

# Continuous Cardinal Incentive Compatible Mechanisms are Ordinal\*

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## Abstract

We show that every cardinal incentive compatible voting mechanism satisfying a mild continuity condition, must be *ordinal*. Our results apply to many standard models in mechanism design without transfers, including the standard voting models with *any* domain restrictions.

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# 1 INTRODUCTION

Many important models in mechanism design preclude the use of monetary transfers. This may be due to ethical (school choice problems, kidney exchange problems) or institutional (voting) reasons. A typical model in such an environment consists of a set of alternatives and ordinal preferences (strict linear orders) of agents over these alternatives. A mechanism or a social choice function (scf) selects a lottery over the set of alternatives. A prominent feature of these models is that they only use ordinal information about preferences over alternatives. In this paper, we ask if there is any loss of generality in restricting attention to ordinal mechanisms in such environments.

If the solution concept is dominant strategy and the mechanism is deterministic (i.e., lotteries produced by the scf are degenerate), then incentive compatibility immediately implies that the scf cannot be sensitive to cardinal information.<sup>1</sup> However, the answer to this question is not obvious if either (a) we allow for randomization and/or (b) we use the weaker solution concept of Bayesian incentive compatibility.<sup>2</sup> The central issue in such cases is that an agent has to evaluate lotteries based on cardinal utilities over alternatives. We follow the standard practice of evaluating lotteries using the expected utility criterion. Consequently, the evaluation of lotteries uses cardinal preference information of agents. Moreover, it implicitly imposes a domain restriction on agent preferences over the outcomes of the mechanism (i.e. lotteries). It seems natural therefore to believe that this cardinal preference information can be used by the mechanism designer to determine the value of the scf at various profiles.

Our main result shows this to be false. We establish that every dominant strategy incentive compatible, cardinal and random scf that satisfies some version of continuity, is ordinal. We discuss two notions of ordinality and, correspondingly, two notions of continuity. To understand them, note that the *type* of an agent in our cardinal model is a utility vector over alternatives and agents use expected utility to evaluate lotteries. Now, consider an agent  $i$  and fix the utility vectors of other agents at  $u_{-i}$ .

Our first notion of ordinality requires that we treat two utility vectors representing the same ordinal ranking over *lotteries* in the same manner. In other words, we say that an scf is *vNM-ordinal* if  $u_i$  and  $u'_i$  are two utility vectors which are affine transformations of each

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<sup>1</sup> To see this, suppose the scf picks alternative  $a$  when agent  $i$  reports some utility representation of his true preference ordering  $P_i$  (a strict ordering), but picks an alternative  $b$  for some other utility representation. If  $a \neq b$ , then agent  $i$  will clearly manipulate to the utility representation that picks the higher ranked alternative according to  $P_i$ .

<sup>2</sup> The demanding requirement of dominant strategies with deterministic mechanisms often leads to impossibility results in standard voting models (Gibbard, 1973; Satterthwaite, 1975). Using randomization or the weaker solution concept of Bayesian incentive compatibility expands the set of incentive compatible mechanisms. Randomization is also a natural device for achieving fairness.

other, then the outcome of the scf is the same at  $(u_i, u_{-i})$  and  $(u'_i, u_{-i})$ .

To understand vNM-ordinality better, note that the outcome of an scf in our model is a lottery over the set of alternatives. Agents evaluate lotteries over alternatives using the expected utility criterion. By the expected utility theorem, the set of all utility vectors over alternatives can be partitioned into equivalence classes where utility vectors within each equivalence class rank lotteries in the same way. vNM-ordinality requires that scfs assign the same lottery to all utility vectors within an equivalence class.

Note however that even though these utility vectors rank the lotteries the same manner, they *do not* assign them the same expected utility values. vNM-ordinality amounts to ignoring the expected values of the lotteries and only considering ordinal ranking of lotteries. Though this is without loss of generality in a model where the designer only cares about ordinal ranking of lotteries (i.e., utility is ordinal), as we show, vNM-ordinality is *not* implied by incentive compatibility. Hence, it is not an assumption that can be imposed without loss of generality when utility is cardinal.

Our second notion of ordinality, which we call *strong ordinality*, requires that the scf be sensitive to only ordinal ranking of *alternatives*. In other words, strong ordinality says that if  $u_i$  and  $u'_i$  are two utility vectors which represent the same ordinal ranking over alternatives, then the outcome of the scf is the same at  $(u_i, u_{-i})$  and  $(u'_i, u_{-i})$ . Strong ordinality implies vNM-ordinality because if  $u_i$  and  $u'_i$  are affine transformations of each other, then they represent the same ordinal ranking over alternatives.

We impose two versions of continuity. Fix the types of all agents except an arbitrary agent  $i$ . If  $i$  varies the cardinal intensities of her preferences over alternatives without changing her ordinal ranking over the alternatives, then the outcome of the scf must vary continuously. Our notion of continuity is clearly weaker than the standard notion in that it applies only to subdomains where the ordinal rankings over alternatives are unchanged. Thus, our formulation allows scfs to be sensitive to cardinal intensities but restricts them to be continuous in a limited sense. We call this requirement *cone continuity* (*c-continuity*). We also consider a stronger version of continuity where we require the outcome of the scf to vary *uniformly* continuously. We call this requirement *uc-continuity*.<sup>3</sup>

Our main result shows that every dominant strategy incentive compatible scf is vNM-ordinal *almost everywhere*. In other words, there is a dense subset of the domain where the scf is vNM-ordinal and the complement of this subset has measure zero. We give an example to illustrate that dominant strategy incentive compatibility does not imply vNM-ordinality (everywhere) - this is in contrast to the trivial conclusion in the deterministic scf case, where

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<sup>3</sup>In general, uniform continuity is stronger than continuity, but they become equivalent if the domain is compact. Since we apply our continuity requirements on subdomains which are open, this equivalence is not true for us.

incentive compatibility implies (strong) ordinality. We use our main result to establish two striking results: (a) every  $c$ -continuous and dominant strategy incentive compatible scf is  $v$ NM-ordinal and (b) every  $uc$ -continuous and dominant strategy incentive compatible scf is strongly ordinal.

It is well known that there are incentive compatible scfs that are not strongly ordinal. For instance, consider an scf that takes some compact and convex subset of lotteries and an agent, and at every profile of types, it chooses the best lottery in that subset for this agent. It is not difficult to see that such an scf will be Bayesian incentive compatible and  $c$ -continuous. Hence, it is a  $v$ NM-ordinal scf. However, as the cardinal value of all the alternatives approach zero for the agent, the value of the scf does not converge to any single value. A consequence of  $uc$ -continuity is that the scf must converge to a single value in this case. This means that such an scf is not  $uc$ -continuous.

If the solution concept is Bayesian incentive compatibility, our conclusions are in terms of interim allocation probabilities.<sup>4</sup> In particular, we show that every cardinal Bayesian incentive compatible mechanism satisfying  $c$ -continuity property must be such that an agent's interim allocation probability of each alternative does not change whenever she changes her type by using an affine transformation. Similarly, every cardinal Bayesian incentive compatible mechanism satisfying  $uc$ -continuity property must be such that an agent's interim allocation probability of each alternative does not change whenever she changes her type such that the ordinal ranking over alternatives do not change. We note that the results for the Bayesian incentive compatibility case requires no assumptions on the priors of the agents.

Our results apply to standard voting models but can also be applied to private good allocation problems (for instance, matching models) with the additional assumption of *non-bossiness*. Non-bossiness requires the following: a type change by an agent that does not change her allocation also leaves the allocations of all other agents unchanged. Introduced by [Satterthwaite and Sonnenschein \(1981\)](#), non-bossiness is a commonly used condition in the literature ([Papai, 2000](#); [Ehlers, 2002](#)).

A critical feature of our model is the cardinality assumption, i.e. the assumption that every agent can associate a real number (or value) with every alternative.<sup>5</sup> We believe this is a reasonable assumption in many situations of interest. For instance, the alternatives could be possible locations of a public facility. The utility associated with each location by an agent is the (privately observed) transportation or time cost to each location. Or the alternatives

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<sup>4</sup>The interim allocation probability of an alternative being chosen at a type of agent  $i$  is the expected probability of that alternative being chosen, where the expectation is the conditional expectation over all possible types of other agents conditional on the type of agent  $i$ .

<sup>5</sup>There is a long standing debate in utility theory on this issue ([Alchian, 1953](#); [Strotz, 1953](#); [Baumol, 1958](#)).

could be production plans relevant to several managers of a firm. The value of each plan to a manager is the profit associated with it. Moreover, each manager’s profit is private information because of non-publicly observed costs. We note that the use of cardinal utilities in models without transfers, is widespread - recent instances include [Schmitz and Troger \(2012\)](#); [Nandeibam \(2013\)](#); [Ambrus and Egorov \(2013\)](#); [Ben-Porath et al. \(2014\)](#); [Ashlagi and Shi \(2014\)](#); [Gershkov et al. \(2014\)](#).<sup>6</sup> Finally, our results can be interpreted in the following way: *even if* utility is cardinal, incentive compatibility and continuity imply that this cardinal information must be ignored.

We define the type space of an agent by specifying an arbitrary set of permissible linear orders over alternatives and considering all positive utility functions representing these orders. Thus our specification requires maximal cardinal richness consistent with an arbitrary set of ordinal restrictions. Our results therefore hold for standard unrestricted ordinal preferences, single-peaked ordinal preferences, and all standard domain restrictions studied in strategic social choice theory.

We believe that the paper contributes to the literature on mechanism design without transfers in several ways. It provides a foundation for the use of ordinal mechanisms. Moreover, we believe that our paper makes a methodological contribution. We use techniques from multidimensional mechanism design with transfers, particularly subgradient techniques used in that literature, to prove our results. Related methods have been used in some restricted one dimensional problems in the voting literature recently ([Borgers and Postl, 2009](#); [Goswami et al., 2014](#); [Gershkov et al., 2014](#); [Hafalir and Miralles, 2014](#)). To the best of our knowledge, ours is the first paper to use the multidimensional versions of these results in such a setting.

## 2 THE MODEL

We first present a simple one-agent model. We show later how the single-agent results can be extended to the multiple agent framework. There is a finite set of alternatives denoted by  $A = \{a, b, c, \dots\}$  with  $|A| \geq 2$ . Let  $\mathcal{P}$  be the set of all strict linear orderings over the set of alternatives  $A$ . For every  $P \in \mathcal{P}$ , we say a utility function  $v : A \rightarrow \mathbb{R}_+$  represents  $P$  if for all  $a, b \in A$ ,  $aPb$  if and only if  $v(a) > v(b)$ . For every  $P \in \mathcal{P}$ , let  $V^P$  be the set of all utility functions that represent  $P$ . Note that  $V^P$  is a cone in  $\mathbb{R}_+^{|A|}$ .

Throughout the paper, we will fix a domain  $\mathcal{D} \subseteq \mathcal{P}$  and let  $V = \cup_{P \in \mathcal{D}} V^P$ . The type of the agent is a vector  $v \in V$ . A **social choice function (scf)** is a map  $f : V \rightarrow \mathcal{L}(A)$ , where

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<sup>6</sup> [Gershkov et al. \(2014\)](#) consider deterministic dominant strategy incentive compatible mechanisms, where restricting attention to ordinal mechanisms is without loss of generality. However, they consider cardinal utilities to compute an *optimal* mechanism.

$\mathcal{L}(A)$  is the set of lotteries over  $A$ . Notice that our scfs allow for randomization. For every  $v \in V$ , we denote the probability that the scf  $f$  chooses  $a \in A$  at  $v$  by  $f^a(v)$ .

We investigate two notions of incentive compatibility: dominant strategy and Bayesian. Given an scf  $f$ , we define a map  $\Pi^f : V \rightarrow \mathcal{L}(A)$  where at every  $v \in V$ , the component corresponding to alternative  $a$  of this map will be denoted by  $\Pi^f(v, a)$ . In the dominant strategy set up, for every  $v \in V$  and every  $a \in A$ ,  $\Pi^f(v, a) = f^a(v)$ . In the Bayesian set up,  $\Pi^f(v, a)$  denotes the expected probability of scf  $f$  choosing alternative  $a$ , where the expectation is taken over types of other agents (in a multi-agent model) using the prior information of the agent. We state the precise definition later. Notice that we do not put any restriction on the prior - in particular, the priors across agents may be correlated.

**DEFINITION 1** *An scf  $f$  is **incentive compatible** if for all  $v, v' \in V$ , we have*

$$v \cdot \Pi^f(v) \geq v \cdot \Pi^f(v').$$

Depending on the solution concept, incentive compatibility here may refer to dominant strategy incentive compatibility or Bayesian incentive compatibility. The incentive compatibility of an ordinal scf is straightforward to define if the solution concept is dominant strategies and the scf is deterministic. If the solution concept is Bayesian or if we consider randomized scfs, then incentive compatibility in an ordinal model is usually defined in terms of a first-order stochastic dominance relation (Gibbard, 1977; d'Aspremont and Peleg, 1988; Majumdar and Sen, 2004).

## 2.1 Ordinal Mechanisms

We introduce two notions of ordinality here. The first one is a strong notion of ordinality in this context.

**DEFINITION 2** *An scf  $f$  is **strongly ordinal** if for all  $P \in \mathcal{D}$ , for all  $v, v' \in V^P$ , we have  $\Pi^f(v) = \Pi^f(v')$ .*

Strong ordinality requires that we completely ignore cardinal intensities of agents on the alternatives and only consider ordinal rankings. Note that strong ordinality only requires the scf to produce the same outcome in each *subdomain*. Hence, it does not require the scf to be a constant scf - in particular, it can be sensitive to ordinal ranking over alternatives.

However, since the outcome of the scf is a lottery over alternatives, the appropriate notion of ordinality in this context must ignore cardinal utilities of lotteries and only consider ordinal ranking of lotteries. Since we are using expected utility to compute cardinal utilities of lotteries, the expected utility theorem defines a specific ordinal ranking of lotteries.

For any  $u, v \in V$ ,  $u$  is an affine transformation of  $v$  if there exists  $\alpha > 0$  and  $\beta \in \mathbb{R}$  such that  $u(a) = \alpha v(a) + \beta$  for all  $a \in A$ . Any utility function  $v$  induces a ranking over lotteries in  $\mathcal{L}(A)$  via the expected utility theorem. If  $u$  is an affine transformation of  $v$ , then  $u$  and  $v$  induce the same ranking over lotteries. Consequently, the set of utility functions  $V$  can be partitioned into equivalence classes, each representing a unique ranking of lotteries in  $\mathcal{L}(A)$ . This gives rise to the following notion of ordinality.

**DEFINITION 3** *An scf  $f$  is **vNM-ordinal** if for every  $u, v \in V$  such that  $u$  is an affine transformation of  $v$ , we have  $\Pi^f(u) = \Pi^f(v)$ .*

A vNM-ordinal scf does not distinguish between utility functions belonging to the same indifference class. It should be noted that vNM ordinality is *much weaker* than strong ordinality that we have defined.

If we use a vNM-ordinal scf, then we can restrict our domain in the following way.

**DEFINITION 4** *A domain  $V^{vNM} \subseteq \mathbb{R}^{|A|}$  is a **vNM domain** if for every  $v \in V$ ,  $v(a) = 1$  if  $v(a) > v(b)$  for all  $b \neq a$  and  $v(a') = 0$  if  $v(a') < v(b)$  for all  $b \neq a'$ .*

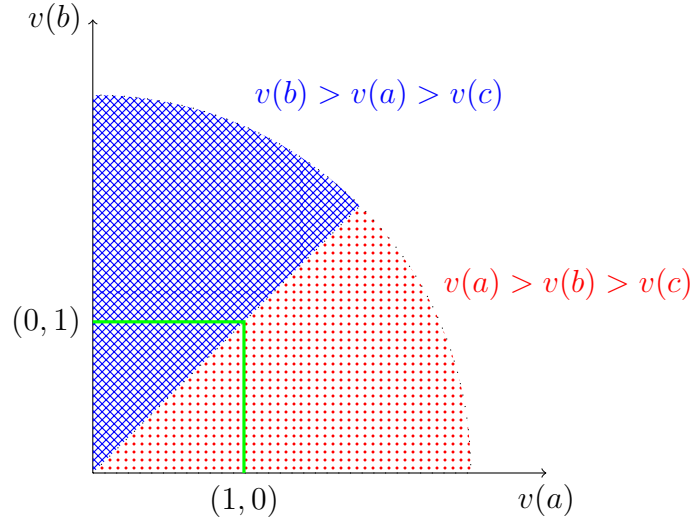


Figure 1: Our domain and the vNM domain

Consider an example with three alternatives  $A = \{a, b, c\}$  and two possible strict linear orders in  $\mathcal{D}$ : for every  $v \in V$ , either  $v(a) > v(b) > v(c)$  or  $v(b) > v(a) > v(c)$ . The projection of  $V$  to the hyperplane  $v(c) = 0$  is shown in Figure 1, which has two cones (above and below the 45-degree line), each representing the strict linear orders in  $\mathcal{D}$ . The vNM domain in this case consists of two lines as shown in Figure 1.

If an scf  $f$  is vNM-ordinal, then its domain can be restricted to  $V^{vNM}$  in the following sense: for every  $v \in V$ , there exists a unique  $v' \in V^{vNM}$  such that  $v'$  is an affine transformation of  $v$  and  $\Pi^f(v) = \Pi^f(v')$ . Moreover, any  $f$  defined on the vNM domain can be uniquely extended to a vNM-ordinal scf over the entire domain.

### 3 INCENTIVE COMPATIBILITY AND vNM-ORDINALITY

We are not assuming vNM-ordinality to begin with - we do not see any compelling reason to do so, particularly in a cardinal setting. By not assuming vNM-ordinality we are allowing the scf to be sensitive to the *expected value* of lotteries. Moreover, incentive compatibility does not imply vNM-ordinality. The following example illustrates that we can have incentive compatible scfs that are not vNM-ordinal.

**EXAMPLE 1** Let  $A = \{a, b, c\}$ . Fix a strict linear order  $P$ , and let the top, middle, and bottom ranked alternatives are  $a, b$ , and  $c$  respectively. We now define an scf  $f$  as follows. For any  $v \in V^P$ , we define  $\Pi^f(v) \equiv (\Pi^f(v, a), \Pi^f(v, b), \Pi^f(v, c))$  as:

$$\Pi^f(v) = \begin{cases} (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) & \text{if } [v(a) - v(b)] > [v(b) - v(c)] \\ (0, \frac{3}{4}, \frac{1}{4}) & \text{if } [v(a) - v(b)] < [v(b) - v(c)] \\ (0, \frac{3}{4}, \frac{1}{4}) & \text{if } [v(a) - v(b)] = [v(b) - v(c)] \text{ and } v(a) - v(c) > 1 \\ (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) & \text{if } [v(a) - v(b)] = [v(b) - v(c)] \text{ and } v(a) - v(c) \leq 1 \end{cases}$$

Figure 2 shows the example in  $\mathbb{R}^2$  by projecting the type space onto the hyperplane  $v(c) = 0$ . The regions that assign the lottery  $(0, \frac{3}{4}, \frac{1}{4})$  are shown above the line  $v(a) = 2v(b)$  (in red color with dashes), while the region that assign the lottery  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$  is shown below the line  $v(a) = 2v(b)$  (in blue color with dots). On the line  $v(a) = 2v(b)$ ,  $f$  assigns the lottery  $(0, \frac{3}{4}, \frac{1}{4})$  for  $v(a) > 1$  and the lottery  $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$  for  $v(a) \leq 1$ .

We first claim that  $f$  is incentive compatible. Pick  $v, v' \in V^P$ . We consider two cases.

**CASE 1.** Suppose  $\Pi^f(v) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ . Notice that, by definition of  $f$ ,  $[v(a) - v(b)] \geq [v(b) - v(c)]$ . If  $\Pi^f(v') = \Pi^f(v)$ , then the agent cannot manipulate from  $v$  to  $v'$ . So, suppose that  $\Pi^f(v') = (0, \frac{3}{4}, \frac{1}{4})$ . Then, truthtelling at  $v$  gives the agent an utility equal to

$$\frac{1}{4}[v(a) + v(b)] + \frac{1}{2}v(c).$$

Manipulating to  $v'$  gives the agent an utility equal to

$$\frac{3}{4}v(b) + \frac{1}{4}v(c).$$



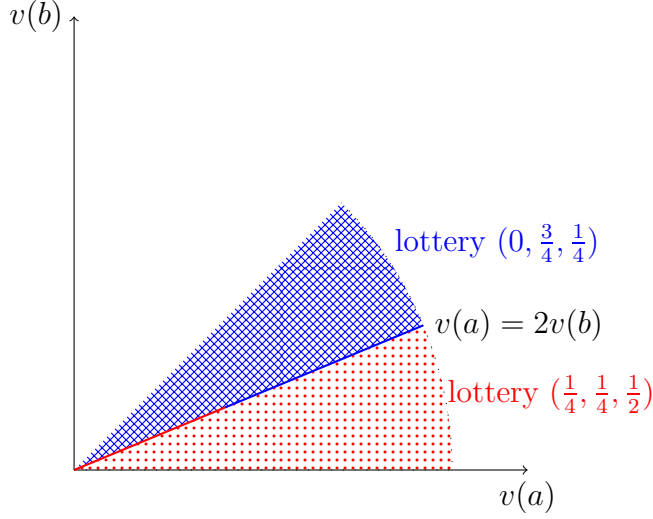


Figure 2: An example of an scf that violates vNM-ordinality

The difference in utility between truthtelling and manipulation is

$$\frac{1}{4}[v(a) - 2v(b) + v(c)] = \frac{1}{4}[[v(a) - v(b)] - [v(b) - v(c)]] \geq 0,$$

where the inequality followed from our earlier conclusion that  $[v(a) - v(b)] \geq [v(b) - v(c)]$ . Hence,  $f$  is incentive compatible.

CASE 2. Suppose  $\Pi^f(v) = (0, \frac{3}{4}, \frac{1}{4})$ . Notice that, by definition of  $f$ ,  $[v(a) - v(b)] \leq [v(b) - v(c)]$ . If  $\Pi^f(v') = \Pi^f(v)$ , then the agent cannot manipulate from  $v$  to  $v'$ . So, suppose that  $\Pi^f(v') = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ . Then, truthtelling at  $v$  gives the agent an utility equal to

$$\frac{3}{4}v(b) + \frac{1}{4}v(c).$$

Manipulating to  $v'$  gives the agent an utility equal to

$$\frac{1}{4}[v(a) + v(b)] + \frac{1}{2}v(c).$$

The difference in utility between truthtelling and manipulation is

$$\frac{1}{4}[-v(a) + 2v(b) - v(c)] = \frac{1}{4}[[v(b) - v(c)] - [v(a) - v(b)]] \geq 0,$$

where the inequality followed from our earlier conclusion that  $[v(a) - v(b)] \leq [v(b) - v(c)]$ . Hence,  $f$  is incentive compatible.

To verify that  $f$  is not vNM-ordinal, pick  $v \equiv (2, 1, 0)$  and  $v' \equiv (1, 0.5, 0)$ . Note that  $\Pi^f(v') = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$  and  $\Pi^f(v) = (0, \frac{3}{4}, \frac{1}{4})$ , but  $v$  is an affine scaling of  $v'$ . This implies that  $f$  is **not** vNM-ordinal.

### 3.1 Almost vNM-ordinality

Though Example 1 showed that incentive compatibility does not imply vNM-ordinality, one can see that vNM-ordinality is violated in that example in a small subset of the domain. We formalize this intuition and show that incentive compatibility implies vNM-ordinality *almost* everywhere.

**DEFINITION 5** *A social choice function  $f$  is **almost vNM-ordinal** if for every  $P \in \mathcal{D}$ , there exists a cone  $\bar{V}^P \subseteq V^P$  such that  $\bar{V}^P$  is dense in  $V^P$ ,  $V^P \setminus \bar{V}^P$  has measure zero, and for all  $u, v \in \bar{V}^P$  such that  $u$  and  $v$  are affine transformations of each other, we have  $\Pi^f(v) = \Pi^f(u)$ .*

We emphasize that  $\bar{V}^P$  is required to be a cone in Definition 5. The main result of the paper is the following.

**THEOREM 1** *Every incentive compatible scf is almost vNM-ordinal.*

*Proof:* Define the indirect utility function  $\mathcal{U}^f$  for any incentive compatible scf  $f$  as follows: for all  $v \in V$ ,  $\mathcal{U}^f(v) = v \cdot \Pi^f(v)$ . Fix a  $P \in \mathcal{D}$ . Incentive constraints inside  $V^P$  reduce to: for all  $v, v' \in V^P$ ,

$$\mathcal{U}^f(v) \geq \mathcal{U}^f(v') + (v - v') \cdot \Pi^f(v').$$

Hence,  $\Pi^f(v')$  is a subgradient of  $\mathcal{U}^f$  at  $v'$ . It is well known that incentive compatibility of  $f$  implies that  $\mathcal{U}^f$  is a convex function. For a quick proof, consider  $v, v' \in V$  and  $\lambda \in (0, 1)$ . Let  $v'' = \lambda v + (1 - \lambda)v'$ . Incentive compatibility implies that

$$\begin{aligned} \mathcal{U}^f(v) &\geq \mathcal{U}^f(v'') + (v - v'') \cdot \Pi^f(v'') \\ \mathcal{U}^f(v') &\geq \mathcal{U}^f(v'') + (v' - v'') \cdot \Pi^f(v''). \end{aligned}$$

Multiplying the first constraint by  $\lambda$  and the second by  $(1 - \lambda)$  gives us  $\lambda \mathcal{U}^f(v) + (1 - \lambda) \mathcal{U}^f(v') \geq \mathcal{U}^f(v'')$ , which shows  $\mathcal{U}^f$  is convex.

Let  $\bar{V}^P$  be the set of points in  $V^P$  at which  $\mathcal{U}^f$  is differentiable. Using [Rockafellar \(1970\)](#), a convex function is differentiable almost everywhere, and hence, the set  $\bar{V}^P$  is dense in  $V^P$  and its complement  $V^P \setminus \bar{V}^P$  has measure zero. We show below that  $\bar{V}^P$  is a cone and  $f$  restricted to  $\bar{V}^P$  is vNM-ordinal.

Consider any  $v \in \bar{V}^P$  and consider  $u \in V^P$  such that  $u(a) = \alpha v(a) + \beta$  for all  $a \in A$ , where  $\alpha > 0$  and  $\beta \in \mathbb{R}$ . Using vector notation, we will write  $u = \alpha v + \mathbf{1}_\beta$ , where  $\mathbf{1}_\beta$  is the vector in  $\mathbb{R}^{|A|}$  whose every component is  $\beta$ . We will show that (a)  $\Pi^f(u) = \Pi^f(v)$  and (b)  $u \in \bar{V}^P$ . We do this in three steps.

STEP 1 -  $\mathcal{U}^f$  IS AFFINE. Incentive constraints imply that

$$v \cdot \Pi^f(v) \geq v \cdot \Pi^f(u),$$

$$\alpha v \cdot \Pi^f(u) + \beta = u \cdot \Pi^f(u) \geq u \cdot \Pi^f(v) = \alpha v \cdot \Pi^f(v) + \beta.$$

The second inequality implies that  $v \cdot \Pi^f(u) \geq v \cdot \Pi^f(v)$ , and this along with the first inequality implies that

$$v \cdot \Pi^f(v) = v \cdot \Pi^f(u). \quad (1)$$

Now,

$$\begin{aligned} \mathcal{U}^f(u) &= u \cdot \Pi^f(u) = \alpha v \cdot \Pi^f(u) + \beta \\ &= \alpha v \cdot \Pi^f(v) + \beta \\ &= \alpha \mathcal{U}^f(v) + \beta, \end{aligned}$$

where the third equality follows from Equation 1. In other words,  $\mathcal{U}^f$  is an affine function.

STEP 2. Now, for every  $\delta > 0$ , for every  $a \in A$ , we denote by  $\mathbf{1}_\delta^a$  as the vector in  $\mathbb{R}^{|A|}$  whose all components are zero except the component of alternative  $a$ . Let  $\delta' = \frac{\delta}{\alpha}$ . We note using Step 1 that

$$\begin{aligned} \mathcal{U}^f(u + \mathbf{1}_\delta^a) &= \mathcal{U}^f(\alpha v + \mathbf{1}_\beta + \mathbf{1}_\delta^a) \\ &= \mathcal{U}^f(\alpha(v + \mathbf{1}_{\delta'}^a) + \mathbf{1}_\beta) \\ &= \alpha \mathcal{U}^f(v + \mathbf{1}_{\delta'}^a) + \beta, \end{aligned}$$

where we used the affine property of  $\mathcal{U}^f$  in the last equality.

STEP 3. We can now conclude the proof. Consider any  $\delta > 0$  and  $a \in A$ . Let  $\delta' = \frac{\delta}{\alpha}$ . Now, since  $v \in \bar{V}^P$ ,  $\Pi^f(v)$  is the gradient of  $\mathcal{U}^f$  at  $v$ . Using this, we can write

$$\begin{aligned} \Pi^f(v, a) &= \lim_{\delta' \rightarrow 0} \frac{\mathcal{U}^f(v + \mathbf{1}_{\delta'}^a) - \mathcal{U}^f(v)}{\delta'} \\ &= \lim_{\delta \rightarrow 0} \frac{\alpha \mathcal{U}^f(v + \mathbf{1}_{\delta'}^a) - \alpha \mathcal{U}^f(v)}{\delta} \\ &= \lim_{\delta \rightarrow 0} \frac{[\alpha \mathcal{U}^f(v + \mathbf{1}_{\delta'}^a) + \beta] - [\alpha \mathcal{U}^f(v) + \beta]}{\delta} \\ &= \lim_{\delta \rightarrow 0} \frac{\mathcal{U}^f(u + \mathbf{1}_\delta^a) - \mathcal{U}^f(u)}{\delta} \\ &= \Pi^f(u, a), \end{aligned}$$

where we used the affine property for the fourth equality. This also shows that  $\mathcal{U}^f$  is differentiable at  $v$ . Hence,  $u \in \bar{V}^P$ . ■

### 3.2 Continuous Mechanisms

We now show that Theorem 1 can be strengthened if assume some form of continuity property of our scf. We present two versions of continuity and show their implications.

For any scf  $f$  and any  $P \in \mathcal{D}$ , consider the map  $\Pi_P^f : V^P \rightarrow \mathcal{L}(A)$ , i.e. the restriction of  $\Pi^f$  to  $V^P$ . Every continuity property that we impose will be on  $\Pi_P^f$  for every  $P \in \mathcal{D}$ . Hence, it will not be for the whole of  $\Pi^f$  - indeed,  $\Pi^f$  may not be continuous. The first continuity we impose is the standard continuity.

**DEFINITION 6** *An scf  $f$  is **cone continuous (c-continuous)** if for every  $P \in \mathcal{D}$ ,  $\Pi_P^f$  is continuous in  $V^P$ , i.e., for every  $v \in V^P$ , for every  $\epsilon > 0$ , there exists  $\delta$  such that for all  $v' \in V^P$  with  $\|v - v'\| < \delta$ , we have  $\|\Pi^f(v) - \Pi^f(v')\| < \epsilon$ .*

Fix an scf  $f$  and  $P \in \mathcal{P}$ . By allowing for cardinal scfs, we are allowing  $\Pi_P^f$  to be a non-constant map, i.e., for any  $v, v' \in V^P$ ,  $\Pi^f(v)$  can be different from  $\Pi^f(v')$ . Note that an ordinal scf trivially satisfies c-continuity since for every  $P$ ,  $\Pi_P^f(v) = \Pi_P^f(v')$  for all  $v, v' \in V^P$ . Notice that c-continuity does not impose any restriction between  $\Pi_P^f$  and  $\Pi_{P'}^f$  if  $P \neq P'$ .

The c-continuity assumption strengthens Theorem 1 in the following manner.

**THEOREM 2** *Every incentive compatible and c-continuous scf is vNM-ordinal.*

*Proof:* Let  $f$  be an incentive compatible and c-continuous scf. By Theorem 1,  $f$  is almost vNM-ordinal. Fix a  $P \in \mathcal{D}$ . Almost vNM-ordinality implies that there exists a cone  $\bar{V}^P \subseteq V^P$  such that  $\bar{V}^P$  is dense in  $V^P$  and  $V^P \setminus \bar{V}^P$  has measure zero. Further, for every  $u, v \in \bar{V}^P$  such that  $u = \alpha v + \mathbf{1}_\beta$ , we have  $\Pi^f(v) = \Pi^f(u)$ , where  $\alpha > 0$  and  $\beta \in \mathbb{R}$ .

Now, consider  $v \in V^P \setminus \bar{V}^P$ . Since  $\bar{V}^P$  is dense, there exists a sequence  $\{v^k\}_k$  such that each  $v^k \in \bar{V}^P$  and this sequence converges to  $v$ . By continuity of  $f$ , the sequence  $\{\Pi^f(v^k)\}_k$  converges to  $\Pi^f(v)$ .

Now, consider  $u = \alpha v + \mathbf{1}_\beta$ , where  $\alpha > 0$  and  $\beta \in \mathbb{R}$ . For every  $v^k$  in the sequence  $\{v^k\}_k$ , define  $u^k = \alpha v^k + \mathbf{1}_\beta$ . By definition, the sequence  $\{u^k\}_k$  converges to  $u$ . Since  $\bar{V}^P$  is a cone, for every  $v^k \in \bar{V}^P$ , we have  $u^k \in \bar{V}^P$ . By continuity of  $f$ , the sequence  $\{\Pi^f(u^k)\}_k$  converges to  $\Pi^f(u)$ . By Theorem 1,  $\Pi^f(v^k) = \Pi^f(u^k)$ . Hence, the sequence  $\{\Pi^f(u^k)\}_k$  also converges to  $\Pi^f(v)$ . Hence,  $\Pi^f(v) = \Pi^f(u)$ . ■

We now strengthen continuity by requiring uniform continuity as follows.

**DEFINITION 7** An scf  $f$  is **uniformly cone continuous (uc-continuous)** if for every  $P \in \mathcal{D}$ ,  $\Pi_P^f$  is uniformly continuous in  $V^P$ , i.e., for every  $\epsilon > 0$ , there exists  $\delta$  such that for all  $v, v' \in V^P$  with  $\|v - v'\| < \delta$ , we have  $\|\Pi^f(v) - \Pi^f(v')\| < \epsilon$ .

A uniformly continuous function is continuous - the converse is true if the domain is compact (Royden, 1968). The difference between uniform continuity and continuity is that the choice of  $\delta$  in Definition 7 depends only on  $\epsilon$  and not on the point in the domain considered as in Definition 6. Given that  $\Pi_P^f$  is a bounded function, it only rules out continuous functions that behave badly (for instance, oscillate) at the boundary of  $V^P$ .

The uc-continuity assumption strengthens Theorem 1 even further.

**THEOREM 3** Every incentive compatible and uc-continuous scf is strongly ordinal.

*Proof:* We do the proof in two steps. In Step 1, we prove a stronger result and show that the theorem follows from this step in Step 2.

**STEP 1.** Consider the following regularity condition.

**DEFINITION 8** The scf  $f$  is **0-regular**, if for all  $P \in \mathcal{D}$  and all sequences  $\{v_k\}, \{v'_k\}$  in  $V^P$ , we have

$$\lim_{v_k \rightarrow \mathbf{0}} \Pi^f(v_k) = \lim_{v'_k \rightarrow \mathbf{0}} \Pi^f(v'_k).$$

The **0-regularity** condition applies to type sequences converging to the origin in a given subdomain. It requires the scf to converge to the same value along all such sequences. We first show that every incentive compatible, c-continuous, and **0-regular** scf is strongly ordinal.

**LEMMA 1** Every incentive compatible, c-continuous, and **0-regular** scf is strongly ordinal.

*Proof:* Let  $f$  be a incentive compatible, c-continuous, and **0-regular** scf. By Theorem 2,  $f$  is a vNM-ordinal scf. Pick any  $v, v' \in V^P$ , where  $P \in \mathcal{D}$  and  $v, v'$  represent  $P$ . By vNM-ordinality, for every  $\epsilon > 0$ , we have  $\Pi^f(v) = \Pi^f(\epsilon v)$  and  $\Pi^f(v') = \Pi^f(\epsilon v')$ . Hence,  $\Pi^f(v) = \lim_{\epsilon \rightarrow 0} \Pi^f(\epsilon v)$  and  $\Pi^f(v') = \lim_{\epsilon \rightarrow 0} \Pi^f(\epsilon v')$ . By c-continuity,  $\lim_{\epsilon \rightarrow 0} \Pi^f(\epsilon v) = \lim_{\epsilon \rightarrow 0} \Pi^f(\epsilon v')$ . Therefore,  $\Pi^f(v) = \Pi^f(v')$  as required. ■

**STEP 2.** In this step, we show that every uc-continuous scf is c-continuous and **0-regular**. Clearly, a uc-continuous scf is c-continuous (since uniform continuity implies continuity). Let  $f$  be an incentive compatible and uc-continuous scf.

**CLAIM 1** If  $\Pi_P^f$  is uniformly continuous, then for every sequence of types  $\{v_k\}_{k \in \mathbb{N}}$  in  $V^P$  converging to  $\mathbf{0}$ , the sequence  $\{\Pi_P^f(v_k)\}_{k \in \mathbb{N}}$  must converge to a point which is independent of the sequence of types  $\{v_k\}_{k \in \mathbb{N}}$  chosen in  $V^P$ .

The proof of this claim is provided in the Appendix. The proof follows from the fact that every uniformly continuous function can be extended to its closure in a uniformly continuous manner, and moreover, such an extension is unique (Royden, 1968). It is then easy to verify that Claim 1 is true. We provide a self contained proof in the Appendix.

Claim 1 shows that if  $f$  is uc-continuous, then it is  $\mathbf{0}$ -regular, and this concludes our proof. ■

The following example illustrates that Theorem 3 is not true if we only use c-continuity (instead of uc-continuity).

**EXAMPLE 2** Let  $A = \{a, b, c\}$ . Fix any strict ordering  $P$  of  $\{a, b, c\}$  such that the top, middle, and bottom ranked alternatives are  $a$ ,  $b$ , and  $c$  respectively.

We give an example of a vNM-ordinal scf that is incentive compatible but not strongly ordinal. We specify the scf over the vNM domain - its extension to the entire domain follows from vNM-ordinality. In the vNM domain, any utility function  $v$  representing  $P$  will have  $v(a) = 1, v(c) = 0$ , and  $v(b) \in (0, 1)$ . We will describe an scf for the preference ordering  $P$ . Let  $\theta$  be the value of  $v(b)$ . Now, consider the following scf  $f$ :

$$\begin{aligned}\Pi_P^f(\theta, a) &= \frac{1}{2} - \frac{\theta^2}{4} \\ \Pi_P^f(\theta, b) &= \frac{\theta}{2} \\ \Pi_P^f(\theta, c) &= \frac{1}{2} - \frac{\theta}{2} + \frac{\theta^2}{4},\end{aligned}$$

where we have abused notation to write  $\Pi_P^f(\theta, \cdot)$  instead of  $\Pi_P^f(v, \cdot)$ . Note that if the agent has a type  $v$  with  $v(b) = \theta$  and reports a type  $v'$  with  $v'(b) = \theta'$ , her net utility is

$$\frac{1}{2} - \frac{\theta'^2}{4} + \frac{\theta\theta'}{2}.$$

On the other hand, truth telling gives a net utility of

$$\frac{1}{2} - \frac{\theta^2}{4} + \frac{\theta^2}{2} = \frac{1}{2} + \frac{\theta^2}{4}.$$

The difference between truth telling and deviating to  $\theta'$  is thus given by

$$\frac{\theta^2}{4} + \frac{\theta'^2}{4} - \frac{\theta\theta'}{2} = \left(\frac{\theta}{2} - \frac{\theta'}{2}\right)^2 \geq 0.$$

Hence,  $f$  is incentive compatible. Note that the extension of  $f$  to the entire domain is c-continuous, but it is not  $\mathbf{0}$ -regular. Hence, it is not uc-continuous.

We also note that if  $|A| = 2$ , Theorem 3 holds without any further assumptions regarding uc-continuity or domains.<sup>7</sup> To see this, suppose  $A = \{a, b\}$  and  $f$  is an incentive compatible scf. Note that  $\Pi^f(v, a) + \Pi^f(v, b) = 1$  for all  $v \in V$ . Using this, for any  $v, v' \in V$ , incentive compatibility implies that

$$\begin{aligned} [v(a) - v(b)]\Pi^f(v, a) + v(b) &\geq [v(a) - v(b)]\Pi^f(v', a) + v(b) \\ [v'(a) - v'(b)]\Pi^f(v', a) + v'(b) &\geq [v'(a) - v'(b)]\Pi^f(v, a) + v'(b). \end{aligned}$$

Combining these two inequalities we get,  $\Pi^f(v, a) = \Pi^f(v', a)$ . This also highlights the fact that we can always fix the lowest ranked alternative value at zero and work with the rest of the alternatives, essentially reducing the dimensionality of the problem.

In cardinal environment, a well studied scf is the utilitarian scf that chooses an alternative that maximizes the sum of utilities of all the agents at every type profile.<sup>8</sup> It is easy to verify that the utilitarian scf is c-continuous. Clearly, it is not a vNM-ordinal scf. Hence, by Theorem 2, it is not incentive compatible.

## 4 A GENERAL CHARACTERIZATION

Theorems 2 and 3 show that if we accept some form of continuity, then a corresponding form of ordinality will be implied by incentive compatibility. However, it does not say anything about the class of such incentive compatible mechanisms. A characterization of the set of incentive compatible mechanisms or the set of continuous incentive compatible mechanisms is a difficult task - later, we discuss some related literature that has addressed this question in a limited way.

Below, we provide an indirect characterization of incentive compatible scfs. Our characterization is inspired by similar characterizations in the multidimensional mechanism design literature that we discuss later. Also, similar characterizations have been used in specific simpler settings (in particular, with one-dimensional types) in the literature (Borgers and Postl, 2009; Goswami et al., 2014; Gershkov et al., 2014; Hafalir and Miralles, 2014).

To state this result, we define a standard notion of monotonicity of scfs.

**DEFINITION 9** *An scf  $f$  is **monotone** if for every  $v, v' \in V$ , we have*

$$(v - v') \cdot [\Pi^f(v) - \Pi^f(v')] \geq 0.$$

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<sup>7</sup>The two alternatives case occupies an important place in the strategic voting literature because the Gibbard-Satterthwaite theorem does not hold in this case. (Schmitz and Troger, 2012) considers an optimal mechanism design problem in a two-alternative model with cardinal intensities.

<sup>8</sup>A random version of the utilitarian scf will choose at every type profile, a probability distribution of alternatives that maximize the sum of utilities of all the agents.

For every  $v, v' \in V$ , define the map  $\psi^{v',v} : [0, 1] \rightarrow \mathbb{R}$  as follows. For every  $t \in [0, 1]$ ,

$$\psi^{v',v}(t) = (v - v') \cdot \Pi^f(v' + t(v - v')).$$

Notice that if  $V$  is convex, for any  $v, v' \in V$ ,  $\int_0^1 \psi^{v',v}(t)dt$  is the line integral of  $\Pi^f$  from  $v'$  to  $v$ .

**THEOREM 4** *Let  $V \subseteq \mathbb{R}^{|A|}$  be a convex set and  $f : V \rightarrow \mathcal{L}(A)$  be an scf defined on  $V$ . Then, the following statements are equivalent.*

1.  $f$  is incentive compatible.
2.  $f$  is monotone and for every  $v, v' \in V$ , we have

$$v \cdot \Pi^f(v) - v' \cdot \Pi^f(v') = \int_0^1 \psi^{v',v}(t)dt.$$

3. For every  $v, v' \in V$ ,  $\psi^{v',v}$  is a non-decreasing function and

$$v \cdot \Pi^f(v) - v' \cdot \Pi^f(v') = \int_0^1 \psi^{v',v}(t)dt.$$

The proof of Theorem 4 is in the Appendix. The implicit characterization provided in Theorem 4 has counterparts in the multidimensional mechanism design literature with transfers and quasi-linear utilities. In these models, a mechanism consists of an *allocation rule* and a *transfer rule*. An allocation rule is *implementable* if there exists a transfer rule such that the resulting mechanism is incentive compatible. Implementable allocation rules can be characterized by a strong monotonicity property called *cycle monotonicity* (Rockafellar, 1970; Rochet, 1987) and the transfer rules by a property called revenue/payoff equivalence principle (Jehiel et al., 1999; Krishna and Maenner, 2001; Milgrom and Segal, 2002; Chung and Olszewski, 2007; Heydenreich et al., 2009). The separability of allocation rule and payoff characterization is possible because transfers are allowed. In our model, a characterization of incentive compatible scfs (Theorem 4) combines elements of both allocation rule and payoff characterizations in the model with transfers.

## 5 EXTENSION TO MANY AGENTS

In this section, we discuss extensions of our result to a multi-agent model. Let  $N = \{1, \dots, n\}$  be the set of agents. The type space of agent  $i$  will be denoted as  $V_i$ , and, as before, it is the set of all utility functions consistent with some set of strict orderings  $\mathcal{D}_i \subseteq \mathcal{P}$ . Let



$\mathcal{V} = V_1 \times \dots \times V_n$  be the set of all profiles of types. As before,  $A$  is the set of alternatives. A social choice function  $f$  is a map

$$f : \mathcal{V} \rightarrow \mathcal{L}(A)^n.$$

Observe that  $f$  picks  $n$  lotteries at each type profile - one for each agent. This formulation allows us to capture both public good and private good problems. We assume **no externalities**, i.e., the utility of an agent only depends on the lottery chosen for her. For any scf  $f$ ,  $f_i(v)$  and  $f_i^a(v)$  will denote the lottery chosen for agent  $i$  at type profile  $v$  and the corresponding probability of choosing alternative  $a$  respectively.

We distinguish between two kinds of models. A **voting** model is one where every scf  $f$  satisfies  $f_i(v) = f_j(v)$  for every pair of agents  $i, j \in N$  and every type profile  $v$ . In these models, an scf must choose the *same* lottery for all the agents. This covers the standard strategic voting models. Any model that is not a voting model will be called a **private good** model. In these models, there is no requirement of choosing the same lottery for all agents although there may be restrictions on the choices. For instance, in the one-sided matching model (here  $A$  is the set of objects), an scf need not choose the same lottery over objects for each agent, but the lotteries must satisfy feasibility conditions.

As before, we consider two notions of incentive compatibility.

**Dominant Strategy Incentive Compatibility.** Dominant strategy incentive compatibility requires that for every  $i \in N$ ,  $v_i, v'_i \in V_i$  and  $v_{-i} \in V_{-i}$ , we have  $v_i \cdot f_i(v_i, v_{-i}) \geq v_i \cdot f_i(v'_i, v_{-i})$ . We can now define  $\Pi_i^f(v) := f_i(v)$  for every  $i \in N$  and for every  $v \in \mathcal{V}$ . The definitions of c-continuity and uc-continuity can now be straightforwardly adapted with additional qualifiers for each  $i \in N$  and each  $v_{-i}$ .

To define the notions of ordinality here, we say that two type profiles  $v, v' \in \mathcal{V}$  are ordinally equivalent if for every  $i \in N$ ,  $v_i$  and  $v'_i$  are affine transformations of each other. Similarly, we say two profiles  $v, v' \in \mathcal{V}$  are strongly ordinally equivalent if for every  $i \in N$ ,  $v_i$  and  $v'_i$  represent the same strict ordering in  $\mathcal{D}_i$ . An scf  $f$  is **vNM-ordinal** if for every pair of type profiles  $v, v' \in \mathcal{V}$  such that  $v$  and  $v'$  are ordinally equivalent, we have  $\Pi_i^f(v) = \Pi_i^f(v')$  for all  $i \in N$ . An scf  $f$  is **strongly ordinal** if for every pair of type profiles  $v, v' \in \mathcal{V}$  such that  $v$  and  $v'$  are strongly ordinally equivalent, we have  $\Pi_i^f(v) = \Pi_i^f(v')$  for all  $i \in N$ .

We can now extend Theorems 2 and 3. Note here that we do not require any assumption on  $\{\mathcal{D}_i\}_{i \in N}$ .

**THEOREM 5** *In the voting model, (a) every dominant strategy incentive compatible and c-continuous scf is vNM-ordinal and (b) every dominant strategy incentive compatible and uc-continuous scf is strongly ordinal.*

*Proof:* Let  $f$  be a dominant strategy incentive compatible and c-continuous scf. By virtue of the voting model assumption,  $\Pi_i^f(v'') = \Pi_j^f(v'')$  for all  $i, j \in N$  and  $v''$ . Fix any pair of ordinally equivalent type profiles  $v, v' \in \mathcal{V}$ . We move from  $v$  to  $v'$  by changing the types of agents one at a time and apply Theorem 2 at each step. This immediately implies that  $f$  is vNM-ordinal. A similar proof using Theorem 3 establishes (b). ■

This argument does not work in the private good models for well-known reasons. In the move from  $(v_i, v_{-i})$  to  $(v'_i, v_{-i})$  (where  $(v_i, v_{-i})$  and  $(v'_i, v_{-i})$  are ordinally equivalent), Theorem 2 guarantees  $\Pi_i^f(v_i, v_{-i}) = \Pi_i^f(v'_i, v_{-i})$  but not  $\Pi_j^f(v_i, v_{-i}) = \Pi_j^f(v'_i, v_{-i})$  for any  $j \neq i$ . This can be restored by the familiar non-bossiness condition adapted to our model. An scf  $f$  is **non-bossy** if for all  $i \in N$ ,  $v_i, v'_i \in V_i$  and  $v_{-i} \in V_{-i}$ ,  $\Pi_i^f(v_i, v_{-i}) = \Pi_i^f(v'_i, v_{-i})$  implies  $\Pi_j^f(v_i, v_{-i}) = \Pi_j^f(v'_i, v_{-i})$  for all  $j \neq i$ . It is now easy to see that Theorem 5 holds in private good problem with the additional non-bossiness condition.

**Bayesian Incentive Compatibility.** Each agent has a conditional distribution over the types of other agents. Thus agent  $i$  of type  $v_i$  has a distribution with cdf  $G_i(\cdot|v_i)$ , i.e. for any  $v_{-i}$ ,  $G_i(v_{-i}|v_i)$  is the cumulative probability that others are of type  $v_{-i}$  given that  $i$  is of type  $v_i$ . We do not impose any restriction on the distributions and allow for arbitrary correlation. For any scf  $f$ ,  $\Pi_i^f(v_i)$  reflects the interim allocation probability vector at  $v_i$  for agent  $i$ . Formally, for every  $a \in A$ ,  $i \in N$  and  $v_i \in V_i$ ,

$$\Pi_i^f(v_i, a) := \int_{V_{-i}} f_i^a(v_i, v_{-i}) dG_i(v_{-i}|v_i).$$

An scf  $f$  is **Bayesian incentive compatible** if for every  $i \in N$  and for every  $v_i, v'_i \in V_i$ , we have  $v_i \cdot \Pi_i^f(v_i) \geq v_i \cdot \Pi_i^f(v'_i)$ . With this interpretation of  $\Pi^f$ , it is straightforward to extend the definition of c-continuity and uc-continuity.

The scf  $f$  is **vNM-ordinal in expectation (ev-ordinal)** if for all  $i \in N$  and pairs of ordinally equivalent types  $v_i, v'_i \in V_i$ , we have  $\Pi_i^f(v_i) = \Pi_i^f(v'_i)$ . Similarly, the scf  $f$  is **strongly ordinal in expectation (es-ordinal)** if for all  $i \in N$  and pairs of strongly ordinally equivalent types  $v_i, v'_i \in V_i$ , we have  $\Pi_i^f(v_i) = \Pi_i^f(v'_i)$ . It is important to observe that the notion of e-ordinality is different from that of ordinality in the dominant strategy case because interim probabilities for an agent depend only on her type rather than on the entire type profile.

With these modifications, Theorems 2 and 3 can be directly extended as follows: every BIC and c-continuous scf is ev-ordinal and every BIC and uc-continuous scf is es-ordinal. In view of the notion of ordinality in the Bayesian model (see our earlier remark), the distinction between the voting and private good models disappears and the non-bossiness assumption is redundant. Hence, an identical proof as in Theorem 5 gives us the following theorem.

**THEOREM 6** *Every Bayesian incentive compatible and  $c$ -continuous scf is ev-ordinal. Every Bayesian incentive compatible and uc-continuous scf is es-ordinal.*

## 6 RELATIONSHIP TO THE LITERATURE

The primary focus of the literature on mechanism design without monetary transfers has been on deterministic models and dominant strategies. As we have noted earlier, the issue of cardinal information has no bearing in these cases and is of relevance only on (a) models involving randomization (b) models using Bayes-Nash rather than dominant strategies as a solution concept. We comment briefly on the literature in these areas and their relationship with our work.

The seminal paper in randomized mechanism design for voting models is [Gibbard \(1977\)](#). The paper considered an explicitly ordinal problem in a dominant strategy framework when the preferences of agents is unrestricted. It proposed a demanding notion of incentive-compatibility where the truth-telling lottery was required to stochastically dominate the lottery arising from any manipulation. This approach has been extended to several restricted domains of ordinal preferences in voting models ([Chatterji et al. \(2012\)](#), [Peters et al. \(2014\)](#), [Chatterji et al. \(2014\)](#) and [Pycia and Unver \(2014\)](#)) and in ordinal matching models ([Bogomolnaia and Moulin \(2001\)](#), and [Erdil \(2014\)](#)).

Random mechanism design with cardinal utilities and dominant strategies has received far less attention. [Hylland \(1980\)](#), [Barbera et al. \(1998\)](#); [Dutta et al. \(2007\)](#); [Nandeibam \(2013\)](#) revisit Gibbard’s voting model with cardinal utilities in the unrestricted ordinal domain while [Zhou \(1990\)](#) considers the one-sided matching problem. Significantly, all the papers use the vNM domain except [Nandeibam \(2013\)](#). While [Barbera et al. \(1998\)](#); [Dutta et al. \(2007\)](#) show that every randomized DSIC scf in the vNM domain with unrestricted ordinal preferences must be a random dictatorship under unanimity and other additional conditions, [Nandeibam \(2013\)](#) shows a weaker version of this result without assuming vNM domain but still requiring unrestricted ordinal preferences. [Zhou \(1990\)](#) shows incompatibility of Pareto efficiency, DSIC, and *symmetry* in the one-sided matching problem.

[Majumdar and Sen \(2004\)](#) consider deterministic Bayesian incentive compatible mechanisms in an ordinal model employing a solution concept developed for this framework in [d’Aspremont and Peleg \(1988\)](#). [Borgers and Postl \(2009\)](#) consider a model with three alternatives and two agents in the vNM domain. The two agents have fixed but completely opposed ordinal preferences. The type of each agent is the utility of the common second-ranked or “compromise” alternative. In the vNM domain, this model gives rise to a special one-dimensional mechanism design problem with transfers. They characterize the set of cardinal Bayesian incentive compatible scfs using Myersonian techniques. The characterization

is further extended in [Postl \(2011\)](#). [Miralles \(2012\)](#) considers a model of allocating two objects to agents without monetary transfers and Bayesian incentive compatibility. A recent paper by [Kim \(2014\)](#) considers the vNM domain with Bayesian incentive compatibility. He shows that every ordinal mechanism is *dominated* (in terms of utilitarian social welfare) by a suitable cardinal mechanism in the vNM domain.

Almost all the papers on cardinal mechanisms cited above, consider vNM domains. Many of them highlight the fact that there is an expansion in the set of incentive compatible scfs relative to the strongly ordinal model. Our paper provides a foundation for the use of vNM domains - in a cardinal model, c-continuity and incentive compatibility implies vNM-ordinality. However, strengthening c-continuity to uc-continuity brings us to a completely ordinal model giving a precise description of the boundary between vNM domain and the strongly ordinal model. Moreover, our conclusions are completely independent of the underlying ordinal structure of preferences.

[Gershkov et al. \(2014\)](#) considers the design of expected welfare maximizing mechanism by considering cardinal utilities when agents have particular ordinal preferences. Since they consider deterministic DSIC mechanisms, these mechanisms must be ordinal. But they still consider cardinal utilities to compute expected welfare maximizing mechanism. Similarly, [Ashlagi and Shi \(2014\)](#) consider the *optimal* design of randomized BIC mechanisms for matching problems by assuming that agents have cardinal utilities.

From a methodological standpoint, our paper is related to ([Borgers and Postl, 2009](#); [Goswami et al., 2014](#); [Gershkov et al., 2014](#); [Hafalir and Miralles, 2014](#)). These papers either explicitly ([Goswami et al., 2014](#); [Gershkov et al., 2014](#)) or indirectly ([Borgers and Postl, 2009](#); [Hafalir and Miralles, 2014](#)) work in a model with one dimensional types. As a result, they can use the machinery developed in [Myerson \(1981\)](#) for one-dimensional type spaces. On the other hand, agents in our model have multidimensional types, and we use results from the multidimensional mechanism design literature - see [Vohra \(2011\)](#) for a comprehensive treatment of this topic.

## APPENDIX: OMITTED PROOFS

### Proof of Theorem 4

1  $\Rightarrow$  2. Suppose  $f$  is incentive compatible and pick  $v, v' \in V'$ . Incentive compatibility implies that

$$\begin{aligned} v \cdot \Pi^f(v) &\geq v \cdot \Pi^f(v') \\ v' \cdot \Pi^f(v') &\geq v' \cdot \Pi^f(v). \end{aligned}$$

Adding these two implies that  $(v - v') \cdot (\Pi^f(v) - \Pi^f(v')) \geq 0$ , and hence,  $f$  is monotone.

Define the function  $\mathcal{U}^f : V' \rightarrow \mathbb{R}$  as  $\mathcal{U}^f(v) := v \cdot \Pi^f(v)$ . By [Krishna and Maenner \(2001\)](#), we know that if  $f$  is incentive compatible and the type space  $V'$  is convex, then payoff equivalence holds, and, moreover, for any two types  $v, v' \in V'$  the difference  $\mathcal{U}^f(v) - \mathcal{U}^f(v')$  equals the line integral of  $\Pi^f$  from  $v'$  to  $v$  - see also [Milgrom and Segal \(2002\)](#); [Jehiel et al. \(1999\)](#)<sup>9</sup>:

$$\mathcal{U}^f(v) - \mathcal{U}^f(v') = \int_0^1 (v - v') \cdot \Pi^f(v' + t(v - v')) dt.$$

Since  $\psi^{v',v}(t) = (v - v') \cdot \Pi^f(v' + t(v - v'))$  for all  $t \in [0, 1]$ , we have  $v \cdot \Pi^f(v) - v' \cdot \Pi^f(v') = \int_0^1 \psi^{v',v}(t) dt$ .

2  $\Rightarrow$  3. Consider any  $v, v' \in V'$  and  $t, t' \in [0, 1]$  with  $t > t'$ . Now, notice that monotonicity between  $v' + t(v - v')$  and  $v' + t'(v - v')$  implies that

$$(t - t')(v - v') \cdot (\Pi^f(v' + t(v - v')) - \Pi^f(v' + t'(v - v'))) \geq 0.$$

Since  $t > t'$ , this implies that

$$(v - v') \cdot (\Pi^f(v' + t(v - v')) - \Pi^f(v' + t'(v - v'))) \geq 0. \quad (2)$$

But then,

$$\psi^{v',v}(t) - \psi^{v',v}(t') = (v - v') \cdot (\Pi^f(v' + t(v - v')) - \Pi^f(v' + t'(v - v'))) \geq 0,$$

where the last inequality followed from Inequality 2. Hence,  $\psi^{v',v}$  is a non-decreasing function.

3  $\Rightarrow$  1. Consider any  $v, v' \in V'$ . We know that

$$v \cdot \Pi^f(v) - v' \cdot \Pi^f(v') = \int_0^1 \psi^{v',v}(t) dt \leq \psi^{v',v}(1) = (v - v') \cdot \Pi^f(v),$$

where the inequality followed from non-decreasingness of  $\psi^{v',v}$ . Rearranging, we get  $v' \cdot \Pi^f(v') \geq v' \cdot \Pi^f(v)$ , i.e., agent with type  $v'$  cannot manipulate by announcing  $v$ . This shows that  $f$  is incentive compatible.

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<sup>9</sup> [Krishna and Maenner \(2001\)](#) showed a stronger result that the difference in payoffs at  $v$  and  $v'$  equals path integral of  $\Pi^f$  from  $v'$  to  $v$  along *any* smooth path. The weaker version that we use follows from facts in convex analysis on one dimensional convex functions ([Royden, 1968](#); [Rockafellar, 1970](#)). Essentially, every convex function is *absolutely continuous* and every absolute function on an interval can be written as the definite integral of its subgradient map. Now, for any  $v, v' \in V^P$ , the restriction of  $\mathcal{U}^f$  to the line segment joining  $v$  and  $v'$  is a one dimensional convex function. Thus, we get the desired result.

## Proof of Claim 1

Fix a  $P \in \mathcal{D}$  and consider a sequence of types  $\{v_k\}_{k \in \mathbb{N}}$  in  $V^P$  such that it converges to  $\mathbf{0}$ . Since  $\{v_k\}_{k \in \mathbb{N}}$  is convergent in the closure of  $V^P$ , it must be a Cauchy sequence. We argue that since  $\Pi_P^f$  is uniformly continuous, the sequence  $\{\Pi_P^f(v_k)\}_{k \in \mathbb{N}}$  is a Cauchy sequence. To see this, for every  $\delta > 0$ , since  $\{v_k\}_{k \in \mathbb{N}}$  is Cauchy sequence, there exists a number  $J$  such that for all  $j, j' > J$  we have  $\|v_j - v_{j'}\| < \delta$ . But by uniform continuity, for every  $\epsilon > 0$ , there exists a number  $\delta > 0$  such that if  $\|v_j - v_{j'}\| < \delta$  then  $\|\Pi_P^f(v_j) - \Pi_P^f(v_{j'})\| < \epsilon$ . This shows that the sequence  $\{\Pi_P^f(v_k)\}_{k \in \mathbb{N}}$  is a Cauchy sequence.

As a consequence,  $\{\Pi_P^f(v_k)\}_{k \in \mathbb{N}}$  must converge. Denote this limit point as  $L_1 \in [0, 1]^{|A|}$ . Similarly, pick another sequence of types  $\{v'_k\}_{k \in \mathbb{N}}$  such that it converges to  $\mathbf{0}$ . By the same argument,  $\{\Pi_P^f(v'_k)\}_{k \in \mathbb{N}}$  must also converge to  $L_2 \in [0, 1]^{|A|}$ . We will show that  $L_1 = L_2$ .

To do this, pick an arbitrary  $\epsilon > 0$ . Now, using the definition of convergence, since  $\{\Pi_P^f(v_k)\}_{k \in \mathbb{N}}$  converges to  $L_1$ , there must exist a number  $n_1$  such that for all  $k > n_1$ , we have

$$\|\Pi_P^f(v_k) - L_1\| < \frac{\epsilon}{3}. \quad (3)$$

Similarly, since  $\{\Pi_P^f(v'_k)\}_{k \in \mathbb{N}}$  converges to  $L_2$ , there must exist a number  $n_2$  such that for all  $k > n_2$ , we have

$$\|\Pi_P^f(v'_k) - L_2\| < \frac{\epsilon}{3}. \quad (4)$$

By uniform continuity of  $\Pi_P^f$ , we get that there exists  $\delta > 0$  such that for all  $v, v' \in V^P$  with  $\|v - v'\| < \delta$ , we have

$$\|\Pi_P^f(v) - \Pi_P^f(v')\| < \frac{\epsilon}{3}. \quad (5)$$

Since both the sequences  $\{v_k\}_{k \in \mathbb{N}}$  and  $\{v'_k\}_{k \in \mathbb{N}}$  are converging to  $\mathbf{0}$ , there must exist a number  $n_3$  such that for all  $k > n_3$  such that  $\|v_k\| < \frac{\delta}{2}$  and  $\|v'_k\| < \frac{\delta}{2}$ , and hence,  $\|v_k - v'_k\| < \delta$ . By Inequality 5, we get for all  $k > n_3$ ,

$$\|\Pi_P^f(v_k) - \Pi_P^f(v'_k)\| < \frac{\epsilon}{3}. \quad (6)$$

Now, pick a number  $K > \max(n_1, n_2, n_3)$  and note that

$$\|L_1 - L_2\| \leq \|L_1 - \Pi_P^f(v_K)\| + \|\Pi_P^f(v_K) - \Pi_P^f(v'_K)\| + \|\Pi_P^f(v'_K) - L_2\| < \epsilon,$$

where the first inequality followed from the Euclidean norm property and the second one followed from Inequalities 3, 4, and 6. Since  $\epsilon$  can be chosen arbitrarily small, we conclude that  $L_1 = L_2$ .

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