THE PRICE OF "ONE PERSON, ONE VOTE"

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Abstract. The principle of 'One Person, One Vote' is viewed by many as one of the cornerstones of Democracy. But in environments where agents have heterogenous stakes, a utilitarian social planner would typically prefer non-anonymous rules, i.e. rules that violate this principle. We consider a simple Bayesian voting environment in which agents have private valuations for the alternatives and a voting mechanism is used to determine the outcome. The *price of 'One Person, One Vote'* is defined as the fraction reduction in expected welfare when the optimal anonymous mechanism is used relative to the optimal (not necessarily anonymous) mechanism. We analyze how this price behaves as a function of the environment (distribution of preferences). In particular, it is shown that if the inequality of stakes in the population increases, in a well-defined sense, then the price of 'One Person, One Vote' is higher. A similar comparative statics holds if the uncertainty about agents' preferences increases. Our results may be useful for social planners who weigh the moral argument (or other arguments) in favor of 'One Person, One Vote' with the loss of efficiency that may result by sticking to this principle.

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1. Introduction

The principle of 'One Person, One Vote' is viewed by many as one of the cornerstones of Democracy. In an influential U.S. supreme court case (Gray v. Sanders, 1963) dealing with equal representation of voters in the American electoral system, Justice William O. Douglas argued that

> "The conception of political equality from the Declaration of Independence, to Lincoln's Gettysburg Address, to the Fifteenth, Seventeenth, and Nineteenth Amendments can mean only one thing - one person, one vote."

There are environments however in which this principle stands in conflict to other potentially desirable goals. Specifically, a utilitarian social planner facing an environment in which agents have heterogenous stakes in the decision at hand can increase social welfare by assigning higher voting weights (i.e., give more than one vote) to agents with higher stakes in the decision. To see this, consider an extreme situation in which everyone except one agent are indifferent (or almost indifferent) between two possible alternatives. Clearly, it would be optimal from a utilitarian point of view to let that one agent be the dictator and make the decision by herself.[1]

Environments with stakes' heterogeneity abound. As a first example, consider an academic department who needs to decide whether to give a job offer to a particular candidate. Faculty members whose research is in the same field as the candidate typically have much more to gain or lose by the decision than those who work in other fields. Second, consider a citizen of country X who resides in country Y. Arguably, when there are general elections in country X the stakes of this citizen are lower than the stakes of citizens of X who also live in X; and indeed different countries have different rules regarding eligibility to vote of non-resident citizens. Third, consider a community who votes on whether to build a new park at location A or at location B. Members who live close to one of the proposed locations would typically have more at stake than those whose home is in about the same distance from A and from B.

Another important example is that of a representative democracy with heterogeneous district sizes. The most famous instance of this type is probably the U.S. Senate, to

[1]This example and indeed our model assumes that we can somehow measure intensity of preferences and make comparisons of this measure across agents. We interpret the intensity of preferences of an agent for alternative A over alternative B as her willingness to pay for A to be chosen rather than B. Thus, each agent has quasi-linear preferences over pairs of an alternative and money, but the voting mechanisms we consider do not use transfers between agents as in reality such transfers are often excluded due to ethical reasons.

which each state sends two representatives regardless of its population. If we think of the stakes of a state in the decision over a certain senate bill as an increasing function of the number of its residents, then California has much more to gain or lose than South Dakota, even though the votes of senators from these two states get equal weight.[2]

There are also examples of institutions in which the voting mechanism does take into account the heterogeneity of the environment. In publicly held firms the weight assigned to a share holder's vote is usually proportional to the number of shares she holds. In ancient Rome the votes of richer citizens counted more than those of the poor, the logic being that if one has more money then one also has more at stake. And at the United Nations Security Council the five permanent members have veto power, which means their votes get higher weight than those of other members of the council.

Our goal in this paper is to quantify the welfare loss incurred due to the anonymity constraint that 'One Person, One Vote' imposes on the voting mechanism, and to analyze how the size of this loss depends on the distribution of preferences in the society. We suggest a simple and tractable theoretical model that captures the potential heterogeneity of the voters by allowing their preferences over the two alternatives to be drawn from different distributions. A voting mechanism is used to aggregate the ballots of the voters into a decision, and such a mechanism is called anonymous if it depends on the profile of ballots only through the number of agents supporting each alternative. We then define the price of 'One Person, One Vote' (price of OPOV, for short) as the fraction reduction in expected welfare when the optimal anonymous mechanism is used relative to the optimal (not necessarily anonymous) mechanism.

Our analysis starts with a characterization of the optimal and anonymous–optimal mechanisms. This gives an explicit formula for the price of OPOV as a function of the environment (distribution of preferences). We then study the behavior of this function as certain parameters of the environment vary. Specifically, the relevant parameters are the probabilities p_i that each agent i prefers alternative A over B, and the expected intensities of preference for A over B conditional on preferring A (a_i) and for B over A conditional on preferring B (b_i). We refer to these expected intensities as the *stakes* that agents have in the decision.

Consider first the case where all p_i's are equal, and where $a_i = b_i$ for all i. Proposition 1 shows that in such environments the price of OPOV increases in the inequality of stakes, where inequality is measured by the majorization partial order. That is, if we smooth

[2]In this example the voting mechanism satisfies 'One State, One Vote' rather than 'One Person, One Vote', but our analysis can still be applied to measure the welfare loss due to this mechanism relative to the optimal mechanism.

out the stakes in the decision of the different agents, then the welfare cost associated with using an anonymous mechanism decreases. In the limit when all agents have the same stakes the optimal mechanism (without the anonymity constraint) is anonymous and the price of OPOV is zero. In the same class of environments, Proposition 2 shows that the price of OPOV increases in the uncertainty about preferences, i.e. it is higher the closer is the probability of preferring A over B to 0.5.

This type of results are useful since they can inform social planners who weigh the moral argument (or other arguments) in favor of 'One Person, One Vote' with the loss of efficiency that may result by sticking to this principle. For example, we learn from Proposition 1 that when choosing a voting system one should pay attention to the level of inequality of stakes in the society; if that level is high then giving the same weight to all votes will be expensive in terms of efficiency loss. And from Proposition 2 we learn that in environments where landslide victory is likely the price of OPOV is relatively small.

For more general environments the analysis becomes trickier. In non-neutral environments ($a_i \neq b_i$) it is not obvious how to compare the stakes of different agents, and thus it is not clear what "smoothing out" stakes means. Still, in Proposition 3 we generalize Proposition 1 to this type of environments by introducing a way to compare inequality between two pairs of vectors (a, b) and (a', b'). This result strengthen the conclusion that the inequality of stakes is a key property in determining the relative efficiency of anonymous mechanisms. We note that Proposition 2 does not hold in non-neutral environments.

The most challenging environments are those in which the probabilities p_i are heterogenous. When this happens it is no longer true that increased inequality of stakes results in higher price of OPOV; it is easy to construct counter examples. However, it is still true that smoothing out stakes between agents with the same probabilities p_i reduces the price of OPOV. Thus, if society is composed of groups, where agents in each group have the same probability p_i, then the price of OPOV increases in the inequality of stakes within each group.

1.1. **Relation to the literature.** The theoretical study of weighted majority rules (with heterogenous weights) dates back at least to von-Neumann and Morgenstern's book [17]. Much of the literature deals with the measurement of the power of players in the cooperative game generated by the rule (e.g., [26]). The evaluation of majority rules based on the expected welfare they generate first appears in [22], and the analysis has been later extended in [4] and [10]. A more modern treatment appears in [25] for the case of symmetric environments and anonymous rules, and in [3] for the case asymmetric environments and general rules. Other papers on similar issues include [12] and [6]. All of the above papers, as well as the current paper, deal with the case of voting over two alternatives. Papers studying welfare properties of voting rules with more than two alternatives include [2], [13], [14], [18], [19], and [20].

There are also papers who take a similar approach to ours and estimate the loss of efficiency in mechanism design when an additional constraint is imposed on the mechanism. In the voting literature such papers include [11], which measures the loss due to an incentive compatibility constraint, and [24] which measures the loss due to the constraint that the decision must be made by a majority rule. In the auctions literature such papers include [16], [23] and [27], which bound the loss of efficiency or revenue to the seller when the auction format is restricted to be of a certain type. In matching, [21] shows that the loss in efficiency that results from restricting attention to ordinal mechanisms can be arbitrarily large. In a more abstract setting, [7] analyze the 'price of fairness', which is defined as one minus the ratio between the fair allocation welfare and the utilitarian allocation welfare. The concepts they use to define fair allocations are proportional fairness and max-min fairness.

Our paper is different than the works mentioned in the previous paragraph in that we are not only interested in bounding the loss of welfare, but we would also like to gain understanding regarding the characteristics of the environment that determine whether this loss is large or small. In particular, our results imply that inequality of stakes across agents is an important characteristic in that respect. The measurement of inequality has been extensively studied in Economics, see e.g. [9] for an extensive account. The recent paper [1] shows an interesting connection between inequality and efficiency in the context of an allocation game.

2. The voting environment

We consider the simplest framework that allows to quantify the loss of welfare due to the requirement of treating all agents equally. There are $n \geq 2$ agents in the society, indexed by $i \in [n] := \{1, \ldots, n\}$. The society faces a binary decision problem of choosing an alternative from the set $\{A, B\}$.

Agents have private valuations for the alternatives. Namely, for each agent i there is a real-valued random variable $\tilde{v}_i$, which determines the intensity by which agent i prefers alternative A over alternative B (or vice versa when $\tilde{v}_i$ is negative). For example, $\tilde{v}_i = 17$ means that i prefers A over B, and moreover that his willingness to pay for alternative

A to be chosen rather than B is 17. If on the other hand $\tilde{v}_i = -3.5$ then i prefers B over A and 3.5 is his willingness to pay for B to be chosen rather than A.

We denote by π the distribution of $\tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_n)$. We assume throughout the paper that $\tilde{v}_1, \ldots, \tilde{v}_n$ are independent, so that π is the product of its marginals. It will also be assumed that each $\tilde{v}_i$ has a finite expectation, that is $\mathbb{E}|\tilde{v}_i| < \infty$.[3] For expositional reasons we further assume that agents are never indifferent between the alternatives ($\mathbb{P}(\tilde{v}_i = 0) = 0$ for all i), and that each of the alternatives may be the favorite of each agent ($\mathbb{P}(\tilde{v}_i > 0) > 0$ and $\mathbb{P}(\tilde{v}_i < 0) > 0$ for all i). We often refer to π as the *environment*.

Each agent casts a vote for one of the alternatives, and a voting mechanism is used to aggregate the ballots into a decision.It will be convenient to identify each profile of ballots with the subset of players (coalition) who voted for alternative A, so that we can view a voting mechanism as a mapping $f : 2^{[n]} \to \{A, B\}$. The next standard definition formalizes the idea that a mechanism treats agents equally, i.e. respects the "One Person, One Vote" principle.

Definition 1. A voting mechanism f is *anonymous* if its outcome depends only on the number of agents who vote for each alternative and not on their names, that is if whenever S, S' are such that $|S| = |S'|$ we have that $f(S) = f(S')$.

We say that agents vote truthfully if they vote for A when they prefer A (i.e. when $\tilde{v}_i > 0$) and for B when they prefer B (when $\tilde{v}_i < 0$). Denote by $S(\tilde{v}) = \{i : \tilde{v}_i > 0\}$ the set of agents who prefer alternative A. The *utilitarian welfare* associated with a voting mechanism f is (the random variable) given by

$$U_f(\tilde{v}) = \begin{cases} \sum \tilde{v}_i, & \text{if } f(S(\tilde{v})) = A, \\ -\sum \tilde{v}_i, & \text{if } f(S(\tilde{v})) = B. \end{cases}$$

Thus, if the realization of $\tilde{v}$ is such that A is the chosen alternative (under truthful voting) then $\sum \tilde{v}_i$ is the total benefit (or loss if this sum is negative) of society relative to the other alternative B. Similarly, if B is chosen then $-\sum \tilde{v}_i$ is the total benefit relative to the alternative A. Let $W_\pi(f) = \mathbb{E}\left(U_f(\tilde{v})\right)$ be the expected utilitarian welfare associated with the mechanism f in π.

Given the environment π, we say that f is an optimal mechanism if it is a maximizer of W_π among all voting mechanisms, and we denote by W^*_π the expected welfare associated with an optimal mechanism. Similarly, f is an optimal anonymous mechanism

[3]Expectations and probabilities of events are always computed relative to the distribution π.

if it maximizes W_π among all anonymous mechanisms. The welfare associated with an optimal anonymous mechanism is denoted by W_π^{**}

Definition 2. 1. The price of "One Person, One Vote" (price of OPOV, for short) in environment π is

$$\rho(\pi) = 1 - \frac{W_\pi^{**}}{W_\pi^{*}}.$$

2. The price of OPOV in a family of environments Π is

$$\rho(\Pi) = \sup_{\pi\in\Pi} \rho(\pi).$$

Thus, $\rho(\pi)$ is the fraction reduction in expected welfare due to the additional anonymity constraint imposed on the voting mechanism. If the environment π is not precisely known, but it is known that π belongs to some class Π of environments, then $\rho(\Pi)$ measures the largest possible such reduction.

3. Optimal mechanisms and welfare

As a first step in the analysis we would like to characterize the optimal mechanisms with and without the anonymity constraint, and to obtain formulas for the associated welfare W_π^* and W_π^{**}. It will be useful to introduce a few more pieces of notation. For each agent i we let $a_i = \mathbb{E}\left(\tilde{v}_i | \tilde{v}_i > 0\right)$ and $b_i = \mathbb{E}\left(-\tilde{v}_i | \tilde{v}_i < 0\right)$, so that a_i is i's expected intensity of preference for alternative A conditional on i actually preferring A, and b_i is i's expected intensity of preference for alternative B conditional on him preferring B. For any coalition $S \subseteq [n]$ denote $a(S) = \sum_{i\in S} a_i$ and $b(S) = \sum_{i\in S} b_i$. Finally, for each i let $p_i = \mathbb{P}(\tilde{v}_i > 0)$ be the probability that i prefers A, and for a coalition S set $p(S) = \prod_{i\in S} p_i \prod_{i\in S^c}(1 - p_i)$ the probability that the members of S are exactly those who support A.[4]

Consider first the problem of maximizing expected welfare without the anonymity constraint. By the law of iterated expectations we have that for any mechanism f

$$\begin{aligned}
\mathbb{E}\left(U_f(\tilde{v})\right) &= \mathbb{E}\left[\mathbb{E}\left(U_f(\tilde{v}) | S(\tilde{v})\right)\right] = \sum_{S\subseteq[n]} p(S)\mathbb{E}\left(U_f(\tilde{v}) | S(\tilde{v}) = S\right) \\
&= \sum_{\{S:f(S)=A\}} p(S)\mathbb{E}\left(\sum_{i\in[n]} \tilde{v}_i \,\middle|\, S(\tilde{v}) = S\right) + \sum_{\{S:f(S)=B\}} p(S)\mathbb{E}\left(-\sum_{i\in[n]} \tilde{v}_i \,\middle|\, S(\tilde{v}) = S\right).
\end{aligned}$$

[4]Note that p_i and $p(\{i\})$ are different.

Using the assumption of independent values and the notation introduced above we can further write that

$$\text{(1)} \qquad \mathbb{E}\left(U_f(\tilde{v})\right) = \sum_{\{S: f(S)=A\}} p(S)\left[a(S) - b(S^c)\right] + \sum_{\{S: f(S)=B\}} p(S)\left[b(S^c) - a(S)\right].$$

It follows that a mechanism f is optimal if and only if it selects alternative A when $a(S) > b(S^c)$ and chooses B when the reverse strict inequality holds. If $a(S) = b(S^c)$ then any alternative can be chosen. Note that this rule is in fact a weighted majority rule in which each agent i has a weight of $a_i + b_i$ and the quota required for alternative A to win is $b([n])$. The following lemma summarizes the above discussion.

Lemma 1. A mechanism f is optimal if and only if it satisfies

$$f(S) = \begin{cases} A, & \text{if } a(S) + b(S) > b([n]), \\ B, & \text{if } a(S) + b(S) < b([n]). \end{cases}$$

Furthermore, the welfare associated with an optimal mechanism is

$$W_\pi^* = \sum_{S \subseteq [n]} p(S)\left|a(S) - b(S^c)\right|.$$

Moving now to the anonymous case, if f is an anonymous voting rule and k is an integer, $0 \le k \le n$, we write $f(k)$ for the alternative chosen by f when k voters support A. From (1) it follows that for any anonymous f,

$$\begin{aligned} \mathbb{E}\left(U_f(\tilde{v})\right) &= \sum_{\{k: f(k)=A\}} \sum_{\{S: |S|=k\}} p(S)\left(a(S) - b(S^c)\right) \\ &+ \sum_{\{k: f(k)=B\}} \sum_{\{S: |S|=k\}} p(S)\left(b(S^c) - a(S)\right). \end{aligned}$$

Therefore, an anonymous mechanism f is optimal among anonymous mechanisms if and only if it chooses A when $\sum_{\{S:|S|=k\}} p(S)\left(a(S) - b(S^c)\right) > 0$ and B if the reverse strict inequality holds. It is not hard to check that $\sum_{\{S:|S|=k\}} p(S)\left(a(S) - b(S^c)\right)$ is a strictly increasing function of k (see Theorem 4 in [3] for the details), so any optimal rule is a qualified majority rule in which alternative A is chosen if and only if the number of supporters of this alternative exceeds a certain (environment-specific) threshold. We thus have the following.

Lemma 2. An anonymous mechanism f is optimal among anonymous mechanisms if and only if it satisfies

$$f(k) = \begin{cases} A, & \text{if } \sum_{\{S:|S|=k\}} p(S)\,(a(S) - b(S^c)) > 0, \\ B, & \text{if } \sum_{\{S:|S|=k\}} p(S)\,(a(S) - b(S^c)) < 0. \end{cases}$$

Furthermore, the welfare associated with an optimal anonymous mechanism is

$$W_\pi^{**} = \sum_{k=0}^{n} \left| \sum_{\{S:|S|=k\}} p(S)(a(S) - b(S^c)) \right|.$$

We end this section with two remarks. First, in both cases with and without anonymity the optimal rules are incentive compatible. Indeed, it is a weakly dominant strategy for each agent to vote for his preferred alternative. Second, the optimal rules as well as the welfare values W_π^* and W_π^{**} depend on the environment π only through the parameters $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ and $p = (p_1, \dots, p_n)$, so for our purposes no additional information about π is needed. In the sequel we therefore view the price of OPOV ρ as a function of these three parameters. Note that given Lemmas 1 and 2 we have

$$\rho(a, b, p) = 1 - \frac{\sum_{k=0}^{n} \left| \sum_{\{S:|S|=k\}} p(S)(a(S) - b(S^c)) \right|}{\sum_{S \subseteq [n]} p(S)\,|a(S) - b(S^c)|}.$$

4. Intensity neutral and probability homogeneous environments

In this section we study the price of OPOV in environments that satisfy the following two restrictions.

Definition 3. An environment (a, b, p) is *Intensity Neutral (IN)* if $a = b$, that is $a_i = b_i$ for each agent i. Let Π_n^{IN} denote the family of IN environments with n agents.

Definition 4. An environment (a, b, p) is *Probability Homogenous (PH)* if the vector p is constant, that is $p_i = p_j$ for any two agents i, j. Let Π_n^{PH} denote the family of PH environments with n agents.

An environment which is both IN and PH is characterized by the vector a and by a single number $p \in (0, 1)$,[5] and we therefore write $W_{a,p}^*$ and $W_{a,p}^{**}$ for the welfare associated with the optimal and optimal anonymous mechanisms, respectively. We denote by $\rho(a, p)$ the price of OPOV in such environments.

[5] We use the same notation p for the vector of probabilities $(p_1, \dots, p_n)$ and for the probability of a single agent preferring A, in case this is constant across agents. No confusion should result.

Lemma 3. For IN and PH environments the optimal anonymous rule is simple majority (with arbitrary tie-breaking rule). Moreover,

$$W^{**}_{a,p} = \left(\sum_{i=1}^{n} a_i\right) \frac{1}{n} \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} |2k-n|$$

and

$$W^{*}_{a,p} = \sum_{k=0}^{n} p^k (1-p)^{n-k} \sum_{\{S:|S|=k\}} |a(S) - a(S^c)|.$$

Proof. Fix $0 \leq k \leq n$ and consider the sum $\sum_{\{S:|S|=k\}} p(S)\left(a(S) - a(S^c)\right)$. By the PH property for each coalition S of size k we have $p(S) = p^k(1-p)^{n-k}$, so we get that this sum is equal to $p^k(1-p)^{n-k} \sum_{\{S:|S|=k\}} \left(a(S) - a(S^c)\right)$. In this latter sum each a_i appears with a positive sign $\binom{n-1}{k-1}$ times and with a negative sign $\binom{n-1}{k}$ times, so the sum becomes $p^k(1-p)^{n-k}\left(\binom{n-1}{k-1} - \binom{n-1}{k}\right)\sum_{i=1}^{n} a_i$. Finally, the difference between these two binomial coefficients is equal to $\frac{1}{n}\binom{n}{k}(2k-n)$. Substituting this into the sum we obtain $\left(\sum_{i=1}^{n} a_i\right)\frac{1}{n}\binom{n}{k} p^k(1-p)^{n-k}(2k-n)$. By Lemma 2 the optimal anonymous rule is simple majority, and

$$W^{**}_{a,p} = \left(\sum_{i=1}^{n} a_i\right) \frac{1}{n} \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} |2k-n|.$$

Also, it follows immediately from Lemma 1 that

$$W^{*}_{a,p} = \sum_{k=0}^{n} p^k (1-p)^{n-k} \sum_{\{S:|S|=k\}} |a(S) - a(S^c)|.$$

□

4.1. **Stake inequality.** In an IN environment the vector a represents the stakes that agents have in the decision: If a_i is large then agent i has (on average) a lot to gain or lose by the decision, while if a_i is close to zero then i is almost indifferent between the two alternatives. We would now like to formulate a result that says that the price of OPOV becomes smaller when the stakes of the various agents become more equal. For this we first need to define what it means that stakes are more equal in one environment than in another.

For a vector $x \in \mathbb{R}^n$ and $k \in \{1, \ldots, n\}$ we denote by $x_{[k]}$ the kth largest coordinate of x, so that $x_{[1]} \geq \ldots \geq x_{[n]}$.

Definition 5. For $a, a' \in \mathbb{R}^n$, we say that a *majorizes* a' if $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} a'_i$ and $\sum_{i=1}^{k} a_{[i]} \geq \sum_{i=1}^{k} a'_{[i]}$ for every $k = 1, \ldots, n-1$.

It is not immediately obvious that the above definition captures the idea that a' is more equal than a. To see that this is indeed the case, we can characterize majorization using the familiar concept of Pigou-Dalton (PD) transfers.

Definition 6. For $a, a' \in \mathbb{R}^n$, a' is obtained from a by a PD transfer if there are i, j with $a_i > a_j$ and $0 \leq \delta \leq a_i - a_j$ such that $a'_j = a_j + \delta$, $a'_i = a_i - \delta$, and $a_k = a'_k$ for every $k \neq i, j$.

Lemma 4. [15, Proposition A.1, page 155] a majorizes a' if and only if a' can be obtained from a by a finite sequence of PD transfers.

Thus, a majorizing a' means that we can get from a to a' by smoothing out the stakes of the various agents, sequentially splitting the difference in stakes between high stakes and low stakes agents. The following is the main result of this section.

Proposition 1. For a fixed $p \in (0, 1)$, the price of OPOV $\rho(a, p)$ is (weakly) increasing relative to the majorization partial order on a. That is, if a majorizes a' then $\rho(a, p) \geq \rho(a', p)$.

Remark. Non-decreasing functions relative to the majorization partial order are known as Schur-convex functions. Proposition 1 thus states that ρ is a Schur-convex function of a.

Proof. By Lemma 4 it is enough to show that if a' can be obtained from a by a PD transfer then $\rho(a, p) \geq \rho(a', p)$. Since a PD transfer does not change the sum of the coordinates of a vector it follows immediately from Lemma 3 that $W^{**}_{a,p} = W^{**}_{a',p}$. The proposition will therefore be proved if we can show that $W^*_{a,p} \geq W^*_{a',p}$ whenever a' is obtained from a by a PD transfer.

Assume therefore that there are i, j with $a_i > a_j$ and $0 \leq \delta \leq a_i - a_j$ such that $a'_j = a_j + \delta$, $a'_i = a_i - \delta$, and $a_k = a'_k$ for every $k \neq i, j$. We first claim that we may assume without loss of generality that $\delta \leq (a_i - a_j)/2$, so that $a'_i \geq a'_j$. Indeed, a transfer of δ and a transfer of $a_i - a_j - \delta$ induce vectors that are different only in that their ith and jth coordinates are exchanged. But it is clear from Lemma 3 that a permutation of the coordinates of a does not change $W^*_{a,p}$. Thus, if $\delta \in ((a_i - a_j)/2, a_i - a_j]$ then we can replace it by $a_i - a_j - \delta \in [0, (a_i - a_j)/2)$ without affecting the resulting $W^*_{a',p}$

Consider the difference

$$W^*_{a,p} - W^*_{a',p} = \sum_S p(S)(|a(S) - a(S^c)| - |a'(S) - a'(S^c)|).$$

If S is a coalition such that $i, j \in S$ or such that $i, j \in S^c$ then $|a(S) - a(S^c)| = |a'(S) - a'(S^c)|$. If S is such that $i \in S$ and $j \in S^c$ then $(a(S) - a(S^c)) - (a'(S) - a'(S^c)) = 2\delta$, so if $a'(S) - a'(S^c) \geq 0$ then $|a(S) - a(S^c)| - |a'(S) - a'(S^c)| = 2\delta$ and if $a'(S) - a'(S^c) < 0$ then $|a(S) - a(S^c)| - |a'(S) - a'(S^c)| \geq -2\delta$. Similarly, if S is a coalition with $j \in S$ and $i \in S^c$ then $(a(S) - a(S^c)) - (a'(S) - a'(S^c)) = -2\delta$, so that if $a'(S) - a'(S^c) \leq 0$ then $|a(S) - a(S^c)| - |a'(S) - a'(S^c)| = 2\delta$ and if $a'(S) - a'(S^c) > 0$ then $|a(S) - a(S^c)| - |a'(S) - a'(S^c)| \geq -2\delta$. Therefore,

$$W^*_{a,p} - W^*_{a',p} \geq 2\delta \left(\sum_{\{S:\ i \in S,\ j \in S^c,\ a'(S) - a'(S^c) \geq 0\}} p(S) - \sum_{\{S:\ i \in S,\ j \in S^c,\ a'(S) - a'(S^c) < 0\}} p(S) + \right.$$
$$\left. \sum_{\{S:\ j \in S,\ i \in S^c,\ a'(S) - a'(S^c) \leq 0\}} p(S) - \sum_{\{S:\ j \in S,\ i \in S^c,\ a'(S) - a'(S^c) > 0\}} p(S) \right).$$

For any coalition S in the second sum consider the coalition $T = (S \cup \{j\}) \setminus \{i\}$. We have that $p(T) = p(S)$, and since $a'_i \geq a'_j$ it follows that $a'(T) - a'(T^c) < 0$ so T is in the third sum. Since this mapping from S to T is one-to-one it follows that the third sum is at least as large as the second. Similarly, any coalition S in the fourth sum we map to the coalition $T = (S \cup \{i\}) \setminus \{j\}$ which has the same probability as S and appears in the first sum because $a'_i \geq a'_j$. This implies that the first sum is at least as large as the fourth. This proves that $W^*_{a,p} \geq W^*_{a',p}$. □

4.2. **The effect of p.** The following proposition describes the behavior of the price of OPOV as a function of the probability p that agents prefer alternative A.

Proposition 2. For any fixed a, $\rho(a, p)$ is (weakly) decreasing in the distance of p from 0.5. That is, if $|p - 0.5| \leq |p' - 0.5|$ then $\rho(a, p) \geq \rho(a, p')$.

Proof. Fix a. First, it is easy to see from Lemma 3 that $W^*_{a,p} = W^*_{a,1-p}$ and $W^{**}_{a,p} = W^{**}_{a,1-p}$, so $\rho(a, p) = \rho(a, 1 - p)$. To prove the proposition we will show that $\rho(a, p)$ is increasing for $p \in (0.0.5)$, or equivalently that the ratio $W^{**}_{a,p}/W^*_{a,p}$ is decreasing on this interval.

To simplify the notation we write $g(p) = W^*_{a,p}$ and $h(p) = W^{**}_{a,p}$. Both g and h are differentiable on (0,0.5) and g is bounded away from zero, so the ratio h/g is differentiable. The sign of its derivative is determined by the sign of $h'g - g'h$. The proposition will therefore be proved if we can show that $h'(p) \leq g'(p)$ and $h'(p) \leq 0$ for any $p \in (0.0.5)$, since then $h'g - g'h \leq h'g - h'h = h'(g - h) \leq 0$, where in the first inequality we used $g \geq 0$ and in the last one we used $g \geq h$.

To compute the derivatives g' and h' notice that both g and h are of the form $\sum_{k=0}^{n} p^k(1-p)^{n-k} d_k$ for some vector $d = (d_0, \dots, d_n)$ with $d_k = d_{n-k}$ for all k. The derivative of such a function is given by

$$\begin{aligned}\frac{d}{dp}\left(\sum_{k=0}^{n} p^k(1-p)^{n-k} d_k\right) &= \sum_{k=0}^{n} d_k \left(kp^{k-1}(1-p)^{n-k} - (n-k)p^k(1-p)^{n-k-1}\right) = \\ &\sum_{k=0}^{n-1}((k+1)d_{k+1} - (n-k)d_k)p^k(1-p)^{n-k-1} = \\ &\sum_{k=0}^{\lfloor (n-2)/2 \rfloor} ((k+1)d_{k+1} - (n-k)d_k)p^k(1-p)^k[(1-p)^{n-2k-1} - p^{n-2k-1}],\end{aligned}$$

where in the last equality we used the symmetry of d around $n/2$. Thus, to prove that $h' \leq 0$ and $h' \leq g'$ on $(0, 0.5)$ it is sufficient to show that for every $k = 0, \dots, \lfloor (n-2)/2 \rfloor$

$$(k+1)\left|\sum_{|S|=k+1}(a(S) - a(S^c))\right| - (n-k)\left|\sum_{|S|=k}(a(S) - a(S^c))\right| \leq 0,$$

and

$$\begin{aligned}(k+1)&\left|\sum_{|S|=k+1}(a(S) - a(S^c))\right| - (n-k)\left|\sum_{|S|=k}(a(S) - a(S^c))\right| \leq \\ (k+1)&\sum_{|S|=k+1}|a(S) - a(S^c)| - (n-k)\sum_{|S|=k}|a(S) - a(S^c)|.\end{aligned}$$

Using the fact that $\left|\sum_{|S|=k}(a(S) - a(S^c))\right| = (\sum_i a_i)\frac{1}{n}\binom{n}{k}|n-2k|$ (see the proof of Lemma 3) and that $k \leq n/2$ we get after some simple manipulations that

$$\begin{aligned}(k+1)\left|\sum_{|S|=k+1}(a(S) - a(S^c))\right| - (n-k)\left|\sum_{|S|=k}(a(S) - a(S^c))\right| &= \\ -2\binom{n-1}{k}\left(\sum_i a_i\right) &< 0,\end{aligned}$$

which proves the first inequality. For the second inequality, note that $(k+1)\sum_{|S|=k+1}|a(S)-a(S^c)| = \sum_{|S|=k}\sum_{i\in S^c}|a(S\cup\{i\}) - a((S\cup\{i\})^c)|$. We can therefore write

$$(k+1)\sum_{|S|=k+1}|a(S)-a(S^c)| - (n-k)\sum_{|S|=k}|a(S)-a(S^c)| =$$
$$\sum_{|S|=k}\sum_{i\in S^c}\left(|a(S\cup\{i\}) - a((S\cup\{i\})^c)| - |a(S)-a(S^c)|\right).$$

For each coalition S and an agent $i \in S^c$ the above difference is at least $-2a_i$. Since each agent i appears $\binom{n-1}{k}$ times as the additional player, the total sum is at least $-2\binom{n-1}{k}\left(\sum_i a_i\right)$, which proves the second inequality. □

4.3. **Worst case analysis.** An immediate implication of Propositions 1 and 2 is that, in the class of IN and PH environments, the highest possible price of OPOV is obtained when $p = 0.5$ and all agents except one are (asymptotically) indifferent between the alternatives. For this type of environment we can explicitly calculate the price of OPOV. Furthermore, we can also look at the asymptotic behavior of this worst-case bound as the number of agents increases.

Corollary 1. The maximal price of OPOV in the class of n-agents IN and PH environments is

$$\rho\left(\Pi_n^{IN}\cap\Pi_n^{PH}\right) = 1 - \frac{1}{n2^n}\sum_{k=0}^{n}\binom{n}{k}|2k-n| = 1 - \binom{n-1}{\lfloor\frac{n-1}{2}\rfloor}\frac{1}{2^{n-1}}.$$

In particular, $\rho\left(\Pi_n^{IN}\cap\Pi_n^{PH}\right)$ is increasing in n and converges to 1 as $n\to\infty$. Furthermore,

$$\lim_n \sqrt{n}\left(1-\rho\left(\Pi_n^{IN}\cap\Pi_n^{PH}\right)\right) = \sqrt{\frac{2}{\pi}}.$$

Thus, without restricting further the family of possible environments, the worst-case price of OPOV approaches 100% as the size of society increases. The rate of convergence is of the order of $\frac{1}{\sqrt{n}}$, so if for instance $n = 10^4$ is the number of agents then at the worst case the price of OPOV is approximately 99%.

5. Non-neutral intensities

In this section we relax the neutrality assumption of the stakes in the environment. In IN environments the stakes of each agent i in the decision are characterized by a single number a_i, so it is clear what it means that one agent has higher stakes than another. Without neutrality agent i stakes are described by two numbers a_i, b_i and therefore the comparison between agents becomes more ambiguous. Nevertheless, we can generalize Proposition 1 as follows.

Definition 7. Let $a, b, a', b' \in \mathbb{R}^n$. We say that the pair (a, b) *sum-majorizes* the pair (a', b') if $a+b$ majorizes $a'+b'$, and in addition $\sum_{i=1}^n a_i = \sum_{i=1}^n a'_i$ and $\sum_{i=1}^n b_i = \sum_{i=1}^n b'_i$.

Note that (a, b) sum-majorizing (a', b') is stronger than simply $a+b$ majorizing $a'+b'$. It is possible to characterize sum-majorization in terms of PD transfers in a similar way to Lemma 4: (a, b) sum-majorizes (a', b') means that we can get from (a, b) to (a', b') by a sequence of transfers of stakes from higher to lower stakes agents, where the level of the stakes of agent i is measured by the sum $a_i + b_i$. The transfers should be such that they keep the sums $\sum_{i=1}^n a_i$ and $\sum_{i=1}^n b_i$ unchanged, which means that if we reduce a_i by δ then we must also increase a_j by the same δ. In other words, the transfers occur within the vectors a and b and not across.

Proposition 3. Consider two PH environments (a, b, p) and (a', b', p') with $p = p'$. If (a, b) sum-majorizes (a', b') then $\rho(a, b, p) \geq \rho(a', b', p')$.

Remark. It is immediate to see that (a, a) sum-majorizes (a', a') if and only if a majorizes a'. Therefore, Proposition 3 is a generalization of Proposition 1.

Proof. To be added. □

A natural question is whether some version of Proposition 2 still holds in non-neutral environments. From numerical simulations we know that for fixed a, b the maximal value of $\rho(a, b, p)$ is attained not necessarily at $p = 0.5$. Rather, the maximum is attained at a point p that is a function of the ratio $\sum_i a_i / \sum_i b_i$, as well as the number of agents n. Nevertheless, we can still prove that from the worst-case point of view the relaxation of intensity neutrality does not matter.

Corollary 2. The maximal price of OPOV in the class of n-agents PH environments is

$$\rho\left(\Pi_n^{PH}\right) = \rho\left(\Pi_n^{IN} \cap \Pi_n^{PH}\right) = 1 - \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor} \left(\frac{1}{2}\right)^{n-1}.$$

Proof. To be added. □

6. Heterogenous probabilities

Once the restriction to PH environments is relaxed the analysis becomes much more challenging. Specifically, we are able to construct examples showing that Proposition 1 is no longer true when agents differ in the likelihood they prefer A over B: There are IN environments in which a PD transfer that makes the stakes more equal according to the majorization order actually increases the price of OPOV. However examining the proof of Proposition 1 one can see that the following holds.

Proposition 4. Let (a, p) (where p is a vector of probabilities) be an IN but not necessarily PH environment. If $p_i = p_j$ for some two agents i and j then a PD transfer between these agents which makes their stakes more equal reduces the price of OPOV.

Thus, we can partition the agents into groups, where each group contains agents with the same probability p_i of preferring A, such that the price of OPOV increases in the inequality of stakes within each group. This is useful since many applications of interest involve environments in which the agents can be partitioned into homogenous groups; in several of the examples in the introduction (academic hiring, non-residents citizens) the society can be naturally divided into two such groups.

7. Discussion

To be added

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