A Characterization of Rationalizable Consumer Behavior*

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Abstract

For an arbitrary dataset $D = \{(p, x)\} \subseteq (\mathbb{R}^m_+ \backslash \{0\}) \times \mathbb{R}^m_+$, finite or infinite, it is shown that the following three conditions are equivalent. (i) D satisfies GARP; (ii) D can be rationalized by a utility function; (iii) D can be rationalized by a strictly increasing, quasiconcave utility function. Examples of infinite datasets satisfying GARP are provided for which every utility rationalization fails to be lower semicontinuous, upper semicontinuous, or concave. Thus condition (iii) cannot be substantively improved upon.

1. Introduction.

We revisit the classical problem of recovering, from a single consumer's demand data, a utility function rationalizing that data. When such a utility function exists, we say that the dataset is rationalizable.

While the rationalizability question has been extensively studied, no condition is known to be both necessary and sufficient for an arbitrary dataset, finite or infinite, to be rationalizable. We provide such a condition here.

The requisite condition is the familiar generalized axiom of revealed preference (GARP) first introduced by Afriat (1967). Afriat showed that any *finite* dataset is rationalizable if and only if it satisfies GARP.[1] However, for the large literature in which datasets are infinite (for example, when summarized by a demand function), no such complete characterization of rationalizability has been given.[2]

The absence of such a characterization has, for example, left the following basic question unanswered. Suppose that a demand function describing a consumer's behavior is unknown to an observer. If the demand function is inconsistent with the utility maximization hypothesis, are

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[1]See also Diewert (1973), Varian (1982), and Chiappori and Rochet (1987). The term "GARP" was coined by Varian (1982).

[2]For sufficient conditions, see, e.g., Houthakker (1950), Uzawa (1971), Richter (1966), Hurwicz and Richter (1971), Hurwicz and Uzawa (1971), Mas-Colell (1978), Sondermann (1982), Fuchs-Seliger (1983, 1996), and Jackson (1986). In several of these papers it is assumed, as in Samuelson (1938), that the dataset is the entire graph of a demand function.

there finitely many choices that, if observed, would reveal the inconsistency? According to our main result the answer is, "yes." In fact, some finite number of choices must violate GARP.[3]

In terms of the analysis, one might hope that Afriat's (1967) techniques extend to infinite datasets, but they do not.[4] In fact, Afriat's conclusion that a finite dataset satisfying GARP is rationalized by a concave utility function is false for infinite datasets, which can sometimes be rationalized only by utility functions that are not concave.[5]

The approach taken here centers around a new and elementary method for constructing a utility function to rationalize a given finite dataset. The construction resembles Afriat's (1967) in that it defines utility in terms of sums of income differences. Importantly however, we restrict these income differences to be nonpositive and we do not require any multipliers to act as weights. Without the need to find appropriate multipliers, our direct and explicit construction is considerably simpler than Afriat's, and it has the additional advantage that it extends from finite to infinite datasets through an appropriate limiting argument.[6]

The next section introduces notation and preliminary material. Section 3 contains the new utility construction and presents a simple result on rationalizing finite datasets satisfying GARP as well as a result about maintaining GARP when adding points to a dataset. Section 4 contains the main result and Section 5 contains examples suggesting that the result cannot be substantively improved upon. Section 6 contains several additional remarks.

2. Preliminaries.

A *dataset* is any nonempty subset, finite or infinite, of $(\mathbb{R}^m_+ \backslash \{0\}) \times \mathbb{R}^m_+$, with typical element (p, x). Thus, a dataset is an arbitrary nonempty set of points, where each point (p, x) consists of a nonzero and nonnegative price vector p and a nonnegative consumption bundle x.

The revealed preference interpretation is that points in D correspond to the choices of a single consumer. We assume throughout that the consumer always spends her entire income. Hence, $(p, x) \in D$ means that, with an income of px and facing prices p, the consumer chose the bundle x.[7]

Note that we do not require the dataset to be complete; there may be prices and/or incomes at which no choice has been observed.

[3] We do not wish to suggest that our main result has empirical significance beyond what is already contained in Afriat's theorem. Empirical datasets are, after all, always finite. The point of our main result is to settle the rationalizability question in its most basic form and to unify the finite and infinite dataset approaches to the problem. But see Remark 9 for a practical consequence of one of our auxiliary results.

[4] But see Remark 7 in Section 6. See also Theorem 1 in Mas-Colell (1978) for an application of Afriat's theorem to infinite datasets generated by *continuous* demand functions. Continuity of demand, an important special case, rules out many continuous preference relations (e.g., perfect substitutes and any continuous preference relation that is not convex) that we do not wish to rule out, a priori, here.

[5] See Remark 5.3.

[6] Fostel et. al. (2004) provide two simple proofs of Afriat's theorem. Both proofs simplify the argument for the existence of a solution to the finite system of inequalities that is central to Afriat's proof. Consequently, neither proof extends to infinite datasets.

[7] Other interpretations are possible. For example, a researcher might wonder whether a particular demand function (possibly even restricted to a subset of prices) can be generated by utility maximizing behavior. The "dataset" D would then consist of the graph of the demand function (over the restricted set of prices), even though no consumer choices are actually observed.

A dataset, D, satisfies *GARP* if for every finite sequence $(p_1, x_1), ..., (p_n, x_n)$ of points in D,

$$p_1(x_2 - x_1) \leq 0,\ p_2(x_3 - x_2) \leq 0, ...,\ p_{n-1}(x_n - x_{n-1}) \leq 0 \Rightarrow p_n(x_1 - x_n) \geq 0.^8 \tag{2.1}$$

A (utility) function $u : \mathbb{R}^m_+ \to \mathbb{R}$ *rationalizes* the dataset D if for every $(p, x) \in D$ and every $y \in \mathbb{R}^m_+$,

$$p(y - x) \leq 0 \Rightarrow u(y) \leq u(x)$$

and if the second inequality is strict whenever the first is strict.[9]

A function $u : \mathbb{R}^m_+ \to \mathbb{R}$ is *strictly increasing* if $x \geq y$ implies $u(x) \geq u(y)$ and $x >> y$ implies $u(x) > u(y)$.[10]

It is straightforward to show that if u rationalizes D, then D satisfies GARP. Our objective is the converse, where, in light of Afriat's (1967) theorem, the interesting cases are those in which D is infinite, though we permit D to be finite as well as infinite.

3. A Utility Construction

Fix, throughout the paper, $\phi : \mathbb{R}^m_+ \to \mathbb{R}$ to be any continuous, strictly increasing, quasiconcave function taking values in $[0, 1]$.[11]

For any $x \in \mathbb{R}^m_+$, say that an arbitrarily long but finite sequence $(p_1, x_1), ..., (p_n, x_n)$ of points in $(\mathbb{R}^m_+ \backslash \{0\}) \times \mathbb{R}^m_+$ is *x-feasible* if each of the quantities $p_1(x - x_1)$, $p_2(x_1 - x_2)$, ..., $p_n(x_{n-1} - x_n)$ is nonpositive.

For any *finite* dataset, F, and any $x \in \mathbb{R}^m_+$ define $u_F(x)$ as follows:

- If $p'(x - x') \leq 0$ for some $(p', x') \in F$, then

$$u_F(x) := \inf p_1(x - x_1) + p_2(x_1 - x_2) + ... + p_n(x_{n-1} - x_n), \tag{3.1}$$

  where the infimum is taken over all x-feasible sequences of points in F, i.e., over all finite sequences $(p_1, x_1), ..., (p_n, x_n)$ of points in F such that each term $p_1(x - x_1)$, $p_2(x_1 - x_2)$, ..., $p_n(x_{n-1} - x_n)$ appearing in the sum (3.1) is nonpositive.

- If $p'(x - x') > 0$ for every $(p', x') \in F$, then $u_F(x) := \phi(x)$.

The quantity $u_F(x)$ is well-defined because the set of x-feasible sequences of points in F is nonempty if and only if some $(p', x') \in F$ satisfies $p'(x - x') \leq 0$. Indeed, any such (p', x') is

[8] If the first $n-1$ weak inequalities always imply that that the n-th inequality is strict, then the dataset satisfies SARP.

[9] Requiring strictly affordable bundles to yield strictly less utility rules out trivial rationalizations such as utility functions that are everywhere constant, or that are equal to 1 for all chosen bundles and equal to zero for all other bundles, etc. The literature often imposes instead the slightly more restrictive requirement of locally non-satiated utility. Given our eventual conclusion that rationalizing utility functions can always be chosen to be strictly increasing, our less restrictive requirement serves to strengthen our result.

[10] The notation $x \geq y$ ($x >> y$) means that, for every i, the i-th coordinate of x is no smaller than (is greater than) the i-th coordinate of y.

[11] For example, $\phi(x) = 1 - e^{-\mathbf{1} \cdot x}$, where $\mathbf{1} \cdot x$ is the sum of the coordinates of x.

an x-feasible sequence of length 1, and the first term (p_1, x_1) in any x-feasible sequence satisfies $p_1(x - x_1) \leq 0$.

Remark 1.

(a) Since there is no bound in (3.1) on the sequence length n, it can happen that $u_F(x) = -\infty$. On the other hand, because $u_F(x)$ either is the sum of nonpositive terms or is equal to $\phi(x) \in [0, 1]$, $u_F(x) \leq \phi(x) \leq 1$ for every x.

(b) Note that u_F is not the pointwise infimum of a fixed collection of functions, because the collection of functions over which the infimum in (3.1) is taken depends upon x.

Our first result shows that if F is a finite dataset satisfying GARP, then u_F is a utility function that rationalizes F, a conclusion that will be particularly useful in the sequel.[12]

Proposition 3.1. *If a finite dataset, F, satisfies GARP, then $u_F(x) > -\infty$ for every $x \in \mathbb{R}^m_+$. Moreover, $u_F : \mathbb{R}^m_+ \to (-\infty, 1]$ rationalizes F and is strictly increasing, quasiconcave, and lower semicontinuous.*

Proof. We first establish the following.

For any $x \in \mathbb{R}^m_+$, whenever $p'(x - x') \leq 0$ for some $(p', x') \in F$, the infimum in (3.1) can be achieved with an x-feasible sequence of *distinct* points in F. (3.2)

Indeed, if in any x-feasible sequence some point appears more than once, GARP implies that the consecutive terms involved in the resulting cycle together contribute a sum of zero to the right-hand side of (3.1). Hence, these terms can be eliminated without affecting the overall sum or x-feasibility.

For example, consider an x-feasible sequence (p_1, x_1), (p_2, x_2), (p_3, x_3), (p_4, x_4), (p_2, x_2), (p_5, x_5) in which the point (p_2, x_2) appears twice. The resulting sum in (3.1) is,

$$p_1(x - x_1) + p_2(x_1 - x_2) + p_3(x_2 - x_3) + p_4(x_3 - x_4) + p_2(x_4 - x_2) + p_5(x_2 - x_5), \tag{3.3}$$

By x-feasibility, all six terms in the sum are nonpositive. This is true, in particular, for the three consecutive terms involved in the cycle: $p_3(x_2 - x_3) \leq 0$, $p_4(x_3 - x_4) \leq 0$, and $p_2(x_4 - x_2) \leq 0$. Hence, by GARP, each one of these three terms is zero, and so the sum in (3.3) is equal to

$$p_1(x - x_1) + p_2(x_1 - x_2) + p_5(x_2 - x_5),$$

where the corresponding subsequence (p_1, x_1), (p_2, x_2), (p_5, x_5), is x-feasible. By applying this technique finitely often, any x-feasible sequence can be reduced to an x-feasible subsequence of *distinct* points without changing the resulting sum on the right-hand side of (3.1).

An obvious implication of (3.2) is that $u_F(x)$ is finite and nonpositive if $p'(x - x') \leq 0$ for some $(p', x') \in F$, and otherwise $u_F(x) = \phi(x) \in [0, 1]$. Hence, $u_F : \mathbb{R}^m_+ \to (-\infty, 1]$.

To see that u_F is lower semicontinuous, suppose that $x^k \to_k x$ and that $u_F(x^k)$ converges (possibly to $-\infty$). We must show that $\lim_k u_F(x^k) \geq u_F(x)$. Without loss of generality (i.e., by

[12] A version of this result was first reported as Exercise 2.12 in Jehle and Reny (2011).

considering a subsequence), either (i) there exists $(p', x') \in F$ such that $p'(x^k - x') \leq 0$ for all k, or (ii) $p'(x^k - x') > 0$ for every $(p', x') \in F$ and every k. If (i) holds, then, because there are only finitely many sequences of distinct points in F, (3.2) implies that, without loss of generality (i.e., by considering a further subsequence), there is a fixed finite sequence of distinct points $(p_1, x_1), ..., (p_n, x_n)$ in F that is x^k-feasible for every k and such that $u_F(x^k) = p_1(x^k - x_1) + p_2(x_1 - x_2) + ... + p_n(x_{n-1} - x_n)$ for every k. In particular, $p_1(x - x_1) = \lim_k p_1(x^k - x_1) \leq 0$. Consequently, this fixed sequence of distinct points is x-feasible and,

$$\begin{aligned} u_F(x) &\leq p_1(x - x_1) + p_2(x_1 - x_2) + ... + p_n(x_{n-1} - x_n) \\ &= \lim_k p_1(x^k - x_1) + p_2(x_1 - x_2) + ... + p_n(x_{n-1} - x_n) \\ &= \lim_k u_F(x^k). \end{aligned}$$

On the other hand, if (ii) holds, then $u_F(x^k) = \phi(x^k)$ for every k and so $\lim_k u_F(x^k) = \phi(x) \geq u_F(x)$ (see Remark 1(b)). Hence, u_F is lower semicontinuous.

To see that u_F is strictly increasing, suppose that $x \geq y$. There are two cases. Either $u_F(x) = \phi(x)$, in which case $u_F(x) = \phi(x) \geq \phi(y) \geq u_F(y)$, with $\phi(x) > \phi(y)$ if $x >> y$; or $u_F(x) = p_1(x - x_1) + p_2(x_1 - x_2) + ... + p_n(x_{n-1} - x_n)$ for some x-feasible sequence $(p_1, x_1), ..., (p_n, x_n)$, in which case

$$\begin{aligned} u_F(x) &= p_1(x - x_1) + p_2(x_1 - x_2) + ... + p_n(x_{n-1} - x_n) \\ &\geq p_1(y - x_1) + p_2(x_1 - x_2) + ... + p_n(x_{n-1} - x_n), \text{ a strict inequality if } x >> y \\ &\geq u_F(y), \end{aligned}$$

where the last inequality follows because $x \geq y$ implies that $(p_1, x_1), ..., (p_n, x_n)$ is also y-feasible. In either case, $u_F(x) \geq u_F(y)$, with the inequality strict if $x >> y$, and so u_F is strictly increasing.

To see that u_F is quasiconcave, let $z = \lambda x + (1 - \lambda)y$. If $u_F(z) = \phi(z)$, then $u_F(z) = \phi(z) \geq \min(\phi(x), \phi(y)) \geq \min(u_F(x), u_F(y))$. Otherwise, $u_F(z) = p_1(z - x_1) + p_2(x_1 - x_2) + ... + p_n(x_{n-1} - x_n)$ for some z-feasible sequence $(p_1, x_1), ..., (p_n, x_n)$, where we may suppose without loss of generality that $p_1 x \leq p_1 y$ and so $p_1 x \leq p_1 z$. But then,

$$\begin{aligned} u_F(z) &= p_1(z - x_1) + p_2(x_1 - x_2) + ... + p_n(x_{n-1} - x_n) \\ &\geq p_1(x - x_1) + p_2(x_1 - x_2) + ... + p_n(x_{n-1} - x_n) \\ &\geq u_F(x), \end{aligned}$$

where the last inequality follows because $p_1 x \leq p_1 z$ implies that $(p_1, x_1), ..., (p_n, x_n)$, being z-feasible, is also x-feasible. In either case, $u_F(z) \geq \min(u_F(x), u_F(y))$, and so u_F is quasiconcave.

Finally, to see that u_F rationalizes F, suppose that $(p, x) \in F$ and $p(y - x) \leq 0$. Since the singleton sequence $(p, x) \in F$ satisfies $p(x - x) \leq 0$, (3.2) implies that

$$u_F(x) = p_1(x - x_1) + p_2(x_1 - x_2) + ... + p_n(x_{n-1} - x_n)$$

for some x-feasible sequence $(p_1, x_1), ..., (p_n, x_n)$. But then, because $p(y - x) \leq 0$, the sequence

$(p,x),(p_1,x_1),...,(p_n,x_n)$ is y-feasible and therefore,

$$\begin{aligned} u_F(y) &\le p(y-x)+p_1(x-x_1)+p_2(x_1-x_2)+...+p_n(x_{n-1}-x_n) \\ &= p(y-x)+u_F(x). \end{aligned}$$

Therefore, $u_F(y) \le u_F(x)$, with the inequality strict if $p(y-x)<0$. Q.E.D.

Remark 2. *The proof above establishes that if F is finite and satisfies GARP, then either $u_F(x) = \phi(x) \ge 0$ or $u_F(x) = p_1(x-x_1)+p_2(x_1-x_2)+...+p_n(x_{n-1}-x_n)$ for some x-feasible sequence $(p_1,x_1),...,(p_n,x_n)$ of distinct points in F, in which case $u_F(x) \ge -(p_1x_1+...+p_nx_n)$. In either case, $u_F(x) \ge -I_F$, where $I_F := \sum_{(p,x)\in F} px$ is defined to be the total income of all the data points in F. That is, u_F is bounded below by $-I_F$.*

The next lemma states that if a finite dataset satisfies GARP, then to any consumption bundle $x >> 0$ we may associate a price vector p so that when (p,x) is added to the dataset, the new dataset also satisfies GARP.[13]

Lemma 3.2. *Let F be a finite dataset satisfying GARP and let x_0 be any point in $\mathbb{R}^m_{++}$. Then there is a price vector $p_0 \in \mathbb{R}^m_+$ such that $p_0x_0 = 1$ and $F \cup \{(p_0,x_0)\}$ satisfies GARP.*

Proof. By Proposition 3.1, u_F is quasiconcave and strictly increasing. Hence, the set $C = \{x : u_F(x) \ge u_F(x_0)\}$ is convex with x_0 on its boundary. By the separating hyperplane theorem, and since $x_0 >> 0$, there exists $p_0 \in \mathbb{R}^m$ such that $p_0x_0 = 1$ and such that $p_0x_0 \le p_0x$ for every $x \in C$, implying also that $p_0 \ge 0$ since (because u_F is strictly increasing) $x_0 + \lambda e_i \in C$ for every $\lambda > 0$ and every i, where e_i is the i-th unit vector. In particular, if $p_0y < p_0x_0$, then $y \notin C$ and so $u_F(y) < u_F(x_0)$. It suffices now to show that $p_0y = p_0x_0$ implies $u_F(y) \le u_F(x_0)$ since then u_F rationalizes $F \cup \{(p_0,x_0)\}$ which must then satisfy GARP.

So suppose that $p_0y = p_0x_0$. Then, for every $\lambda \in (0,1)$, $p_0(\lambda y) < p_0x_0$ (since $p_0x_0 > 0$), and so $u_F(\lambda y) < u_F(x_0)$. Because, by Proposition 3.1, u_F is strictly increasing and lower semicontinuous, $\lim_{\lambda\uparrow 1} u_F(\lambda y) = u_F(y)$ and so $u_F(y) \le u_F(x_0)$. Q.E.D.

Remark 3. *Lemma 3.2 can, of course, also be proven with the aid of Afriat's (1967) theorem instead of Proposition 3.1. But not having to rely on Afriat's more involved proof, in addition to remaining self-contained is, we believe, advantageous.*

4. The Main Result

Theorem 4.1. *For an arbitrary dataset $D = \{(p,x)\} \subseteq (\mathbb{R}^m_+\backslash\{0\}) \times \mathbb{R}^m_+$, finite or infinite, the following three conditions are equivalent. (i) D satisfies GARP; (ii) D can be rationalized by a utility function; (iii) D can be rationalized by a strictly increasing and quasiconcave utility function.*

[13] There need be no such p when the given dataset is infinite.

Proof. Since (iii)⇒(ii)⇒(i) is clear, it suffices to show (i)⇒(iii).

Let U denote the set of strictly increasing and quasiconcave functions from $\mathbb{R}^m_+$ into $[-1,1]$, and let $y_1, y_2, ...$ be a dense sequence of points in $\mathbb{R}^m_{++}$.

For every positive integer k, and every finite subset F of $(\mathbb{R}^m_+\backslash\{0\}) \times \mathbb{R}^m_+$, define the subset $U_{F,k}$ of U as follows, where for any vector x, $x(j)$ denotes its j-th coordinate.

$$U_{F,k} := \left\{ u \in U : \begin{array}{l} \text{(i) } u \text{ rationalizes } F, \text{ and} \\ \text{(ii) } u(x) \leq \frac{1}{2^{k'}} \max_j \left(\frac{x(j)}{y_{k'}(j)} - 1 \right) + u(y_{k'}),\ \forall k' \leq k, \forall x \in \mathbb{R}^m_+ \text{ s.t. } x << y_{k'} \end{array} \right\}.$$

Clearly, the set $U_{F,k}$ becomes smaller as F and k become larger. That is, if $F \supseteq F'$ and $k \geq k'$, we have $U_{F,k} \subseteq U_{F',k'}$. Consequently,

$$\inf_{u \in U_{F,k}} u(x) \geq \inf_{u \in U_{F',k'}} u(x) \text{ for every } x \in \mathbb{R}^m_+. \tag{4.1}$$

Before taking advantage of this useful property, let us show that $U_{F,k}$ is nonempty whenever F is a finite subset of D.

Let $F = \{(p'_1, x'_1), ..., (p'_n, x'_n)\}$ be any finite subset of D and let k be any positive integer. Since D satisfies GARP, F satisfies GARP. Therefore, since each $y_1, ..., y_k$ is in $\mathbb{R}^m_{++}$, k successive applications of Lemma 3.2 yields k nonnegative and nonzero price vectors, $q_1, ..., q_k$, such that each $q_i y_i = 1$ and the finite set $\{(q_1, y_1), ..., (q_k, y_k), (p'_1, x'_1), ..., (p'_n, x'_n)\}$ satisfies GARP. Since GARP is not affected when prices are multiplied by positive constants, the finite set

$$F' := \left\{ \left(\frac{1}{2} q_1, y_1\right), ..., \left(\frac{1}{2^k} q_k, y_k\right), \left(\frac{\alpha_1}{2^{k+1}} p'_1, x'_1\right), ..., \left(\frac{\alpha_n}{2^{k+n}} p'_n, x'_n\right) \right\},$$

satisfies GARP, where the $\alpha_i > 0$ are chosen so that $\alpha_i p'_i x'_i \leq 1$ for every i.[14] (Note that F' need not be a subset of D.) Moreover, $I_{F'}$, the total income of the data points in F' (see Remark 2), is no greater than 1 because each $q_i y_i = 1$ and each $\alpha_i p'_i x'_i \leq 1$. Consequently, by Proposition 3.1 and Remark 2, $u_{F'} : \mathbb{R}^m_+ \to [-1,1]$ is quasiconcave, strictly increasing and rationalizes the dataset F'. In particular, $u_{F'} \in U$ and $u_{F'}$ rationalizes F since every datapoint in F can be obtained by multiplying the price of some datapoint in F' by a positive scalar. Consequently, $u_{F'}$ satisfies (i) of the definition of $U_{F,k}$. We next show that $u_{F'}$ also satisfies (ii), which implies that $U_{F,k}$ is nonempty since $u_{F'}$ is then a member.

Suppose that $k' \leq k$ and that $x \in \mathbb{R}^m_+$ is such that $x << y_{k'}$. To show that $u_{F'}$ satisfies (ii) of the definition of $U_{F,k}$, we must show that $u_{F'}(x) \leq \frac{1}{2^{k'}} \max_j \left(\frac{x(j)}{y_{k'}(j)} - 1 \right) + u_{F'}(y_{k'})$.

Because $\left(\frac{1}{2^{k'}} q_{k'}, y_{k'}\right)$ is a member of F', $u_{F'}(y_{k'})$ is defined by (3.1). Hence, as shown in the proof of Proposition 3.1, $u_{F'}(y_{k'}) = p_1(y_{k'} - x_1) + p_2(x_1 - x_2) + ... + p_n(x_{n-1} - x_n)$ for some $y_{k'}$-feasible sequence $(p_1, x_1), ..., (p_n, x_n)$ of points in F'. But then $x << y_{k'}$ implies that $q_{k'}(x - y_{k'}) < 0$ and

[14] The α_i cannot necessarily be chosen so that each $\alpha_i p'_i x'_i = 1$ since we allow datapoints with zero income.

that $\left(\frac{1}{2^{k'}}q_{k'}, y_{k'}\right), (p_1, x_1), ..., (p_n, x_n)$ is x-feasible. Consequently, $u_{F'}(x)$ is defined by (3.1) and,

$$\begin{aligned} u_{F'}(x) &\leq \frac{1}{2^{k'}}q_{k'}(x - y_{k'}) + p_1(y_{k'} - x_1) + p_2(x_1 - x_2) + ... + p_n(x_{n-1} - x_n) \\ &= \frac{1}{2^{k'}}q_{k'}(x - y_{k'}) + u_{F'}(y_{k'}) \\ &\leq \frac{1}{2^{k'}} \max_j \left(\frac{x(j)}{y_{k'}(j)} - 1\right) + u_{F'}(y_{k'}).^{15} \end{aligned}$$

Hence, $u_{F'} \in U_{F,k}$, which establishes that $U_{F,k}$ is nonempty.

We can now exhibit a utility function that we will show is strictly increasing, quasiconcave, and rationalizes D.

Define $u^* : \mathbb{R}^m_+ \to [-1, 1]$ by

$$u^*(x) := \sup_{F,k} \inf_{u \in U_{F,k}} u(x),$$

where the sup is taken over all positive integers k, and all finite subsets F of D.

That u^* is well-defined and takes values in $[-1, 1]$ follow because all the $U_{F,k}$ are nonempty and contain only functions taking values in $[-1, 1]$. It remains to show that u^* is strictly increasing, quasiconcave, and rationalizes D. We consider each in turn.

For the remainder of the proof, F, F', F'' etc. will always denote finite subsets of D, and k, k', k'' etc. will always denote positive integers.

I. u^* is strictly increasing.

Suppose first that $x \leq y$. We must show that $u^*(x) \leq u^*(y)$.

For any F and k, every $u \in U_{F,k}$ is strictly increasing (recall that $U_{F,k}$ is a subset of U). Therefore,

$$u(x) \leq u(y), \text{ for every } u \in U_{F,k}.$$

Hence,

$$\inf_{u \in U_{F,k}} u(x) \leq \inf_{u \in U_{F,k}} u(y).$$

Since this holds for every F and k, it follows that

$$u^*(x) = \sup_{F,k} \inf_{u \in U_{F,k}} u(x) \leq \sup_{F,k} \inf_{u \in U_{F,k}} u(y) = u^*(y).$$

Thus, we have shown that $x \leq y$ implies that $u^*(x) \leq u^*(y)$.

Suppose next that $x << y$. We must show that $u^*(x) < u^*(y)$. Since $\{y_1, y_2, ...\}$ is a dense subset of $\mathbb{R}^m_{++}$, there exists k' such that $x << y_{k'} << y$. Fix this k'. By what we have just shown, $u^*(y_{k'}) \leq u^*(y)$. Hence it suffices to show that $u^*(x) < u^*(y_{k'})$.

Consider any F and any $k \geq k'$. Since $x << y_{k'}$, part (ii) of the definition of $U_{F,k}$ implies that,

$$u(x) \leq \frac{1}{2^{k'}} \max_j \left(\frac{x(j)}{y_{k'}(j)} - 1\right) + u(y_{k'}), \text{ for every } u \in U_{F,k}.$$

[15] We have used the fact that, for any $y >> 0$ and any $x, q \geq 0$ such that $qy = 1$, $qx = \sum_j (q(j)y(j))\frac{x(j)}{y(j)} \leq \max_j \frac{x(j)}{y(j)}$.

Hence,

$$\inf_{u \in U_{F,k}} u(x) \leq \frac{1}{2^{k'}} \max_j \left(\frac{x(j)}{y_{k'}(j)} - 1 \right) + \inf_{u \in U_{F,k}} u(y_{k'}).$$

Since this holds for every F and every $k \geq k'$, we have,

$$\sup_{F, k \geq k'} \inf_{u \in U_{F,k}} u(x) \leq \frac{1}{2^{k'}} \max_j \left(\frac{x(j)}{y_{k'}(j)} - 1 \right) + \sup_{F, k \geq k'} \inf_{u \in U_{F,k}} u(y_{k'}). \tag{4.2}$$

For any $z \in \mathbb{R}^m_+$, (4.1) implies that

$$\sup_{F, k \geq k'} \inf_{u \in U_{F,k}} u(z) \geq \sup_{F,k} \inf_{u \in U_{F,k}} u(z).$$

Since the reverse inequality is obvious (the sup on the left is over a smaller set of k's than that on the right), we have,

$$\sup_{F, k \geq k'} \inf_{u \in U_{F,k}} u(z) = \sup_{F,k} \inf_{u \in U_{F,k}} u(z) = u^*(z).$$

Applying this to (4.2) yields,

$$u^*(x) \leq \frac{1}{2^{k'}} \max_j \left(\frac{x(j)}{y_{k'}(j)} - 1 \right) + u^*(y_{k'}).$$

Finally, because $x << y_{k'}$ implies that $\max_j \left(\frac{x(j)}{y_{k'}(j)} - 1 \right) < 0$, we conclude that $u^*(x) < u^*(y_{k'})$, as desired.

II. u^* is quasiconcave.

For any F and k, every $u \in U_{F,k}$ is in U and is therefore quasiconcave. Consequently, the function $u_{F,k}$ defined by $u_{F,k}(x) = \inf_{u \in U_{F,k}} u(x)$ is quasiconcave, being the pointwise infimum of a collection of quasiconcave functions. Also, by the definition of u^*, we have $u^*(x) = \sup_{F,k} u_{F,k}(x)$ for every x.

Fix any $x, y \in \mathbb{R}^m_+$ and any $\lambda \in [0,1]$, and let $z = \lambda x + (1-\lambda) y$. For any F and k, the definition of u^* and the quasiconcavity of $u_{F,k}$ imply that

$$u^*(z) \geq u_{F,k}(z) \geq \min(u_{F,k}(x), u_{F,k}(y)).$$

Hence, for all F, F', k, k', if we let $F'' = F \cup F'$ and $k'' = \max(k, k')$, then

$$\begin{aligned} u^*(z) &\geq \min(u_{F'',k''}(x), u_{F'',k''}(y)) \\ &\geq \min(u_{F,k}(x), u_{F',k'}(y)), \end{aligned}$$

where the second inequality follows by (4.1). Since F, F', k, and k' are arbitrary, this implies that

$$\begin{aligned} u^*(z) &\geq \min(\sup_{F,k} u_{F,k}(x), \sup_{F',k'} u_{F',k'}(y)) \\ &= \min(u^*(x), u^*(y)), \end{aligned}$$

and we conclude that u^* is quasiconcave.

III. u^* rationalizes D.

Fix any $(p, x) \in D$. For any $y \in \mathbb{R}^m_+$, we must show that if $p(y - x) \leq 0$, then $u^*(y) \leq u^*(x)$ and that if $p(y - x) < 0$, then $u^*(y) < u^*(x)$.

Suppose first that $p(y - x) \leq 0$. For any F and k, every $u \in U_{F,k}$ rationalizes F. Therefore, every $u \in U_{F,k}$ will satisfy $u(y) \leq u(x)$ so long as F contains (p, x). That is, for every k and every F containing (p, x),

$$u(y) \leq u(x), \text{ for every } u \in U_{F,k}.$$

Hence, for every k and every F containing (p, x),

$$\inf_{u \in U_{F,k}} u(y) \leq \inf_{u \in U_{F,k}} u(x).$$

It follows that

$$\sup_{F,k:(p,x)\in F} \inf_{u \in U_{F,k}} u(y) \leq \sup_{F,k:(p,x)\in F} \inf_{u \in U_{F,k}} u(x). \tag{4.3}$$

For every $z \in \mathbb{R}^m_+$, (4.1) implies that

$$\sup_{F,k:(p,x)\in F} \inf_{u \in U_{F,k}} u(z) \geq \sup_{F,k} \inf_{u \in U_{F,k}} u(z).$$

But because the reverse inequality is obvious (the sup on the left is over a smaller set of F's than that on the right), we have,

$$\sup_{F,k:(p,x)\in F} \inf_{u \in U_{F,k}} u(z) = \sup_{F,k} \inf_{u \in U_{F,k}} u(z) = u^*(z).$$

Applying this to (4.3), we conclude that $u^*(y) \leq u^*(x)$.

Thus, we have so far shown that $u^*(y) \leq u^*(x)$ for every $y \in \mathbb{R}^m_+$ such that $p(y - x) \leq 0$.

Suppose next that $y \in \mathbb{R}^m_+$ is such that $p(y - x) < 0$. Then there exists y' such that $y << y'$ and $p(y' - x) < 0$. By what we have already shown, $u^*(y') \leq u^*(x)$. But because u^* is strictly increasing, $u^*(y) < u^*(y')$. Hence, $u^*(y) < u^*(x)$, as desired. Q.E.D.

5. Examples.

We present four examples of infinite datasets, each satisfying GARP. The first example admits no lower semicontinuous rationalization, the second no upper semicontinuous rationalization, the third no concave rationalization, and the fourth, whose dataset satisfies SARP, admits no strict rationalization.[16] In each example, there are two goods, i.e., $m = 2$, and a typical bundle will be denoted by $x = (a, b) \in \mathbb{R}^2_+$.

Example 5.1. ***(No lower semicontinuous rationalization)*** *Suppose that, for every $n \geq 2$, the bundle $x^0 = (1, 0)$ is chosen at the price vector $p^n = (1 - \frac{1}{n}, 1)$, and the bundle $x^1 = (0, 1)$ is*

[16] A utility function, u, *strictly rationalizes* the data if for each data point (p, x), $u(x) > u(y)$ holds for every $y \neq x$ such that $py \leq px$.

chosen at the price vector $p = (1, 2)$*. Let* D *denote the resulting dataset. It is easily checked that* D *satisfies GARP because only two bundles,* x^0 *and* x^1*, are ever chosen and* x^1 *is never affordable when* x^0 *is chosen. For each* $k \geq 1$*, let* $x^k = (0, 1 - \frac{1}{k})$*. If* v *is any utility function rationalizing* D*, then we must have* $v(x^0) < v(x^1)$ *because* $px^0 < px^1$*, and for each* k *we must have* $v(x^k) < v(x^0)$ *because* $p^n x^k < p^n x^0$ *for* $n > k$*. Consequently,* $\underline{\lim}_k v(x^k) \leq v(x^0) < v(x^1)$*. Since* x^k *converges to* x^1*,* v *is not l.s.c. at* x^1*.*

Example 5.2. ***(No upper semicontinuous rationalization)*** *Suppose that bundles* $x^0 = (1, 0)$ *and* $x^1 = (0, 1)$ *are each chosen at the price vector* $p^0 = (1, 1)$*, and for each* $n \geq 2$ *the bundle* $x^n = (\frac{3}{n}, 1 - \frac{2}{n})$ *is chosen at the price vector* $p = (1, 2)$*. Let* D *denote the resulting dataset. It is easily checked that* D *is rationalized by the increasing utility function* u *defined by* $u(a, b) = a + b$ *if* $a + b \leq 1$*, and* $u(a, b) = a + 2b$ *otherwise. Consequently,* D *satisfies GARP. If* v *is any utility function rationalizing* D*, then we must have* $v(x^0) < v(x^2) < v(x^n)$ *for every* n *because* $p^0 x^0 < px^2 < px^n$*, and we must have* $v(x^1) = v(x^0)$ *because* $p^0 x^0 = p^0 x^1$*. Consequently,* $v(x^1) = v(x^0) < v(x^2) \leq \overline{\lim}_n v(x^n)$*. Since* x^n *converges to* x^1*,* v *is not u.s.c. at* x^1*.*

Example 5.3. ***(No concave rationalization)*** *Consider the strictly increasing and quasiconcave utility function* $u(a, b) = b + \sqrt{a + b^2}$*. Its indifference curves are straight lines connecting the axes, though they are not parallel. Their slopes decrease as one moves outward from the origin. It is well known (Fenchel, 1953; Arrow and Enthoven, 1961; Aumann, 1975) that this utility function cannot be concavified.*[17] *That is, there is no strictly increasing function* f *defined on the range of* u *such that* $f \circ u$ *is concave. Consequently, if* D *consists of all* (p, x) *such that* p *is the gradient of* u *at* x*, then* u *rationalizes* D*. Hence* D *satisfies GARP but has no concave rationalization.*

Example 5.4. ***(SARP and no strict rationalization)*** *Suppose that, for every* $\lambda > 0$ *and every* $n \geq 2$*, the bundle* $(\lambda, 2\lambda)$ *is chosen at the price vector* $p^n = (1 + \frac{1}{n}, 1)$ *and the bundle* $(2\lambda, \lambda)$ *is chosen at the price vector* $p = (1, 1)$*. Let* D *be the resulting dataset. To see that* D *satisfies SARP, note that if* (a, b) *is affordable when the distinct bundle* (a', b') *is chosen, then* $a' + b' > a + b$ *in all cases but those in which* $(a, b) = (\lambda, 2\lambda)$ *and* $(a', b') = (2\lambda, \lambda)$*.*[18] *This is sufficient to ensure that, within the dataset, there can be no cycles in the "directly revealed preferred to" relation, proving SARP.*[19] *Let* v *rationalize* D*. We must show that* v *does not strictly rationalize* D*. If* $\lambda' > \lambda$*, then* $(2\lambda, \lambda)$ *is strictly affordable when* $(2\lambda', \lambda')$ *is chosen and so we must have* $v(2\lambda', \lambda') > v(2\lambda, \lambda)$*. Hence,* $v(2\lambda, \lambda)$ *is a strictly increasing function of* $\lambda > 0$ *and so it is continuous except at perhaps countably many points. Let* $\lambda^* > 0$ *be a continuity point. Since* $(2\lambda^*, \lambda^*)$ *is chosen at the price vector* $(1, 1)$ *when* $(\lambda^*, 2\lambda^*)$ *could have been chosen, we must have* $v(2\lambda^*, \lambda^*) > v(\lambda^*, 2\lambda^*)$ *if* v *is to strictly rationalize* D*. Thus, it suffices to show that* $v(2\lambda^*, \lambda^*) \leq v(\lambda^*, 2\lambda^*)$*. For any* $\lambda < \lambda^*$*,* $p^n(2\lambda, \lambda) < p^n(\lambda^*, 2\lambda^*)$ *for some* n *large enough. Consequently,* $(2\lambda, \lambda)$ *is affordable when* $(\lambda^*, 2\lambda^*)$ *is chosen at the price vector* p^n *and so* $v(2\lambda, \lambda) \leq v(\lambda^*, 2\lambda^*)$*. Since this inequality holds for any*

[17] See Reny (2013) for a general nonconcavifiability result that includes this function as a special case.

[18] The only nontrivial cases are those in which $(a, b) = (2\lambda, \lambda)$ is affordable when $(a', b') = (\lambda', 2\lambda')$ is chosen at some price $p = (1 + \frac{1}{n}, 1)$. Then, $(1 + \frac{1}{n})2\lambda + \lambda \leq (1 + \frac{1}{n})\lambda' + 2\lambda'$, which, after factoring out λ and λ' on each side, implies that $\lambda < \lambda'$. Then, adding $\frac{1}{n}\lambda$ to the lefthand side and the larger $\frac{1}{n}2\lambda'$ to the righthand side gives $(1 + \frac{1}{n})(2\lambda + \lambda) < (1 + \frac{1}{n})(\lambda' + 2\lambda')$ and division by $(1 + \frac{1}{n})$ yields the desired conclusion.

[19] One bundle is *directly revealed preferred* to another if the one is affordable at a price vector at which the other is chosen.

$\lambda < \lambda^*$, *we may take the limit as* λ *converges to* λ^* *from below. The continuity of* $v(2\lambda, \lambda)$ *at* λ^* *then yields* $v(2\lambda^*, \lambda^*) \le v(\lambda^*, 2\lambda^*)$.

6. Additional Remarks

Remark 4. *Suppose that* D *satisfies GARP and is* income-continuous *in the sense that for every* $(p, x) \in D$ *there is a sequence* $(p, x_n) \in D$ *with* $x_n \to x$ *such that* $px_n > px$ *for every* n*. By our main result,* D *is rationalized by a strictly increasing and quasiconcave* u *taking values in* $[-1, 1]$*. For every* $x \in \mathbb{R}^m_+$*, let* $u^*(x) = \lim_n u(x + \mathbf{1}/n)$*, where* $\mathbf{1} = (1, ..., 1) \in \mathbb{R}^m$*. It is straightforward to show that* u^* *is upper semicontinuous, strictly increasing and quasiconcave. To see that* u^* *rationalizes* D*, suppose that* $(p, x) \in D$ *and* $py \le px$*. Then there is a sequence* $(p, x_n) \in D$ *with* $x_n \to x$ *such that* $py \le px < px_n$ *for every* n*. For any* k*, we may first choose* n *sufficiently large so that* $x_n << x + \mathbf{1}/k$*, and then choose* m *sufficiently large so that* $p(y + \mathbf{1}/m) < px_n$*. Hence,* $u(y + \mathbf{1}/m) \le u(x_n) \le u(x + \mathbf{1}/k)$*. That is, for every* k*,* $u(y + \mathbf{1}/m) \le u(x + \mathbf{1}/k)$ *holds for all* m *large enough. Consequently,* $u^*(y) \le u^*(x)$*, as desired. This differs from Mas-Colell (1978) Theorem 1 in two ways. First, our dataset needn't be a complete dataset generated by a demand function. Second, we do not require continuity in prices. A consequence of the latter is that our upper semicontinuous rationalization, unlike that obtained in Mas-Colell (1978) is not necessarily strict.*[20]

Remark 5. *Suppose that* D *satisfies SARP and, as in Sondermann (1982), is* connected *in the sense of Richter (1966), i.e., if* (p, x)*,* $(q, y) \in D$ *then* $(p', tx + (1-t)y) \in D$ *for some price vector* p' *and some* $t \in (0, 1)$*. By our main result,* D *is rationalized by a strictly increasing and quasiconcave* u *taking values in* $[-1, 1]$*. Moreover, if for* $(p, x), (q, y) \in D$ *we have* $y \ne x$ *and* $py = px$*, we claim that* $u(y) < u(x)$*. Indeed, by connectedness* $(p', tx + (1-t)y) \in D$ *for some price* p' *and some* $t \in (0, 1)$*. Letting* $\bar{x} = tx + (1-t)y$*,* $p\bar{x} = px$ *implies that* $p'(tx + (1-t)y) < p'x$ *by SARP and that* $u(\bar{x}) \le u(x)$*. Hence,* $p'y < p'\bar{x}$ *and so* $u(y) < u(\bar{x}) \le u(x)$*. So, redefining* u *to be* -1 *for bundles that are never chosen, this redefined* u *strictly rationalizes* D*. This generalizes Sondermann (1982) and, because we obtain a utility representation, Theorem 1 in Hurwicz and Richter (1971) under our maintained hypothesis (which they do not impose) that all income is always spent since strict rationalization implies that the redefined* u *is strongly increasing and strictly quasiconcave on the set of chosen bundles.*[21] *In particular, if the set of of chosen bundles is all of* $\mathbb{R}^m_+$*,* u *is everywhere strongly increasing and strictly quasiconcave, and strictly rationalizes* D*.*

Remark 6. *If a dataset can be rationalized by a preference relation then the dataset must satisfy GARP.*[22] *Consequently, Theorem 4.1 implies that a dataset can be rationalized by a preference relation if and only if it can be rationalized by a strictly increasing and quasiconcave utility function.*[23]

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[20] The conditions in Hurwicz and Richter (1971) and in Sondermann (1982) suffice to obtain a strict rationalization that is upper semicontinuous on the set of *chosen* bundles, but are insufficient to ensure upper semicontinuity on the entire consumption set (which, in our case, is the nonnegative orthant).

[21] A utility funciton u is *strongly increasing* if $u(x) > u(y)$ whenever $x \ge y$ and $x \ne y$.

[22] By *preference relation*, we mean a complete, reflexive, and transitive binary relation. A preference relation, $\succsim$, *rationalizes* a dataset D if $(p, x) \in D$ implies $x \succsim y$ for every y such that $py \le px$ and also not $y \succsim x$ if $py < px$.

[23] So, for example, it is not a coincidence that the demand behavior induced by lexicographic preferences, as

Remark 7. *Afriat's theorem can be used to prove rather easily that an arbitrary dataset D satisfying GARP can be rationalized by a preference relation on $\mathbb{R}^m_+$ that is strictly increasing and convex.*[24] *The idea is to view a preference relation as an indicator function, i.e., as an element σ of $\{0,1\}^{R^m_+ \times R^m_+}$, where $\sigma(x,y) = 1$ means "x is at least as good as y" and $\sigma(x,y) = 0$ means "y is strictly preferred to x." For every finite subset F of D, define $\Sigma_F := \{\sigma \in \{0,1\}^{R^m_+ \times R^m_+} : \sigma$ is strictly increasing, convex, and rationalizes $F\}$. Each Σ_F is compact in the product topology. By Afriat's theorem, each Σ_F is nonempty and so $\{\Sigma_F\}_{F \subseteq D}$ has the finite intersection property. Then, by Tychonoff's theorem, $\cap_{F \subseteq D} \Sigma_F$ is nonempty and any member, σ^*, is increasing, convex, and rationalizes D. However, the preference relation σ^* need not have a utility representation and so this otherwise very simple proof technique is unhelpful in establishing the existence of a utility function that rationalizes the dataset D.*[25] *Indeed, there can be many strictly increasing and convex preference relations that rationalize the given data and some may not have a utility representation (consider, e.g., lexicographic preferences and the dataset they generate). Our main result shows that at least one of these preference relations has a utility representation.*

Remark 8. *A similarly simple argument as in Remark 7 above, but using Matzkin and Richter (1991) instead of Afriat's theorem, establishes that if D satisfies SARP, then D can be rationalized by a strictly increasing and strictly convex preference relation. But again, not all such rationalizations need have a utility representation. Indeed, as our fourth example above shows, it can happen that no such strictly increasing and strictly convex preference relation has a utility representation.*

Remark 9. *Proposition 3.1 suggests yet another order-n^3 method for testing whether a finite dataset, $F = \{(p_1, x_1), ..., (p_n, x_n)\}$, is rationalizable.*[26] *Define $a_{ij} := p_j(x_i - x_j)$ and initialize $u^0_i := 0$ for $i = 1, ..., n$. Define u^n_i inductively so that $u^k_i := \min_j(a_{ij} + u^{k-1}_j)$ holds for $i, k = 1, ..., n$ and where the minimum is over all j such that $a_{ij} \leq 0$. It is not difficult to argue that the vector $u^n := (u^n_1, ..., u^n_n)$ is computable in a number of steps that is of the order n^3. If F satisfies GARP, then it is straightforward to show that $u^n_i = u_F(x_i)$ for every i and therefore that $u^n_i \leq a_{ij} + u^n_j$ whenever $a_{ij} \leq 0$. Hence the dataset F is rationalizable if and (by Proposition 3.1) only if the vector u^n rationalizes the data in the sense that $u^n_i \leq u^n_j$ whenever $a_{ij} \leq 0$, with the first inequality strict whenever the second is strict.*[27]

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is well-known, can be rationalized by a utility function even though the preferences themselves have no utility representation.

[24] A preference relation, $\succsim$, is *strictly increasing* if $x \geq y$ implies $x \succsim y$ and $x >> y$ implies $x \succ y$, and is *convex* if $x \succsim y$ implies $tx + (1-t)y \succsim y$ for $t \in [0,1]$.

[25] In addition, because Tychonoff's theorem is employed, this proof technique invokes the axiom of choice, whereas the proof of our main result does not.

[26] It is well-known (see Varian, 2006) that Warshall's (1962) algorithm for computing the transitive closure of a binary relation furnishes an order-n^3 method for checking GARP. I am grateful to Hal Varian for a helpful discussion on the usefulness of Warshall's algorithm. See Fostel et. al. (2004) for an order-n^2 algorithm under SARP.

[27] For the "if" direction, define $u(x_i) := u^n_i$ for each i and define $u(x) := -1 + \min_i u^n_i$ for all $x \notin \{x_1, ..., x_n\}$.

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