# Revenue Management Without Commitment: Dynamic Pricing and Periodic Fire Sales*

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**Abstract**

We consider a market with a profit-maximizing monopolist seller who has $K$ identical goods to sell before a deadline. At each time, the seller posts a price and the quantity available. Over time, buyers privately enter the market, and they strategically time their purchases. The equilibrium prices decline smoothly over the time period between sales and jump up immediately after a transaction occurs. Crucially, the seller may periodically liquidate part of his stock via fire sales before the deadline in order to secure a higher price in the future.

**Keywords:** revenue management, commitment power, dynamic pricing, fire sales, inattention frictions.

**JEL Classification Codes: D82, D83**

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# 1 Introduction

Many markets share the following characteristics: (1) goods for sale are (almost) identical, and all expire and must be consumed by a certain time, (2) the initial number of goods for sale is fixed in advance, and (3) consumers have heterogeneous reservation values and enter the market sequentially over time. Such markets include the airline, cruise-line, hotel and entertainment industries. The revenue management literature studies the pricing of goods in these markets, and these techniques are reported to be quite valuable in many industries, such as airlines (Davis (1994)), retailers (Friend and Walker (2001)), etc. The standard assumptions in this literature are that sellers have perfect commitment power and buyers are impatient. That is, buyers cannot time their purchases and sellers can commit to the future price path or mechanism. In contrast, this paper studies a revenue management problem in which buyers are *patient* and sellers are endowed with *no commitment power*. [1]

We consider the profit-maximizing problem faced by a (male) monopolist seller who has $K$ identical goods to sell before a deadline. At any time, the seller posts a price and the quantity available (capacity control) but cannot commit to future offers. Over time, potential (female) buyers with different reservation values (either high or low) privately enter the market. Each buyer has a single unit of demand and can time her purchase. Goods are consumed at the fixed deadline, and all trades happen at or before that point.

Our goal is to show that the seller can sometimes use fire sales to credibly reduce his inventory and so charge higher prices in the future. We accordingly consider settings where the seller does not find it profitable to sell only at the deadline and then only to high-value buyers, with the accompanying possibility of unsold units. In such settings, we explore the properties of a pricing path in which, at the deadline, if the seller still has unsold goods, he sets the price sufficiently low that all remaining goods are sold for certain. For most of the time, the seller posts the highest price consistent with high-value buyers purchasing immediately upon arrival, and he occasionally posts a fire sale price that is affordable to low-value buyers. By holding fire sales, the seller reduces his inventory quickly, and therefore, he can induce high-value buyers to accept a higher price in the future. Intuitively, these sales allow the seller to 'commit' to high prices going forward. Once the transaction happens, whether at the discount price or not, the seller's inventory is reduced, and the price jumps up instantaneously. Hence, in general, a highly fluctuating path of realized

[1] In many industries including the airline industry, although computer systems based on revenue management algorithms are widely used, revenue managers frequently adjust prices based on their information and personal experience instead of mechanically following the pricing strategy suggested by the algorithm. Hence, it is reasonable to investigate the case where the seller has limited commitment power. See "Confessions of an Airline Revenue Manager" by George Hobica. http://www.foxnews.com/travel/2011/12/08/confessions-airline-revenue-manager/.

sales prices will appear, which is in line with the observations in many relevant industries.[2]

The sub-optimality of selling only at the deadline to high-value buyers could occur for many reasons. For example, at the deadline, the seller may expect that there will be little effective high-value demand in the market. This may be because the arrival rate of high-value buyers is low, or because buyers may also leave the market without making a purchase, or because buyers face inattention frictions and may therefore miss the deadline. We discuss the latter possibility in detail below.

The equilibrium price path relies on the seller's lack of commitment and buyers' intertemporal concerns. An intuitive explanation is as follows. At the deadline, due to the insufficient effective demand, the seller holding unsold goods sets a low price to clear his inventory, creating what is known as the last-minute deal.[3] Before the deadline, because a last-minute deal is expected to be posted shortly, buyers have an incentive to wait for the discount price. However, waiting for a deal is risky due to competition at the low price from both newly arrived high-value buyers and low-value ones who are only willing to pay a low price. To avoid future competition, a high-value buyer is willing to make her purchase immediately at a price higher than the discount price. We name the highest price she is willing to pay to avoid the competition as her reservation price. The reservation price of any such high-value buyer decreases over time because the arrival of competition shrinks as the deadline approaches. Her reservation price is decreasing in the current inventory size because the probability that she will be rationed at deal time depends on the amount of remaining goods. To maximize his profit, the seller posts the high-value buyer's reservation price for most of the time and, at certain times before the deadline, may hold fire sales in order to reduce his inventory and to be able to charge a higher price in the future.

Figure 1 illustrates this idea in the simplest case with only two items for sale at the beginning. Suppose the seller serves only high-value buyers before the deadline, allowing discounts only at the deadline. Conditional on the inventory size, the price declines over time. The high-value buyer's reservation price in the two-unit case is lower than her reservation price in the one-unit case, and the price difference indicates the difference in the probability that a high value buyer is rationed at the last minute in different cases. If a high-value buyer enters the market early and buys a unit immediately, the seller can sell it at a relatively high price and earn a higher profit than he could earn from running fire sales. However, if no such buyer shows up, then the time will eventually come when selling one unit via a fire sale and then following the one-unit pricing strategy is more

[2] For example, McAfee and te Velde (2008) find that the fluctuation of airfares is too high to be explained by the standard monopoly pricing models.

[3] The last-minute deal or clearance sale is optimal in many dynamic pricing settings. See Nocke and Peitz (2007) as an example. In reality, the last minute deal strategy is commonly used in many industries. See *Wall Street Journal*, March 15, 2002, "Airlines now offer 'last-minute' fare bargains weeks before flights," by Kortney Stringer.

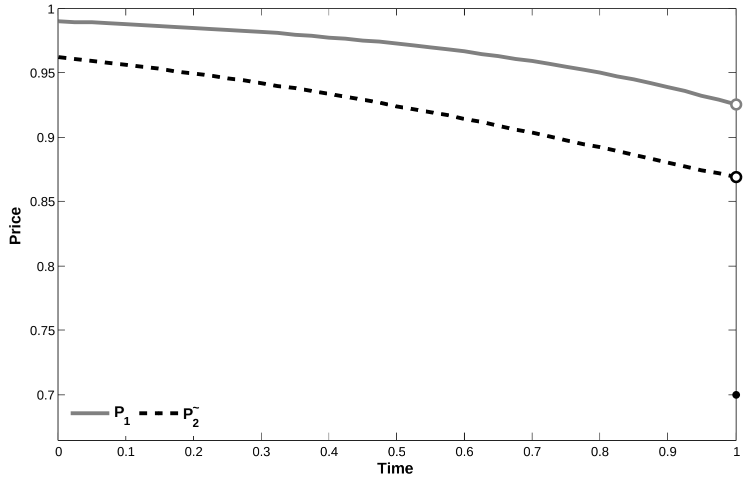


Figure 1: Necessity of Fire Sales Before the Deadline in the Two-Unit Case. The solid (dashed) line shows how the list price will change in the case of one unit (two units) of initial stock if low-value buyers are served only at the deadline.

profitable to the seller. To see the intuition, consider the seller's benefit and cost of liquidating the first unit via a fire sale. The *benefit* is that, by reducing stock by one unit,, the seller can charge the high-value buyer who arrives next a higher price for his last unit. On the other hand, the (opportunity) *cost* is that, if more than one high-value buyers arrive before the deadline, the seller cannot serve the second one, who is willing to pay a price higher than the fire sale price. As the deadline approaches, it is less likely that multiple high-value buyer arrives, thus, the opportunity cost is negligible compared to the benefit, and therefore, the seller has the incentive to liquidate the first unit via a fire sale.

Analyzing a dynamic pricing game with private arrivals is difficult for the following reason. Because the seller can choose both the price and quantity available at any time, he may want to sell his inventory unit by unit. Thus, some buyers may be rationed before the game ends when demand is less than supply. Suppose a buyer was rationed at time $t$ and the seller still holds unsold units. The rationed buyer privately learns that demand is greater than supply at time $t$ and uses the information to update her belief about the number of remaining buyers. Buyers who arrive after this transaction have no such information. As a result, belief heterogeneity among buyers naturally occurs following their private histories, and buyers' strategies will depend non-trivially on their private beliefs. Such belief heterogeneity evolves over time and becomes more complicated as transactions happen one after another, making the problem intractable.

To overcome this technical challenge, we assume that buyers face inattention frictions. That is, in each "period" having a positive measure of time, instead of assuming that buyers can observe offers all the time, we assume that each buyer notices the seller's offer and makes her purchase decision only at her attention times. In each "period," a buyer independently draws one attention time from an atom-less distribution. In addition, buyers' attention can be attracted by an offer having a sufficiently low price, that is, a fire sale.[4] This implies that (1) at any particular time, the probability that a buyer observes a non-fire sale offer is zero, (2) the probability that more than one buyer observes a non-fire-sale offer at the same time is also zero, and (3) all buyers observe a fire sale offer when it is posted. As a result, high-value buyers will not be rationed except at deal time. Furthermore, we focus on equilibria where high-value buyers make their purchases upon arrival. Therefore, a high-value buyer who is rationed at deal time attributes the failure of her purchase to competition with low-value buyers instead of with other high-value buyers, so she cannot infer extra information about the number of buyers in the market. As we will show, there is an equilibrium in which buyers' strategies do not depend on their private histories.[5]

As we described earlier, we are interested in the environment where the seller finds selling only at the deadline and serving only high-value buyers to be suboptimal. In the presence of inattention frictions, the seller cannot guarantee that the high-value buyers will be available at the deadline. Hence, at the deadline, to maximize his profit, the seller has to post a sufficiently low price for all remaining inventory to draw the full attention of the market. As a result, by serving only high-value buyers at the deadline, the seller cannot extract any rent from the high-value buyers, so he has to start selling early.[6]

[4]In practice, this extra chance is justified by consumers' attention being attracted by advertisements of deals sent by a third party or by the seller himself. Low price airfare alert emails, for example, may originate from intermediary websites such as http://www.orbitz.com and http://www.faredetective.com. In addition, the Internet allows sellers to use "cookies" to track consumers who look but do not buy, and to lure them back later by targeting them with a low-price advertisement.

[5]The idea that decision times arrive randomly in a continuous-time environment is not new. For examples, see the bargaining models described by Perry and Reny (1993) and Ambrus and Lu (2014). However, neither of those papers employ such an assumption to avoid the complexities of private beliefs.

[6]Notice that our economic prediction on the price path does not depend on the presence of inattention frictions. As we mentioned before, a low arrival rate of buyers or the disappearance of present buyers can also exclude the trivial case where the seller is willing to sell at the deadline only. We explore the possibility of disappearing buyers in the extension.

## 1.1 The Literature

There is a large body of *revenue management* literature that has examined markets having sellers who need to sell finitely many goods before a deadline and impatient buyers who arrive sequentially.[7] Gershkov and Moldovanu (2009) extend the benchmark model to address the case of heterogeneous objects. The standard assumption maintained in these works is that buyers are impatient, and therefore cannot strategically time their purchases. However, as argued by Besanko and Winston (1990), mistakenly treating forward-looking customers as myopic may have an important impact on sellers' revenue. Board and Skrzypacz (2013) characterize the revenue-maximizing mechanism in a model where agents arrive in the market over time. In the continuous time limit, the revenue-maximizing mechanism is implemented via a price-posting mechanism with an auction for the last unit at the deadline.[8]

In the works mentioned above, perfect commitment of the seller is typically assumed. There has been little discussion of the case in which a monopolist with scarce supply and no commitment power sells to forward-looking customers. To our knowledge, Chen (2012) and Hörner and Samuelson (2011) have made the first attempts to address the non-commitment issue in a revenue management environment using a multiple-period game-theoretic model.[9] They assume that the seller faces a fixed number of buyers who strategically time their purchases. They show that the seller either replicates a Dutch auction or posts unacceptable prices up to the very end and charges a static monopoly price at the deadline. However, as argued by McAfee and te Velde (2008), the arrival of new buyers seems to be an important driving force of many observed phenomena in a dynamic environment. As we will show, the sequential arrival of buyers plays a critical role in the seller's optimal pricing and in his decision on whether to hold fire sales.

Additionally, our model is also related to the *durable goods* literature in which the seller without capacity constraint sells durable goods to strategic buyers over an infinite horizon. As Hörner and Samuelson (2011) show, the deadline endows the seller with considerable commitment power, and the scarcity of the good changes the issues surrounding price discrimination, with the impetus for buying early at a high price now arising out of the fear that another buyer will snatch the good in the meantime. In the standard durable goods literature, the number of buyers is fixed. However, some papers consider the arrival of new buyers. Conlisk *et al.* (1984) allow a new cohort of buyers with binary valuation to enter the market in each period and show that the seller will

[7]See the book by Talluri and van Ryzin (2004).

[8]Also see Li (2013), Mierendorff (2014) and Pai and Vohra (2013).

[9]Aviv and Pazgal (2008) consider a two-period case where the seller lacks of commitment, and so do Jerath, Netessine, and Veeraraghavan (2010).

vary the price over time.[10] In most periods, he charges a price just to sell immediately to high-value buyers. Periodically, he charges a sale price to sell to accumulated low-value buyers. Even though, similarly to Conlisk *et al.* (1984), new arrivals and heterogeneous valuation are also the drivers behind fire sales in our model, the economic channels are very different. In their papers, the seller has a discounting cost, so he charges a low price to sell to accumulated low-value buyers in order to reap some profit and avoid delay costs. However, in our model, the seller does not discount the future and can ensure a unit profit for all inventory from the fire sales income at the deadline. Since the buyers face scarcity, the seller liquidates some goods to convince future buyers to accept higher prices. Garrett(2013) considers a durable good model where (1) the seller has perfect commitment power, (2) the buyer arrives privately, and (3) the present buyer's value changes stochastically. He shows that in such a stationary environment, the seller periodically charge a low price. In these papers, the seller does not face the capacity constraint nor a deadline. As a result, the equilibrium price has to smoothly decline as the fire sale time approaches, and buyers accept non-fire sale offers to avoid the cost of discounting. However, in our model, the seller faces an inventory constraint, and he can choose the quantity in addition to the price. The high-value buyer is willing to trade at a non-fire sale price to avoid future competition, and the price jumps down at the fire sale times.

The rest of this paper is organized as follows. In Section 2, we present the model setting and define the solution concept we are going to use. In Section 3, we derive an equilibrium in the single-unit case. In Section 4, the multi-unit case is studied. In Section 5, we provide two applications of our model. Section 6 discusses some modeling choices and possible extensions of the baseline model. Section 7 concludes. All proofs are collected in the Appendices.

# 2 Model

**Environment**. We consider a dynamic pricing game between a single (male) seller who has $K$ identical and indivisible items for sale and many (female) buyers. Goods are consumed at a fixed time that we normalize to 1, and deliver zero value after. Time is continuous. The seller has the interval $[0, 1]$ of time in which to trade with buyers. There is a parameter $\Delta$ such that $1/\Delta \in \mathbb{N}$. The time interval $[0, 1]$ is divided into periods: $[0, \Delta)$,$[\Delta, 2\Delta)$, ...$[1 - \Delta, 1]$. The seller and the buyers do not discount.

**Seller**. The seller can adjust the price and supply at each moment: at time $t$, the seller posts the price $P(t) \in \mathbb{R}$, and capacity control $Q(t) \in \{1, 2, ..K(t)\}$, where $K(t) \in \mathbb{N}$ represents the

[10] Also see Sobel (1991).

number of goods remaining at time $t$, and $K(0) = K$.[11] The seller has a zero reservation value on each item, so his payoff is the summation of all transaction prices.

**Buyers**. There are two kinds of buyers: low-value buyers and high-value buyers. Each buyer has a single unit of demand. Let $v_L$ denote an low-value buyer's reservation value for the unit, and $v_H$ that of an high-value buyer, where $v_H > v_L > 0$. A buyer who buys an item at price $p$ gets payoff $v - p$ where $v \in \{v_L, v_H\}$.

**Population Dynamics**. The population structure of buyers changes differently over time. At the beginning, there is no high-value buyer in the market. As time progresses, high-value buyers arrive privately at a constant rate $\lambda > 0$. Let $N(t)$ be the number of high-value buyers at time $t$. Without loss of generality, we normalize the initial number of high-value buyers to be zero: $N(0) = 0$. An high-value buyer leaves only if her demand is satisfied.[12] For tractability, we assume that the population structure of low-value buyers is relatively predictable and stationary: at any moment, it is common knowledge that there are $M$ low-value buyers in the market regardless of the history.[13] The assumption can be justified as follows. At the beginning of the game $M$ low-value buyers arrive in the market, where $M \in \mathbb{N}$ is common knowledge. Once an low-value buyer's demand is satisfied, she leaves the market immediately, and another low-value buyer immediately arrives. For simplicity, we assume $M \geq K(0)$, which means the seller can liquidate all inventory by serving low-value buyers at any time.[14]

**Transaction Mechanism**. If the amount of demand at price $P(t)$ is less than or equal to $Q(t)$, all demands are satisfied; otherwise, $Q(t)$ randomly selected buyers are able to make purchases, and the rest are rationed. A price lower than $v_L$ is always dominated by $v_L$, so the seller's optimal price is bounded by $v_L$ and the low-value buyers' payoff is zero in any non-trivial model. Consequently, the low-value buyer's purchase decision is effectively myopic, so we do not model the low-value buyers as strategic players but assume that they will accept any price no higher than $v_L$. We define such a price as a *deal*.

**Definition 1.** *A* ***deal*** *is an offer with* $P(t) \leq v_L$.

If $i \leq Q(t)$ goods are sold at time $t$, the seller's inventory goes down. In other words, $\lim_{t' \searrow t} K(t') = K(t) - i$. Over time, as buyers make purchases, the inventory decreases after

[11] We assume $Q(t) \neq 0$. However, the seller can post a price sufficiently high to block any transactions.

[12] Our results continue to hold when high-value buyers leave the market at a rate $\rho > 0$.

[13] An added value of this assumption is that it allows us to highlight our channel to generate fire sales. In Conlisk *et al* (1984), the presence of periodic sales is driven by the arrival and accumulation of low-value buyers. Because we assume that the population structure of low-value buyers is stationary, their classical explanation of a price cycle does not work in our model.

[14] We make this assumption to simplify the presentation of the paper. Because the replacement of a departed low-value buyer takes no time, the seller can sequentially sell all inventory as soon as he wants.

purchases. Hence, the realized path of the inventory process $K(t)$ is left continuous and non-increasing over time. Once $K(t)$ hits zero or time reaches the deadline, the game ends.

**Inattention Frictions**. We assume that buyers, regardless of their reservation value and arrival times, face inattention frictions. At the beginning of each period, all buyers, regardless of their value, randomly draw an attention time $\tau$. These attention times are independently and uniformly distributed in the time interval of the current period.[15] For an high-value buyer who arrives in the period, her attention time in the current period is her arrival time. In the period where the seller posts a *deal* at time $\tau$, each buyer has an additional attention time at time $\tau$ in the current period. In the rest of this paper, we call these random attention times exogenously assigned by Nature *regular attention times*, while we call the additional attention times *deal attention times*. Each buyer observes the offer posted, $P(t), Q(t)$ and the seller's inventory size, $K(t)$ at her attention time *only*. At that time, she can decide to accept or reject the offer. Rejection is observed neither by the seller nor by other buyers. Since, without deal announcements, each buyer draws her attention time independently, once a buyer observes and decides to take an available offer $P(t) > v_L$, she will not be rationed. Thus the competition among buyers is always intertemporal when $P(t) > v_L$. At deal times when $P(t) \leq v_L$, buyers observe the offer at the same time, so there is direct competition among buyers.

**History**. A non-trivial *seller history* at time $t$, $h_S^t = \{\{P(\tau), Q(\tau)\}_{0\leq\tau<t}, \{K(\tau)\}_{0\leq\tau\leq t}\}$ summarizes all transaction information about offers in the past and all inventory information until time $t$. Let $\mathcal{H}_S$ be the set of all seller's histories. The seller's strategy $\sigma_S$ determines a price $P(t)$ and capacity control $Q(t)$ given a seller history $h_S^t$. Due to the buyers' inattention frictions, at any time before the deadline, the seller believes that more than one buyer notices an offer with probability zero. As a result, we focus on the seller's strategy space in which $Q(t) = 1$ for $P(t) > v_L$ without loss of generality.

Let $a(t)$ be an index function such that its value is 1 at an high-value buyer's attention times, and 0 otherwise. Thus, $a^t = \{a(\tau)\}_{\tau=0}^t$ records the history of an high-value buyer's past attention times up to $t$. A non-trivial *buyer history*, $h_B^t = \left\{a^t, \{P(\tau), Q(\tau), K(\tau)\}_{\tau:a(\tau)=1 \text{ and } \tau\in[0,t]}\right\}$. In other words, a buyer remembers the prices, capacity and inventory size she observed at her past attention times. Let $\mathcal{H}_B$ denote the set of all history of an high-value buyer. Following Chen (2012) and Hörner and Samuelson (2011), we focus on symmetric equilibria in which an high-value buyer's strategy depends only on her history are not on her identity. That is to say, the high-value buyer's strategy $\sigma_B$ determines the probability that she will accept the current price $P(t)$ given a buyer's history $h_B^t$. We focus on a pure strategy profile, so the seller does not randomize ($\sigma_S(\mathcal{H}_S) \in \mathbb{R}_+ \times \mathbb{N}$) and so do not the buyers ($\sigma_B(\mathcal{H}_B) \in \{0, 1\}$).

[15] Our results hold for any atom-less distribution with full support.

## 2.1 Admissible Strategies

The determination of the optimal timing for fire sales is in fact an optimal stopping time problem, which is easy to characterize in a continuous time setting; thus we study a continuous time model. However, continuous time raises obstacles to the analysis of dynamic games. First, it is well known that, in a continuous time game, a well-defined strategy may not induce a well-defined outcome. [16]

To make the game well-defined, we must impose additional restrictions on the set of strategies. Following Bergin and MacLeod (1993), we restrict the seller's choices in the admissible strategy space. Specifically, to construct the set of admissible strategies, we first restrict the strategy to the inertia strategy space. Intuitively speaking, an inertia strategy is a strategy that disallows instantaneous responses, instead allowing a player to change her decision only after a very short time lag; hence, such strategy cannot be conditional on very recent information. The set of all inertia strategies includes strategies with arbitrarily short lags, so it may not be complete. To capture the instantaneous response of players, we complete the set and use the completion as the feasible strategy set of our game. We identify the associated outcome of each instantaneous response strategy as follows. First, we find a sequence of inertia strategies converging to the instantaneous strategy. In such a sequence, each inertia strategy has a well-defined outcome, which gives us a sequence of outcomes. Second, we identify the limit of the outcome sequence as the outcome of this instantaneous response strategy. Lastly, because of the presence of inattention frictions, there is zero probability that multiple buyers observe a price higher than $v_L$ at the same time. Hence, without any loss of generality, we can restrict the strategy space such that $Q(t) = 1$ for $P(t) > v_L$. Let $\Sigma_S^*$ represent the admissible strategy space of the seller. Because high-value buyers face inattention frictions, they cannot revise their decisions instantaneously, so we do not need to impose any restrictions on their strategies; let $\Sigma_B^*$ denote the set of strategies of high-value buyers, and let $\Sigma^* = \Sigma_S^* \times \Sigma_B^*$ be the strategy space we study.

## 2.2 Payoff and Solution Concept

In general, a player's strategy depends on his or her private history. A perfect Bayesian equilibrium (PBE) in our game is a strategy profile of the seller and the buyers such that, given other players' strategies, each player has no incentive to deviate, and players update their belief via Bayes' rule where possible. However, the set of all perfect Bayesian equilibria of this game is hard

[16] The reason is that there is no well-defined "last" or "next" period in a continuous time game; hence, players' actions at time $t$ may depend on information arriving instantaneously before $t$. See Simon and Stinchcombe (1989) and Bergin and MacLeod (1993) for additional discussion.

to characterize because buyers may play private strategies depending on their private histories. We instead look for simple but intuitive equilibria: *(no-waiting) weak Markov perfect equilibria* (weak MPE), which is commonly used in dynamic pricing and Coase conjecture literature.[17] A (no waiting) weak MPE satisfies the following properties. First, the equilibrium strategy profile must be simple; that is, buyers' equilibrium strategies can be described as functions of two *public* state variables specified later. Second, on the path of play, high-value buyers do not delay their trades but make their purchases upon arrival. Third, we impose a restriction on buyers' beliefs about the underlying history off the path of play: each high-value buyer believes that there are no other previous high-value buyers in the market. Notice that the last restriction on buyers' beliefs off the equilibrium path implicitly rules out the possibility that the seller can manipulate buyers' beliefs and therefore their willing to pay by charging high price. The restriction is necessary to obtain weak Markov equilibria. Otherwise, after some histories, buyers who saw different deviating prices may have heterogenous private beliefs about the number of high-value buyers present and therefore heterogenous willingness to pay so that their strategies have to be non-Markovian.

Note that, off the path of play, high-value buyers may wait because of the deviation of the seller: the seller can post an unacceptable price for a time period in which high-value buyers have to wait for future offers. However, each buyer can observe offers at her past attention times and, for the rest of time, she has to form a belief about the underlying history. The perfect Bayesian equilibrium concept does not impose any restrictions on those beliefs where the Bayes' rule does not apply. To support a (no-waiting) weak Markov equilibrium, we assume that each high-value buyer believes that the seller follows the equilibrium pricing strategy in such time periods. Since each buyer can only sample finitely many times, she can have seen at most finitely many deviating prices at her past attention times. However, the total measure of such time periods is zero. Since each high-value buyer draws her attention time independently, the probability that other high-value buyers also observe these deviating prices is zero. Consequently, each high-value buyer believes that no other high-value buyers are waiting in the market either on or off the path of play, which confirms her belief that the seller charges the equilibrium price in the time period in which she is inattentive.

### 2.2.1 Payoff

To define the equilibrium, we need to specify an high-value buyer's payoff given that she believes that no previous high-value buyers are waiting in the market. Given a seller's continuation strategy $\tilde{\sigma}_S \in \Sigma_S^*$, other high-value buyers' symmetric continuation strategy $\tilde{\sigma}_B \in \Sigma_B^*$, and a buyer's history $h_B^t$, an high-value buyer's expected continuation payoff at time $t$ from choosing a continuation

[17] See Fudenberg *et al.* (1985), Gul *et al.* (1986) and Ausubel and Deneckere (1989)

strategy $\tilde{\sigma}'_B \in \Sigma^*_B$ at her attention time is defined as

$$U\left(\tilde{\sigma}'_B, \tilde{\sigma}_B, \tilde{\sigma}_S, h^t_B\right) = \mathbb{E}_{\tau \in (t,1]}\left[v_H - P\left(\tau\right)\right] \Pr(\tau \in (t, 1]),$$

whenever $\tau > t$ is high-value buyers' believed transaction time and $P(\tau)$ is the transaction price. Both $\tau$ and $P(\tau)$ depend on the other players' strategies and the population dynamics of buyers. When $\tau > 1$, the buyer does not obtain the good and gets zero payoff. At her attention time $t$, an high-value buyer's payoff is

$$\max\{v_H - p_t, U\left(\tilde{\sigma}'_B, \tilde{\sigma}_B, \tilde{\sigma}_S, h^t_B\right)\},$$

Notice that the buyer believes that her continuation payoff does not depend on the current price. Hence, the buyer employs a cutoff strategy where she accepts a price if it is less than or equal to some reservation price $p$, and this reservation price is pinned down by the buyer's indifference condition

$$v_H - p = U\left(\tilde{\sigma}'_B, \tilde{\sigma}_B, \tilde{\sigma}_S, h^t_B\right).$$

Suppose all high-value buyers play a symmetric $\tilde{\sigma}_B \in \Sigma^*_B$. The payoff to the seller with stock $k$ from a strategy $\tilde{\sigma}_S \in \Sigma^*_S$ is given by

$$\Pi_k\left(\tilde{\sigma}_B, \tilde{\sigma}_S, h^t_S\right) = \mathbb{E}_\tau\left[P\left(\tau\right) + \Pi_{k-1}\left(\tilde{\sigma}_B, \tilde{\sigma}_S, h^\tau_S\right)\right],$$

where $h^t_S$ is the seller's history, $\Pi_0 = 0$. Because buyers face inattention frictions, by posting any price $P\left(1\right) > v_L$, the seller believes that no buyer will notice the offer so his expected profit is zero; on the other hand, by posting a deal price, the seller can sell as many goods as he wants. Apparently, the seller's dominant strategy is to sell all of his inventory by charging $v_L$. Hence, at the deadline, we have

$$\Pi_k\left(\tilde{\sigma}_B, \tilde{\sigma}_S, h^1_S\right) = kv_L.$$

Note that the seller may or may not believe that there are previously arrived high-value buyers waiting in the market. His belief about the number of high-value buyers depends on his previous posted prices.

### 2.2.2 Weak Markov Perfect Equilibrium

We define a weak MPE to be a PBE where an high-value buyer makes her purchase decision based on the current price $p_t$ and two state variables, calendar time $t$ and inventory size $K(t)$ and she makes the purchase upon her arrival on the equilibrium path. The seller's equilibrium strategy depends on the entire private history, but can be described as a function of two state variables on the path of play. Nonetheless, note that potential deviations can be either Markovian or non-Markovian.

Notice that in the equilibrium, the high-value buyer's strategy is a function of the calendar time and the seller's inventory size, but this does not imply that the number of other high-value buyers is payoff-irrelevant to an high-value buyer in general. In fact, an high-value buyer's continuation value does depend on her belief about the number of other high-value buyers. However, we focus on *no-waiting equilibria* where each high-value buyer believes that no other high-value buyer is waiting in the market; thus, her strategy does not depend non-trivially on her belief about the number of other high-value buyers. Hence, the high-value buyer's strategy is Markovian both on and off the path of play. Since high-value buyer's belief cannot be manipulated by the seller, an high-value buyer believes that her continuation value after any history does not depend on the current price. As a result, an high-value buyer's equilibrium strategy can be characterized by a cutoff price, which depends on two state variables. In particular, the reservation price $p_k(t)$ is pinned down by the buyer's indifference condition

$$v_H - p_k(t) = U\left(\tilde{\sigma}_B, \tilde{\sigma}_S, h_B^t\right).$$

where $U\left(\tilde{\sigma}_B, \tilde{\sigma}_S, h_B^t\right)$ denotes the buyer's equilibrium continuation payoff in equilibrium, $(\tilde{\sigma}_B, \tilde{\sigma}_S)$ is the believed continuation equilibrium strategy profile, and the current inventory $K(t) = k$ is consistent with the current buyer's history $h_B^t$. However, the seller's equilibrium strategy is non-Markovian as in the standard Coase conjecture literature.[18] Henceforth, we refer to a (no-waiting) weak MPE as an equilibrium unless it is specified otherwise.

# 3 Single Unit

We start by analyzing the game where $K(0) = 1$. The seller only has one unit to sell, so $Q(t) = 1$. We first provide an intuitive conjecture on an equilibrium of this game and verify our conjecture. Furthermore, we show that the equilibrium we proposed is the unique equilibrium.

The first observation is that the seller can ensure a profit $v_L$ because there are M low-value buyers at any time. An intuitive conjecture of the seller's strategy is to serve the high-value buyers only before the deadline to obtain a profit higher than $v_L$ and charge $v_L$ at the deadline if no high-value buyer arrives. Since an high-value buyer would like to avoid competition with (1) low-value buyers at the deadline, and (2) other high-value buyers who may arrive before the

[18] A more natural solution concept is Markov perfect equilibrium. However, it does not exist in general. Off the path of play, and similarly to the standard Coase conjecture models, the seller's continuation strategy is history-dependent. See Fudenberg and Tirole (1983) and Gul *et al.* (1983) for detail.

deadline, she is willing to forgo some surplus and accept a price higher than $v_L$. Moreover, as the deadline approaches, the competition coming from newly arrived high-value buyers becomes less and less intense, and therefore the high-value buyer's reservation price declines.

Specifically, we conjecture that in equilibrium, the seller charges a price such that: (1) high-value buyers accept it upon arrival, and (2) low type buyers make their purchases only at the deadline if the good is still available. The optimality of the seller's pricing rule implies that, before the deadline, an high-value buyer is indifferent between purchasing at time $t$ and waiting: on the one hand, if the high-value buyer strictly prefers to purchase the good immediately, the seller can raise the price a little bit to increase his profit; on the other hand, if the price is so high that the high-value buyer strictly prefers to wait, the transaction will not happen at time $t$ and all high-value buyers will wait in the market. Furthermore, we will show that accumulating high-value buyers is suboptimal for the seller because the high-value buyer's reserve price is declining over time. At the deadline, the seller will charge the price $v_L$ to clean out his stock because he believes that there are high-value buyers left.

We give a heuristic description of the equilibrium in the main text and leave the formal analysis to the Appendix. At the deadline, the high-value buyer's reservation price is $v_H$. However, the probability that an high-value buyer's regular attention time is at the deadline is zero; thus, the dominant pricing strategy for the seller is to post a deal price $v_L$ to obtain a positive profit. As a result, in any equilibrium, $P(1) = v_L$. For the rest of the time, we denote $p_1(t)$ as an high-value buyer's reservation price at her attention time $t < 1$ and the inventory size $K(t) = 1$. Consider an high-value buyer with an attention time $t \in [1 - \Delta, 1)$; thus, the probability that new high-value buyers arrive before the deadline is $1 - e^{-\lambda(1-t)}$. Suppose this high-value buyer understands that on the path of play, no high-value buyer who has arrived before her waited. Therefore, she believes that she is the only high-value buyer in the market. She then faces the following trade-off:

1. if she accepts the current offer, she gets the good for sure at a price that is higher than $v_L$;
2. if she does not accept the current offer, the seller will believe that no high-value buyer have arrived and to obtain a positive profit, he will charge a price $v_L$ to liquidate the good at the deadline. In the latter situation, the high-value buyer has to compete with M low-value buyers for the item, and the probability she is not rationed is $\frac{1}{M+1}$.

These considerations can pin down an high-value buyer's reservation price $p_1(t)$, the price at which she is indifferent between accepting the offer or not at time $t$. Specifically, the indifference condition of an high-value buyer whose attention time is $t$ is given as follows:

$$v_H - p_1(t) = e^{-\lambda(1-t)} \frac{1}{M+1} (v_H - v_L). \tag{1}$$

The left-hand side represents the high-value buyer's payoff if she purchases the good now; the right-hand side represents the expected payoff if she waits, which is risky because (1) other high-value buyers may arrive in $(t, 1)$ with a probability $1 - e^{-\lambda(1-t)}$, and (2) she has to compete with M low-value buyers at the deadline.

Letting $t \to 1$, we obtain the limit price right before the deadline,

$$p_1 \left(1^-\right) = \frac{M}{M+1} v_H + \frac{1}{M+1} v_L. \tag{2}$$

Hence, if $M$ is large, the limit price right before the deadline is very close to $v_H$. Note that $p_1 \left(1^-\right)$ is different from the high-value buyer's actual reservation price at the deadline, $v_H$. Let $U_{1-\Delta}$ denote an high-value buyer's expected utility in the last period.[19] Since her attention time, $\tilde{t}$, is a random variable, we have

$$\begin{aligned} U_{1-\Delta} &= \int_{1-\Delta}^{1} \frac{1}{\Delta} e^{-\lambda\left(\tilde{t}-1+\Delta\right)} \left[v_H - p_1\left(\tilde{t}\right)\right] d\tilde{t} \\ &= \int_{1-\Delta}^{1} \frac{1}{\Delta} \left[e^{-\lambda\Delta} \frac{v_H - v_L}{M+1}\right] d\tilde{t}. \end{aligned} \tag{3}$$

Notice that, for each $\tilde{t}$, the high-value buyer's ex ante payoff, by considering the risk of the arrival of new buyers and the price declining until $\tilde{t}$, is $e^{-\lambda\Delta} \frac{v_H - v_L}{M+1}$, which is independent of $\tilde{t}$. Hence, $U_{1-\Delta} = v_H - p_1 (1 - \Delta)$.[20]

Now, consider the high-value buyer's reservation price at an earlier time. Note that, when $K(0) = 1$, the seller can ensure a profit $v_L$ at any time by charging the fire sale price. However, he expects to charge a higher price to high-value buyers who arrive early and want to avoid competition with high-value buyers who arrive in the future and low-value buyers. As a result, the fire sale price $v_L$ is charged only at the deadline. At any other time $t$, the seller targets high-value buyers only and offers a price $p_1(t)$. Consider an high-value buyer whose attention time is $t \in [1 - 2\Delta, 1 - \Delta)$. Her indifference condition is given by

$$v_H - p_1(t) = e^{-\lambda(1-\Delta-t)} U_{1-\Delta}, \tag{4}$$

where the left-hand side represents the high-value buyer's payoff if she purchases the good now; the right-hand side represents the expected payoff if she waits. With probability $e^{-\lambda(1-\Delta-t)}$, she

[19]Notice that $U_{1-\Delta}$ is not the buyer's continuation value at the beginning of the last period, but rather the expected continuation value in the last period over the buyer's attention time, which is equal to the left-limit of the buyer's continuation value at $1 - \Delta$, i.e., $U_{1-\Delta} = \lim_{t \nearrow 1-\Delta} U\left(\tilde{\sigma}_B, \tilde{\sigma}_S, h_B^t\right)$.

[20]This implies that (1) at the beginning of the last period, an high-value buyer is indifferent between all possible assigned attention times in the current period, and (2) the high-value buyer's equilibrium continuation value is continuous with respect to $t$ at $1 - \Delta$.

is still in the market at the beginning of the next period and the good is still available; so she can draw a new attention time in the last period and expect a payoff $U_{1-\Delta}$. As $t$ goes to $1-\Delta$, $v_H - p_1(t)$ converges to $U_{1-\Delta}$, so the buyer's equilibrium continuation value is continuous at $1-\Delta$. As a result, $p_1(t)$ is differentiable in $[1-2\Delta, 1)$. Repeating the argument above for $1/\Delta$ times, one can construct the high-value buyer's reservation price $p_1(t)$ for $t \in [0,1)$. It can be shown that $p_1(t)$ satisfies the following differential equation (ODE, henceforth)

$$\dot{p}_1(t) = -\lambda(v_H - p_1(t)) \text{ for } t \in [0,1), \tag{5}$$

with a boundary condition (2). Since the high-value buyer will not accept any price higher than $v_H$, $p_1(t) \leq v_H$ for any $t$, $p(t)$ declines over time. The speed at which the price declines is determined by the arrival rate of high-value buyers, which measures the risk of competition. In our conjectured equilibrium, the seller charges $p_1(t)$ for time $t \in [0,1)$ and charges $v_L$ at time $t=1$.

Similarly, we can derive the seller's payoff $\Pi_1(t)$. At the deadline, $\Pi_1(1) = v_L$ because the good is sold for sure at the fire sale price. Before the deadline, for a small $dt > 0$, the profit follows the following recursive equation:

$$\begin{aligned} \Pi_1(t) &= p_1(t)\lambda dt + (1-\lambda dt)\,\Pi_1(t+dt) + o(dt), \\ &= p_1(t)\lambda dt + (1-\lambda dt)\left[\Pi_1(t) + \dot{\Pi}_1(t)\,dt\right] + o(dt), \end{aligned}$$

where an high-value buyer arrives and purchases the good at time $t$ with probability $\lambda dt$, and no high-value buyer arrives with a complementary probability. By taking $dt \to 0$, the seller's profit must satisfy the following ODE:

$$\dot{\Pi}_1(t) = \lambda\left[\Pi_1(t) - p_1(t)\right], \tag{6}$$

with a boundary condition $\Pi_1(1) = v_L$. Note that even though the equilibrium price is not continuous in time at the deadline, the seller's profit is because the probability that the transaction happens at a price higher than $v_L$ goes to zero as $t$ approaches the deadline.

In short, in our conjectured equilibrium, high-value buyers accept a price not higher than their reservation price $p_1(t)$, the seller posts such price for any $t < 1$, and he posts $v_L$ at the deadline. No high-value buyer waits on the path of play. The next question is whether players have any incentive to follow the conjectured equilibrium strategies. A simple observation is that no high-value buyer has the incentive to deviate since she is indifferent between taking and leaving the offer at any attention time. What about the seller? Does the seller have the incentive to do so and accumulate high-value buyers for a while before the deadline? The answer again is no. This is because each buyer believes that no high-value buyers are waiting in the market, and the seller is going to follow the equilibrium pricing rule in the continuation play. Since the high-value buyer's reservation price declines over time, the seller always wants to serve the earliest high-value buyer. Formally,

**Proposition 1.** *Suppose $K = 1$. There is a unique equilibrium in which,*

1. *for any non-trivial seller's history, the seller posts a price, $P(t)$ such that*

$$P(t) = \begin{cases} p_1(t), & \text{if } t \in [0,1), \\ v_L, & \text{if } t = 1, \end{cases}$$

*where*

$$p_1(t) = v_H - \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)},$$

2. *an high-value buyer accepts a price at her attention time $t \in [0,1)$ if and only if it is less than or equal to $p_1(t)$ and she accepts any price no higher than $v_H$ at the deadline.*

Figure 2 presents the simulation of the equilibrium price path. Given the equilibrium strategy profile, the seller's equilibrium expected payoff at $t$ can be calculated as follows:

$$\Pi_1(t) = \int_t^1 e^{-\lambda(s-t)} \lambda p_1(s)\, ds + e^{-\lambda(1-t)} v_L.$$

Notice that neither $p_1(t)$ nor $\Pi_1(t)$ depends on $\Delta$! This is because at any time each high-value buyer is indifferent to purchasing the good or not purchasing it. Fire sales appear with positive probability at the deadline only, effectively creating a last-minute deal. With probability $e^{-\lambda}$, no high-value buyer arrives in the market and the seller posts the last-minute deal. The good is not allocated to an low-value buyer unless no high-value buyer arrives. As a result, the allocation rule is efficient.

# 4 Multiple Units

In this section we consider the general case in which the seller has $K > 1$ units to sell. Since most of the intuition can be explained using the two-unit case, we provide a heuristic description of the equilibrium in a two-unit case, and we then state the equilibrium for $K > 2$.

## 4.1 The Two-Unit Case

Consider the case where $K = 2$. A simple observation is that, after the first transaction at time $\tau$, $K(t) \leq 1$ for $t \in (\tau, 1]$, and what happens afterwards is characterized by Proposition

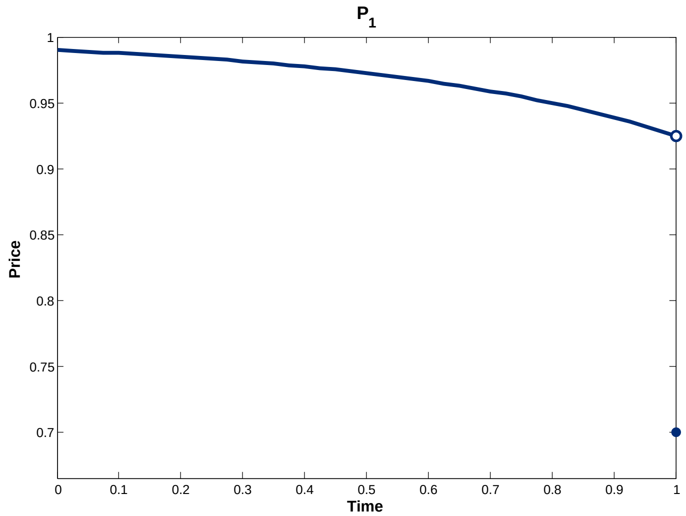


Figure 2: The equilibrium price path in the single-unit case, $K = 1$. The parameter values are $v_H = 1, v_L = 0.7, M = 3$, and $\lambda = 2$.

1. The question is how the first transaction happens: what is the sale price and when does the high-value buyer accept the offer? Note that the seller has a choice to post a price $v_L$ at any $t$. Since this price is low enough for low-value buyers to afford, a transaction will happen for sure and the seller's stock switches to $K(t^+) = K(t) - 1$. In equilibrium, the earliest time at which the seller charges price $v_L$ for the ?rst item is denoted by $t_1^*$. Formally, fix an equilibrium,

$$t_1^* = \inf\{t \geq 0 | K(t) = 2, P(t) = v_L\},$$

where $P(t)$ is consistent with the seller's equilibrium strategy. In principle, when $K(t) = 2$, $t_1^*$ can be any time before or at the deadline. As we have shown in Proposition 1, in any continuation game with $K(t) = 1$, on the equilibrium path, the seller charges the price $v_L$ only at the deadline; hence, the last equilibrium fire sale time is always $t_0^* = 1$. However, it is not clear yet whether $t_1^*$ is greater than or equal to 1. Note that, because of the scarcity of the goods at the price $v_L$, an high-value buyer may be rationed at $t_1^*$. Consequently, she is willing to pay a higher price before $t_1^*$.

We conjecture that the unique equilibrium should satisfy the following properties. Before $t_1^*$, the seller posts a price such that an high-value buyer is willing to purchase the good upon her arrival. When an high-value buyer buys the good, the amount of stock held by the seller jumps to one. From that moment on, the equilibrium is described by Proposition 1. Similar to equation (5) in the single-unit case, when $K(t) = 2$, an high-value buyer's reservation price at $t \leq t_1^*$, $p_2(t)$, satisfies the following ODE:

$$\dot{p}_2(t) = -\lambda [p_1(t) - p_2(t)] \text{ for } t \in [0, t_1^*) \tag{7}$$

The intuition is as follows. Suppose, at $t < t_1^*$, an high-value buyer sees the price $p_2(t)$. It is risky for her to wait because a new high-value buyer arrives at rate $\lambda$ and gets the first good at price $p_2(t)$, in which case the earlier H-buyer can get the second good only at price $p_1(t)$. At her attention time $t$, the high-value buyer is indifferent between taking the current offer and waiting only if the effect of the declining price, which is measured by $\dot{p}_2(t)$, can compensate the possible loss.

Since the seller may obtain a higher per-unit-profit by selling a good to an high-value buyer instead of to an low-value buyer, a reasonable conjecture is as follows. In equilibrium, the seller does not run any fire sales prior to the deadline. In other words, the first fire sale time is $t_1^* = 1$, and the seller's optimal price path, $P(t)$, is such that (1) $P(t) > v_L$ for $t < 1$, (2) an high-value buyer takes the offer when she arrives, and (3) the seller holds fire sales to sell the remaining inventory at the deadline. Consequently, when $K(t) = 2$, the equilibrium price satisfies the ODE (7) with $t_1^* = 1$. At the deadline, the seller has to post $v_L$, and an high-value buyer can obtain a good at the deal price with probability $\frac{2}{M+1}$; thus, the boundary condition of the ODE (7) at $t = 1$ is $p_2(1^-) = \frac{2}{M+1}v_H + \frac{M-1}{M+1}v_L$. However, this conjectured price path cannot be supported in an equilibrium.

**Lemma 1.** *In any equilibrium, $t_1^* < 1$.*

Lemma 1 rules out the aforementioned conjecture. We explain the intuition by using apagoge. Suppose there is an equilibrium support the aforementioned price path. In such an equilibrium, $p_2(t) < p_1(t)$ for $t < 1$ because an high-value buyer is more likely to get the good when the inventory is 2. As $t$ approaches the deadline, the probability that a new high-value buyer arrives before the deadline becomes smaller and smaller. The probability that only one high-value buyer arrives before the deadline is approximated by $\lambda(1-t)$. In this case,

1. if the seller follows the equilibrium strategy and posts price $p_2(t)$, his profit is $p_2(\tau) + v_L$ where $\tau$ is the high-value buyer's arrival time.

2. Alternatively, if the seller deviates and runs a one-unit fire sale before the arrival time, he can ensure a payoff of $v_L$ immediately and expect a price $p_1(\tau) > p_2(\tau)$ in future.

and thus, the *benefit* of the deviation is $[p_1(\tau) + v_L] - [p_2(\tau) + v_L]$ in this event, which can be approximated by $p_1(1) - p_2(1)$. On the other hand, there is an *opportunity cost* to holding a fire sale before the deadline. Multiple high-value buyers may arrive before the deadline and the probability of this event is approximated by $\lambda^2 (1-t)^2$. In this case, if the seller naively posts price $p_2(t)$ and $p_1(t)$ to the end but does not post $v_L$, his profit is approximated by $p_2(1) + p_1(1)$. Thus the opportunity cost of the fire sale is approximated by $p_2(1) - v_L$ when $t$ is close to the deadline. As $t$ goes to 1, $\lambda^2 (1-t)^2$ goes to zero at a higher speed than $\lambda(1-t)$; thus, the cost is dominated by the benefit for $t$ sufficiently close to 1, and therefore, the seller will post the fire sale price $v_L$ to liquidate one unit at $t_1^* < 1$ to raise future high-value buyers' reservation price. In other words, the fire sale plays the role of a commitment device.

We leave the formal equilibrium construction to the Appendix but provide the intuition here. Suppose buyers believe that the fire sale time is $t_1^*$. For $t < t_1^*$, and $K(t) = 2$, an high-value buyer's reservation price satisfies the ODE (7); for $t \in [t_1^*, 1)$ and $K(t) = 2$, high-value buyers believe that the seller is going to post $v_L$ immediately, and thus their reservation prices satisfy the following equation

$$v_H - p_2(t) = \frac{1}{M+1}(v_H - v_L) + \frac{M}{M+1}[v_H - p_1(t)],$$

where the left-hand side of the equation is the high-value buyer's payoff from accepting her reservation price and obtaining the good now, and the right-hand side is her expected payoff from rejecting the current offer.[21] With probability $\frac{1}{M+1}$, the high-value buyer gets the good at the deal price right after time $t$, and with a complementary probability, an low-value buyer gets the deal and the high-value buyer has to take $p_1(t)$ at her next attention time. Since $\Delta$ is small, one can ignore the arrivals and the time difference between two adjacent attention times of the high-value buyer, and therefore, the high-value buyer's reservation price at $t \in [t_1^*, 1)$ is given by

$$p_2(t) = \frac{1}{M+1} v_L + \frac{M}{M+1} p_1(t). \tag{8}$$

Since $p_1(t)$ is continuous on $[0,1)$, $p_2(t)$ must be right continuous at $t_1^*$. On the other hand, the incentive-compatibility condition of the high-value buyer implies that $p_2(t)$ must be left continuous at $t_1^*$, and thus the boundary condition of the ODE (7) is

$$p_2(t_1^*) = \frac{1}{M+1} v_L + \frac{M}{M+1} p_1(t_1^*). \tag{9}$$

[21]In a continuous time game, one cannot define the "next period" appropriately. Hence, there is no well-defined fire sale time right after time $t$. However, in our model, to make sure the buyer's strategy is well-defined, we only need to appropriately define the buyer's continuation value if she decides to wait. Since the strategy space is the completion of the space of inertia strategy, the buyer's continuation payoff can be uniquely specified by the limit of a sequence of payoffs induced by a sequence of associated inertia strategies.

As a result, an high-value buyer's reservation price at $t$ when $K(t) = 2$ critically depends on her belief about $t_1^*$.

Given high-value buyers' common beliefs about $t_1^*$, and their reservation prices when $K(t) = 2$, the seller's problem is to choose his optimal fire sale time to maximize his profit; i.e.:

$$\Pi_2(t) = \max_{t_1} \int_t^{t_1} e^{-\lambda(s-t)} \lambda [p_2(s) + \Pi_1(s)] \, ds + e^{-\lambda(t_1 - t)} [v_L + \Pi_1(t_1)] .$$

In equilibrium, buyers' beliefs are correct, so the seller's optimal fire sale time is $t_1^*$ itself. The first-order condition of the seller's problem at $t_1^*$ is:

$$\lambda [p_2(t_1^*) - v_L] + \dot{\Pi}_1(t_1^*) = 0. \tag{10}$$

At $t_1^*$, a transaction happens at price $v_L$ for sure, so we have

$$\Pi_2(t_1^*) = \Pi_1(t_1^*) + v_L, \tag{11}$$

which is the well-known *value-matching condition*.

For $t < t_1^*$, and $K(t) = 2$, the seller posts the high-value buyer's reservation price, $p_2(t)$, and his expected profit is given by

$$\Pi_2(t) = \lambda dt [p_2(t) + \Pi_1(t + dt)] + (1 - \lambda dt) \Pi_2(t + dt) + o(dt) .$$

Taking $dt \to 0$, the seller's profit satisfies the following *Hamilton-Jacobi-Bellman* (henceforth, HJB) equation

$$\dot{\Pi}_2(t) = -\lambda [p_2(t) + \Pi_1(t) - \Pi_2(t)] . \tag{12}$$

Combining (10), (11) and (12) at $t_1^*$ yields

$$\dot{\Pi}_2(t_1^*) = \dot{\Pi}_1(t_1^*) , \tag{13}$$

which is known as the *smooth-pasting condition*.

As a result, at the equilibrium fire sale time $t_1^*$, three necessary conditions (9), (11), and (13) must hold. The necessity of the value-matching condition (11) and the smooth-pasting condition (13) comes from the optimal stopping time property of the interior fire sale time, and condition (9) results from the high-value buyers' incentive-compatible condition. When time is arbitrarily close to $t_1^*$, the probability that new high-value buyers arrive before $t_1^*$ shrinks, and the high-value buyer needs to choose between taking the current offer and waiting to compete with the low-value buyers for the deal. Therefore, her reservation price must make the high-value buyer indifferent between taking it and rejecting it. If $t$ is not close to $t_1^*$, the competition from newly arrived high-value

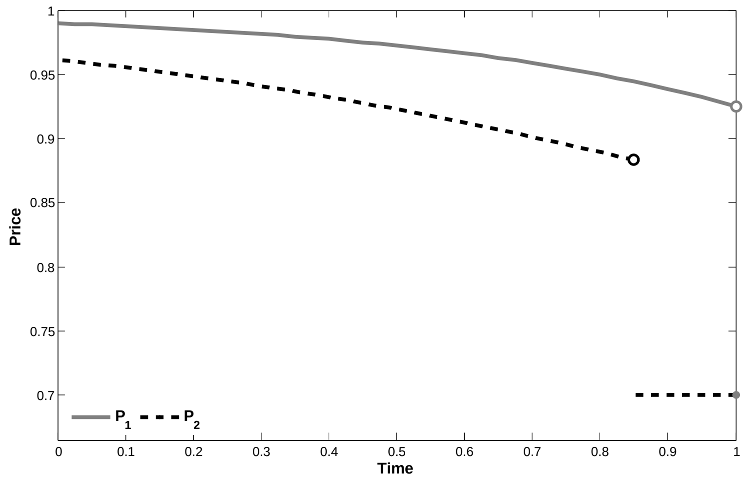


Figure 3: The equilibrium price path for the two-unit case. The solid line is the equilibrium price when $K(t) = 1$, and the dashed line is that when $K(t) = 2$. The first fire sale time is $t_1^* = 0.84$. When $t \geq t_1^*$ and $K(t) = 2$, the seller posts the deal price, $v_L$, to liquidate the first unit immediately. The parameter values are $v_H = 1, v_L = 0.7, M = 3$, and $\lambda = 2$.

buyers before $t_1^*$ is non-trivial, and therefore, to convince an high-value buyer to accept the price, her reservation price must satisfy the ODE (7) with a boundary condition (9) at $t_1^*$. Figure 3 shows a simulated equilibrium price path. The following proposition formalizes our heuristic description of the equilibrium.

**Proposition 2.** *Suppose $K(0) = 2$. There exists a unique equilibrium. The high-value buyer's equilibrium strategy is characterized by a reservation price: $p_1(t)$ and $p_2(t)$ when $t < 1$, $K(t) = 1$ and 2, respectively where $p_1(t)$ is specified in Proposition 1 and*

$$p_2(t) = \begin{cases} v_H - \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)} \left[ e^{\lambda\left(1-t_1^*\right)} + \frac{M}{M+1} + \lambda\left(t_1^* - t\right) \right], & t \in [0, t_1^*), \\ \frac{1}{M+1} v_L + \frac{M}{M+1} p_1(t), & t \in [t_1^*, 1), \end{cases}$$

*On the path of play, the seller posts price*

$$P(t) = \begin{cases} p_1(t), & \text{if } t < 1 \text{ and } K(t) = 1, \\ p_2(t), & \text{if } t < t^* \text{ and } K(t) = 2, \\ v_L, & \text{if } t = t_1^* \text{ and } K(t) = 2, \text{ or } t\text{=}1, \end{cases}$$

*where the fire sale time satisfies $t_1^* < 1$, and the associated quantity is $Q(t) = 1$ for each $t \in [0, 1]$.*

On the path of play, the seller's profit when $K(t) = 2$ is given by

$$\Pi_2(t) = \begin{cases} \Pi_1(t) + v_L, & t \geq t_1^*, \\ \int_t^{t_1^*} e^{-\lambda(s-t)} \lambda \left[p_2(s) + \Pi_1(s)\right] ds + e^{-\lambda\left(t_1^* - t\right)} \left[v_L + \Pi_1(t_1^*)\right], & t < t_1^*. \end{cases}$$

where $t_1^*$ satisfies conditions (9),(11) and (13), $\Pi_1(t)$ is characterized in Proposition 1, and $p_2(t)$ satisfies ODE (7) with a boundary condition (9).

In the equilibrium, for $t < t_1^*$, the price is $p_2(t)$ and it jumps up to $p_1(t)$ when a transaction happens. If there is no transaction before $t_1^*$, the price jumps down to $v_L$, and one unit is sold immediately; it then jumps up to the path of $p_1(\cdot)$. The first fire sale actually happens at $t_1^*$ with probability $e^{-\lambda\left(1-t_1^*\right)}$. Simple algebra yields that

$$\frac{dt_1^*}{d\lambda} = \frac{1 - t_1^*}{\lambda} > 0,$$

where the strict inequality comes from the fact that $t^* < 1$ in the equilibrium. The intuition is that the higher the arrival rate of high-value buyers, the higher the probability that multiple high-value buyers arrive before the deadline; and thus the seller's opportunity cost by serving low-value buyers is lower. Hence, the seller would postpone the fire sales.

Since two or more high-value buyers arrive after $t_1^*$ with positive probability, the allocation is *inefficient*. However, in contrast to the standard monopoly pricing game where the inefficiency results from the seller's withholding, the inefficiency in this game arises from the scarce good being misallocated to low-value buyers when many high-value buyers arrive late.[22]

It is worth noting that our equilibrium prediction on the fire sale critically depends on two assumptions: (1) the high-value buyers are forward-looking, and (2) that the number of low-value buyers is finite. First, suppose each high-value buyer can draw at most one attention time, and thus she cannot strategically time her purchase. As a result, for any $t \in [0,1]$ and $k \in \mathbb{N}$, the high-value buyers' reservation price is always $p_k(t) = v_H$ for any $k$. Hence, the optimal price path $P(t) = v_H$ when $t < 1$ and $P(t) = v_L$ when $t = 1$ for any $k \in \mathbb{N}$. In this particular model, the price is constant until $t = 1$. In a more general model, for example, buyers may have a heterogeneous reservation value $v \in [v_L, v_H]$. Talluri and van Ryzin (2004) consider many variations of this model. In these models, the result does not depend on the seller's commitment power. Second, when the number of low-value buyers, $M$, is finite, an high-value buyer can get a good at the deal price with positive probability. However, if $M$ is infinity, the probability that an high-value buyer can get a good at the deal price is zero. Hence, the difference between $p_1(t)$ and $p_2(t)$ disappears. In fact, an high-value buyer cannot expect any positive surplus and is willing to accept a price $v_H$ at any time.

[22] One way to improve the efficiency is to allow the seller to overbook and reallocate goods in the end. See Ely *et al.* (2013) and Fu *et al.* (2012) for more analysis in a different environment.

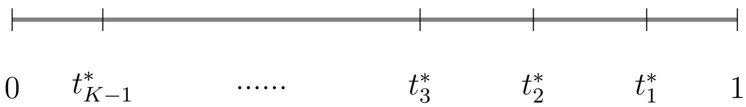


Figure 4: Equilibrium Fire Sales

## 4.2 The General Case

In general, the seller has $K$ units where $K \in \mathbb{N}$. In the equilibrium, the seller may periodically post a deal price before the deadline. Specifically, there is a sequence of fire sale times, $\{t^*_k\}_{k=1}^{K-1}$, such that $t^*_{k+1} \leq t^*_k$ for $k \in \{1, 2, ..K-1\}$. Each fire sale time $t^*_k$ represents the time at which the seller finds it optimal to hold at most $k$ units of inventory. At any time $t$, if $t \geq t^*_k$ and $K(t) > k$, the seller immediately posts a fire sale offer such that $P(t) = v_L$ and $Q(t) = K(t) - k$. See Figure 4 as an illustration. When the initial inventory $K$ is small, we can show that $t^*_{K-1} > 0$ and $t^*_k$ is strictly decreasing. As a result, on the path of play, when $t \in [0, t^*_{K-1})$, the seller serves high-value buyers only by charging the high-value buyer's reservation price $p_k(t)$ if the current inventory $K(t) = k \in \{1, 2, ...K\}$. If no transaction occurs before $t^*_{K-1}$, the seller holds a fire sale at $t^*_{K-1}$ in order to liquidate one unit so that his inventory $K(t) \leq K - 1$ afterward. By the same logic, for any $k \in \{2, ...K-1\}$, when $t \in [t^*_k, t^*_{k-1})$, the seller's equilibrium inventory $K(t)$ cannot be greater than $k$. We will show that, as long as $t^*_{k-1} > 0$, $t^*_k < t^*_{k-1}$, thus at each fire sale time $t^*_k$, the seller puts at most one unit on sale. However, when the initial inventory size $K$ is so large that there exists a $K^* < K$ such that $t^*_k = 0$ for each $k \in \{K^*, K^*+1, ...K\}$, the seller holds multiple-unit fire sales at time $t = 0$, i.e., $Q(0) = K - K^*$ and $P(0) = v_L$.

We derive the equilibrium by induction. Suppose that in the $(K-1)$-unit case, high-value buyers' reservation price is $p_k(t)$ for $k \in \{1, 2, ..K-1\}$, and the seller's equilibrium strategy is consistent with the description above. The seller's equilibrium profit is represented by $\Pi_k(t)$ for $k \in \{1, 2, ..K-1\}$. Now we construct the high-value buyers' reservation price and the seller's pricing strategy and payoff in the $K$-units case. To satisfy the high-value buyers' incentive-compatibility condition, the equilibrium price at $t$ when $K(t) = K \in \mathbb{N}$ satisfies the following differential equation:

$$\dot{p}_K(t) = -\lambda \left[p_{K-1}(t) - p_K(t)\right] \text{ for } t \in [0, t^*_{K-1}), \tag{14}$$

where $t^*_{K-1}$ is the first equilibrium fire sale time when $K(t) = K$, and

$$p_K(t) = \frac{i}{M+1} v_L + \frac{M+1-i}{M+1} p_{K-i}(t) \text{ for } t \in [t^*_{K-i}, t^*_{K-i-1}), \tag{15}$$

where $i = 1, 2, ...K - 1$ and $t_0^* := 1$. Similarly to the two-unit case, the incentive-compatible condition of the high-value buyer implies that $p_K(t)$ must be continuous at $t_{K-1}^*$; thus, the boundary condition of the ODE (14) is given by $p_K\left(t_{K-1}^*\right) = \frac{1}{M+1} v_L + \frac{M}{M+1} p_{K-1}\left(t_{K-1}^*\right)$, and therefore, the high-value buyer's best response is specified for any $t \in [0, 1]$ and $k \in \{1, 2, ...K\}$.[23]

The seller's problem is to choose the optimal fire sale time and quantity to maximize his profit. Formally,

$$\Pi_K(t) \quad = \quad \max_{t_{K-1} \in [0,1]} \int_t^{t_{K-1}} e^{-\lambda(\tau - t)} \lambda \left[p_K(s) + \Pi_{K-1}(s)\right] ds \; + \; e^{-\lambda(t_{K-1} - t)} \left[v_L + \Pi_{K-1}(t_{K-1})\right].$$

In equilibrium, buyers' beliefs are correct, so the seller's optimal fire sale time when $K(t) = K$ is $t_{K-1}^*$, which satisfies the value-matching and the smooth-pasting conditions.

If there exists an interior solution, $t_{K-1}^*$ is pinned down as follows. At $t_{K-1}^*$,

$$p_K\left(t_{K-1}^*\right) = \frac{1}{M+1} v_L + \frac{M}{M+1} p_{K-1}\left(t_{K-1}^*\right), \tag{16a}$$

$$\Pi_K\left(t_{K-1}^*\right) = \Pi_{K-1}\left(t_{K-1}^*\right) + v_L, \tag{16b}$$

$$\dot{\Pi}_K\left(t_{K-1}^*\right) = \dot{\Pi}_{K-1}\left(t_{K-1}^*\right). \tag{16c}$$

In equilibrium, we have $t_{K-1}^* \leq t_{K-2}^*$. The intuition is simple. In a no-waiting equilibrium, no previously-arrived high-value buyers are waiting in the market; thus, the demand from high-value buyers shrinks as the deadline approaches. What is more, the probability that more than $k$ high-value buyers arrive before the deadline is approximated by $\lambda^k (1 - t)^k$ when the current time $t$ is close to the deadline. Apparently, the higher $k$ is, the smaller the probability is. Hence, the larger the seller's inventory, the earlier he has an incentive to liquidate some of his units.

The following proposition formalizes our heuristic equilibrium description.

**Proposition 3.** *Suppose $K \in \mathbb{N}$. There is a unique equilibrium in which there is a sequence of fire sale times $\{t_k^*\}_{k=1}^{K-1}$ such that:*

1. $0 \leq t_{k+1}^* < t_k^* < 1$ *if* $t_k^* > 0$ *for* $k = 1, 2, ...K - 2$,

2. $t_{k+1}^* = 0$ *if* $t_k^* = 0$ *for* $k = 1, 2, ...K - 2$,

[23] Notice that $p_K(t)$ is not left continuous at $t_{K-i}^*$ for $i > 1$. At any $t$ which is arbitrarily close to but less than $t_{K-i}^*$, the high-value buyer's expected continuation value for waiting is $\frac{M+i}{M+1}(v_H - v_L) + \frac{i-1}{M+1} p_{K-i+1}(t)$. Intuitively, at $t < t_{K-i}^*$, the high-value buyer believes the seller with $K$ units of inventory is going to put $i - 1$ units on sale "immediately", which is before $t_{K-i}^*$. On the other hand, at $t_{K-i}^*$, the high-value buyer believes that the seller is going to put $i$ units on sale "immediately" so her continuation value for waiting is $\frac{M+1+i}{M+1}(v_H - v_L) + \frac{i}{M+1} p_{K-i}(t)$.

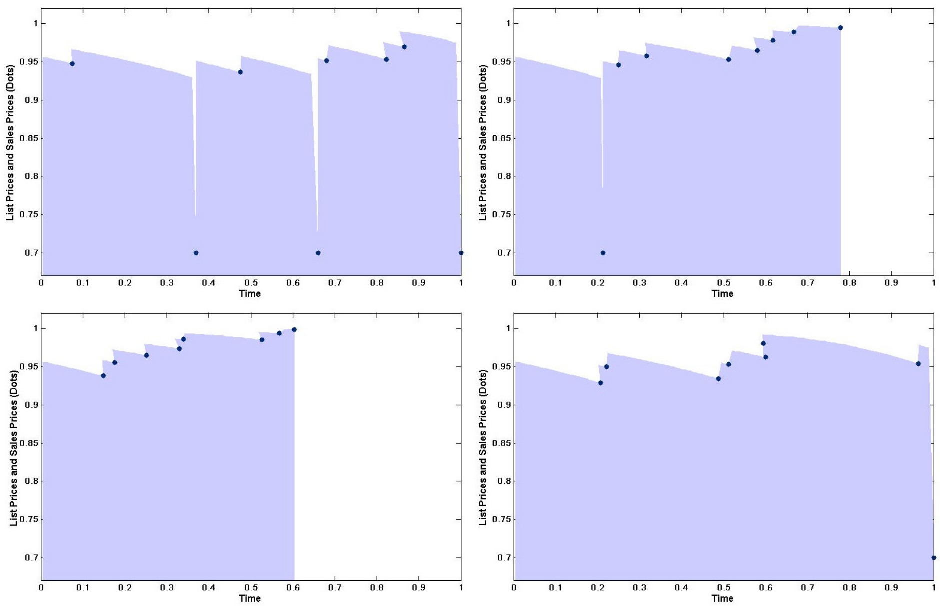


Figure 5: Simulated price paths for different realizations of high-value buyers' arrival in the 8-unit case. The upper edge of the shaded area describes the equilibrium list price, and dots indicate transactions. The parameter values are $v_H = 1$, $v_L = 0.7$, $M = 10$, $K = 8$ and $\lambda = 7$.

3. *the high-value buyers' reservation price is* $p_k(t)$ *for* $t < 1$ *and* $K(t) = k \in \{1, 2, ...K(0)\}$*; and it is* $v_H$ *at* $t = 1$ *where for each* $k = 1, 2, ...K(0)$*,* $p_k(\cdot)$ *such that*

$$\dot{p}_k(t) = -\lambda [p_{k-1}(t) - p_k(t)] \;\; for \; t \in [0, t^*_{k-1})$$

*and*

$$p_k(t) = \frac{i}{M+1} v_L + \frac{M+1-i}{M+1} p_{k-i}(t) \;\; for \; t \in [t^*_{k-i}, t^*_{k-i-1})$$

4. *on the path of play, when* $K(t) = k$*, the seller posts price*

$$P(t) = \begin{cases} p_k(t), & if \; t < t^*_{k-1}, \; and \; K(t) = k, \\ v_L, & if \; t = t^*_{k-1}, \; and \; K(t){=}k, \end{cases}$$

*with the associated quantity* $Q(t)$ *such that*

$$Q(t) = \begin{cases} K - K^*, & if \; t = 0 \; and \; \exists K^* = min\{k \in \mathbb{N} | t^*_k = 0, k < K\}, \\ 1, & otherwise. \end{cases}$$

In equilibrium, when $K(t) = k$, the price is $p_k(t)$ for $t < t^*_{k-1}$. If no transaction occurs, the price smoothly declines and jumps up to $p_{k-1}(t)$ when a transaction happens at $t$. If there is no transaction before $t^*_{k-1}$, the seller holds fire sale at time $t^*_{k-1}$ so that the price jumps down to $v_L$ and the transaction takes place immediately. After time $t^*_{k-1}$, the inventory size is reduced to $k-1$, and the price path jumps back to $p_{k-1}(\cdot)$. Consequently, a highly fluctuating price path can be generated. In Figure 5, we provide some simulations of the equilibrium price path.

Notice that, our model predicts that the price declines as the deadline approaches given the inventory size. However, on the equilibrium path, transactions also occur as the deadline approaches, so the inventory size declines, which pushes the price up. Hence, when the initial inventory is not too large and the arrival rate of the high-value buyer is not too small, the equilibrium price will statistically rise as the deadline approaches. These implications are consistent with the empirical studies by Escobari (2012). He studies price patterns in the airline industry and shows that, as the departure time approaches, the unconditional airfare rises. However, once the number of available seats is controlled, the airfare declines as the departure time approaches.

Given the equilibrium strategy profile, we can calculate the seller's equilibrium profit when $K(t) = k$ and $t \leq t^*_{k-1}$ which is given by

$$\Pi_k(t) = \begin{cases} \{\int_t^{t^*_{k-1}} e^{-\lambda(s-t)} \lambda \left[p_k(\tau) + \Pi_{k-1}(\tau)\right] d\tau & \text{if } t < t^*_{k-1}, \\ \quad + e^{-\lambda\left(t^*_{k-1}-t\right)} \left[v_L + \Pi_{k-1}\left(t^*_{k-1}\right)\right]\}, & \\ v_L + \Pi_{k-1}(t), & \text{if } t = t^*_{k-1}, \end{cases}$$

where $t^*_{k-1}$ satisfies conditions (16a), (16b) and (16c).

# 5 Applications

In this section, we apply the baseline model to consider simple applications.

## 5.1 Optimal Inventory Decision

In the baseline model, we treat the seller's initial inventory $K$ as a parameter. In short the run, it is a reasonable assumption for most relevant industries. However, in the long run, the seller can adjust his inventory size. To consider the seller' optimal inventory choice, we assume that before time 0, the seller can choose an initial inventory size by paying some cost. For simplicity, the marginal cost of inventory production is assumed to be constant, and greater than $v_L$.[24] The following proposition shows that the seller optimal initial inventory exists.

[24]Otherwise, the solution of the optimal inventory size is not well-defined in our model since the seller can make profit from producing and selling infinitely many goods to low-value buyers.

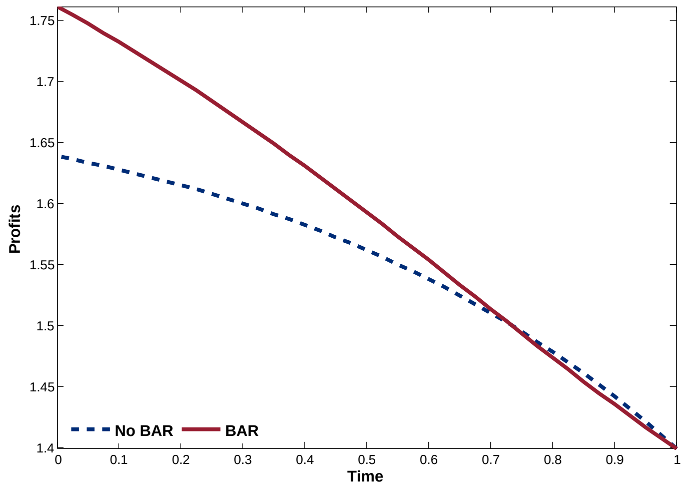


Figure 6: The solid line is the profit with BAR, while the dashed line is that without BAR. When $t$ is close to 0, the profit with BAR is higher than that without BAR. The parameter values are $v_H = 1, v_L = 0.7, M = 3$, and $\lambda = 2$.

**Proposition 4.** *Suppose that the constant marginal cost of production $c$ is strictly greater than $v_L$. There exists a finite natural number $K_c = \arg\max_{k \in \mathbb{N}} \Pi_k(0) - kc$.*

The intuition behind the proposition is very simple. The marginal benefit of adding one more unit of the inventory is $\Pi_{k+1}(t) - \Pi_k(t)$ for $t = 1$ for each $k$, while the marginal cost is $c$ which is independent of $k$. We can show that the marginal benefit is strictly decreasing in $k$, and there is a finite $K^*$ s.t. the marginal $\Pi_{K^*+1}(t) - \Pi_{K^*}(t) = v_L$. Since $c > v_L$, we can find a finite $K_c \leq K^*$ s.t. the marginal surplus of adding one more unit of inventory is positive only if $k < K_c$.

## 5.2 Best Available Rate

In the baseline model, we assume the seller has no commitment power. What if the seller has partial commitment power? In practice, sellers in both the airline and the hotel industries sometimes employ a best available rate (BAR) policy and commit to not posting price lower than this best rate in the future. Does the seller have the incentive to do so in our model? Suppose the seller can commit to not posting a deal before the deadline. Then the seller may benefit. The intuition is as follows. An high-value buyer's reservation price depends on the next fire sale time. If there is a deal soon, the reservation price is low, since there is a non-trivial probability that an low-value buyer can obtain a good at the fire sale price. At the beginning of the game, if the seller can employ a BAR and commit to not posting $v_L$ before the deadline, he can charge a higher price conditional on the inventory size. To illustrate the idea, we can consider the two-unit case. The seller's payoff by committing $P(t) > v_L$ for $t < 1$ is

$$\Pi_2^{BAR} = \int_0^1 e^{-\lambda s} \lambda \left[ p_2(s) + \Pi_1(s) \right] ds + e^{-\lambda} 2 v_L,$$

such that $p_2(t)$ satisfies the ODE (7) with a boundary condition $p_2(1^-) = \frac{M-1}{M+1} v_H + \frac{2}{M+1} v_L$. By committing to no fire sale before the deadline, the seller can ask a higher price when $K(t) = 2$. As a result, $\Pi_2^{BAR} > \Pi_2(0)$ for certain parameters. In Figure 6, we plot the profit with BAR, $\Pi_2^{BAR}(t)$ and that without it, $\Pi_2(t)$. In the beginning $\Pi_2^{BAR}(t) > \Pi_2(t)$. As time goes on, the difference between them vanishes and becomes negative when the time is very close to the deadline.

# 6 Discussion

This section discusses the role of each assumption used in our model.

## 6.1 Inattention Frictions

We assume that buyers face inattention frictions. There are two kinds of attention times: regular attention, which is randomly and independently drawn by Nature in each period, and deal attention, which is triggered by the deal alert. Buyers observe the price and make their purchase decisions at their attention times. The assumption provides some technical convenience: it ensures the existence of weak Markov perfect equilibria. However, a natural question is whether it is restrictive to assume that the regular attention time is completely exogenous. After all, people can endogenously choose their attention times to some extent. We believe the exogeneity of regular attention is less restrictive than it appears in our model. First, in a no-waiting equilibrium, the high-value buyer is indifferent between accepting the price immediately or not at almost all times. As a result, even though an high-value buyer is allowed to choose her regular attention time endogenously in each period, she has no strong preference among all the feasible choices, i.e., an high-value buyer has no incentive to disobey the regular attention time "randomly assigned by Nature" in any period. Second, in a no-waiting equilibrium, an high-value buyer would make her purchase upon her arrival. Thus, the seller has no incentive to draw extra regular attention.[25]

[25]Similarly, in a no-waiting equilibrium, the seller also has no incentive to manipulate buyers' deal attention times (or cutoff). The reason hold a fire sale is to quickly "throw away" some inventory to low-value buyers because the seller believes that there are no high-value buyers in the market. However, low-value buyers would only make their purchases when the price is not greater than $v_L$. On the one hand, sending a deal alert for a price higher than $v_L$

The presence of the deal attention also plays a critical role in our model. It has two implications: (1) once the seller holds fire sales, the goods can be sold immediately, and (2) high-value buyers' equilibrium reservation price is smooth. The first implication is easy to understand, while the second deserves more explanation. Briefly speaking, without the extra attention drawn by the deal alert, the high-value buyer's reservation price is not continuous but jumps up at the beginning of the fire sale period, which undermines the existence of the no-waiting equilibrium. Specifically, consider a two-unit model where buyers observe offers only at their regular attention times. Suppose that the equilibrium fire sale time is in the $lth$ period: $t_1^* \in [l\Delta, (l+1)\Delta)$ for some $l = 0, 1, 2, ....$ At time $t_1^*$, the seller posts a fire sale price until one unit is sold. Notice that the buyer strictly prefers to draw an attention time at or right after $t_1^*$ since the probability that she takes the fire sale offer is close to 1. The high-value buyers whose attention time in the $lth$ period is before time $t^*$, can get the fire sale offer only if there is no other buyer noticing the fire sale offer in the $lth$ period. One can show that the buyer's continuation value is discontinuous at $l\Delta$: $\lim_{t \nearrow l\Delta} U(\tilde{\sigma}_B, \tilde{\sigma}_S, h_B^t) = U_{l\Delta} > \lim_{t \searrow l\Delta} U(\tilde{\sigma}_B, \tilde{\sigma}_S, h_B^t)$. The reason is as follows. Before the realization of her attention time in the $lth$ period, the high-value buyer believes that her attention time may be right after the fire sale time $t^*$. If so, she is more likely to get the fire sale offer and obtain a high payoff. Once an high-value buyer realizes her attention time in the $lth$ period is before $t^*$, say $l\Delta^+$, she will immediately adjust the belief that she can get the fire sale offer significantly downward. As a result, the buyer's reservation price jumps at $l\Delta$, which gives the seller incentive to post an unacceptable price to "save" some buyers arriving in the end of the $(l-1)th$ period and serve them in the $lth$ period. The seller's incentive to accumulate buyers will destroy the existence of the no-waiting equilibrium, even though we believe the economic insight we delivered still exists in other PBE. The presence of the deal attention ensures that the high-value buyer's believed probability that she will take the fire sale offer smoothly declines over time.[26]

While the assumption of inattention frictions is made for technical convenience, we believe it is very natural in many dynamic settings. First, it is natural to assume that buyers do not pay

is meaningless since low-value buyers would not make their purchases. On the other hand, the seller also has no incentive to lower the deal alert cutoff price because it will delay the transaction and increase the possibility that a newly arrived high-value buyer gets the bargain.

[26] An alternative assumption is to eliminate the deal alert assumption but assume that each buyer has infinitely many regular attention times in each period: the first attention time, $\tau_1$ is uniformly drawn from the support $[l\Delta, (l+1)\Delta)$, the second attention time, $\tau_2$ is uniformly drawn from $(\tau_1, (l+1)\Delta)$, and the third $\tau_3$ is from $(\tau_2, (l+1)\Delta)$... In such a model, a buyer can draw infinitely many attention times, and at each of them, the buyer believes that the conditional probability that she takes the fire sales offer is $\frac{1}{M+1}$, so her continuation value is continuous at the beginning of each period so that the seller has no incentive to accumulate buyers, and therefore, we can construct a no-waiting equilibrium. Due to the lack of deal attention, it takes time to sell goods at the fire sale price.

attention to the price all the time due to their other daily responsibilities. In reality, average consumers can only spend a limited time on shopping activities. Recently, it has been shown that, in online markets, the limited attention of consumers significantly affects the revenue of different sales mechanisms.[27] Second, in most markets, especially online markets, buyers cannot coordinate their purchase times, but make their own purchase decisions independently of each other. Thus, our assumption that buyers independently "draw" their attention times is reasonable. Ambrus *et al.* (2014) consider an eBay-like dynamic auction model in which bidders have limited attentions and can only place bids at random times. They show that bidders' limited attention induces many different equilibria which involve gradual bidding, periods of inactivity, and delays in bidding until the end of the auction approaches. Last, we assume that the seller can draw extra attention from the market by charging an extremely low price, which is consistent with the real life experience. Recently, a number of studies have explored the role of consumers' attention in a pricing setting. For example, de Clippel *et al.* (2014) analyze the impact of consumers' limited attention in a competition setting. They show that firms' prices can deflect or draw attention to their markets, and therefore, limited attention introduces a new dimension of competition across markets. Another example is Öry (2013) who assumes that the seller can draw extra attention from the consumers by holding a costly advertised sale.

## 6.2 Asymmetry in Population Dynamics

We assume that the number of low-value buyers is constant: once an low-value buyer leaves, another low-value buyer replace her "immediately". This assumption, together with the deal alert assumption, implies that at any time the seller can get ride of "excess" inventory as soon as he wants. This assumption is critical to the existence of the weak Markov equilibrium. It ensures that all buyers share common belief about the number of low-value buyers, which determines the probability that an high-value buyer can get the fire sale offer at a fire sale time, $\frac{1}{M+1}$. As a result, all high-value buyers have identical continuation values if they decide to wait, and thus, they have the identical purchase strategy: a reservation price that depends on two state variables after any history. In a model where low-value buyers also arrive privately, one has to deal with private strategy unless making assumptions on the observability of the history of fire sales. To see the reason, suppose that low-value buyers also arrive market according to a Poisson process. The number of present low-value buyers $M(t)$ will be a random variable, and players form beliefs on $M(t)$ since it determines the probability that an high-value buyer takes the fire sale offer and

[27] Einav *et al.* (2013) show that the decline of the attention of buyers on eBay auction makes the online seller prefer price posting mechanism to eBay auction.

therefore determines her continuation value. The law of motion of $M(t)$ is determined by the inflow process (arrival) and the outflow process (departure after trade). The former is common knowledge, while the latter relies on the previous transactions at the fire sale price and it is not observed by buyers symmetrically: in a no-waiting equilibrium, if a good is purchased at a fire sales price, it is almost surely purchased by an low-value buyer. At her attention time $t$, a buyer knows that $K(0) - K(t)$ units were sold, but she does not know how many of them were sold to low-value buyers via fire sales, thus she does not knows how many low-value buyers have departed. Due to the their private heterogenous histories, buyers may have heterogenous beliefs about $M(t)$ and therefore they have to play private strategies after some histories. In particular, suppose that an high-value buyer deviates from the no-waiting equilibrium by waiting in the market. The seller holds a fire sale at time $t^*$, the deviating high-value buyer attempts to make the purchase but she is rationed. She would update her belief on $M(t^*)$ according to her private experience of being rationed. Her continuation strategy would also depend on her private history via the private belief on $M(t')$ for $t' > t^*$; thus we cannot sustain a weak Markov equilibrium in the continuation play. To accommodate the assumption of arriving low-value buyers, one has to make further assumptions on the observability of previous transactions. One feasible choice is to assume that all previous deal alerts are publicly observable. Doing so means that the number of low-value buyers who have traded and left is also common knowledge, so buyers share symmetric beliefs about the law of motion of $M(t)$ and symmetric strategy.

## 6.3 No-Waiting Equilibrium

We focus on no-waiting weak MPE, in which an high-value buyer makes her purchase upon her arrival. A natural question is whether there are other equilibria involving waiting. While the full characterization of the set of PBE is extremely difficult, we can show that there is no other weak MPE in which the high-value buyer's strategy depends on only the two state variables. The key idea is as follows. To sustain a weak Markov equilibrium, an high-value buyer's equilibrium strategy must be a function of two public state variables after every history, which implicitly implies that the seller cannot manipulate high-value buyers' beliefs on the number of present buyers in the market. In the no-waiting equilibrium, each high-value buyer believes that she is the only high-value buyer in the market both on and off the path of play. If there are other weak Markov equilibria, the high-value buyer's belief about the number of present buyers cannot be affected by the seller's price non-trivially, i.e., it can only be a function of the state variables. However, such an weak MPE does not exist. The reason is that, in any weak MPE, conditional on the inventory size, the high-value buyer's reservation price is non-increasing over time and strictly decreasing during the time interval in which the seller is supposed to serve the high-value buyers, thus the seller always has an incentive to serve high-value buyers when he is supposed to charge an unacceptable price in order to accumulate buyers.

## 6.4 Observability of Inventory

In our baseline model, the inventory size is observable. The seller's incentive to hold fire sales critically depends on this assumption. We believe that our model may be so stylized that it does not perfectly match any real market because in practice, the inventory size may not be perfectly observed by buyers. However, we still believe that our mechanism is illustrative for the reasons described hereafter.

First, in many industries, buyers observe some imperfect but informative signal of the real inventory. For example, in the airline industry, the remaining available seats on each flight can be observed online. The number of available seats is not the real inventory itself because the airline seller sometimes blocks some seats for elite passengers, but it is an informative proxy. Escobari (2012) uses the number of available seats as a proxy of the real inventory and empirically studies the price patterns in the airline industry. He finds that the price significantly increases as the number of available seats decreases.

Second, even though the inventory size is the seller's private information, the seller may be able to use price to signal his inventory. If the inventory size is small, the seller has an incentive to charge a high price and not to hold fire sales. However, if his inventory size is large, it is too costly to do so due to the risk that some inventory would not be sold in the end. Hence, we can imagine that some partial separating equilibria may exist. However, full investigation of such a dynamic signaling game is beyond the scope of this paper.

## 6.5 Disappearing Buyers and Discounting

In the baseline model, we assume that an high-value buyer leaves the market only when her demand is satisfied. Our results do not qualitatively change if buyers leave at a non-trivial rate over time. Suppose a buyer leaves the market at a rate $\rho > 0$ at any time, and her payoff by leaving the market without making a purchase is zero. If a buyer chooses to wait in the market, she faces the risk of exogenous departure. In particular, when $K = 1$, an high-value buyer's reservation price satisfies the following ODE

$$\dot{p}_1(t) = -(\lambda + \rho)\left[v_H - p_1(t)\right] \text{ for } t \in [0, 1),$$

with the boundary condition (2). By rejecting the current offer, an high-value buyer needs to take into account two risks: (1) another high-value buyer arrives and purchases the first units before her next attention time, and (2) her exogenous departure. Her payoff is zero if either happens.

In the two-unit case, for $t < t_1^*$, the high-value buyer's reservation price follows

$$\dot{p}_2(t) = -\lambda [p_1(t) - p_2(t)] - \rho [v_H - p_2(t)],$$

and for $t \geq t_1^*$, the form of $p_2(t)$ is identical to that in the baseline model. The intuition behind it is as follows. For $t < t_1^*$, by rejecting a current offer, an high-value buyer needs to take into account the risk that (1) another high-value buyer arrives before her next attention time, and (2) she exogenously leaves the market. In the former case, she has to pay $p_1(\tilde{t})$ instead of $p_2(\tilde{t})$ at her next attention time $\tilde{t} > t$; in the latter case, she obtains a payoff of zero, which is equivalent to paying a price $v_H$. Since the risk of exogenous departure will only change the high-value buyer's reservation price qualitatively, our main results still hold.

Similarly, the presence of discounting will also only change our main results quantitatively. As long as players have the same discount rate, one can simply normalize the price to take the discounting into account.

## 6.6 Multiple Types

In general, considering buyers' multiple reservation values is complicated in our model. The reason is that on the path of play the seller may accumulate some types of buyers, and players have to form beliefs about the number of such types of buyers, making it impossible to construct a no-waiting equilibrium. However, we believe the key mechanism in our binary-type model still works in a multi-type model. In a multi-type model, in order to eliminate the information rent of buyers in the future, the seller has an incentive to reduce his inventory quickly, just as he has in the binary-type model. Thanks to the well-known skimming property, buyers must employ a cutoff strategy. Hence, to quickly reduce his inventory, the seller has to charge a price that is significantly lower than the current price in order to increase demand by a large amount. Once the reduction of the inventory is accomplished, the seller can charge a higher price again because the remaining and future buyers' willingness to pay becomes higher. As a result, we expect the price to jump up and down over time on the equilibrium path.

# 7 Conclusion

This paper makes two contributions. First, we highlight a new channel for generating the periodic fire sales. When the deadline is approaching, the seller, if he still has a large inventory, does not expect many high-value buyers to arrive, so he has an incentive to liquidate part of his stock via a sequence of fire sales to increase future high-value buyers' reservation price. This insight can justify the price fluctuations in industries such as airlines, cruise-lines and hotel services. In a recent paper, Deb and Said (2013) study a two-period problem where a seller faces buyers who arrive in each period. They show that the seller's optimal contract pools low-value buyers, separates high-value ones, and induces intermediate ones to delay their purchases. The seller deliberately postpones the transaction with some buyers because he wants to commit to a high price in the second period. Their insight shares a similar spirit to ours except that we focus on the optimal timing of multiple fire sales. Second, by introducing the inattention frictions of buyers, we provide a tractable framework for studying dynamic pricing problems in which the seller lacks commitment power. We believe that the assumption of the inattention frictions is very natural in many dynamic pricing settings and it can be applied in other environments.

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# A Appendices (Not for Publication)

## A.1 Proofs for the Single-Unit Case

### Equilibria Construction

We construct an equilibrium such that the following conditions hold: (1) the seller posts a price such that an high-value buyer is indifferent between taking it or not, (2) an high-value buyer makes the purchase once she arrives, and (3) $P(1) = v_L$ is posted at the deadline.

Consider the last period first. At the deadline, an high-value buyers' reservation price is $v_H$. However, in the presence of inattention frictions, it is the seller's dominant strategy to posts $P(1) = v_L$ in order to obtain positive profit. Hence, in any equilibrium, the seller posts $P(1) = v_L$. At $t \in [1-\Delta, 1)$, the high-value buyers' reservation price is

$$v_H - p_1(t) = e^{-\lambda(1-t)} \frac{v_H - v_L}{M+1},$$

As $t \to 1$, $p_1(t) \to p_1(1^-)$. Differentiating $p_1(t)$ yields

$$\dot{p}_1(t) = -\lambda e^{-\lambda(1-t)} \frac{v_H - v_L}{M+1} = -\lambda \left[v_H - p_1(t)\right],$$

with the boundary condition $p_1(1^-)$ at $t = 1$.

By the equilibrium hypothesis, the seller posts the high-value buyer's reservation price $p(t)$ at any time $t \in [1-\Delta, 1)$. Let $U_{1-\Delta}$ be an high-value buyer's expected payoff at the beginning of the last period. The expectation is over the random attention time, and the risk of arrival of new buyers. Hence

$$\begin{aligned} U_{1-\Delta} &= \int_{1-\Delta}^{1} \frac{1}{\Delta} e^{-\lambda(s-1+\Delta)} \left[v_H - p_1(s)\right] ds \\ &= e^{-\lambda\Delta} \frac{v_H - v_L}{M+1} = v_H - p_1(1-\Delta). \end{aligned}$$

Suppose an high-value buyer whose regular attention time in the last period $t$ is smaller than but arbitrarily close to $1-\Delta$. At time $t$, the high-value buyer's reservation price is

$$v_H - p_1(t) = e^{-\lambda(1-\Delta-t)} U_{1-\Delta},$$

and we also have

$$\dot{p}_1(t) = -\lambda \left[v_H - p_1(t)\right].$$

As $t \to 1-\Delta$, $\lim_{t \nearrow 1-\Delta} p_1(t) = p_1(1-\Delta)$, so $p_1(t)$ is differentiable at $1-\Delta$. Repeating the above argument for $1/\Delta$ times, the reservation price $p_1(t)$ is differentiable in $[0, 1)$ and satisfies the ODE (5) with the boundary condition (2).

By the equilibrium hypothesis, the deal price is posted at the deadline only, and high-value buyers do not delay their purchases, so neither the high-value buyers' reservation price nor the seller's equilibrium profit depends on $\Delta$. The closed-form solution of $p_1(t)$ and $\Pi_1(t)$ are given by

$$\begin{aligned} p_1(t) &= v_H - \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)}, \\ \Pi_1(t) &= \left[1 - e^{-\lambda(1-t)}\right] v_H + e^{-\lambda(1-t)} v_L - \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)} \lambda (1-t). \end{aligned}$$

In sum, the equilibrium strategy profile $(\sigma_S^*, \sigma_B^*)$ is given as follows.

- After any private history, an high-value buyer's reservation price is equal to $p(t)$ for $t < 1$ and $v_H$ at $t = 1$.
- After any seller's history, the seller posts $p(t)$ for $t < 1$ and $v_L$ at $t = 1$.

### The Proof of Proposition 1

We prove Proposition 1 step by step. A simple observation is that, given the seller's equilibrium strategy, high-value buyers do not have an incentive to deviate since they are indifferent everywhere. To ensure the existence of the conjectured equilibrium, we only need to rule out deviations by the seller.

First of all, it is obvious that it is the seller's dominant strategy to post $P(1) = v_L$ at the deadline after any history. Hence, the seller has no profitable deviation at $t = 1$. Second, we need to rule out all possible deviation at $t < 1$ after any history. Formally, after a seller's private history $h_S^t$, the seller's profit by following equilibrium strategy $\sigma_s(t) = p_1(t)$ is given by

$$\Pi_1(\sigma_S, \sigma_B, h_S^t) = \int_t^{t+dt} [g(s|h_S^t) p_1(s)] ds + (1 - G(t+dt|h_S^t)) \Pi_1(\sigma_S, \sigma_B, h_S^{t+dt}),$$

where $dt > 0$ and the probability measure $G(s|h_S^t)$ is the seller's subjective belief about the earliest (present or newly arriving) high-value buyer's regular attention time after $t$ being not greater than $s$ given his private history $h_S^t$, and $g(\cdot|h_S^t)$ is the associated Radon-Nikodym derivative. Notice that

$$\Pi_1(\sigma_S, \sigma_B, h_S^t) = \int_t^1 [g(s|h_S^t) p_1(s)] ds + (1 - G(1|h_S^t)) v_L,$$

and $p_1(s)$ is strictly decreasing, so $\dot{\Pi}_1 < 0$ after any history. We want to show that, after any history, the seller would not be better off by deviation from $p_1(t)$ at time $t$. There are three kinds of deviation we need to rule out: (1) charging $v_L$, (2) charging a price between $v_L$ and $p_1(t)$ in a time period $[t, t+dt)$, and (3) charging "unacceptable price" which is higher than $p_1(s)$ for $s \in [t, t+dt)$ for any $dt > 0$. We rule out them one-by-one.[28]

1. $\Pi_1(\sigma_S, \sigma_B, h_S^t) > v_L$ for any $t < 1$, so the seller has no incentive to deviate by charging $v_L$ at any time $t < 1$.

2. Consider a deviation strategy $\hat{\sigma}_S$: charging a price $\tilde{p}(s) \in (v_L, p_1(s))$ for $s \in [t, t+dt)$ and then switch back to the equilibrium strategy after $t + dt$. The seller's profit is given by

$$\Pi_1(\hat{\sigma}_S, \sigma_B, h_S^t) = \int_t^{t+dt} [g(s|h_S^t)\tilde{p}(s)]ds + (1 - G(t+dt|h_S^t))\Pi_1(\sigma_S, \sigma_B, h_S^{t+dt}).$$

   Notice that by charging $\tilde{p}(s)$, the seller would not change the continuation history after $t+dt$. Hence, the deviation is not profitable.

3. Consider another deviation strategy $\tilde{\sigma}_S$: charging a price strictly greater than $p_1(s)$ for $s \in [t, t+dt)$. Since the price is higher than the buyer's reservation price, there would be no trade even some high-value buyers' attention times are in $[t+dt)$, so the seller's profit is given by

$$\Pi_1(\tilde{\sigma}_S, \sigma_B, h_S^t) = \int_t^{t+dt} [g(s|h_S^t)\Pi(\sigma_S, \sigma_B, \tilde{h}_S^{t+dt})]ds + (1 - G(t+dt|h_S^t))\Pi_1(\sigma_S, \sigma_B, h_S^{t+dt}, )$$

   where $h_S^{t+dt}$ denotes the history until $t + dt$ in which no high-value buyer's attention time is in $[t, t+dt)$, and $\tilde{h}_S^{t+dt}$ denotes the history in which some (present or newly arrived) high-value buyer's attention time is in $[t, t+dt)$. Since $p_1(t)$ is strictly decreasing, we must have $\Pi_1(t+dt, \tilde{h}_S^{t+dt}) \leq p_1(t+dt) < p_1(s)$ for $s \in [t, t+dt)$, which implies that charging a price higher than $p_1(s)$ is not profitable for $s \in [t, t+dt)$.

Consequently, it is the seller's best response to post $p_1(t)$ for $t < 1$, and $v_L$ at $t = 1$, and our conjectured equilibrium is an equilibrium. By construction, $p_1(t)$ is unique, so there is no other equilibrium. *Q.E.D.*

## A.2 Proofs for the Two-Unit Case

### The Proof of Lemma 1

Suppose not. Since $v_L$ is posted only at the deadline, the seller's equilibrium profits at the deadline are given by

$$\Pi_k(1) = kv_L, k = 1, 2.$$

[28]Similar to Sannikov (2007), the analogue of our argument in discrete time is the one-shot deviation principle.

and $p_k(t)$, the reservation price at $k = 1, 2$, is post to serve high-value buyers only at any $t < 1$. Specifically,

$$\begin{aligned} p_2(t) &= v_H - \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)} \left[2 + \lambda(1-t)\right], \text{ and} \\ p_1(t) &= v_H - \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)}. \end{aligned}$$

Define $\tilde{\Pi}_2(t)$ as the seller's profit if $p_2(t)$ is always posted when $t < 1$ and $K(t) = 2$, then

$$\begin{aligned} \tilde{\Pi}_2(t) &= \int_t^1 \lambda e^{-\lambda(s-t)} \left[p_2(s) + \Pi_1(s)\right] ds + 2v_L e^{-\lambda(1-t)} \\ &= 2v_H - 2(v_H - v_L) e^{-\lambda(1-t)} \\ &\quad - \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)} \left[\lambda(1-t)(M+3) + \lambda^2(1-t)^2\right]. \end{aligned}$$

Immediately,

$$\begin{aligned} & \tilde{\Pi}_2(t) - \left[v_L + \Pi_1(t)\right] \\ = & (v_H - v_L)\left(1 - 2e^{-\lambda(1-t)}\right) \\ & + \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)} \left[M + 1 - \lambda(1-t)(M+2) - \lambda^2(1-t)^2\right]. \end{aligned}$$

Though this difference is not monotone, using a Taylor expansion and algebra, there are two cases: (i) either $\tilde{\Pi}_2(t) - [v_L + \Pi_1(t)] < 0$ for all $t < 1$ when $\tilde{\Pi}_2(0) < v_L + \Pi_1(0)$, (ii) or, if $\tilde{\Pi}_2(0) > v_L + \Pi_1(0)$, $\exists\, t_1^* < 1$ s.t. $\tilde{\Pi}_2(t_1^*) = v_L + \Pi_1(t_1^*)$ and $\tilde{\Pi}_2(t) < v_L + \Pi_1(t)$ for $t \in (t_1^*, 1)$. *Q.E.D.*

### The Proof of Proposition 2

The proof of Proposition 2 is divided into several steps. To make the proof clear to the reader, we note that we will be following this road map:

1. For an arbitrary fire sales time $t_1^*$, we construct the high-value buyer's reservation price $p_2(\cdot)$ on $t \in [0, 1)$.

2. Given the high-value buyer's equilibrium strategy, we show that the seller has no incentive to charge prices that an high-value buyer does not accept after any history. Name, the seller either charges the high-value buyer's reservation price $p_k(t)$ or $v_L$ for $t \in [0, 1]$ regardless his private history $h_S^t$ since any price between $v_L$ and $p_k(t)$ is strictly dominated by $p_k(t)$.

3. We solve and characterize the seller's unique optimal fire sales time $t_1$ which is equal to the buyers believed fire sales time $t_1^*$ (fixed point).

**High-value buyers' Best Response**. By the equilibrium hypothesis, all buyers believe that

- there are no other high-value buyers in the market,
- there is a fire sale time $t_1^* < 1$ s.t. the seller posts a deal offer at the fire sale price $v_L$ with quantity $Q(t) = 1$ for $t \in [t_1^*, 1)$ if $K(t) = 2$,[29]
- the seller posts a price equaling $p_1(t)$ for $t < 1$ when $K(t) = 1$, and
- at the deadline $t = 1$, the seller posts a clearance fire sale at a price $v_L$ and supply $Q(t) = K(t)$.

Notice that, on the path of play, $K(t) < 2$ for $t > t_1^*$. However, we also impose buyers' belief on the seller's choice off the path of play when $K(t) = 2$ for $t \in (t_1^*, 1)$. The motivation is as follows. In order to obtain a weak MPE, the buyer's equilibrium strategy must be Markovian after any history including deviation history. When $K(t) = 2$ for $t \in (t_1^*, 1)$, the buyer understands that the seller deviates by not holding fire sales at time $t_1^*$. However, the buyer believes the seller posts $p_2(\tilde{t})$ for $\tilde{t} \in [t_1^*, t)$ so that she believes that (the seller also believes that) there is no other high-value buyer in the market. Thus, the seller's best response is to hold fire sales "immediately".

Given such a belief, we construct the high-value buyers' reservation price. At the deadline an high-value buyer's reservation price is $v_H$ regardless of the inventory $K(1)$. Now we consider the high-value buyers' reservation price for $t < 1$. Once, $K(t)$ becomes 1, the high-value buyers' reservation price becomes $p_1(t)$, which is given in equation **(1)**. For $K(t) = 2$ and a given $t_1^* \in [0, 1)$, there are two cases.

**Case 1**. When $K(t) = 2$ and $t \in [t_1^*, 1)$, an high-value buyer's believed continuation value if she waits is $\frac{1}{M+1}(v_H - v_L) + \frac{M}{M+1} e^{-\lambda(l\Delta - t)} U_{l\Delta}^1$, which is justified by the following belief on the continuation play: the buyer believes that (1) she is the only high-value buyer in the market, and (2) the seller would post fire sale "immediately" if no transaction occurs currently. Notice that, in a continuous-time game, the time right after $t$ or the "next period" is not well-defined. However, since each outcome in our game is identified by the limit outcome of a sequence of inertia strategies, we can appropriately define the buyer's believed continuation value as the limit of a sequence of continuation payoff determined by the associated inertia strategy sequence.

As a result, the high-value buyer's reservation price $p_2(t)$ when $t \in [t_1^*, 1)$ satisfies:

$$v_H - p_2(t) = \frac{1}{M+1}(v_H - v_L) + \frac{M}{M+1} e^{-\lambda(l\Delta - t)} U_{l\Delta}^1, \tag{A.1}$$

[29] Lemma 1 implies that there is no equilibrium where $t_1^* = 1$.

where, as in the single-unit case, $U^1_{l\Delta} = e^{-\lambda(1-l\Delta)}\frac{v_H - v_L}{M+1}$ and $t \in [(l-1)\Delta, l\Delta)$ for some $l < 1/\Delta$, hence

$$\begin{aligned} p_2(t) &= \frac{Mv_H + v_L}{M+1} - \frac{M}{(M+1)^2}(v_H - v_L)e^{-\lambda(1-t)} \\ &= \frac{M}{M+1}p_1(t) + \frac{1}{M+1}v_L, \text{ for } t \in [t_1^*, 1). \end{aligned}$$

Observe that $\dot{p}_2 = M/(M+1)\dot{p}_1$ for $t \in [t_1^*, 1)$.

**Case 2**. When $K(t) = 2$ and $t \in [0, t_1^*)$, the high-value buyer's reservation price $p_2(t)$ is pinned down as follows. There are some $l \in \mathbb{N}$ s.t. $[(l-1)\Delta, l\Delta) \cap [0, t_1^*) \neq \emptyset$.

In the "fire sales period", $l\Delta \geq t_1^* \geq (l-1)\Delta$, and $p_2(t)$ for $t \in [(l-1)\Delta, t^*)$ satisfies:

$$v_H - p_2(t) = e^{-\lambda\left(t_1^*-t\right)}U^2_{t_1^*} + \lambda(t_1^* - t)e^{-\lambda\left(t_1^*-t\right)}e^{-\lambda\left(l\Delta-t_1^*\right)}U^1_{l\Delta}, \tag{A.2}$$

The right-hand-side of the equation (A.2 ) denotes the buyer's continuation value if she decides to wait, which is consist of several terms

1. with probability $e^{-\lambda(t_1^*-t)}$, there is no new high-value buyer arriving before time $t_1^*$ so that the seller will hold fire sales,

2. $U^2_{t_1^*} = \frac{1}{M+1}(v_H - v_L) + \frac{M}{M+1}e^{-\lambda\left(l\Delta-t_1^*\right)}U^1_{l\Delta}$ is the buyer's expected continuation value at $t_1^*$. By the definition of $p_2(\cdot)$ on $[t_1^*, 1)$, $U^2_{t_1^*} = v_H - p_2(t_1^*)$ where $p_2(t_1^*)$ is the high-value buyer reservation price at $t_1^*$ if the seller posts a non fire sales price, and it is pinned down by equation (A.1) in case 1.

3. $\lambda(t_1^* - t)e^{-\lambda(t_1^*-t)}$ denotes the probability that a new high-value buyer arrives before $t_1^*$ so that the seller does not hold fire sales at time $t_1^*$. With probability $e^{-\lambda(l\Delta-t6*_1)}$, no other high-value buyer arrives before time $l\Delta$, so the high-value buyer can draw her regular attention time in the next period to make her purchase and her expected continuation value at time $l\Delta$ is $U^1_{l\Delta}$. With a complementary probability, others arrive before time $l\Delta$ and make the purchase, so the high-value buyer's payoff is zero.

In a non "fire sale period", $l\Delta < t_1^*$, and the high-value buyer's reservation price is given by

$$v_H - p_2(t) = e^{-\lambda(l\Delta-t)}U^2_{l\Delta} + \lambda(l\Delta - t)e^{-\lambda(l\Delta-t)}U^1_{l\Delta}, \tag{A.3}$$

where $U^2_{l\Delta} = v_H - p_2(l\Delta)$. In either of the two cases, we have

$$p_2(t) = v_H - \frac{v_H - v_L}{M+1}e^{-\lambda(1-t)}\left[e^{\lambda\left(1-t_1^*\right)} + \frac{M}{M+1} + \lambda(t_1^* - t)\right] < p_1(t),$$

for $t \in [0, t_1^*)$, and $\dot{p}_2(t) = -\lambda(p_1(t) - p_2(t))$ for $t \in [0, t_1^*)$.

The high-value buyer's incentive compatibility condition implies that $p_2(t)$ must be left continuous at $t_1^*$, so $p_2(\cdot)$ is continuous on $[0, 1)$.

In the hypothetic equilibrium, high-value buyers have no incentive to deviate after any history since each of them always believes that the seller has not charged unacceptable price for a positive measure of time, and the continuation price is either her equilibrium reservation price or $v_L$.

**The Seller's Problem: No Accumulation Result**. Now we consider the seller's problem. First, we claim that, given the high-value buyer's best response, after any history with $K(t) = 2$ for $t \in [0, 1]$, the seller's best response is to post either $p_2(t)$ or $v_L$ so that on the equilibrium path the seller does not post a price which is strictly higher than the reservation price of the high-value buyer.(No high-value buyer accumulates on the path of play.)

**Lemma A.1.** *After any seller's history $h_S^t$ with $K(t) = 2$ and $t < 1$, the seller's best response given buyer's equilibrium strategy $\sigma_B(t, 2)$ is either $\sigma_S(h_B^t) = (p_2(t), 1)$ or $\sigma_S(h_B^t) = (v_L, 1)$.*

*Proof.* By Proposition 1, once the inventory becomes $K(t) = 1$, the seller's best response is to post $p_1(t)$ for $t < 1$ and $v_L$ at the deadline. Obviously, any price $p \in (v_L, p_2(t)) \cup (-\infty, v_L)$ is strictly dominated by either posting $p_2(t)$ or $v_L$, so we only need to rule out the deviation strategy $\tilde{\sigma}_B$ that the seller posts unacceptable price $p > p_2(s)$ for $s \in [t, t + dt)$ and switch back to $\sigma_B$ after time $t + dt$ for any $dt > 0$. Denote the history after such a deviation as $\tilde{h}_S^{t+dt}$. There are two cases.

1. Suppose $\sigma_S(\tilde{h}_S^{t+dt}) = (v_L, 1)$. Namely, the seller charges unacceptable price for $[t, t + dt)$ and posts fire sale price at $t + dt$, and follows the equilibrium strategy afterwards. We are going to show that, the seller is better off by posting a fire sale price at $t$, and following the equilibrium strategy afterwards. The seller's deviation payoff is

$$\Pi_2(\tilde{\sigma}_S, \sigma_B, h_S^t) = v_L + \int_t^{t+dt} g(s|h_S^t) ds \Pi_1(\sigma_S, \sigma_B, \tilde{h}_S^{t+dt})] + (1 - G(t + dt|h_S^t))\Pi_1(\sigma_S, \sigma_B, h_S^{t+dt}),$$

   where $G(\cdot|h_S^t)$ again denotes the seller's subjective belief that the first regular attention time of high-value buyers after time $t$ is not greater than $s$, $h_S^{t+dt}$ denotes the history until $t + dt$ in which no high-value buyer's attention time is in $[t, t + dt)$ and the seller holds fire sales at time $t + dt$ with an associated continuation value $v_L + \Pi_1(\sigma_S, \sigma_B, h_S^{t+dt})$, and $\tilde{h}_S^{t+dt}$ denotes the history in which some (present or newly arrived) high-value buyer's attention time is in $[t, t + dt)$ and the seller holds fire sales at time $t + dt$, with an associated continuation value $v_L + \Pi_1(\sigma_S, \sigma_B, \tilde{h}_S^{t+dt})$. Now suppose the seller uses the following strategy $\bar{\sigma}_S$: posts the fire sale offer at $t$ with $Q(t) = 1$ and switch back the equilibrium strategy then. The seller's profit is given by

$$\Pi_2(\bar{\sigma}_S, \sigma_B, h_S^t) = v_L + \int_t^{t+dt} g(s|h_S^t)p_1(s)ds + (1 - G(t+dt|h_S^t))\Pi_1(\sigma_S, \sigma_B, h_S^{t+dt}).$$

Since $p_1(s) > \Pi_1(\sigma_S, \sigma_B, \tilde{h}_S^{t+dt})$ for $s < t+dt$, $\bar{\sigma}_S$ dominates the deviation strategy $\tilde{\sigma}_S$.

2. Suppose $\sigma_S(\tilde{h}_S^{t+dt}) = (p_2(t+dt), 1)$. Namely, the seller charges unacceptable price for $[t, t+dt)$, posts $p_2(t)$ at $t+dt$ and follows the equilibrium strategy afterwards. The seller's deviation payoff is

$$\Pi_2(\tilde{\sigma}_S, \sigma_B, h_S^t) = \int_t^{t+dt} g(s|h_S^t)ds\Pi_2(\sigma_S, \sigma_B, \tilde{h}_S^{t+dt}) + (1 - G(t+dt|h_S^t))\Pi_2(\sigma_S, \sigma_B, h_S^{t+dt}),$$

where $\Pi_2(\sigma_S, \sigma_B, h_S^{t+dt})$ denotes the history until $t+dt$ in which no high-value buyer's attention time is in $[t, t+dt)$, and $\Pi_2(\sigma_S, \sigma_B, \tilde{h}_S^{t+dt})$ denotes the history in which some (present or newly arrived) high-value buyer's attention time is in $[t, t+dt)$. Now suppose the seller uses the following strategy $\hat{\sigma}_S$: charging $p_2(s) \in [t, t+dt)$ and switch back to the equilibrium strategy $\sigma_S$ if either $K(s) = 1$ for $s \in [t, t+dt)$ or $s > t+dt$. The seller's profit is given by

$$\Pi_2(\hat{\sigma}_S, \sigma_B, h_S^t) = \int_t^{t+dt} g(s|h_S^t)[p_2(s) + \Pi_1(\sigma_S, \sigma_B, h_S^s)]ds + (1 - G(t+dt|h_B^t))\Pi_2(\sigma_S, \sigma_B, h_S^{t+dt}),$$

where $\Pi_1(\sigma_S, \sigma_B, h_S^s)$ denotes the seller's profit by following equilibrium strategy at time $s$ given the history $h_S^s$ and the inventory size $K(s) = 1$.

By the hypothesis, following a history $h_S^{t+dt}$, the seller's equilibrium profit is given by

$$\begin{aligned}
\Pi_2(\sigma_S, \sigma_B, h_S^{t+dt}) &= \int_{t+dt}^{\tau} g(s|h_S^{t+dt})[p_2(s) + \Pi_1(\sigma_S, \sigma_B, h_S^s)]ds \\
&+ (1 - G(\tau|h_S^{t+dt})ds)[v_L + \Pi_1(\sigma_S, \sigma_B, h_S^\tau)] \\
&\leq p_2(t+dt) + \Pi_1(\sigma_S, \sigma_B, h_S^{t+dt}).
\end{aligned}$$

where the last inequality comes from the facts that both $p_2(\cdot)$ and $\Pi_1(\sigma_S, \sigma_B, h_S)$ is strictly decreasing over time after any history and $\Pi_2(\sigma_S, \sigma_B, h_S^s) \geq v_L + p_2(s)$ and $\Pi_1(\sigma_S, \sigma_B, h_S^s)$ for $s < \tau$. Consequently, we must have

$$\Pi_2(\hat{\sigma}_S, \sigma_B, h_S^t) \geq \Pi_2(\tilde{\sigma}_S, \sigma_B, h_S^t).$$

In sum, the seller will be better off by either posting $v_L$ or $p_2(t)$ immediately and obtain a payoff as follows.

$$\Pi_2(\sigma_S, \sigma_B, h_S^t) = \max\{\Pi_2(\bar{\sigma}_S, \sigma_B, h_S^t), \Pi_2(\hat{\sigma}_S, \sigma_B, h_S^t)\},$$

so the seller has no incentive to deviate from $\sigma_S$ to $\tilde{\sigma}_S$.

Last, since $\Pi_2(\sigma_S, \sigma_B, h_S^t)) \geq 2v_L$ for $t < 1$, it is profitable to deviate to post a fire sales with $Q(t) = 2$ before the deadline. $\square$

Lemma A.1 implies that after any history, the seller has no incentive to charge an unacceptable price. As a result, on the path of play, at any time, the seller believes that there is no high-value buyer waiting in the market. The only remaining problem for the seller when $K(t) = 2$ is to decide when to charge $p_2(t)$ and $v_L$.

**The Seller's Problem: Optimal Fire Sale Time**. Given the buyer's reservation price $p_2(\cdot)$ based on the belief of $t_1^*$, the seller chooses the actual fire sale time, with $p_2(\cdot)$ forced to be the pricing strategy before the fire sale time. Hence,

$$\Pi_2(t) = \max_{t_1 \in [t,1]} \int_t^{t_1} e^{-\lambda(s-t)} \lambda \left[p_2(s) + \Pi_1(s)\right] ds + e^{-\lambda(t_1 - t)} \left[v_L + \Pi_1(t_1)\right]. \tag{A.4}$$

In equilibrium, the buyers' belief is correct, so the seller's optimal choice is indeed $t_1^*$. The first derivative w.r.t. $t_1$ at $t_1^*$ is

$$\begin{aligned}
& e^{-\lambda\left(t_1^*-t\right)}\lambda[p_2(t_1^*) - v_L] + e^{-\lambda\left(t_1^*-t\right)}\dot{\Pi}_1(t_1^*) \\
= \; & \lambda e^{-\lambda\left(t_1^*-t\right)}\left[p_2(t_1^*) - v_L - p_1(t_1^*) + \Pi_1(t_1^*)\right] \leq 0.
\end{aligned}$$

If an interior equilibrium fire sales time $t_1^*$ exists, it must satisfy the following first-order-condition (FOC)

$$p_2(t_1^*) - v_L - p_1(t_1^*) + \Pi_1(t_1^*) = 0. \tag{A.5}$$

at some $t_1^* \geq 0$. The remaining is to show that there is a unique $t_1^*$ s.t equation (A.5) and solves the seller's problem (A.4). Notice that equation (A.5) is not only the seller's optimality condition, but also the equilibrium consistent condition: the seller's optimal fire sales time is equal to the buyers' believed fire sales time.

Define $f(\cdot)$ on $[0,1]$ as follows:

$$f(t) = p_2(t) - v_L - p_1(t) + \Pi_1(t).$$

For $t \geq t_1^*$, we have $p_2(t_1) - p_1(t_1) = \frac{1}{M+1}\left[v_L - p_1(t_1)\right]$. Let

$$\begin{aligned}
f^0(t) &= \Pi_1(t) - v_L - \frac{p_1(t) - v_L}{M+1} \\
&= \frac{v_H - v_L}{M+1}\left\{M - e^{-\lambda(1-t)}\left[M + \frac{M}{M+1} + \lambda(1-t)\right]\right\}.
\end{aligned}$$

Obviously, $\dot{f}^0(t) < 0$ and $f^0(1) = -M/(M+1) < 0$. Define $t_1^*$ as the unique solution to $f^0(t) = 0$ if it exists, otherwise define $t_1^* = 0$. By construction, for $t \in (t_1^*, 1)$, the optimal solution of (A.4) is $t$; thus, the seller does not have any incentive to choose a fire sale time later than $t_1^*$. If $t_1^* > 0$ i.e. $f^0(t_1^*) = 0$, and let $f(t) = f^0(t)$ for $t \in [t_1^*, 1]$. For $t < t_1^*$, $\dot{p}_2(t) = -\lambda(p_1(t) - p_2(t))$, hence

$$f(t) = \frac{1}{\lambda}\dot{p}_2(t) + \Pi_1(t) - v_L,$$

and thus $\dot{f}(t) = \dot{p}_2(t) + \dot{\Pi}_1(t) - \dot{p}_1(t)$ where $\dot{\Pi}_1(t) - \dot{p}_1(t) = \lambda e^{-\lambda(1-t)}\frac{v_H - v_L}{M+1}[1 - \lambda(1-t) - M] < 0$ and $\dot{p}_2(t) < 0$, therefore $\dot{f}(t) < 0$ for $t < t_1^*$. Since $p_2$, $p_1$ and $\Pi_1$ are all continuous over $[0,1]$, we have a continuous $f(t)$ and $\lim_{t \nearrow t_1^*} f(t) = f(t_1^*) = 0$, consequently $f(t) > 0$ for $t < t_1^*$; thus, the FOC (A.5) is not only necessary but also sufficient to establish the equilibrium fire sale time, and therefore the seller does not have any incentive to choose a fire sale time earlier than $t_1^*$. Since $t_1^*$ is uniquely constructed, there is no other equilibrium. *Q.E.D.*

**Deriving $dt_1^*/d\lambda$**

At $t_1^*$, $f^0(t_1^*) = 0$. Since $\frac{v_H - v_L}{M+1} > 0$, we must have

$$\left\{M - e^{-\lambda\left(1-t_1^*\right)}\left[M + \frac{M}{M+1} + \lambda\left(1 - t_1^*\right)\right]\right\} = 0. \tag{A.6}$$

Differentiating equation (A.6) with respect to $\lambda$ yields $Me^{\lambda(1-t^*)}[(1 - t_1^* - \lambda t')] = 1 - t_1^* - \lambda\frac{t_1^*}{d\lambda}$, or

$$\frac{t_1^*}{d\lambda} = \frac{1 - t_1^*}{\lambda} > 0.$$

*Q.E.D.*

## A.3 Proofs for the Multi-Unit Case

### Proof of Proposition 3

We construct the equilibrium by induction. Suppose there is a unique equilibrium for the game where $K(0) = K - 1$ in which

1. there exists a sequence of $\{t_k^*\}_{k=1}^{K-2}$, such that $t_0^* := 1$, $t_k^* \leq t_{k-1}^*$ and $t_{K-2}^* \geq 0$,
2. the seller posts price $P(t) = p_k(t)$ if $K(t) = k$ and $t < t_{k-1}^*$ where $p_k(t)$ for $k = \{1, 2, ...K-1\}$ such that $p_{k+1}(t) < p_k(t)$ for any $t$, and $\dot{p}_k(t) < 0$ where differentiable, and
3. the seller holds fire sales at $t = t_{k-1}^*$ for $K(t) > k$ to liquidate redundant inventory, i.e. $P(t) = v_L$ and $Q(t) = K(t) - k$.

Consequently, by the indifference conditions of an high-value buyer's reservation price and uniform distributed attention time in a period, we can define $U_{l\Delta}^{k} = v_H - p_k(l\Delta)$ as the expected utility of an high-value buyer if her next attention time is in next period starting from $l\Delta < t_{k-1}^{*}$ and $K(l\Delta) = k$, and $U_{t_{k-1}^{*}}^{k} = v_H - p_k\left(t_{k-1}^{*}\right)$ the expected utility if the next attention time is $t_{k-1}^{*}$ and $K\left(t_{k-1}^{*}\right) = k$. We construct the unique candidate equilibrium for the game where $K(0) = K$, which includes: the high-value buyers reservation price $p_K(t)$ when $K(t) = K$, the equilibrium fire sale time $t_{K-1}^{*}$ when the inventory is $K$, and the seller' pricing strategy.

**High-value buyers' Best Response: Construction**. When $t_{K-2}^{*} = 0$, $t_{K-1}^{*} = 0$ as well. When $t_{K-2}^{*} > 0$, similar to the two-unit case, we can construct a fire sale time $t_{K-1}^{*} \in [0, t_{K-2}^{*}]$. Suppose buyers believe that the seller posts deals at $0 \le t_{K-1}^{*} \le t_{K-2}^{*} < ... < t_1^{*} < 1$ when $K(t_k^{*}) > k$ and posts the high-value buyer's reservation price $p_k(t)$ when $K(t) = k$, $k = 1, ..., K$.

Consider the case where $t \ge t_{K-1}^{*}$. Trivially, the seller will post $p_K(1) = v_L$ at the deadline and the reservation price of an high-value buyer is $v_H$. When $t \in [t_{K-i}^{*}, t_{K-i-1}^{*})$, and $K(t) = K$, an high-value buyer understands that seller deviates by charging the high-value buyer's reservation price when he is supposed to hold fire sales. Hence the buyer believes that no other high-value buyers are in the market and the seller believes so too and thus expects the seller to post a fire sales offer $(P(t), Q(t)) = (v_L, i)$ immediately and to reduce his inventory to $K - i$, hence

$$p_K(t) = \frac{i}{M+1} v_L + \frac{M+1-i}{M+1} p_{K-i}(t),$$

for $i = 1, ..., K-1$ .

Note that, when $t > t_{K-1}^{*}$, $p_K(t)$ is decreasing but it is not continuous for the following reason. At time $t < t_k^{*}$ and $K(t) = K$, the high-value buyer believes that the seller will "immediately" put $K-(k+1)$ units good for sale. With probability $\frac{K-k-1}{M+1}$, she gets the fire sale offer, and with the complementary probability she does not. In the later event, her continuation value is $v_H - p_{k+1}(t)$. As $t \to t_k^{*}$, $p_{k+1} \to \frac{1}{M+1} v_L + \frac{M}{M+1} p_k(t_k^{*})$. Formally,

$$\begin{aligned}
\lim_{t \nearrow t_k^{*}} p_K(t) &= \frac{K-k-1}{M+1} v_L + \frac{M+2-K+k}{M+1} \lim_{t \nearrow t_k^{*}} p_{k+1}(t) \\
&= \frac{K-k-1}{M+1} v_L + \frac{M+2-K+k}{M+1} \lim_{t \nearrow t_k^{*}} [\frac{1}{M+1} v_L + \frac{M}{M+1} p_k(t_k^{*})].
\end{aligned}$$

On the other hand, $p_K(t_k^{*}) = \frac{K-k}{M+1} v_L + \frac{M+1-K+k}{M+1} p_k(t_k^{*})$, so $\lim_{t \nearrow t_k^{*}} p_K(t) > p_K(t_k^{*})$, $\forall k < K-1$ and simple algebra implies that $\dot{p}_K = (M+1-i)/(M+1)\, \dot{p}_{K-i} < 0$ where it exists.

Now consider $t < t^*_{K-1}$. In the "fire sales period": $(l-1)\Delta \leq t < t^*_{K-1} < l\Delta$, the high-value buyer's indifference condition is:

$$\begin{aligned} v_H - p_K(t) &= e^{-\lambda\left(t^*_{K-1}-t\right)} U^K_{t^*_{K-1}} \\ &\quad + \sum_{k=1}^{K-1} \lambda^k e^{-\lambda(l\Delta - t)} \sum_{i=1}^{k} \frac{\left(t^*_{K-1}-t\right)^i}{i!} \frac{\left(l\Delta - t^*_{K-1}\right)^{k-i}}{(k-i)!} U^{K-k}_{l\Delta}; \end{aligned}$$

In the "no fire sales period": $(l-1)\Delta \leq t < l\Delta \leq t^*_{K-1}$, the condition becomes:

$$v_H - p_K(t) = \sum_{k=0}^{K-1} e^{-\lambda(l\Delta - t)} \frac{\lambda^k (l\Delta - t)^k}{k!} U^{K-k}_{l\Delta}. \tag{A.7}$$

The expected continuation values $U^k_{l\Delta}$ and $U^K_{t^*_{K-1}}$, defined in the same fashion as before, are the expected utilities of an high-value buyer if her next attention time is in the next period or at $t^*_{K-1}$, whichever comes first. The analytical expression for $p_K(t)$ is then obtained using the continuation values in a recursive way. It is straightforward to show that $p_K(t)$ is continuous at $t^*_{K-1}$. In addition, we have

$$\dot{p}_K(t) = -\lambda\left(p_{K-1}(t) - p_K(t)\right) \text{ for } t < t^*_{K-1}. \tag{A.8}$$

**High-value buyers' Best Response: Characterization**.

**Lemma A.2.** *For each $t \in [0,1)$, $p_{k+1} < p_k < 0$ where $k = \{1, 2, ...K-1\}$.*

*Proof.* First, for each $t \geq t^*_k$, by equation (15 ), $p_{k+1}(t) < p_k(t)$. Second, for $t < t^*_k$, by equation (A.7), we have

$$p_{K-1}(t) - p_K(t) = \sum_{k=0}^{K-2} e^{-\lambda(l\Delta - t)} \frac{\lambda^k (l\Delta - t)^k}{k!} [U^{K-k}_{l\Delta} - U^{K-1-k}_{l\Delta}] + e^{-\lambda(l\Delta - t)} \frac{\lambda^{K-1}(l\Delta - t)^{K-1}}{(K-1)!} U^1_{l\Delta},$$

where $U^k_{l\Delta} = v_H - p_k(l\Delta) > 0$ is the high-value buyer's expected payoff at time $l\Delta$ when $K(l\Delta) = k$. The strictly inequality comes from the fact that, an high-value buyer can always wait for the last minute low price and obtain $v_H - v_L$ with a positive probability. Moreover, we have

$$U^{K-k}_{l\Delta} - U^{K-1-k}_{l\Delta} = p_{K-1-k}(l\Delta) - p_{K-k}(l\Delta).$$

We already know that $p_1(t) > p_2(t)$ for each $t \in [0, t^*_1)$, so $p_2(t) > p_3(t)$ for $t \in [0, t^*_2)$. A simple induction argument implies our desired result. □

By the construction of $p_K(\cdot)$, we immediately have the following result.

**Corollary 1.** *$\dot{p}_K(t) < 0$ whenever $p_K(\cdot)$ is differentiable.*

Notice that $p_K(\cdot)$ is not continuous at $t_k^*$ for $k < K-1$.

**Lemma A.3.** *For $t < t_k^*$, $\dot{p}_{k+1} - \dot{p}_k < 0$ where $k = \{1, 2, ...K-1\}$.*

*Proof.* We solve the closed-form solution of $p_{k+1} - p_k$ for $t < t_k^*$. Simple algebra implies that

$$\dot{p}_{k+1}(t) - \dot{p}_k(t) = \lambda(p_{k+1} - p_k) + \lambda(p_{k-1} - p_k),$$

which is equivalent to

$$\begin{aligned} & \left[\dot{p}_{k+1}(t) - \dot{p}_k(t) - \lambda(p_{k+1} - p_k)\right] e^{-\lambda t} \\ = & \frac{d}{dt}\left[(p_{k+1} - p_k) e^{-\lambda t}\right] = -\lambda(p_k - p_{k-1}) e^{-\lambda t}. \end{aligned}$$

Recursively, we have

$$\frac{d^k}{dt^k}\left[(p_{k+1} - p_k) e^{-\lambda t}\right] = -(-\lambda)^k \frac{v_H - v_L}{M+1} e^{-\lambda},$$

so

$$p_{k+1} - p_k = -(-\lambda)^k \frac{v_H - v_L}{M+1} e^{-\lambda} \frac{t^k}{k!} e^{\lambda t} + \sum_{i=1}^{k} C_i \frac{t^{k-i}}{(k-i)!} e^{\lambda t},$$

where $C_i$ is a constant number for each $i$, and

$$\begin{aligned} \dot{p}_{k+1} - \dot{p}_k &= -(-\lambda)^k \frac{v_H - v_L}{M+1} e^{-\lambda} \frac{t^{k-1}}{(k-1)!} e^{\lambda t} + \sum_{i=1}^{k-1} C_i \frac{t^{k-i-1}}{(k-i-1)!} e^{\lambda t} + \lambda(p_{k+1} - p_k) \\ &= (p_k - p_{k-1}) + \lambda(p_{k+1} - p_k) + (\lambda+1)(-\lambda)^{k-1} \frac{v_H - v_L}{M+1} e^{-\lambda} \frac{t^{k-1}}{(k-1)!} e^{\lambda t}. \end{aligned}$$

Hence, when $k \in \{2, 4, 6, 8, ...\}$, we have $\dot{p}_{k+1} - \dot{p}_k < 0$. By the same logic, we have

$$p_2 - p_1 = \frac{\lambda(v_H - v_L)}{M+1} e^{-\lambda(1-t)} t + e^{\lambda t} C_1 < 0, \tag{A.9}$$

and

$$\frac{d^{k-1}}{dt^{k-1}}\left[(p_{k+1} - p_k) e^{-\lambda t}\right] = (-\lambda)^{k-1}\left[\frac{\lambda(v_H - v_L)}{M+1} e^{-\lambda} t + C_1\right],$$

so

$$p_{k+1} - p_k = (-\lambda)^{k-1}\left[\frac{\lambda(v_H - v_L)}{M+1} e^{-\lambda} \frac{t^{k-1}}{(k-1)!} + C_1 \frac{t^{k-2}}{(k-2)!}\right] e^{\lambda t} + \sum_{i=2}^{k} C_i \frac{t^{k-i}}{(k-i)!} e^{\lambda t},$$

and

$$\begin{aligned} \dot{p}_{k+1} - \dot{p}_k = & \lambda(p_{k+1} - p_k) + (p_k - p_{k-1}) \\ & -(\lambda+1)(-\lambda)^{k-2} \frac{t^{k-3}}{(k-2)!}\left[\frac{\lambda(v_H - v_L)}{M+1} e^{-\lambda} t + C_1(k-2)\right] e^{\lambda t}, \end{aligned}$$

where the first two terms are negative by Lemma A.2, and the last term is negative when $k \in \{3, 5, 7, 9...\}$ by the inequality (A.9), $\dot{p}_{k+1} - \dot{p}_k < 0$. In short, $\dot{p}_{k+1} - \dot{p}_k < 0$ for any $k \in \mathbb{N}$. □

**The Seller's Problem: No Accumulation Result**. Similar to the two-unit case, we claim the following lemma is true.

**Lemma A.4.** *After any seller's history $h_S^t$ with $K(t) = k$ and $t < 1$, the seller's best response to the high-value buyer's equilibrium strategy $\sigma_B(t, k)$ specifies an offer either $\sigma_S(h_B^t) = (p_2(t), 1)$ or $\sigma_S(h_B^t) = (v_L, Q)$ where $Q \in \{1, 2, ..K(t)\}$.*

The proof is similar to the proof of lemma A.1, so it is omitted here. As a result, after any history, the seller has no incentive to accumulate buyers, so the remaining problem is to pin down the optimal fire sale times $t_{k-1}^*$ for each $k = 1, 2, ...K$ provided that the seller believes there is no high-value buyer waiting in the market.

**The Seller's Problem: Optimal Fire Sale Time**. We employ the mathematical induction again to prove the result. Suppose for $k = 1, 2, ...K - 1$, both the buyers and the seller follow the equilibrium strategy. Given the high-value buyer's best response, we first prove the following lemma.

**Lemma A.5.** *For each $t < t_{K-2}^*$, $\dot{\Pi}_{K-1}(t) - \dot{\Pi}_{K-2}(t) < 0$.*

*Proof.* At $t_{K-1}^*$, $\dot{\Pi}_{K-1}\left(t_{K-1}^*\right) - \dot{\Pi}_{K-2}\left(t_{K-1}^*\right) = 0$, and the $\lim_{t \searrow t_{K-1}^*}\left[\ddot{\Pi}_{K-1}(t) - \ddot{\Pi}_{K-2}(t)\right] = -\lambda\left[\dot{p}_{K-1}(t) - \dot{p}_{K-2}(t) + \dot{\Pi}_{K-2}(t) - \dot{\Pi}_{K-3}(t)\right] > 0$. Hence $\dot{\Pi}_{K-1}(t) - \dot{\Pi}_{K-2}(t) < 0$ for $t \in \left(t_{K-1}^* - \varepsilon, t_{K-1}^*\right)$ where $\varepsilon$ is small but positive. If $\dot{\Pi}_{K-1}(t) - \dot{\Pi}_{K-2}(t) > 0$ for some $t$, by continuity of $\dot{\Pi}_{K-1}(\cdot) - \dot{\Pi}_{K-2}(\cdot)$, there must be a $\hat{t}$ s.t. $\dot{\Pi}_{K-1}\left(\hat{t}\right) - \dot{\Pi}_{K-2}\left(\hat{t}\right) = 0$ and $\dot{\Pi}_{K-1}(t) - \dot{\Pi}_{K-2}(t) < 0$ for any $t \in \left(\hat{t}, t_{K-1}^*\right)$. However, $\ddot{\Pi}_{K-1}\left(\hat{t}\right) - \ddot{\Pi}_{K-2}\left(\hat{t}\right) > 0$, which is a contradiction! $\square$

When $K(t) = K$, we can formulate the problem faced by the seller as follows.

$$\Pi_K(t) = \max_{t_{K-1} \in [t,1]} \int_t^{t_{K-1}} e^{-\lambda(s-t)}\lambda\left[p_K(s) + \Pi_{K-1}(s)\right] ds + e^{-\lambda(t_{K-1}-t)}[v_L + \Pi_{K-1}(t_{K-1})], \quad \text{(A.10)}$$

where $\Pi_k(\cdot)$ is defined recursively for each $k = 1, 2, ...., K$. By the induction hypothesis, we have

$$\Pi_{K-1}(t_{K-1}) = \begin{cases} (i-1)v_L + \Pi_{K-i}(t_{K-1}), & t_{K-1} \in [t_{K-i}^*, t_{K-1-i}^*) \\ (K-2)v_L, & t_{K-1} = 1 \end{cases}$$

for $i = 2, 3, ...K - 1$. We are going to show that as long as $t_{K-2}^* > 0$, the optimal fire sale time is strictly greater than $t_{K-2}^*$.

In equilibrium, the high-value buyers' belief is correct, so the seller's optimal choice is indeed $t_{K-1}^*$. If $t_{K-1}^* \neq t_k^*$, the equilibrium first-order condition (FOC) must hold at $t = t_{K-1}^*$:

$$\lambda[p_K\left(t_{K-1}^*\right) - v_L] + \dot{\Pi}_{K-1}\left(t_{K-1}^*\right) \leq 0, \quad \text{(A.11)}$$

and when $t_{K-1}^* > 0$ the weak inequality is replaced by an equality.

**Lemma A.6.** *Suppose that $t^*_{K-2} > 0$. There is a unique $t^*_{K-1} < t^*_{K-2}$ s.t.*

$$\lambda[p_K(t_{K-1}) - v_L] + \dot{\Pi}_{K-1}(t_{K-1}),$$

1. *is strictly negative at any $t_{K-1} \in (t^*_{K-1}, 1)$,*
2. *is strictly positive at any $t_{K-1} \in [0, t^*_{K-1})$ when $t^*_{K-1} > 0$, and*
3. *is equal to zero at $t_{K-1} = t^*_{K-1}$.*

*Proof.* The function form of the left-hand side of the FOC, $\lambda[p_K(t_{K-1}) - v_L] + \dot{\Pi}_{K-1}(t_{K-1})$, depends on the seller's continuation value after the fire sale, $\Pi_{K-1}(t^*_{K-1})$, so we need to consider its value case by case.

First, consider $t_{K-1} \in [0, t^*_{K-2})$. Let

$$f^1_K(t) = v_L + \Pi_{K-1}(t) - \Pi_{K-2}(t) - \frac{p_{K-1}(t) - v_L}{M+1} \text{ for } t \in [0, t^*_{K-2}).$$

Similar to the two-unit case, a simple observation is that $\lim_{t \to t^*_{K-2}} \Pi_{K-1}(t) - \Pi_{K-2}(t) \to v_L$ by the induction hypothesis and $\lim_{t \to t^*_{K-2}} \left[\mathbb{E}_{\tilde{t}|t}\left[e^{-\lambda(\tilde{t}-t)} p_{K-1}(t)\right] - v_L\right] > 0$ and both $\Pi_{K-1}(t) - \Pi_{K-2}(t)$ and $p_{K-1}(t)$ are continuous functions, so $f^1_K(t) < 0$ for $t$ close to $t^*_{K-2}$. If $f^1_K(t) < 0$ for any $t \in [0, t^*_{K-2})$, we claim that $t^*_{K-1} = 0$. Otherwise, we let $t^*_{K-1} = \sup\{t | t \leq t_{K-2},\ f^1_K(t) = 0$ and $\exists \varepsilon > 0$ s.t. $f^1_K(t') > 0$ for $t' \in (t - \varepsilon, t)\}$.

If $t^*_{K-1} > 0$, let

$$f^0_K(t) = p_K(t_{K-1}) - v_L - p_{K-1}(t_{K-1}) + \Pi_{K-1}(t_{K-1}) - \Pi_{K-2}(t_{K-1}),$$

for $t \in [0, t^*_{K-1})$ where $p_K(t)$ is defined in equation (A.8 ). Similar to the two-unit case, we can show that $f^0_K(t) > 0$ for $t < t^*_{K-1}$ and $\lim_{t \to t^*_{K-1}} f^0_K(t) = 0$. Notice that, when $t < t^*_{K-1}$, we have

$$\dot{f}^0_K(t) = \dot{p}_K - \dot{p}_{K-1} + \dot{\Pi}_{K-1} - \dot{\Pi}_{K-2}$$

We claim that $\dot{f}^0_K(t) < 0$ for $t < t^*_{K-1}$ since $\dot{p}_K - \dot{p}_{K-1} < 0$ by Lemma A.3 and $\dot{\Pi}_{K-1} - \dot{\Pi}_{K-2} < 0$ by Lemma A.5.

Let

$$f_K(t) = \begin{cases} f^0_K(t), & t < t^*_{K-1}, \\ f^1_K(t), & t \in [t^*_{K-1}, t^*_{K-2}). \end{cases}$$

Hence, the FOC becomes $f^0(t^*_{K-1}) = 0$, which has a unique solution in the interval $[0, t^*_{K-2})$. Further more $f_K(t)$ is strictly positive for $t < t^*_{K-1}$ and strictly negative for $t \in (t^*_{K-1}, t^*_{K-2})$.

Now consider the case in which $t_{K-1} \in \left(t^*_{K-1-i}, t^*_{K-2-i}\right)$ for $i = 1, 2, ..K-2$. The first derivative of the seller's objective function is given by

$$p_K\left(t_{K-1}\right) + \Pi_{K-1}\left(t_{K-1}\right) - iv_L - \Pi_{K-1-i}\left(t_{K-1}\right) - \left[p_{K-1-i}\left(t\right) + \Pi_{K-2-i} - \Pi_{K-1-i}\right],$$

for $t_{K-1} \in \left(t^*_{K-1-i}, t^*_{K-2-i}\right)$. By construction of $p_k$, we have

$$p_K\left(t_{K-1}\right) = \frac{i+1}{M+1}v_L + \frac{M-i}{M+1}p_{K-1-i}\left(t\right),$$

where $\tilde{t}$ is the high-value buyer's next regular attention time. Let

$$\begin{aligned} f^i_K\left(t\right) &= \Pi_{K-1}\left(t_{K-1}\right) - iv_L - \Pi_{K-2-i}\left(t_{K-1}\right) + \left[p_K\left(t_{K-1}\right) - p_{K-1-i}\right] \\ &= \left[\Pi_{K-1}\left(t_{K-1}\right) - iv_L - \Pi_{K-2-i}\left(t_{K-1}\right)\right] \\ &\quad + \left\{\frac{i+1}{M+1}v_L + \frac{M-i}{M+1}p_{K-1-i}\left(t\right) - p_{K-1-i}\left(t\right)\right\} \\ &= \left[\Pi_{K-1}\left(t_{K-1}\right) - iv_L - \Pi_{K-2-i}\left(t_{K-1}\right)\right] + \frac{i+1}{M+1}\left[v_L - p_{K-1-i}\left(t\right)\right]. \end{aligned}$$

And by the construction of $\Pi_k$, $\Pi_{K-1}\left(t_{K-1}\right) - iv_L - \Pi_{K-2-i}\left(t_{K-1}\right) = 0$ for $t_{K-1} \in \left(t_{K-1-i}, t_{K-2-i}\right)$. So $f^i_K < 0$. For $t_{K-1} \in \left(t^*_{K-1-i}, t^*_{K-2-i}\right)$, let $f_K(t) = f^i_K(t)$, and the FOC is negative in these time interval. □

Lemma A.6 implies that there is at most one $t^*_{K-1} < t^*_{K-2}$ at which the FOC holds. Hence, there is a unique equilibrium fire sale time $t^*_{K-1} \geq 0$. When $t^*_{K-1} > 0$, it satisfies the equilibrium FOC in an equality. In short, the equilibrium of the $K$-unit game exists and it is unique.

*Q.E.D.*

## A.4 Proofs for the Applications

We first prove the following lemma which is useful to prove Proposition 4.

**Lemma A.7.** *For any $k \in \mathbb{N}$ and $t \in [0, t^*_k]$, $v_L \leq \Pi_{k+1}(t) - \Pi_k(t) < \Pi_k(t) - \Pi_{k-1}(t)$*

*Proof.* Fix a $k \in \mathbb{N}$ and $t < t^*_k$. Consider the law of motion of $\Pi_{k+1}$ and $\Pi_k(t)$: $\dot{\Pi}_{k+1}(t) = \lambda[\Pi_{k+1}(t) - p_{k+1}(t) - \Pi_k(t)]$ and $\dot{\Pi}_k(t) = \lambda[\Pi_k(t) - p_k(t) - \Pi_{k-1}(t)]$, hence we have

$$\lambda\{[\Pi_{k+1}(t) - \Pi_k(t)] - [\Pi_k(t) - \Pi_{k-1}(t)]\} = [\dot{\Pi}_{k+1}(t) - \dot{\Pi}_k(t)] + \lambda[p_{k+1}(t) - p_k(t)]. \tag{A.12}$$

Since both term on the right-hand-side of equation (A.12) are strictly negative for $t < t^*_k$ by Lemma A.3 and Lemma A.5, we have the desired result. □

**Proof of Proposition 4**

Lemma A.7 implies that the marginal benefit of adding inventory is decreasing. For some $\bar{c} \geq v_L$, we always find $c > \bar{c}$ and an associated $K_c$ s.t. the $\Pi_{K_c+1}(t) - \Pi_{K_c}(t) < c \leq \Pi_{K(c)}(t) - \Pi_{K_c-1}(t)$ so that $K_c$ is the optimal initial inventory size to the seller. The remaining is to show that $\bar{c} = v_L$ so that for any $c \geq v_L$ the optimal initial inventory to the seller $K_c$ is finite. Notice that when $c > v_H$, $K_c = 0$.

Suppose not and $\bar{c} > v_L$. Then for some $c \in (v_L, \bar{c})$, for any $K$, $\Pi_{K+1}(0) - \Pi_K(0) > c$, and the seller's inventory choice problem has no solution.

Notice that for a given $K \in \mathbb{N}$, the seller's profit has a upper bound as follows.

$$\begin{aligned}
\Pi_K(0) - Kc &< \sum_{k=0}^{K-1} e^{-\lambda} \frac{\lambda^k}{k!}[kv_H + (K-k)v_L] + \sum_{k=K}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} K v_H - Kc \\
&= \sum_{k=0}^{K-1} e^{-\lambda} \frac{\lambda^k}{k!}[kv_H + (K-k)v_L] + \sum_{k=K}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} K v_H - K v_L + K(v_L - c) \\
&= \sum_{k=0}^{K-1} e^{-\lambda} \frac{\lambda^k}{k!} k(v_H - v_L) + \sum_{k=K}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} K(v_H - v_L) + K(v_L - c) \\
&< (v_H - v_L) \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} k + K(v_L - c),
\end{aligned}$$

where $\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} k = \frac{1}{\lambda} < \infty$ is independent from $K$ but $K(v_L - c) < 0$ if $c > v_L$. More importantly, as $K$ goes to infinity, $K(v_L - c)$ goes to negative infinite, so for sufficient big $K$, the seller's profit is negative, which is suboptimal. *Q.E.D.*