Awareness of Unawareness: A Theory of Decision Making in the Face of Ignorance

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Abstract

In the wake of growing awareness, decision makers anticipate that they might acquire knowledge that, in their current state of ignorance, is unimaginable. Supposedly, this anticipation manifests itself in the decision makers' choice behavior. In this paper we model the anticipation of growing awareness, lay choice-based axiomatic foundations to subjective expected utility representation of beliefs about likelihood of discovering unknown consequences, and assign utility to consequences that are not only unimaginable but may also be nonexistent. In so doing, we maintain the flavor of reverse Bayesianism of Karni and Vierø (2013, 2014).

Keywords: Awareness, unawareness, ignorance, reverse Bayesianism, utility of undescribable consequences

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1 Introduction

Habituation to technologies and ideas that, prior to their discovery, were unimaginable or, for lack of appropriate language, indescribable, is an important aspect of human experience. The anticipation of additional such discoveries shapes our future outlook and manifests itself in our choice behavior.

In this paper, which builds upon Karni and Vierø (2013, 2014), we propose a dynamic, choice-based, theory designed to capture a decision maker's anticipation of becoming aware of consequences which she is currently unaware of, and analyze the behavioral implications of such anticipation. Although a decision maker cannot know what it is that she might be unaware of, she can entertain the belief that there are unimaginable aspects of the universe yet to be discovered. In this paper we model the anticipation of such discoveries and its manifestations in the decision maker's choice behavior. Because we adhere to the revealed preference methodology, we require that the decision maker's choice set consists only of objects that are well-defined given her level of awareness. In other words, we insist that, when uncertainty resolves, it must be possible to meaningfully settle any bet or trade that the decision maker may have engaged in.

The main thrust of Karni and Vierø (2013, 2014) is the evolution of decision makers' beliefs as they become aware of new acts, consequences, and the links among them. In these models, however, decision makers are myopic, believing themselves, at every stage, to be fully aware of the scope of their universe. Formally, in these models, decision makers consider the state space that resolves the uncertainty associated with the feasible courses of action and consequences of which they are aware, to be a sure event. Consequently, even though it has happened before, decision makers fail to anticipate the possibility of discoveries that would require expansions of the state space. In a major break with our earlier work, this paper extends the analytical framework to incorporate decision makers' awareness of their potential ignorance, and the anticipation that new discoveries may reveal consequences that were unspecified in the original formulation of the decision problem. The resulting state space is partitioned into a set of fully describable states and a set of states that are only partially describable.

This work also departs from the analytical framework we employed before. In particular, in Karni and Vierø (2013, 2014) the state space was constructed from the set of feasible consequences, while the choice space consisted of a set of conceivable Anscombe-Aumann

(1963) acts, (that is, mappings from the appropriate state space to the set of *lotteries on* the feasible consequences). This formulation is based on the tacit assumption that the decision maker can conceive of acts whose state-contingent payoffs are lotteries on the set of feasible consequences. While analytically convenient, this assumption is unsatisfactory. It is inconsistent to suppose that decision makers can conceive of lotteries over consequences when contemplating conceivable acts, but not when constructing the state space. If the lotteries on feasible consequences are taken to be the consequences used to construct the conceivable state space, then our approach implies that the state space is infinite, which would complicate the analysis. To avoid the aforementioned inconsistency and, at the same time, to maintain the finiteness of the state space, in this work we redefine conceivable acts to be functions from states to feasible consequences. We then assume that the choice space is the set of probability distributions over the conceivable acts, dubbed mixed conceivable acts. This approach seems more satisfactory because, analogously to the use of mixed strategies in games, it is natural to suppose that decision makers can imagine choosing among conceivable acts randomly.

Within the new analytical framework we develop an axiomatic model of choice under uncertainty and analyze the behavioral implications of a decision maker's awareness of her unawareness. Moreover, we also analyze the evolution of her beliefs about her ignorance in the wake of discovery of new consequences.

Depending on the nature of the discoveries, the sense of ignorance, or the 'residual' unawareness, may shrink, grow, or remain unchanged. For instance, as unsuspected regions of the Earth or the solar system were discovered (or rediscovered), fewer regions remained to be discovered, and the sense of ignorance diminished. By contrast, some scientific discoveries, such as relativity, atoms, or the structure of the DNA, resolved certain outstanding issues in physics and biology and, at the same time, opened up new vistas. These discoveries enhanced the sense that our ignorance is, in fact, greater than what was previously believed. Our model is designed to accommodate all the aforementioned possibilities of evolution of the sense of ignorance.

The sense that there might be consequences, lurking in the background, of which one is unaware may inspire fear or excitement, and affect individual choice behavior. By assigning utility to the unknown consequences, our model captures the decision maker's attitudes towards discovering indescribable consequences, and the emotions it evokes. For instance, if the predominant emotion evoked by the unknown is fear, then confidence that one is unlikely to encounter unknown consequences would beget boldness of action while the lack of it would induce more prudent behavior. To represent the attitude towards the unknown, we need further enrichment of our framework. In particular, because we require that bets should be meaningfully describable using the decision maker's current language, and actually settled once uncertainty resolves, decision makers cannot conceive of acts that assign indescribable consequences to fully describable states. Thus, to represent the attitudes towards indescribable consequences we expand the set of conceivable acts to include acts that assign to partially describable states, consequences that will be discovered if these states obtain. The resulting model is a generalization of subjective expected utility that captures the decision maker's "utility of the unknown." A high utility value will reflect excitement, or optimism, about potential unknown aspects of the universe, while a low value will reflect fear, or pessimism. The representation thus allows us to explicitly and formally express this attitude.

On a more mundane level, decision makers are routinely confronted with the need to make a decision in specific situations. For example, a decision maker about to embark on a trip must choose how to get from here to there. Another example, following a diagnosis of illness, a decision maker must decide which treatment to seek. It is natural to approach such decision problems by identifying the relevant courses of action and the outcomes that these actions may produce. It might happen, however, that due to lack of imagination or insufficient attention, the chosen course of action results in an outcome that the decision maker failed to consider. Therefore, when facing a specific decision, a decision maker worries that she might fail to consider all the relevant outcomes. This awareness of the possibility that an outcome that should have been considered is, inadvertently, neglected, bears resemblance to awareness of possible ignorance. Awareness of one's potential failure to consider all the relevant outcomes of one's actions affects individual choice behavior similarly to the anticipation of discoveries of new consequences. We discuss this similarity between awareness of unawareness and "small worlds" in further detail in the concluding remarks.

In the next section we present the analytical framework and the basic preference structure. In section 3 we introduce additional axioms linking distinct levels of unawareness and a representation theorem that assigns probability to making new discoveries and provides rules for updating beliefs in the wake of new discoveries. We also axiomatize the phenomena of shrinking and growing sense of ignorance. In section 4, we extend the analysis to allow for the assignment of utility to unknown consequences. In section 5, we provide concluding remarks and place our results in the context of the related literature. The proofs are collected in section 6.

2 The Analytical Framework

In Karni and Vierø (2013, 2014), we modeled and analyzed the evolution of a decision maker's beliefs when her universe, formalized as a state space, expands in the wake of discoveries of new acts and/or consequences. There is, however, a fundamental difference between discoveries of acts and consequences. Discovery of new acts, such as portfolio positions made possible by the introduction of new financial instruments (e.g., derivatives), or faster ways of traveling made possible by the introduction of new means of transportation (e.g., jet-propelled airplanes), are the result of innovative designs. By contrast, the discovery of new consequences, such as new diseases, (e.g., the discovery of syphilis by the Europeans), the beneficial effects of drugs (e.g., the effect of Penicillium fungi (penicillin) in fighting certain bacterial infections), is arrived at coincidentally and/or through systematic observation and experimentation. From a modeling perspective, there is a crucial difference between the two types of discoveries: while the discovery of new acts expands the state space by refining it, the discovery of new consequences expands the state space by augmenting it through the addition of unthinkable states. Put differently, when a new feasible act is designed, each element of the prior state space (the state space that existed before the introduction of the new act) becomes a non-degenerate event in the posterior state space (the state space following the introduction of the new act). By contrast, when a new consequence is discovered the prior state space is augmented by the addition of states whose existence was not apparent.[1]

In this work, we study some behavioral and cognitive aspects of awareness of unawareness. Our investigation focuses on the effects of the anticipation of discovering unexpected consequences on decision makers' choice behavior, and the evolution of decision makers' sense of ignorance following such discoveries. Due to the differences in both the nature of the discoveries and in the evolution of the state space, we leave the investigation of the anticipation of discovery of new feasible acts for future work.

[1] For details, see Karni and Vierø (2013).

2.1 Conceivable states and acts

Let F be a finite, nonempty, set of *feasible acts* and C_0 be a finite, nonempty, set of *feasible consequences.* We define $x_0 = \neg C_0$ to be the abstract "consequence" that has the interpretation "none of the above".[2] Let $\hat{C}_0 = C_0 \cup \{x_0\}$. Together these sets determine the *augmented conceivable state space*, $\hat{C}_0^F := \{s : F \to \hat{C}_0\}$, that is, the set of all functions from F to $\hat{C}_0$, which is, by definition, exhaustive.[3] They also determine the subset of *fully describable states*, $C_0^F := \{s : F \to C_0\}$. To illustrate, let there be two feasible acts, $F = \{f_1, f_2\}$, and two feasible consequences, $C_0 = \{c_1, c_2\}$. The resulting augmented conceivable state space consists of nine states as depicted in the following matrix:

$$\begin{array}{cccccccccc} F/\hat{C}_0^F & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & s_7 & s_8 & s_9 \\ f_1 & c_1 & c_2 & c_1 & c_2 & x_0 & x_0 & c_1 & c_2 & x_0 \\ f_2 & c_1 & c_1 & c_2 & c_2 & c_1 & c_2 & x_0 & x_0 & x_0 \end{array} \tag{1}$$

The subset of fully describable states in this example is $C_0^F = \{s_1, ..., s_4\}$.

The set of *conceivable acts*, consists of all the mappings from the augmented conceivable state space to the set of feasible consequences. Formally,

$$\hat{F}_0 := \{f : \hat{C}_0^F \to C_0\}. \tag{2}$$

We restrict the payoffs of conceivable acts to *feasible* consequences because for a given level of awareness, we require that bets can be both meaningfully described using current language and settled once uncertainty resolves. Under the level of awareness depicted by $\hat{C}_0^F$, the only payoffs that can be meaningfully specified in every state are the consequences in C_0.[4]

[2]Since there is no universal set of consequences in the background, the addition of the abstract consequence x_0 to the set C_0 generates a set of consequences that is, by definition, universal. The element x_0 is defined "negatively"using the set of feasible consequences. If x_0 is the empty set, then C_0 is the universal set of consequences.

[3]This method of constructing the state space from the primitive sets of feasible acts and consequences appears in Schmeidler and Wakker (1987) and Karni and Schmeidler (1991). It was used in Karni and Vierø (2013, 2014). The augmentation due to "none of the above" is specific to the present paper.

[4]We revisit this assertion in Section 4. Note also that the definition of conceivable acts in the present paper differs from the definition of conceivable acts in Karni and Vierø (2013, 2014). In our previous work, conceivable acts were functions from conceivable states to lotteries over consequences, i.e. Anscombe-Aumann (1963) acts. For the reasons discussed in the introduction, the approach taken here is more satisfactory.

Because conceivable acts are functions whose domain is the state space, adding them to the list of acts does not require further expansion of the state space. In other words, once the state is known, all uncertainty regarding the outcome of a conceivable act is resolved and no new states are created. By contrast, if a new *feasible act* is discovered then, by definition, it assigns all the consequences to each state in $\hat{C}_0^F$. Thus, each state in $\hat{C}_0^F$ becomes an event in the newly defined state space. Consider the example in which there are two feasible acts and two feasible consequences. If a new feasible act is discovered, the state (c_i, c_j) becomes the event $\{(c_i, c_j, c_1), (c_i, c_j, c_2), (c_i, c_j, x_0)\}, i, j = 1, 2$. That is, the discovery of a new feasible act means that state (c_i, c_j) no longer resolves the uncertainty since the payoff of the newly discovered act in this state can be c_1, c_2 or x_0. By contrast, if the conceivable constant act that pays off c_1 in every state is added, then the state (c_i, c_j) completely resolves the uncertainty since the payoff of every act, including the new conceivable acts becomes known.[5]

Next we assume that decision makers can choose a conceivable act at random. More formally, denote by $\Delta(\hat{F}_0)$ the set of all probability distributions on $\hat{F}_0$. A generic element $\mu \in \Delta(\hat{F}_0)$ selects a conceivable act in $\hat{F}_0$ according to the distribution μ. We refer to the elements of $\Delta(\hat{F}_0)$ by the name *mixed conceivable acts*. The set $\Delta(\hat{F}_0)$ of all such randomizations is the choice set. Decision makers are supposed to be able to form and express preferences over $\Delta(\hat{F}_0)$.

Suppose that a new consequence, $c' \notin C_0$, is discovered. This discovery expands the set of feasible consequences to $C_1 = C_0 \cup \{c'\}$. At the same time, the abstract "consequence" that has the interpretation "none of the above" becomes $x_1 = \neg C_1$, and the augmented set of consequences becomes $\hat{C}_1 = C_1 \cup \{x_1\}$. The posterior conceivable state space is $\hat{C}_1^F$. In our illustrating example, if a new consequence c_3 is discovered, the augmented conceivable state space becomes

$$\begin{array}{cccccccccccccccccc}
F/\hat{C}_1^F & s_1 & s_2 & s_3 & s_4 & s_5' & s_5 & s_6' & s_6 & s_7' & s_7 & s_8' & s_8 & s_9' & s_9'' & s_9''' & s_9 \\
f_1 & c_1 & c_2 & c_1 & c_2 & c_3 & x_1 & c_3 & x_1 & c_1 & c_1 & c_2 & c_2 & c_3 & c_3 & x_1 & x_1 \\
f_2 & c_1 & c_1 & c_2 & c_2 & c_1 & c_1 & c_2 & c_2 & c_3 & x_1 & c_3 & x_1 & c_3 & x_1 & c_3 & x_1
\end{array} \tag{3}$$

The set of fully describable states also expands and is now $C_1^F = C_0^F \cup \{s_5', s_6', s_7', s_8', s_9'\}$. Thus, when a new feasible consequence is discovered, each of the prior fully describable

[5] For more detailed discussion of the implications of discovering new feasible acts, see Karni and Vierø (2013).

states remains as before, while each of the prior imperfectly describable states is split into a fully describable state and one, or more, posterior imperfectly describable states. Hence, elements are added to the subset of fully describable states and, simultaneously, the number of imperfectly describable states increases. As the decision maker's augmented conceivable state space expands, so does the set of conceivable acts, $\hat{F}_1 := \{f : \hat{C}_1^F \to C_1\}$. The corresponding set of mixed conceivable acts is $\Delta(\hat{F}_1)$.

We abuse notation and denote by c also the constant act that assigns c to every state in $\hat{C}_i^F$, and by f also the degenerate mixed conceivable act that assigns unit probability mass to the conceivable act f. For all $\mu, \mu' \in \Delta(\hat{F}_i)$ and $\alpha \in [0,1]$, let $\alpha\mu + (1-\alpha)\mu' \in \Delta(\hat{F}_i)$ be defined as pointwise mixtures on the support of the mixed conceivable acts (that is, $(\alpha\mu + (1-\alpha)\mu')(f) = \alpha\mu(f) + (1-\alpha)\mu'(f)$, for all $f \in \hat{F}_i$). Then $\Delta(\hat{F}_i)$ is a convex set.

Because the set of conceivable acts is a variable in our model, a decision maker is characterized by a collection of preference relations, one for each level of awareness over the corresponding set of mixed conceivable acts. We denote the strict preference relation on $\Delta(\hat{F}_i)$ by $\succ_i$, $i = 0, 1$. In particular, the prior preference relation is denoted by $\succ_0$ on $\Delta(\hat{F}_0)$ and the posterior preference relation is denoted by $\succ_1$ on $\Delta(\hat{F}_1)$.

2.2 Basic preference structure

When the state space expands in the wake of discoveries of new consequences, the set of conceivable acts expands and the preference relation must be redefined on the extended domain. Consider next a decision maker whose choices are characterized by a preference relation $\succ_i$ on $\Delta(\hat{F}_i)$, $i = 0, 1$. We assume that, for each $\hat{F}_i$, $\succ_i$ adheres to the well-known axioms of expected utility theory.

(A.1) (**Preorder**) For $i = 0, 1$, the preference relation $\succ_i$ is asymmetric and negatively transitive.[6]

(A.2) (**Archimedean**) For $i = 0, 1$, for all $\mu, \mu', \mu'' \in \Delta(\hat{F}_i)$, if $\mu \succ_i \mu'$ and $\mu' \succ_i \mu''$ then there are $\alpha, \beta \in (0,1)$ such that $\alpha\mu + (1-\alpha)\mu'' \succ_i \mu'$ and $\mu' \succ_i \beta\mu + (1-\beta)\mu''$.

(A.3) (**Independence**) For $i = 0, 1$, for all $\mu, \mu', \mu'' \in \Delta(\hat{F}_i)$ and $\alpha \in (0,1]$, $\mu \succ_i \mu'$ if and only if $\alpha\mu + (1-\alpha)\mu'' \succ_i \alpha\mu' + (1-\alpha)\mu''$.

[6] This implies that $\succ_i$ is irreflexive and transitive (see Kreps [1988], proposition 2.3).

Define the weak preference relation, $\succcurlyeq_i$, to be the negation of the corresponding strict preference relation, (i.e., $\succcurlyeq_i = \neg(\succ_i)$), and the indifference relation, $\sim_i$, to be the symmetric part of $\succcurlyeq_i$. Then, $\succcurlyeq_i$ is a weak order (i.e., complete and transitive) satisfying the corresponding version of independence. For any $f, g \in \hat{F}_i$ and $E \subset \hat{C}_i^F$, let $g_E f$ be the act in $\hat{F}_i$ defined by $(g_E f)(s) = g(s)$ for all $s \in E$ and $(g_E f)(s) = f(s)$ otherwise.

We now extend the indifference relation on $\Delta(\hat{F}_i)$.[7] Consider the mapping $\varphi^i : \Delta(\hat{F}_i) \to (\Delta(C_i))^{\hat{C}_i^F}$, where, for all $s \in \hat{C}_i^F$, $c \in C_i$, and $\mu \in \Delta(\hat{F}_i)$,

$$\varphi_s^i(\mu)(c) := \sum_{\{f \in Supp(\mu) | f(s) = c\}} \mu(f). \tag{4}$$

The mapping φ^i transforms each mixed conceivable act into an Anscombe-Aumann (1963) act. More specifically, for each $s \in \hat{C}_i^F$, the vector $\varphi_s^i(\mu) \in \Delta(C_i)$ is the lottery that $\varphi^i(\mu)$ assigns to the state s. Henceforth, we also denote by $\varphi_s^i(\mu)$ the mixed conceivable act that assigns the probability $\varphi_s^i(\mu)(c)$ to the constant conceivable act c. Under this convention, the set $\Delta(C_0)$ also denotes the subset of mixed conceivable acts whose supports are restricted to the constant conceivable acts (that is, $\Delta(C_0) \subset \Delta(\hat{F}_0)$).

While the mapping φ^i yields a unique Anscombe-Aumann act for each $\mu \in \Delta(\hat{F}_i)$, in general every Anscombe-Aumann act corresponds to multiple mixed conceivable acts. The next axiom asserts that the decision maker is indifferent among mixed conceivable acts whose images under φ^i are the same (that is, the decision maker is indifferent among mixed conceivable acts that are transformed to the same Anscombe-Aumann act).

(A.4) (**Extended Indifference**) For $i = 0, 1$, and all $\mu, \mu' \in \Delta(\hat{F}_i)$, if $\varphi^i(\mu) = \varphi^i(\mu')$ then $\mu \sim_i \mu'$.

The next Lemma shows that preference relations satisfying (A.1) - (A.4) have expected utility (over acts) and additively separable (across states) representations. To state the Lemma we invoke the following definition: A set of real-valued functions $\{W_s^i\}_{s \in \hat{C}_i^F}$ on C_i, representing a preference relation $\succ_i$ on $\Delta(\hat{F}_i)$ is *unique up to cardinal unit-comparable transformation*, if and only if the set $\{\hat{W}_s^i\}_{s \in \hat{C}_i^F}$ on C_i, where $\hat{W}_s = bW_s + a_s$, $b > 0$, also represents the same preference relation.

[7]Here we follow a procedure mentioned in Kreps (1988), Chapter 7. The next axiom is suggested there.

Lemma 1. *A preference relation $\succ_i$ on $\Delta(\hat{F}_i)$ satisfies (A.1) - (A.4) if and only if there exist real-valued functions $\{W_s^i\}_{s\in\hat{C}_i^F}$ on C_i, unique up to cardinal unit-comparable transformation, such that, for all $\mu, \mu' \in \Delta(\hat{F}_i)$,*

$$\mu \succ_i \mu' \Leftrightarrow \sum_{f\in\hat{F}_i} \mu(f) \sum_{s\in\hat{C}_i^F} W_s^i(f(s)) > \sum_{f\in\hat{F}_i} \mu'(f) \sum_{s\in\hat{C}_i^F} W_s^i(f(s)). \tag{5}$$

Following Savage (1954), a state $s \in \hat{C}_i^F$ is said to be *null* if $c_{\{s\}}f \sim_i c'_{\{s\}}f$, for all $c, c' \in C_i$, for all $f \in \hat{F}_i$. A state is said to be *nonnull* if it is not null. To state the next axiom we use the following notations: Given $f \in \hat{F}_i$ and $s \in \hat{C}_i^F$, define $\hat{F}_i(f,s) := \{c_{\{s\}}f \in \hat{F}_i \mid c \in C_i\}$ (that is, $\hat{F}_i(f,s)$ is the subset of acts that agree with f outside s). Let $\Delta(\hat{F}_i(f,s))$ be the subset of mixed conceivable acts whose support is $\hat{F}_i(f,s)$.[8]

(A.5) (**Monotonicity**) For $i = 0, 1$, all $f \in \hat{F}_i$, all nonnull $s \in \hat{C}_i^F$, all $\mu, \mu' \in \Delta(\hat{F}_i(f,s))$, and the corresponding $\varphi_s^i(\mu), \varphi_s^i(\mu') \in \Delta(C_i) \subset \Delta(\hat{F}_i)$, it holds that $\mu \succ_i \mu'$ if and only if $\varphi_s^i(\mu) \succ_i \varphi_s^i(\mu')$.

In the Monotonicity axiom, the mixed conceivable acts μ and μ' have as their supports conceivable acts whose payoffs differ in a single state s. The Anscombe-Aumann acts induced by μ and μ', return the same lotteries in every state except s, in which they yield $\varphi_s^i(\mu)$ and $\varphi_s^i(\mu')$, respectively. The axiom states that the direction of preference between μ and μ' is the same as the direction of preference between the mixed conceivable acts that have distributions $\varphi_s^i(\mu)$ and $\varphi_s^i(\mu')$ over the constant conceivable acts. Thus, our Monotonicity axiom has the same spirit as it has in the Anscombe-Aumann framework, but the expression of it is different because the decision maker's choice set consists of mixed conceivable acts.

(A.6) (**Nontriviality**) For $i = 0, 1$, $\succ_i$ on $\Delta(\hat{F}_i)$ is nonempty.

Note that (A.6) implies the existence of consequences, $c_i^*, c_*^i \in C_i$, such that $c_i^* \succ_i c_*^i$, $i = 0, 1$. The implication of the next axiom is that there are $c_0^* = c_1^* = c^*$ and $c_*^0 = c_*^1 = c_*$. Hence, for this particular purpose, we can suppress the index i and simply write c^*, c_*.

[8] Recall that $\varphi_s^i(\mu)$ also denotes the mixed conceivable act that assigns the probability $\varphi_s^i(\mu)(c)$ to the constant conceivable act c.

Thus far the axiomatic structure characterized the preference relations for each given level of awareness. To link the preference relations across expanding sets of mixed conceivable acts, we invoke the relevant part of the invariant risk preferences axiom introduced in Karni and Vierø (2013). This axiom asserts the commonality of risk attitudes across levels of awareness. Recall that $\Delta(C_0) \subset \Delta(\hat{F}_0)$ also denotes the subset of mixed conceivable acts whose supports are the constant conceivable acts in $\hat{F}_0$, and note that for $C_1 \supset C_0$, we also have that $\Delta(C_0) \subset \Delta(\hat{F}_1)$.

(A.7) (**Invariant risk preferences**) For all $C_0 \subset C_1$ and $\succ_i$ on $\Delta(\hat{F}_i)$ for $i = 0, 1$, and for all $\mu, \mu' \in \Delta(C_0)$, it holds that $\mu \succ_0 \mu'$ if and only if $\mu \succ_1 \mu'$.

3 Preference Representation and the Evolution of Beliefs

3.1 The main result

The following two axioms further link the preference relations across different levels of awareness. The first axiom, dubbed Refinement Consistency I, asserts that the decision maker's ranking of objective versus subjective uncertainty, conditional on the initial set of fully describable states, remains unchanged in the wake of discovery of new consequences. The intuition is that, while the discovery of new consequences may change the decision maker's sense of ignorance, such discoveries do not affect the part of his preferences that only concerns the fully describable and well-understood part of his universe.

(A.8) (**Refinement Consistency I**) For all $C_0 \subset C_1$ and the corresponding sets of mixed conceivable acts $\Delta(\hat{F}_0)$ and $\Delta(\hat{F}_1)$, for all $s \in C_0^F$ and $\eta \in [0, 1]$, if $\lambda = \eta c^* + (1 - \eta) c_* \in \Delta(\hat{F}_0)$, $\lambda' = \eta c^* + (1 - \eta) c_* \in \Delta(\hat{F}_1)$, $\mu = \eta\big(c^*_{\{s\} \cup (\hat{C}_0^F \setminus C_0^F)} c_*\big) + (1 - \eta)\big(c^*_{\{s\}} c_*\big) \in \Delta(\hat{F}_0)$ and $\mu' = \eta\big(c^*_{\{s\} \cup (\hat{C}_1^F \setminus C_0^F)} c_*\big) + (1 - \eta)\big(c^*_{\{s\}} c_*\big) \in \Delta(\hat{F}_1)$, then it holds that $\mu \succ_0 \lambda$ if and only if $\mu' \succ_1 \lambda'$.

The mixed conceivable act λ assigns probability η to the constant act c^* and probability $1 - \eta$ to the constant act c_*. The mixed conceivable act μ assigns probability η and $1 - \eta$ to two acts, both of which pay c^* in state s and c_* in $C_0^F \setminus \{s\}$, and the former of which pays c^* in $\hat{C}_0^F \setminus C_0^F$ while the latter pays c_* in that event. Thus, the Anscombe-Aumann acts induced by μ and λ agree on the event $\hat{C}_0^F \setminus C_0^F$ that consists of imperfectly describable states. The mixed conceivable acts λ' and μ' satisfy a similar relationship, but in $\Delta(\hat{F}_1)$. Thus, the Anscombe-Aumann acts induced by μ' and λ' agree on $\hat{C}_1^F \setminus C_0^F$. When a new consequence is discovered the event on which the mixed conceivable acts μ and λ as well as μ' and λ' agree is partitioned more finely. The axiom asserts that such refinement does not alter preferences conditional on the event that is unaffected by the change. In other words, Refinement Consistency I ensures robustness of the decision maker's preferences, conditional on the a-priori fully describable event, with respect to discovery of new consequences.

The second axiom, dubbed Refinement Consistency II, asserts that, in the wake of discovery of new consequences, and conditional on the set of a-priori imperfectly describable states, a decision maker's ranking of objective uncertainty versus subjective uncertainty regarding a state in the prior state space is the same as that of objective uncertainty versus subjective uncertainty regarding the corresponding event in the posterior state space. To state this idea formally we introduce the following additional notations: If $C_0 \subset C_1$ then for each $s \in \hat{C}_0^F \setminus C_0^F$ there corresponds an event $E(s) \subset \hat{C}_1^F \setminus C_0^F$, defined by $E(s) = \{\hat{s} \in \hat{C}_1^F \setminus C_0^F \mid \forall f \in F$, if $f(s) \in C_0$, then $f(\hat{s}) = f(s)$, and if $f(s) = x_0$ then $f(\hat{s}) \in \{x_1\} \cup (C_1 \setminus C_0)\}$.[9]

(A.9) (**Refinement Consistency II**) For all $C_0 \subset C_1$ and the corresponding sets of mixed conceivable acts $\Delta(\hat{F}_0)$ and $\Delta(\hat{F}_1)$, for all $s \in \hat{C}_0^F \setminus C_0^F$ and $\eta \in [0,1]$, if $\lambda = \eta c^* + (1-\eta)c_* \in \Delta(\hat{F}_0)$, $\lambda' = \eta c^* + (1-\eta)\, c_* \in \Delta(\hat{F}_1)$, $\mu = \eta \left(c^*_{\{s\}\cup C_0^F} c_*\right) + (1-\eta)\left(c^*_{\{s\}} c_*\right) \in \Delta(\hat{F}_0)$, and $\mu' = \eta \left(c^*_{E(s)\cup C_0^F} c_*\right) + (1-\eta)\,(c^*_{E(s)} c_*) \in \Delta(\hat{F}_1)$, then it holds that $\mu \succ_0 \lambda$ if and only if $\mu' \succ_1 \lambda'$.

The intuition underlying axiom (A.9) is that, conditional on the event that is not fully describable a-priori, the decision maker views the relative likelihoods of a-priori measurable sub-events as being independent of the extent to which he can describe the events.[10]

Theorem 1 below asserts the existence and describes the uniqueness properties of a subjective expected utility representation for each level of awareness, of preference relations satisfying the aforementioned axioms. In addition, it describes the evolution of beliefs about

[9] It may be helpful to look at the matrices (1) and (3) to see what this notation captures.

[10] An event E is measurable with respect to the prior state space if there is an act, $f \in \hat{F}_0$ and consequence $c \in \hat{C}_0$ such that $f^{-1}(c) = E$.

the relative likelihoods of fully describable events and the relative likelihoods of imperfectly describable events in the wake of increasing awareness.

Theorem 1. *For each $C_0 \subset C_1$ and the corresponding preference relations $\succcurlyeq_0$ on $\hat{F}_0$ and $\succcurlyeq_1$ on $\hat{F}_1$, the following two conditions are equivalent:*

(i) *The preference relations $\succ_0$ and $\succ_1$ each satisfy (A.1) - (A.6), and jointly they satisfy (A.7) - (A.9).*

(ii) *There exist real-valued, continuous, nonconstant, affine functions, U_0 on $\Delta(C_0)$ and U_1 on $\Delta(C_1)$, and probability measures, π_0 on $\hat{C}_0^F$ and π_1 on $\hat{C}_1^F$, such that for all $\mu, \lambda \in \Delta(\hat{F}_0)$,*

$$\mu \succcurlyeq_0 \lambda \Leftrightarrow \sum_{s \in \hat{C}_0^F} \pi_0(s)\, U_0\left(\varphi_s^0\,(\mu)\right) \geq \sum_{s \in \hat{C}_0^F} \pi_0(s)\, U_0\left(\varphi_s^0\,(\lambda)\right). \tag{6}$$

and, for all $\mu', \lambda' \in \Delta(\hat{F}_1)$,

$$\mu' \succcurlyeq_1 \lambda' \Leftrightarrow \sum_{s \in \hat{C}_1^F} \pi_1(s)\, U_1\left(\varphi_s^1\,(\mu')\right) \geq \sum_{s \in \hat{C}_1^F} \pi_1(s)\, U_1\left(\varphi_s^1\,(\lambda')\right). \tag{7}$$

The functions U_0 and U_1 are unique up to positive linear transformations and $U_0\,(p) = U_1\,(p)$ for all $p \in \Delta\,(C_0)$, the probability measures π_0 and π_1 are unique and, for all $s, s' \in C_0^F$,

$$\frac{\pi_0\,(s)}{\pi_0\,(s')} = \frac{\pi_1\,(s)}{\pi_1\,(s')} \tag{8}$$

and, for all $s, s' \in \hat{C}_0^F \setminus C_0^F$,

$$\frac{\pi_0\,(s)}{\pi_0\,(s')} = \frac{\pi_1\,(E\,(s))}{\pi_1\,(E(s'))}. \tag{9}$$

By the affinity of U_i, $U_i(\varphi_s^i(\mu)) = \Sigma_{c \in Supp(\varphi_s^i(\mu))}\varphi_s^i(\mu)(c)u_i(c)$, where u_i is a real-valued function on C_i, for $i = 0, 1$. That $U_0\,(p) = U_1\,(p)$ for all $p \in \Delta\,(C_0)$ follows from axiom (A.7). Property (8) follows from axiom (A.8) and asserts that, in the wake of discoveries of new consequences, conditional of the initial set of fully describable states, the decision maker's subjective beliefs about the relative likelihoods of fully describable states remain unchanged. Property (9) follows from axiom (A.9) and asserts that the decision maker's subjective beliefs about the relative likelihood of a-priori measurable sub-events, conditional of the set of states that he cannot fully describe a-priori, remains unchanged in the wake of discoveries of new consequences. Property (8) is reverse Bayesian updating following the discovery of a new consequence as in Karni and Vierø (2013, 2014). Thus, insofar as the discovery of new consequences is concerned, the model of Karni and Vierø (2013, 2014) is nested within the present one and correspond to the special case when $\pi_i(C_i^F) = 1$ for all i. That is, in Karni and Vierø (2013, 2014), for any level of awareness, the decision maker assigns probability zero to future expansions of his awareness.

3.2 Decreasing and increasing sense of ignorance

A decision maker can respond to the discovery of a new consequence in one of three different ways: First, she could think that fewer consequences remain to be discovered. Second, if the discovery of new consequences poses new questions, she could think that more consequences are waiting to be discovered. Third, she could consider the current discovery as having no effect on the likelihood of future discoveries. Thus, the discovery of new consequences expands the decision maker's universe and, depending on their nature, may be accompanied by diminishing, growing, or unchanged sense of ignorance. These reactions have revealed preference manifestations that can be expressed axiomatically.

The next axiom captures the preferential expression of a *decreasing* sense of ignorance. The case of an *increasing* sense of ignorance is symmetric and can be treated formally in the same way. For both decreasing and increasing sense of ignorance, the axioms describe the decision maker's willingness to bet on or against making discoveries of new consequences.

(A.10) (**Decreasing Sense of Ignorance**) For all $C_0 \subset C_1$, the corresponding sets of mixed conceivable acts $\Delta(\hat{F}_0)$ and $\Delta(\hat{F}_1)$, and $\eta \in [0,1]$, $\lambda = \eta c_* + (1-\eta)\, c^* \in \Delta(\hat{F}_0)$, $\lambda' = \eta c_* + (1-\eta)\, c^* \in \Delta(\hat{F}_1)$, $\mu = c_{*C_0^F} c^*$, and $\mu' = c_{*C_1^F} c^*$, if $\lambda \sim_0 \mu$ then $\lambda' \succcurlyeq_1 \mu'$.

Note that this is a decreasing sense of ignorance in the weak sense. It includes the cases of strictly decreasing sense of ignorance ($\lambda' \succ_1 \mu'$) and constant sense of ignorance ($\lambda' \sim_1 \mu'$) as special instances. The mixed conceivable acts λ and λ' only involve objective uncertainty, while μ and μ' are bets on discovering new consequences. A decision maker has a constant sense of ignorance if she is equally inclined to bet on something unforeseen arising before and after the discovery of a new consequence. She has a strictly decreasing sense of ignorance if she is less inclined to bet on the realization of imperfectly describable states after the discovery.

Theorem 2 below quantifies decreasing sense of ignorance by subjective probabilities. Specifically, if growing awareness is accompanied by decreasing sense of ignorance, the subjective probability assigned to the 'residual' unawareness diminishes.

Theorem 2. *For each pair $C_0 \subset C_1$ and the corresponding preference relations $\succ_0$ on $\Delta(\hat{F}_0)$ and $\succ_1$ on $\Delta(\hat{F}_1)$, the following statements are equivalent:*

(i) *$\succ_0$ and $\succ_1$ each satisfy (A.1) - (A.6), and jointly they satisfy (A.7) - (A.10).*

(ii) *There exists a representation as in Theorem 1 and, in addition,*

$$\pi_0(\hat{C}_0^F \setminus C_0^F) \geq \pi_1(\hat{C}_1^F \setminus C_1^F). \tag{10}$$

Inequality (10) includes the case of strictly decreasing ignorance, $\pi_0(\hat{C}_0^F \setminus C_0^F) > \pi_1(\hat{C}_1^F \setminus C_1^F)$, and the case of constant ignorance, $\pi_0(\hat{C}_0^F \setminus C_0^F) = \pi_1(\hat{C}_1^F \setminus C_1^F)$, as special instances.

Clearly, it is possible to formulate the notion of a strictly *increasing sense of ignorance* by changing the conclusion of Axiom (A.10) as follows:

(A.10') (**Increasing Sense of Ignorance**) For all $C_0 \subset C_1$, the corresponding sets of mixed conceivable acts $\Delta(\hat{F}_0)$ and $\Delta(\hat{F}_1)$, and $\eta \in [0,1]$, $\lambda = \eta c_* + (1-\eta)\, c^* \in \Delta(\hat{F}_0)$, $\lambda' = \eta c_* + (1-\eta)\, c^* \in \Delta(\hat{F}_1)$, $\mu = c_{*C_0^F} c^*$, and $\mu' = c_{*C_1^F} c^*$, if $\mu \sim_0 \lambda$ then $\mu' \succcurlyeq_1 \lambda'$.

A decision maker has an increasing sense of ignorance if she is more inclined to bet on a future increase in awareness after a new consequence is discovered. Correspondingly, we have the following:

Corollary 1. *For all $C_0 \subset C_1$ and the corresponding preference relations $\succ_0$ on $\Delta(\hat{F}_0)$ and $\succ_1$ on $\Delta(\hat{F}_1)$, the following statements are equivalent:*

(i) *$\succ_0$ and $\succ_1$ each satisfy (A.1) - (A.6), and jointly they satisfy (A.7) - (A.9) and (A.10').*

(ii) *There exists a representation as in Theorem 1 and, in addition,*

$$\pi_0(\hat{C}_0^F \setminus C_0^F) \leq \pi_1(\hat{C}_1^F \setminus C_1^F). \tag{11}$$

Constant or strictly increasing sense of ignorance necessitates that the decision maker views the world as infinite. There will, in her view, always be more consequences to discover. On the other hand, with a decreasing sense of ignorance, both finite and infinite views of the universe are possible.

The model of Karni and Vierø (2013, 2014) is the special case of growing awareness in which the decision maker exhibits a constant sense of ignorance, assigning zero probability to discovery of new consequences. In those works, new discoveries were outside of the decision maker's outlook. The same behavior arises if the decision maker can, in fact, conceive of new discoveries, but considers them impossible.

4 Utility of Unknown Consequences

4.1 Extended conceivable acts

The specification of conceivable acts whose range are feasible consequences is sufficient to obtain a subjective expected utility representation on such acts. In this framework it is possible to assign probabilistic beliefs to discovery of new consequences but not to assign utility to unspecified consequences, that may not even exist. Presumably, the sentiments associated with the potential discovery of unknown and unimaginable aspects of the universe, such as fear or excitement, affect individual choice behavior. Assigning utility to the unspecified consequence, x_0, would allow to explicitly and formally represent the decision maker's sentiments evoked by the prospect of discovering consequences of which she is currently unaware.

Conceivable acts are mappings from the set of states to the set of feasible consequences.[11] This specification is the most general possible, if we require that in every state one must be able to settle any bet over conceivable acts once uncertainty resolves. In other words, including the abstract consequence "none of the above," or x_0, in the range of the conceivable acts would create a conceptual problem in fully describable states (e.g., the states $s_1, ..., s_4$ in the first example in Section 2.1). In these states, x_0 remains abstract, so a conceivable act that pays off x_0 cannot be settled in those states and is, therefore, meaningless. While the decision maker could potentially describe such acts (as we just did), it is meaningless to suppose that she could express preferences over them.

[11] That is, their range is C_i, which does not include x_i.

However, the argument in the preceding paragraph only applies if we require the acts to map into the same set of consequences in all states. If we give up this requirement, we can expand the set of acts that preferences can meaningfully be expressed over. In particular, in states whose partial or complete descriptions include x_0, this abstract consequence takes a concrete meaning ex post, and mixed conceivable acts that pay off x_0 in one or more of these states can be settled. In the first example in Section 2.1, with the state space depicted in (1), a conceivable act that assigns a consequence, which is not yet known but is neither c_1 nor c_2, and will be discovered in the event $\{s_5, ..., s_9\}$, is well-defined in the states $s_5, \ldots, s_9$. In other words, the decision maker can promise to deliver a newly discovered consequence, whatever it may be, if such a consequence is discovered, and she will be able to keep her promise only if such a discovery is made.

To explore the possibility of assigning utility to unknown consequences, x_0, we extend the range of the set of acts by including x_0 as payoff in the imperfectly describable states. Formally, let $\hat{F}_0$ be the set of conceivable acts defined in (2) and let

$$\tilde{F} := \{\tilde{f} : \hat{C}_0^F \setminus C_0^F \to \hat{C}_0\}.$$

That is, $\tilde{F}$ is the set of all functions from the set of imperfectly describable states to the set of *extended* consequences. Define the set of *extended conceivable acts* F^* as follows:

$$F^* := \{\tilde{f}_{\hat{C}_0^F \setminus C_0^F} f \mid f \in \hat{F}_0, \tilde{f} \in \tilde{F}\}.$$

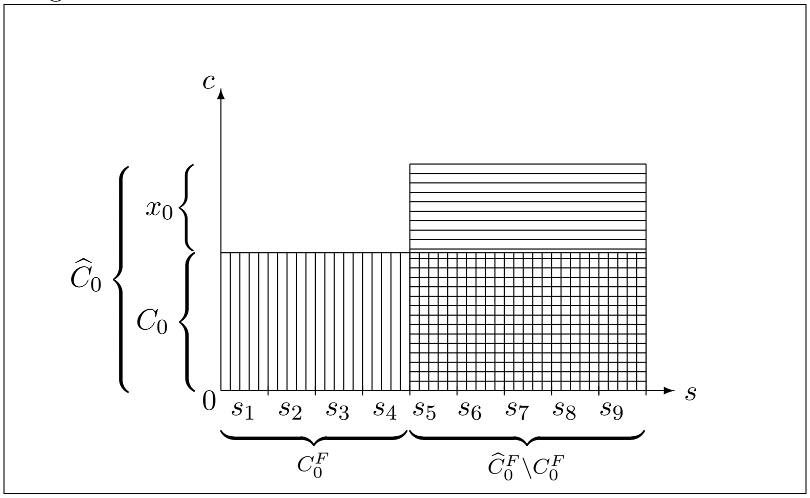
A schematic illustration in the context of the example in matrix (1) is given in Figure 1. Note that $F^* \supset \hat{F}_0$ and that F^* does not include, among others, the constant act whose payoff is x_0. Given the decision maker's awareness, the set of extended conceivable acts F^* is the most of what can be both meaningfully expressed and be settled ex post. In what follows, the choice set is the set of *extended mixed conceivable acts* $\Delta(F^*)$ (that is, the set of distributions over the set F^* of extended conceivable acts).

4.2 Extended preferences and their representation

Let $\succ^*$ be a preference relation on $\Delta(F^*)$ satisfying the axioms (A.1)-(A.6) and denote by $\succcurlyeq^*$ the corresponding weak preference relation (that is, $\succcurlyeq^*$ is the negation of $\succ^*$ including the extended indifference relation).[12] Since $\Delta(\hat{F}_0) \subset \Delta(F^*)$, we assume that the restriction

[12]Note that $\Delta(F^*)$ is a convex set and that the definitions of null and nonnull events from Section 2.1 still applies.

Figure 1: Illustration of the set of extended conceivable acts



of $\succ^*$ to $\Delta(\hat{F}_0)$ agrees with $\succ_0$ (that is, on $\Delta(\hat{F}_0)$, $\succ^* = \succ_0$). By Theorem 1, $\succ_0$ on $\Delta(\hat{F}_0)$ is a preorder satisfying the Archimedean, Independence, Extended Indifference, Monotonicity and Nontriviality axioms if and only if there exists a non-constant, real-valued, affine function, U_0 on $\Delta\left(C_0\right)$, unique up to positive linear transformation, and a unique probability measure π on $\hat{C}_0^F$ such that for all $\mu, \lambda \in \hat{F}_0$,

$$\mu \succ_0 \lambda \Leftrightarrow \sum_{s \in \hat{C}_0^F} \pi(s) U_0\big(\varphi_s^0(\mu)\big) > \sum_{s \in \hat{C}_0^F} \pi(s) U_0\big(\varphi_s^0(\lambda)\big). \tag{12}$$

The representation (12) yields utilities of feasible consequences and a probability measure over the augmented conceivable state space. To extend the representation so that it also yields a utility of the abstract consequence x_0, we define sets of *conditional extended conceivable acts* as follows: For every $f \in \hat{F}_0$, let

$$F_{\hat{C}_0^F \backslash C_0^F}(f) := \{\tilde{f}_{\hat{C}_0^F \backslash C_0^F} f \in F^* \mid \tilde{f} \in \tilde{F}\}.$$

That is, $F_{\hat{C}_0^F \backslash C_0^F}(f)$ is the set of all acts in F^* that are extensions of a particular $f \in \hat{F}_0$. Note that $\cup_{f \in \hat{F}_0} F_{\hat{C}_0^F \backslash C_0^F}(f) = F^*$.

We denote by $\Delta\big(F_{\hat{C}_0^F \backslash C_0^F}(f)\big)$ the corresponding set of mixed conditional extended conceivable acts. With this we can obtain a subjective expected utility representation on each of the sets of conditional extended conceivable acts, that is, one for each given $f \in \hat{F}_0$.

In Proposition 1 and Theorem 3 below, the preference relation satisfies (A.1) - (A.6) with F^* replacing F_0 in the original statement of the axioms.

Proposition 1. *For every given* $f \in \hat{F}_0$ *the restriction of* $\succ^*$ *to* $\Delta\big(F_{\hat{C}_0^F \setminus C_0^F}(f)\big)$ *satisfies (A.1) - (A.6) if and only if there exist a real-valued, non-constant, affine function* U_f^* *on* $\Delta(\hat{C}_0)$ *and a probability measure* ϕ *on* $\hat{C}_0^F \setminus C_0^F$ *such that, for all* μ *and* λ *in* $\Delta(F_{\hat{C}_0^F \setminus C_0^F}(f))$,

$$\mu \succ^* \lambda \Leftrightarrow \sum_{s \in \hat{C}_0^F \setminus C_0^F} \phi(s) U_f^*(\varphi_s^0(\mu)) > \sum_{s \in \hat{C}_0^F \setminus C_0^F} \phi(s) U_f^*(\varphi_s^0(\lambda)), \tag{13}$$

where U_f^* *is unique up to positive affine transformation,* ϕ *is unique and* $\phi(s) = 0$ *if and only if* s *is null.*

The proof is an immediate implication of Theorem 1 and is omitted.

Since $\succ^*$ agrees with $\succ_0$ on $\Delta(\hat{F}_0)$, the representations in (12) and (13) together imply that $U_f^*(p) = U(p)$, for all $f \in \hat{F}_0$ and $p \in \Delta(C_0)$, and that $\phi(s) = \pi(s)/\pi(\hat{C}_0^F \setminus C_0^F)$, for all $s \in \hat{C}_0^F \setminus C_0^F$. However, the utility of the abstract consequence x_0, $U_f^*(x_0)$, may depend on the act f. The axiom below, which we call Separability, links the different conditional representations in Proposition 1. The axiom requires that the ranking of mixed conceivable acts whose supports are conceivable acts that agree on the set of fully describable states, C_0^F, and are constant on the set of partially describable states, $\hat{C}_0^F \setminus C_0^F$, be independent of the part on which the conceivable acts in the support of the mixtures agree. This separability is not implied by the independence axiom because the payoff x_0 is not defined on the subset of fully describable states. To state the axiom formally, we invoke the following notation. For each $f \in \hat{F}_0$ and $p \in \Delta(\hat{C}_0)$, denote by $p_{\hat{C}_0^F \setminus C_0^F} f$ the distribution in $\Delta(F^*)$ that for all $c \in \hat{C}_0$ assigns the probability $p(c)$ to the extended conceivable act $c_{\hat{C}_0^F \setminus C_0^F} f$.

(A.11) **(Separability)** For all $f, g \in \hat{F}_0$ and $p, q \in \Delta(\hat{C}_0)$, $q_{\hat{C}_0^F \setminus C_0^F} f \succ^* p_{\hat{C}_0^F \setminus C_0^F} f$ if and only if $q_{\hat{C}_0^F \setminus C_0^F} g \succ^* p_{\hat{C}_0^F \setminus C_0^F} g$.

In the next theorem we use the separability axiom to combine the representations (12) and (13). This allows us to obtain a general subjective expected utility representation that includes an assignment of utility to the consequence x_0.

Theorem 3. *The following conditions are equivalent:*

(i) The preference relation $\succ^*$ on $\Delta(F^*)$ satisfies axioms (A.1) - (A.6) and (A.11).

(ii) There exist real-valued, non-constant, affine functions, U on $\Delta(C_0)$ and U^* on $\Delta(\hat{C}_0)$, and a probability measure π on $\hat{C}_0^F$ such that, for all $\mu,\lambda \in \Delta(F^*)$, $\mu \succ^* \lambda$ if and only if

$$\sum_{s\in C_0^F} \pi(s)U(\varphi_s^0(\mu)) + \sum_{s\in \hat{C}_0^F\setminus C_0^F} \pi(s)U^*(\varphi_s^0(\mu)) > \sum_{s\in C_0^F} \pi(s)U(\varphi_s^0(\lambda)) + \sum_{s\in \hat{C}_0^F\setminus C_0^F} \pi(s)U^*(\varphi_s^0(\lambda)). \tag{14}$$

Moreover, the functions U and U^* are unique up to positive linear transformation and they agree on $\Delta(C_0)$. Also, the probability measure is unique, with $\pi(s) = 0$ if and only if s is null.

Note also that for all $s \in \hat{C}_0^F \setminus C_0^F$, $\pi(s)/\pi(\hat{C}_0^F \setminus C_0^F) = \phi(s)$, where ϕ is the measure from Proposition 1.

The framework of sections 2 and 3 allowed us to obtain the decision maker's beliefs, including those assigned to the less than fully describable event and its measurable sub-events. Enriching the framework to include extended conceivable acts further allows us to obtain the utility of the unknown. This utility reflects whether the decision maker faces the unknown with fear, excitement, or indifference.

4.3 An Example

A strength of our framework is that it distinguishes between states in which different feasible acts result in new consequences, as illustrated in the matrix (1). It therefore allows the decision maker to view different acts as being more or less likely to increase awareness. If familiarity begets boldness while lack of it begets prudence, acts that are perceived as less likely to result in unforeseeable consequences are expected be preferred over similar acts that are more likely to result in unforeseeable consequences. Consider, for example, the matrix (1) in Section 2.1. Suppose that the decision maker is confident that the act f_1 is unlikely to reveal an unforeseen consequence. Specifically, let f_1 be taking a familiar route from Spain to India around the Cape of Good Hope, and suppose that the decision maker believes that if she chooses f_1 either the consequence c_1 "getting to India safely" or c_2 "ending the trip in the bottom of the ocean" will obtain. In other words, on the basis of past experience, the decision maker believes that if f_1 is implemented it is impossible that "neither c_1 nor c_2" (that is, x_0) will obtain. Formally, she considers the event $\{s_5, s_6, s_9\}$ to be null. By contrast, she considers x_0 to be a real possibility if f_2, a route that was not tried before, such as going to India by sailing westward, is chosen. Thus, the event $\{s_7, s_8\}$ is assigned positive probability. By the representation (14),

$$f_1 \mapsto U(c_1)\left[\pi_0(s_1) + \pi_0(s_3) + \pi_0(s_7)\right] + U(c_2)\left[\pi_0(s_2) + \pi_0(s_4) + \pi_0(s_8)\right].$$

and

$$f_2 \mapsto U(c_1)\left[\pi_0(s_1) + \pi_0(s_2)\right] + U(c_2)\left[\pi_0(s_3) + \pi_0(s_4)\right] + U^*(x_0)\left[\pi_0(s_7) + \pi_0(s_8)\right].$$

Therefore, a choice of f_2 over f_1 yields a higher probability of encountering an "unknown" consequence, x_0. If $U(c_1) > U^*(x_0)$ and $\pi_0(s_3) \geq \pi_0(s_2) + \pi_0(s_8)$, then $f_1 \succ f_2$.

5 Concluding Remarks

5.1 Small Worlds

The definitions of the state space and the set of conceivable acts derived from the entire sets of feasible acts and consequences depict the *grand world*. When facing specific decisions, however, it is natural to suppose that the decision maker constructs the relevant choice space as follows: First, she identifies the *relevant* courses of action, or feasible acts, available (e.g., lists the means of transportation and routes to go from here to there, lists the available treatments of an illness). Second, she identifies the *relevant* consequences of the relevant acts (e.g., getting there late or not at all, allergic reaction to medication or bad outcome of surgery). Third, she constructs the relevant state space. For a given specific decision problem, let $F_r \subset F_0$ and $C_r \subset C_0$ denote, respectively, the relevant set of feasible acts and consequences. Using these primitives, construct the *relevant* state space, $C_r^{F_r}$. The set of *relevant* conceivable acts, $\hat{F}_r$, (that is, the set of all mappings from $C_r^{F_r}$ to C_r) constitutes the *relevant* choice set.[13] Suppose that the decision maker's preferences on $\hat{F}_r$ is the restriction of $\succ_0$ to $\hat{F}_r$.

[13]For the sake of simplicity, we restrict the discussion to the prior sets of acts and consequences and the corresponding set of conceivable acts. Clearly, it is possible to follow the same procedure with the posterior sets.

In this context, unawareness corresponds to failure (e.g., due to lack of attention, forgetfulness) to consider some relevant consequences when constructing the choice set for the decision problem at hand. In other words, some consequences that the decision maker is aware of and should have been included in the set of relevant consequences are neglected.

Analogously to awareness of unawareness, the decision maker may anticipate that she might have neglected to include in her deliberation some relevant consequences. Applying our results to this small world context, it is straightforward to obtain the probability the decision maker assigns to the possibility of failing to include relevant consequences by defining an abstract consequence $x_r = \neg C_r$ that captures neglected relevant consequences, and proceeding as in section 2. If we apply the model that includes extended conceivable acts, we could also assign utility to the concern that relevant consequences are left out of consideration.

5.2 The evolution of beliefs about describable events

Theorem 1 concerns the evolution of the relative likelihoods of fully describable (and also of the relative likelihoods of imperfectly describable) events in the wake of discovery of new consequences, but is silent on the absolute likelihoods. By contrast, Theorem 2 concerns the evolution of the absolute likelihood of the imperfectly describable event. Therefore, combining the results of the two theorems makes it possible to discuss the magnitude of the change in beliefs about the likelihoods of fully describable events. For instance, suppose that a new discovery is accompanied by a sense of constant unawareness. By Theorem 2, $\pi_0(\hat{C}_0^F \setminus C_0^F) = \pi_1(\hat{C}_1^F \setminus C_1^F)$. But

$$\Sigma_{s \in C_0^F} \pi_0(s) + \pi_0(\hat{C}_0^F \setminus C_0^F) = 1$$

and

$$\Sigma_{s \in C_0^F} \pi_1(s) + \Sigma_{s \in (C_1^F \setminus C_0^F)} \pi_1(s) + \pi_1(\hat{C}_1^F \setminus C_1^F) = 1.$$

Hence, probability mass must be shifted from the set of originally fully describable states C_0^F to $C_1^F \setminus C_0^F$, proportionally (that is, the probabilities of all the states in C_0^F must be reduced equiproportionally). Similarly, an increasing sense of unawareness requires that probability mass must be shifted from C_0^F to $C_1^F \setminus C_0^F$ proportionally, and that some of this probability must be shifted to $\hat{C}_1^F \setminus C_1^F$. Finally, decreasing sense of unawareness implies that some probability mass of the event $\hat{C}_0^F \setminus C_0^F$ is shifted towards the newly describable event $C_1^F \setminus C_0^F$. In the latter instance, the effect of growing awareness on the subjective probability assigned to the set of originally fully describable states, C_0^F, is unpredictable.

This paper shows that the model of Karni and Vierø (2013) is, in fact, the special case of growing awareness in which decision makers exhibits not only a constant sense of ignorance, but a constant sense of ignorance assigning zero probability to discovering new consequences. Such decision makers display myopia regarding growing awareness, believing at every point that they are fully aware of the scope of their universe.

Clearly, the model of this paper can be applied to sequentially occurring increases in awareness. After each discovery, the posterior augmented conceivable state space, beliefs, etc. become the prior state space, beliefs, etc. for further increases in awareness.

5.3 Related literature

The exploration of the issue of (un)awareness in the literature has invoked at least three different approaches; the epistemic approach, the game-theoretic or interactive decision making approach, and the choice-theoretic approach.

The epistemic approach is taken in Fagin and Halpern (1988), Dekel, Lipman, and Rustichini (1998), Modica and Rustichini (1999), Halpern (2001), Heifetz, Meier, and Schipper (2006, 2008), Li (2009), Hill (2010), Board and Chung (2011), Walker (2011) and Halpern and Rego (2009, 2013a). Of these, Board and Chung (2011), Walker (2011) and Halpern and Rego (2009, 2013a) consider awareness of unawareness. Schipper (2013a) provides an excellent overview of the epistemic literature as well as of the literature on awareness and unawareness more generally.

The game-theoretic, or interactive decision making, approach is taken in Halpern and Rego (2008, 2013b), Heifetz, Meier, and Schipper (2013a, 2013b), Heinsalu (2014), and Grant and Quiggin (2013). The latter develops a model of games with awareness in which inductive reasoning may cause an individual to entertain the possibility that her awareness is limited. Individuals thus have inductive support for propositions expressing their own unawareness. In this paper, we implicitly assume inductive reasoning to motivate considering awareness of unawareness.

The choice-theoretic approach to unawareness or related issues is taken in Li (2008), Ahn and Ergin (2010), Schipper (2013b), Lehrer and Teper (2014), Kochov (2010), Walker and Dietz (2011), and Alon (2014). The former four are discussed in detail in Karni and

Vierø (2013). Walter and Dietz (2011) and Kochov (2010) consider decision makers who are aware of their potential unawareness, and are thus the papers closest related to the present paper.

Walker and Dietz (2011) take a choice theoretic approach to static choice under "conscious unawareness." In their model, unawareness materializes in the form of coarse contingencies (that is, their state space does not resolve all uncertainty). Their representation is similar to Klibanoff, Marinacci, and Mukerji's (2005) smooth ambiguity model. The model of Walker and Dietz (2011) differs from ours in several respects: theirs is a static model and thus does not consider the issue of updating when awareness increases, their approach to modeling the state space differs from ours, and in their model a decision maker's beliefs are not represented by a single probability measure.

Kochov (2010) develops an axiomatic model of dynamic choice in which the decision maker knows that her perception of the environment may be incomplete. This causes the decision maker's beliefs to be represented by a set of priors, with prior by prior Bayesian updating as the decision maker's perception of the universe becomes more precise. Kochov's work differs from ours in the way the state space and its evolution are modeled, and in the representation of decision makers' beliefs.

Alon (2014) considers a decision maker in a Savage framework. The axioms she imposes imply that the decision maker acts as if he completes the state space with an extra state to which he assigns the worst consequence obtainable from every act. The decision maker is a subjective expected utility maximizer over the set of extended acts. An interpretation of the model is that the decision maker acts as if she faces some unforeseen event. Unlike the model of this paper, Alon's model is static and thus begs the issue of updating. Moreover, since the range of acts is simply the standard set of consequences, Alon's model does not extend the utility to unknown consequences.

In the framework of preferences over menus, Dekel, Lipman and Rustichini (2001) propose "... a model that allows for unforeseen contingencies in the sense that the agent does not have an exogenously given list of all possible states of the world." (p. 893). The agent in their model knows that there may be considerations that she cannot specify. While this sounds similar, the content is completely different from the model of this paper. Specifically, the states in Dekel, Lipman and Rustichini are alternative preferences that the decision maker might entertain at the time he has to choose from the menu. These "mental states" resolve the uncertainty concerning the decision maker's own preferences rather than the payoffs of the feasible acts.

Our modeling of small worlds is concerned with the possibility that a decision maker fails to pay attention to some relevant aspects of a decision problem. In this respect, our theory is related to the recent literature on revealed attention (see Masatlioglu, Nakajima, and Ozbay, 2012). Ortoleva (2012) models non-Bayesian reactions to unexpected news.

Statistical theories of inductive inference have long wrestled with the problem of how to deal with the potential existence of unknown and unsuspected phenomena and how, once such phenomena occur, to incorporate the new knowledge into the corpus of the decision maker's prior beliefs. Zabell (1992) describes a particular instance of this issue, known as the sampling of species problem, involving repeated sampling which might result in an observation whose existence was not suspected (e.g., a new species):[14] "On the surface there would appear to be no way of incorporating such new information into our system of beliefs, other than starting from scratch and completely reassessing our subjective probabilities. Coherence of old and new makes no sense here: there are no old beliefs for the new to cohere with." (Zabell [1992], p. 206). Zabell proceeds to detail a process, anticipated by De Morgan, that accommodates situations in which the possible species to be observed is not supposed to be known ahead of time. The process is based on the idea of exchangeability of random partitions and it yields a representation theorem, a distinguished class of random partitions, and a rule of succession, describing the updated beliefs following the discovery of new species.[15]

Despite the similarity of the objectives, and to some extent structure (think of repeated sampling as different acts and observed species as consequences) the solution for the sampling of species problem and the conclusion of our approach, dubbed 'reverse Bayesianism', are quite distinct. Perhaps the most important distinction is the specification of the prior. In the solution to the sampling of species problem, the prior is induced by exchangeability applied to the distinguished class of random partitions. In other words, it is implied by the stochastic structure of the problem and, as a result, loses its subjective flavor. For instance, the De Morgan rule creates an additional category: "new species not yet observed" and assigns it the probability $(N+t+1)^{-1}$, where N is the number of observations

[14]Zabell (1992) emphasizes that this is not the same thing as observing a phenomenon whose existence is taken into consideration and is judged impossible (that is, a zero probability event). Rather, it is observing a phenomenon whose possibility was not previouly considered.

[15]We are grateful to Teddy Seidenfeld for calling our attention to Zabell's work.

and t the number of known species.[16] By contrast, in 'reverse Bayesianism' the prior is a representation of the decision maker's subjective beliefs, which includes an assignment of subjective probability to the event of observing an indescribable consequence. Moreover, unlike our model of 'reverse Bayesianism', the solution to the sampling of species problem neither requires, nor does it yield, a utility valuation of the newly observed species or of the anticipated, yet indescribable, species.

6 Proofs

6.1 Proof of Lemma 1

Sufficiency Since $\Delta(\hat{F}_i)$ is a convex set and $\succ_i$ satisfies (A.1) - (A.3), by the expected utility theorem, there exists a real-valued function, $V^i : \hat{F}_i \to \mathbb{R}$, such that $\succ_i$ on $\Delta(\hat{F}_i)$ is represented by expected utility: For all $\mu, \mu' \in \Delta(\hat{F}_i)$,

$$\mu \succ_i \mu' \Leftrightarrow \sum_{f \in \hat{F}_i} V^i(f)\mu(f) > \sum_{f \in \hat{F}_i} V^i(f)\mu'(f) \tag{15}$$

Moreover, V^i is unique up to positive linear transformation.

To show that $V^i(f) = \sum_{s \in \hat{C}_i^F} W_s^i(f(s))$, fix $f^* \in \hat{F}_i$. For each $f \in \hat{F}_i$ and $s \in \hat{C}_i^F$, let $f^s \equiv f_{\{s\}}f^*$, i.e. f^s is the conceivable act obtained from f^* by replacing its $s-$coordinate with $f(s)$. Formally, $f^s(s) = f(s)$ and $f^s(t) = f^*(t)$ if $t \neq s$.

Let $\mid \hat{C}_i^F \mid = n$. Consider the mixed conceivable acts, $\mu \in \Delta(\hat{F}_i)$ that assigns probability $1/n$ to f and probability $(n-1)/n$ to f^*, and $\mu' \in \Delta(\hat{F}_i)$ that assigns probability $1/n$ to each f^s, $s \in \hat{C}_i^F$. Then, by the definition in (4), $\varphi^i(\mu) = \varphi^i(\mu')$. Thus, by (A.4), $\mu \sim_i \mu'$. Hence, by the representation in (15), the last indifference is equivalent to

$$\frac{1}{n}V^i(f) + \frac{n-1}{n}V^i(f^*) = \frac{1}{n}\sum_{s \in \hat{C}_i^F} V^i(f^s). \tag{16}$$

For each $s \in \hat{C}_i^F$, define $W_s^i(\cdot) : C_i \to \mathbb{R}$ as follows:[17]

$$W_s^i(c) = V^i(c_{\{s\}}f^*) - \frac{n-1}{n}V^i(f^*),$$

[16] See Zabell (1992), p. 209.

[17] Recall that c denotes both the outcome c and the constant act whose payoff is c in every state.

Thus, for $f \in \hat{F}_i$, $W_s^i(f(s)) = V^i(f^s) - \frac{n-1}{n} V^i(f^*)$. This implies that $\sum_{s \in \hat{C}_i^F} W_s^i(f(s)) = \sum_{s \in \hat{C}_i^F} V^i(f^s) - (n-1)V^i(f^*)$. Multiplying by $1/n$ on both sides together with (16) implies that

$$V^i(f) = \sum_{s \in \hat{C}_i^F} W_s^i(f(s)). \tag{17}$$

Plugging (17) into (15), we get

$$\mu \succ_i \mu' \Leftrightarrow \sum_{f \in \hat{F}_i} \mu(f) \sum_{s \in \hat{C}_i^F} W_s^i(f(s)) > \sum_{f \in \hat{F}_i} \mu'(f) \sum_{s \in \hat{C}_i^F} W_s^i(f(s)). \tag{18}$$

Necessity This is immediate.

Uniqueness The uniqueness of $\{W_s^i\}_{s \in \hat{C}_i^F}$ follows from that of V^i. To see this, define $\hat{W}_s^i(\cdot) = bW_s^i(\cdot) + a_s$, $b > 0$, for all $s \in \hat{C}_i^F$. By definition, for all $s \in \hat{C}_i^F$ and $c \in C_i$, $\hat{W}_s^i(c) = b\left[V^i(c_{\{s\}}f^*) - \frac{n-1}{n} V^i(f^*)\right] + a_s$. Hence,

$$\sum_{s \in \hat{C}_i^F} \hat{W}_s^i(f(s)) = b \sum_{s \in \hat{C}_i^F} W_s^i(f(s)) + \sum_{s \in \hat{C}_i^F} a_s = bV^i(f) + a,$$

where $a = \sum_{s \in \hat{C}_i}^F$. Since V_i is unique up to positive linear transformation, $\hat{V}^i = bV^i + a$ represents the same preferences as V^i. Hence, $\{\hat{W}_s^i\}_{s \in \hat{C}_i^F}$ represents the same preferences as $\{W_s^i\}_{s \in \hat{C}_i^F}$. It is easy to show that $\hat{W}_s^i(c) = \hat{V}^i(c_{\{s\}}f^*) - \frac{n-1}{n} \hat{V}^i(f^*)$. ♠

6.2 Proof of Theorem 1

Sufficiency By (A.1) - (A.4) and Lemma 1,

$$\mu \succ_i \mu' \Leftrightarrow \sum_{f \in \hat{F}_i} \mu(f) \sum_{s \in \hat{C}_i^F} W_s^i(f(s)) > \sum_{f \in \hat{F}_i} \mu'(f) \sum_{s \in \hat{C}_i^F} W_s^i(f(s)).$$

By definition (4),

$$\sum_{f \in \hat{F}_i} \mu(f) \sum_{s \in \hat{C}_i^F} W_s^i(f(s)) = \sum_{s \in \hat{C}_i^F} \sum_{c \in C_i} \varphi_s^i(\mu)(c) W_s^i(c).$$

Fix a non-null $s' \in \hat{C}_i^F$ (that such s' exists is an implication of (A.6)), and define, for $p \in \Delta(C_i)$, $U_i(p) = \sum_{c \in C_i} W_{s'}^i(c)p(c)$. By (A.5), for any $p, q \in \Delta(C_i)$,

$$\sum_{c \in C_i} W_{s'}^i(c)p(c) > \sum_{c \in C_i} W_{s'}^i(c)q(c)$$

if and only if

$$\sum_{c\in C_i} W_s^i(c)p(c) > \sum_{c\in C_i} W_s^i(c)q(c)$$

for all non-null $s \in \hat{C}_i^F$.

Thus, standard arguments imply that, for $i = 0, 1$,

$$\mu \succcurlyeq_i \mu' \Leftrightarrow \sum_{s\in\hat{C}_i^F} U_i\left(\varphi_s^i(\mu)\right)\pi_i(s) \geq \sum_{s\in\hat{C}_i^F} U_i\left(\varphi_s^i(\mu')\right)\pi_i(s),$$

where U_i is continuous, non-constant, affine, real-valued, and unique up to positive linear transformations, and the the probability measure π_i is unique. This completes the proof of the representations (6) and (7).

By (6) and (7), the restriction of $\succcurlyeq_0$ and $\succcurlyeq_1$ to the mixed conceivable acts in $\Delta(C_0)$ whose support is the subset of constant conceivable acts in $\hat{F}_0$, implies that, for any $p, q \in \Delta(C_0)$, $U_0(p) \geq U_0(q)$ if and only if $p \succcurlyeq_0 q$ and that $U_1(p) \geq U_1(q)$ if and only if $p \succcurlyeq_1 q$. By (A.7), $p \succcurlyeq_0 q$ if and only if $p \succcurlyeq_1 q$. Thus, by the uniqueness of the representations, U_0 and U_1 can be chosen so that $U_0 = U_1$ on $\Delta(C_0)$.

For some $s \in C_0^F$ let μ, μ',λ and λ' be as in Axiom (A.8) and suppose that $\mu \sim_0 \lambda$. But $\mu \sim_0 \lambda$ if and only if

$$\eta\left(c^*_{\{s\}\cup(\hat{C}_0^F\setminus C_0^F)}c_*\right) + (1-\eta)\left(c^*{}_{\{s\}}c_*\right) \sim_0 \eta c^* + (1-\eta)\,c_*. \tag{19}$$

By the representation in (6) the last indifference holds if and only if

$$\begin{aligned} U_0(c^*)\left(\pi_0(s) + \eta(1-\pi_0(C_0^F)\right) + U_0(c_*)\left(1-\pi_0(s) - \eta(1-\pi_0(C_0^F)\right) \\ = U_0(c^*)\,\eta + U_0(c_*)\,(1-\eta). \end{aligned} \tag{20}$$

Since $U_0(c^*) > U_0(c_*)$, (20) holds if and only if $\pi_0(s) + (1-\pi_0(C_0^F))\eta = \eta$. Hence,

$$\eta = \frac{\pi_0(s)}{\pi_0(C_0^F)}. \tag{21}$$

By Axiom (A.8), $\mu \sim_0 \lambda$ if and only if $\mu' \sim_1 \lambda'$. The latter indifference is equivalent to

$$\eta c^*_{\{s\}}(c^*_{\{s\}\cup(\hat{C}_1^F\setminus C_1^F)}c_*) + (1-\eta)\left(c^*_{\{s\}}c_*\right) \sim_1 \eta c^* + (1-\eta)\,c_*. \tag{22}$$

By the representation in (7), (22) holds if and only if

$$U_1(c^*)\left(\pi_1(s) + \eta(1-\pi_1(C_0^F))\right) + U_1(c_*)\left(1-\pi_1(s) - \eta(1-\pi_1(C_0^F))\right) \tag{23}$$

$$= U_1 \left(c^*\right) \eta + U_1 \left(c_*\right) (1 - \eta).$$

But (23) holds if and only if $\pi_1(s) + (1 - \pi_1(C_0^F)\eta = \eta$. Thus, $\mu' \sim_1 \lambda'$ if and only if

$$\eta = \frac{\pi_1(s)}{\pi_1(C_0^F)}. \tag{24}$$

By (21) and (24) we have that

$$\frac{\pi_0(s)}{\pi_0(C_0^F)} = \frac{\pi_1(s)}{\pi_1(C_0^F)}. \tag{25}$$

An analogous argument applies for any $s' \in C_0^F$. We therefore also have that, for any $s' \in C_0^F$,

$$\frac{\pi_0(s')}{\pi_0(C_0^F)} = \frac{\pi_1(s')}{\pi_1(C_0^F)}. \tag{26}$$

Together, (25) and (26) imply that

$$\frac{\pi_1(s)}{\pi_1(s')} = \frac{\pi_0(s)}{\pi_0(s')}. \tag{27}$$

For some $s \in \hat{C}_0^F \setminus C_0^F$, let μ, μ', λ and λ' be as in Axiom (A.9) and suppose that $\mu \sim_0 \lambda$. But $\mu \sim_0 \lambda$ if and only if

$$\eta \left(c^*_{\{s\} \cup C_0^F} c_*\right) + (1 - \eta) \left(c^*_{\{s\}} c_*\right) \sim_0 \eta c^* + (1 - \eta)\, c_*. \tag{28}$$

By the representation in (6) and the fact that $\pi_0(\hat{C}_0^F \setminus C_0^F) = 1 - \pi_0(C_0^F)$, the last indifference holds if and only if

$$U_1 \left(c^*\right) \left(\pi_0(s) + \eta\pi_0(C_0^F))\right) + U_1 \left(c_*\right) \left(1 - \pi_0(s) - \eta\pi_0(C_0^F))\right) = U_1 \left(c^*\right) \eta + U_1 \left(c_*\right) (1 - \eta)\,. \tag{29}$$

But (29) holds if and only if $\pi_0(s) + \eta\pi_0(C_0^F) = \eta$. Hence,

$$\eta = \frac{\pi_0(s)}{1 - \pi_0(C_0^F)}. \tag{30}$$

By Axiom (A.9), $\mu \sim_0 \lambda$ if and only if $\mu' \sim_1 \lambda'$. The latter indifference is equivalent to

$$\eta \left(c^*_{E(s) \cup C_0^F} c_*\right) + (1 - \eta) \left(c^*_{E(s)} c_*\right) \sim_1 \eta c^* + (1 - \eta)\, c_*. \tag{31}$$

By the representation in (7), (31) holds if and only if

$$\begin{aligned} & U_1(c^*)\left(\pi_1(E(s))+\eta\pi_1(C_0^F)\right)+U_1(c_*)\left(1-\pi_1(E(s))-\eta\pi_1(C_0^F)\right) \\ = \; & U_1(c^*)\,\eta+U_1(c_*)\,(1-\eta) \end{aligned} \tag{32}$$

But (32) holds if and only if $\pi_1(E(s))+\eta\pi_1(C_0^F)=\eta$. Thus, $\mu' \sim_1 \lambda'$ if and only if

$$\eta=\frac{\pi_1(E(s))}{1-\pi_1(C_0^F)}. \tag{33}$$

By (30) and (33) we have that

$$\frac{\pi_0(s)}{1-\pi_0(C_0^F)}=\frac{\pi_1(E(s))}{1-\pi_1(C_0^F)}. \tag{34}$$

An analogous argument applies for any $s' \in \hat{C}_1^F \setminus C_0^F$. We therefore also have that, for any $s' \in \hat{C}_1^F \setminus C_0^F$,

$$\frac{\pi_0(s')}{1-\pi_0(C_0^F)}=\frac{\pi_1(E(s'))}{1-\pi_1(C_0^F)}. \tag{35}$$

Together (34) and (35) imply that

$$\frac{\pi_1(E(s))}{\pi_1(E(s'))}=\frac{\pi_0(s)}{\pi_0(s')}. \tag{36}$$

Necessity That $\succ_0$ and $\succ_1$ satisfy (A.1) - (A.6) is an implication of Lemma 1 and the theorem of Anscombe and Aumann (1963). Invariant risk preferences, (A.7), follows from the equality of U_0 and U_1 on $\Delta(C_0)$.

To show that (A.8) holds, let $\mu, \lambda \in \Delta(\hat{F}_0)$ and $\mu', \lambda' \in \Delta(\hat{F}_1)$ be as in (A.8). By (6), $\mu \succcurlyeq_0 \lambda$ if and only if

$$U_0(c^*)\,\pi_0(s)+U_0(c_*)\left(\pi_0(C_0^F)-\pi_0(s)\right)+(1-\pi_0(C_0^F))\,(U_0(c^*)\,\eta+U_0(c_*)\,(1-\eta))$$

$$\geq U_0(c^*)\,\eta+U_0(c_*)\,(1-\eta)\,.$$

But $U_0(c^*)>U_0(c_*)$. Hence, the last inequality holds if and only if

$$\frac{\pi_0(s)}{\pi_0(C_0^F)}\geq\eta. \tag{37}$$

Suppose that $\lambda' \succ_1 \mu'$. By (7), $\lambda' \succ_1 \mu'$ if and only if

$$U_1(c^*)\,\pi_1(s)+U_1(c_*)\left(\pi_1(C_0^F)-\pi_1(s)\right)+(1-\pi_1(C_0^F)\,(U_1(c^*)\,\eta+U_1(c_*)\,(1-\eta))$$

$$< U_1\left(c^*\right)\eta + U_1\left(c_*\right)(1-\eta).$$

By the same argument as above, this holds if and only if $\pi_1(s)+(1-\pi_1(C_0^F))\eta<\eta$. Hence,

$$\eta > \frac{\pi_1(s)}{\pi_1(C_0^F)}. \tag{38}$$

Now, expressions (37) and (38) imply that

$$\frac{\pi_0(s)}{\pi_0(C_0^F)} > \frac{\pi_1(s)}{\pi_1(C_0^F)}. \tag{39}$$

However, by (8),

$$\frac{\pi_0\left(s'\right)}{\pi_0\left(s\right)} = \frac{\pi_1\left(s'\right)}{\pi_1\left(s\right)} \tag{40}$$

for all $s, s' \in C_0^F$. Summing over $s' \in C_0^F$ and rearranging, (40) implies that

$$\frac{\pi_0(s)}{\pi_0(C_0^F)} = \frac{\pi_1(s)}{\pi_1(C_0^F)}$$

which contradicts (39).

To show that (A.9) holds, let $\mu, \lambda \in \Delta(\hat{F}_0)$ and $\mu', \lambda' \in \Delta(\hat{F}_1)$ be as in (A.9). By (6), $\mu \succcurlyeq_0 \lambda$ if and only if

$$U_0\left(c^*\right)\pi_0(s) + U_0\left(c_*\right)\left(1-\pi_0(C_0^F)-\pi_0(s)\right) + \pi_0(C_0^F)\left(U_0\left(c^*\right)\eta + U_0\left(c_*\right)(1-\eta)\right)$$

$$\geq U_0\left(c^*\right)\eta + U_0\left(c_*\right)(1-\eta).$$

The last inequality holds if and only if

$$\frac{\pi_0(s)}{1-\pi_0(C_0^F)} \geq \eta. \tag{41}$$

Suppose that $\lambda' \succ_1 \mu'$. By (7), $\lambda' \succ_1 \mu'$ if and only if

$$U_1\left(c^*\right)\left(\pi_1(E(s)) + \eta\pi_1(C_0^F)\right) + U_1\left(c_*\right)\left(1-\pi_1(E(s))-\eta\pi_1(C_0^F)\right)$$

$$< U_1\left(c^*\right)\eta + (1-\eta)\,U_1\left(c_*\right).$$

By the same argument as above, this holds if and only if $\pi_1(E(s))+\eta\pi_1(C_0^F)<\eta$. Hence,

$$\eta > \frac{\pi_1(E(s))}{1-\pi_1(C_0^F)}. \tag{42}$$

Now, expressions (41) and (42) imply that

$$\frac{\pi_0(s)}{1-\pi_0(C_0^F)} > \frac{\pi_1(E(s))}{1-\pi_1(C_0^F)}. \tag{43}$$

However, by (8),

$$\frac{\pi_0\left(s'\right)}{\pi_0\left(s\right)} = \frac{\pi_1\left(E(s')\right)}{\pi_1\left(E(s)\right)} \tag{44}$$

for all $s, s' \in \hat{C}_0^F \setminus C_0^F$. Summing over $s' \in \hat{C}_0^F \setminus C_0^F$ and rearranging, (44) implies that

$$\frac{\pi_0(s)}{1-\pi_0(C_0^F)} = \frac{\pi_1(E(s))}{1-\pi_1(C_0^F)}$$

which contradicts (43). ♠

6.3 Proof of Theorem 2

Sufficiency That the axioms imply existence of a representation as in Theorem 1 follows from the proof of Theorem 1. Let $\lambda, \mu \in \Delta(\hat{F}_0)$ and $\lambda', \mu' \in \Delta(\hat{F}_1)$ be as in Axiom (A.10). Suppose that $\mu \sim_0 \lambda$. But $\mu \sim_0 \lambda$ if and only if

$$c_{*C_0^F} c^* \sim_0 \eta c_* + (1-\eta)\, c^*. \tag{45}$$

By the representation in (6) the last indifference holds if and only if

$$U_0\left(c_*\right)\pi_0(C_0^F) + U_0\left(c^*\right)\left(1-\pi_0(C_0^F)\right) = U_0\left(c_*\right)\eta + U_0\left(c^*\right)\left(1-\eta\right). \tag{46}$$

But, $U_0\left(c^*\right) > U_0\left(c_*\right)$. Hence, (46) holds if and only if

$$\eta = \pi_0(C_0^F). \tag{47}$$

By Axiom (A.10), $\mu \sim_0 \lambda$ implies that $\lambda' \succcurlyeq_1 \mu'$, which is equivalent to

$$\eta c_* + (1-\eta)\, c^* \succcurlyeq_1 c_{*C_1^F} c^*. \tag{48}$$

By the representation in (7), (48) holds if and only if

$$U_1\left(c_*\right)\eta + U_1\left(c^*\right)\left(1-\eta\right) \geq U_1\left(c_*\right)\pi_1(C_1^F) + U_1\left(c^*\right)\left(1-\pi_1(C_1^F)\right). \tag{49}$$

Hence, by the same argument as above, (49) holds if and only if

$$\pi_1(C_1^F) \geq \eta. \tag{50}$$

By (47) and (50) we have that

$$\pi_1(C_1^F) \geq \pi_0(C_0^F), \tag{51}$$

which is equivalent to $\pi_1(\hat{C}_1^F \setminus C_1^F) \leq \pi_0(\hat{C}_0^F \setminus C_0^F)$. The inequality in (51) is strict if and only if $\lambda' \succ_1 \mu'$ in Axiom (A.10), and holds with equality if and only if $\lambda' \sim_1 \mu'$ in Axiom (A.10).

Necessity The necessity of axioms (A.1)-(A.9) follows from the proof of Theorem 1. To show that (A.10) holds, let $\mu, \lambda \in \Delta(\hat{F}_0)$ and $\mu', \lambda' \in \Delta(\hat{F}_1)$ be as in (A.10). By (6), $\mu \sim_0 \lambda$ if and only if

$$U_0\left(c_*\right)\pi_0(C_0^F) + U_0\left(c^*\right)\left(1 - \pi_0(C_0^F)\right) = U_0\left(c_*\right)\eta + U_0\left(c^*\right)\left(1-\eta\right). \tag{52}$$

Since $U_0\left(c^*\right) > U_0\left(c_*\right)$, (52) holds if and only if

$$\eta = \pi_0(C_0^F). \tag{53}$$

Suppose now that $\mu' \succ_1 \lambda'$. By (7), $\mu' \succ_1 \lambda'$ if and only if

$$U_1\left(c_*\right)\pi_1(C_1^F) + U_1\left(c^*\right)\left(1 - \pi_1(C_1^F)\right) > U_1\left(c_*\right)\eta + U_1\left(c^*\right)\left(1-\eta\right).$$

Since $U_0\left(c^*\right) > U_0\left(c_*\right)$, this holds if and only

$$\pi_1(C_1^F) < \eta. \tag{54}$$

Now, expressions (53) and (54) imply that

$$\pi_0(C_0^F) > \pi_1(C_1^F). \tag{55}$$

However, by (10), $\pi_0(C_0^F) \leq \pi_1(C_1^F)$, which contradicts (55). ♠

6.4 Proof of Theorem 3

Sufficiency We give the part of the proof that does not follow directly from (12) or Proposition 1. The agreement of $\succ^*$ and $\succ_0$ on $\Delta(\hat{F}_0)$ and the representations (12) and (13) imply that, for all $f \in \hat{F}_0$ and for all $\tilde{f}, \tilde{g} \in \tilde{F}$ such that $\tilde{f} = p$ and $\tilde{g} = q$, where $p, q \in \Delta\left(C_0\right)$, $\tilde{f}_{\hat{C}_0^F \setminus C_0^F} f \succ^* \tilde{g}_{\hat{C}_0^F \setminus C_0^F} f$ if and only if $U_f^*\left(p\right) > U_f^*\left(q\right)$. Hence, with appropriate normalization, for all $p \in \Delta(C_0)$, $U_f^*(p) = U(p)$, for all $f \in \hat{F}_0$. Therefore $U_f^*(p)$ is independent of f.

Suppose that $c^* \succ^* x_0 \succ^* c_*$, let $\hat{p} = \alpha c^* + (1-\alpha) c_*$ be such that $\hat{p}_{\hat{C}_0^F \backslash C_0^F} f \sim^* x_0 {}_{\hat{C}_0^F \backslash C_0^F} f$. By representation (13), this is equivalent to $U_f^*(\hat{p}) = U_f^*(x_0)$. Then, by axiom (A.11) and representation (13) we have that $U_g^*(\hat{p}) = U_g^*(x_0)$ for all $g \in \hat{F}_0$. But $U_f^*(\hat{p}) = U_f^*(x_0)$ is equivalent to

$$U_f^*(x_0) = \alpha U(c^*) + (1-\alpha) U(c_*)$$

and $U_g^*(\hat{p}) = U_g^*(x_0)$ is equivalent to

$$U_g^*(x_0) = \alpha U(c^*) + (1-\alpha) U(c_*).$$

Hence, $U_f^*(x_0) = U_g^*(x_0) \equiv u(x_0)$, for all $f, g \in \hat{F}_0$.

Suppose instead that $x_0 \succcurlyeq^* c^* \succ^* c_*$, and let $\hat{p} = \alpha x_0 + (1-\alpha) c_*$ be such that $\hat{p}_{\hat{C}_0^F \backslash C_0^F} f \sim^* c^*_{\hat{C}_0^F \backslash C_0^F} f$. By representation (13), this is equivalent to $U_f^*(\hat{p}) = U_f^*(c^*)$. Then, by axiom (A.11) and representation (13) we have that $U_g^*(\hat{p}) = U_g^*(c^*)$ for all $g \in \hat{F}_0$. But $U_f^*(\hat{p}) = U_f^*(c^*)$ is equivalent to

$$\alpha U_f^*(x_0) + (1-\alpha) U(c_*) = U(c^*),$$

and $U_g^*(\hat{p}) = U_g^*(x_0)$ is equivalent to

$$\alpha U_g^*(x_0) + (1-\alpha) U(c_*) = U(c^*).$$

Solving for $U_f^*(x_0)$ and $U_g^*(x_0)$ we get,

$$U_f^*(x_0) = U_g^*(x_0) = \frac{U(c^*) - U(c_*)}{\alpha} + U(c_*) \equiv u(x_0)$$

for all $f, g \in \hat{F}_0$.

Finally, if $c^* \succ^* c_* \succcurlyeq^* x_0$ let $\hat{p} = \alpha c^* + (1-\alpha) x_0$ such that $\hat{p}_{\hat{C}_0^F \backslash C_0^F} f \sim^* c_{*\hat{C}_0^F \backslash C_0^F} f$ then, by the same argument,

$$u_f^*(x_0) = u_g^*(x_0) = \frac{U(c_*) - \alpha U(c^*)}{1-\alpha} \equiv u(x_0)$$

for all $f, g \in \hat{F}_0$.

It follows that $U^*(\hat{p}) = \sum_{c \in C_0} \hat{p}(c) U(c) + \hat{p}(x_0) u(x_0)$ for any $\hat{p} \in \Delta(\hat{C}_0)$.

The uniqueness of the subjective probabilities is implied by the uniqueness of the subjective probabilities in (12).[18]

Necessity Necessity of axioms (A.1) - (A.6) on the respective domains follows from Theorem 1. The necessity of (A.11) is immediate. ♠

[18]The uniqueness of π in conjunction with Proposition 1 imply that $\mu(s) = \pi(s) / \pi\left(\hat{C}_0^F \setminus C_0^F\right)$ for all $s \in \hat{C}_0^F \setminus C_0^F$.

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