MECHANISM DESIGN WITH EX-POST VERIFICATION AND NO TRANSFERS

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Abstract. We study a problem of allocating a good among several agents in an environment with asymmetric information, no monetary transfers, and ex-post verifiability. We show that optimal allocation has a number of anti-competitive features: participation might be restricted to a select group of agents; allocation is stochastic, occasionally favoring low-value agents. If the number of agents is high, the optimal mechanism is a shortlisting procedure. Otherwise, the optimal mechanism is a restricted bid auction, augmented by a shortlisting procedure for the agents with minimal bids. From a methodological perspective, the paper provides a solution to an interesting mechanism design problem without transfers that features a nontrivial interaction of incentive and feasibility constraints.

Keywords: mechanism design without transfers, matching with asymmetric information, stochastic mechanism, auction, feasibility constraint, shortlisting procedure

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1. Introduction

This paper studies a novel mechanism design problem without transfers and with ex-post verification. The headquarters of a company would like to allocate a new office space to one of its departments. The demand for the office space and its fit to the needs of a department is the department's private information. The headquarters can learn the true value of allocating the space to a department only ex-post, after the decision has been made, and only for the department that gets the space. The headquarters can ask the departments to submit the proposals outlining the benefit for the company of allocation the space to them, and penalize the department ex-post if their proposal is revealed to be insincere. The departments, however, have limited liability that restricts the severity of punishment.

Another application of the model is labor markets in which applicants submit their CVs, but the true qualification of the hired applicant becomes revealed over time. Also, the model can be applied to organic search in which websites provide descriptions of their content for a search engine algorithm and the search engine can penalize the websites for unfair search engine optimization practices by dropping their future search rankings.

In our model, there is a principal who has to choose one agent from a pool of ex-ante identical agents. The agents' value for the principal is their private information and all agents would like to be selected. The utility is not transferrable, but the principal can learn the value of the selected agent ex-post and there are (limited) penalties that can be imposed on this agent.

Since the penalty is limited, the low value agents will lie and exaggerate their value for the principal if competition is fierce and their chances of winning by truthful reporting are slim. Consequently, the optimal rules must maximize the probability of selecting high-value agents subject to the constraint that the low-value agents are chosen frequently enough so that they do not want to misreport their information.

If the number of agents is sufficiently large, an optimal allocation rule is a *shortlisting procedure*: each agent is shortlisted with probability that is increasing in her report about her value and is equal to one if the report is above some bar, and the principal chooses an agent at random from the shortlist.[1] Surprisingly, there is a neutrality result: the maximal attainable payoff for the government is constant after the number of agents reaches some threshold, which depends on the penalty size that can be imposed on the agents.

If the number of agents is small, an optimal allocation rule is a combination of a restricted-bid auction and a shortlisting procedure. There are two thresholds: low bar and high bar. The principal puts all agents whose reports are below the low bar onto the waiting list and classifies as superstars all agents whose reports are above the high bar. The winner is selected among the superstars if there are any. Otherwise, the principal selects an agent with the highest report among those who pass the low bar. Finally, if all the agents are below the low bar, the principal shortlists an agent with a probability which depends on her report and randomly selects an agent from the shortlist.

[1]Naturally, after the government observes the true value of the selected agent, it imposes the maximum feasible penalty whenever the agent's report is inconsistent with the realized value.

Methodologically, the principal's problem in our model is a maximization problem subject to incentive compatibility constraints which are expressed in interim probabilities of being selected. These interim probabilities are subject to generalized Matthews-Border feasibility condition. We solve the principal's problem by separating the incentive compatibility and the feasibility constraints into sub-constraints and solving two auxiliary maximization problems subject to different subsets of incentive compatibility and feasibility sub-constraints. We hope that this approach might prove useful in other mechanism design environments where the interaction of incentive compatibility and feasibility constraints poses challenges.

The rest of the paper is organized as follows. We discuss the related literature in Section 2. The model is presented in Section 3. Section 4 characterizes optimal allocation in reduced forms. We derive an upper bound on the principal's payoff in our environment in Section 5 and show that it can achieved if the number of agents is sufficiently large. In Section 6, we consider the case of small number of agents. Some of the proofs are in the Appendix.

2. Literature

Ben-Porath, Dekel and Lipman (2013) (henceforth, BDL) study a related model, with the key difference in modeling verification of agents' types by the principal. In BDL the principal can pay a cost and acquire information about agents' types before making an allocation decision, that is, types can be verified *ex ante*.[2] Our paper takes a different approach: after having selected an agent, the principal learns the type and can impose a penalty on that agent, that is, types are verified *ex post*.[3]

There is a recent literature on mechanism design with partial transfers; in this literature the agents' information is non-verifiable.[4] In Chakravarty and Kaplan (2013) and Condorelli (2012b), a benevolent principal would like to allocate an object to the agent with the highest valuation, and the agents signal their private types by exerting socially wasteful effort. Condorelli (2012b) studies a general model with heterogeneous objects and agents and characterizes optimal allocation rules where a socially wasteful cost is a part of mechanism design. Chakravarty and Kaplan (2013) restrict attention to homogeneous objects and agents, and consider environments in which socially wasteful cost has two components: an exogenously given type and a component controlled by

[2]A growing literature studies environments in which evidence that can be presented *before* an allocation decision is made, e.g., Townsend (1979), Grossman and Hart (1980), Grossman (1981), Milgrom (1981), Green and Laffont (1986), Postlewaite and Wettstein (1989), Lipman and Seppi (1995), Seidmann and Winter (1997), Glazer and Rubinstein (2004, 2006, 2012, 2013), Forges and Koessler (2005), Bull and Watson (2007), Severinov and Deneckere (2006), Deneckere and Severinov (2008), Kartik, Ottaviani and Squintani (2007), Kartik (2009), Sher (2011), Sher and Vohra (2011), Ben-Porath and Lipman (2012), Dziuda (2012), and Kartik and Tercieux (2012).

[3]In our model, the utility is not transferable. Optimal mechanism design with transfers that can depend on ex-post information has been studied in, e.g., Mezzetti (2004), DeMarzo, Kremer and Skrzypacz (2005), Eraslan and Yimaz (2007), Dang, Gorton and Holmström (2013), Deb and Mishra (2013), and Ekmekci, Kos and Vohra (2013). This literature is surveyed in Skrzypacz (2013).

[4]An exception is Bar and Gordon (forthcoming), discussed below, who consider an extension with ex-post verifiable types.

the principal. In particular, they demonstrate conditions under which, surprisingly, the uniform lottery is optimal.[5]

Bar and Gordon (forthcoming) consider a problem of project selection. For each project, the principal's and the project manager's values of the potential match are privately known to the manager. Transfers are permitted in one direction: the principal can subsidize but cannot tax projects.

In Manelli and Vincent (1995), a principal would like to procure a good from suppliers whose quality is uncertain. In their environment, a trading mechanism that selects the bidder with the lowest price might result in only low-quality goods being offered for sale, so competitive mechanisms might price out high quality suppliers. Che, Gale and Kim (2013) consider a problem of efficient allocation of resource to budget constrained agents and show that a random allocation with resale can outperform competitive market allocation. In an allocation problem in which the private and the social values of the agents' are private information, Condorelli (2012a) characterizes conditions under which optimal mechanism is stochastic and does not employ payments.

The literature has identified multiple reasons for restricting participation in allocation mechanisms. In auctions with entry costs, large number of bidders might be inefficient, as low-value agents have low probability to win and thus lack incentives to participate (Levin and Smith 1994, Gilbert and Klemperer 2000, Ye 2007). Compte and Jehiel (2002) study auctions in affiliated value environments and show that the uncertainty about the common value component might imply that more bidders need not lead to higher welfare.

If the value of surplus is endogenous and is determined by the actions of the agents prior to the allocation decision, excessive thickness of the market might weaken their incentives to undertake costly actions that increase the total surplus. For example, in research and development contests, it might be optimal to limit the number of participants to improve their incentives to invest in developing new technology (Taylor 1995, Fullerton and McAfee 1999, Che and Gale 2003). In financial settings, it may also be desirable to limit the number of banks to keep their incentives to screen loan applicants (Cao and Shi 2001).

3. Model

There is a principal who has to select one of $n \geq 2$ agents. The principal's payoff from a match with agent i is $x_i \in X \equiv [a, b]$, where x_i is private to agent i. The values of x_i's are i.i.d. random draws, with continuously differentiable c.d.f. F on X, whose density f is positive almost everywhere on X.

Each agent i makes a statement $y_i \in X$ about his type x_i, then the principal chooses an agent according to a specified rule. If an agent is not selected, his payoff is 0. Otherwise, he obtains a payoff of $v(x_i) > 0$. In addition, we assume that if the agent is selected, the

[5] See also McAfee and McMillan (1992), Hartline and Roughgarden (2008), Yoon (2011) for environments without transfers and money burning. In addition, money burning is studied in Ambrus and Egorov (2012) in the context of a delegation model.

principal observes x_i and can impose a penalty $c(x_i) \geq 0$ on the agent.[6] Our primary interpretation of c is the upper bound on the (expected) penalty that can be imposed on the agent after his type has been verified.[7] Functions v and c are bounded and almost everywhere continuous on $[a, b]$. Note that $v - c$ can be non-monotonic.

The principal has full commitment power and can choose any stochastic allocation rule that determines a probability of selecting each agent conditional on the report profile and the penalty conditional on the report profile and the type of the selected agent after it is verified ex-post. An allocation rule (p, ξ) associates with every profile of statements $y = (y_1, \ldots, y_n)$ a probability distribution $p(y)$ over $\{1, 2, \ldots, n\}$ and a family of functions $\xi_i(x_i, y) \in [0, 1]$, $i = 1, \ldots, n$, which determine the probability that agent i is penalized if he is selected given his type and the report profile. The allocation rule is common knowledge among the agents. The solution concept is perfect Bayesian equilibrium.

By the revelation principle, it is sufficient to consider allocation rules in which it is a perfect Bayesian equilibrium for all agents to make truthful statements. Furthermore, since type x_i of the selected agent is verifiable, it is optimal to penalize the selected agent whenever he lies, $y_i \neq x_i$, and not to penalize him otherwise. Hence we set $\xi_i(x_i, y) = 0$ if $y_i = x_i$ and 1 otherwise and drop ξ in the description of the allocation rules. Thus, the payoff of agent i whose type is x_i and who reports y_i is equal to[8]

$$V_i(x_i, y_i) = \int_{x_{-i} \in X^{n-1}} p_i(y_i, x_{-i})(v(x_i) - \mathbf{1}_{y_i \neq x_i} c(x_i)) \mathrm{d}\bar{F}_{-i}(x_{-i}).$$

The principal wishes to maximize the expected payoff,

$$\max_p \int_{x \in X^n} \sum_{i=1}^{n} x_i p_i(x) \mathrm{d}\bar{F}(x) \tag{P_0}$$

subject to the incentive compatibility constraints

$$V_i(x_i, x_i) \geq V_i(x_i, y_i) \quad \text{for all } x_i, y_i \in X \text{ and all } i = 1, \ldots, n. \tag{IC_0}$$

Denote by $h(x)$ the share of the surplus retained by a selected agent after deduction of the penalty (truncated at zero):

$$h(x) = \frac{\max\{v(x) - c(x), 0\}}{v(x)}.$$

Proposition 1. *Allocation rule p satisfies* (IC_0) *if and only if for every $i = 1, \ldots, n$ there exists $r_i \in [0, 1]$ such that for all $x_i \in X$*

$$r_i h(x_i) \leq \int_{x_{-i} \in X^{n-1}} p_i(x_i, x_{-i}) \mathrm{d}\bar{F}_{-i}(x_{-i}) \leq r_i. \tag{1}$$

[6]We could allow c to depend on the entire type profile, $c(x_1, x_2, \ldots, x_n)$, where n is the number of agents, without affecting any of the results. In that case, $c(x_i)$ should be thought of as the expected value of $c(x_1, x_2, \ldots, x_n)$ conditional on x_i.

[7]The assumption that x_i is verified with certainty can be relaxed; if $\alpha(x_i)$ is the probability that x_i is verified and $L(x_i)$ is the limit on i's liability, then set $c(x_i) = \alpha(x_i)L(x_i)$.

[8]Denote by $\bar{F}$ the joint c.d.f. of all n agents and by $\bar{F}_{-i}$ the joint c.d.f. of all agents except i. For each agent i, $p_i(y_i, y_{-i})$ stands for the probability of choosing i as a function of the profile of reports.

Proof. Each agent i's best deviation is the one that maximizes the probability of i being chosen, so (IC_0) is equivalent to

$$v(x_i)\int_{x_{-i}} p_i(x_i,x_{-i})\mathrm{d}\bar{F}_{-i}(x_{-i}) \geq (v(x_i)-c(x_i))\sup_{y_i\in X}\int_{x_{-i}} p_i(y_i,x_{-i})\mathrm{d}\bar{F}_{-i}(x_{-i}). \tag{2}$$

Thus, (IC_0) implies (1) by setting $r_i = \sup_{y_i\in X}\int_{x_{-i}} p_i(y_i,x_{-i})\mathrm{d}\bar{F}_{-i}(x_{-i})$. Conversely, if (1) holds with some $r_i\in[0,1]$, then it also holds with $r_i' = \sup_{y_i\in X}\int_{x_{-i}} p_i(y_i,x_{-i})\mathrm{d}\bar{F}_{-i}(x_{-i})$ $\leq r_i$, which implies (2). ■

Problem in reduced form. We will approach problem (P_0) by formulating and solving its reduced form. Denote by $g_i(y)$ the probability that agent i is selected conditional on reporting y,

$$g_i(y) = \int_{x_{-i}\in X^{n-1}} p_i(y,x_{-i})\mathrm{d}\bar{F}_{-i}(x_{-i}), \quad y\in X,$$

and define $g : X\to[0,n]$ by

$$g(y) = \sum_{i=1}^{n} g_i(y), \quad y\in X.$$

We will now formulate the principal's problem in terms of g:

$$\max_g \int_{x\in X} xg(x)\mathrm{d}F(x) \tag{P}$$

subject to the incentive compatibility constraint

$$v(x)g(x)\geq (v(x)-c(x))\sup_{y\in X} g(y) \quad \text{for all } x\in X, \tag{IC}$$

the feasibility condition due to $\sum_i p_i(y)=1$ for all $y\in X^n$,

$$\int_X g(x)\mathrm{d}F(x) = 1, \tag{F_0}$$

and a generalization of Matthews-Border feasibility criterion (Matthews 1984, Border 1991, Hart and Reny 2013) that guarantees existence of an allocation rule p that induces a given g (see Lemma 3 in the Appendix):

$$\int_{\{x:g(x)\geq t\}} g(x)\mathrm{d}F(x)\leq 1-\Big(F(\{x: g(x)<t\})\Big)^n \quad \text{for all } t\in[0,n]. \tag{F}$$

Proposition 2.
(i) If p is a solution of (P_0)*, then the reduced form of p is a solution of* (P)*.*
(ii) If g is a solution of (P)*, then it is the reduced form of some solution of* (P_0)*.*

Problem (P) is interesting because of its constraints. First, incentive compatibility constraints (IC) are global rather than local as is often the case in mechanism design. Second, feasibility constraint (F) is substantive and will bind at the optimum iff incentive compatibility constraint (IC) slacks.

If g is increasing, then (F) takes a simple form,

$$\int_t^b g(x)\mathrm{d}F(x) \le 1 - F^n(t) \quad \text{for all } t \in X.$$

Therefore, a possible way to verify (F) is by "reordering" types in X in the ascending order in the image of g.

A *measure-preserving monotonization of function* g is a weakly increasing function $\tilde{g} : X \to [0, n]$ such that for every $t \in [0, 1]$

$$\int_{\{x:g(x)\ge t\}} g(x)\mathrm{d}F(x) = \int_{\{x:\tilde{g}(x)\ge t\}} \tilde{g}(x)\mathrm{d}F(x).$$

Proposition 1′. *Function g satisfies* (IC) *and* (F) *if and only if there exists $r \in [0, 1]$ such that*

$$\text{(IC}_1\text{)} \qquad g(x) \ge h(x)r, \quad x \in X,$$

$$\text{(IC}_2\text{)} \qquad g(x) \le r, \quad x \in X,$$

$$\text{(F}'\text{)} \qquad \int_x^b \tilde{g}(t)\mathrm{d}F(t) \le 1 - F^n(x), \quad x \in X,$$

where $\tilde{g}$ is a measure-preserving monotonization of g.

4. Optimal allocation rules in reduced form

By Proposition 1', we can write the principal's problem (P) as

$$\text{(P}^*\text{)} \qquad \max_{g,r} \int_{x\in X} xg(x)\mathrm{d}F(x)$$

subject to (IC_1), (IC_2), (F_0), and (F).

For each $r \in [0, n]$ let us find the lowest trajectory of $G(x) := \int_a^x g(t)\mathrm{d}F(t)$ subject to g satisfying (IC_1) and (F):

$$\text{(P}_{\min}\text{)} \qquad \min_g \int_X G(x)\mathrm{d}F(x)$$

$$\text{(IC}_{\min}\text{)} \qquad \text{s.t. } g(x) \ge rh(x) \quad \text{for almost all } x \in X,$$

$$\text{(F}_{\min}\text{)} \qquad \int_{\{x:g(x)<t\}} g(x)\mathrm{d}F(x) \ge (F(\{x : g(x) < t\}))^n \quad \text{for all } t \in [0, n].$$

We also find the highest trajectory of $\bar{G}(x) := \int_x^b g(t)\mathrm{d}F(t)$ subject to g satisfying (IC_2) and (F):

$$\text{(P}_{\max}\text{)} \qquad \max_g \int_X \bar{G}(x)\mathrm{d}F(x)$$

$$\text{(IC}_{\max}\text{)} \qquad \text{s.t. } g(x) \le r \quad \text{for almost all } x \in X,$$

$$\text{(F}_{\max}\text{)} \qquad \int_{\{x:g(x)\ge t\}} g(x)\mathrm{d}F(x) \le 1 - (F(\{x : g(x) < t\}))^n \quad \text{for all } t \in [0, n].$$

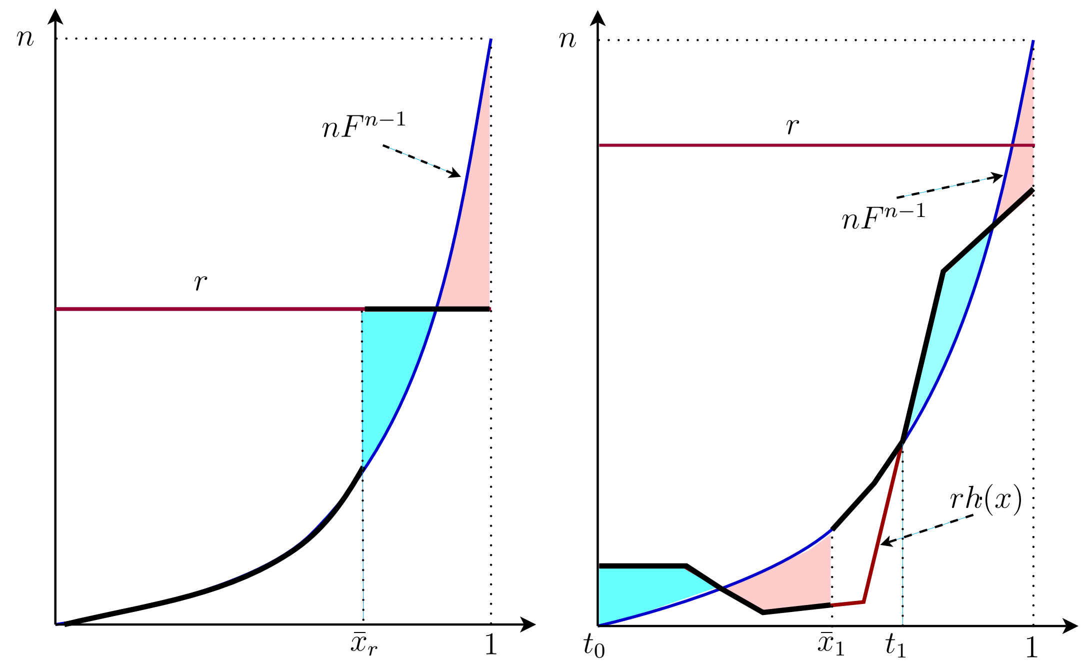

FIG. 1. Examples of solutions of $\mathrm{P_{max}}$ (left) and $\mathrm{P_{min}}$ (right).

Let $\underline{g}_r$ and $\overline{g}_r$ be solutions of problems ($\mathrm{P_{min}}$) and ($\mathrm{P_{max}}$), respectively. For every $z \in X$ define $g_z^* : X \to [0, n]$ as the concatenation of $\underline{g}_r$ and $\overline{g}_r$ at point z:

$$g_z^*(x) = \begin{cases} \underline{g}_r(x), & x \le z, \\ \overline{g}_r(x), & x > z, \end{cases} \tag{3}$$

where $r \in [0, n]$ is the smallest solution of

$$\int_a^z \underline{g}_r(x)\mathrm{d}F(x) + \int_z^b \overline{g}_r(x)\mathrm{d}F(x) = 1. \tag{4}$$

Theorem 1. *Mechanism g^* is a solution of* (P^*) *if and only if $g^* = g_z^*$, where z solves*

$$\max_{z \in X} \int_a^b x g_z^*(x)\mathrm{d}F(x). \tag{5}$$

The idea behind the result is as follows. Observe that for every g that satisfies ($\mathrm{F_0}$), $G(x) = \int_a^x g(t)\mathrm{d}F(t)$ is a c.d.f. In these notations, the objective of the principal is to choose a c.d.f. G that maximizes $\int_X x\mathrm{d}G(x)$ subject to (IC) and (F). To prove the result, we show that the set $\{g_z^*\}_{z\in X}$ contains all functions that are maximal w.r.t. first order stochastic dominance order (FOSD) among all feasible and incentive compatible ones, in the sense that for every feasible and incentive compatible g there exists $z \in X$ such that $G_z^*(x) = \int_a^x g_z^*(t)\mathrm{d}F(t)$ first order stochastically dominates $G(x) = \int_a^x g(t)\mathrm{d}F(t)$. Thus, optimizing on set $\{g_z^*\}_{z\in X}$ yields a solution of (P^*).

The solutions $\underline{g}_r$ and $\overline{g}_r$ of problems ($\mathrm{P_{min}}$) and ($\mathrm{P_{max}}$) are illustrated by Fig. 1. The left diagram depicts $\overline{g}_r$ (cf. Lemma 1 below). The blue curve is $nF^{n-1}(x)$ and the red curve is r; the black curve depicts $\overline{g}_r(x)$. Starting from the highest type $x = b$, the black line follows r so long as constraint ($\mathrm{F}_{\max}$) slacks. Down from point $\bar{x}_r$ constraint ($\mathrm{F}_{\max}$) is binding, and the highest trajectory of $\bar{G}(x)$ that satisfies this constraint is exactly $1 - F^n(x)$. Since $\bar{G}(x) = \int_x^b \overline{g}_r(t)\mathrm{d}F(t)$, the solution $\overline{g}_r(t)$ is equal to $nF^{n-1}(t)$ for $t < \bar{x}_r$.

The right diagram on Fig. 1 depicts an example of $\underline{g}_r$. The blue curve is $nF^{n-1}(x)$ and the red curve is $rh(x)$; the black curve depicts $\underline{g}_r(x)$. Initially we have $r_0 = a$, and the black line follows $rh(x)$ up to the point $\bar{x}_1$ where blue area is equal to red area, and then jumps to $nF^{n-1}(x)$. Then, the black line follows $nF^{n-1}(x)$ so long as it is above $rh(x)$. After the crossing point, t_1, the black line again follows $rh(x)$, etc.

5. Upper bound

There are two qualitatively distinct cases: when feasibility constraint (F) is binding at the optimum and when it is not. Recall that (F) is necessary (and, together with (F_0), sufficient) for existence of allocation rule p that induces a desired reduced-form mechanism g. This constraint becomes weaker as n increases and, eventually, permits all nondegenerate distributions as $n \to \infty$. On the other hand, incentive compatibility constraint (IC) does not depend on n. So when n is large enough, the shape of the optimal distribution is determined entirely by (IC) and does not depend on n.

Let x^* be the unique[9] solution of

$$z^* \left(\int_a^{x^*} xh(x)\mathrm{d}F(x) + \int_{x^*}^b x\mathrm{d}F(x) \right) = x^* \tag{6}$$

where z^* is the normalizing constant:

$$z^* = \left(\int_a^{x^*} h(x)\mathrm{d}F(x) + \int_{x^*}^b \mathrm{d}F(x) \right)^{-1}. \tag{7}$$

Theorem 2. *For every number of agents, in any allocation rule the principal's payoff is at most x^*. Moreover, if a solution of* (P) *achieves the payoff of x^*, then its reduced form must be (almost everywhere) equal to*

$$g^*(x) = \begin{cases} z^*h(x), & x \le x^*, \\ z^*, & x \ge x^*. \end{cases} \tag{8}$$

We obtain the bound on the principal's payoff by solving (P) subject to (IC) and (F_0), while ignoring constraint (F). It is evident that the relaxed problem does not depend on n.

Since the principal's objective is linear, the solution is (almost everywhere) boundary. The principal's payoff is maximized by a cutoff rule that maximizes the probability of selecting agents with types above x^* subject to the constraint that the types below x^* are selected with high enough probability to provide incentives for truthful reporting,

[9]To show uniqueness of x^*, rewrite (6) and (7) as $\int_a^{x^*}(x^*-x)h(x)\mathrm{d}F(x) = \int_{x^*}^b (x-x^*)\mathrm{d}F(x)$ and observe that the left-hand side is strictly increasing, while the right-hand side is strictly decreasing in x^*.

$g(x) \geq h(x) \sup g$. So, the solution is given by (8), where $z^* = \sup g^*$ is a constant determined by (F_0):

$$\int_a^{x^*} z^* h(x)\mathrm{d}F(x) + \int_{x^*}^b z^* \mathrm{d}F(x) = 1,$$

that yields (7).

The incentive constraint thus pins down the distribution of selected types, $G^*(x) = \int_a^x g(s)\mathrm{d}F(s)$ and determines the best attainable expected payoff $\int_a^b x\mathrm{d}G^*(x)$ absent feasibility constraint (F). Equation (6) is equivalent to equation $x^* = \int_a^b x\mathrm{d}G^*(x)$. The left-hand side of (6) is the marginal incentive cost as a function of x^* due to the rents that have to be given to the agents below x^*, while the right-hand side of (6) is the marginal value of selecting agents with types above x^*.

Attainment of the upper bound. Let $E_t = \{x : z^* h(x) \leq t\} \cap [a, x^*]$. Denote by $\bar{n}$ the smallest number that satisfies

$$\int_{E_t} z^* h(x)\mathrm{d}F(x) \geq \Big(F(E_t)\Big)^{\bar{n}} \quad \text{for all } t \in [0, z^*]. \tag{9}$$

This is a condition on primitives: F and h determine x^* and z^* and, consequently, $\bar{n}$.

Proposition 4. *There exists an allocation rule that attains the payoff of x^* if and only if $n \geq \bar{n}$.*

Condition (9) is not particularly elegant. Here is a sufficient condition that is simple and independent of F and x^*. Let $\tilde{n}$ the smallest number such that[10]

$$\frac{c(x)}{v(x)} \leq 1 - \frac{1}{\tilde{n}} \quad \text{for all } x \in X. \tag{10}$$

Corollary 1. *There exists an allocation rule that attains the payoff of x^* if $n \geq \tilde{n}$.*

Note that in some cases $\bar{n}$ and $\tilde{n}$ need not be very large. For example, $\tilde{n} \leq 2$ if $c(x) \leq \frac{1}{2} v(x)$ for all x, i.e., agents can be penalized by at most half of their gross payoff.

Remark 1. Theorem 2 implies that the optimal rule with $\bar{n}$ agents is weakly superior to any rule with $n > \bar{n}$ agents. That is, the value of competition is limited and expanding the pool of agents beyond $\bar{n}$ confers no benefit to the principal.

Implementation of the upper bound. Consider the following *shortlisting procedure.* Let each agent $i = 1, \dots, n$ be short-listed with some probability $q(y_i)$ given report y_i. The rule chooses an agent from the shortlist with equal probability. If the short list is empty, then the choice is made at random, uniformly among all n agents.

Proposition 5. *Let $n \geq \bar{n}$. Then the shortlisting procedure with*

$$q(x) = \begin{cases} \frac{Kh(x)-1}{K-1}, & x < x^*, \\ 1, & x \geq x^*. \end{cases} \tag{11}$$

[10]Note that $\tilde{n}$ exists if and only if $\sup_{x \in X} h(x) < 1$.

attains the upper bound x^*, where K is the unique solution of

$$\frac{(K-1)^{n-1}}{K^n} = \frac{(z^*-1)^{n-1}}{(z^*)^n}, \quad K > z^*. \tag{12}$$

6. Small number of agents

If the number of agents is small, $n < \bar{n}$, then feasibility constraint (F) is binding at the optimum. The upper bound cannot be attained, and Theorem 2 is not applicable. To find an optimal allocation rule as described by Theorem 1, we must solve problems ($\mathrm{P}_{\min}$) and ($\mathrm{P}_{\max}$) subject to both incentive compatibility and feasibility constraints.

The solution of ($\mathrm{P}_{\max}$) is easy.

Lemma 1. *For every $r \in [0, n]$, the solution of* ($\mathrm{P}_{\max}$) *is equal to*

$$\overline{g}_r(x) = \begin{cases} nF^{n-1}(x), & x \in [a, \bar{x}_r), \\ r, & x \in [\bar{x}_r, b], \end{cases}$$

where $\bar{x}_r$ is implicitly defined by

$$\int_{\bar{x}_r}^{b} r \mathrm{d}F(x) = 1 - F^n(\bar{x}_r) \quad \text{for each } r \geq 1, \tag{13}$$

and $\bar{x}_r = a$ for $r < 1$.

The solution of ($\mathrm{P}_{\min}$) is more complex, as it involves function $h(x)$ in the constraints. To obtain tractable results, we make the following assumption.

Assumption 1. *$h(F^{-1}(t))$ is weakly concave.*

Examples that satisfy Assumption 1:

(a) Let the penalty be proportional to the value, $c(x) = \alpha v(x)$, $0 < \alpha < 1$. Then h is constant,
$$h(x) = 1 - \alpha, \quad x \in X.$$

(b) Suppose that the principal is benevolent and wishes to maximize the agents' surplus, that is, $v(x) = x$. Let the penalty be constant, $c(x) = \bar{c} < 1$, and let $X = [\bar{c}, 1]$ (so that $v(x) \geq c(x)$ for all $x \in X$). Then
$$h(F^{-1}(t)) = 1 - \frac{\bar{c}}{F^{-1}(t)}, \quad t \in [0, 1]$$
is concave, provided $f(x)x^2$ is weakly increasing.

(b) Suppose that the principal and the chosen agent share a unit surplus: the principal's payoff is x and the agent's payoff is $v(x) = 1 - x$. Let the penalty be constant, $c(x) = \bar{c} < 1$ and let $X = [0, 1 - \bar{c}]$ (so that $v(x) \geq c(x)$ for all $x \in X$). Then
$$h(F^{-1}(t)) = 1 - \frac{\bar{c}}{1 - F^{-1}(t)}, \quad t \in [0, 1]$$
is concave, provided $f(x)(1 - x)^2$ is weakly decreasing.

Lemma 2. *Let Assumption 1 hold. Then for every $r \in [0, n]$ the solution of problem* ($\mathrm{P}_{\min}$) *is equal to*

$$\underline{g}_r(x) = \begin{cases} rh(x), & x \in [a, \underline{x}_r], \\ nF^{n-1}(x), & x \in (\underline{x}_r, b], \end{cases}$$

where $\underline{x}_r$ is implicitly defined by

$$\int_a^{\underline{x}_r} rh(x)\mathrm{d}F(x) = F^n(\underline{x}_r) \quad \text{for each } r \leq \bar{r}, \tag{14}$$

and $\underline{x}_r = b$ for $r > \bar{r}$, where

$$\bar{r} = \left(\int_a^b h(x)\mathrm{d}F(x) \right)^{-1}.$$

Proposition 6. *Let $n < \bar{n}$ and suppose that Assumption 1 holds. Then g^* is a solution of* (P^*) *if and only if*

$$g^*(x) = \begin{cases} rh(x), & x \leq \underline{x}_r, \\ nF^{n-1}(x), & \underline{x}_r < x \leq \bar{x}_r, \\ r, & x > \bar{x}_r, \end{cases} \tag{15}$$

where $\bar{x}_r$ and $\underline{x}_r$ are defined by (13) *and* (14), *respectively, and $r \in [1, \bar{r}]$ is the solution of*

$$\int_a^{\underline{x}_r} (\underline{x}_r - x)h(x)\mathrm{d}F(x) = \int_{\bar{x}_r}^b (x - \bar{x}_r)\mathrm{d}F(x). \tag{16}$$

Solution g^* can be implemented as a combination of a restricted-bid auction and a shortlisting procedure. The principal asks each agent to report a number in $[a, \bar{x}_r]$; a report y by agent with type x is considered truthful if $y = \min\{x, \bar{x}_r\}$. Then, the agents are divided into two groups: regular candidates (with $x > \underline{x}_r$) and "waiting list" candidates (with $x \leq \underline{x}_r$). If there is at least one regular candidate available, the principal chooses the one with the highest report. Otherwise, if only waiting list candidates are available, the principal applies to them a shortlisting procedure similar to the one describe in the previous section.

Interestingly, the interval $[\underline{x}_r, \bar{x}_r]$ where agents' types are fully separated shrinks as n increases and disappears for $n \geq \bar{n}$.

Proposition 7. *Suppose that Assumption 1 holds. Then, at the solution of* (P^*), *difference $\bar{x}_r - \underline{x}_r$ is decreasing in n; moreover, $\underline{x}_r \geq \bar{x}_r$ for $n \geq \bar{n}$.*

The proof is straightforward by definition of $\underline{x}_r$ and $\overline{x}_r$.

Example (Proportional penalty). Consider the example with the penalty proportional to the value, $c(x) = \alpha v(x)$.

A *restricted bid* auction is a rule that (i) allows the agents to report $y_i \in [\underline{y}, \overline{y}]$, where $a \leq \underline{y} < \overline{y} \leq b$, (ii) selects an agent with the highest report, splitting ties randomly, and (iii) penalizes the selected agent unless he makes a report closest to the truth $y_i = \max\{\min\{x_i, \overline{y}\}, \underline{y}\}$.

Proposition 8. *If $n < \bar{n}$ and $\frac{c(x)}{v(x)}$ is constant, then a restricted bid auction is optimal.*

Proof. Let g^* be the solution described in Proposition 6 and consider the restricted bid auction with $\underline{y} = \underline{x}_r$ and $\overline{y} = \overline{x}_r$. Since $h(x)$ is constant, g^* pools the types above $\overline{x}_r$ and below $\underline{x}_r$, and hence the restricted bid auction implements g^*. ■

As the number of agents increases, the range of allowed reports $\overline{y} - \underline{y} \to 0$. If n is sufficiently large, the optimal mechanism converges to the following degenerate two bid auction with limited participation: it dismisses a fixed number of agents, asks the remaining agents to report whether their type is above or below a threshold, and chooses at random any agent whose reported type is above the threshold if there is at least one such report, and any agent at random otherwise.

Appendix

Proof of Proposition 2. Observe that for every p and its reduced form g, objective functions in (P_0) and (P) are identical. We now verify that every solution of (P_0) is admissible for (P), and for every solution of (P) there is an admissible solution for (P_0).

Feasibility condition (F) is the criterion for existence of p that implements g. This condition is due to the following lemma, which is a generalization of Matthews-Border feasibility criterion (e.g., Border 1991, Proposition 3.1) to asymmetric mechanisms.

Let $\mathcal{Q}_n$ be the set of functions $q : X^n \to [0,1]^n$ such that $\sum q_i \leq 1$ and let λ be a measure on X. We say that $Q : X \to [0, n]$ is a reduced form of $q \in \mathcal{Q}_n$ if $Q(z) = \sum_i \int_{X^{n-1}} q_i(z, x_{-i}) \mathrm{d}\lambda^{n-1}(x_{-i})$ for all $z \in X$.

Lemma 3. *$Q : X \to [0, n]$ is the reduced form of some $q \in \mathcal{Q}_n$ if and only if*

$$\int_{\{x:Q(x)\geq z\}} Q(x)\mathrm{d}\lambda(x) \leq 1 - \Big(\lambda(\{x : Q(x) < z\})\Big)^n \quad \textit{for all } z \in [0, n]. \tag{17}$$

Proof. Sufficiency is due to Proposition 3.1 in Border (1991) implying that if Q satisfies (17), then there exists a *symmetric* q whose reduced form is Q. To prove necessity, consider $q \in \mathcal{Q}_n$ and let Q be its reduced form. For every $t \in [0, n]$ denote $E_t = \{x \in X : Q(x) \geq t\}$. Then

$$\begin{aligned}
\int_{y\in E_t} Q(y)\mathrm{d}\lambda(y) &= \int_{y\in X} \left[\sum_{i=1}^n \int_{x_{-i}\in X^{n-1}} q_i(y, x_{-i})\mathrm{d}\lambda^{n-1}(x_{-i})\right] \mathbf{1}_{\{y\in E_t\}}\mathrm{d}\lambda(y) \\
&= \sum_{i=1}^n \left[\int_{(x_i,x_{-i})\in X^n} q_i(x_i, x_{-i})\mathbf{1}_{\{x_i\in E_t\}}\mathrm{d}\lambda^n(x_i, x_{-i})\right] \\
&\leq \sum_{i=1}^n \left[\int_{(x_i,x_{-i})\in X^n} q_i(x_i, x_{-i})\mathbf{1}_{\cup_j\{x_j\in E_t\}}\mathrm{d}\lambda^n(x_i, x_{-i})\right] \\
&= \int_{x\in X^n} \left(\sum_{i=1}^n q_i(x)\right) \mathbf{1}_{\cup_j\{x_j\in E_t\}}\mathrm{d}\lambda^n(x) \leq \int_{x\in X^n} \mathbf{1}_{\cup_j\{x_j\in E_t\}}\mathrm{d}\lambda^n(x) \\
&= 1 - \int_{x\in X^n} \mathbf{1}_{\cap_j\{x_j\in X\setminus E_t\}}\mathrm{d}\lambda^n(x) = 1 - \big(\lambda(X\setminus E_t)\big)^n.
\end{aligned}$$

■

Feasibility condition (F_0) is due to $\sum_i p_i(x) = 1$:

$$\int_{y\in X} g(y)\mathrm{d}F(y) = \int_{y\in X}\left[\sum_{i=1}^{n}\int_{x_{-i}\in X^{n-1}} p_i(y,x_{-i})\mathrm{d}\bar{F}_{-i}(x_{-i})\right]\mathrm{d}F(y) \tag{18}$$
$$= \int_{x\in X^n}\left(\sum_{i=1}^{n} p_i(x)\right)\mathrm{d}\bar{F}(x) = 1.$$

Let p be a solution of (P_0). Then it's reduced form satisfies feasibility conditions (F) by Lemma 3 and (F_0) by (18). Incentive constraint (IC) is satisfied as well, since (IC_0) applies separately for each i and thus, in general, is stronger than (IC).

Conversely, let g be a solution of (P). Since g satisfies (F) and (F_0), by Proposition 3.1 in Border (1991) there exists a symmetric p whose reduced form is g. This p will satisfy incentive constraint (IC_0), since for symmetric mechanisms (IC) and (IC_0) are equivalent. ■

Proof of Theorem 1. Denote $G(x) = \int_a^x g(t)\mathrm{d}F(t)$, where g satisfies (F_0). Note that by (F_0) $G(x)$ is a c.d.f. In these notations, the objective of the principal is to choose a c.d.f. G that maximizes $\int_X x\mathrm{d}G(x)$ subject to (IC) and (F). A necessary condition for G to be a solution is that it is maximal w.r.t. first-order stochastic dominance order (FOSD) subject to (IC) and (F).

Below we prove that the set of FOSD maximal functions is $\{g_z^*\}_{z\in X}$, where g_z^* is defined by (3) and (4). It is immediate that optimization on the set of these functions yields the set of solutions of (P^*).

Lemma 4. *For every $z \in X$, g_z^* is well defined and satisfies* (IC), (F_0), *and* (F).

Proof. First, let us show that for every $z \in X$, g_z^* is well defined, that is, there exists r that satisfies (4). Fix $z \in X$ and let

$$\bar{r} = \frac{1-F^n(z)}{1-F(z)}.$$

Observe that $z = \bar{x}_{\bar{r}}$, where $\bar{x}_{\bar{r}}$ is given by (13). Lemma 1 then implies that $\overline{g}_r(x) = r$ for all $x \geq z$ and all $r \leq \bar{r}$.

Consider $r = 0$. Then $\overline{g}_r(x) = 0$ for all $x \geq z$, while ($\mathrm{IC}_{\min}$) is vacuous, thus $\underline{g}_r$ fits ($\mathrm{F}_{\min}$) everywhere, $\underline{g}_r(x) = nF^{n-1}(x)$ for all x. Hence for $r = 0$

$$\int_a^z \underline{g}_r(x)\mathrm{d}F(x) + \int_z^b \overline{g}_r(x)\mathrm{d}F(x) = F^n(z) \leq 1.$$

Next, consider $r = \bar{r}$. In this case $\overline{g}_r(x) = \bar{r}$ for all $x \geq z$, and thus

$$\int_a^z \underline{g}_r(x)\mathrm{d}F(x) + \int_z^b \overline{g}_r(x)\mathrm{d}F(x) \geq F^n(z) + \int_z^b \bar{r}\mathrm{d}F(x) = F^n(z) + 1 - F^n(z) = 1,$$

where we used ($\mathrm{F}_{\min}$) and (13). Finally, $\int_a^z \underline{g}_r(x)\mathrm{d}F(x)$ is weakly increasing in r, since a greater r implies a stronger constraint ($\mathrm{IC}_{\min}$). Also $\int_z^b \overline{g}_r(x)\mathrm{d}F(x) = \int_z^b r\mathrm{d}F(x)$ is strictly increasing in r. Consequently, $\int_a^z \underline{g}_r(x)\mathrm{d}F(x) + \int_z^b \overline{g}_r(x)\mathrm{d}F(x)$ is strictly increasing in r. Thus there exists a unique solution of (4) on $[0, \bar{r}]$, and this is also the smallest solution on $[0, n]$.

Now let us show that g_z^* is feasible and incentive compatible. Note that g_z^* satisfies (F_0) and (F) by construction, due to constraints ($\mathrm{F}_{\min}$) and ($\mathrm{F}_{\max}$). To prove that g_z^* satisfies (IC), we need to verify that $\underline{g}_r(x)$ satisfies (IC_2) for $x < z$ (while (IC_1) holds by ($\mathrm{IC}_{\min}$)), and $\overline{g}_r(x)$ satisfies (IC_1) for $x \geq z$ (while (IC_2) holds by ($\mathrm{IC}_{\max}$)). For $x \geq z$, we have shown that $\overline{g}_r(x) = r > rh(x)$, hence ($\mathrm{IC}_1$) is satisfied. For $x < z$, it must be that $\underline{g}_r(x) \leq r$, as otherwise r is not a solution of (4). Assume by contradiction that $\underline{g}_r(x) > r$ for some $x < z$. Since $rh(x) < r$, and we are optimizing a linear objective in ($\mathrm{P}_{\min}$), constraint ($\mathrm{F}_{\min}$) must be binding. Further, the optimal trajectory such that ($\mathrm{F}_{\min}$) binds is increasing. Thus $\underline{g}_r(x') > r$ for all for all $x' > x$, and ($\mathrm{F}_{\min}$) remains binding on $[x, b]$, implying that $\int_a^b \underline{g}_r(t)\mathrm{d}F(t) = 1$. Hence,

$$\int_a^z \underline{g}_r(t)\mathrm{d}F(t) = 1 - \int_z^b \underline{g}_r(t)\mathrm{d}F(t) < 1 - (1 - F(z))r.$$

But then

$$\int_a^z \underline{g}_r(t)\mathrm{d}F(t) + \int_z^b \overline{g}_r(t)\mathrm{d}F(t) < 1 - (1 - F(z))r + \int_z^b r\mathrm{d}F(t) = 1,$$

contradicting that r is a solution of (4). ■

Lemma 5. *C.d.f.* $G(x) = \int_a^x g(t)\mathrm{d}F(t)$ *is FOSD maximal subject to* (IC) *and* (F) *if and only if there exists* $z \in Z$ *such that* $g = g_z^*$.

Proof. Consider an arbitrary $\tilde{g}$ that satisfies (IC), (F), and (F_0), and let $\tilde{G} = \int_a^x \tilde{g}(t)\mathrm{d}F(t)$. Let $r = \sup_X \tilde{g}(x)$. Then $\tilde{g}$ satisfies (IC_1) and (IC_2) with this r. Consider now $G_z^*(x) = \int_a^x g_z^*(t)\mathrm{d}F(t)$, where g_z^* is a concatenation of $\underline{g}_r$ and $\overline{g}_r$ at point z such that (4) holds. By Lemma 4, G_z^* is a c.d.f. that satisfies (IC) and (F).

Since $\int_a^x \underline{g}_r(t)\mathrm{d}F(t)$ describes the lowest trajectory $G(x)$ that satisfies (IC_2) and (F), we have for all $x \leq z$

$$G_z^*(x) = \int_a^x \underline{g}_r(t)\mathrm{d}F(t) \leq \int_a^x \tilde{g}(t)\mathrm{d}F(t) = \tilde{G}(x).$$

Also, since $\int_x^b \overline{g}_r(t)\mathrm{d}F(t)$ describes the highest trajectory $\bar{G}(x)$ that satisfies (IC_1) and (F), we have for all $x > z$

$$1 - G_z^*(x) = \int_x^b \overline{g}_r(t)\mathrm{d}F(t) \geq \int_x^b \tilde{g}(t)\mathrm{d}F(t) = 1 - \tilde{G}(x).$$

Hence, G_z^* FOSD $\tilde{G}$. ■ ■

Proof of Theorem 2. To derive the upper bound on the principal's payoff we solve (P^*) subject to (IC_1), (IC_2) and (F_0), while ignoring constraint (F).

Solving ($\mathrm{P}_{\min}$) subject to ($\mathrm{IC}_{\min}$), and solving ($\mathrm{P}_{\max}$) subject to ($\mathrm{IC}_{\max}$) (we ignore ($\mathrm{F}_{\min}$) and ($\mathrm{F}_{\max}$)) yields for every $r \in [0, n]$

$$\underline{g}_r(x) = rh(x) \quad \text{and} \quad \overline{g}_r(x) = r, \quad x \in X.$$

By Lemma 5, each concatenation of $\underline{g}_r$ and $\overline{g}_r$ at z,

$$g_z^* = \begin{cases} rh(x), & x \le z, \\ r, & x > z, \end{cases}$$

with r satisfying $\int_a^z rh(x)\mathrm{d}F(x) + \int_z^b r\mathrm{d}F(x) = 1$ is FOSD maximal. Solving for r yields

$$r = \left(\int_a^z h(x)\mathrm{d}F(x) + \int_z^b \mathrm{d}F(x) \right)^{-1} = (H(z) + 1 - F(z))^{-1}, \tag{19}$$

where we denote

$$H(x) = \int_a^x h(t)\mathrm{d}F(t).$$

Substituting g_z^* and (19) into the principal's objective function yields

$$\max_{z \in X} \frac{1}{H(z) + 1 - F(z)} \left(\int_a^z xh(x)\mathrm{d}F(x) + \int_z^b x\mathrm{d}F(x) \right)$$

The first-order condition is equivalent to

$$z(h(z)-1)f(z)(H(z)+1-F(z)) - (h(z)-1)f(z) \left(\int_a^z xh(x)\mathrm{d}F(x) + \int_z^b x\mathrm{d}F(x) \right) = 0.$$

By assumption, $f(x) > 0$ and $h(x) < 1$, hence the above is equivalent to

$$\int_a^z xh(x)\mathrm{d}F(x) + \int_z^b x\mathrm{d}F(x) = z(H(z) + 1 - F(z)). \tag{20}$$

Observe that (19) and (20) are identical to (6) and (7) with $x^* = z$ and $r^* = r$, and thus g_z^* is precisely (8). First-order condition (20) is also sufficient by the argument provided in Footnote 9. ■

Proof of Proposition 4. By Theorem 2, x^* can be achieved if and only if g^* given by (8) is feasible. Thus we need to verify that g^* satisfies (F) if and only if (9) holds.

Denote $E_t = \{x \in X : g^*(x) \le t\}$. Since the image of g^* is $\{r^* h(x) : x \in [0, x^*]\} \cup \{r^*\}$, we have $E_t = \{x : r^* h(x) \le t\}$ when $t \in \{r^* h(x) : x \in [0, x^*]\}$ and $E_t = X$ when $t = r^*$. So, for $g = g^*$, (F) is equivalent to:

$$\int_{x \in E_t} r^* h(x)\mathrm{d}F(x) \ge \left(F(E_t)\right)^{\bar{n}} \quad \text{for all } t \in \{r^* h(x) : x \in [0, x^*]\}, \tag{21}$$

and

$$\int_{x \in X} g^*(x)\mathrm{d}F(x) \ge 1. \tag{22}$$

Observe that (21) is equivalent to (9), while (22) is redundant by (F_0). ■

Proof of Corollary 1. We need to verify (9) under the assumption that $n \geq \tilde{n}$, which is equivalent to

$$h(x) \geq \frac{1}{n}, \quad x \in X. \tag{23}$$

Let $E_t = \{x \in X : z^* h(x) \leq t\}$. Note that (9) is equivalent to

$$\int_{A_t} z^* h(x) \mathrm{d}F(x) \geq \Big(F(A_t)\Big)^{\bar{n}} \quad \text{for all } t \leq z^* \max_{x \in [a, x^*]} h(x).$$

Denote $F_t = F(E_t)$ and $H_t = \int_{x \in E_t} h(x) \mathrm{d}F(x)$, and denote $H^* = H([a, x^*])$ and $F^* = F([a, x^*])$. In these notations we have $z^* = (H^* + 1 - F^*)^{-1}$, and the above inequality is equivalent to

$$H_t \geq (H^* + 1 - F^*) F_t^n,$$

or

$$H_t(1 - F_t^n) \geq \Big(1 - F_t + (H^* - H_t) - (F^* - F_t)\Big) F_t^n.$$

Since $(H^* - H_t) - (F^* - F_t) \leq 0$ by $h(x) \leq 1$ and since $H_t \geq \frac{1}{n} F_t$ by (23), the above inequality holds if

$$\frac{1}{n} F_t (1 - F_t^n) \geq (1 - F_t) F_t^n.$$

This is true, since

$$\frac{1 - F_t^n}{F_t^{n-1}(1 - F_t)} = \frac{F^{-n} - 1}{F^{-1} - 1} = 1 + F^{-1} + F^{-2} + \ldots + F^{-(n-1)} \geq n.$$

■

Proof of Proposition 5. Consider q defined by (11). Let $Q = \int_X q(x) \mathrm{d}F(x)$ be the ex-ante probability to be short-listed, and let A and B be the expected probabilities to be chosen conditional on being shortlisted and conditional on not being short-listed, respectively:

$$A = \sum_{k=1}^{n} \frac{1}{k} \binom{n-1}{k-1} Q^{k-1} (1 - Q)^{n-k} \quad \text{and} \quad B = \frac{1}{n} (1 - Q)^{n-1}.$$

Then an agent's probability to be chosen conditional on reporting x is equal to $g_i(x) = q(x)A + (1 - q(x))B$. Set $K = A/B$ and evaluate

$$g(x) \equiv \sum_i g_i(x) = n \left(q(x)A + (1 - q(x))B\right) = \begin{cases} nAh(x), & x < x^*, \\ nA, & x \geq x^*. \end{cases}$$

By Theorem 2, the shortlisting procedure achieves x^* if $g(x) = g^*(x)$ for all x, where g^* is given by (8). This holds if $nA = z^*$. Thus we need to verify that condition (12) implies $A = z^*/n$. We have

$$Q = \int_X q(x)\mathrm{d}F(x) = \frac{K}{K-1}\left(\int_a^{x^*} h(x)\mathrm{d}F(x) + \int_{x^*}^b \mathrm{d}F(x)\right) - \frac{1}{K-1} \tag{24}$$
$$= \frac{K}{K-1}\frac{1}{z^*} - \frac{1}{K-1} = \frac{K-z^*}{z^*(K-1)}, \quad \text{thus} \quad 1-Q = \frac{K(z^*-1)}{z^*(K-1)},$$

where we used (7). Also,

$$A = \sum_{k=1}^n \frac{1}{k}\frac{(n-1)!}{(k-1)!(n-k)!}Q^{k-1}(1-Q)^{n-k} = \frac{1}{nQ}\sum_{k=1}^n \frac{n!}{k!(n-k)!}Q^k(1-Q)^{n-k}$$
$$= \frac{1}{nQ}\left(1-(1-Q)^n\right).$$

Substituting (24) into the above yields

$$A = \frac{z^*(K-1)}{n(K-z^*)}\left(1-\left(\frac{K}{z^*}\right)^n\left(\frac{z^*-1}{K-1}\right)^n\right).$$

By (12),

$$A = \frac{z^*(K-1)}{n(K-z^*)}\left(1-\frac{z^*-1}{K-1}\right) = \frac{z^*(K-1)}{n(K-z^*)}\frac{K-z^*}{K-1} = \frac{z^*}{n}.$$

■

Proof of Lemma 1. Observe that, as the incentive constraint ($\mathrm{IC}_{\max}$) permits monotonic solutions, the feasibility constraint ($\mathrm{F}_{\max}$) reduces to $\bar{G}(x) \le 1 - F^n(x)$, or equivalently

$$\int_x^b g(t)\mathrm{d}F(t) \le \int_x^b nF^{n-1}(t)\mathrm{d}F(t) \quad \text{for all } x \in X.$$

It is then straightforward that incentive constraint ($\mathrm{IC}_{\max}$) is binding so long as $x > \bar{x}_r$, and hence $\overline{g}_r(x) = r$ for $x > \bar{x}_r$, while ($\mathrm{F}_{\max}$) is binding when $x < \bar{x}_r$, and hence $\overline{g}_r(x) = nF^{n-1}(x)$ for $x < \bar{x}_r$. For $r \in [1, n]$ equation (13) has a unique solution. For $r < 1$ we have $\int_{x'}^b r\mathrm{d}F(x) < 1 - F^n(x')$ for all x', that is, ($\mathrm{F}_{\max}$) is nowhere binding. Hence $\bar{x}_r = a$ in this case. ■

Proof of Lemma 2. First, we show that $\underline{g}_r$ given by (**??**) is well defined and unique. In order to do that, we prove that $\underline{x}_r$ is well defined for each $r \in [0, n]$.

As $h(F^{-1}(t))$ is concave by Assumption 1, it follows that for every $n \ge 2$, $h(F^{-1}(t)) - nt^{n-1}$ is concave (and strictly concave for $n > 2$). Hence, by monotonicity of F, for all $r \ge 0$

$$rh(x) - nF^{n-1}(x) \quad \text{is quasiconcave.} \tag{25}$$

Denote by $\tilde{x}$ the greatest solution of $rh(\tilde{x}) = nF^{n-1}(\tilde{x})$. Note that such a solution always exists, since $rh(0) \ge nF^{n-1}(0) = 0$ and $rh(1) \le nF^{n-1}(1) = n$ by $h(1) \le 1$ and $r \le n$.

Thus we have $rh(t) - nF^{n-1}(t)$ nonnegative on $[a, \tilde{x}]$ and negative on $(\tilde{x}, b]$. As a result,

$$\int_a^x (rh(t) - nF^{n-1}(t))\mathrm{d}F(t)$$

is positive and increasing for $x < \tilde{x}$ and strictly decreasing for $x > \tilde{x}$. Consequently, either there exists an $\underline{x}_r$ that satisfies[11] (14), or $\int_a^b (rh(t) - nF^{n-1}(t))\mathrm{d}F(t) > 0$, equivalently,

$$r \int_a^b h(t)\mathrm{d}F(t) > \int_a^b nF^{n-1}(t)\mathrm{d}F(t) = 1.$$

This holds if and only if $r > \left(\int_a^b rh(t)\mathrm{d}F(t)\right)^{-1} \equiv \bar{r}$.

Next, we argue that $\underline{g}_r(x)$ is the solution of problem ($\mathrm{P_{min}}$) subject to ($\mathrm{IC_{min}}$) and a weakening of ($\mathrm{F_{min}}$):

$$\int_a^x g(t)\mathrm{d}F(t) \geq F^n(x), \quad x \in [\underline{x}_r, b]. \tag{26}$$

This is indeed true, since $\underline{g}_r$ defines the lowest trajectory of $G(x) = \int_a^x g(t)\mathrm{d}F(t)$ that satisfies both ($\mathrm{IC_{min}}$) and (26), with ($\mathrm{IC_{min}}$) binding on $[a, \underline{x}_r]$ and (26) binding on $(\underline{x}_r, b]$.

Thus it remains to show that $\underline{g}_r$ satisfies ($\mathrm{F_{min}}$). Suppose that h is weakly increasing. Then $\underline{g}_r$ satisfies ($\mathrm{F_{min}}$) if and only if

$$\int_a^x rh(t)\mathrm{d}F(t) \geq F^n(x) \text{ for all } x \leq \underline{x}_r, \text{ and}$$
$$\int_a^{\underline{x}_r} rh(t)\mathrm{d}F(t) + \int_{\underline{x}_r}^x nF^{n-1}(t)\mathrm{d}F(t) \geq F^n(x) \text{ for all } x > \underline{x}_r.$$

The second inequality holds as equality by (14). The first inequality can be rewritten as

$$\int_a^x \left(rh(t) - nF^{n-1}(t)\right)\mathrm{d}F(t) \geq 0,$$

which holds by (14) and (25).

To handle the case of nonmonotonic h, we apply the above argument to a transformation that monotonizes $h \circ F^{-1}$ without affecting ($\mathrm{F_{min}}$) as follows.

Fix r and define $\bar{t} = F(\underline{x}_r)$. A *measure-preserving monotonization of function* $h \circ F^{-1}$ on domain $[0, \bar{t}]$ is a weakly increasing function $\phi : [0, \bar{t}] \to [0, 1]$ such that for every $s \in [0, 1]$

$$\int_{\{t \in [0,\bar{t}]: h(F^{-1}(t)) \geq s\}} h(F^{-1}(t))\mathrm{d}t = \int_{\{t \in [0,\bar{t}]: \phi(t) \geq s\}} \phi(t)\mathrm{d}t.$$

Since $h \circ F^{-1}$ is concave by Assumption 1, the monotonization procedure is straightforward. Let $\bar{s} = \max_{t \in [0,\bar{t}]} rh(F^{-1}(t))$. For every $s \in [0, \bar{s}]$ define $E_s = \{t \in [0, \bar{t}] : h(F^{-1}(t)) \geq s\}$. Observe that $\sup E_s - \inf E_s$ is strictly decreasing and concave, due to concavity of $h \circ F^{-1}$. Let $\phi(t)$ be the inverse of $\bar{t} - \sup E_s + \inf E_s$. Thus, $\phi(t)$ is

[11]Note for $n = 2$, $rh(x) - 2F(x)$ is *weakly* quasiconcave. So it is possible that there is an interval $[x', x'']$ of solutions of (14), but in that case $rh(x) = 2F(x)$ for all $x \in [x', x'']$, and hence every $\underline{x}_r \in [x', x'']$ defines the same function in (14).

strictly increasing and concave, and hence $r\tilde{h}(x) - nF^{n-1}(x)$ is quasiconcave, where $\tilde{h}$ is implicitly defined by $\phi(t) = \tilde{h}(F(t))$, $t \in [0, \bar{t}]$. ■

Proof of Proposition 6. By Theorem 1, the solution g^* is chosen among concatenations of $\underline{g}_r$ and $\overline{g}_r$. By Lemmata 1 and 2, g^* is given by (15) for some r if $\underline{x}_r < \overline{x}_r$ and by (8) if $\underline{x}_r \geq \overline{x}_r$. The latter case is ruled out by the assumption that $n < \bar{n}$: (9) is violated and the feasibility constrain must be binding for a positive measure of types. Hence, g^* is as in (15), and r is chosen to maximize the payoff of the principal:

$$\int_a^{\underline{x}_r} xrh(x)\mathrm{d}F(x) + \int_{\underline{x}_r}^{\overline{x}_r} xnF^{n-1}(x)\mathrm{d}F(x) + \int_{\overline{x}_r}^{b} xr\mathrm{d}F(x).$$

Taking the derivative w.r.t. r yields the first-order condition that is precisely (16). Since $\underline{x}_r$ is strictly increasing and $\overline{x}_r$ is strictly decreasing in r, the solution of (16) is unique. ■

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