Effort-Maximizing Contests

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Abstract

We study a contest environment with a large number of players and prizes that accommodates complete and incomplete information, and heterogeneity among players and prizes. We characterize the effort-maximizing prize structure when players may differ in their marginal valuations for prizes and when the valuation may differ from the designer's cost of providing the prizes. We also provide such a characterization when players' cost of effort differs from the designer's benefit from the effort, as in Moldovanu and Sela (2001).

Contest design with a discrete number of agents and prizes has proven difficult, because even for a given set of prizes: (a) In the models which have been solved in the existing literature, equilibria have complicated structure; (b) In some other settings studied in the literature, the authors were able to provide only an algorithm for deriving equilibria; (c) In some relevant settings, there is no existing characterization of equilibria. In addition, contests can have multiple equilibria, so it is not obvious whether optimal means for the best equilibrium, the worst, or something else.

Because, or perhaps despite of these difficulties, Moldovanu and Sela (2001) obtained some interesting but only partial characterization of the optimal prize structure in discrete contests. We avoid these difficulties by studying the limits of equilibria of discrete contests as the number of players and prizes grow large. This analysis is

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possible due to the methods developed in Olszewski and Siegel (2014). We characterize the optimal prize structure in large (limit) contests. We confirm Moldovanu and Sela's results in our setting, and establish some additional features of the optimal prize structure.

This is an incomplete paper. In particular, it contains no introduction, only an extended abstract prepared for the conference submission. However, the draft contains all results I would present. Alternatively, I can present a closely related and completed paper, entitled "Large Contests," which can be found at http://faculty.wcas.northwestern.edu/~wol737/Cons.pdf

1 Model

1.1 Heterogenous contests

The purpose of this paper is to study contest design in a setting in which players and prizes are heterogenous. In our model of such a contest, a player is characterized by a type $x \in X = [0, 1]$, and a prize is characterized by a number $y \in Y = [0, 1]$. Prize 0 is "no prize." Prize 1 corresponds to the maximal possible prize, which is fixed exogenously. Section 3.6 discusses how changing the maximal possible prize affects our results.

The utility of a type x player from bidding $t \geq 0$ and obtaining prize y is

$$U(x, y, t) = xh(y) - t, \tag{1}$$

where $h(0) = 0$ and h is continuously differentiable and strictly increasing.[1]

In a contest, n players compete for n known prizes $y_1^n \leq y_2^n \leq ... \leq y_n^n$ (some of which may be 0, i.e., no prize). Player i's privately-known type x_i^n is distributed according to a *cdf* F_i^n, and these distributions are commonly known and independent across players.[2] In the special case of complete information, each *cdf* corresponds to a Dirac measure. Since we

[1] In Section 4.5, we consider (1) with $c(t)$ in place of t, where $c(0) = 0$ and c is continuously differentiable and strictly increasing, as in M&S.

[2] All probability measures are defined on the σ-algebra of Borel sets.

study some limits of sequences of contests when n diverges to infinity, we refer to a contest with n players and n prizes as the "n-th contest." In the contest, each player chooses a bid, the player with the highest bid obtains the highest prize, the player with the second-highest bid obtains the second-highest prize, and so on. Ties are resolved by a fair lottery. Every contest has at least one mixed-strategy Bayesian Nash equilibrium.[3]

2 Mechanism-design approach to studying contests

The optimal design of heterogeneous contests of the kind described in Section 1.1 is difficult or impossible, because no method currently exists to characterize their equilibria for most type and prize distributions. And even in the few cases for which a characterization exists, the equilibria have a complicated form, or can be derived only by means of algorithms. Therefore, we will use the mechanism-design approach to studying the equilibria of large-contest, developed in Olszewski and Siegel (2014). We now describe this approach.

2.1 Limit distributions

Let $F^n = \left(\sum_{i=1}^n F_i^n\right)/n$, so $F^n(x)$ approximates the expected percentile ranking of type x in the n-th contest given the vector of players' types. We denote by G^n the empirical prize distribution, which assigns a mass of $1/n$ to each prize y_j^n (recall that there is no uncertainty about the prizes). We assume that F^n converges in weak*-topology to a distribution F that has a continuous, strictly positive density f, and G^n converges to some (not necessarily continuous) distribution G.[4]

The convergence of F^n and G^n to limit distributions F and G accommodates as special, extreme cases complete-information contests with asymmetric players in which for some distributions F and G, player i's type is $x_i^n = F^{-1}(i/n)$ and prize j is $y_j^n = G^{-1}(j/n)$, where

$$G^{-1}(r) = \inf\{z : G(z) \geq r\}.$$

[3]This follows from a slight adaptation of the proof of Corollary 1 in Siegel (2009) when each player's set of possible types is finite, and from Corollary 5.2 in Reny (1999) for general distributions F_i^n.

[4]Convergence in weak*-topology can be defined as convergence of *cdf*s at points at which the limit *cdf* is continuous.

One example is contests with identical prizes and players who differ in their valuations for a prize. For this, consider $h(y) = y$, F uniform, and G that has $G(y) = 1 - p$ for all $y \in [0, 1)$ and $G(1) = 1$, where $p \in (0, 1)$ is the limit ratio of the number of prizes to the number of players. Then $x_i^n = i/n$ and $y_j^n = 0$ if $j/n \leq 1 - p$ and $y_j^n = 1$ if $j/n > 1 - p$. The n-th contest is an all-pay auction with n players and $m \equiv \ulcorner pn \urcorner$ identical (non-zero) prizes, and the value of a prize to player i is i/n. These contests were studied by C&R, who considered competitions for promotions, rent seeking, and rationing by waiting in line.

Another example is contests with heterogeneous prizes and players who differ in their constant marginal valuation for a prize. For this, consider $h(y) = y$ and F and G uniform. Then $x_i^n = i/n$ and $y_j^n = j/n$. The n-th contest is an all-pay auction with n players and n prizes, and the value of prize j to player i is ij/n^2. These contests were studied by B&L, who considered hospitals that have a common ranking for residents and compete for them by offering identity-independent wages. Hospitals are players, their posted wages are bids, and residents are prizes.[5]

Many other complete-information contests with asymmetric players can be accommodated, including contests for which no equilibrium characterization exists. One example is contests with a combination of heterogeneous and identical prizes. Such contests are modeled by a limit prize distribution G in which there is at least one atom at a positive prize that does not include all of the mass allocated to positive prizes.

Another special, extreme case of the convergence of F^n and G^n is incomplete-information contests with ex-ante symmetric players that have the same *iid* type distributions $F_i^n = F$. This case includes the setting of M&S with linear costs. Beyond these extreme cases, our setting obviously accommodates numerous incomplete-information contests with ex-ante asymmetric players. No equilibrium characterization exists for such contests.

[5] Xiao (2013) presents another model with complete information and heterogenous prizes, in which players have increasing marginal utility for a prize. He considers quadratic and exponential specifications, which are obtained in our model by setting $h(y) = y^2$ and $h(y) = e^y$ and F and G uniform.

2.2 Assortative allocation and transfers

The *assortative allocation* assigns to each type x prize $y^A(x) = G^{-1}(F(x))$. It is well-known that the unique, incentive-compatible mechanism that implements the assortative allocation and gives type $x = 0$ a utility of 0 specifies for every type x bid

$$t^A(x) = xh\left(y^A(x)\right) - \int_0^x h\left(y^A(z)\right) dz. \tag{2}$$

For example, in the setting corresponding to C&R the assortative allocation assigns a prize to each type higher than $1-p$, and the associated bids are $t^A(x) = 0$ for $x \leq 1-p$ and $t^A(x) = 1-p$ for $x > 1-p$. In the setting corresponding to B&L, the assortative allocation assigns prize x to type x, and the associated bids are $t^A(x) = x^2/2$.[6]

By integrating by parts, we obtain the following expression for the aggregate bids in the mechanism that implements the assortative allocation:

$$\int_0^1 t^A(x) f(x) dx = \int_0^1 h\left(y^A(x)\right)\left(x - \frac{1-F(x)}{f(x)}\right) f(x) dx. \tag{3}$$

For the remainder of our analysis, we make the following assumption, which is standard in the mechanism design literature:[7]

Assumption $x - (1-F(x))/f(x)$ strictly increases in x.

For many of our results it is convenient to rewrite (3) using the change of variable $z = F(x)$ to obtain

$$\int_0^1 h\left(G^{-1}(z)\right)\left(F^{-1}(z) - \frac{1-z}{f\left(F^{-1}(z)\right)}\right) dz = \int_0^1 h\left(G^{-1}(z)\right) k(z) dz, \tag{4}$$

where $k(z) = F^{-1}(z) - (1-z)/f\left(F^{-1}(z)\right)$.[8] Thus, by our assumption, $k(z)$ strictly increases in z.

[6] In the setting corresponding to Xiao (2013), the assortative allocation assigns prize x to type x, and the associated bids are $t(x) = xh(x) - \int_0^x h(y) dy$ for $h(y) = y^2$ or $h(y) = e^y$.

[7] The assumption is implied, for example, by a monotone hazard rate.

[8] Even though G^{-1} may be discontinuous, it is monotonic, so the change of variable applies.

2.3 The approximation result

Corollary 2 in Olszewski and Siegel (2014), which we state as Theorem **??** below, shows that the equilibria of a contest with many players and prizes can be approximated by the unique mechanism that implements the assortative allocation.

Theorem 1 *For any $\varepsilon > 0$ there is an N such that for all $n \geq N$, in any equilibrium of the n-th contest each of a fraction of at least $1 - \varepsilon$ of the players i obtains with probability at least $1 - \varepsilon$ a prize that differs by at most ε from $y^A(x_i^n)$, and bids with probability at least $1 - \varepsilon$ within ε of $t^A(x_i^n)$.*

Theorem 1 implies that the aggregate bids in large contests can be approximated by (3). More precisely, we refer to the aggregate bids in an equilibrium of the n-th contest divided by n as the *average bid*. We then have the following corollary of Theorem **1.**

Corollary 1 *For any $\varepsilon > 0$ there is an N such that for all $n \geq N$, in any equilibrium of the n-th contest the average bid is within ε of (3).*

To gain some intuition for why (3) approximates the aggregate bids in large contests, observe that (3) coincides with the expected revenue from a bidder in a single-object independent private-value auction if we replace $h\left(y^A(x)\right)$ with the probability that the bidder wins the object when his type is x (Myerson (1981)). In the auction setting, increasing the probability that type x obtains the object along with the price the type is charged allows the auctioneer to capture the entire increase in surplus for this type, but requires a decrease in the price that higher types are charged to maintain incentive compatibility. In a large contest, increasing the prize that type x obtains also allows the auctioneer to capture the entire increase in surplus for this type, because the higher prize increases the competition with slightly lower types until the gain from the higher prize is exhausted, but decreases the competition and bids of higher types for their prizes, since the prize of type x becomes more attractive to them.

3 Optimal Contest Design

We now turn to analyzing the prize structures in large contests that maximizes the aggregate bids. Proposition 1 below shows that we can focus on identifying the prize distributions that maximize (3). To formulate this result, consider a sequence of contests in which distributions F_i^n of players' types are given exogenously and their averages F^n converge to a distribution F with a continuous, strictly positive density f.

The corresponding empirical prize distributions $G_{\max}^n$ are not exogenously given, but maximize the aggregate bids. That is, $G_{\max}^n$ describes a set of n prizes that lead to some equilibrium with maximal aggregate bids (possibly subject to the budget constraint that the average prize does not exceed a certain value C).[9]

In addition, denote by $\mathcal{M}$ the set of prize distributions that maximize (3) (possibly subject to the budget constraint that the expected prize does not exceed C). An upper hemi-continuity argument, given in the Appendix, shows that $\mathcal{M}$ is not empty. Denote by M the corresponding maximal value of (3). Finally, consider any metrization of the weak*-topology on the space of prize distributions.

Proposition 1 *1. For any $\varepsilon > 0$, there is an N such that for every $n \geq N$, $G_{\max}^n$ is within ε of some distribution in $\mathcal{M}$. In particular, if there is a unique prize distribution $G_{\max}$ that maximizes (3), then $G_{\max}^n$ converges to $G_{\max}$ in weak*-topology. 2. $M_{\max}^n/n$ converges to M. 3. For any $\varepsilon > 0$ and any G in $\mathcal{M}$, there are an N and a $\delta > 0$ such that for any $n \geq N$ and any empirical prize distribution G^n of n prizes that is within δ of G, the average bid in any equilibrium of the n-th contest with empirical prize distribution G^n is within ε of $M_{\max}^n/n$.*

Part 1 of Proposition 1 shows that the optimal empirical prize distributions in large contests are approximated by the prize distributions that maximize (3). Part 2 shows that

[9] That a maximizing set of prizes exists can be shown by a straightforward upper hemi-continuity argument of the kind used, for example, to prove Corollary 2 in Siegel (2009). We note, however, that our results do not depend on the existence of such a maximizing set of prizes. For example, none of the analysis changes if $G_{\max}^n$ is instead chosen to corresponds to a set of n prizes that lead to some equilibrium with aggregate equilibrium bids that are within $1/n$ of the supremum of the aggregate equilibrium bids over all sets of n prizes (possibly subject to the budget constraint that the average prize does not exceed a certain value C) and all equilibria for any given set of prizes.

the maximal aggregate equilibrium bids are approximated by the maximal value of (3). Part 3 shows that any empirical prize distribution that is close to a prize distribution that maximize (3) generates aggregate equilibrium bids (in any equilibrium) that are close to maximal. For example, given a prize distribution G that maximizes (3), the set of n prizes defined by $y_j^n = G^{-1}(j/n)$ for $j = 1, ..., n$ generates, for large contests, aggregate equilibrium bids that are close to maximal.

Therefore, to determine the optimal prize structures in large contests and the corresponding maximal aggregate equilibrium bids, it suffices to characterize the set of distributions that maximize (3) and the corresponding maximal value of (3).

3.1 Unrestricted budget

Suppose first that the prize budget is unrestricted. Denote by $x^* \in (0,1)$ the unique type that satisfies

$$x^* - \frac{(1 - F(x^*))}{f(x^*)} = 0;$$

such a type exists because $x - (1 - F(x))/f(x)$ strictly increases in x and f is continuous and strictly positive on $[0,1]$. For types $x < x^*$, the value of the integrand in (3) is negative, and for $x > x^*$, the value is positive. Let $z^* = F(x^*) \in (0,1)$, so $k(z^*) = 0$. Then, optimizing the integrand in (4) leads to $G^{-1}(z) = 0$ if $z \leq z^*$ and $G^{-1}(z) = 1$ if $z > z^*$. This G^{-1} is left-continuous and monotonic, so G is a prize distribution and is therefore optimal. We thus obtain the following result.

Proposition 2 *If the prize budget is unrestricted, then for any function h the optimal prize distribution assigns mass $1 - F(x^*) \in (0,1)$ to the highest possible prize and mass $F(x^*)$ to prize 0.*

Proposition 2 shows that an all-pay auction with identical prizes, as studied by C&R, is optimal when the budget is unrestricted.

3.2 Restricted budget

In many applications the prize budget is limited. We model this by introducing the budget constraint

$$\int_0^1 ydG(y) \leq C.$$

The parameter $C > 0$ should be interpreted as the budget per competitor, denominated in units of the highest possible prize. Similarly, prizes are denominated in units of the highest possible prize, that is, prize y costs y. Thus, the expected prize cannot exceed C.

The following result is an immediate implication of Proposition 2.

Proposition 3 *If $C \geq 1 - F(x^*)$, then the optimal prize distribution coincides with the one in the unrestricted budget case.*

Proposition 3 shows that when the prize budget is large some of it is optimally left unused. This is analogous to a monopolist limiting the quantity sold.

Now, consider perhaps the most interesting case, a budget $C < 1 - F(x^*)$. To analyze this case, we first transform the budget constraint to a more convenient form. Because G is a probability measure on $[0,1]$, we have $\int_0^1 ydG(y) = \int_0^1 (1 - G(y))\, dy$ (by integrating by parts) and $\int_0^1 G(y)\, dy + \int_0^1 G^{-1}(z)\, dz = 1$ (by looking at the areas below the graphs of G and G^{-1} in the square $[0,1]^2$). Thus, the budget constraint can be rewritten as

$$\int_0^1 G^{-1}(z)\, dz \leq C. \tag{5}$$

This is a substantial simplification, because maximizing (4) subject to (5) is a calculus of variations problem with respect to variable G^{-1}.

Note that since we now consider the case $C < 1 - F(x^*)$, the budget constraint (5) holds with equality.

3.3 Conditions describing the solution

Consider an optimal G^{-1}. Because it is non-decreasing, left-continuous, and takes values in $[0,1]$, there are $z_{\min} \leq z_{\max}$ in $[0,1]$ such that $G^{-1}(z) = 0$ for $z \leq z_{\min}$, $G^{-1}(z) = 1$ for $z > z_{\max}$, and $G^{-1}(z) \in (0,1)$ for $z \in (z_{\min}, z_{\max})$.

There are two cases:

Case 1 ($z_{\min} < z_{\max}$): Then, there exists a $\lambda \geq 0$ such that $h'(G^{-1}(z))\,k(z) = \lambda$ for $z \in (z_{\min}, z_{\max}]$; in addition, $h'(0)\,k(z_{\min}) \leq \lambda$ and $h'(1)k(z_{\max}) \geq \lambda$ if $z_{\max} < 1$.

Case 2 ($z_{\min} = z_{\max}$): Then, $h'(0)\,k(z_{\min}) \leq h'(1)\,k(z_{\max})$.

A rigorous proof that G^{-1} satisfies the conditions described in the two cases is provided in the Appendix. To gain some intuition for Case 1, note that $h'(G^{-1}(z))\,k(z)$ is the derivative of the integrand of (4) with respect to $G^{-1}(z)$ for a given z. Thus, if $h'(G^{-1}(\underline{z}))\,k(\underline{z}) < h'(G^{-1}(\overline{z}))\,k(\overline{z})$ for some $\underline{z}, \overline{z} \in (z_{\min}, z_{\max})$, then an infinitesimal increase in $G^{-1}(\overline{z})$ accompanied by a simultaneous decrease in $G^{-1}(\underline{z})$ of the same infinitesimal size would raise the value of the objective function (4) without affecting the budget constraint (5). At $z = z_{\min}$ or $z > z_{\max}$, we have only inequalities, as the value of $G^{-1}$is 0 or 1, respectively, and cannot be decreased or increased. Since k is increasing and continuous, the inequality $h'(1)\,k(z) \geq \lambda$ for $z > z_{\max}$ is equivalent to $h'(1)k(z_{\max}) \geq \lambda$.[10] The intuition for Case 2 is analogous.

3.4 Convex and concave functions h

From the conditions provided in the previous section, we can easily derive the form of the optimal prize distribution for convex and concave functions h.

Proposition 4 *If $C < 1 - F(x^*)$ and h is weakly convex, then the optimal prize distribution assigns mass C to the highest possible prize and mass $1 - C$ to prize 0.*

Proof: In this case, we have $z_{\min} = z_{\max}$. Indeed, since h' is weakly increasing and k is strictly increasing, we would have that $h'(G^{-1}(z'))\,k(z') < h'(G^{-1}(z''))\,k(z'')$ for any $z' < z''$ in $(z_{\min}, z_{\max})$.

Proposition 4 shows that an all-pay auction with identical prizes remains optimal when the prize budget is small, provided that agents' marginal prize utility is non-decreasing. An

[10]Finally, $h'\left(G^{-1}(z)\right)k(z) = \lambda$ at $z = z_{\max}$ by left-continuity of G^{-1} and continuity of h' and k.

immediate implication of Proposition 4 is that increasing the budget increases the optimal quantity of prizes. This increases the resulting aggregate bids, since the optimal prize distribution is unique and differs from the optimal prize distribution associated with the smaller budget, which is also a feasible prize distribution with the larger budget.

The next result shows that with concave h the optimal prize distribution includes a range of prizes.

Proposition 5 *If $C < 1 - F(x^*)$ and h is weakly concave (but not linear), then the optimal prize distribution assigns positive mass to intermediate prizes $y \in (0, 1)$.*

Proof: In this case, we have $z_{\min} < z_{\max}$. Indeed, since $h'(0) > h'(1)$, we cannot have that $z_{\min} = z_{\max}$ and $h'(0)\,k(z_{\min}) \leq h'(1)\,k(z_{\max})$, unless $k(z_{\min}) = k(z_{\max}) \leq 0$. But $k(z_{\max}) \leq 0$ implies that $z_{\max} \leq z^*$, so $G^{-1}(z) = 1$ for $z > z_{\max}$ violates the budget constraint (5).

We now show that when h is strictly concave our constrained maximization problem has an explicit, closed-form solution. Since $h'(G^{-1}(z))\,k(z) = \lambda$ for all $z \in (z_{\min}, z_{\max}]$ and h' is decreasing, $h'(0)\,k(z) \geq \lambda$, and since k is continuous, $h'(0)\,k(z_{\min}) \geq \lambda$. Since we also have $h'(0)\,k(z_{\min}) \leq \lambda$ (because we are in Case 1), we obtain $h'(0)\,k(z_{\min}) = \lambda$. Thus,

$$z_{\min} = k^{-1}(\lambda/h'(0)). \tag{6}$$

Since $h'(G^{-1}(z_{\max}))\,k(z_{\max}) = \lambda$ and h' is decreasing, $h'(1)\,k(z_{\max}) \leq \lambda$. If $z_{\max} < 1$, then we also have $h'(1)k(z_{\max}) \geq \lambda$ (because we are in Case 1), so we obtain $h'(1)\,k(z_{\max}) = \lambda$. Thus,

$$z_{\max} = 1 \text{ or } k^{-1}(\lambda/h'(1)). \tag{7}$$

Finally,

$$G^{-1}(z) = (h')^{-1}(\lambda/k(z)) \text{ for } z \in (z_{\min}, z_{\max}], \tag{8}$$

$$G^{-1}(z) = 0 \text{ for } z \leq z_{\min}, \text{ and } G^{-1}(z) = 1 \text{ for } z > z_{\max}.$$

Thus, G^{-1} is pinned down by λ. The value of λ is determined by the binding budget constraint.

3.5 Example

To illustrate this solution, let distribution F be the uniform and let $h(y) = \sqrt{y}$.[11] Then, $k(z) = 2z - 1$, $x^* = z^* = 1/2$, $h'(0) = \infty$, $h'(1) = 1/2$, and $(h')^{-1}(r) = 1/(4r^2)$. The budget constraint is binding if $C < 1 - F(x^*) = 1/2$. By (6), $z_{\min} = 1/2$. Suppose first that $z_{\max} = 1$. By (8) and the binding version of (5), $\int_{1/2}^{1} (2z-1)^2/(4\lambda^2)dz = C$. Solving for λ, we obtain $\lambda = 1/\sqrt{24C}$. This yields $G^{-1}(z) = 6C(2z-1)^2$; in particular, $C \leq 1/6$. Thus, we have that

$$G(y) = \begin{cases} \frac{1}{2} + \sqrt{\frac{y}{24C}} & y \in [0, 6C] \\ 1 & y \in [6C, 1] \end{cases}.$$

This is a continuous distribution over an interval of positive intermediate prizes (along with a mass $1/2$ of "no prize"). The resulting aggregate bids, given by (4), are $\sqrt{6C}/6$.

Suppose now that $z_{\max} < 1$. By (7), $z_{\max} = \lambda + 1/2$. The binding version of (5) implies that $\int_{1/2}^{1/2+\lambda} (2z-1)^2/(4\lambda^2)dz + \int_{1/2+\lambda}^{1} 1dz = C$. Solving for λ, we obtain $\lambda = 3/4 - (3/2)C$. This implies that $G^{-1}(z) = 0$ for $z \in [0, 1/2]$, $G^{-1}(z) = 4(2z-1)^2/(9(1-2C)^2)$ for $z \in (1/2, (5-6C)/4]$, and $G^{-1}(z) = 1$ for $z \in ((5-6C)/4, 1]$. Since $z_{\max} = 1/2 + \lambda < 1$, we have $C > 1/6$. Thus,

$$G(y) = \begin{cases} \frac{1}{2} + \frac{3\sqrt{y}(1-2C)}{4} & y \in [0, 1) \\ 1 & y = 1 \end{cases}.$$

This is a continuous distribution over an interval of positive intermediate prizes, along with a mass $(6C-1)/4$ of the highest possible prize (and a mass $1/2$ of "no prize"). The resulting aggregate bids are $(12C(1-C)+1)/16$.

[11] Although $h'(0) = \infty$, it is straightforward to show that a slight modification of our characterization of the solution from the previous subsection still applies.

The following figure depicts these results.

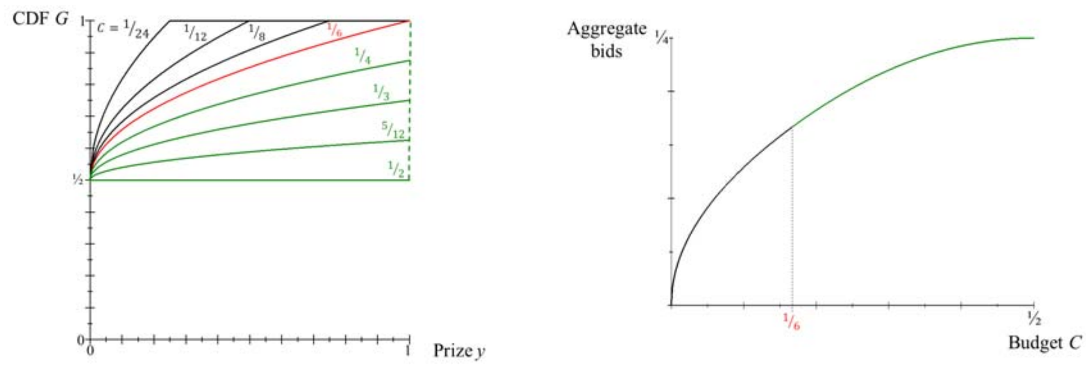

Figure 1: The optimal prize distribution as C increases from 0 to 1/2 (left), and the resulting aggregate bids (right)

To summarize the example, for any budget the optimal prize distribution has a mass 1/2 of zero prize. As C increases from 0 to 1/6, the maximal prize awarded increases from the lowest possible (0) to the highest possible (1), and the prize distribution is continuous above 0. Once C reaches 1/6, the maximal prize awarded is the highest possible prize, and as C increases from 1/6 to 1/2, the mass of the highest possible prize increases from 0 to 1/2, so the prize distribution is discontinuous at 1.

3.6 Varying the maximal prize

We have assumed that the value of the maximal possible prize is exogenously given. The bids and costs of prizes were normalized to be in units of this maximal prize, which corresponds to $y = 1$. Now suppose that the maximal possible prize is increased to $m > 1$ (in units of the original maximal prize) and suppose that the utility is still given by (1) for $y \in [0, m]$.

We can renormalize the units in which the bids and costs are measured so that $y \in [0, 1]$ and $y = 1$ corresponds to the higher maximal possible prize. Denoting the resulting utility function by U^m, we obtain

$$U^m(x, y, t) = U(x, my, mt) = xh(my) - mt.$$

Dividing by m, we obtain

$$\hat{U}^m(x, y, t) = x\hat{h}(y) - t,$$

where $\hat{h}(y) = h(my)/m$.

We can now apply the analysis to the utility function $\hat{U}^m(x, y, t)$. With an unrestricted budget, the solution is still to award the maximal prize to all types $x > x^*$. Thus, the same mass of the highest possible prize is awarded as with the original maximal possible prize. The resulting aggregate bids must be multiplied by m to be in the units of the original maximal prize, and since $m > 1$, the aggregate bids increase by a factor of m.

When the budget is restricted, renormalizing the cost of prizes to be in units of the higher maximal prize (without increasing the budget) changes the budget constraint to

$$\int_0^1 y dG(y) \leq C/m.$$

That is, the budget is decreased when denominated in units of the higher maximal prize. If $C/m \geq 1 - F(x^*)$, then the budget constraint does not bind, and the solution is as in the case of unrestricted budget.

Suppose that $C/m < 1 - F(x^*)$. If h is convex, then so is $\hat{h}$, so our analysis above shows that the uniquely optimal prize distribution awards mass C/m of the higher maximal possible prize. Thus, increasing the maximal possible prize decreases the optimal mass of prizes and increases their value. As m grows large, the optimal mass of prizes shrinks to 0. This corresponds, in the limit, to "awarding the entire budget as a single prize." Increasing the maximal possible prize also increases the maximal aggregate bids. This follows because the optimal prize structure is uniquely optimal, and therefore leads to higher aggregate bids than any other feasible prize structure; and one feasible prize structure is awarding a mass of the original maximal prize.

If h is concave, then so is $\hat{h}$, so our earlier analysis can be used to characterize the solution. If this solution differs from the one with the original maximal possible prize, then the aggregate bids increase (because the prize structure that was optimal with the original maximal possible prize is still feasible). In the example, the aggregate bids increase for $m > 1$ if and only if $C > 1/6$. Indeed, repeating the analysis of the example with budget constraint C/m and $\hat{h}(y) = \sqrt{my}/m$ shows that if $C/m \leq 1/6$, then the optimal prize distribution is

$$G(y) = \begin{cases} \frac{1}{2} + \sqrt{\frac{my}{24C}} & y \in [0, 6C/m] \\ 1 & y \in [6C/m, 1] \end{cases}, \tag{9}$$

where y is denominated in units of the higher maximal possible prize. This coincides with the optimal prize distribution with the original maximal possible prize if $C \leq 1/6$. Thus, if $C \leq 1/6$, then $C/m \leq 1/6$, so the optimal prize distribution does not change as a result of an increase in the maximal possible prize. In contrast, if $C > 1/6$, then regardless of whether $C/m \leq 1/6$ or $C/m > 1/6$ the optimal prize distribution with the higher maximal possible prize differs from the optimal prize distribution with the original maximal possible prize (for example, in the former case, the optimal prize distribution with the higher maximal possible prize does not have an atom at any positive prize). In addition, for any m such that $C/m < 1/6$, further increasing m does not change the optimal prize distribution (when denominated in some fixed units). Thus, the optimal prize distribution with an unrestricted maximal prize coincides with the optimal prize distribution for any $m > 6C$, which is given by (9).

4 Appendix

4.1 Proof of Corollary 1

Theorem 1 shows that for large n, in any equilibrium of the n-th contest the average bid is within $\varepsilon/2$ of

$$\frac{\sum_{i=1}^{n} \int_0^1 t^A(x)\, dF_i^n(x)}{n} = \int_0^1 t^A(x)\, dF^n(x),$$

where the equality follows from the definition of F^n. In addition,

$$\int_0^1 t^A(x)\, dF^n(x) \rightarrow_n \int_0^1 t^A(x)\, dF(x),$$

which completes the proof. The convergence follows from the fact that t^A is monotonic and the assumption that F is continuous, because $\int gdF^n \rightarrow_n \int gdF$ for any bounded and measurable function g for which distribution F assigns measure 0 to the set of points at which function g is discontinuous. (This fact is established as the first claim of the proof of Theorem 25.8 in Billingsley (1995).)

4.2 Proof that $\mathcal{M} \neq \varnothing$

Let $(G^n)_{n=1}^{\infty}$ be a sequence on which (4) converges to its supremum (and which satisfies, if necessary, the budget constraint that the expected prize does not exceed C). By passing to a convergent subsequence (in the weak*-topology) if necessary, assume that G^n converges to some G. We will show below that $(G^n)^{-1}$ converges almost surely to G^{-1}, and since function h is continuous, the value of (4) with $(G^n)^{-1}$ instead of G^{-1} converges to the value of (4). Similarly, if every $(G^n)^{-1}$ satisfies the budget constraint (5), so does G^{-1}.

Thus, it suffices to show that $(G^n)^{-1}$ converges to G^{-1}, except perhaps on the (at most) countable set $R = \{r \in [0,1] : \text{there exist } y' < y'' \text{ such that } G(y) = r \text{ for } y \in (y', y'')\}$.

Suppose first that for some $r \in [0,1]$ and $\delta > 0$ we have that $(G^n)^{-1}(r) \leq G^{-1}(r) - \delta$ for arbitrarily large n. Passing to a subsequence if necessary, assume that the inequality holds for all n, and that $(G^n)^{-1}(r)$ converges to some $y \leq G^{-1}(r) - \delta$. Then, there exists a prize z such that $y < z < G^{-1}(r)$ and G is continuous at z. We cannot have that $G(z) = r$, since this would imply that $G^{-1}(r) \leq z$. Thus, $G(z) < r$. Since $G^n(z)$ converges to $G(z)$, as G is continuous at z, we have that $G^n(z) < r$ for large enough n. This yields $z \leq (G^n)^{-1}(r)$, contradicting the assumption that $(G^n)^{-1}(r)$ converges to $y < z$.

Suppose now that for some $r \in [0,1] - R$ and $\delta > 0$ we have that $(G^n)^{-1}(r) \geq G^{-1}(r) + \delta$ for arbitrarily large n. Passing to a subsequence if necessary, assume that the inequality holds for all n, and that $(G^n)^{-1}(r)$ converges to some $y \geq G^{-1}(r) + \delta$. Then, there exists a prize z such that $G^{-1}(r) < z < y$ and G is continuous at z. We have that $r < G(z)$, as $r \notin R$. Since $G^n(z)$ converges to $G(z)$, as G is continuous at z, we have that $r \leq G^n(z)$ for large enough n. This yields $(G^n)^{-1}(r) \leq z$, contradicting the assumption that $(G^n)^{-1}(r)$ converges to $y > z$.

4.3 Proof of Proposition 1

Since every sequence of distributions has a converging subsequence in weak*-topology, suppose without loss of generality that $G^n_{\max}$ converges to some distribution G. Denote the value of (3) under distribution G by V. If Part 1 is false, then $G \notin M$, so $V < M$.

Consider a distribution $G_{\max} \in \mathcal{M}$, and for every n consider an empirical distribution G^n

of a set of n prizes, such that G^n converges to $G_{\max}$ in weak*-topology. For example, such a set of n prizes is defined by $y_j^n = G_{\max}^{-1}(j/n)$ for $j = 1, ..., n$.

Corollary **1** shows that for large n the average bid in any equilibrium of the n-th contest with empirical prize distribution G^n exceeds $2(V+M)/3$. On the other hand, Corollary **1** also shows that for large n the average bid in any equilibrium of the n-th contest with empirical prize distribution $G_{\max}^n$ falls below $(V+M)/3$. This contradicts the definition of $G_{\max}^n$ for large n.

For Part 2, Corollary **1** applied to the sequence G^n defined above implies that $\liminf M_{\max}^n/n \geq M$. If $\limsup M_{\max}^n/n > M$, then there is a corresponding subsequence of $G_{\max}^n$. A converging subsequence of this subsequence has a limit G. For this G, the value of (3) is by Corollary **1**strictly larger than M, a contradiction.

Part 3 follows from part 2 and the fact that Corollary **1** shows that the average bid in any equilibrium of the n-th contest with empirical prize distribution G^n converges to M.

4.4 A proof for the conditions in Cases 1 and 2

We will show that in Case 1 the condition $h'(G^{-1}(z))\,k(z) = h'(G^{-1}(z'))\,k(z')$ holds for all $z, z' \in (z_{\min}, z_{\max})$. For this, we first approximate G^{-1} by a sequence of inverse distribution functions $((G^n)^{-1})_{n=1}^{\infty}$ that satisfy the budget constraint and whose value of (4) converges to that for G^{-1}. We then show that if the condition fails there exists a sequence of inverse distribution functions $((H^n)^{-1})_{n=1}^{\infty}$ that satisfy the budget constraint such that for large n the value of (4) for $(H^n)^{-1}$ exceeds that for $(G^n)^{-1}$ by a positive constant independent of n, and therefore improves upon G^{-1}. The second condition in Case 1 and the condition in Case 2 are obtained by analogous arguments.

To define $(G^n)^{-1}$, partition interval $[0,1]$ into intervals of size $1/2^n$, and set the value of $(G^n)^{-1}$ on interval $(j/2^n, (j+1)/2^n]$ to be constant and equal to the highest number in the set $\{0, 1/2^n, 2/2^n, ..., (2^n-1)/2^n, 1\}$ that is no higher than $G^{-1}(j/2^n)$. By left-continuity of G^{-1}, $(G^n)^{-1}$ converges pointwise to G^{-1}**,** so the value of (4) for $(G^n)^{-1}$ converges to that for G^{-1}.

Suppose that $h'(G^{-1}(\underline{z}))\,k(\underline{z}) < h'(G^{-1}(\overline{z}))\,k(\overline{z})$ for some $\underline{z}, \overline{z} \in (z_{\min}, z_{\max})$. By left-continuity of G^{-1}, and continuity of h' and k, the previous inequality also holds for points slightly smaller than $\underline{z}$ and $\overline{z}$. Thus, there are $\delta > 0$, N, and intervals $(j/2^N, (j+1)/2^N]$ and $(l/2^N, (l+1)/2^N]$, such that for every $n \geq N$ we have $h'\left((G^n)^{-1}(z)\right)k(z) - h'\left((G^n)^{-1}(z')\right)k(z') > \delta$ for any $z \in (j/2^N, (l+1)/2^N]$ and $z' \in (l/2^N, (j+1)/2^N]$.

Denote the infimum of the values $h'\left((G^n)^{-1}(z)\right)k(z)$ for $n \geq N$ and z in the former interval by I, and the supremum of the values $h'\left((G^n)^{-1}(z)\right)k(z)$ for $n \geq N$ and z in the latter interval by S. Now, define functions $(\widetilde{H}^n)^{-1}$ by increasing the value of $(G^n)^{-1}$ on $(j/2^N, (j+1)/2^N]$ by ε, and decreasing the value of $(G^n)^{-1}$ on $(l/2^N, (l+1)/2^N]$ by ε, so the budget constraint is maintained. For sufficiently small $\varepsilon > 0$, the former change increases (4) at least by $\left(\varepsilon/2^N\right)(I - \delta/3)$, and the latter change decreases (4) at most by $\left(\varepsilon/2^N\right)(S + \delta/3)$. This increases the value of (4) by at least $\delta\varepsilon/2^N$ (for all $n \geq N$).

If functions $(\widetilde{H}^n)^{-1}$ are monotonic, they are inverse distribution functions, so it suffices to set $(H^n)^{-1} = (\widetilde{H}^n)^{-1}$. Otherwise, define $(H^n)^{-1}$ by setting its value on interval $(0, 1/2^n]$ to the lowest value of $(\widetilde{H}^n)^{-1}$ over intervals $(0, 1/2^n], (1/2^n, 2/2^n], ..., ((2^n - 1)/2^n, 1]$, setting its value on interval $(1/2^n, 2/2^n]$ to the second lowest value of $(\widetilde{H}^n)^{-1}$, etc. The value of (4) is higher for $(H^n)^{-1}$ than for $(\widetilde{H}^n)^{-1}$ because k is an increasing function.

4.5 Connection to M&S

We now consider utility function $U(x, y, t) = xy - c(t)$, where $c(0) = 0$ and c is continuously differentiable and strictly increasing. The discrete contests of M&S correspond to this utility function. To simplify the analysis, we assume that distribution F is uniform.[12]

The effort-maximizing prize structure is, in large contests, approximated by the prize distribution that solves the following problem:

$$\max_{G^{-1}} \left\{ \int_0^1 c^{-1}\left(zG^{-1}(z) - \int_0^z G^{-1}(r)\,dr \right) dz \right\}$$

$$\text{subject to } \int_0^1 G^{-1}(r)\,dr \leq C.$$

Indeed, the cost of effort $c(t(x))$ exerted by type x is determined by equation (**??**), which previously determined the bid $t^A(x)$.

[12] Our analysis can be extended to general F and h (instead of $h(y) = y$) without any conceptual difficulty, but such an extension requires more involved notation and calculations.

We first transform the objective function to a more convenient form. By looking at the areas below the graphs of G and G^{-1} in the rectangle $[0, G^{-1}(z)] \times [0, z]$, we have that $\int_0^{G^{-1}(z)} G(y)dy + \int_0^z G^{-1}(r)\,dr = zG^{-1}(z)$. Thus, the objective function can be rewritten as

$$\int_0^1 c^{-1}\left(\int_0^{G^{-1}(z)} G(y)dy\right) dz. \tag{10}$$

Consider an optimal G^{-1}. There exist $z_{\min} \leq z_{\max}$ in $[0,1]$ such that $G^{-1}(z) = 0$ for $z \leq z_{\min}$, $G^{-1}(z) = 1$ for $z > z_{\max}$, and $G^{-1}(z) \in (0,1)$ for $z \in (z_{\min}, z_{\max})$.

There are two cases:

Case 1 ($z_{\min} < z_{\max}$): Then, there exists a $\lambda \geq 0$ such that

$$(c^{-1})'(l(z))z - \int_z^1 (c^{-1})'(l(r))dr = \lambda, \tag{11}$$

where $l(z) = \int_0^{G^{-1}(z)} G(y)dy$, for $z \in (z_{\min}, z_{\max}]$; in addition,

$$(c^{-1})'(0)z_{\min} - \int_{z_{\min}}^1 (c^{-1})'(l(r))dr \leq \lambda \text{ and } (c^{-1})'(l(1))(2z_{\max} - 1) \geq \lambda.$$

Case 2 ($z_{\min} = z_{\max}$): Then,

$$(c^{-1})'(0) \leq (c^{-1})'(l(1)). \tag{12}$$

The proof that G^{-1} satisfies the conditions described in the two cases is analogous to that for the conditions in Section 3.3. The argument is, however, more involved, because the objective function (10) depends on G as well as on G^{-1}. For the argument, it is convenient to extend the functional $l(z)$ to functions G^{-1} which are not monotonic. We define $l(z)$ by adding with the plus sign the area above the graph of G^{-1} between 0 and z and and below the line $y = G^{-1}(z)$, and with the minus sign the area below the graph of G^{-1} between 0 and z and and above the line $y = G^{-1}(z)$. (This is illustrated in Figure 2, where $l(z)$ is equal to the sum of the shaded areas taken with the signs marked on them.)

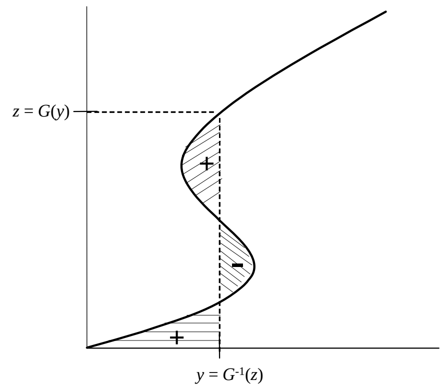

Figure 2: Definition of $l(z)$

To derive the first condition in Case 1, consider some inverse distribution function G^{-1} that takes values only in the set $\{0, 1/2^n, 2/2^n, ..., (2^n - 1)/2^n, 1\}$, and is constant on each interval $(0, 1/2^n], (1/2^n, 2/2^n], ..., ((2^n - 1)/2^n, 1]$. Suppose that we increase the value of G^{-1} on an interval $(l/2^n, (l+1)/2^n]$ by $\varepsilon > 0$. (That is, we move the graph of G^{-1} in Figure 3 to the right, by the shaded square.) This change does not affect the integrand in (10) on intervals $(k/2^n, (k+1)/2^n]$ for $k < l$. It increases $\int_0^{G^{-1}(z)} G(y)dy$ for $z \in (l/2^n, (l+1)/2^n]$ by $\varepsilon(l/2^n)$ (the darkened rectangle in Figure 3), so to a first-order approximation it increases the integrand in (10) on $(l/2^n, (l+1)/2^n]$ by $(c^{-1})'(l(z))\varepsilon(l/2^n)$. For any $k > l$, it decreases $\int_0^{G^{-1}(z)} G(y)dy$ by $\varepsilon(1/2^n)$ (the shaded square in Figure 3) on $(k/2^n, (k+1)/2^n]$, so to a first-order approximation it decreases the integrand in (10) on $(k/2^n, (k+1)/2^n]$ (for all $k > l$) by $(c^{-1})'(l(z))\varepsilon(1/2^n)$. Letting $z = l/2^n$, we have that in total, (10) increases approximately by

$$\varepsilon(1/2^n)\left[(c^{-1})'(l(z))z - \int_z^1 (c^{-1})'(l(r))dr\right].$$

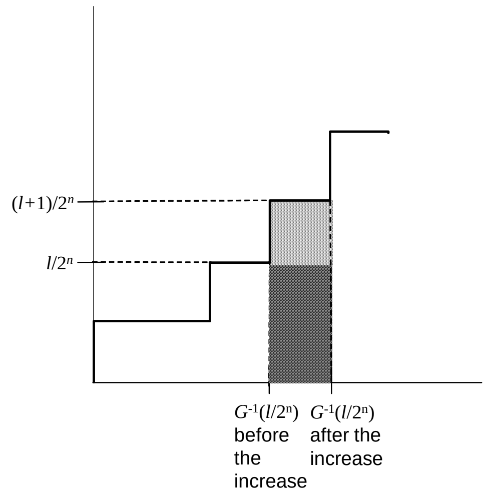

Figure 3: Increasing G^{-1}

Thus, if the first condition in Case 1 is violated for an optimal G^{-1}, we could construct functions $(G^n)^{-1}$ that converge to G^{-1} and functions $(\widetilde{H}^n)^{-1}$, as in Section 4.4. If functions $(\widetilde{H}^n)^{-1}$ are monotonic, we would obtain a contradiction to the optimality of G^{-1}.

If a $(\widetilde{H}^n)^{-1}$ is not monotonic, then there is a monotonic $(H^n)^{-1}$ whose value of (10) is higher than that for $(\widetilde{H}^n)^{-1}$. Indeed, consider two adjacent intervals $(k/2^n, (k+1)/2^n]$ and $(l/2^n, (l+1)/2^n]$ (that is, $k+1 = l$) such that $(\widetilde{H}^n)^{-1}(z) = U$ on $(k/2^n, (k+1)/2^n]$ and $(\widetilde{H}^n)^{-1}(z) = D$ on $(l/2^n, (l+1)/2^n]$, where $D < U$. By changing the value of $(\widetilde{H}^n)^{-1}$ on $(k/2^n, (k+1)/2^n]$ to D, and changing the value of $(\widetilde{H}^n)^{-1}$ on $(l/2^n, (l+1)/2^n]$ to U, we raise the value of (10). This is easy to see in Figure 4, in which the graph of $(H^n)^{-1}$ is obtained from the graph of $(\widetilde{H}^n)^{-1}$ by moving it to the left by the shaded square, and moving it to the right by the darkened square. This makes the value of $l(z)$ on $(k/2^n, (k+1)/2^n]$ higher than its previous value on $(l/2^n, (l+1)/2^n]$ by the shaded area. Similarly, the value of $l(z)$ on $(l/2^n, (l+1)/2^n]$ becomes higher than its previous value on $(k/2^n, (k+1)/2^n]$ by the shaded area. This increases the integrand of **(10)** on $(k/2^n, (l+1)/2^n]$**.** Finally, the value of $l(z)$ and the integrand of (10) on other intervals of the partition stay the same.

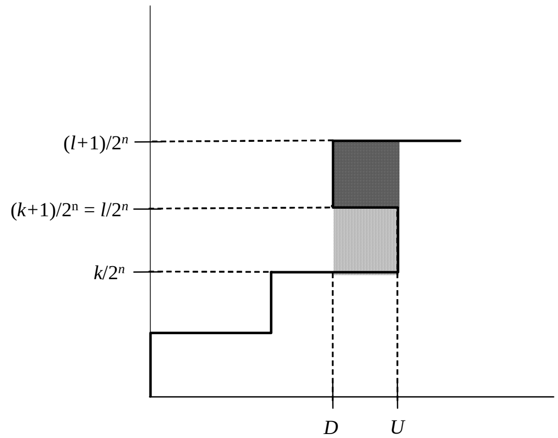

Figure 4: Changing $(\widetilde{H}^n)^{-1}$

For the second condition in Case 1, notice that the inequality $(c^{-1})'(l(z))z - \int_z^1 (c^{-1})'(l(r))dr \geq \lambda$ for $z > z_{\max}$ reduces to $(c^{-1})'(l\,(1))(2z_{\max} - 1) \geq \lambda$ by taking the limit as z tends to $z_{\max}$. For Case 2, notice that the left-hand side of (11) for $z = z_{\min}$ is equal to $(c^{-1})'(0)z_{\min} - \int_{z_{\min}}^1 (c^{-1})'(l(r))dr$, and the limit of the left-hand side of (11) as z tends to $z_{\max}$ is $(c^{-1})'l\,(1)\,z_{\max} - \int_{z_{\max}}^1 (c^{-1})'(l(r))dr$. This yields the condition in Case 2, as $z_{\min} = z_{\max}$.

We can now recover the results from M&S.

Proposition 6 *If the budget constraint binds and c^{-1} is weakly convex, then the optimal prize distribution assigns mass C to the highest possible prize and mass $1 - C$ to prize 0.*

Proof: In this case, we have $z_{\min} = z_{\max}$. Indeed, since $(c^{-1})'$ and l are weakly increasing, $(c^{-1})'(l(z))z$ strictly increases in z; in turn, $\int_z^1 (c^{-1})'(l(r))dr$ weakly decreases in z. Therefore, (11) strictly increases in z.

Proposition 4.5 mirrors Propositions 2 and 4 in M&S, which show that when the cost function is linear or concave it is optimal to award the entire budget as a single prize. The discrepancy between M&S's single prize and the mass of identical highest prizes prescribed by Proposition 4.5 arises because M&S do not consider a limit on the highest possible prize.

Increasing the highest possible prize in our setting, as discussed in Section 3.6, optimally leads to awarding a smaller mass of this prize. This corresponds, in the limit, to "awarding the entire budget as a single prize."

Proposition 7 *If the budget constraint is binding and* $(c^{-1})'(0) > (c^{-1})'(r)$ *for all* $r > 0$*, then the optimal prize distribution assigns a positive mass to intermediate prizes* $y \in (0,1)$*.*

Proof: In this case, it follows directly from (12) that $z_{\min} < z_{\max}$.

Proposition 7 corresponds to Proposition 5 of M&S, which shows that with a convex cost function splitting the budget into two prizes is sometimes better than awarding the entire budget as a single grand. But Proposition 4.5 applies whenever the cost function c satisfies only the mild "convexity" condition that $(c^{-1})'(0) > (c^{-1})'(r)$ for all $r > 0$, whereas M&S require another condition, which involves the number of players and the degree of convexity of the cost function. The seeming discrepancy may be resolved by noting that we consider large contests and observing, as M&S did, that for a given convex cost function their condition is easier to satisfy if there are more players.

While Propositions 4.5 and 7 mirror results in M&S, the set of contests to which they apply are different from those studied by M&S. While M&S studied contests with any finite number of players, the players were restricted to being ex-ante symmetric and having private information about their cost. Our results apply to contests with a large, but finite, number of players, and apply to both ex-ante symmetric and asymmetric players, who may or may not have private information.

Our analysis also makes it possible to obtain results not found in M&S, and which would be difficult (or perhaps even impossible) to obtain by analyzing discrete contests. The following proposition describes one such result.

Proposition 8 *If* $(c^{-1})'(r) > 0$ *for all* r*, then the optimal prize distribution may have atoms only at* 0 *(no prize) and* 1 *(the highest possible prize).*

Proof: An atom at some intermediate prize would mean that Case 1 must hold and $G^{-1}(\underline{z}) = G^{-1}(\overline{z})$ for some $z_{\min} < \underline{z} < \overline{z} < z_{\max}$. Then, however, $l(\underline{z}) = l(\overline{z})$, so $(c^{-1})'(l(\underline{z}))\underline{z} < (c^{-1})'(l(\overline{z}))\overline{z}$; in turn, $\int_{\underline{z}}^{1}(c^{-1})'(l(r))dr \geq \int_{\overline{z}}^{1}(c^{-1})'(l(r))dr$. Thus, (11) for $\overline{z}$ exceeds (11) for $\overline{z}$, which contradicts condition (11).

5 Leftover: Comparative statics

The fact that the average bids in large contests can be approximated by (3) implies that comparative statics relating to aggregate bids in large contest behave as the corresponding comparative statics do with respect to (3). The following result compares the aggregate bids in a contest across different sets of players, and does not require $x - (1 - F(x))/f(x)$ to be increasing.

Proposition 9 *For any prize distribution G, a first-order stochastic dominance shift in the type distribution F increases the limit aggregate bids (3).*

Proof: Integration by parts shows that for all differentiable G^{-1}, the limit aggregate bids are equal to[13]

$$\int_0^1 h'(G^{-1}(z))(G^{-1})'(z)(1-z)F^{-1}(z)dz.$$

A first-order stochastic dominance shift in the type distribution increases this expression, as it increases $F^{-1}(z)$ for all z. For an arbitrary prize distribution G we obtain the result by approximating G with prize distributions whose inverse distributions are differentiable.

Although this result seems intuitive, its validity for all discrete contests is unclear, because of the problems with characterizing discrete-contest equilibria. Perhaps more surprising is the comparative statics concerning function h, which can also be viewed as comparing the aggregate bids in a contest across different sets of players. It turns out that some upward pointwise shifts of function h may decrease the aggregate bids in certain contests. This happens, for example, when $h(y)$ is shifted up at prizes $y > 0$ such that $y = G^{-1}(z)$ and $k(z) < 0$ (by (4)).

[13] In the integration by parts, we use the formula

$$[(1-z)F^{-1}(z)]' = -F^{-1}(z) - \frac{1-z}{f(F^{-1}(z))}.$$

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