Welfare criteria from choice: the sequential solution[*]

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Abstract

We study the problem of deriving a *complete* welfare ordering from a choice function. Under the *sequential solution*, the best alternative is the alternative chosen from the universal set; the second best is the one chosen when the best alternative is removed; and so on.

We show that this is the only completion of Bernheim and Rangel's (2009) welfare relation that satisfies two natural axioms: *neutrality*, which ensures that the names of the alternatives are welfare-irrelevant; and *persistence*, which stipulates that every choice function between two welfare-identical choice functions must exhibit the same welfare ordering.

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1 Introduction

This paper revisits the problem of extending choice-based welfare analysis to settings where agents may not be fully rational. Bernheim and Rangel (2009) observe that choices by boundedly rational agents generally exhibit a substantial degree of consistency that can be exploited to derive acyclic welfare judgements. According to their approach, an agent is better off with alternative x than alternative y if and only if the agent never chooses y from any set where x is available. From a purely choice-theoretic perspective, this Pareto-like criterion is fairly innocuous. Unfortunately, it is incomplete unless the agent is rational. In this paper, we are interested in extracting a complete welfare ordering of the alternatives for any choice behavior.

Much like Bernheim and Rangel, our *model-free* approach is based entirely on observed choices. In contrast with the *model-specific* approaches proposed in the literature (see Rubinstein and Salant (2012) among others), it does not rely on using an underlying model of choice behavior

to help make welfare judgments. We refrain from comparing our model-free approach to these model-specific approaches. The relative merits and shortcomings of each have been debated at length in the literature and are now relatively well understood.

Obviously, we are asking a great deal from very little. If one were to consider the choice behavior of each agent in isolation, as Bernheim and Rangel do, our task would be quite desperate. The task becomes manageable only when one imposes conditions on the relationship among the welfare orderings assigned to different agents. For this purpose, the natural object of study is the class of functions that, for each choice function defined on (the subsets of) a universal set X, assign a particular welfare ordering to the alternatives in X. We call such functions *solutions*.

Our approach is quite unlike previous approaches to choice-based welfare analysis.[1] In terms of practicality, we feel that it has significant appeal: by requiring some amount of "coherence" among the welfare orderings assigned to different agents, the range of solutions can be narrowed tremendously. What is more, we believe that this leads to sound policy: to evaluate aggregate social welfare, it would seem important to make coherent welfare judgments across agents.

The straightforward solution that emerges from our analysis is sequential: the best alternative is the one chosen from the universal set; the second best alternative is the one chosen from the set obtained by deleting the best alternative from the universal set; and so on.[2] We show that this *sequential solution* is the only solution that satisfies admissibility, neutrality and persistence.

Admissibility means that the ordering assigned to a rational choice function must be the one that rationalizes it. This is a very basic condition of non-paternalism: welfare judgements should respect choices when they are rational. In turn, *neutrality* states that the solution covaries with respect to permutations of the alternatives. This condition is innocuous (and even quite desirable) when the nature of the alternatives is unspecified. Finally, *persistence* stipulates that if the same ordering is assigned to two choice functions, it is also assigned to any choice function *between* them. For the purpose of this axiom, a choice function is between two others if, from every set, it picks an alternative that is selected by at least one of them.

By virtue of neutrality, a solution *must* assign the same welfare ordering to a wide range of agents who make different choices.[3] Persistence simply groups agents in a way that displays a natural kind of homogeneity. Though we do not claim that this is the only compelling way to group agents coherently, we do find it quite reasonable as a model-free approach. Clearly, a different notion of coherence might be more appropriate if one started from a particular model of choice behavior. While we certainly find this model-specific approach intriguing, we do not pursue it here.

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[1]Nishimura (2014) studies functions that, for each complete binary relation on X, assign a reflexive and transitive (but potentially incomplete) welfare relation on X. While this bears some resemblance to our approach, it is closer to the vast literature on extracting orderings from tournaments. See Bouyssou (2004) for a recent survey.

[2]This is a "folk" procedure for extracting an ordering from individual (Moulin (1988), Ex. 11.9) or group (Arrow and Raynaud (1986), Ch. 7) choice data. See Bouyssou (2004) for a survey of other work on *ranking by choosing*.

[3]For $|X| = n$, any neutral solution must assign the same welfare ordering to exactly $\prod_{k=1}^{n} k^{\binom{n}{k}-1}$ choice functions.

In our view, the result illustrates the inherent power and richness of our approach. For one, it shows that a few natural axioms in our setting can uniquely determine a straightforward solution. Given the formidable range of possibilities, we find this surprising.[4] What is more, it shows that axioms in our setting can combine in ways that are quite unexpected. Even though our solution is inherently sequential, for instance, none of our axioms appear to have this feature.

With this in mind, we are more inclined to view our work as the first step towards a coherent theory of choice-based welfare rather than the final word on the matter. Indeed, we feel that other natural axioms in our setting, like those discussed in Section 4, merit further consideration.

2   Definitions and Axioms

Let $X = \{1, ..., n\}$ denote a finite (universal) set of alternatives such that $n \geq 2$. For any $A \subseteq X$, let $\mathcal{A} = \{B \subseteq A : |B| \geq 2\}$ denote the subsets of A with two or more alternatives.

A choice function on A is a function $C : \mathcal{A} \to A$ such that $C(B) \in B$ for every $B \in \mathcal{A}$. In words, a choice function on A selects a single alternative from every subset of A that contains more than one alternative. Let $\mathcal{C}(A)$ denote the set of choice functions on A.

Let $\mathcal{R}(A)$ denote the set of (linear) orderings on A. Given an ordering $R \in \mathcal{R}(A)$, we use interchangeably the standard notations $(x, y) \in R$ and xRy. When convenient, we also denote $R \in \mathcal{R}(A)$ by listing the elements of A in decreasing order according to R. For instance, the natural ordering $R^1 := \{(x, y) \mid 1 \leq x \leq y \leq n\}$ on X can also be written as $R^1 = 1, ..., n$.

Our object of interest is a function that assigns an ordering to every choice function. Formally, a **solution** on A is a function $f : \mathcal{C}(A) \to \mathcal{R}(A)$. Let $\mathcal{F}(A)$ denote the set of solutions on A.

We consider three natural axioms on solutions: admissibility, neutrality and persistence.

To formalize the first axiom, let $A \in \mathcal{X}$. For all $R \in \mathcal{R}(A)$, let $\max_R \in \mathcal{C}(A)$ denote the choice function that selects from every $B \in \mathcal{A}$ the best alternative in B according to the ordering R. We call such a choice function *rational*. A solution $f \in \mathcal{F}(A)$ is **admissible** if

$$f(\max_R) = R \text{ for all } R \in \mathcal{R}(A).$$

To formalize the second axiom, let $\mathcal{P}(A)$ denote the set of permutations (or bijections) on A. For all $\pi \in \mathcal{P}(A)$, $R \in \mathcal{R}(A)$ and $C \in \mathcal{C}(A)$, define the ordering $\pi R \in \mathcal{R}(A)$ by $\pi R := \{(\pi(x), \pi(y)) : (x, y) \in R\}$; and define the choice function $\pi C \in \mathcal{C}(A)$ by $\pi C(B) := \pi(C(\pi^{-1}(B)))$ for all $B \in \mathcal{A}$. Then, a solution $f \in \mathcal{F}(A)$ is **neutral** if

$$f(\pi C) = \pi f(C) \text{ for all } C \in \mathcal{C}(A) \text{ and all } \pi \in \mathcal{P}(A).$$

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[4]To get a sense of the sheer magnitude, there are $n!^{K(n)}$ possible solutions for $|X| = n$ where $K(n) := \prod_{k=1}^{n} k^{\binom{n}{k}}$.

To formalize the last axiom, define a choice function $C'' \in \mathcal{C}(A)$ to be *between* $C \in \mathcal{C}(A)$ and $C' \in \mathcal{C}(A)$ if $C''(B) = C(B)$ or $C''(B) = C'(B)$ for all $B \in \mathcal{A}$.[5] To denote this relationship, we write $C'' \in [C, C']$ when C'' is between C and C'. Then, a solution $f \in \mathcal{F}(A)$ is **persistent** if

$$f(C'') = f(C) = f(C') \text{ for all } C, C', C'' \in \mathcal{C}(A) \text{ such that } f(C) = f(C') \text{ and } C'' \in [C, C'].$$

Finally, the solution described in the introduction can be defined recursively. For all $C \in \mathcal{C}(A)$:

$$\text{let } A_1^C := A; \text{ and, let } A_k^C := A_{k-1}^C \setminus \left\{ C(A_{k-1}^C) \right\} \text{ for } k = 2, ..., |A|.$$

Using these definitions, the **sequential solution** on A is the solution $\varphi_A \in \mathcal{F}(A)$ given by

$$\varphi_A(C) := C(A_1^C), ..., C(A_{|A|}^C) \text{ for all } C \in \mathcal{C}(A).$$

By convention, let $C(\{x\}) := x$ for all $x \in A$ so that $C(A_{|A|}^C)$ is well-defined.

3   Result

Theorem. *A solution $f \in \mathcal{F}(X)$ is admissible, neutral and persistent if and only if $f = \varphi_X$.*

It is straightforward to show that the sequential solution is admissible, neutral and persistent. Proving that it is the *only* solution with these properties is considerably more involved. To illustrate the kinds of arguments that our proof exploits, it is instructive to consider the special case of three alternatives where $X = \{1, 2, 3\}$. The general proof is postponed to Section 5.

Fix a solution f that is admissible, neutral and persistent. Think of a choice function C as an element of the Cartesian product $\{1, 2, 3\} \times \{1, 2\} \times \{1, 3\} \times \{2, 3\}$. By neutrality, it is enough to show that the set of choice functions to which f assigns the natural ordering $R^1 = 1, 2, 3$ coincides with the set of choice functions to which the sequential solution assigns the natural ordering.

The key observation is that the former defines a Cartesian product: for each set of alternatives A, there exists a subset of alternatives $\Gamma(A) \subseteq A$ such that

$$f^{-1}(R^1) = \Gamma(\{1, 2, 3\}) \times \Gamma(\{1, 2\}) \times \Gamma(\{1, 3\}) \times \Gamma(\{2, 3\}).$$

This separability property is precisely the formal content of the persistence axiom.

Since admissibility requires that the rational choice function generated by R^1 belongs to $f^{-1}(R^1)$, it then follows that $1 \in \Gamma(\{1, 2, 3\}) \cap \Gamma(\{1, 2\}) \cap \Gamma(\{1, 3\})$ and $2 \in \Gamma(\{2, 3\})$. Since admissibility also requires that the rational choice function generated by the ordering $1, 3, 2$ cannot belong to $f^{-1}(R^1)$, it is moreover the case that $\Gamma(\{2, 3\}) = \{2\}$.

[5]Note that this implies $C''(B) = C(B) = C'(B)$ whenever $C(B) = C'(B)$.

The rest of the argument exploits the power of neutrality. Because there are $3 \times 2^3 = 24$ choice functions and $3! = 6$ orderings on the universal set, exactly $24/6 = 4$ choice functions must be assigned the natural ordering R^1. In other words,

$$|\Gamma(\{1, 2, 3\})| \times |\Gamma(\{1, 2\})| \times |\Gamma(\{1, 3\})| = 4.$$

As a result, $|\Gamma(\{1, 2, 3\})|$ is either 1 or 2. To rule out the latter possibility, consider the sub-class of choice functions for which some alternative is chosen from *both* two-element sets to which it belongs. Because there are $3^2 \times 2 = 18$ such choice functions, exactly $18/6 = 3$ must be assigned the natural ordering R^1. If $|\Gamma(\{1, 2, 3\})| = 2$ however, the set $f^{-1}(R^1)$ must contain either 2 or 4 choice functions from this sub-class. Therefore $|\Gamma(\{1, 2, 3\})| = 1$. We conclude that

$$\Gamma(\{1, 2, 3\}) = \{1\}, \ \Gamma(\{1, 2\}) = \{1, 2\}, \ \Gamma(\{1, 3\}) = \{1, 3\} \ \text{and} \ \Gamma(\{2, 3\}) = \{2\}.$$

So, $f^{-1}(R^1)$ coincides with the set of choice functions to which the sequential solution assigns R^1.

4   Discussion

(1) It is natural to strengthen admissibility. Given a choice function $C \in \mathcal{C}(A)$, define the binary relation R_C on A by $(x, y) \in R_C$ if and only if $C(B) \neq y$ for all $B \in \mathcal{A}$ such that $x, y \in B$. This is the *unambiguous choice* welfare relation proposed by Bernheim and Rangel (2009).

Call a solution $f \in \mathcal{F}(A)$ **consistent** if

$$R_C \subseteq f(C) \ \text{for all} \ C \in \mathcal{C}(A).$$

By definition, consistency implies admissibility. As a direct corollary of our theorem, the sequential solution is the only solution that is consistent, neutral and persistent. In other words, it is the only neutral and persistent way to complete Bernheim and Rangel's welfare relation.

(2) It is equally natural to weaken persistence. Call a solution $f \in \mathcal{F}(A)$ **weakly persistent** if

$$C' \in \left[C, \max_{f(C)}\right] \ \text{implies} \ f(C') = f(C) \ \text{for all} \ C, C' \in \mathcal{C}(A).$$

This means that the ordering assigned to a choice function C is also assigned to any choice function that lies between C and the rational choice function generated by the ordering assigned to C.[6] By definition, persistence implies weak persistence.

The sequential solution is consistent, neutral and weakly persistent. However, it is not the *only* solution with these properties. For $n = 3$, consider the binary relation $\alpha(C)$ defined by

$$(x, y) \in \alpha(C) \iff \begin{cases} |\{A \in \mathcal{X} : C(A) = x\}| > |\{A \in \mathcal{X} : C(A) = y\}| \, ; \ \text{or} \\ |\{A \in \mathcal{X} : C(A) = x\}| = |\{A \in \mathcal{X} : C(A) = y\}| \ \text{and} \ C(\{x, y\}) = x. \end{cases}$$

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[6]Can and Storcken's (2013) *update monotonicity* is a similar condition in the preference aggregation context.

According to this binary relation, x is welfare preferred to y if: x is chosen more frequently than y; or both alternatives are chosen equally frequently and x is pairwise-chosen over y.

To see that α does indeed define a solution, it is helpful to re-write it using the Cartesian product notation $C := (C(\{1,2,3\}), C(\{1,2\}), C(\{1,3\}), C(\{2,3\}))$ (described in Section 3):

$$
\alpha(C) = \begin{cases}
1,2,3 & \text{if} \quad C \in \{(1,1,1,2),(2,1,1,2),(1,1,3,2),(3,1,1,2)\} \\
1,3,2 & \text{if} \quad C \in \{(1,1,1,3),(3,1,1,3),(1,2,1,3),(2,1,1,3)\} \\
2,1,3 & \text{if} \quad C \in \{(2,2,1,2),(1,2,1,2),(2,2,1,3),(3,2,1,2)\} \\
2,3,1 & \text{if} \quad C \in \{(2,2,3,2),(3,2,3,2),(2,1,3,2),(1,2,3,2)\} \\
3,1,2 & \text{if} \quad C \in \{(3,1,3,3),(1,1,3,3),(3,1,3,2),(2,1,3,3)\} \\
3,2,1 & \text{if} \quad C \in \{(3,2,3,3),(2,2,3,3),(3,2,1,3),(1,2,3,3)\}
\end{cases}
$$

Written this way, it is straightforward to see that α is consistent, neutral and weakly persistent.

(3) Our three axioms are independent. The solution α in (2) shows that persistence is essential.

Neutrality cannot be dropped either. To see this, consider the tournament T_C defined on X by pairwise choices, namely $(x,y) \in T_C$ if and only if $C(\{x,y\}) = x$. Using this tournament, one can define a variety of solutions on X that depend only on pairwise choices. When $n = 3$, for instance, consider the following:

$$
\tau(C) := \begin{cases}
1,2,3 & \text{if} \quad 1T_C 2 T_C 3 T_C 1 \\
1,3,2 & \text{if} \quad 1T_C 3 T_C 2 T_C 1 \\
T_C & \text{otherwise}
\end{cases}
$$

This solution uses the tournament T_C if it is acyclic. Otherwise, it breaks the cycle in T_C in favor of the alternative that comes first in the natural ordering.

Since it gives an inherent advantage to alternative 1, τ is not neutral. However, it is admissible and persistent. To see this, simply re-write the solution using the Cartesian product notation:

$$
\tau(C) = \begin{cases}
1,2,3 & \text{if} \quad C \in \{1,2,3\} \times \{1\} \times \{1,3\} \times \{2\} \\
1,3,2 & \text{if} \quad C \in \{1,2,3\} \times \{1,2\} \times \{1\} \times \{2\} \\
2,1,3 & \text{if} \quad C \in \{1,2,3\} \times \{2\} \times \{1\} \times \{2\} \\
2,3,1 & \text{if} \quad C \in \{1,2,3\} \times \{2\} \times \{3\} \times \{2\} \\
3,1,2 & \text{if} \quad C \in \{1,2,3\} \times \{1\} \times \{3\} \times \{3\} \\
3,2,1 & \text{if} \quad C \in \{1,2,3\} \times \{2\} \times \{3\} \times \{3\}
\end{cases}
$$

Finally, it is clear that admissibility is also essential: the *anti-sequential* solution that assigns to every choice function C the inverse of the ordering $\varphi_X(C)$ is both neutral and persistent.

(4) Using the sequential solution, one can extend a collection $\mathcal{F}_{n-k}$ of solutions on subsets of cardinality $n-k$ into a solution on X. Given $\mathcal{F}_{n-k} := \{f_A \in \mathcal{F}(A) : A \in \mathcal{X}$ such that $|A| = n-k\}$, the idea is to define a solution $\varphi_X \otimes \mathcal{F}_{n-k} \in \mathcal{F}(X)$ that, on the "top" k alternatives, coincides with $\varphi_X \in \mathcal{F}(X)$ and, on the "tail" of $n-k$ alternatives, coincides with the appropriate solution in $\mathcal{F}_{n-k}$. To formalize:

$$(x,y) \in \varphi_X \otimes \mathcal{F}_{n-k}(C) \iff \begin{cases} x \in X \setminus X_{k+1}^C \quad \text{and} \quad (x,y) \in \varphi_X(C); \text{ or} \\ x,y \in X_{k+1}^C \quad \text{and} \quad (x,y) \in f_{X_{k+1}^C}(C|_{X_{k+1}^C}). \end{cases}$$

In this formulation, X_{k+1}^C denotes the "tail" of $n-k$ alternatives according to C (as per the definition in Section 2); and $C|_{X_{k+1}^C}$ denotes the restriction of C to (the subsets of) X_{k+1}^C. Following this approach, one can extend the solutions α and τ (defined for $n=3$) in a natural way. In either case, the extension to $n \geq 4$ inherits the properties of the base solution. Intuitively, this follows from the separability between the "top" and the "tail" of the extension.

(5) We conclude with some potential directions for future research. Which solutions are consistent, neutral and weakly persistent? Does some version of our theorem remain valid on: the restricted domain of *welfare-relevant* choice sets (see Bernheim and Rangel (2009) for the definition)? or the restricted space of choice functions suggested by various theories of bounded rationality? Finally, what can one recommend when: a solution is only required to extract a *weak* ordering from a choice function? there are infinitely many alternatives? or choice behavior defines a correspondence?

5   Proof of Uniqueness

Fix an admissible, neutral and persistent rule $f \in \mathcal{F}(X)$. We claim that $f = \varphi_X$.

The proof is by induction on n, the size of X. The claim is trivially true if $n = 2$. For the induction step, suppose $n \geq 3$ and suppose that, for all $x \in X$, the only admissible, neutral and persistent solution on $X \setminus \{x\}$ is $\varphi_{X\setminus\{x\}}$. Recall that $R^1 = 1, ..., n$ denotes the natural ordering on X. Because f is neutral, it is sufficient to show that $f^{-1}(R^1) = \varphi_X^{-1}(R^1)$.

For any $R \in \mathcal{R}(X)$ and $C \in \mathcal{C}(X)$, let $R|_{X\setminus\{1\}} \in \mathcal{R}(X \setminus \{1\})$ denote the restriction of the ordering R to $X \setminus \{1\}$ and let $C|_{X\setminus\{1\}} \in \mathcal{C}(X \setminus \{1\})$ denote the restriction of the choice function C to (the subsets of) $X \setminus \{1\}$. Finally, define

$$f^{-1}(R)|_{X\setminus\{1\}} := \{C \in \mathcal{C}(X \setminus \{1\}) : \exists C' \in f^{-1}(R) \text{ such that } C = C'|_{X\setminus\{1\}}\}.$$

Step 1. We show that $f^{-1}(R^1)|_{X\setminus\{1\}} = \varphi_X^{-1}(R^1)|_{X\setminus\{1\}}$.

For any $C \in \mathcal{C}(X \setminus \{1\})$, first define the choice function $C^1 \in \mathcal{C}(X)$ by

$$C^1(A) := \begin{cases} 1 & \text{if } 1 \in A \\ C(A) & \text{otherwise.} \end{cases}$$

For any $C \in f^{-1}(R^1)$, observe that $\max_{R^1} \in f^{-1}(R^1)$ by admissibility. Since $(C|_{X \setminus \{1\}})^1 \in [C, \max_{R^1}]$, persistence then implies $(C|_{X \setminus \{1\}})^1 \in f^{-1}(R^1)$. In other words:

$$C \in f^{-1}(R^1) \Rightarrow (C|_{X \setminus \{1\}})^1 \in f^{-1}(R^1). \tag{1}$$

Next, define

$$\begin{aligned} \mathcal{R}^1(X) &:= \left\{ R \in \mathcal{R}(X) : \max_R(X) = 1 \right\} \text{ and} \\ \mathcal{C}^1(X) &:= \{ C \in \mathcal{C}(X) : C(A) = 1 \text{ for all } A \in \mathcal{X} \text{ such that } 1 \in A \}. \end{aligned}$$

Observe that

$$C \in \mathcal{C}^1(X) \Rightarrow f(C) \in \mathcal{R}^1(X). \tag{2}$$

If $f(C) \notin \mathcal{R}^1(X)$, consider the ordering R obtained from $f(C)$ by pushing alternative 1 to the first rank without altering the relative ranks of the other alternatives. Since $\max_R \in \left[\max_{f(C)}, C \right]$ and $f(\max_{f(C)}) = f(C)$, we obtain $f(\max_R) = f(C) \neq R$, contradicting admissibility.

Finally, define the solution $f_1 \in \mathcal{F}(X \setminus \{1\})$ by $f_1(C) := f(C^1)$ for all $C \in \mathcal{C}(X \setminus \{1\})$. It is straightforward to check that f_1 is an admissible, neutral and persistent solution on $X \setminus \{1\}$. By the induction hypothesis,

$$f_1 = \varphi_{X \setminus \{1\}}. \tag{3}$$

To complete Step 1, notice that:

$$\begin{aligned} C \in f^{-1}(R^1)|_{X \setminus \{1\}} &\Leftrightarrow \exists C' \in f^{-1}(R^1) \text{ such that } C = C'|_{X \setminus \{1\}} \\ &\Leftrightarrow C^1 \in f^{-1}(R^1) \text{ [by implication (1)]} \\ &\Leftrightarrow f(C^1) = R^1 \\ &\Leftrightarrow f(C^1)|_{X \setminus \{1\}} = R^1|_{X \setminus \{1\}} \text{ [by implication (2)]} \\ &\Leftrightarrow f_1(C) = R^1|_{X \setminus \{1\}} \text{ [by definition of } f_1] \\ &\Leftrightarrow \varphi_{X \setminus \{1\}}(C) = R^1|_{X \setminus \{1\}} \text{ [by identity (3)]} \\ &\Leftrightarrow C \in \varphi_{X \setminus \{1\}}^{-1}(R^1_{X \setminus \{1\}}) \\ &\Leftrightarrow C \in \varphi_X^{-1}(R^1)|_{X \setminus \{1\}} \text{ [by definition of } \varphi_{X \setminus \{1\}} \text{ and } \varphi_X]. \blacksquare \end{aligned}$$

Because f is persistent, $f^{-1}(R^1)$ is a Cartesian product set. For each $A \in \mathcal{X}$, there exists a nonempty set $\Gamma(A) \subseteq A$ such that $f^{-1}(R^1) = \prod_{A \in \mathcal{X}} \Gamma(A)$. Moreover, $\max_{R^1} \in f^{-1}(R^1)$ by

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admissibility. Hence,

$$\max_{R^1}(A) \in \Gamma(A) \text{ for all } A \in \mathcal{X}. \tag{4}$$

Denoting the cardinality of the set $\Gamma(A)$ by $\gamma(A)$, we have

$$\left|f^{-1}(R^1)\right| = \prod_{A \in \mathcal{X}} \gamma(A). \tag{5}$$

From Step 1, $\Gamma(\{x, ..., n\}) = \{x\}$ for each $x \in \{2, ..., n\}$ and $\Gamma(A) = A$ for every other set $A \in \mathcal{X}$ that does not contain 1. To prove that $f^{-1}(R^1) = \varphi_X^{-1}(R^1)$, it remains to be shown that $\Gamma(X) = \{1\}$ and $\Gamma(A) = A$ for every set $A \in \mathcal{X} \setminus \{X\}$ such that $1 \in A$.

Note: For ease of notation from now on, we drop any reference to X unless this causes confusion. Thus, we write $\mathcal{R}$ instead of $\mathcal{R}(X)$, $\mathcal{C}$ instead of $\mathcal{C}(X)$, $\mathcal{P}$ instead of $\mathcal{P}(X)$ and φ instead of φ_X.

Step 2. We show that $\gamma(A) = n - 1$ for every set A such that $|A| = n - 1$ and $A \neq \{2, ..., n\}$.

Let us call a set $\mathcal{D} \subseteq \mathcal{C}$ *symmetric* if, for all $C \in \mathcal{D}$ and $\pi \in \mathcal{P}$, we have $\pi C \in \mathcal{D}$. Because f is neutral, it is easy to see that, for every symmetric set $\mathcal{D} \subseteq \mathcal{C}$,

$$\left|f^{-1}(R^1) \cap \mathcal{D}\right| = \frac{|\mathcal{D}|}{|\mathcal{R}|} = \frac{|\mathcal{D}|}{n!}. \tag{6}$$

It is straightforward to compute[7] that

$$\frac{|\mathcal{C}|}{n!} = \prod_{k=2}^{n-1} k^{\binom{n}{k}-1}. \tag{7}$$

Since $\mathcal{C}$ is a symmetric set, (5) and (6) imply

$$\prod_{A \in \mathcal{X}} \gamma(A) = \frac{|\mathcal{C}|}{n!}. \tag{8}$$

For $x \in X$, define $\mathcal{C}_{x, n-1} := \{C \in \mathcal{C} : C(A) = x \text{ if } |A| = n - 1 \text{ and } x \in A\}$; and let $\mathcal{C}_{n-1} := \cup_{x \in X} \mathcal{C}_{x, n-1}$. The symmetric set $\mathcal{C}_{n-1}$ contains all the choice functions on X where *some* alternative $x \in X$ is selected from every set of size $n - 1$ that contains it. It is easy to compute that

$$\frac{|\mathcal{C}_{n-1}|}{n!} = n \times \prod_{k=2}^{n-2} k^{\binom{n}{k}-1}. \tag{9}$$

Since R^1 ranks alternative 1 first, (4) implies that $1 \in \Gamma(A)$ for all A such that $|A| = n - 1$ and $x \in A$. Therefore alternative 1 may be chosen from every set of size $n - 1$ which contains it. In other words, $\mathcal{C}_{1, n-1} \subseteq f^{-1}(R^1)$. Suppose $f^{-1}(R^1) \cap \mathcal{C}_{n-1} = f^{-1}(R^1) \cap \mathcal{C}_{1, n-1}$ so that 1 is the

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[7]An easy way is to check that $|\varphi^{-1}(R^1)| = \prod_{k=2}^{n-1} k^{\binom{n}{k}-1}$ and note that $|\varphi^{-1}(R^1)| = |\mathcal{C}|/n!$ because φ is neutral.

only such alternative. Since $\gamma(\{2, ..., n\}) = 1$, it then follows that

$$\left| f^{-1}(R^1) \cap \mathcal{C}_{n-1} \right| = \gamma(X) \times 1 \times \prod_{|A|=2}^{n-2} \gamma(A). \tag{10}$$

Denote the last factor by G^{n-2}. Since $\mathcal{C}_{n-1}$ is a symmetric set, (6) and (10) imply

$$\gamma(X) \times G^{n-2} = \frac{|\mathcal{C}_{n-1}|}{n!}.$$

Dividing (8) by this equation and simplifying using (7) and (9) gives

$$\prod_{|A|=n-1} \gamma(A) = \frac{|\mathcal{C}|}{|\mathcal{C}_{n-1}|} = \frac{(n-1)^{n-1}}{n}.$$

Denote the term on the left side of this expression by G_{n-1}. Since n and $n-1$ are co-prime, we conclude that G_{n-1} is not an integer, which is a contradiction.

So, it must be that some alternative other than 1 may be chosen from every set of size $n-1$ to which it belongs. Since $\Gamma(\{2, ..., n\}) = \{2\}$, this other alternative must be 2. In other words, $f^{-1}(R^1) \cap \mathcal{C}_{n-1} = f^{-1}(R^1) \cap (\mathcal{C}_{1,\,n-1} \cup \mathcal{C}_{2,\,n-1})$. Since there are $\gamma(\{1, 3, ..., n\})$ ways to guarantee that 2 is chosen from every set of size $n-1$ that contains it,

$$\left| f^{-1}(R^1) \cap \mathcal{C}_{n-1} \right| = \gamma(X) \times (1 + \gamma(\{1, 3, ..., n\})) \times G^{n-2}. \tag{11}$$

Since $\mathcal{C}_{n-1}$ is a symmetric set, (6) and (11) imply

$$\gamma(X) \times (1 + \gamma(\{1, 3, ..., n\})) \times G^{n-2} = \frac{|\mathcal{C}_{n-1}|}{n!}.$$

Dividing (8) by this equation and using (7) and (9) gives

$$\frac{G_{n-1}}{1 + \gamma(\{1, 3, ..., n\})} = \frac{|\mathcal{C}|}{|\mathcal{C}_{n-1}|} = \frac{(n-1)^{n-1}}{n} \quad \text{or} \quad G_{n-1} = \frac{(n-1)^{n-1}}{n} \times [1 + \gamma(\{1, 3, ..., n\})].$$

Since G_{n-1} is an integer and n and $n-1$ are co-prime, it must be that $n = 1 + \gamma(\{1, 3, ..., n\})$ or, equivalently, $\gamma(\{1, 3, ..., n\}) = n - 1$. Plugging this back into the above formula establishes that $G_{n-1} = (n-1)^{n-1}$. Since $\gamma(\{2, ..., n\}) = 1$, we conclude that $\gamma(A) = n - 1$ for every set A of size $n-1$ other than $\{2, ..., n\}$. This completes Step 2. ∎

Note: If $n = 3$, Steps 1 and 2 imply that $\Gamma(\{1, 2\}) = \{1, 2\}$, $\Gamma(\{1, 3\}) = \{1, 3\}$ and $\Gamma(\{2, 3\}) = \{2\}$. From (8), it then follows that $\gamma(\{1, 2, 3\}) = 1$. Hence, $\Gamma(\{1, 2, 3\}) = \{1\}$ by (4). This means that $f^{-1}(R^1) = \varphi^{-1}(R^1)$. So, f is the sequential solution. From now on, we assume that $n \geq 4$.

Step 3. We show that $\gamma(X) = 1$ or $\gamma(X) = n$.

Using Step 2, we can rewrite (8) as

$$\gamma(X) \times (n-1)^{n-1} \times G^{n-2} = \frac{|\mathcal{C}|}{n!}. \tag{12}$$

Define $\mathcal{C}_X^{n-1} := \{C \in \mathcal{C} : C(A) \neq C(X) \text{ if } |A| = n-1\}$. This is the symmetric set of choice functions where the alternative selected from X is *never* chosen from any set of size $n-1$. It is straightforward to compute that

$$\frac{|\mathcal{C}_X^{n-1}|}{n!} = (n-2)^{n-1} \times \prod_{k=2}^{n-2} k^{\binom{n}{k}-1}. \tag{13}$$

On the other hand,

$$\left| f^{-1}(R^1) \cap \mathcal{C}_X^{n-1} \right| = [(n-2)^{n-1} + (\gamma^*(X)-1)(n-1)(n-2)^{n-2}] \times G^{n-2}$$

$$\text{where} \quad \gamma^*(X) := \begin{cases} \gamma(X) - 1 & \text{if } 2 \in \Gamma(X) \\ \gamma(X) & \text{otherwise.} \end{cases} \tag{14}$$

This is because there are:

(i) $(n-2)^{n-1}$ ways of not choosing 1 from any set of size $n-1$;

(ii) no ways of not choosing 2 from any set of size $n-1$ (because $\Gamma(\{2, ..., n\}) = \{2\}$); and,

(iii) $(n-1)(n-2)^{n-2}$ ways of not choosing any other alternative from any set of size $n-1$.

Since $\mathcal{C}_X^{n-1}$ is a symmetric set, (6) and (14) imply

$$[(n-2)^{n-1} + (\gamma^*(X)-1)(n-1)(n-2)^{n-2}] \times G^{n-2} = \frac{|\mathcal{C}_X^{n-1}|}{n!}.$$

Dividing (12) by this equation and simplifying using (7) and (13) gives

$$\frac{\gamma(X) \times (n-1)^{n-1}}{(n-2)^{n-1} + (\gamma^*(X)-1)(n-1)(n-2)^{n-2}} = \frac{|\mathcal{C}|}{|\mathcal{C}_X^{n-1}|} = \frac{(n-1)^{n-1}}{(n-2)^{n-1}}.$$

Further simplifying this expression gives $(\gamma^*(X)-1)(n-1) = (\gamma(X)-1)(n-2)$. Since $n-1$ and $n-2$ are co-prime: (i) $\gamma^*(X)-1 = \gamma(X)-1 = 0$; or (ii) $\gamma^*(X)-1 = n-2$ and $\gamma(X)-1 = n-1$. In case (i), $\gamma(X) = 1$; and, in case (ii), $\gamma(X) = n$. This completes Step 3. ∎

Step 4. We show that $\gamma(X) = 1$.

For any $k \in \{2, ..., n\}$, define $\mathcal{C}_{-k}^* := \{C \in \mathcal{C} : \exists R \in \mathcal{R} \text{ such that } C(A) = \max_R(A) \text{ if } |A| \neq k\}$. This is the symmetric set of choice functions that are rational *except* possibly on sets of size k. It is straightforward to compute that

$$\frac{|\mathcal{C}_{-k}^*|}{n!} = k^{\binom{n}{k}}. \tag{15}$$

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By way of contradiction, suppose $\gamma(X) = n$. Let $R^2 := 2, 1, 3, ..., n$. Since $\max_{R^1} \in f^{-1}(R^1)$ and $\max_{R^2} \notin f^{-1}(R^1)$, there exists some $\widehat{A} \in \mathcal{X}$ such that $1 \in \Gamma(\widehat{A})$ and $2 \in \widehat{A} \setminus \Gamma(\widehat{A})$. Let $\hat{k} := |\widehat{A}|$. From Step 2 and $\gamma(X) = n$, $\hat{k} \in \{2, ..., n-2\}$. To simplify the notation, let $G_k := \prod_{|A|=k} \gamma(A)$.

Substep 4.1. We claim that $G_k = k^{\binom{n}{k}-1}$ for all $k \in \{2, ..., n-1\} \setminus \{\hat{k}\}$ when $\gamma(X) = n$.

Fix $k \in \{2, ..., n-1\} \setminus \{\hat{k}\}$. By Step 2, $G_{n-1} = (n-1)^{n-1} = (n-1)^{\binom{n}{n-1}-1}$. This proves the claim if $n = 4$ since in that case $\{2, ..., n-1\} \setminus \{\hat{k}\} = \{2, 3\} \setminus \{2\} = \{3\} = \{n-1\}$. Next, assume $n \geq 5$ and $k \neq n-1$. We claim that

$$\left| f^{-1}(R^1) \cap \mathcal{C}^*_{-k} \right| \leq k G_k. \tag{16}$$

To see why this is the case, consider a choice function $C \in f^{-1}(R^1) \cap \mathcal{C}^*_{-k}$. By definition of $\mathcal{C}^*_{-k}$, there exists an ordering $R \in \mathcal{R}$ such that $C(A) = \max_R(A)$ whenever $|A| \neq k$. Since by Step 1 $\Gamma(\{i, ..., n\}) = \{i\}$ for $i = 2, ..., n-1$, it follows that we must have $C(\{i, ..., n\}) = i$ for $i = 2, ..., (n-k), (n-k+2), ..., (n-1)$. Therefore

$$2 \ R \ ... \ R \ (n-k) \ R \ (n-k+2) \ R \ ... \ R \ n \quad \text{and} \quad (n-k) \ R \ (n-k+1). \tag{17}$$

Since $1 \in \Gamma(\widehat{A})$ and $2 \in \widehat{A} \setminus \Gamma(\widehat{A})$, it must be that

$$1R2. \tag{18}$$

Exactly k orderings R on X satisfy (17) and (18): these are obtained from R_1 by pushing the alternative $n-k+1$ to any rank lower than or equal to $n-k+1$. This proves (16).

Since $\mathcal{C}^*_{-k}$ is a symmetric set, $\left| f^{-1}(R^1) \cap \mathcal{C}^*_{-k} \right| = \left| \mathcal{C}^*_{-k} \right| / n!$. Using (15) and (16), it then follows that $G_k \geq k^{\binom{n}{k}-1}$. But, since $\gamma(\{n-k+1, ..., n\}) = 1$ and $\prod_{|A|=k} |A| = k^{\binom{n}{k}}$, we also know that $G_k \leq k^{\binom{n}{k}-1}$. Combining these two inequalities gives $G_k = k^{\binom{n}{k}-1}$. This completes Substep 4.1.

Substep 4.2. To complete the proof of Step 4, we derive a contradiction from $\gamma(X) = n$.

Given the assumption that $G_n := \gamma(X) = n$, Step 1 and Substep 4.1 imply

$$\left| f^{-1}(R^1) \right| = n \times G_{\hat{k}} \times \prod_{k \neq \hat{k}, n} k^{\binom{n}{k}-1}. \tag{19}$$

Since $\mathcal{C}$ is a symmetric set, (6), (7) and (19) then imply

$$G_{\hat{k}} = \frac{\hat{k}^{\binom{n}{\hat{k}}-1}}{n}. \tag{20}$$

For each $x \in X$, define $\mathcal{C}_x^{-\hat{k}} := \{C \in \mathcal{C} : C(A) \neq x \text{ if } |A| \neq \hat{k}\}$ and let $\mathcal{C}^{-\hat{k}} = \cup_{x \in X} \mathcal{C}_x^{-\hat{k}}$. This is the symmetric set of choice functions where some alternative is *never* chosen *except* possibly from

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sets of size $\hat{k}$. It is straightforward to compute that

$$\frac{|\mathcal{C}^{-\hat{k}}|}{n!} = \left[\prod_{k=\hat{k}+1}^{n-1} k^{\binom{n-1}{k}}(k-1)^{\binom{n-1}{k-1}-1}\right] \times \hat{k}^{\binom{n}{\hat{k}}-1} \times \left[\prod_{k=2}^{\hat{k}-1} k^{\binom{n-1}{k}-1}(k-1)^{\binom{n-1}{k-1}}\right].$$

This simplifies to

$$\frac{|\mathcal{C}^{-\hat{k}}|}{n!} = \widehat{\Pi} \times \hat{k}^{\binom{n}{\hat{k}}-1} \times \left[\frac{(n-1)!}{\hat{k} \times (\hat{k}-1)}\right], \tag{21}$$

where

$$\widehat{\Pi} := \frac{\prod_{k=2}^{n-1} k^{\binom{n-1}{k}-1}(k-1)^{\binom{n-1}{k-1}-1}}{\hat{k}^{\binom{n-1}{\hat{k}}-1}(\hat{k}-1)^{\binom{n-1}{\hat{k}-1}-1}}.$$

Since, by Step 1, $\Gamma(\{x, ..., n\}) = \{x\}$ for each $x \neq 1$, alternatives 1 and $(n - \hat{k} + 1)$ are the only two alternatives that can be never chosen from any set of size other than $\hat{k}$. That is, $f^{-1}(R^1) \cap \mathcal{C}^{-\hat{k}} = f^{-1}(R^1) \cap (\mathcal{C}_1^{-\hat{k}} \cup \mathcal{C}_{n-\hat{k}+1}^{-\hat{k}})$. Therefore,

$$\begin{aligned}
\left|f^{-1}(R^1) \cap \mathcal{C}^{-\hat{k}}\right| &= (n-1) \times \left[\prod_{k=\hat{k}+1}^{n-1} k^{\binom{n-1}{k}-1}(k-1)^{\binom{n-1}{k-1}}\right] \times G_{\hat{k}} \times \left[\prod_{k=2}^{\hat{k}-1} k^{\binom{n-1}{k}-1}(k-1)^{\binom{n-1}{k-1}}\right] \\
&+ (n-1) \times \left[\prod_{k=\hat{k}+1}^{n-1} k^{\binom{n-1}{k}}(k-1)^{\binom{n-1}{k-1}-1}\right] \times G_{\hat{k}} \times \left[\prod_{k=2}^{\hat{k}-1} k^{\binom{n-1}{k}-1}(k-1)^{\binom{n-1}{k-1}}\right].
\end{aligned}$$

Given (20), this simplifies to

$$\left|f^{-1}(R^1) \cap \mathcal{C}^{-\hat{k}}\right| = \widehat{\Pi} \times \hat{k}^{\binom{n}{\hat{k}}-1} \times \left[\frac{(n-1)!}{\hat{k} \times (\hat{k}-1)}\right] \times \left[\frac{\hat{k}-1+n}{n}\right]. \tag{22}$$

Since $\mathcal{C}^{-\hat{k}}$ is a symmetric set, (6), (21) and (22) establish that $\hat{k} = 1$. Since it must be the case that $\hat{k} \in \{2, ..., n-2\}$, this is a contradiction. This completes Substep 4.2 and, hence, Step 4. ∎

Steps 1 and 4 establish that $\gamma(\{x, ..., n\}) = 1$ for each $x \in X$. It follows from (4) and (8) that $\Gamma(X) = \{1\}$ and $\Gamma(A) = A$ for every set $A \in \mathcal{X} \setminus \{X\}$ such that $1 \in A$. Together with Step 1, this implies that $\Gamma(\{x, ..., n\}) = \{x\}$ for each $x \in X$ and $\Gamma(A) = A$ for every other set $A \in \mathcal{X}$. In turn, this establishes that $f^{-1}(R^1) = \varphi^{-1}(R^1)$, which completes the proof.

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