Time Horizons, Lattice Structures, and Welfare in Multi-period Matching Markets

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Abstract

Consider a T-period, bilateral matching economy without monetary transfers. Under natural restrictions on agents' preferences, which accommodate switching costs, status-quo bias, and other forms of inter-temporal complementarity, dynamically-stable matchings exist. Generally, "optimal" dynamically-stable matchings may not exist, but under a suitable partial order the stable set forms a lattice. The welfare properties of different stable outcomes is ascertained and the implications for normative market-design are discussed. The robustness of dynamically-stable matchings with respect to the market's time horizon is examined.

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The passage of time is a key component of many social and economic interactions. Wedding anniversaries are celebrated—or forgotten. Employees are recognized for years of service. And students receive an education at a succession of institutions, from pre-school to high school to, possibly, graduate school. Sometimes persistence is held in high regard. At other times, change is eagerly anticipated.

Though time is an ingredient of many economic models, it is largely absent from typical studies of bilateral matching markets, as originally formulated by Gale and Shapley (1962). In such a market agents are partitioned into two groups—men and women, firms and workers, schools and students—and seek to match together to realize benefits. Crucially, each agent has preferences defined over potential partners and these preferences often display conflicting assessments. In Gale and Shapley's classic terminology, which we adopt solely for its simplicity, men and women sometimes (dis)agree in their evaluations of each other. Likewise, firms value a particular worker's skills differently and he may hold an unconventional ranking of employers. That "stable" matchings, where no agent or no pair can pursue a mutually-preferable arrangement to a proposed aggregate assignment, are possible is both surprising and profound.

Extending a matching market's time horizon, forces one to confront several real-world complications. First, with the passage of time agents frequently change their partners.[1] The possibility of change introduces a degree of complexity not encountered in the single-period, one-shot case. The order of interactions matters and inter-temporal trade-offs assume a prominent role. Second, practical decision making over a time horizon is difficult. Complexity and psychology mix to promote habit formation and path-dependence into significant behavioral features. Unlike theory, practice often conflicts with Samuelson's (1937) influential discounted-utility framework. Finally, when time horizons are long, agents' commitment ability is imperfect and dynamic incentives matter.

Kadam and Kotowski (2014) study a two-period generalization of Gale and Shapley's (1962) bilateral matching market. Their analysis lets agents change partners between periods and accommodates inter-temporal preference complementarities. Kadam and Kotowski (2014) employ their model to examine the role of financial transfers and uncertainty in multi-

[1]Change occurs in all celebrated applications of matching theory. Only 54.4 percent of couples married between 1975 and 1979 celebrated their 25th anniversary (Kreider and Ellis, 2011). Similarly, the U.S. Bureau of Labor Statistics (2012) notes that the average person born between 1957 and 1964 held 11.3 jobs between the ages of 18 and 46. Non-employment accounted for 22 percent of weeks during this period. Finally, the U.S. public high school adjusted cohort graduation rate for school year 2011–12 was 80 percent (Stetser and Stillwell, 2014). Thus, twenty percent of students "re-match" with some alternative(s) to a public high school education. This may range from dropping out entirely to pursuing an alternative credential.

period markets. While a two-period market offers insights into those questions, it may be restrictive in some normative applications. For example, the assignment of students to high schools has emerged as a celebrated application of matching theory (Abdulkadiroğlu and Sönmez, 2003). Traditional models of this problem have been static, one-period affairs. Of course, students attend high school for (typically) four years and, therefore, this is a multi-period assignment problem. A very coarse conflation of the successive school years yields the typical one-shot model; however, an unbundling of successive years, semesters, or quarters yields a rich family of multi-period alternatives. In fact, assignment length and re-matching frequency are in principle design variables open to refinement.[2] Clearly, going beyond two periods is necessary, first, to understand and, second, to leverage these dimensions of matching problems.

Though we are mindful of applications, the primary purpose of our study is to examine the positive theory of T-period, bilateral matching economies where agents' preferences exhibit non-trivial, inter-temporal complementarities. We focus on dynamically-stable matchings, where agents are limited in their commitment ability and must be continually incentivized to continue with a proposed assignment plan. Importantly, our study differs from those of Damiano and Lam (2005), Kurino (2009), or Pereyra (2013) where agents' preferences are "time separable." In a multi-period matching market, such a condition is a substantive restriction since status-quo bias, switching costs, preference reversals, and other forms of preference inertia are common. To accommodate these features, we first generalize a class of preferences studied by Kadam and Kotowski (2014). Such preferences merge a ranking of potential partners with a bias toward more persistent assignments. In the two-period case, they satisfy the "rankability" condition of Kennes et al. (2014a), which is among the few other studies incorporating inter-temporal preference complementarities.[3] Contrary to nomenclatural connotation, inertia need not reinforce a matching's stability. Interim preference reversals render the matching problem more nuanced than a sequence of independent, single-period markets.

Despite a bias toward persistent matchings, dynamic stability and volatility in assignments are not only compatible, but surprisingly common. Stable matchings may involve change at the agent level at every opportunity. At times, stability may necessitate periods of

[2]For example, educational programs may be able to alleviate capacity concerns by building-in re-matchings or rotations into initial assignments.

[3]Kennes et al. (2014a) define rankability only for the two-period case. Therefore, our analysis may be interpreted as a (possible) generalization of their condition to a T-period setting. However, our construction and motivation for such preferences is closer to the exposition of Kadam and Kotowski (2014).

sacrifice where agents pair temporarily with (extremely) low-quality partners in anticipation of a better pairing later on. Similarly, agents may experience isolated periods of being unmatched. In a labor market application, we often call such cases job-hopping,[4] internships, and unemployment, respectively. Our model accommodates all of them and can form a basis for further study of labor market dynamics and career planning. We show how many of these outcomes require a long time horizon to be stable market arrangements. Importantly, they need not follow from an intrinsic "preference for variety" and may emerge as a compromise among otherwise conflicting interests.

Though our analysis in Section 2 identifies sufficient conditions for the existence of dynamically-stable matchings, existence alone is not our primary focus. Instead, we address two questions hitherto unexplored in multi-period bilateral matching economies. Both are salient for normative applications as they address welfare and robustness. First, in Section 3, we investigate the lattice structure of the set of dynamically-stable matchings.[5] Such a structure provides economic insight concerning the way markets mediate conflicting interests, facilitates welfare comparisons, and frequently simplifies formal arguments. Though characterized in static applications,[6] the nature of the lattice of stable matchings in dynamic markets has not been explored. Our analysis yields subtle conclusions and qualifications. First, we identify cases where dynamically-stable matchings are not Pareto optimal. Moreover, we show that man- or woman-optimal stable matchings do not always exist.[7] Man- and woman-optimal matchings exist in a wide range of one-period models (Gale and Shapley, 1962; Roth, 1984, 1985b). Therefore, our environment is a substantive departure from a one-period setting and our assumptions do not "reduce" matters to a static market. As a counterpoint to these intriguing though negative observations, we propose a refinement on agents' preferences and a new ordering of the set of dynamically-stable matchings ensuring that this set is a lattice. The new ordering maintains a link to agents' preferences, thereby accommodating comparative welfare analysis.

While the set of stable matchings offers a rich collection of welfare implications, this set's robustness with respect to slight changes in the environment remains unexplored. As our analysis emphasizes the multi-period nature of agents' interactions, it is important to

[4]"Job hopping" refers to the rapid movement of workers between firms. Fallick et al. (2006) document this phenomenon among technology-sector workers in Silicon Valley.

[5]A lattice is a partially-ordered set where each subset has a supremum and an infimum.

[6]Knuth (1976), Blair (1984, 1988), Roth (1984, 1985b), and Alkan (2001, 2002), study the lattice structure of the stable set in an array of matching models. See also Sotomayor (1999).

[7]A stable matching is man-optimal if each man prefers his assignment in that matching to his assignment in all other stable matchings. A woman-optimal stable matching is defined analogously.

understand how the set of stable matchings may change with the market's time horizon, the parameter T in our notation. This is the second focus of our analysis and it is significant for two reasons. First, in many positive analyses the selection and interpretation of "T" is at the analyst's discretion. Thus, a hint of robustness along this dimension is desirable. In Section 4 we show that many conclusions are robust to changes in T. Namely, intuitive projections and embeddings of stable outcomes are possible as T changes. Such conclusions are not immediate since, for instance, adjusting a market's time horizon increases/decreases the number of blocking opportunities. Second, in normative analyses, the market's time horizon provides a new design variable open to refinement and fine-tuning. Understanding the consequences of more or fewer re-matching opportunities, for instance, can help guide design.

We view our focus on the above questions as complementary and providing a preliminary tool-kit for future applications. Both are relevant for analyses of markets where the time horizon is long but agents' commitment ability is limited. As we discuss before the conclusion, specific applications can include studies of student-school or job-worker assignments.

Immediately below we briefly survey the related literature. For brevity of exposition, we relegate many proofs to Appendix A. Appendix B is available as an online supplement.

Related Literature Dynamic models occupy a nook within the expansive literature following Gale and Shapley (1962). Some papers, of which Kurino (2014) is a recent example, consider one-sided problems. In contrast, we consider a two-sided market where agents on both sides of the market, men and women in our nomenclature, have preferences over their partner's identity. Specific applications have motivated previous studies in this vein. For example, Kennes et al. (2014a) study the assignment of children to Danish daycares while Dur (2012) and Pereyra (2013) consider school-choice applications. Though these models incorporate features that are absent from our analysis, our model is not a special case of any of them.[8]

As our model generalizes Gale and Shapley's (1962) original analysis, we maintain their pairwise focus in our preferred stability concept, which we term dynamic stability. Admittedly there are multiple plausible definitions of stability for dynamic matching economies. Some authors, such as Damiano and Lam (2005) and Kurino (2009), favor coalition-based

[8]The two-period case our model satisfies the rankability condition of Kennes et al. (2014a). However, the asymmetry in the model of Kennes et al. (2014a) imposes a more restrictive condition on some agents' preferences. Hence, our model is not subsumed by their analysis. Furthermore, Kadam and Kotowski (2014) show that a stable matching in the sense of Kennes et al. (2014a) may not satisfy our preferred stability notions. Thus, the stability concepts studied are also distinct.

definitions of stability with additional credibility qualifications. Independently, Doval (2014) also examines “dynamically-stable” matchings in her model, which differs from our usage of the term. As elaborated upon by Kadam and Kotowski (2014), the pairwise notions of stability used below are intuitive generalizations of well-known concepts and they provide an accessible benchmark for further analysis.

A link exists between a dynamic, one-to-one matching market and a static, many-to-many matching market. Every agent is matched to many partners, though in succession. Our analysis is not a special case of the most general treatments of many-to-many matching markets lodged in matching with contracts framework (Hatfield and Milgrom, 2005). Notably, our model does not satisfy the substitutability condition stressed by Hatfield and Kominers (2012).[9]

While dynamics have been underemphasized in the literature on two-sided matching following Gale and Shapley (1962), they are a central pillar of alternative examinations of bilateral markets. For example, the fact that workers shuffle between jobs has been formalized in the literature on matching within the search-theoretic paradigm. In the work of Mortensen and Pissarides (1994), to cite but one well-known example, workers move between employment and unemployment; job opportunities are created and destroyed. Micro-level change and churn are features of an economy’s steady state. We identify comparably volatile stable outcomes in our setting and we hope our analysis serves as a useful step toward integrating insights from both “matching” literatures.[10]

1 Model

Let M and W be finite, disjoint sets of agents—men and women, respectively—who interact over T periods. In every period, each man (woman) can be matched with one woman (man) or remain single. Following convention, a single agent is said to be “matched to themselves.” Thus, the set of potential partners for $m \in M$ is $W_m \equiv W \cup \{m\}$. $M_w \equiv M \cup \{w\}$ is the set of potential partners for $w \in W$. We use m’s and w’s to denote specific men and women when this distinction is helpful. Otherwise, i, j, k, l are generic agents.

Remark 1. For brevity, we define some concepts only from the perspective of a typical man.

[9] Hatfield and Kominers (2012, p. 3–4) illustrate the failure of (contract) substitutability in multi-period matching problems. When substitutability fails, they propose bundling complementary periods together into longer-term contracts. In our nomenclature, this entails focusing on ex ante stable outcomes. Ex ante stability is not our preferred solution concept as we emphasize limited commitment.

[10] The links between these literatures was recognized by Crawford and Knoer (1981, p. 437–8).

Our model is symmetric and all definitions apply to women with obvious notational changes.

Over a lifetime each agent encounters a sequence of partners, called a partnership plan. We denote a plan by $x = (x_1, x_2, \ldots, x_T)$ where x_t is the assigned partner in period t. When confusion is unlikely, we write $x = x_1 x_2 \cdots x_T$ with truncations or subsets of x denoted as follows:

$$\begin{array}{lll} x_{\leq t} = x_1 x_2 \cdots x_t & x_{\geq t} = x_t x_{t+1} \cdots x_T & x_{[t,t']} = x_t \cdots x_{t'} \\ x_{<t} = x_1 x_2 \cdots x_{t-1} & x_{>t} = x_{t+1} x_{t+2} \cdots x_T & x_{(t,t')} = x_{t+1} \cdots x_{t'-1} \end{array}$$

Often we combine and remix the above notations, i.e. $x = (x_{<t}, x_{[t,t']}, x_{>t'})$.[11] A constant plan is written as $\bar{i} \equiv (i, i, \ldots, i)$.

Each agent i has a strict, rational preference $\succ_i$ defined over feasible partnership plans. If i prefers plan x to plan y, we write $x \succ_i y$. If $x \succ_i y$ or $x = y$, then $x \succsim_i y$. We occasionally summarize $\succ_i$ by listing plans in preferred order, i.e. $\succ_i \colon x, y, z, \ldots$.

A *matching* $\mu \colon M \cup W \to (M \cup W)^T$ assigns a partnership plan to each agent. It is a function comprising of T one-period matchings.

Definition 1. $\mu_t \colon M \cup W \to M \cup W$ is a *one-period matching (for period t)* if (i) for all $m \in M$, $\mu_t(m) \in W_m$; (ii) for all $w \in W$, $\mu_t(w) \in M_w$; and, (iii) for all i, $\mu_t(i) = j \implies \mu_t(j) = i$.

Thus, $\mu(i) = (\mu_1(i), \ldots, \mu_T(i))$ is i's plan under the matching μ.

A matching is stable if it cannot be blocked by any agent or pair. We first define admissible blocking actions. Thereafter we propose two definitions of stability.

Definition 2. Agent i can *period-t block* the matching μ if $(\mu_{<t}(i), \bar{i}_{\geq t}) \succ_i \mu(i)$.

At period t, $(\mu_{<t}(i), \bar{i}_{\geq t})$ is the best outcome that agent i can guarantee himself independent of others' behavior or of the market's future development. The matching μ is ex ante individually rational if it cannot be period-1 blocked by i. If i cannot period-t block μ for all t, then it is dynamically individually rational.

A pair can block a matching in period t if they can form a (continuation) partnership plan only among themselves for periods $t, t+1, t+2, \ldots$ that they both prefer given the elapsed history. Such a plan involves a sequence of arrangements only among the blocking pair.

[11] By convention, $x_{<1} = \emptyset$ and $x_{>T} = \emptyset$.

Definition 3. $\mu_t^{\{m,w\}}\colon \{m,w\} \to \{m,w\}$ is a *one-period matching (for period t) among* $\{m,w\}$ if (i) $\mu_t^{\{m,w\}}(m) = w$ and $\mu_t^{\{m,w\}}(w) = m$; or, (ii) $\mu_t^{\{m,w\}}(m) = m$ and $\mu_t^{\{m,w\}}(w) = w$.

Definition 4. $\{m,w\}$ can *period-t block* the matching μ if there exists a matching among $\{m,w\}$, $\mu^{\{m,w\}} = (\mu_1^{\{m,w\}}, \ldots, \mu_T^{\{m,w\}})$, such that $(\mu_{<t}(i), \mu_{\geq t}^{\{m,w\}}(i)) \succ_i \mu(i)$ for all $i \in \{m,w\}$.

All else equal, when agents have more blocking opportunities supporting a stable outcome becomes more difficult. Thus, we introduce two definitions of stability that bracket all others.

Definition 5. The matching μ is *ex ante stable* if it cannot be period-1 blocked by any agent or by any pair. The matching μ is *dynamically stable* if it cannot be period-t blocked by any agent or by any pair for all t.

Kadam and Kotowski (2014) analyze ex ante and dynamically-stable matchings in a two-period model and they provide additional context and motivation for these definitions. Ex ante stability is appropriate when agents can credibly commit to a proposed partnership plan. Ex ante stable matchings always exist (see Appendix B) and they provide a preliminary, if weak, benchmark for any multi-period matching market. With a longer time horizon, as assumed here, we would like to refine the set of ex ante stable matchings by deemphasizing the role of commitment to long-term plans. Thus, dynamic stability is our preferred solution concept. It allows for a richer, more realistic set of dynamic blocking opportunities.

Before examining dynamically-stable matchings, we briefly comment on stability notions in multi-period bilateral matching markets. Above we noted that other studies favor coalition-based definitions of stability. Ours are pair-wise notions though their coalition-based extensions are straightforward.[12] Another class of definitions admits additional blocking actions. For instance, dynamic stability does not allow m and w to form a one-period partnership and then to return to the original matching, μ, as if nothing ever happened. A moment of reflection suggests that this may be a fanciful outcome. A priori there is no reason to presume that m's and w's partners under μ will welcome their return after an unexpected, temporary deviation or departure. Preferences need not be time separable and a deviation may trigger a cascade of further deviations and a market realignment.[13] Dynamic stability posits that agents evaluate a blocking opportunity conjecturing a worst-case outcome in

[12]Replacing $\{m,w\}$ in Definitions 4 and 5 with a coalition $C \subset M \cup W$ leads to definitions of the ex ante and dynamic cores (Kadam and Kotowski, 2014). Conditions ensuring the non-emptiness of the core are more restrictive than those proposed in this study.

[13]Matters hinge on what non-deviating agents can do in the interim and how their preferences evolve conditional on the deviation. Damiano and Lam (2005) propose the concept of strict self-sustaining stability ("S^4") which allows agents to return to an original plan following a temporary deviation. However, they acknowledge that this presumes a non-deviating agent accepts a deviator's return.

which they cannot reliably return to the market as a whole and may only arrange a continuation plan among themselves. Thus, agents do not need to rely on the accommodative behavior of others to determine whether a blocking plan is worthwhile. As several negative results hinge on our definition, they continue to apply if more blocking opportunities are allowed.

2 Characteristics of Dynamically-Stable Matchings

When $T = 1$, our model reduces to Gale and Shapley's (1962) analysis. Hence, their deferred acceptance algorithm identifies a stable matching. As we call upon this algorithm below, we summarize it here.

Definition 6 (DA)**.** The *(one-period, man-proposing[14]) deferred acceptance* algorithm identifies a matching μ^* as follows:

1. For each $m \in M$, let $X_m^0 = W$. Initially, no partners in X_m^0 have been rejected.
2. In round $t \geq 1$:
   (a) Let $X_m^t \subset X_m^0$ be the subset of women who have not rejected m in some round $t' < t$. If $X_m^t = \emptyset$ or $m \succ_m w$ for all $w \in X_m^t$, then m does not make any proposals. Otherwise, m proposes to his most-preferred potential partner in X_m^t.
   (b) Let X_w^t be the set of proposals received by w. If $w \succ_w m$ for all $m \in X_w^t$, w rejects all proposals. Otherwise, w (tentatively) accepts her most preferred suitor and rejects the others.
3. The above process continues until no rejections occur. If w accepts m's proposal in the final round, define $\mu^*(m) = w$ and $\mu^*(w) = m$. If i does not make/receive any proposals in the final round, set $\mu^*(i) = i$.

Once $T \geq 2$, the existence of a dynamically-stable matching is not guaranteed.[15] To ensure existence, Kadam and Kotowski (2014) restrict agents' preferences by introducing a condition called weak sequential complementarity. In the two-period case, a tractable and behaviorally-relevant class of preferences satisfy this requirement. This class satisfies the

[14] The woman-proposing deferred acceptance algorithm is defined analogously.

[15] Kadam and Kotowski (2014) propose the following example. The man's preferences are $\succ_m: wm, ww, mm, \ldots$ and the woman's preferences are $\succ_w: mm, ww, \ldots$.

rankability condition of Kennes et al. (2014a) and Kadam and Kotowski (2014) show that it merges a ranking of potential partners and a slight bias for more-persistent plans, which we term preference inertia.[16] Below we develop the T-period generalization of this class of preferences. As we explain below, this class affords sufficient tractability to investigate our questions of interest while remaining sufficiently rich for applications.

2.1 Spot Rankings

Imagine an agent holds a strict ranking of potential partners abstracting from all dynamic considerations. We call such a ranking a *spot ranking* and we denote it by P_i. If j ranks above k, we write jP_ik. If jP_ik or $j = k$, then jR_ik. The following definition proposes a link between P_i and the preference $\succ_i$.

Definition 7. The preference $\succ_i$ *reflects* P_i if for all x and t,

$$jP_ik \iff (x_1, \ldots, x_{t-1}, j, x_{t+1}, \ldots, x_T) \succ_i (x_1, \ldots, x_{t-1}, k, x_{t+1}, \ldots, x_T).$$

Let $\mathcal{S}_i$ be the set of preferences for agent i that reflect a spot ranking.

When $\succ_i \in \mathcal{S}_i$, i prefers the plan with the higher-ranked partner if two plans differ only in their assignment for one period. Of course, many distinct preferences may reflect the same spot ranking. Definition 7 is similar to the commonly-encountered assumption that preferences are "responsive" introduced by Roth (1985a); however, it is additionally sensitive to the order of encountered partners.

2.2 Preference Inertia

Though $\mathcal{S}_i$ is an appealing class of preferences, it precludes many plausible situations. For example, if kP_ij and $\succ_i \in \mathcal{S}_i$, then agent i must believe that $jkj \succ_i jjj$. However, status-quo bias due to explicit or implicit switching costs may tilt i's preference toward the jjj plan in lieu of plans incorporating many changes (Samuelson and Zeckhauser, 1988). In practice, plans economizing on switching may be held in relative favor even if the assignments are intrinsically less desirable in some time-independent sense.

To address the associated nuances, we first introduce a vocabulary describing partnership plan variability. The plan $x = x_1 \cdots x_T$ is *maximally persistent* if $x_t = x_{t'}$ for all t and t'.

[16]Kennes et al. (2014a) define rankability only for two periods.

Otherwise it is *volatile*. It is *maximally volatile* if $x_t \neq x_{t+1}$ for all t. Finally, the following comparison of plans' relative persistence will prove crucial.

Definition 8. The plan x is *more persistent than* plan y, denoted as $x \trianglerighteq y$, if for all $t \leq t'$, $y_t = \cdots = y_{t'} \implies x_t = \cdots = x_{t'}$.

Definition 8 defines a preorder of partnership plans. It is reflexive ($x \trianglerighteq x$) and transitive ($x \trianglerighteq y, y \trianglerighteq z \implies x \trianglerighteq z$), but not anti-symmetric. If $x \trianglerighteq y$, $y \trianglerighteq x$, but $x \neq y$, then x *is equally persistent to* y and we write $x \bowtie y$. $x \triangleright y$ when $x \trianglerighteq y$ and $y \ntrianglerighteq x$. Finally, we write $x \parallel y$ if x and y are not $\trianglerighteq$-comparable. To illustrate, $iij \triangleright ijk$, $iij \bowtie jjk$, and $iij \parallel ijj$.

To model a bias toward more persistent plans, we allow such plans to rise in an agent's preference ranking relative to a benchmark preference relation.

Definition 9. The preference $\succ_i$ *exhibits inertia relative to* $\succ'_i$ if

1. $x \succ'_i y$ and $x \trianglerighteq y \implies x \succ_i y$; and,
2. for all x and y such that $x \parallel y$, $x \succ'_i y \iff x \succ_i y$.

Let $\Upsilon(\succ'_i)$ be the set of preferences that exhibit inertia relative to $\succ'_i$.

Condition 2 in Definition 9 says that the relative rankings of non-comparable plans does not change vis-à-vis to the $\succ'_i$ baseline. A similar conclusion to equally-persistent plans. The proof of the following lemma is in Appendix A.

Lemma 1. *If* $\succ_i \in \Upsilon(\succ'_i)$ *and* $x \bowtie y$, *then* $x \succ'_i y \iff x \succ_i y$.

The following example illustrates the preceding definition and lemma.

Example 1. Suppose $W = \{w_1, w_2\}$ and consider the following preferences for $m \in M$:

$$\begin{aligned}
&\succ_m : \underline{w_1w_1w_1w_2}, w_1w_1w_2w_2, \boxed{w_2w_2w_1w_1}, w_1w_2mw_1, w_1mmw_1, \ldots \\
&\succ'_m : w_1w_1w_2w_2, w_1w_2mw_1, \underline{w_1w_1w_1w_2}, \boxed{w_2w_2w_1w_1}, w_1mmw_1, \ldots
\end{aligned}$$

$\succ_m$ exhibits inertia relative to $\succ'_m$. The plans $\underline{w_1w_1w_1w_2}$ and $\boxed{w_2w_2w_1w_1}$ advanced in the preference list. Since $\boxed{w_2w_2w_1w_1} \bowtie w_1w_1w_2w_2$, these plans' relative ranking under $\succ_m$ and $\succ'_m$ is the same.

2.3 Dynamically Stable Matchings

Formally, the ideas of a spot ranking and preference inertia are independent. However, together they combine to define a tractable family of preferences. Specifically, given $\mathcal{S}_i$ we can define the set

$$\bar{\mathcal{S}}_i = \bigcup_{\succ'_i \in \mathcal{S}_i} \Upsilon(\succ'_i).$$

As shown by Theorem 1 below, when $\succ_i \in \bar{\mathcal{S}}_i$ for all i, a dynamically stable matching exists. Theorem 1's proof draws on a technical lemma that we prove in Appendix A.

Lemma 2. *Suppose the matching μ is dynamically individually rational for $m \in M$ and $w \in W$. Suppose this couple can period-t block μ. If $\succ_m \in \bar{\mathcal{S}}_m$ and $\succ_w \in \bar{\mathcal{S}}_w$, then $(\mu_{<t}(m), \bar{w}_{\geq t}) \succ_m \mu(m)$ and $(\mu_{<t}(w), \bar{m}_{\geq t}) \succ_w \mu(w)$.*

Theorem 1. *If $\succ_i \in \bar{\mathcal{S}}_i$ for all i, there exists a dynamically-stable matching.*

Proof. We construct a dynamically-stable matching using the *(man-proposing) ex ante deferred acceptance* (E-DA) procedure. Kadam and Kotowski (2014) study the same procedure in their two-period model. To specify this procedure, define the *ex ante spot ranking induced by $\succ_i$,*[17] denoted $P_{\succ_i}$, as $jP_{\succ_i}k \iff \bar{j} \succ_i \bar{k}$. If $\succ_i \in \Upsilon(\succ'_i)$, then $P_{\succ_i} = P_{\succ'_i}$. The E-DA procedure defines the matching μ^* as follows: For each t, μ^*_t is the one-period matching identified by the (man-proposing) DA algorithm when each agent i makes/accepts proposals according to his/her ex ante spot ranking induced by $\succ_i$, $P_{\succ_i}$. Below, we show that μ^* is dynamically stable.

First, we verify that μ^* cannot be period-t blocked by i alone. Suppose the contrary. By Lemma A.2 agent i can period-1 block μ^*, i.e. $\bar{i} \succ_i \mu^*(i)$. But this implies $iP_{\succ_i}\mu^*_t(i)$. Thus, the one-period deferred acceptance algorithm assigned i to a partner whom he like less than being single—a contradiction.

It remains to verify that no pair of agents can period-t block μ^*. Again we argue by contradiction. Suppose m and w can period-t block μ^*. By Lemma 2, this implies $\tilde{\mu}(m) = (\mu^*_{<t}(m), \bar{w}_{\geq t}) \succ_m \mu^*(m)$ and $\tilde{\mu}(w) = (\mu^*_{<t}(w), \bar{m}_{\geq t}) \succ_w \mu^*(w)$. Since $\succ_m \in \bar{\mathcal{S}}_m$, $\succ_m \in \Upsilon(\succ'_m)$ for some $\succ'_m \in \mathcal{S}_m$. As $\mu^*(m) \trianglerighteq \tilde{\mu}(m)$ and $\tilde{\mu}(m) \succ_m \mu^*(m)$, it follows that $\tilde{\mu}(m) \succ'_m \mu^*(m)$. This implies $wP_{\succ'_m}\mu^*_t(m)$ and thus, $wP_{\succ_m}\mu^*_t(m)$. Likewise, $mP_{\succ_w}\mu^*_t(w)$. However, this implies m and w can block the one-period matching identified by the deferred acceptance

[17] Kennes et al. (2014a) introduce a similar concept that they call the isolated preference relation. They use it to define their DA-IP mechanism. The DA-IP mechanism may generate dynamically-unstable outcomes when $\succ_i \in \bar{\mathcal{S}}_i$ for all i (Kadam and Kotowski, 2014).

algorithm when agents' preferences are given by $P_{\succ_i}$. This is a contradiction as the deferred acceptance algorithm identifies a stable one-period matching. As μ^* cannot be period-t blocked by any agent or by any pair, it is dynamically stable. $\square$

As noted above, the E-DA procedure was initially studied in a two-period setting. The procedure's continued effectiveness at identifying a dynamically-stable matching suggests that our T-period preference generalization retains the relevant stability-assuring characteristics. Furthermore, it is reassuring that a simple procedure leads to at least one stable matching in an otherwise complex setting. (As apparent from many examples to follow, communicating non-time separable preferences over very long partnership plans may be difficult.)

Technical similarities notwithstanding, there are notable economic differences between the $T = 2$ and $T > 2$ cases as illustrated by the following example.

Example 2. Suppose $M = \{m_1, m_2\}$, $W = \{w_1, w_2\}$, and $T = 3$. Table 1 summarizes agents' preferences, which satisfy the $\succ_i \in \bar{\mathcal{S}}_i$ restriction. Table 2 lists all dynamically-stable matchings in this economy. To read this table, $\mu^1(m_1) = w_1w_1w_1$. The man- and woman-proposing variants of the E-DA procedure identify the μ^1 matching.

Table 1: Agent's preferences in Example 2.

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{w_1}$	$\succ_{w_2}$
$w_1w_1w_1$	$w_2w_2w_2$	$m_2m_2m_2$	$m_1m_1m_1$
$w_1m_1w_1$	$w_2m_2w_2$	$m_1m_2m_2$	$m_2m_1m_1$
$w_1w_2w_1$	$w_2w_1w_2$	$m_2m_2m_1$	$m_1m_1m_2$
$m_1m_1w_1$	$m_2m_2w_2$	$m_1m_2m_1$	$m_2m_1m_2$
$w_1m_1m_1$	$w_2m_2m_2$	$w_1m_2m_2$	$w_2m_1m_1$
$m_1w_2w_1$	$m_2w_1w_2$	$m_2m_2w_1$	$m_1m_1w_2$
$m_1m_1m_1$	$m_2m_2m_2$	$w_1m_2m_1$	$w_2m_1m_2$
		$m_1m_1m_1$	$m_2m_2m_2$
		$w_1w_1w_1$	$w_2w_2w_2$

To conclude this section, we make four observations motivated by Example 2. *First, maximally-volatile plans can be dynamically-stable, even when agents' preferences exhibit inertia.* In fact, agents may prefer volatile outcomes among all dynamically-stable matchings. For instance, w_1 and w_2 prefer μ^2 among the stable set. This observation may be surprising as preference inertia seemingly nudges agents to prefer more persistent plans.

Table 2: All dynamically-stable matchings in Example 2.

Matching	m_1	m_2	w_1	w_2
μ^1	$w_1w_1w_1$	$w_2w_2w_2$	$m_1m_1m_1$	$m_2m_2m_2$
μ^2	$w_1w_2w_1$	$w_2w_1w_2$	$m_1m_2m_1$	$m_2m_1m_2$
μ^3	$m_1w_2w_1$	$m_2w_1w_2$	$w_1m_2m_1$	$w_2m_1m_2$

Second, a stable matching may incorporate a period of sacrifice. In Example 2, μ^2 and μ^3 assign both men to a partner whom they rank *below* single-hood in an ex ante sense: $m_1 P_{\succ_{m_1}} w_2$ and $m_2 P_{\succ_{m_2}} w_1$. Both men are rewarded with a highly-ranked partner in the final period. Thus, with a long enough time horizon costly actions can be adequately incentivized as part of a stable outcome. Importantly, "per-period" characterizations of individual rationality may fail to capture the importance of inter-temporal linkages.

Third, stable outcomes may include periods of temporary single-hood. The matching μ^3 in Example 2 has the surprising property that *all* agents are unmatched in period 1 but matched in later periods. This outcome is not possible when $T = 2$.

Theorem 2. *Suppose $\succ_i \in \bar{\mathcal{S}}_i$ for all i. Let μ be a dynamically-stable matching when $T = 2$. Then agent i either has a partner in each period or remains unmatched in both periods.*

Proof. See Appendix A. □

Example 2 and Theorem 2 qualify Roth's (1986) rural hospital theorem. When the time horizon is short ($T \leq 2$), Roth's (1986) conclusion applies, but the relationship breaks down when $T \geq 3$.[18] Example 2 shows that if an agent is unmatched in some period in one dynamically-stable matching, s/he may be matched in that period in some other dynamically-stable matching.

It is also instructive to relate the preceding observations to contemporary labor markets. With this interpretation, the μ^3 matching exhibits a simultaneous bout of temporary unemployment when all agents are unmatched. One interpretation for this outcome is that of a business cycle due to market mis-coordination. Furthermore, institutional features implicit in our definition of blocking serve to reinforce this outcome. For example, m_1 and w_1 would prefer to be partnered in period 1, but they cannot period-1 block μ^3. w_1 is not keen on the long-term relationship with m_1 that period-1 blocking implies (Lemma 2). Though stylized,

[18] Of course, this qualification applies to the case where $\succ_i \in \bar{\mathcal{S}}_i$ for all i.

the implicit persistence of blocking actions captures some common labor-market rigidities, such as those presently found in many European counties.

Finally, dynamically-stable matchings may not be Pareto optimal.[19] When $T = 1$, stable matchings are Pareto optimal (Gale and Shapley, 1962). Example 2 shows that this conclusion is not true at a longer time horizon.[20] For all i, $\mu^2(i) \succ_i \mu^3(i)$. Several peculiarities of this example—such as the temporary single-hood in μ^3—suggest that non-Pareto optimal dynamically-stable matchings occur under unusual circumstances. Such outcomes can occur in more ordinary cases as well.

Theorem 3. *If $T \geq 2$ and $\succ_i \in \bar{\mathcal{S}}_i$ for all i, a dynamically-stable matching may be Pareto-dominated by another dynamically-stable matching.*

Proof. See Example 5. $\square$

3 Welfare and Conflicting Interests

Conventional wisdom suggests that agents on different sides of a market have opposing interests. A seller prefers a price hike while a buyer desires a discount. Similarly, agents on the same side of the market compete for lucrative trading opportunities. A competitor's success is often interpreted as one's own failure. As markets mediate such conflicts, these intuitions have been thoroughly investigated in an array of matching models (Roth, 1984, 1985b). Their extent and intensity in dynamic matching markets remains an open question.

To describe agents' collective interests, we let $\succsim_M$ be the men's *common preference.* Given the matchings μ and μ', $\mu \succ_M \mu'$ if and only if $\mu(m) \succsim_m \mu'(m)$ for all $m \in M$ and $\mu(m) \succ_m \mu'(m)$ for some $m \in M$. If $\mu \succ_M \mu'$ or $\mu = \mu'$, then $\mu \succsim_M \mu'$. Women's common preference, $\succsim_W$, is defined analogously.

When $T = 1$, a surprisingly rich set of conclusions has been identified.[21]

1. There exists a conflict of interest among agents on opposing sides of the market. If men collectively prefer one stable matching over another, the market's women hold the opposite opinion, i.e. if μ and μ' are stable, then $\mu \succsim_M \mu' \iff \mu' \succsim_W \mu$ (Knuth, 1976; Roth, 1985b).

[19] As usual, we call a matching μ *(strongly) Pareto optimal* if there does not exist a matching μ' such that $\mu'(i) \succsim_i \mu(i)$ for all i and $\mu'(i) \succ_i \mu(i)$ for some i.

[20] In one-period many-to-many matching markets, stable matchings may not be Pareto optimal (Roth and Sotomayor, 1990, Proposition 5.23).

[21] Roth and Sotomayor (1990) provide an overview of these properties.

2. Agents on the same side of the market express consensus concerning which stable outcomes are preferable. If μ is a stable matching, it is *M-optimal* if for every other stable matching μ', $\mu \succsim_M \mu'$. When $T = 1$, the matching identified by the man-proposing deferred acceptance algorithm is M-optimal (Gale and Shapley, 1962).

3. Observations 1 and 2 stem from the stable set's lattice structure when ordered by $\succsim_M$ (attributed to John Conway by Knuth, 1976). A *lattice* is a partially-ordered set where any two elements have a greatest lower bound and a least upper bound (Birkhoff, 1940). Thus, welfare comparisons among stable matchings are relatively straightforward.[22]

Once $T \geq 2$, the above observations do not apply without further qualifications. Theorem 3 showed that a dynamically-stable matching may be Pareto-dominated by another dynamically-stable matching. Therefore, interests are no longer necessarily opposed. Moreover, as illustrated by the following example, a multi-period market may lack an M-optimal matching. Thus, the set of dynamically-stable matchings (when ordered by $\succsim_M$) cannot be a lattice.

Example 3. There are three men and women. Their preferences are:

$$\succ_{m_1}: w_2w_2, w_2w_1, w_2w_3, w_1w_1, w_3w_3, m_1m_1, \ldots$$
$$\succ_{m_2}: w_3w_3, w_3w_2, w_3w_1, w_2w_2, w_1w_1, m_2m_2, \ldots$$
$$\succ_{m_3}: w_1w_1, w_1w_3, w_3w_3, w_1w_2, w_2w_2, m_3m_3, \ldots$$

$$\succ_{w_1}: m_2m_2, m_1m_1, m_2m_1, m_1m_2, w_1m_2, m_3m_2, w_1w_1, m_3m_3, \ldots$$
$$\succ_{w_2}: m_3m_3, m_2m_2, m_3m_2, m_2m_3, w_2m_3, m_1m_3, w_2w_2, m_1m_1, \ldots$$
$$\succ_{w_3}: m_1m_1, m_1m_3, m_3m_1, w_3m_1, m_2m_1, m_3m_3, w_3w_3, m_2m_2, \ldots$$

In this market there are three dynamically-stable matchings (Table 3). m_1 and m_2 like μ^3 the most. m_3 prefers μ^1. Therefore, there does not exist an M-optimal stable matching. Furthermore, in many matching models, derivatives of the DA algorithm consistently pin-down an optimal matching (Roth, 1984). In this case, the E-DA procedure succeeds in identifying a dynamically-stable outcome, but that outcome may not be universally preferred by all men or women.

[22]Given the stable matchings μ and μ', there exists a stable matching μ'' such that $\mu'' \succsim_M \mu, \mu'$. Hence, men gain when the market moves to μ'' from μ or μ'. By observation 1, women lose from this transition.

Table 3: All dynamically-stable matchings in Example 3.

Matching	m_1	m_2	m_3	w_1	w_2	w_3
μ^1	w_1w_1	w_2w_2	w_3w_3	m_1m_1	m_2m_2	m_3m_3
μ^2	w_3w_3	w_1w_1	w_2w_2	m_2m_2	m_3m_3	m_1m_1
μ^3	w_2w_3	w_3w_1	w_1w_2	m_3m_2	m_1m_3	m_2m_1

To transform the preceding negative conclusions into positive claims we can either restrict attention to a subset of dynamically-stable matchings or further refine agents' preferences. In the analysis to follow, we do both beginning with the former.

3.1 Persistent Matchings and Lattice Structures

Fix an economy and let $\mathbb{D}$ be the set of dynamically-stable matchings. We sometimes write $(\mathbb{D}, \succsim_M)$ to emphasize this set's ordering by $\succsim_M$. The matching μ is maximally persistent if $\mu(i)$ is maximally persistent for each i. Let $\mathbb{P} \subset \mathbb{D}$ be the set of maximally-persistent, dynamically-stable matchings. There is a close connection between $\mathbb{P}$ and stable matchings in a one-period economy.

Lemma 3. *Suppose $\succ_i \in \bar{\mathcal{S}}_i$ for all i. Let μ be a maximally-persistent matching. The matching μ is dynamically-stable if and only if μ_t is a stable matching in a one-period market where agent i's preference coincides with $P_{\succ_i}$.*

The applicability of observations 1–3 on this restricted domain follows as a corollary.

Corollary 1. *Suppose $\succ_i \in \bar{\mathcal{S}}_i$ for all i. (i) $(\mathbb{P}, \succsim_M)$ is a lattice. (ii) The matching identified by the man-proposing E-DA procedure is M-optimal among matchings in $\mathbb{P}$. (iii) For all $\mu, \mu' \in \mathbb{P}$, $\mu \succsim_M \mu' \iff \mu' \succsim_W \mu$.*

Proof. By Lemma 3, each maximally-persistent, dynamically-stable matching in a T-period economy corresponds to a stable matching in a one-period economy and vice versa. The set of stable matchings in a one-period economy is a lattice when ordered by men's common preference (Knuth, 1976). Thus, $(\mathbb{P}, \succsim_M)$ is a lattice as well. Points (ii) and (iii) follow from Gale and Shapley (1962) and Knuth (1976), as noted in observations 1–3 above. $\square$

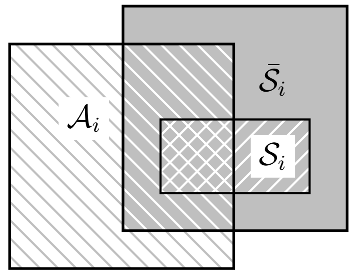

Figure 1: Preference domains. $\mathcal{S}_i$ – preferences that reflect a spot ranking; $\bar{\mathcal{S}}_i$ – preferences that exhibit inertia relative to $\mathcal{S}_i$; $\mathcal{A}_i$ – sacrifice averse preferences.

3.2 Volatile Matchings and Lattice Structures

To extend the preceding conclusions beyond $\mathbb{P}$ it is necessary to restrict agents' preferences. In light of Example 3, we first resolve the non-existence of an M-optimal matching. The restriction we propose is the following:

Definition 10. The preference $\succ_i$ is *sacrifice averse* if for all partnership plans x and $\bar{j}$, $x \succ_i \bar{j} \implies \overline{x_t} \succsim_i \bar{j}$. Let $\mathcal{A}_i$ be the set of preferences for i that are sacrifice averse.

Figure 1 sketches the relationships among $\mathcal{S}_i$, $\bar{\mathcal{S}}_i$, and $\mathcal{A}_i$. The following lemma rationalizes the "sacrifice averse" nomenclature though the condition's implications are broader.[23] In a stable matching, an agent never accepts a partner worse than being single. When $\succ_i \notin \mathcal{A}_i$, this may not be true (Example 2).

Lemma 4. *If $\mu \in \mathbb{D}$ and $\succ_i \in \bar{\mathcal{S}}_i \cap \mathcal{A}_i$, then $\mu_t(i) R_{\succ_i} i$ for all t.*

When we restrict agents' preferences to $\bar{\mathcal{S}}_i \cap \mathcal{A}_i$, the stable set gains added structure and an M-optimal stable matching exists.

Theorem 4. *Suppose $\succ_i \in \bar{\mathcal{S}}_i \cap \mathcal{A}_i$ for all i. The matching identified by the man-proposing E-DA procedure is M-optimal.*

We prove Theorem 4 as a corollary to Theorem 5 below, which characterizes the stable set's lattice structure.

[23]In words the condition can read as follows: If a programmer prefers successive, short-term contracts at Google and then at Facebook in lieu of a stable long-term job at Microsoft, then he must prefer long-term employment at Google and Facebook over long-term employment at Microsoft. Examples 4 and 5 show that this condition *does not* preclude volatile dynamically stable matchings.

Given Theorem 4, a tempting conjecture is that when $\succ_i \in \bar{\mathcal{S}}_i \cap \mathcal{A}_i$ for all i, $(\mathbb{D}, \succsim_M)$ is a lattice. Surprisingly, this is need not be the case.

Example 4. Suppose there are two men and two women. Their preferences are:

$$\succ_{m_1}: w_1w_1w_1, w_2w_1w_1, w_1w_1w_2, w_1w_2w_2, w_2w_2w_1, w_2w_2w_2, \ldots$$
$$\succ_{m_2}: w_2w_2w_2, w_2w_2w_1, w_1w_2w_2, w_1w_1w_2, w_2w_1w_1, w_1w_1w_1, \ldots$$

$$\succ_{w_1}: m_2m_2m_2, m_1m_2m_2, m_2m_2m_1, m_2m_1m_1, m_1m_1m_2, m_1m_1m_1, \ldots$$
$$\succ_{w_2}: m_1m_1m_1, m_1m_1m_2, m_2m_1m_1, m_2m_2m_1, m_1m_2m_2, m_2m_2m_2, \ldots$$

This market has six dynamically-stable matchings (Table 4). $(\mathbb{D}, \succsim_M)$ is not a lattice since μ^4 and μ^5 do not have a least upper bound (Figure 2).

Table 4: All dynamically-stable matchings in Example 4.

Matching	m_1	m_2	w_1	w_2
μ^1	$w_1w_1w_1$	$w_2w_2w_2$	$m_1m_1m_1$	$m_2m_2m_2$
μ^2	$w_2w_1w_1$	$w_1w_2w_2$	$m_2m_1m_1$	$m_1m_2m_2$
μ^3	$w_1w_1w_2$	$w_2w_2w_1$	$m_1m_1m_2$	$m_2m_2m_1$
μ^4	$w_1w_2w_2$	$w_2w_1w_1$	$m_1m_2m_2$	$m_2m_1m_1$
μ^5	$w_2w_2w_1$	$w_1w_1w_2$	$m_2m_2m_1$	$m_1m_1m_2$
μ^6	$w_2w_2w_2$	$w_1w_1w_1$	$m_2m_2m_2$	$m_1m_1m_1$

Though further restrictions precluding cases like Example 4 can be proposed, we will instead pursue a different vein.[24] Like Blair (1988), who identifies a lattice structure in a many-to-many matching market, we will arrange $\mathbb{D}$ with a weaker partial order. importantly, the new order has a behavioral foundation and it continues to facilitate meaningful welfare analysis. We define it by first weakening each agent's preference over partnership plans.

Definition 11. Agent i *decisively prefers plan x to plan y*, denoted $x \succ_i^* y$, if (i) $x \succ_i y$ and (ii) $\overline{x_t} \succsim_i \overline{y_{t'}}$ for all t and t'. We write $x \succsim_i^* y$ when $x = y$ or $x \succ_i^* y$.

Though we use $\succsim_i^*$ below to define a common preference for each side of the market, at least two additional motivations support its introduction. Both are justifications for the usefulness of incomplete preferences, like $\succsim_i^*$, in decision analysis.

[24] Kadam and Kotowski (2014) propose a restriction called "strong inertia." It implies that $\mathbb{D} = \mathbb{P}$.

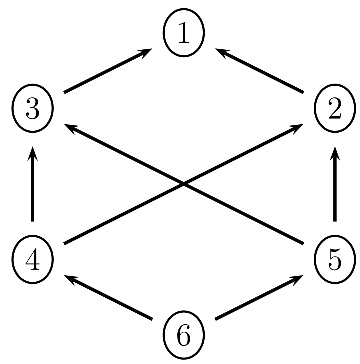

Figure 2: Hasse diagram of $(\mathbb{D}, \succsim_M)$ in Example 4.

First, and stepping back from the assumption that $\succ_i$ describes i's true preferences, we can interpret $\succsim_i^*$ to be i's best ranking of available plans after some reflection, but he retains some indecisiveness among similar options (Aumann, 1962). Notably, $\succsim_i^*$ provides a complete ranking of maximally-persistent plans, which seems plausible. However, the agent's arbitration among volatile plans is more clouded. To illustrate, take $m \in M$ and suppose that

$$w_1w_1 \succ_m w_1w_2 \succ_m w_2w_1 \succ_m w_2w_2 \succ_m w_1w_3 \succ_m w_3w_3 \succ_m w_2w_3. \tag{1}$$

Figure 3 illustrates $\succsim_m^*$ as derived from (1). In this case, m is certain that w_1w_1 dominates w_2w_2, which dominates w_3w_3. However, w_1w_2, w_2w_1, and w_1w_3 are more difficult to compare. Interestingly, $\succsim_m^*$ does not rank w_2w_3 "in between" w_2w_2 and w_3w_3. Thus, $\succsim_m^*$ accounts for preference inertia—w_2w_3 incorporates an assignment to a better partner but w_3w_3 is more persistent. In this case, m is unsure which is better. When an agent is only certain of $\succsim_i^*$, we may interpret $\succ_i$ to be a complete ranking of available outcomes reported by i when pressed for such a response.[25]

While the first case places $\succsim_i^*$'s origin with the agent, an alternative interpretation casts $\succsim_i^*$ as an outside observer's best guess concerning i's preferences (Ok, 2002). The observer is unaware of i's refined opinions given by $\succ_i$ but may be able to use $\succsim_i^*$ as a conservative benchmark for market analysis.

Whether $\succsim_i^*$ is motivated by behavioral concerns or analytic limitations, it extends to an ordering of matchings in the usual way: $\mu \succ_M^* \mu'$ if and only if $\mu(m) \succsim_m^* \mu'(m)$ for all

[25]In this case, stable outcomes would be stable with respect to reported preferences.

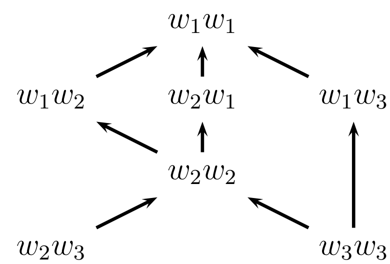

Figure 3: The $\succsim^*_m$ order derived from (1).

$m \in M$ and $\mu(m) \succ^*_m \mu'(m)$ for some $m \in M$. The associated weak relation, $\succsim^*_M$, and the corresponding common preference for women, $\succsim^*_W$, are defined as expected.

Below, Theorem 5 shows that $(\mathbb{D}, \succsim^*_M)$ is a lattice. To shorten the exposition, we precede the argument with six preliminary lemmas, which we prove in Appendix A. Lemmas 5–7 focus on agents' true preferences, $\succ_i$. Of these claims, Lemma 5 is of independent interest. Coupled with Lemma 3, it says that when $\succ_i \in \bar{\mathcal{S}}_i$, the final period assignment in a dynamically-stable matching is a stable assignment in a corresponding one-period economy. This conclusion is similar to the requirement that agents' final-period actions in a Nash equilibrium of a finitely repeated game are also equilibrium actions in the constituent stage game. Lemmas 8–10 focus on decisive preferences and the $\succsim^*_M$ / $\succsim^*_W$ orderings. Lemma 8 shows that $\succsim_M$ and $\succsim^*_M$ coincide on $\mathbb{P}$. Lemma 9 shows that every dynamically-stable matching is dominated by a dynamically-stable matching in $\mathbb{P}$. Finally, Lemma 10 shows that $\succsim^*_W$ is the inverse of $\succsim^*_M$ when restricted to $\mathbb{D}$. Thus, $\succsim^*_M$ and $\succsim^*_W$ characterize the conflict of interest between the market's two sides.

Lemma 5. *Suppose $\succ_i \in \bar{\mathcal{S}}_i$ for all i. If $\mu \in \mathbb{D}$, then $\overline{\mu_T} = (\mu_T, \ldots, \mu_T) \in \mathbb{D}$.*

Lemma 6. *Suppose $\succ_i \in \bar{\mathcal{S}}_i \cap \mathcal{A}_i$ for all i. If $\mu \in \mathbb{D}$, then $\overline{\mu_t} = (\mu_t, \ldots, \mu_t) \in \mathbb{D}$.*

Lemma 7. *Suppose $\succ_i \in \bar{\mathcal{S}}_i \cap \mathcal{A}_i$ for all i. Let $\mu \in \mathbb{D}$. (i) If $\mu(i)$ is volatile, there exists a period $t(i)$ such that $\mu(i) \succ_i \overline{\mu_{t(i)}(i)}$. (ii) If $\mu(i) \succ_i \overline{\mu_t(i)}$, then for all t' such that $\mu_{t'}(i) \neq \mu_t(i)$, $\overline{\mu_{t'}(i)} \succ_i \mu(i)$.*

Lemma 8. *Let μ and μ' be maximally-persistent matchings. For all i, $\mu(i) \succsim_i \mu'(i) \iff \mu(i) \succsim^*_i \mu'(i)$ and $\mu \succsim_M \mu' \iff \mu \succsim^*_M \mu'$. Therefore, $(\mathbb{P}, \succsim_M) = (\mathbb{P}, \succsim^*_M)$.*

Lemma 9. *Suppose* $\succ_i \in \bar{\mathcal{S}}_i \cap \mathcal{A}_i$ *for all* i. *If* $\mu \in \mathbb{D}$, $\exists\ \mu' \in \mathbb{P} \subset \mathbb{D}$ *such that* $\mu' \succsim_M^* \mu$.

Lemma 10. *Suppose* $\succ_i \in \bar{\mathcal{S}}_i \cap \mathcal{A}_i$ *for all* i. *Let* $\mu, \mu' \in \mathbb{D}$. $\mu \succ_M^* \mu' \iff \mu' \succ_W^* \mu$.

The lattice structure of $(\mathbb{D}, \succsim_M^*)$ follows from the preceding results.

Theorem 5. *Suppose* $\succ_i \in \bar{\mathcal{S}}_i \cap \mathcal{A}_i$ *for all* i. $(\mathbb{D}, \succsim_M^*)$ *is a lattice.*

Proof. By Lemma 10, $\succsim_W^*$ is the inverse of $\succsim_M^*$ on $\mathbb{D}$. Thus, to verify that $(\mathbb{D}, \succsim_M^*)$ is a lattice it is sufficient to show that there exists a least upper bound for any two dynamically-stable matchings.

Let $\mu, \mu' \in \mathbb{D}$. If $\mu \succ_M^* \mu'$, then μ is the least upper bound for μ and μ'. Instead, suppose μ and μ' are not ordered by $\succsim_M^*$. From Lemma 9 we know that μ and μ' are bounded above by some $\mu^1, \mu^2 \in \mathbb{D}$, i.e.

$$\left.\begin{matrix}\mu^1\\ \mu^2\end{matrix}\right\} \succ_M^* \left\{\begin{matrix}\mu\\ \mu'\end{matrix}\right. .$$

It is sufficient to show that $\exists\ \tilde{\mu} \in \mathbb{D}$ such that

$$\left.\begin{matrix}\mu^1\\ \mu^2\end{matrix}\right\} \succsim_M^* \tilde{\mu} \succ_M^* \left\{\begin{matrix}\mu\\ \mu'\end{matrix}\right. .$$

Let $\mathbb{P}(\mu^k) = \{\overline{\mu_t^k} \colon t = 1, \ldots, T\}$. By Lemma 6, $\overline{\mu_t^k} \in \mathbb{P}$. Since $(\mathbb{P}, \succsim_M^*)$ is a lattice, $\mathbb{P}(\mu^1) \cup \mathbb{P}(\mu^2) \subset \mathbb{P}$ has a greatest lower bound in $(\mathbb{P}, \succsim_M^*)$. Let $\lambda \in \mathbb{P}$ be this greatest lower bound.

Next, we argue that for each m, $\mu^k \succsim_m^* \lambda$. There are two properties to check:

1. For each t and t', $\overline{\mu_t^k(m)} \succsim_m \overline{\lambda_{t'}(m)}$.

   $\overline{\mu_t^k} \in \mathbb{P}(\mu^k)$. Hence, $\overline{\mu_t^k} \succsim_M^* \lambda$. This implies $\overline{\mu_t^k(m)} \succsim_m^* \lambda(m) = \overline{\lambda_{t'}(m)}$ for all t' as $\lambda \in \mathbb{P}$.

2. $\mu^k(m) \succsim_m \lambda(m)$.

   Assume the contrary. Then $\lambda(m) \succ_m \mu^k(m)$. From part (1) above we know that for all t, $\overline{\mu_t^k(m)} \succsim_m \tilde{\mu}(m) \succ_m \mu^k(m)$. There are two possibilities. If $\mu^k(m) \in \mathbb{P}$, then $\mu^k(m) = \overline{\mu_t^k(m)}$ and we have a contradiction. If instead $\mu^k \notin \mathbb{P}$, then by Lemma 7 $\mu^k(m) \succ_m \overline{\mu_t^k(m)}$ for some t. This too is a contradiction. Hence, our assumption to the contrary was incorrect.

To complete the proof, we verify that $\lambda(m) \succsim_m^* \mu(m)$ for all m. (The case of μ' follows identically.) As a preliminary point we note that since $\mu^k \succsim_M^* \mu$, for all m, t, t', and k, $\overline{\mu_t^k(m)} \succsim_m \overline{\mu_{t'}(m)}$. Suppose that for some m, $\lambda(m) \not\succsim_m^* \mu(m)$. There are two possibilities:

1. For some t, $\overline{\mu_t(m)} \succ_m \lambda(m)$.

   Given the observation made immediately above, $\overline{\mu_t} \in \mathbb{P}$ is a lower bound for $\mathbb{P}(\mu^1) \cup \mathbb{P}(\mu^2)$. But λ is the greatest such lower bound. Therefore, this case cannot be true.

2. $\mu(m) \succ_m \lambda(m)$.

   If $\mu(m) \in \mathbb{P}$, then case (1) above applies. If $\mu(m) \notin \mathbb{P}$, then by Lemma 7 $\overline{\mu_t(m)} \succ_m \lambda(m)$ for some t. But again, this implies case (1) applies.

As neither case (1) nor case (2) applies, we conclude that in fact $\lambda(m) \succsim_m^* \mu(m)\ \forall m$. □

A corollary to Theorem 5 is that an M-optimal matching exists (Theorem 4). If $\mu \in \mathbb{D}$, then there exists $\mu^* \in \mathbb{P}$ such that $\mu^* \succsim_M^* \mu \implies \mu^*(m) \succsim_m^* \mu(m)\ \forall m \implies \mu^*(m) \succsim_m \mu(m)\ \forall m \implies \mu^*(m) \succsim_M \mu(m)$. Thus, the $\succsim_M$-maximal matching in $\mathbb{P}$, which is identified by the E-DA procedure, dominates all matchings in $\mathbb{D}$ and is M-optimal.

Example 5. Let $T = 2$. There are four men and women with the following preferences:

$$\succ_{m_1}: w_1w_1, w_4w_4, w_1w_4, w_4w_1, w_2w_1, w_2w_4, w_1w_2, w_2w_2, w_3w_3, \ldots$$
$$\succ_{m_2}: w_2w_2, w_3w_3, w_2w_3, w_3w_2, w_1w_2, w_1w_3, w_2w_1, w_1w_1, w_4w_4, \ldots$$
$$\succ_{m_3}: w_3w_3, w_2w_2, w_3w_2, w_2w_3, w_4w_3, w_4w_2, w_3w_4, w_4w_4, w_1w_1, \ldots$$
$$\succ_{m_4}: w_4w_4, w_1w_1, w_4w_1, w_1w_4, w_3w_4, w_3w_1, w_4w_3, w_3w_3, w_2w_2, \ldots$$

$$\succ_{w_1}: m_2m_2, m_3m_3, m_2m_3, m_3m_2, m_2m_1, m_3m_1, m_1m_2, m_1m_1, m_4m_4, \ldots$$
$$\succ_{w_2}: m_1m_1, m_4m_4, m_1m_4, m_4m_1, m_1m_2, m_4m_2, m_2m_1, m_2m_2, m_3m_3, \ldots$$
$$\succ_{w_3}: m_4m_4, m_1m_1, m_4m_1, m_1m_4, m_4m_3, m_1m_3, m_3m_4, m_3m_3, m_2m_2, \ldots$$
$$\succ_{w_4}: m_3m_3, m_2m_2, m_3m_2, m_2m_3, m_3m_4, m_2m_4, m_4m_3, m_4m_4, m_1m_1, \ldots$$

This market has 16 dynamically-stable matchings (Table 5). Figure 4 summarizes these matchings when ordered by $\succsim_M$ and by $\succsim_W$. $\succsim_W$ is not the inverse of $\succsim_M$ since a conflict of interest is not maintained throughout the set of dynamically-stable outcomes. For instance, μ^6 strictly Pareto dominates μ^{11}. (This proves Theorem 3.) When ordered by $\succsim_M^*$ or $\succsim_W^*$, the set of dynamically-stable matchings is a lattice and $\succsim_W^*$ is the inverse of $\succsim_M^*$ (Figure 5).

Table 5: All dynamically-stable matchings in Example 5.

Matching	m_1	m_2	m_3	m_4	w_1	w_2	w_3	w_4
μ^1	w_1w_1	w_2w_2	w_3w_3	w_4w_4	m_1m_1	m_2m_2	m_3m_3	m_4m_4
μ^2	w_1w_1	w_2w_2	w_4w_3	w_3w_4	m_1m_1	m_2m_2	m_4m_3	m_3m_4
μ^3	w_1w_1	w_2w_2	w_3w_4	w_4w_3	m_1m_1	m_2m_2	m_3m_4	m_4m_3
μ^4	w_1w_1	w_2w_2	w_4w_4	w_3w_3	m_1m_1	m_2m_2	m_4m_4	m_3m_3
μ^5	w_2w_1	w_1w_2	w_3w_3	w_4w_4	m_2m_1	m_1m_2	m_3m_3	m_4m_4
μ^6	w_2w_1	w_1w_2	w_4w_3	w_3w_4	m_2m_1	m_1m_2	m_4m_3	m_3m_4
μ^7	w_2w_1	w_1w_2	w_3w_4	w_4w_3	m_2m_1	m_1m_2	m_3m_4	m_4m_3
μ^8	w_2w_1	w_1w_2	w_4w_4	w_3w_3	m_2m_1	m_1m_2	m_4m_4	m_3m_3
μ^9	w_1w_2	w_2w_1	w_3w_3	w_4w_4	m_1m_2	m_2m_1	m_3m_3	m_4m_4
μ^{10}	w_1w_2	w_2w_1	w_4w_3	w_3w_4	m_1m_2	m_2m_1	m_4m_3	m_3m_4
μ^{11}	w_1w_2	w_2w_1	w_3w_4	w_4w_3	m_1m_2	m_2m_1	m_3m_4	m_4m_3
μ^{12}	w_1w_2	w_2w_1	w_4w_4	w_3w_3	m_1m_2	m_2m_1	m_4m_4	m_3m_3
μ^{13}	w_2w_2	w_1w_1	w_3w_3	w_4w_4	m_2m_2	m_1m_1	m_3m_3	m_4m_4
μ^{14}	w_2w_2	w_1w_1	w_4w_3	w_3w_4	m_2m_2	m_1m_1	m_4m_3	m_3m_4
μ^{15}	w_2w_2	w_1w_1	w_3w_4	w_4w_3	m_2m_2	m_1m_1	m_3m_4	m_4m_3
μ^{16}	w_2w_2	w_1w_1	w_4w_4	w_3w_3	m_2m_2	m_1m_1	m_4m_4	m_3m_3

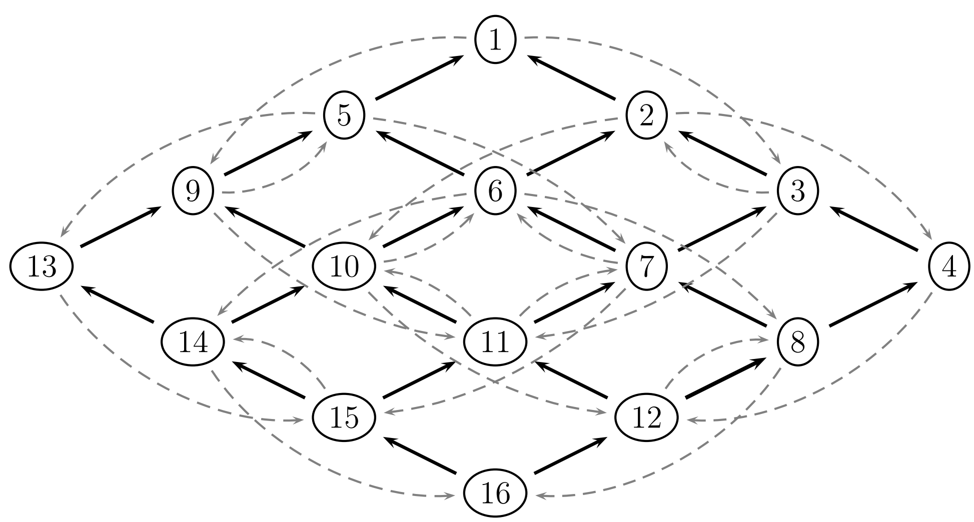

Figure 4: The set of dynamically-stable matchings in Example 5 ordered by $\succsim_M$ (solid) and $\succsim_W$ (dashed).

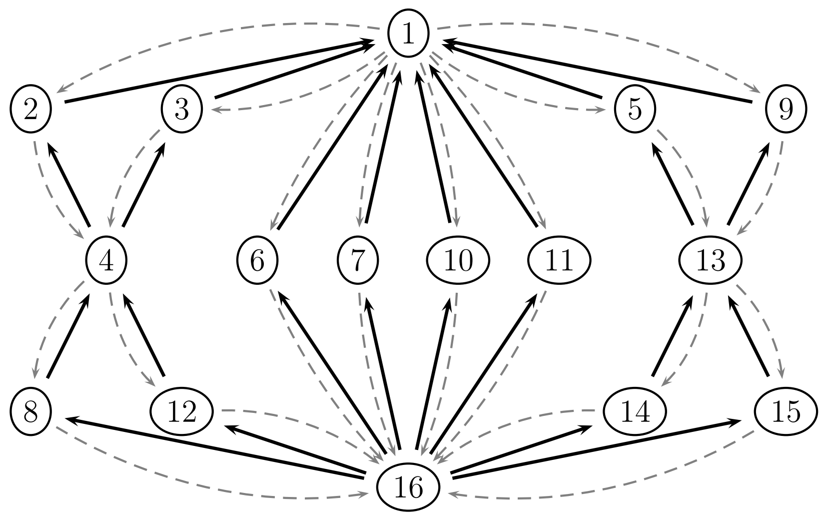

Figure 5: The set of dynamically-stable matchings in Example 5 ordered by $\succsim_M^*$ (solid) and $\succsim_W^*$ (dashed).

4 Temporal Robustness

We have thus far taken the market's time horizon, the parameter T, as given. A natural question, however, concerns the sensitivity of the set of dynamically stable matchings to changes in T. While some applications of matching models have an "obvious" time horizon and a well-specified set of rematching opportunities, many do not. Only a best assessment guides analysis. Thus, it is important to know whether a particular dynamically-stable matching has an easily-identifiable counterpart as the time horizon is slightly perturbed or the frequency of revision opportunities is adjusted.[26]

4.1 Abridgment

Comparing markets with different time horizons involves mapping stable matchings from one case to the other. Thus, the set of agents must be the same and agents' preferences need to be coherently related as the time horizon changes. In this regard, abridgments are simple because matchings can be truncated and preferences can be projected onto a restricted domain. Consider, for example, a T-period market where $x = x_1 \cdots x_T$ is a typical

[26]By rescaling time, both cases are the same type of problem.

partnership plan. Agent i's preference conditional on $x_{\leq t}$, denoted $\succ_i^{x_{\leq t}}$, is a preference over plans of length $T-t$ such that

$$(y_1, \ldots, y_{T-t}) \succ_i^{x_{\leq t}} (z_1, \ldots, z_{T-t}) \iff (x_{\leq t}, y_1, \ldots, y_{T-t}) \succ_i (x_{\leq t}, z_1, \ldots, z_{T-t}).$$

By conditioning preferences on the market's elapsed history, we can relate the final periods of a stable outcome given a longer time horizon to a stable matching in a shorter market.

Theorem 6. *Let μ^* be a dynamically-stable matching in a T-period economy where each agent's preference is $\succ_i$. Then $\mu^*_{>t}$ is a dynamically-stable matching in a $T-t$-period economy where each agent's preference is $\succ_i^{\mu^*_{\leq t}(i)}$.*

Proof. See Appendix A. □

A natural follow-up question asks whether stable matchings can be truncated conditioning on the future, rather than the past? As above, agent i's preference anticipating $x_{>t}$, denoted $\succ_i^{x_{>t}}$, is a preference over plans of length t such that

$$(y_1, \ldots, y_t) \succ_i^{x_{>t}} (z_1, \ldots, z_t) \iff (y_1, \ldots, y_t, x_{>t}) \succ_i (z_1, \ldots, z_t, x_{>t}).$$

In this case, Theorem 6's analogue does not hold. The t-period matching $\mu^*_{\leq t}$ need not be dynamically stable when preferences are $\succ_i^{\mu^*_{>t}(i)}$. As a counterexample, m_1 can block $\mu^3_{\leq 2}$ in Example 2.

4.2 Extension

While every T-period economy can be shortened, introducing additional periods requires greater finesse. An extension involves embedding the set of dynamically-stable matchings into an economy with a longer time horizon. Thus, extensions serve as a robustness check confirming that insights from small-T examples continue to apply when T is large.

Again consider a T-period economy that runs for one additional period. To study this new situation, we first extend agents' preferences to account for the extra period.

Definition 12. The preference $\hat{\succ}_i$ is a *one-period extension* of $\succ_i$ if

$$x_1 \cdots x_T \succ_i y_1 \cdots y_T \iff x_1 \cdots x_T x_T \hat{\succ}_i y_1 \cdots y_T y_T.$$

As a complementary motivation for Definition 12, consider an economy that last two periods of calendar time. However, institutional or legal constraints do not accommodate re-matching between periods. Of course, this restricted economy is equivalent to a one period economy as the two periods are treated as one. To illustrate with a canonical application, a student may express the preference that School A is better than School B *if* he must be enrolled in one school for four years. Suppose, however, that the student is allowed to change schools after two years. Given his opinion in the restricted market, it is reasonable to assume the student will rank a plan where he spends the first two years and the last two years at School A ahead of a plan where he spends all years at School B.

Definition 12 imposes relatively minor restrictions on admissible preference extensions, some of which do not preserve a preference's important analytical properties. Thankfully, as noted in Lemma 11, some extensions do. In Theorem 7 below we use such preferences to embed dynamically stable matchings from a T-period market into a market with a longer time horizon.

Lemma 11. *Let $\mathcal{S}_i$ and $\hat{\mathcal{S}}_i$ be the sets of preferences that reflect a spot ranking in a T-period and a $\hat{T} = T + 1$ -period market, respectively.*

1. *If $\succ_i \in \mathcal{S}_i$, then there exists a one-period extension of $\succ_i$ such that $\hat{\succ}_i \in \hat{\mathcal{S}}_i$.*
2. *If $\succ_i \in \bar{\mathcal{S}}_i$, then there exists a one-period extension of $\succ_i$ such that $\hat{\succ}_i \in \bar{\hat{\mathcal{S}}}_i$.*

Proof. See Appendix A. A lexicographic construction proves (1). A more elaborate construction is required to verify (2) since the dual goals of maintaining a link to a spot ranking while accommodating inertia need to be addressed. $\square$

Theorem 7. *Let $\mu^* = (\mu_1^*, \ldots, \mu_T^*)$ be a dynamically-stable matching in a T-period market where $\succ_i \in \bar{\mathcal{S}}_i$ for all i. Consider a $\hat{T} = T + 1$ -period market with the same set of agents and where each agent's preference $\hat{\succ}_i \in \bar{\hat{\mathcal{S}}}_i$ is a one-period extension of $\succ_i$. The matching $\hat{\mu}^* = (\mu_1^*, \ldots, \mu_T^*, \mu_T^*)$ is dynamically stable in the $\hat{T}$-period market.*

Proof. See Appendix A. $\square$

Theorem 7 embeds all dynamically-stable matchings from a T-period market into the set of stable matchings from a $T + 1$-period market. The converse association is generally not possible. Stable outcomes in longer markets may not have an obvious precursor at a shorter time horizon. To illustrate we adapt an example proposed by Kadam and Kotowski (2014).

Example 6. Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. Consider a one-period market where agents' preferences are

$$\succ_{m_1}: w_1, m_1, w_2 \qquad \succ_{w_1}: m_2, m_1, w_1$$
$$\succ_{m_2}: w_2, m_2, w_1 \qquad \succ_{w_2}: m_1, m_2, w_2$$

This market has one stable matching where $\mu^*(m_1) = w_1$ and $\mu^*(m_2) = w_2$.

Consider a one-period extension where $\hat{\succ}_i \in \bar{\hat{\mathcal{S}}}_i$ for all i and

$$\hat{\succ}_{m_1}: w_1w_1, w_1m_1, w_1w_2, m_1w_1, w_2w_1, m_1m_1, w_2w_2, \ldots$$
$$\hat{\succ}_{m_2}: w_2w_2, w_2m_2, w_2w_1, m_2w_2, w_1w_2, m_2m_2, w_1w_1, \ldots$$

$$\hat{\succ}_{w_1}: m_2m_2, m_2m_1, m_1m_2, m_1m_1, w_1w_1, \ldots$$
$$\hat{\succ}_{w_2}: m_1m_1, m_1m_2, m_2m_1, m_2m_2, w_2w_2, \ldots$$

There are two dynamically-stable matchings in this two-period market (Table 6). The matching $\hat{\mu}^1$ extends μ^* in the sense of Theorem 7. $\hat{\mu}^2$ is not the extension of any stable matching from the one-period market.

Table 6: All dynamically-stable matchings in Example 6.

Matching	m_1	m_2	w_1	w_2
$\hat{\mu}^1$	w_1w_1	w_2w_2	m_1m_1	m_2m_2
$\hat{\mu}^2$	w_2w_1	w_1w_2	m_2m_1	m_1m_2

4.3 Discussion & Applications

Beyond settling questions of robustness, temporal manipulations also have normative applications worthy of emphasis. To illustrate, consider the problem faced by a school district wishing to match students to schools. Recently, many districts have adopted centralized matching procedures to navigate the conflict between schools' physical capacity constraints and students' preferences (Abdulkadiroğlu and Sönmez, 2003). Such problems have a multi-period nature because each student is assigned to a particular school for many years. Though we often emphasize mechanisms that assign students to a school for a focal duration, such as all four years of high school, it is possible to build-in planned revision opportunities at finer time scales. For instance, a student may like a particular school only because it offers a specialized upper-year curriculum and may prefer or be willing to enroll elsewhere in early years. Allowing him to do so in a pre-planned manner may ease a capacity constraint and may better align with his preferences. Under the assumptions of Theorem 7, the introduction of additional re-matching opportunities need not beget instability. The original assignment has a natural analogue under the new regime. This conclusion runs counter to the mechanical effect that more blocking opportunities hearten instability.

Welfare improvements stemming from the fine-tuning of rematching frequencies are possible too. For instance, in Example 6 the introduction of a second period allows for a novel stable matching, $\hat{\mu}^2$, that the market's women prefer to the original outcome's extension, $\hat{\mu}^1$. Though $\hat{\mu}^2$ involves a welfare loss for the men relative to $\hat{\mu}^1$, this tradeoff may be acceptable in applications. In many school-assignment problems the schools' "preferences" are administratively-defined student priorities, which do not have the usual welfare interpretation. Though hypothetical, the preceding discussion emphasizes that the allowed or the anticipated revision frequency and the duration of proposed assignments are manipulable parameters in many market-design applications. Similar principles apply to job-assignment problems or to other matchings accommodating a rotation among distinct positions.

5 Concluding Remarks

Dynamics are an integral feature of many bilateral markets. Though our model is stylized, it accommodates many properties of real-world interactions that unfold over a nontrivial time horizon. While we have maintained Gale and Shapley's (1962) original terminology by speaking of a matching between men and women, the theory developed above captures key features of many labor markets and it can be extended and applied to tackle important allocation problems, such as student-school assignment. We hope the preceding analysis provides a stepping stone toward these applications.

Our analysis points to at least two directions for further research. First, there is considerable scope to refine and extend our analysis' normative implications. The ability to fine-tune relationship length and re-matching frequency may be useful tools in market-design applications. Second, we have focused on "small markets," in the economic sense of the term. An analysis of large markets with an eye toward dynamics is necessary to better link our conclusions with those obtained in parallel matching literatures, such as those emphasizing agents' search behavior.[27] In the latter, it is known that equilibria may not be Pareto optimal due to coordination failures or externalities. We believe that the results presented above, including the qualifications surrounding the lattice of stable outcomes, hint at these implications but more research is required.

[27] In a recent working paper, Kennes et al. (2014b) examine a "large" dynamic matching market focusing on agents' strategic incentives.

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A Proofs

Proof of Lemma 1. ($\Rightarrow$) Suppose $x \succ_i' y$. Since $\succ_i \in \Upsilon(\succ_i')$ and $x \trianglerighteq y$, Definition 9 implies that $x \succ_i y$. ($\Leftarrow$) Suppose $x \succ_i y$. Therefore, $x \neq y$. If $y \succ_i' x$, then $y \trianglerighteq x$ and $\succ_i \in \Upsilon(\succ_i')$ imply that $y \succ_i x$—a contradiction. Therefore, $x \succ_i' y$. $\square$

Lemmas A.1 and A.2 are preliminary results used in some of the arguments to follow.

Lemma A.1. *Suppose $\succ_i \in \bar{\mathcal{S}}_i$ and let x be a volatile partnership plan. $\overline{x_t} \succ_i x$ for some t.*

Proof of Lemma A.1. Suppose the contrary. Since $\succ_i \in \bar{\mathcal{S}}_i$, $\succ_i \in \Upsilon(\succ_i')$ for some $\succ_i' \in \mathcal{S}_i$. Because $\overline{x_t} \triangleright x$, $x \succ_i \overline{x_t} \implies x \succ_i' \overline{x_t}$ for all t. Let x_{t^*} be such that $x_{t^*} R_{\succ_i'} x_t$ $\forall t$. $x \succ_i \overline{x_{t^*}}$ implies that $x_t P_{\succ_i} x_{t^*}$ for some $t \neq t^*$, which is a contradiction. $\square$

Lemma A.2. *Suppose $\succ_i \in \bar{\mathcal{S}}_i$. Suppose agent i can period-t block the matching μ for some $t \geq 2$. (i) If $\mu(i)$ is maximally persistent, then i can period-1 block μ. (ii) If $\mu(i)$ is volatile, then i can also period-t' block μ where $\mu_{t'-1}(i) \neq \mu_{t'}(i)$.*

Proof of Lemma A.2. (i) Suppose $\mu(i) = \bar{j}$ for some $j \neq i$. Suppose i can period-$t$block μ but cannot period-1 block μ. Then $(\bar{j}_{<t}, \bar{i}_{\geq t}) \succ_i \bar{j} \succ_i \bar{i}$. Since $\bar{i} \trianglerighteq (\bar{j}_{<t}, \bar{i}_{\geq t})$, this implies $(\bar{j}_{<t}, \bar{i}_{\geq t}) \succ_i' \bar{i}$ for some $\succ_i' \in \mathcal{S}_i$ such that $\succ_i \in \Upsilon(\succ_i')$. But this implies $jP_{\succ_i} i \implies \bar{j} \succ_i' (\bar{j}_{<t}, \bar{i}_{\geq t})$. As $\bar{j} \trianglerighteq (\bar{j}_{<t}, \bar{i}_{\geq t})$, $\bar{j} \succ_i (\bar{j}_{<t}, \bar{i}_{\geq t})$, which is a contradiction.

(ii) Suppose $\mu_{t-1}(i) = \mu_t(i)$. (Otherwise $t = t'$ satisfies the lemma's claim.) Without loss of generality, we can write $\mu(i)$ as

$$\mu(i) = (\mu_{<s}(i), \bar{j}_{[s,t)}, \underset{\underset{t}{\uparrow}}{j}, \bar{j}_{(t,s']}, \mu_{>s'}(i)).$$

where $\mu_{s-1}(i) \neq j$ and $\mu_{s'+1}(i) \neq j$. By assumption $s < t$. If we let

$$\tilde{\mu}(i) = (\mu_{<s}(i), \bar{j}_{[s,t)}, \underset{\underset{t}{\uparrow}}{i}, \bar{i}_{(t,s']}, \bar{i}_{>s'})$$

then by assumption $\tilde{\mu}(i) \succ_i \mu(i)$. There are three cases.

1. If $j = i$, then $t' = s$.

2. If $jP_{\succ_i} i$, then $(\mu_{<s}(i), \bar{j}_{[s,t)}, j, \bar{j}_{(t,s']}, \bar{i}_{>s'}) \succ_i' \tilde{\mu}(i)$. By inspection, $\tilde{\mu}(i) \parallel \mu(i)$. Hence, $(\mu_{<s}(i), \bar{j}_{[s,t)}, j, \bar{j}_{(t,s']}, \bar{i}_{>s'}) \succ_i \tilde{\mu}(i)$, which implies $(\mu_{<s}(i), \bar{j}_{[s,t)}, j, \bar{j}_{(t,s']}, \bar{i}_{>s'}) \succ_i \tilde{\mu}(i) \succ_i \mu(i)$. Thus, $t' = s' + 1$.

3. If $iP_{\succ_i}j$, then $(\mu_{<s}(i), \bar{i}_{[s,t)}, i, \bar{i}_{(t,s']}, \bar{i}_{>s'}) \succ_i' \tilde{\mu}(i)$. Since $(\mu_{<s}(i), \bar{i}_{[s,t)}, i, \bar{i}_{(t,s']}, \bar{i}_{>s'}) \trianglerighteq \tilde{\mu}(i)$, $(\mu_{<s}(i), \bar{i}_{[s,t)}, i, \bar{i}_{(t,s']}, \bar{i}_{>s'}) \succ_i \tilde{\mu}(i) \succ_i \mu(i)$. Thus, $t' = s$.

$\square$

Proof of Lemma 2. By assumption, $\hat{\mu}(i) = (\mu_{<t}(i), \mu_{\geq t}^{\{m,w\}}(i)) \succ_i \mu(i)$ for $i \in \{m, w\}$. As μ cannot be blocked by m or w alone, there exists $t' \geq t$ such that $\hat{\mu}_{t'}(m) = w$ and $\hat{\mu}_{t'}(w) = m$. Let t' be the smallest such index. Given t' define $\tilde{\mu}(m) \equiv (\mu_{<t}(m), \hat{\mu}_{[t,t')}(m), \bar{w}_{\geq t'})$. By construction, $\tilde{\mu}(m) \trianglerighteq \hat{\mu}(m)$.

Since $\succ_i \in \Upsilon(\succ_i')$ for some $\succ_i' \in \mathcal{S}_i$, $(\mu_{<t}(m), \bar{m}_{\geq t}) \trianglerighteq \hat{\mu}(m)$ and $\hat{\mu}(m) \succ_m \mu(m) \succ_m (\mu_{<t}(m), \bar{m}_{\geq t})$ imply that $\hat{\mu}(m) \succ_m' (\mu_{<t}(m), \bar{m}_{\geq t})$. Hence, $wP_{\succ_m}m$. Thus, $\tilde{\mu}(m) \succsim_m' \hat{\mu}(m)$ and, therefore, $\tilde{\mu}(m) \succsim_m \hat{\mu}(m)$. Likewise, we conclude that $\tilde{\mu}(w) \succsim_w \hat{\mu}(w)$.

If $t' = t$, then the proof is complete. Suppose $t' > t$. In this case $(\mu_{<t}(m), \bar{w}_{\geq t}) \succ_m' \tilde{\mu}(m)$. There are two cases:

1. If $(\mu_{<t}(m), \bar{w}_{\geq t}) \trianglerighteq \tilde{\mu}(m)$, then $(\mu_{<t}(m), \bar{w}_{\geq t}) \succ_m \tilde{\mu}(m)$.

2. If $(\mu_{<t}(m), \bar{w}_{\geq t}) \not\trianglerighteq \tilde{\mu}(m)$, then $\tilde{\mu}(m) \not\trianglerighteq (\mu_{<t}(m), \bar{w}_{\geq t})$.[28] Hence, $(\mu_{<t}(m), \bar{w}_{\geq t}) \succ_m \tilde{\mu}(m)$.

In both cases we see that $(\mu_{<t}(m), \bar{w}_{\geq t}) \succ_m \tilde{\mu}(m) \succsim_m \hat{\mu}(m) \succ_m \mu(m)$. An analogous argument applies to w. $\square$

Proof of Theorem 2. Let μ be a dynamically stable matching. Without loss of generality suppose $m_1 \in M$ is single in period 1 and matched to $w_1 \in W$ in period 2. (The argument when m_1 is matched to w_1 in period 1 and single in period 2 is identical and we omit it for brevity.)

As μ is dynamically stable, $\mu(m_1) = m_1 w_1 \succ_{m_1} m_1 m_1$. Since $\succ_{m_1} \in \bar{\mathcal{S}}_{m_1}$,

$$w_1 w_1 \succ_{m_1} \mu(m_1) = m_1 w_1 \succ_{m_1} m_1 m_1.$$

Furthermore, $\mu(w_1) \succ_{w_1} m_1 m_1$ as otherwise m_1 and w_1 could block μ. As $\mu(w_1) \succ_{w_1} w_1 w_1$ and $\succ w_1 \in \bar{\mathcal{S}}_{w_1}$, there exists $m_2 \in M$, such that

$$m_2 m_2 \succ_{w_1} \mu(w_1) = m_2 m_1 \succ_{w_1} m_1 m_1.$$

[28] This case may occur if $\mu_{t-1}(m) = m$.

As above, $\mu(m_2) \succ_{m_2} w_1 w_1$ as otherwise m_2 and w_1 could block μ. As $\mu(m_2) \succ_{m_2} m_2 m_2$ and $\succ_{m_2} \in \bar{\mathcal{S}}_{m_2}$, there exists $w_2 \in W$ such that

$$w_2 w_2 \succ_{m_2} \mu(m_2) = w_1 w_2 \succ_{m_2} w_1 w_1.$$

Continuing by induction, suppose that for $k \geq 2$ there exists distinct men $m_2, \ldots, m_k$ and distinct women $w_2, \ldots, w_k$ such that for each $k' \leq k$

$$m_{k'} m_{k'} \succ_{w_{k'-1}} \mu(w_{k'-1}) = m_{k'} m_{k'-1} \succ_{w_{k'-1}} m_{k'-1} m_{k'-1}. \tag{A.1}$$

and

$$w_{k'} w_{k'} \succ_{m_{k'}} \mu(m_{k'}) = w_{k'-1} w_{k'} \succ_{m_{k'}} w_{k'-1} w_{k'-1}. \tag{A.2}$$

We will show that we can find a new m_{k+1} and a new w_{k+1} satisfying (A.1) and (A.2).

Given (A.2) it follows that $\mu(w_k) \succ_{w_k} m_k m_k$ and $\mu_2(w_k) = m_k$. Thus, there exists $m_{k+1} \in M$ such that

$$m_{k+1} m_{k+1} \succ_{w_k} \mu(w_k) = m_{k+1} m_k \succ_{w_k} m_k m_k.$$

As m_{k+1} and w_k cannot block μ, $\mu(m_{k+1}) \succ_{m_{k+1}} w_k w_k$. Clearly $m_{k+1} \neq m_1, \ldots, m_k$. Otherwise $\mu(m_{k+1}) = w_k i = \mu(m_{k'}) = w_{k'-1} w_{k'}$, which implies $w_{k'-1} = w_k$. But this contradicts the assumption that $w_1, \ldots, w_k$ are distinct.

It follow, therefore, that there exists $w_{k+1} \in W$ such that

$$w_{k+1} w_{k+1} \succ_{m_{k+1}} \mu(m_{k+1}) = w_k w_{k+1} \succ_{m_{k+1}} w_k w_k.$$

Clearly, $w_{k+1} \neq w_k$. If instead $w_{k+1} = w_{k'}$ for $k' < k$, then $\mu(w_{k+1}) = \mu(w_{k'}) = m_{k'+1} m_{k'}$ implying that $m_{k+1} = m_{k'}$, which contradicts the preceding analysis. Therefore w_{k+1} is distinct from $w_1, \ldots, w_k$.

Thus, continuing in this manner, we can construct a sequence of distinct men $(m_1, \ldots)$, and an sequence of distinct women $(w_1, \ldots)$, satisfying (A.1) and (A.2). However, this is impossible since there is a finite number of men and women in the market. $\square$

Proof of Lemma 3. By definition, $iP_{\succ_i}\mu_t(i) \iff \bar{i} \succ_i \mu(i)$. Thus, μ can be blocked by agent i if and only if μ_t can be blocked by i. If μ can be blocked by m and w, these agents can block it in period 1 (Lemma 2). Thus, m and w can block μ if and only if they can block

μ_t. Thus, μ is stable if and only if μ_t is stable. $\square$

Proof of Lemma 4. As $\mu \in \mathbb{D}$, $\mu(i) \succsim_i \bar{i}$. $\succ_i \in \mathcal{A} \cap \bar{\mathcal{S}}_i$ implies $\overline{\mu_t(i)} \succsim_i \bar{i}$ for every t. Thus, $\mu_t(i) R_{\succ_i} i$. $\square$

Proof of Lemma 5. By Lemma 3 it is sufficient to verify that $\overline{\mu_T} \in \mathbb{D}$. Noting Lemmas A.2 and 2 we need only check a few cases. Fix i and let t be the smallest index such that $\mu_t(i) = \cdots = \mu_T(i)$. We may further assume that $i \neq \mu_t(i)$ as otherwise the conclusion is trivial. Let $\hat{\mu}(i) = (\mu_{<t}, \bar{i}_{\geq t})$. By construction, $\hat{\mu}(i) \trianglerighteq \mu(i)$ and $\mu(i) \succsim_i \hat{\mu}(i)$. As $\succ_i \in \bar{\mathcal{S}}_i$, $\exists \succ_i' \in \mathcal{S}_i$ such that $\succ_i \in \Upsilon(\succ_i')$. Hence, $\mu(i) \succsim_i' \hat{\mu}(i)$. As $\mu(i)$ and $\hat{\mu}(i)$ may differ only in periods $t, t+1, \ldots, T$, and $\mu_t(i) = \cdots = \mu_T(i)$, $\mu_T(i) R_{\succ_i'} i$, which implies $\mu_T(i) R_{\succ_i} i$. Thus, i cannot block $\overline{\mu_T}$.

Choose any pair m and w. Assume that $\mu_T(m) \neq w$ (otherwise, the argument is trivial). As $\mu \in \mathbb{D}$, m and w cannot block it. Thus, without loss of generality, $\mu(m) \succ_m (\mu_{<t}(m), \bar{w}_{\geq t})$ where t is the smallest index such that $\mu_t(m) = \cdots = \mu_T(m)$. As above, $\succ_m \in \bar{\mathcal{S}}_m$ implies that $\mu_T(m) P_{\succ_m} w$. Hence, m is unwilling to block $\overline{\mu_T}$ with w. Therefore, $\overline{\mu_T} \in \mathbb{D}$. $\square$

Proof of Lemma 6. By Lemma 4, $\mu_t(i) R_{\succ_i} i$ for all i. Thus, $\overline{\mu_t}(i) \succsim_i \bar{i}$ and i cannot block μ' alone. Instead, suppose that m and w can block $\overline{\mu_t}$. By Lemma 2, m and w can period-1 block $\overline{\mu_t}$. Thus, $w P_{\succ_m} \overline{\mu_t}(m)$ and $m P_{\succ_w} \overline{\mu_t}(w)$. As $\mu \in \mathbb{D}$, m and w cannot period-1 block μ. Thus, $\mu(m) \succ_m \bar{w}$ or $\mu(w) \succ_w \bar{m}$. Without loss of generality, suppose $\mu(m) \succ_m \bar{w}$. Since $\succ_m \in \mathcal{A}_m$, $\mu_t(m) R_{\succ_m} w$. But then $\overline{\mu_t}(m) R_{\succ_m} w P_{\succ_m} \overline{\mu_t}(m)$, which is a contradiction. Thus, $\overline{\mu_t} \in \mathbb{D}$. $\square$

Proof of Lemma 7. To prove part (i) we argue by contradiction. Without loss of generality, suppose $\mu(m_1)$ is volatile and $\overline{\mu_t(m_1)} \succ_{m_1} \mu(m_1)$ for all t. As $\mu \in \mathbb{D}$, $\mu(m_1) \succ_{m_1} \overline{m_1}$. Thus, $\mu_t(m_1) = w_1 \in W$ for some t and $\overline{w_1} \succ_{m_1} \mu(m_1)$. $\mu(w_1) \succ_{w_1} \overline{m_1}$; else, w_1 and m_1 can block μ.

As m_1 is not matched to w_1 for all t, $\mu(w_1)$ is volatile. By Lemma A.1, $\exists\, m_2 \in M$ such that

$$\overline{m_2} \succ_{w_1} \mu(w_1) \succ_{w_1} \overline{m_1}$$

and $\mu_{t(w_1)}(w_1) = m_2$ for some period $t(w_1)$. Clearly, $m_1 \neq m_2$.

As $\mu \in \mathbb{D}$, $\mu(m_2) \succ_{m_2} \overline{w_1}$; else, m_2 and w_1 could block μ. Since m_2 is not matched to w_1

for all t, $\mu(m_2)$ is volatile. By Lemma A.1, $\exists\, w_2 \in W$ such that

$$\overline{w_2} \succ_{m_2} \mu(m_2) \succ_{m_2} \overline{w_1}$$

and $\mu_{t(m_2)}(m_2) = w_2$ for some period $t(m_2)$. Clearly $w_2 \neq w_1$.

We continue by induction. Suppose that for all $2 \leq k' \leq k$:

1. $\overline{m_{k'}} \succ_{w_{k'-1}} \mu(w_{k'-1}) \succ_{w_{k'-1}} \overline{m_{k'-1}}$;
2. $\overline{w_{k'}} \succ_{m_{k'}} \mu(m_{k'}) \succ_{m_{k'}} \overline{w_{k'-1}}$;
3. $\mu_{t(w_{k'-1})}(w_{k'-1}) = m_{k'}$;
4. $\mu_{t(m_{k'})}(m_{k'}) = w_{k'}$; and,
5. $m_1, \ldots, m_k$ and $w_1, \ldots, w_k$ are all distinct.

We will show that we can find m_{k+1} and w_{k+1} distinct from those already identified satisfying 1–5.

First, consider w_k. As $\mu \in \mathbb{D}$, $\mu(w_k) \succ_{w_k} \overline{m_k}$; else, w_k and m_k can block μ. By Lemma A.1, $\exists\, m_{k+1} \in M$ such that

$$\overline{m_{k+1}} \succ_{w_k} \mu(w_k) \succ_{w_k} \overline{m_k}$$

and $\mu_{t(w_k)}(w_k) = m_{k+1}$ for some period $t(w_k)$.

Clearly $m_{k+1} \neq m_k$. If $m_{k+1} = m_1$, then w_k and m_1 can block μ, which is a contradiction. Finally, if $m_{k+1} = m_{k'}$ for $1 < k' < k$, we know that

$$\overline{w_{k'}} \succ_{m_{k'}} \mu(m_{k'}) \succ_{m_{k'}} \overline{w_{k'-1}}$$

and $\mu_{t(w_{k'-1})}(w_{k'-1}) = m_{k'}$. As $\mu \in \mathbb{D}$, $\mu(m_{k'}) \succ_{m_{k'}} \overline{w_k}$. Since $\succ_{m_{k'}} \in \mathcal{A}_{m_{k'}}$ and $w_{k'-1}$ and w_k are distinct members encountered during the plan $\mu(m_{k'})$,

$$\mu(m_{k'}) \succ_{m_{k'}} \overline{w_{k'-1}} \implies \overline{w_k} \succ_{m_{k'}} \overline{w_{k'-1}}$$

and

$$\mu(m_{k'}) \succ_{m_{k'}} \overline{w_k} \implies \overline{w_{k'-1}} \succ_{m_{k'}} \overline{w_k}.$$

Clearly, this is a contradiction. Therefore $m_{k+1} \neq m_{k'}$. Hence, m_{k+1} is distinct from each $m_1, \ldots, m_k$.

Now consider m_{k+1}. As $\mu \in \mathbb{D}$, $\mu(m_{k+1}) \succ_{m_{k+1}} \overline{w_k}$; else m_{k+1} and w_k can block μ. By Lemma A.1, $\exists\, w_{k+1} \in W$ such that

$$\overline{w_{k+1}} \succ_{m_{k+1}} \mu(m_{k+1}) \succ_{m_{k+1}} \overline{w_k}.$$

and $w_{k+1} = \mu_{t(m_{k+1})}(m_{k+1})$ for some period $t(m_{k+1})$.

Clearly, $w_{k+1} \neq w_k$. Suppose $w_{k+1} = w_{k'}$ for some $k' < k$. We know that

$$\overline{m_{k'+1}} \succ_{w_{k'}} \mu(w_{k'}) \succ_{w_{k'}} \overline{m_{k'}},$$

$\mu_{t(m_{k'})}(w_{k'}) = m_{k'}$ and $\mu_{t(m_{k+1})}(w_{k'}) = m_{k+1}$. As $\mu \in \mathbb{D}$, $\mu(w_{k'}) \succ_{w_{k'}} \overline{m_{k+1}}$; else, m_{k+1} and $w_{k'} = w_{k+1}$ can block μ. Since $\succ_{w_{k'}} \in \mathcal{A}_{w_{k'}}$ and $m_{k'}$ and m_{k+1} are distinct

$$\mu(w_{k'}) \succ_{w_{k'}} \overline{m_{k'}} \implies \overline{m_{k+1}} \succ_{w_{k'}} \overline{m_{k'}}$$

and

$$\mu(w_{k'}) \succ_{w_{k'}} \overline{m_{k+1}} \implies \overline{m_{k'}} \succ_{w_{k'}} \overline{m_{k+1}}$$

Clearly, this is a contradiction. Therefore, $w_{k+1} \neq w_{k'}$ for all $k' \leq k$.

Proceeding by induction we can construct an infinite sequence of men $m_2, m_3, \ldots$ and women $w_2, w_3, \ldots$ satisfying conditions 1–5 above. However, this is not possible as there is a finite number of agents. Thus, if $\mu(m_1)$ is volatile, there exists a period t such that $\mu(m_1) \succ_{m_1} \overline{\mu_t(m_1)}$.

To verify point (ii) it is sufficient to observe that when $\mu_t(i) \neq \mu_{t'}(i)$, $\mu(i) \succ_i \overline{\mu_t(i)} \implies \overline{\mu_{t'}(i)} \succ_i \overline{\mu_t(i)}$ and $\mu(i) \succ_i \overline{\mu_{t'}(i)} \implies \overline{\mu_t(i)} \succ_i \overline{\mu_{t'}(i)}$, which is a contradiction as $\succ_i \in \mathcal{A}_i$. $\square$

Proof of Lemma 8. Consider agent i. If $\mu(i) = \mu'(i)$, the lemma's first part is clearly true. Suppose $\mu(i) \succ_i \mu'(i)$. As both matchings are maximally persistent, $\mu(i) = \overline{\mu_t(i)}$ and $\mu'(i) = \overline{\mu'_{t'}(i)}$ for all t and t'. Thus, $\overline{\mu_t(i)} \succ_i \overline{\mu'_{t'}(i)}$. Therefore, $\mu(i) \succ_i^* \mu'(i)$. Conversely, if $\mu(i) \succ_i^* \mu'(i)$, then $\mu(i) \succ_i \mu'(i)$ follows immediately from the definition of $\succsim_i^*$.

Now consider the collective preference. If $\mu = \mu'$, the lemma's conclusion is again trivial. If $\mu \succ_M \mu'$, then $\mu(m) \succsim_m \mu'(m)$ for all m and $\mu(m') \succ_m \mu'(m')$ for some m'. As μ and μ' are maximally-persistent matchings, $\mu(m) \succsim_m^* \mu'(m)$ for all m and $\mu(m') \succ_m^* \mu'(m')$ for some m'. Hence, $\mu \succ_M^* \mu'$. Conversely, $\mu \succsim_M^* \mu' \implies \mu(m) \succsim_m^* \mu'(m)\ \forall m \in M \implies \mu(m) \succsim_m \mu'(m)\ \forall m \in M \implies \mu \succsim_M \mu'$. $\square$

Proof of Lemma 9. Let μ' be the $\succsim_M$-maximal matching in $\mathbb{P}$. Since $(\mathbb{P}, \succsim_M)$ is a finite lattice, this matching exits and is unique. Let $\mu \in \mathbb{D}$ and suppose $\mu' \not\succsim^*_M \mu$. Hence, $\exists m \in M$ such that $\mu'(m) \not\succsim^*_m \mu(m)$. This implies $\mu(m)$ is volatile. Thus, there exists a period t' such that $\mu(m) \succ_m \overline{\mu_{t'}(m)}$ and for all $t \neq t'$, $\overline{\mu_t(m)} \succ_m \mu(m)$. However, $\overline{\mu_t} \in \mathbb{P}$ for all t. Thus, $\mu'(m) \succsim_m \overline{\mu_t(m)} \succ_m \mu(m) \succ_m \overline{\mu_{t'}(m)}$. But this implies $\mu'(m) \succ^*_m \mu(m)$, which is a contradiction. $\square$

Proof of Lemma 10. Suppose $\mu \succ^*_M \mu'$. Thus, there exists at least one man and at least one woman for whom $\mu(i) \neq \mu'(i)$. If $\mu' \not\succ^*_W \mu$, then $\mu'(w) \not\succsim^*_w \mu(w)$ for some $w \in W$. Thus, $\mu'(w) \neq \mu(w)$ and $\mu'(w) \not\succ^*_w \mu(w)$. There are two cases.

1. There exist t and t' such that $\overline{\mu_t(w)} \succ_w \overline{\mu'_{t'}(w)}$. Hence, there exists $m \in M$ such that $m = \mu_t(w)$ and $\bar{m} = \overline{\mu_t(w)} \succ_w \overline{\mu'_{t'}(w)}$. Since $\mu \succ^*_M \mu'$, it follows that for this agent m, $\mu(m) \succsim^*_m \mu'(m)$ and for t and t', $\bar{w} = \overline{\mu_t(m)} \succsim_m \overline{\mu'_{t'}(m)}$. As $\mu_t(w) = m \neq \mu_{t'}(w)$, it must be the case that $\overline{\mu_t(m)} \succ_m \overline{\mu'_{t'}(m)}$.

   By Lemma 6, $\overline{\mu'_{t'}} \in \mathbb{D}$. However, $\overline{\mu'_{t'}}(w) = \overline{\mu'_{t'}(w)}$ and $\overline{\mu'_{t'}}(m) = \overline{\mu'_{t'}(m)}$. Thus, m and w can period-1 block $\overline{\mu'_{t'}}$, which is a contradiction.

2. Suppose $\mu(w) \succ_w \mu'(w)$. By Lemma A.1, there exist t and t' such that $\overline{\mu_t(w)} \succsim_w \mu(w)$ and $\mu'(w) \succsim_w \overline{\mu'_{t'}(w)}$. Thus, the same argument as case (1) applies.

As neither case applies, we arrive at a contradiction. Hence, $\mu' \succ^*_W \mu$. $\square$

Proof of Theorem 6. Suppose i can block $\mu^*_{>t}$, i.e.

$$(\mu^*_{(t,t')}(i), \bar{i}_{\geq t'}) \succ_i^{\mu^*_{\leq t}(i)} \mu^*_{>t}(i) \iff (\mu^*_{\leq t}(i), \mu^*_{(t,t')}(i), \bar{i}_{\geq t'}) \succ_i (\mu^*_{\leq t}(i), \mu^*_{>t}(i)).$$

Thus, i can block μ^*, which is a contradiction.

If $\{m, w\}$ can block $\mu^*_{>t}$, then

$$(\mu^*_{(t,t')}(i), \mu^{\{m,w\}}_{\geq t'}(i)) \succ_i^{\mu^*_{\leq t}(i)} \mu^*_{>t}(i) \iff (\mu^*_{\leq t}(i), \mu^*_{(t,t')}(i), \mu^{\{m,w\}}_{\geq t'}(i)) \succ_i (\mu^*_{\leq t}(i), \mu^*_{>t}(i))$$

for $i \in \{m, w\}$. Thus, $\{m, w\}$ can block μ^*, which is a contradiction. Hence, $\mu^*_{>t}$ is dynamically stable. $\square$

Proof of Lemma 11.

1. We propose a lexicographic extension of $\succ_i \in \mathcal{S}_i$. For all $x = x_1 \cdots x_T$ and $y = y_1 \cdots y_T$, define $\hat{\succ}_i$ as follows:

$$x_1 \cdots x_T j \;\hat{\succ}_i\; y_1 \cdots y_T k \iff \begin{cases} x \succ_i y \\ \text{or} \\ x = y \;\&\; j P_{\succ_i} k \end{cases} \tag{A.3}$$

It is simple to verify that $\hat{\succ}_i$ reflects $P_{\succ_i} = P_{\hat{\succ}_i}$. Thus, $\hat{\succ}_i \in \hat{\mathcal{S}}_i$.

2. Given $\succ_i \in \bar{\mathcal{S}}_i$, let $\succ_i' \in \mathcal{S}_i$ be such that $\succ_i \in \Upsilon(\succ_i')$. Let $\hat{\succ}_i' \in \hat{\mathcal{S}}_i$ be the one-period lexicographic extension, as in (A.3), of $\succ_i'$. $\hat{\succ}_i'$ reflects $P_{\hat{\succ}_i'} = P_{\succ_i'}$. Define $\hat{\succ}_i$ as follows:

   (a) If $\hat{x}_T \neq \hat{x}_{T+1}$ or $\hat{y}_T \neq \hat{y}_{T+1}$, then $\hat{x} \hat{\succ}_i' \hat{y} \iff \hat{x} \hat{\succ}_i \hat{y}$.

   (b) If $\hat{x}_T = \hat{x}_{T+1}$ and $\hat{y}_T = \hat{y}_{T+1}$, then $\hat{x}_{\leq T} \succ_i \hat{y}_{\leq T} \iff \hat{x} \hat{\succ}_i \hat{y}$.

   $\hat{\succ}_i$ is a one-period extension of $\succ_i$. To confirm that $\hat{\succ}_i \in \Upsilon(\hat{\succ}_i')$, we verify two conditions.

   (a) Suppose $\hat{x} \hat{\succ}_i' \hat{y}$ and $\hat{x} \trianglerighteq \hat{y}$. To work toward a contradiction, assume $\hat{y} \hat{\succ}_i \hat{x}$. Then $\hat{y}_T = \hat{y}_{T+1}$. Since $\hat{x} \trianglerighteq \hat{y}$ then $\hat{x}_T = \hat{x}_{T+1}$. Therefore, $\hat{y}_{\leq T} \succ_i \hat{x}_{\leq T}$ as well. But, $\hat{x} \trianglerighteq \hat{y}$ also implies $\hat{x}_{\leq T} \trianglerighteq \hat{y}_{\leq T}$ and, therefore, $\hat{y}_{\leq T} \succ_i' \hat{x}_{\leq T}$. But then $\hat{y} \hat{\succ}_i' \hat{x}$, which is a contradiction. Therefore, $\hat{x} \hat{\succ}_i \hat{y}$.

   (b) Suppose $\hat{x} \parallel \hat{y}$. This implies $\hat{x}_{\leq T} \neq \hat{y}_{\leq T}$; else, the two partnership plans could be ordered by $\trianglerighteq$.

   First, suppose $\hat{x} \hat{\succ}_i' \hat{y}$. As $\hat{x}_{\leq T} \neq \hat{y}_{\leq T}$, it follows that $\hat{x}_{\leq T} \succ_i' \hat{y}_{\leq T}$. Suppose $\hat{y} \hat{\succ}_i \hat{x}$. This is possible only if $\hat{y}_T = \hat{y}_{T+1}$ and $\hat{y}_{\leq T} \succ_i \hat{x}_{\leq T}$. But if $\hat{x}_{\leq T} \succ_i' \hat{y}_{\leq T}$, then it must be that $\hat{y}_{\leq T} \trianglerighteq \hat{x}_{\leq T}$ as $\succ_i \in \Upsilon(\succ_i')$. This implies $(\hat{y}_{\leq T}, \hat{y}_{T+1}) \trianglerighteq (\hat{x}_{\leq T}, \hat{x}_{T+1})$, which contradicts $\hat{y} \not\trianglerighteq \hat{x}$. Thus, $\hat{x} \hat{\succ}_i \hat{y}$.

   Conversely, let $\hat{x} \hat{\succ}_i \hat{y}$. To derive a contradiction, suppose $\hat{y} \hat{\succ}_i' \hat{x}$. Thus, $\hat{x}_T = \hat{x}_{T+1}$ and $\hat{x}_{\leq T} \succ_i \hat{y}_{\leq T}$. However, since $\hat{x}_{\leq T} \neq \hat{y}_{\leq T}$, $\hat{y}_{\leq T} \succ_i' \hat{x}_{\leq T}$. Thus, $\hat{x}_{\leq T} \trianglerighteq \hat{y}_{\leq T}$. As $\hat{x}_T = \hat{x}_{T+1}$, then $\hat{x} \trianglerighteq \hat{y}$—a contradiction. Therefore, $\hat{x} \hat{\succ}_i' \hat{y}$.

$\square$

Proof of Theorem 7. Let $\succ_i' \in \mathcal{S}_i$ be such that $\succ_i \in \Upsilon(\succ_i')$. Define $\hat{\succ}_i'$ analogously, i.e.

$\hat{\succ}_i \in \Upsilon(\hat{\succ}_i')$. Since μ^* is dynamically stable, i cannot block it. As $\hat{\succ}_i$ extends $\succ_i$,

$$\mu^*(i) \succsim_i (\mu^*_{<t}(i), \bar{i}_{[t,T]}) \implies \underbrace{(\mu^*_1(i), \ldots, \mu^*_T(i), \mu^*_T(i))}_{\hat{\mu}^*(i)} \hat{\succsim}_i \underbrace{(\mu^*_{<t}(i), \bar{i}_{[t,T]}, i)}_{(\hat{\mu}^*_{<t}(i), \bar{i}_{[t,\hat{T}]})}.$$

Hence, i cannot block $\hat{\mu}^*$ in period $t \leq T$.

Suppose i can block $\hat{\mu}^*$ in period $\hat{T}$, i.e. $(\hat{\mu}^*_{\leq T}(i), i) \hat{\succ}_i \hat{\mu}^*(i)$. Thus, $\hat{\mu}^*_T(i) \neq i$ and, therefore, $\hat{\mu}^*(i) \trianglerighteq (\hat{\mu}^*_{\leq T}(i), i)$. This implies $(\hat{\mu}^*_{\leq T}(i), i) \hat{\succ}_i' \hat{\mu}^*(i)$ and, therefore, $iP_{\hat{\succ}_i'} \hat{\mu}^*_{\hat{T}}(i) \implies iP_{\succ_i'} \hat{\mu}^*_T(i)$. Let $t' \leq T$ be the smallest value such that $\mu^*_{t'}(i) = \cdots = \mu^*_T(i)$. Thus,

$$\tilde{\mu}(i) \equiv (\mu^*_{<t'}(i), \bar{i}_{\geq t'}) \succ_i' (\mu^*_{<t'}(i), \mu^*_{t'}(i), \ldots, \mu^*_T(i)) = \mu^*(i).$$

But, $\tilde{\mu}(i) \trianglerighteq \mu^*(i)$ and thus, $\tilde{\mu}(i) \succ_i \mu^*(i)$. Hence, μ^* can be period-t' blocked by i in the T-period economy. This is a contradiction. Therefore, i cannot block $\hat{\mu}^*$.

Consider the pair m and w. They cannot block μ^* in period $t \leq T$. From the definition of stability, $\mu^*(m) \succsim_m (\mu^*_{<t}(m), \bar{w}_{[t,T]})$ and $\mu^*(w) \succsim_w (\mu^*_{<t}(w), \bar{m}_{[t,T]})$. Hence,

$$\underbrace{(\mu^*_{\leq T}(m), \mu^*_T(m))}_{\hat{\mu}^*(m)} \hat{\succsim}_m (\mu^*_{<t}(m), \bar{w}_{[t,T]}, w) \quad \text{and} \quad \underbrace{(\mu^*_{\leq T}(w), m)}_{\hat{\mu}^*(w)} \hat{\succsim}_w (\mu^*_{<t}(w), \bar{m}_{[t,T]}, m).$$

Thus, m and w cannot block $\hat{\mu}^*$ in period-t.

Suppose m and w can period-$\hat{T}$ block $\hat{\mu}^*$:

$$\tilde{\mu}(m) \equiv (\hat{\mu}^*_{\leq T}(m), w) \hat{\succ}_m \hat{\mu}^*(m) \quad \text{and} \quad \tilde{\mu}(w) \equiv (\hat{\mu}^*_{\leq T}(w), m) \hat{\succ}_w \hat{\mu}^*(w).$$

Of course, this implies $\hat{\mu}^*_{T+1}(m) = \mu^*_T(m) \neq w$. By reasoning analogous to the single-agent case above, $\hat{\mu}^*(m) \trianglerighteq \tilde{\mu}(m) \implies \tilde{\mu}(m) \hat{\succ}_m' \hat{\mu}^*(m) \iff wP_{\hat{\succ}_m'} \hat{\mu}^*_{\hat{T}}(m)$. Similarly, $mP_{\hat{\succ}_w'} \mu^*_T(w)$.

Let t_i' be the smallest index such that $\mu^*_{t_i'}(i) = \cdots = \mu^*_T(i)$ and let $t' = \max\{t_m', t_w'\}$. Then,

$$(\mu^*_{<t'}(m), \bar{w}_{[t',T]}, w) \hat{\succ}_m' (\mu^*_{<t'}(m), \mu^*_{[t',T]}(m), w) = \tilde{\mu}(m).$$

Given the definition of t', $(\mu^*_{<t'}(m), \mu^*_{[t',T]}(m), w) \not\trianglerighteq (\mu^*_{<t'}(m), \bar{w}_{[t',T]}, w)$. And so,

$$(\mu^*_{<t'}(m), \bar{w}_{[t',T]}, w) \hat{\succ}_m \tilde{\mu}(m).$$

But then $(\mu^*_{<t'}(m), \bar{w}_{[t',T]}, w) \hat{\succ}_m \hat{\mu}^*(m)$ and likewise for w, $(\mu^*_{<t'}(m), \bar{w}_{[t',T]}, w) \hat{\succ}_w \mu^*(w)$. Thus, m and w can block $\hat{\mu}^*$ in period $t' \leq T$. Above we have already established that this is not

possible. $\square$

The following appendix is intended for online publication only.

B Ex Ante Stability

To prove that an ex ante stable matching exists, we employ an argument proposed by Kadam and Kotowski (2014) for the two-period case. The exposition below follows closely their argument. Introduced modifications account for the T-period setting studied here.

For each $m \in M$ and $w \in W$, let $X_m(w) = \{x_1 \cdots x_T : x_t \in \{m, w\}\} \setminus \{\bar{m}\}$ be the set of all partnership plans for m involving only w. Let $X_m = \cup_{w \in W} X_m(w)$.

Definition B.1 (P-DA)**.** The *(man-proposing) plan deferred acceptance* procedure identifies a matching μ^* as follows:

1. For each m, let $X_m^0 = X_m$. Initially, no plans in X_m^0 have been rejected.
2. In round $t \geq 1$:
   (a) Let $X_m^t \subset X_m^0$ be the subset of plans that have not been rejected in some round $t' < t$. If $X_m^t = \emptyset$ or $\bar{m} \succ_m x$ for all $x \in X_m^t$, then m does not make any proposals. Otherwise, m proposes to the (only) woman identified in his most preferred plan x in X_m^t. More precisely, let $\omega(x)$ be the set of periods in which m and w are to be paired according to x. m proposes to w an arrangement where they are partners for all $t \in \omega(x)$ and unmatched for all $t \notin \omega(x)$.
   (b) Let X_w^t be the set of plans made available to w. If $\bar{w} \succ_w x$ for all $x \in X_w^t$, w rejects all proposals. Otherwise, w (tentatively) accepts her most preferred proposed arrangement in X_w^t and rejects the others.
3. The above process continues until no rejections occur. If w accepts m's proposal in the final round, define $\mu^*(m)$ and $\mu^*(w)$ accordingly. If i does not make or receive any proposals in the final round, set $\mu^*(i) = \bar{i}$.

Theorem B.1. *There exists an ex ante stable matching.*

Proof. Let μ^* be the matching identified by the P-DA procedure. First, according to the procedure, no agent will be assigned to a plan that is worse than being unmatched. Thus,

$\bar{i} \not\succ_i \mu^*(i)$. Suppose instead that μ^* can be period-1 blocked by m and w. Thus, there exists a matching among $\{m, w\}$ such that $\mu^{\{m,w\}}(m) \succ_m \mu^*(m) \succsim_m \bar{m}$ and $\mu^{\{m,w\}}(w) \succ_w \mu^*(w) \succsim_w \bar{w}$. As $\mu^{\{m,w\}}(m)$ implicates at most one women, w, it belongs to X_m. Since $\mu^{\{m,w\}}(m)$ is preferred to $\mu^*(m)$, m must have proposed that arrangement in the P-DA procedure and w must have rejected it. But this implies w must have accepted an alternative plan that she preferred to $\mu^{\{m,w\}}(w)$. Hence, for some m', $\mu^*(w) \succsim_w \mu^{\{m',w\}}(w) \succ_w \mu^{\{m,w\}}(w) \succ_w \mu^*(w)$, which is a contradiction. $\square$