3}}b_i^{\frac{1}{3}}T(b)_i &= \sum_{i=1}^{2}\frac{3}{16}\left(b_i^{\frac{2}{3}}+\frac{2}{3}b_i^{-\frac{1}{3}}t\eta_i-\frac{1}{9}b_i^{-\frac{4}{3}}t^2\eta_i^2\right)b_i^{\frac{1}{3}}T(b)_i+o(t^2)\\ &= \frac{3}{16}b\cdot T(b)+t\frac{1}{8}\eta\cdot T(b)-t^2\frac{1}{48}\sum_{i=1}^{2}\frac{T(b)_i}{b_i}\eta_i^2+o(t^2).\end{aligned}$$

$$\begin{aligned}r_B(b+t\eta)+\frac{1}{2}r_B(b)+\frac{3}{2}r_S(T(b)) &= \frac{3}{16}b\cdot T(b)+r_B(b+t\eta)-r_B(b)\\ &= \frac{3}{16}b\cdot T(b)+t\frac{1}{8}\eta\cdot T(b)+t^2\frac{1}{16}\eta\cdot DT(b)\eta+o(t^2).\end{aligned}$$

Thus, inequality (12) turns into

$$t^2\eta\cdot\left(3DT(b)+\begin{pmatrix}\frac{T(b)_1}{b_1} & 0\\ 0 & \frac{T(b)_2}{b_2}\end{pmatrix}\right)\eta+o(t^2)\geq o(t^2).$$

Letting $t\to 0$ shows that $3DT(b)+\begin{pmatrix}\frac{T(b)_1}{b_1} & 0\\ 0 & \frac{T(b)_2}{b_2}\end{pmatrix}$ must be positive semi-definite. The second claim follows by symmetry (or from explicitly spelling out the second order approximation of (13), using $T(b+t\eta)=T(b)+tDT(b)\eta+\frac{t^2}{2}D^2T(b)(\eta,\eta)+o(t^2)$). □

Proof of Lemma 6. As $DT(b)=8\operatorname{Hess} r_B(b)$ is symmetric, there is a basis of $\mathbb{R}^2$ consisting of orthonormal (w.r.t. the standard inner product) eigenvectors. Since $DT(b)$

is non-singular, all eigenvalues differ from zero. For the purpose of deriving a contradiction, assume that $DT(b)$ has an eigenvalue $\lambda < 0$, with corresponding eigenvector η. From the first bound of Lemma 5 it follows that $3\lambda + \eta_1^2 \frac{T(b)_1}{b_1} + (1-\eta_1^2)\frac{T(b)_2}{b_2} \geq 0$, i.e. $\eta_1^2 \frac{T(b)_1}{b_1} + (1-\eta_1^2)\frac{T(b)_2}{b_2} \geq 3|\lambda|$. The second bound of Lemma 5 implies $3\lambda^{-1} + \eta_1^2 \frac{b_1}{T(b)_1} + (1-\eta_1^2)\frac{b_2}{T(b)_2} \geq 0$, i.e. $\eta_1^2 \frac{b_1}{T(b)_1} + (1-\eta_1^2)\frac{b_2}{T(b)_2} \geq \frac{3}{|\lambda|} = \frac{9}{3|\lambda|}$. It follows

$$\begin{aligned} 9 &\leq \left(\eta_1^2 \frac{T(b)_1}{b_1} + (1-\eta_1^2)\frac{T(b)_2}{b_2}\right)\left(\eta_1^2 \frac{b_1}{T(b)_1} + (1-\eta_1^2)\frac{b_2}{T(b)_2}\right) \\ &\leq 1 + \eta_1^2(1-\eta_1^2)\left(\frac{T(b)_1}{b_1}\frac{b_2}{T(b)_2} + \frac{T(b)_2}{b_2}\frac{b_1}{T(b)_1}\right). \end{aligned}$$

Since $\eta_1^2(1-\eta_1^2) \leq \frac{1}{4}$ this requires $32 \leq \left(\frac{T(b)_1}{b_1}\frac{b_2}{T(b)_2} + \frac{T(b)_2}{b_2}\frac{b_1}{T(b)_1}\right)$. Contradiction. □

Proof of Claim 3. As $\text{Supp}(\mu_B)$ is the closure of an open convex set, Lemma 6 implies that T is the gradient of a convex function on $\text{Supp}(\mu_B)$. Therefore, the matching π_T associated with T is concentrated on a w-cyclically monotone set. Hence, by Theorem 1, it is ex-ante optimal. □

Proofs for Section 5.3

Proof of Lemma 7. Note that $f_1(xy)$ is strictly increasing and strictly supermodular in (x, y), and that $(\max_{i=1,\ldots,K} f_i)(xy) = g(f_1(xy))$ for the strictly increasing, convex function

$$g(t) = \begin{cases} t & \text{for } t \leq \gamma_1 z_{12}^{\alpha_1} \\ \gamma_1^{-\alpha_i/\alpha_1}\gamma_i t^{\alpha_i/\alpha_1} & \text{for } \gamma_1 z_{(i-1)i}^{\alpha_1} < t \leq \gamma_1 z_{i(i+1)}^{\alpha_1},\ i = 2, ..., K-1 \\ \gamma_1^{-\alpha_K/\alpha_1}\gamma_K t^{\alpha_K/\alpha_1} & \text{for } t > \gamma_1 z_{(K-1)K}^{\alpha_1}. \end{cases}$$

The claim thus follows, e.g from an adaptation of Lemma 2.6.4 in Topkis (1998). □

Details for footnote 43

For $s_1' < s_1''$, $(b_1, 0)$ and $(0, b_2)$ with $b_2 s_1'' > \tau$, the expression that must be analyzed to verify 2-cycle monotonicity is

$$w((0, b_2), (s_1'', s_1'')) + w((b_1, 0), (s_1', s_1')) - w((0, b_2), (s_1', s_1')) - w((b_1, 0), (s_1'', s_1'')).$$

Two cases should be distinguished. If $s_1' \geq \frac{\tau}{b_2}$, then

$$\begin{aligned}
&w((0,b_2),(s_1'',s_1'')) + w((b_1,0),(s_1',s_1')) - w((0,b_2),(s_1',s_1')) - w((b_1,0),(s_1'',s_1'')) \\
&= \int_{s_1'}^{s_1''} \kappa\left(\frac{3}{2},\frac{1}{2}\right) 3b_2^3 t^2 - \kappa(1,1) b_1 \, dt > \int_{s_1'}^{s_1''} 3\kappa(1,1) b_2 - \kappa(1,1) b_1 \, dt.
\end{aligned}$$

The inequality holds since $b_2 t > \tau$, so that $\kappa\left(\frac{3}{2},\frac{1}{2}\right) b_2^3 t^3 > \kappa(1,1) b_2 t$. In particular, matching $(0, b_2)$ to the higher seller type is definitely in line with 2-cycle monotonicity if $3b_2 \geq b_1$. If $s_1' < \frac{\tau}{b_2}$ however, an additional term with a potentially opposite sign occurs.

$$\begin{aligned}
&w((0,b_2),(s_1'',s_1'')) + w((b_1,0),(s_1',s_1')) - w((0,b_2),(s_1',s_1')) - w((b_1,0),(s_1'',s_1'')) \\
&= \int_{\tau/b_2}^{s_1''} \kappa\left(\frac{3}{2},\frac{1}{2}\right) 3b_2^3 t^2 - \kappa(1,1) b_1 \, dt + \int_{s_1'}^{\tau/b_2} \kappa(1,1)(b_2 - b_1) dt.
\end{aligned}$$

Proofs for Section 5.4

$$\begin{aligned}
x_{iij} &= \left(\frac{\gamma_i \alpha_i}{4}\right)^{\frac{1}{4-2\alpha_i}} \left(\frac{\kappa_i}{\kappa_j}\right)^{\frac{2-\alpha_j}{4(\alpha_j-\alpha_i)}} \\
&= \gamma_i^{\frac{1}{4-2\alpha_i} + \frac{2-\alpha_j}{2(2-\alpha_i)(\alpha_j-\alpha_i)}} \gamma_j^{-\frac{1}{2(\alpha_j-\alpha_i)}} \left(\frac{\alpha_i}{4}\right)^{\frac{1}{4-2\alpha_i}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}} (2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}} (2-\alpha_j)}\right)^{\frac{2-\alpha_j}{4(\alpha_j-\alpha_i)}} \\
&= \left(\frac{\gamma_i}{\gamma_j}\right)^{\frac{1}{2(\alpha_j-\alpha_i)}} \left(\frac{\alpha_i}{4}\right)^{\frac{1}{4-2\alpha_i}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}} (2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}} (2-\alpha_j)}\right)^{\frac{2-\alpha_j}{4(\alpha_j-\alpha_i)}} \\
&= z_{ij}^{\frac{1}{2}} \left(\frac{\alpha_i}{4}\right)^{\frac{1}{4-2\alpha_i}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}} (2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}} (2-\alpha_j)}\right)^{\frac{2-\alpha_j}{4(\alpha_j-\alpha_i)}}.
\end{aligned}$$

A similar computation yields

$$\begin{aligned}
x_{jij} &= \left(\frac{\gamma_j \alpha_j}{4}\right)^{\frac{1}{4-2\alpha_j}} \left(\frac{\kappa_i}{\kappa_j}\right)^{\frac{2-\alpha_i}{4(\alpha_j-\alpha_i)}} \\
&= \gamma_i^{\frac{1}{2(\alpha_j-\alpha_i)}} \gamma_j^{\frac{1}{4-2\alpha_j} - \frac{2-\alpha_i}{2(2-\alpha_j)(\alpha_j-\alpha_i)}} \left(\frac{\alpha_j}{4}\right)^{\frac{1}{4-2\alpha_j}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}} (2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}} (2-\alpha_j)}\right)^{\frac{2-\alpha_i}{4(\alpha_j-\alpha_i)}} \\
&= z_{ij}^{\frac{1}{2}} \left(\frac{\alpha_j}{4}\right)^{\frac{1}{4-2\alpha_j}} \left(\frac{\left(\frac{\alpha_i}{4}\right)^{\frac{\alpha_i}{2-\alpha_i}} (2-\alpha_i)}{\left(\frac{\alpha_j}{4}\right)^{\frac{\alpha_j}{2-\alpha_j}} (2-\alpha_j)}\right)^{\frac{2-\alpha_i}{4(\alpha_j-\alpha_i)}}.
\end{aligned}$$

It is straightforward to check that ψ_X is a v-convex function with respect to the sets $\mathrm{cl}(\beta(I))$ and $\mathrm{cl}(\sigma(I)) = \mathrm{cl}(\beta(I))$, that ψ_Y is its transform, and that the pure matching of the symmetric attribute measures given by the identity mapping is supported in $\partial_v \psi_X$. This yields a stable and feasible bargaining outcome for the attribute economy.

Given ψ_Y, buyer type b_{13} is indifferent between the option (choose $x = x_{113}$, match with $y = x_{113}$) and the option (choose $x = x_{313}$, match with $y = x_{313}$). Indeed, net payoffs from this are $\gamma_1 x_{113}^{2\alpha_1}/2 - c_B(x_{113}, b_{13}) = w_1(b_{13})/2$ and $\gamma_3 x_{313}^{2\alpha_3}/2 - c_B(x_{313}, b_{13}) = w_3(b_{13})/2$ which are equal by definition of b_{13}. I show next that these are indeed the optimal choices for buyer type b_{13}. Note that for a given y, the conditionally optimal $x(y, b_{13})$ solves

$$\max_{x\in\mathbb{R}_+} \left(v(x,y) - \frac{v(y,y)}{2} - c_B(x, b_{13}) \right),$$

where

$$v(x,y) = \begin{cases} \gamma_1 (xy)^{\alpha_1} & \text{for } x \leq z_{12}/y \\ \gamma_2 (xy)^{\alpha_2} & \text{for } z_{12}/y \leq x \leq z_{23}/y \\ \gamma_3 (xy)^{\alpha_3} & \text{for } z_{23}/y \leq x. \end{cases}$$

Let $y \leq x_{113}$. Then, $x(y, b_{13}) \leq x_{113}$. Indeed,

$$\frac{\partial}{\partial x}\left(\gamma_i (xy)^{\alpha_i} - \frac{x^4}{b_{13}^2} \right) = \gamma_i \alpha_i y^{\alpha_i} x^{\alpha_i - 1} - \frac{4x^3}{b_{13}^2}$$

is strictly positive for $x < \left(\frac{\gamma_i \alpha_i y^{\alpha_i} b_{13}^2}{4}\right)^{\frac{1}{4-\alpha_i}}$ and strictly negative for $x > \left(\frac{\gamma_i \alpha_i y^{\alpha_i} b_{13}^2}{4}\right)^{\frac{1}{4-\alpha_i}}$. For $y \leq x_{113}$, this zero is less than or equal to $\left(\frac{\gamma_i \alpha_i x_{113}^{\alpha_i} b_{13}^2}{4}\right)^{\frac{1}{4-\alpha_i}}$, which for $i = 1$ equals x_{113}. For $i = 2$, $\left(\frac{\gamma_2 \alpha_2 x_{113}^{\alpha_2} b_{13}^2}{4}\right)^{\frac{1}{4-\alpha_2}} = 0.8385 < z_{12}/x_{113} = 0.8391$, so that the derivative is negative on the entire second part of the domain. Similarly, $\left(\frac{\gamma_3 \alpha_3 x_{113}^{\alpha_3} b_{13}^2}{4}\right)^{\frac{1}{4-\alpha_3}} = 0.7599 < z_{23}/x_{113}$. It follows that $\max_{x\in\mathbb{R}_+, y\leq x_{113}} \left(v(x,y) - \frac{v(y,y)}{2} - c_B(x, b_{13}) \right)$ is attained in the domain of definition of v where it coincides with f_1, the first order condition then yields $y = x$ and thus (maximizing $\frac{\gamma_1 x^{2\alpha_1}}{2} - c_B(x, b_{13})$) $x = y = x_{113}$. A completely analogous reasoning applies for $y \geq x_{313}$ (I omit the details), showing that $\max_{x\in\mathbb{R}_+, y\geq x_{313}} \left(v(x,y) - \frac{v(y,y)}{2} - c_B(x, b_{13}) \right)$ is attained at $x = y = x_{313}$.

Therefore, buyer type b_{13} is indifferent between his two optimal choices (choose x_{113}, match with $y = x_{113}$) and (choose x_{313}, match with $y = x_{313}$). Note next that the buyer objective function $v(x,y) - \frac{v(y,y)}{2} - c_B(x,b)$ is supermodular in (x,y) on the lattice $\mathbb{R}_+ \times \mathrm{cl}(\beta(I))$ and has increasing differences in $((x,y),b)$. By Theorem 2.8.1 of Topkis (1998), the solution correspondence is increasing w.r.t. b in the usual set order (see Topkis 1998, Chapter 2.4). Hence, for $b < b_{13}$ there must be an optimum in the domain where v is defined via f_1. First order conditions lead to $y = x$, thus to maximization of $\gamma_1 x^{2\alpha_1}/2 - c_B(x, b)$ and hence to $x = \beta(b)$. The argument for buyer types $b > b_{13}$ is analogous. Since the entire argument applies to sellers as well, this concludes the proof. Q.E.D.

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