A Battle of Informed Traders and the Market Game Foundations for Rational Expectations Equilibrium[*]

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Abstract

Potential manipulation of prices and convergence to rational expectations equilibrium is studied in a game without noise traders. Informed players with initially long and short positions (bulls and bears) seek to manipulate consumer expectations in opposite directions. In equilibrium, period 1 prices reveal the state, so manipulation is unsuccessful. Bears and uninformed consumers sell up to their short-sale limits in period 1. Bulls buy in period 1 but receive arbitrage losses. When the number of bulls and bears approaches infinity, the equilibrium converges to the REE. Without short-sale constraints there is a non-revealing equilibrium but no revealing equilibrium.

1. Introduction

Manipulation of prices by agents with superior information is often thought to be both profitable to the individual and damaging to society. One classic example of price manipulation occurs when an agent ultimately wants to sell a commodity, but first buys in an attempt to influence the price at which she can sell in the future. However, it is difficult to capture such manipulation in a model in which the price formation process is explicit and all agents are fully rational. Auction models can be difficult to work with if buyers at one stage become sellers at another stage. Demand-curve submission games hold promise, but so far the models are either static or assume the presence of noise traders.[1] The present paper adopts a two-period Shapley-Shubik strategic market game. We reach the surprising conclusion that an informed trader may be unable to profitably manipulate prices when all agents are rational. Indeed, an informed trader may be at a disadvantage relative to uninformed traders. As a by-product of our analysis, we show that as the number of informed traders approaches infinity, the equilibrium allocation converges to what would obtain in a fully-revealing Rational Expectations Equilibrium (REE).

In the market game model studied here, the same two goods are traded on two consecutive spot markets, where agents only care about their final holdings. Good y is the numeraire good, which can be thought of as money, and good x is the good whose price is being manipulated, which can be thought of as shares of stock in some firm. There is a continuum of consumers or small traders who are modeled as players in the game, and there is a finite number of large traders who value only the numeraire good. Some large traders are "bulls," who are endowed with a positive position in good x, and benefit from being able to sell their endowment at a high price. Other large traders are "bears," who are endowed with a negative position in good x, and benefit from a low price. Large traders observe the state of nature θ, drawn from a continuous distribution, but ordinary consumers are uninformed. We impose short-sale constraints and show that in a fully revealing equilibrium, the unconstrained traders (bulls) are at a disadvantage. Bulls purchase good x in period 1 in an attempt to manipulate the price and thereby manipulate the inferences of ordinary consumers. Still, the

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[1]Vives (2011a,b) offers an interesting and sophisticated analysis of a supply-function submission game. The model is static, and therefore not suited to the study of the type of situation in which an agent buys or sells at one stage, and then unwinds her trade at a later stage. Cespa and Vives (2012) study a dynamic trading model containing noise traders.

period 1 price in state θ, $p^1(\theta)$, reveals the state and the period 2 price in state θ, $p^2(\theta)$, is what would prevail in a fully revealing REE, so consumers are not fooled in equilibrium. However, $p^1(\theta) > p^2(\theta)$ always holds, so the bulls lose money on the round trip transaction of buying in period 1 and then selling in period 2. The bears and uninformed consumers sell up to their short-sale limits in period 1, and make arbitrage profits as a result.

Here is the intuition for why the bulls are willing to purchase good x in period 1. If a bull decided not to purchase, consumers would see a lower p^1 and infer that the state is lower than it actually is, resulting in a lower p^2. Thus, the bull avoids losing money on the round trip transaction of buying in period 1 and selling in period 2, but he reduces the money he receives from selling his endowment in period 2. As it turns out, these effects offset, so the bull's purchase in period 1 is optimal.

Consider a sequence of games in which the short sale constraint approaches infinity, in the sense that larger and larger short sales are allowed. Then in the sequence of fully revealing equilibria we construct, each bear's short sale constraint is binding everywhere along the sequence, so the quantity sold by each bear in period 1 approaches infinity. The quantity purchased by each bull in period 1 also approaches infinity. However, the limiting strategy profile with infinite bids and offers is not well defined and cannot constitute an equilibrium for the game without short sale constraints. Intuitively, the resulting battle between bulls and bears attempting to manipulate the market in opposite directions would lead each side to try to outleverage the other, with no fully revealing equilibrium possible.

What would happen if bears were not subject to short-sale constraints? Besides the fully revealing REE, the competitive economy has a rich set of partially revealing REE and a completely non-revealing REE. Without short-sale constraints, the market game has an equilibrium that implements the non-revealing REE allocation. If consumers believe that the period 1 price reveals nothing about the state, then there is no incentive for informed traders to try to manipulate the price. In equilibrium, the bulls and bears refrain from trading in period 1, and the price in each period equals the non-revealing REE price.[2]

Noise trader models have been used extensively in the finance literature. Noise trading refers to exogenously specified trading, usually independent of prices, but

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[2]With short sale constraints, the non-revealing and partially revealing REE allocations obtain as an equilibrium allocation in the game, as the number of bears and bulls approaches infinity. Without short sale constraints, my conjecture is that the non-revealing REE is the only REE whose allocation is obtainable in an equilibrium of the market game.

there is disagreement in the literature as to whether noise traders should be interpreted as irrational or constrained by liquidity needs. It is a tautology that any noise trader model can be recast as a model with fully rational but constrained traders. However, the constraints are usually implicit, with the utility maximization problems faced by these constrained traders rarely modeled formally. More importantly, the constraints implicitly assumed in the finance literature are very special. "Liquidity traders" choose their trade in a given period independently of their previous trades, any fundamental information, and (in many cases) prices. As a result, noise traders, whether irrational or rational but constrained, are set up to absorb losses and provide profit opportunities for other traders in the model. By modeling traders as rational with explicitly specified constraints, it becomes apparent that a wide range of new and interesting structures could be studied, yielding new outcomes. In the present paper, there are no liquidity traders in the sense of being constrained to trade in any one period. However, bulls and bears receive no utility from consumption of good x, and act as if they are constrained to close out their positions by the end of period 2. Although we can consider bulls and bears to be constrained to close out their positions, the constraints act over two periods and are affected by fundamental information. The main model also imposes short sale constraints, but these are again very different from the constraints implicit in noise trader models. Bulls and bears want to push consumer beliefs in opposite directions, and short sale constraints allow prices to be revealing with neither side having an incentive to deviate, since one side is already at a corner. Unlike any of the noise trader models, bears actually benefit from being constrained, and *uninformed* consumers succeed in selling high and buying low. Also, it is important to note that we consider pure strategy equilibria in which the magnitude of the "noise" is zero.

In Section 2, we discuss the relevant literatures. Section 3 presents the model with short-sale restrictions, and contains the equilibrium construction and result on convergence to REE. Section 4 contains the result of nonexistence of fully revealing equilibrium for the model without short-sale restrictions. Section 5 considers partially revealing and non-revealing equilibria. Section 6 offers some concluding remarks. Proofs are given in the appendix.

2. Literature Review

There is a large literature on manipulation in noise-trader models, which will not be reviewed here.[3] For papers discussing whether noise traders should be interpreted as irrational or deriving utility directly from the trading process, versus rational but constrained by liquidity needs, see Black (1986), Shleifer and Summers (1990), and the interesting discussion in Bloomfield et al (2009).

An early paper on manipulation involving fully rational agents is Allen and Gale (1992). A large trader called a manipulator makes profits in equilibrium, because the market cannot distinguish between trade by a manipulator and trade by an informed trader. The informed trader knows the content of an announcement that will take place about the stock value (either good or bad). Unlike the present paper, price formation is not explicitly modeled in Allen and Gale (1992). Allen, Litov, and Mei (2006) present a model of price manipulation with small arbitrageurs and a large manipulator who threatens to corner the market. The settlement price in the event of a corner is exogenously specified, and there is a random amount of supply that emerges at date 1 that might prevent the corner. The present paper looks at more routine environments in which cornering the market is not a possibility, and models competing (potential) manipulators who seek to push the price in opposite directions. However, uninformed traders are not fooled in equilibrium. Attempts to manipulate prices, and thereby to manipulate expectations of uninformed traders, are ultimately unsuccessful.

There are well-known paradoxes associated with REE, stemming from the fact that price-formation is not explicitly modeled.[4] The Shapley-Shubik market game model[5] employed in this paper explicitly models prices. Offers to sell a commodity and bids of money to purchase the commodity are placed on a "trading post," with the price set to clear the market. This structure closely corresponds to a market in which all traders place market orders. It has a desirable feature for

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[3]Some of these papers build on Kyle (1985). Some papers on manipulation in noise-trader models include Vila (1989), Kumar and Seppi (1992), Bagnoli and Lipman (1996), Chakraborty and Yilmaz (2004), and Goldstein and Guembel (2008). See also Kyle and Viswanathan (2008) for a discussion of what constitutes price manipulation.

[4]For an early REE paper, see Radner (1979). Also see the discussions and references in Jordan and Radner (1982), Milgrom (1981), and DGS (1987).

[5]A partial list of related papers in this literature includes Shapley and Shubik (1977), Dubey and Shubik (1978), Postlewaite and Schmeidler (1978), Peck and Shell (1990), Peck, Shell, and Spear (1992), Jackson and Peck (1999), Ghosal and Morelli (2004), and McLean, Peck, and Postlewaite (2005).

a model of price manipulation, that the more a trader buys (respectively, sells), the higher (respectively, lower) the price. The paper by Dubey, Geanakoplos, and Shubik (1987, henceforth DGS), like the present paper, considers a two-period market game model with agents who are asymmetrically informed about the state of nature. DGS forcefully makes the point that being too small to affect the price is different from being a price taker. Although prices reveal the state of nature, this revelation occurs after bids and offers have been submitted, so informed agents are able to benefit from their information and receive higher utility than otherwise-identical uninformed agents. Thus, the REE allocation is not obtained. One of the models considered by Forges and Minelli (1997) is an infinitely repeated market game in which agents receive information at the beginning and the state of nature remains constant. For any REE, they construct an equilibrium in which period 1 behavior reveals information (partial or full, depending on the REE) and the REE allocation is obtained in all subsequent periods. In DGS and Forges and Minelli (1997), consumption occurs in (and utility is derived from) each period. In the present paper on the other hand, the same goods (in terms of providing utility) are traded in both periods, so uninformed agents have the option to delay their trade until the state is revealed. Hu and Wallace (2012) considers a two-stage market game with a more general information structure than the present paper, in which the state of nature is revealed through stage-1 trading, and the equilibrium converges to the REE. Unlike the present paper, a small fraction of the traders are exogenously "inactive" for the main market at stage 2, and instead trade at stage 1 at fixed non-market-clearing prices. The authors show that the fraction of restricted traders and the degree of inefficiency can be made arbitrarily small.

The game theoretic foundations of REE have been studied in models outside of the Shapley-Shubik market game context. Reny and Perry (2006) consider a large double-auction model, and show that there exists an equilibrium arbitrarily close to the fully revealing REE. It is assumed that agents consume either zero or one unit. The matching literature also addresses this issue. Wolinsky (1990) and Gottardi and Serrano (2005) consider matching models with two states of nature and unit demands. Wolinsky (1990) finds that the REE does not obtain as an equilibrium to the matching game. Gottardi and Serrano (2005) show that the REE allocation is consistent with equilibrium for the market with clienteles (where a buyer is matched to one seller in a period), and that the REE allocation obtains in all equilibria for the market without clienteles (where a buyer can purchase from

any seller).[6] Vives (2011a) considers a static game in which traders submit supply functions and demonstrates that the equilibrium converges to the fully revealing REE. The present paper adds to this literature in several ways. As in Vives (2011a), but unlike the auction and matching papers, the model has a continuum of states and does not assume unit demands;[7] while Vives (2011a) corresponds to limit orders, the present model corresponds to market orders. Interestingly, the present model considers what Vives (2011a) calls the pure common value case, where his revealing equilibrium collapses. Also, the competitive economy here has partially revealing and non-revealing REE, which are obtained as equilibrium allocations of the game.

3. The Model

There are two goods, x and y, and a continuum of states of nature, θ, where the state is drawn from a continuous and strictly increasing c.d.f., $G(\theta)$, with support $[\underline{\theta}, \bar{\theta}]$. The economy is comprised of two types of potential manipulators who observe θ, labeled bulls and bears, and a continuum of uninformed consumers. The set of consumers is denoted by the unit interval, $C = [0, 1]$, where consumer $h \in C$ has the endowment vector, (ω_h^x, ω_h^y), and is a von Neumann-Morgenstern expected utility maximizer with the twice continuously differentiable and quasi-linear Bernoulli utility function in state θ given by $u_h(x_h, \theta) + y_h$.[8] We make the

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[6]The difference in the results in these two papers is related to the fact that Wolinsky (1990) considers steady state equilibria and Gottardi and Serrano (2005) consider a nonstationary economy with a finite number of agents. Gale (1987) considers an economy without uncertainty, and he shows that as search and bargaining frictions approach zero, the equilibrium allocation converges to the competitive equilibrium allocation. Gale (1987) studies a non-stationary game with a finite measure of agents, which is appropriate to compare to the competitive economy; he points out that steady-state analysis implicitly assumes an infinite measure of agents passing through the economy, leading to paradoxical results.

[7]Gale (1986) generalizes the assumption of unit demand in a model where the competitive economy has no uncertainty (one state).

[8]Negative consumption of good x is outside of the consumption set, yielding utility of negative infinity. We allow for negative consumption of good y, although this will not happen in equilibrium if it does not occur in the REE. The quasi-linearity assumption is discussed in the concluding remarks.

following additional regularity assumptions for all $\theta > \underline{\theta}$:

$$\frac{\partial^2 u_h(x_h, \theta)}{\partial (x_h)^2} < 0 \text{ (concavity)}$$

$$\lim_{x_h \downarrow 0} \frac{\partial u_h(x_h, \theta)}{\partial x_h} = \infty \text{ and } \lim_{x_h \to \infty} \frac{\partial u_h(x_h, \theta)}{\partial x_h} = 0 \text{ (Inada)}$$

$$\frac{\partial^2 u_h(x_h, \theta)}{\partial x_h \partial \theta} > 0 \text{ (single crossing)}.$$

In addition to consumers, there are n bulls and n bears, who care only about consumption of good y. That is, utility is linear in consumption of good y and does not depend on consumption of good x as long as it is nonnegative; negative consumption of good x yields utility of negative infinity. Denote the set of bulls as A_+ and the set of bears as A_-. We impose no nonnegativity restrictions on consumption of good y.[9] Bulls begin the game with a positive endowment or long position of good x, $\omega \geq 0$, and bears begin the game with a negative endowment or short position of good x, $-\omega$. For the special case of $\omega = 0$, bulls and bears are identical except for their labels.

It follows from the regularity conditions on consumer utility and the implicit function theorem that the competitive economy corresponding to the model has a unique fully revealing REE, where the price of good x in terms of good y is given by the strictly increasing and differentiable function $f(\theta)$. We also assume that $f(\theta)$ is continuous at $\underline{\theta}$ and that $f(\underline{\theta}) = 0$ holds.[10] Denote the REE consumption for consumer h in state θ as $(\widehat{x}_h(\theta), \widehat{y}_h(\theta))$. To interpret the model as bulls and bears manipulating an asset market, think of good x as the asset and good y as money or general wealth, and think of $(\widehat{x}_h(\theta), \widehat{y}_h(\theta))$ as the desired portfolio of an ordinary consumer. For another interpretation that better connects the model with the REE literature, think of good x as the nonnumeraire commodity and good y as the numeraire commodity, where consumer utility depends on the consumption bundle and the realized state. Then the bulls and bears are informed consumers whose aggregate net demands are exactly zero in every state in the REE.

To model the price formation process, and to allow for the possibility of prices revealing information that can later be used by consumers, we consider a dynamic

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[9]Alternatively, we could assume that each bull or bear has an endowment of good y that is large enough to make all desired purchases. See Remark 2.

[10]See Remark 2 for a discussion of how this assumption simplifies the construction of equilibrium. For an example in which it is satisfied along with the consumer regularity conditions, suppose all consumers have the utility function $u_h(x_h, \theta) = \theta \ln(x_h)$, are endowed with one unit of good x, and we have $\underline{\theta} = 0$. Then $f(\theta) = \theta$ and $f(\underline{\theta}) = 0$ hold.

market structure with two periods. All agents make bids (denoted by the letter "b") of good y and offers (denoted by the letter "q") of good x in each period, with prices determined according to a Shapley-Shubik clearing house. For simplicity, we assume that agents cannot both bid and offer in the same period. Bulls and bears observe the action profile in period 1 before making their choices for period 2, but consumers are only assumed to observe the period 1 price (and their period 1 action). We think of this trading process as occurring within a narrow window of real time, with consumption occurring afterwards, so that utility depends only on the net position achieved at the end of period 2.

Consumers can offer and bid up to their endowment, but no more. Thus, consumer h's action set in period 1 is given by

$$\{(b_h^1, q_h^1) \in \Re_+^2 : q_h^1 \leq \omega_h^x, b_h^1 \leq \omega_h^y, q_h^1 b_h^1 = 0\}.$$

Consumers observe the period 1 price but not the state or the bids or offers themselves. Thus, consumer h's bids and offers in period 2 can depend on the observed period 1 price, and cannot exceed her period 2 holdings.

Bulls and bears cannot carry a short position of good x greater than $\overline{\omega}$ into period 2, where we have $\overline{\omega} > \omega$. We do not limit the size of bids. Thus, bull i in state θ has an action set in period 1 given by

$$\{(b_i^1(\theta), q_i^1(\theta)) \in \Re_+^2 : q_i^1 b_i^1 = 0, q_i^1(\theta) \leq \overline{\omega} + \omega\},$$

and bear j in state θ has an action set in period 1 is given by

$$\{(b_j^1(\theta), q_j^1(\theta)) \in \Re_+^2 : q_j^1 b_j^1 = 0, q_j^1(\theta) \leq \overline{\omega} - \omega\}.$$

Denoting a strategy profile for the entire game as σ, the price of good x in period 1 when the state is θ is given by

$$p^1(\theta, \sigma) = \frac{\sum_{i \in A_+} b_i^1(\theta) + \sum_{j \in A_-} b_j^1(\theta) + \int_{h \in C} b_h^1 dh}{\sum_{i \in A_+} q_i^1(\theta) + \sum_{j \in A_-} q_j^1(\theta) + \int_{h \in C} q_h^1 dh}. \tag{3.1}$$

In period 2, a consumer's bid and offer can depend on the period 1 price and the chosen period 1 action, and a bull's or a bear's bid and offer can depend on the period 1 action profile, s^1, and the observed state. For simplicity, we require that an agent cannot offer a positive bid and a positive offer in period 2, and we require that offers do not exceed an agent's period 2 holdings. We denote period 2 actions as $(b_h^2(p^1, b_h^1, q_h^1), q_h^2(p^1, b_h^1, q_h^1))$ for consumer h, $(b_i^2(\theta, s^1), q_i^2(\theta, s^1))$ for bull i, and $(b_j^2(\theta, s^1), q_j^2(\theta, s^1))$ for bear j.

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Since we must evaluate deviations from σ, we will need notation for the price in period 2 in state θ under strategy profile σ, following an arbitrary action profile in period 1, s^1, which determines a period 1 price, p^1. This price is the sum of the bids divided by the sum of the offers, given by

$$p^2(\theta, \sigma; s^1) = \frac{\sum_{i \in A_+} b_i^2(\theta, s^1) + \sum_{j \in A_-} b_j^2(\theta, s^1) + \int_{h \in C} b_h^2(p^1, b_h^1, q_h^1) dh}{\sum_{i \in A_+} q_i^2(\theta, s^1) + \sum_{j \in A_-} q_j^2(\theta, s^1) + \int_{h \in C} q_h^2(p^1, b_h^1, q_h^1) dh}. \tag{3.2}$$

The final allocation is determined according to the standard Shapley-Shubik market clearing rules. Offers sent to the market are subtracted from a player's holdings of good x, and bids sent to the market are subtracted from a player's holdings of good y. However, offers are then sold, returning an amount of good y equal to the offer multiplied by the price; bids are used to purchase a quantity of good x equal to the amount of the bid divided by the price.[11] For consumer h, final consumption in state θ under strategy profile σ is given by[12]

$$x_h(\theta, \sigma) = \omega_h^x + \frac{b_h^1}{p^1(\theta, \sigma)} - q_h^1 + \frac{b_h^2(p^1, b_h^1, q_h^1)}{p^2(\theta, \sigma)} - q_h^2(p^1, b_h^1, q_h^1) \tag{3.3}$$

$$y_h(\theta, \sigma) = \omega_h^y - b_h^1 + q_h^1 p^1(\theta, \sigma) - b_h^2(p^1, b_h^1, q_h^1) + q_h^2(p^1, b_h^1, q_h^1) p^2(\theta, \sigma). \tag{3.4}$$

For bull i and bear j, the final allocation, net of the endowment of good y, is given by

$$x_i(\theta, \sigma) = \omega + \frac{b_i^1(\theta)}{p^1(\theta, \sigma)} - q_i^1(\theta) + \frac{b_i^2(\theta, s^1)}{p^2(\theta, \sigma)} - q_i^2(\theta, s^1)$$

$$y_i(\theta, \sigma) = -b_i^1(\theta) + q_i^1(\theta) p^1(\theta, \sigma) - b_i^2(\theta, s^1) + q_i^2(\theta, s^1) p^2(\theta, \sigma)$$

$$x_j(\theta, \sigma) = -\omega + \frac{b_j^1(\theta)}{p^1(\theta, \sigma)} - q_j^1(\theta) + \frac{b_j^2(\theta, s^1)}{p^2(\theta, \sigma)} - q_j^2(\theta, s^1)$$

$$y_j(\theta, \sigma) = -b_j^1(\theta) + q_j^1(\theta) p^1(\theta, \sigma) - b_j^2(\theta, s^1) + q_j^2(\theta, s^1) p^2(\theta, \sigma).$$

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[11] This specification presumes that prices are nonzero and that the final consumption of good x is nonnegative. If all bids are zero, leading to a price of zero, then we adopt the convention that $0/0=0$ (i.e., all offers are lost). If some agent receives negative consumption of good x, we cancel all trades. This harsh penalty ensures that no one is unable to meet their obligations in equilibrium. We also cancel all trades if either p^1 or p^2 is not well defined because aggregate bids or offers are not Lebesgue measurable functions.

[12] To consider consumption that would arise following a deviation, we will sometimes use the notation $x_h(\theta, \sigma; s^1)$ or $y_h(\theta, \sigma; s^1)$ to denote consumption in state θ under strategy profile σ, following the period 1 action profile s^1.

Our solution concept is weak perfect Bayesian equilibrium (WPBE), in which the only consistency requirement is that beliefs satisfy Bayes' rule wherever possible.[13] We focus exclusively on *type-symmetric equilibria*, defined to be a WPBE in which all bulls choose the same strategy and all bears choose the same strategy. The relevant beliefs are the beliefs of consumers about the state conditional on their information in period 2.[14] In the WPBE constructed below, beliefs assign probability one to a single state, denoted by $\theta^e(p^1)$. We construct a type-symmetric equilibrium in which bulls offer zero and make positive bids in period 1, while consumers and bears bid zero and offer up to their short-sale constraints in period 1. The maximum possible offer by consumers and bears in period 1, denoted by $\overline{x}^1$, is given by

$$\overline{x}^1 = n(\overline{\omega} - \omega) + \int_{h \in C} \omega_h^x dh.$$

Proposition 1: *The following strategy profile[15] and beliefs constitute a type-*

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[13]See Mas-Colell et al (1995) for a definition. All of the WPBE are in the spirit of sequential equilibrium and PBE as well, because updating is simple: there are only two periods and agents are either fully informed or fully uninformed. However, the standard definition of sequential equilibrium requires a finite action set and the standard definition of PBE requires observable actions and independent types. To avoid the need for technical clarifications in defining sequential equilibrium or PBE for this game, we adopt the simpler notion of WPBE.

[14]Given the strategy profile, beliefs about the state determine beliefs about the period 1 actions of each player.

[15]See Remark 1 below.

symmetric, fully revealing WPBE:

$$\text{bull } i \quad : \quad b_i^1(\theta) = \frac{f(\theta)(\bar{x}^1 + \omega)}{n}, \qquad q_i^1(\theta) = 0, \tag{3.5}$$

$$b_i^2(\theta, s^1) \;=\; 0, \quad q_i^2(\theta, s^1) = \omega + \frac{b_i^1}{p^1} - q_i^1, \tag{3.6}$$

$$\text{bear } j \quad : \quad b_j^1(\theta) = 0, \qquad q_j^1(\theta) = \bar{\omega} - \omega, \tag{3.7}$$

$$b_j^2(\theta, s^1) \;=\; (q_j^1 + \omega - \frac{b_j^1}{p^1})f(\theta^e(p^1)), \qquad q_j^2(\theta, s^1) = 0, \tag{3.8}$$

$$\text{consumer } h \quad : \quad b_h^1 = 0, \qquad q_h^1 = \omega_h^x, \tag{3.9}$$

$$b_h^2(p^1, b_h^1, q_h^1) \;=\; \left[\widehat{x}_h(\theta^e(p^1)) + q_h^1 - \omega_h^x - \frac{b_h^1}{p^1} \right] f(\theta^e(p^1)), \tag{3.10}$$

$$q_h^2(p^1, b_h^1, q_h^1) \;=\; 0.$$

$$\theta^e(p^1) \;=\; f^{-1}(\frac{p^1 \bar{x}^1}{\bar{x}^1 + \omega}) \quad \text{if} \quad p^1 \le \frac{\bar{x}^1 + \omega}{\bar{x}^1} f(\bar{\theta}) \tag{3.11}$$

$$\theta^e(p^1) \;=\; \bar{\theta} \quad \text{if} \quad p^1 > \frac{\bar{x}^1 + \omega}{\bar{x}^1} f(\bar{\theta}) \tag{3.12}$$

Along the equilibrium path, prices are given by

$$p^1(\theta) \;=\; \left(\frac{\bar{x}^1 + \omega}{\bar{x}^1} \right) f(\theta) \; \text{and} \tag{3.13}$$

$$p^2(\theta) \;=\; f(\theta). \tag{3.14}$$

Remark 1. The specified actions in period 2 presume that agents did not deviate by so much in period 1 to put them on the opposite side of the market in period 2. For completeness, if bull i deviates to offer good x in period 1 and offers more than his holding,

$$\omega - q_i^1 \le 0,$$

then his action in period 2 is

$$b_i^2(\theta, s^1) = (q_i^1 - \omega)f(\theta^e(p^1)), \quad q_i^2(\theta, s^1) = 0.$$

If, bear j deviates to bid in period 1 and winds up purchasing more than his initial short position,

$$\omega - \frac{b_j^1}{p^1} \le 0,$$

then her action in period 2 is

$$b_j^2(\theta, s^1) = 0, \qquad q_j^2(\theta, s^1) = \frac{b_j^1}{p^1} - \omega.$$

If consumer h deviates to bid in period 1 and winds up with holdings of good x greater than $\widehat{x}_h(\theta^e(p^1))$,

$$\widehat{x}_h(\theta^e(p^1)) - \omega_h^x - \frac{b_h^1}{p^1} \le 0,$$

then consumer h's action in period 2 is

$$b_h^2(p^1, b_h^1, q_h^1) = 0, \ \ q_h^2(p^1, b_h^1, q_h^1) = -\left[\widehat{x}_h(\theta^e(p^1)) - \omega_h^x - \frac{b_h^1}{p^1}\right].$$

The following lemma demonstrates that, both on and off the equilibrium path, the price in period 2 equals the REE price corresponding to state $\theta^e(p^1)$. The intuition is that if the price were different from what consumers are expecting, then it would be impossible for bulls and bears to completely close out their positions as sequential rationality requires. Extending the proof of Proposition 1 to establish sequential rationality for the deviations discussed in Remark 1 requires a straightforward adjustment in Lemma 1 below and its proof, so we omit the details.

Lemma 1: *Assume that agents did not deviate by so much in period 1 to put them on the opposite side of the market in period 2. That is, for each bull i, bear j, and consumer h, we have*

$$\omega + \frac{b_i^1}{p^1} - q_i^1 \ > \ 0,$$

$$q_j^1 + \omega - \frac{b_j^1}{p^1} \ > \ 0,$$

$$\widehat{x}_h(\theta^e(p^1)) + q_h^1 - \omega_h^x - \frac{b_h^1}{p^1} \ > \ 0.$$

Then for arbitrary period 1 actions s^1 satisfying the above inequalities, the period 2 actions given by (3.6), (3.8), (3.10), and (3.11) yield $p^2(\theta, \sigma; s^1) = f(\theta^e(p^1))$.

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One might think that bulls can take advantage of their market power and their ability to manipulate the beliefs of consumers, but this is not the case. Bulls increase their long position in period 1 in an effort to manipulate the price and convince consumers that good x is more valuable, but consumers understand this incentive and correct their beliefs accordingly. Bulls actually bid up the price in period 1 above the price in period 2, and lose money as a result as bears and consumers take the other side of the transactions. Still, if the bulls do not bid up the price, consumers would mistakenly think that the state is lower than it actually is. Notice the similarity to the signaling in the Milgrom and Roberts (1982) model of limit pricing. In that model, the equilibrium exhibits a wedge between the established firm's pre-entry price and the monopoly price corresponding to its cost type. The pre-entry price is decreasing in the cost type, thereby revealing the cost. Nonetheless, the pre-entry price is below the monopoly price, balancing a tradeoff between the effects of deviating to a lower pre-entry price: (i) fooling the entrant into thinking that the cost type is lower, thereby reducing the level of entry, and (ii) sacrificing profits in the pre-entry period by charging too low a price.

The side of the market that is not constrained in its ability to manipulate prices is actually worse off than they would be under full information. In the appendix, the profits of a bull as a function of his choice in period 1, given the equilibrium strategies of the other players, is simplified to

$$\pi_i^{bull}(\theta, b_i^1, q_i^1) = \frac{n-1}{n} \omega f(\theta). \tag{3.15}$$

Under full information, the bull would receive $\omega f(\theta)$. As the number of bulls approaches infinity, this difference approaches zero.

When considering deviations, we see from (3.15) that a bull is indifferent as to how much to bid or offer in period 1. Increasing his bid increases p^2 and his net sales revenue (from selling ω units in period 2), but increasing his bid also increases his arbitrage losses, and these effects exactly balance.[16]

It is interesting to consider how the equilibrium characterized in Proposition 1 behaves as the short sale constraint is relaxed. The bulls' bids and the bears'

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[16]Here is a rough intuition for the indifference. From the strategies specified in Proposition 1 and equation (7.8) from the proof, we can compute that increasing b_i^1 by one unit increases p^1 by $1/\overline{x}^1$, increases bull i's holding of good x by $\overline{x}^1$, and increases p^2 by $1/(\overline{x}^1 + \omega)$. Therefore, the sales revenue from all purchases goes up by $\overline{x}^1/(\overline{x}^1 + \omega)$ and the sales revenue from the initial holdings ω goes up by $\omega/(\overline{x}^1 + \omega)$, exactly balancing the increased bid.

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offers in period 1 approach infinity, and the period 1 price converges to the REE price. The limiting strategy profile, however, does not exist. See Sections 4 and 5 for an analysis of the game without short sale constraints.

Corollary to Proposition 1: *Consider a sequence of economies indexed by the short sale constraint, $\overline{\omega}$, and consider the sequence of equilibria characterized in Proposition 1. Then as $\overline{\omega}$ approaches infinity, $b_i^1(\theta) \to \infty$ holds for each bull i and for all $\theta > \underline{\theta}$; $q_j^1 \to \infty$ holds for each bear j and for all θ; and $p^1(\theta) \to f(\theta)$ holds for all θ.*

The proof of the preceeding corollary follows immediately from the fact that $\overline{x}^1$ approaches infinity as $\overline{\omega}$ approaches infinity. The following proposition shows that the WPBE prices and allocation converge to the REE price and allocation (for fixed $\overline{\omega}$), as we replicate the economy.

Proposition 2: *Consider an r-fold replication of the economy. If $\omega = 0$ holds, then for the equilibrium characterized in Proposition 1, the prices and allocation coincide with the REE for all r. If $\omega > 0$ holds, then the equilibrium converges to the REE as $r \to \infty$, in the following sense. For all θ, $p^1(\theta)$ converges to the REE price, $p^2(\theta)$ is exactly the REE price, and the allocation (uniformly across consumers) converges to the REE allocation.*

It is interesting to note that the convergence result in Proposition 2 applies to the most paradoxical environment discussed in the REE literature, in which the net trades of all informed agents do not depend on the state of nature.[17] In the competitive economy, each bull is a net seller of ω units of good x, and each bear is a net buyer of ω units of good x, independent of θ. Thus, the question might arise as to how that information can become embedded in the price. The resolution of this paradox is that we model the price formation process with a dynamic sequence of markets. It turns out that the overall net trades of good x by informed agents (across the two markets) do not depend on the state, but the net trades of good x on the first market are state dependent and revealing.

Remark 2. According to the construction in Proposition 1, beliefs off the equilibrium path play essentially no role in the analysis, because a bull who reduces his bid yields a period 1 price that is on the equilibrium path and he receives utility of $\frac{n-1}{n}\omega f(\theta)$. However, if $f(\underline{\theta}) > 0$ holds, then the equilibrium is more

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[17]See the discussion and references in Vives (2011).

complicated. A deviation to bid zero cannot bring θ^e below $\underline{\theta}$, yielding the bull a utility of at least $\omega f(\underline{\theta})$. For θ close enough to $\underline{\theta}$, $\omega f(\underline{\theta})$ will now exceed $\frac{n-1}{n}\omega f(\theta)$, so the construction in Proposition 1 must now be modified along the lines of the partially pooling equilibrium construction in Section 5. As $n \to \infty$, the partially pooling range of states can shrink to zero, so the convergence to the fully revealing REE in Proposition 2 continues to hold.

4. The Model without Short-Sale Restrictions

From the Corollary to Proposition 1, we see that relaxing short-sale restrictions with a higher $\overline{x}^1$ leads to a thicker market and smaller price distortions.[18] What would happen without any short-sale restrictions at all? Although we show in the next section that there is a pooling equilibrium corresponding to the non-revealing REE, it turns out that there is no revealing equilibrium. Bulls and bears want to manipulate consumer beliefs by trading more than the other side, with no solution with finite bids and offers. To see this, suppose that there is a fully revealing type-symmetric equilibrium in which bulls choose a bid function $b^1(\theta)$ and bears choose an offer function $q^1(\theta)$ in period 1. Letting $\int_{h \in C} b_h^1 dh$ be denoted by B^1 and $\int_{h \in C} q_h^1 dh$ be denoted by Q^1, the equilibrium period 1 price in state θ is then given by

$$p^1(\theta) = \frac{nb^1(\theta) + B^1}{nq^1(\theta) + Q^1}.$$

The period 1 price in state θ, given that the other players are employing their equilibrium strategies and bull i chooses the period 1 bid b_i^1, and the period 1 price in state θ, given that the other players are employing their equilibrium strategies and bear j chooses the period 1 offer q_j^1, are given by

$$p^1(b_i^1, \theta) = \frac{b_i^1 + (n-1)b^1(\theta) + B^1}{nq^1(\theta) + Q^1}$$

$$p^1(q_j^1, \theta) = \frac{nb^1(\theta) + B^1}{q_j^1 + (n-1)q^1(\theta) + Q^1}$$

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[18] See Peck and Shell (1990).

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Lemma 2. *In the model without short-sale restrictions, suppose we have an open-market,[19] fully revealing, type-symmetric WPBE with bulls choosing the bid function $b^1(\theta)$ and bears choosing the offer function $q^1(\theta)$ in period 1. Then for all p^1 such that $p^1 = p^1(\theta)$ holds for some θ, we have:*

(i) consumer beliefs assign probability one to a single state, denoted by $\theta^e(p^1)$, which is the unique solution to $p^1 = p^1(\theta)$, and

(ii) for all period-1 action profiles s^1 leading to the price p^1, the final holdings of good x for each bull and bear is zero, and the price in period 2 is $p^2(\theta, \sigma; s^1) = f(\theta^e(p^1))$.

Lemma 2 allows us to establish the following result.

Proposition 3. *In the model without short-sale restrictions and $\omega > 0$, there does not exist an open-market, fully revealing, type-symmetric WPBE with bid and offer functions that are continuously differentiable in θ.*

Remark 3. Throughout the analysis, we have maintained the assumption, standard in the partial equilibrium literature, that agents do not face non-negativity constraints on their consumption of good y. One interpretation is that agents have a large enough endowment of good y to make all desired purchases, but that interpretation is no longer valid when there are no short-sale constraints. Reading between the lines of the proof of Proposition 3, we see that bulls could have an incentive to bid more than their endowment of good y, no matter how large that endowment is, in response to the short selling of bears. We should think of the model without short-sale restrictions as also imposing no restrictions on bulls and bears regarding bidding above one's endowment of good y. The model without short-sale restrictions but with bid restrictions (a sort of margin requirement) has not been solved, but a reasonable conjecture is that there is a fully revealing WPBE in which bulls bid up to their limit in period 1, and bears submit offers such $p^1(\theta) < p^2(\theta) = f(\theta)$ holds. The unconstrained party, now the bears, are at a disadvantage.

5. Pooling and Partially Pooling Equilibria

The competitive economy associated with this model, besides having a fully revealing REE, also has a non-revealing REE and an infinite number of partially

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[19] Open-market means that aggregate bids and offers are positive on all markets.

revealing REE.[20] From the non-revealing and partially revealing REE, we can construct pooling and partially pooling WPBE of the market game with short-sale restrictions. Also, from the non-revealing REE, we will construct a pooling WPBE of the market game without short-sale restrictions.

In the non-revealing REE, the optimization problem of consumer h is given by

$$\max \int_{\underline{\theta}}^{\bar{\theta}} u_h(x_h, \theta) dG(\theta) + y_h$$

$$\text{subject to}$$

$$px_h + y_h = p\omega_h^x + \omega_h^y,$$

with interior first order condition

$$\int_{\underline{\theta}}^{\bar{\theta}} \frac{\partial u_h(x_h, \theta)}{\partial x_h} dG(\theta) = p. \tag{5.1}$$

Due to our regularity assumptions on consumer utility, there is a unique solution to this optimization problem, yielding a continuously differentiable and strictly decreasing demand function we will denote as $x_h^{nr}(p)$. Then the non-revealing REE is characterized by the unique price, which we denote by p^{nr}, solving

$$\int_{h \in C} x_h^{nr}(p) dh = \int_{h \in C} \omega_h^x dh.$$

Based on the non-revealing REE, we can construct a pooling WPBE in which only consumers participate on the period 1 market and consumers believe that the state is distributed according to $G(\theta)$, no matter what price is observed in period 1.

Proposition 4: *The following strategy profile[21], and beliefs given by $G(\theta)$ following all period 1 prices, constitute a type-symmetric, non-revealing WPBE for*

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[20]The reason for the multiplicity of REE is that each informed trader's final consumption of good x is zero, independent of the state and the price.

[21]See Remark 1.

all $A \in (0,1)$:

$$\text{bull } i \quad : \quad b_i^1(\theta) = q_i^1(\theta) = 0,$$

$$b_i^2(\theta, s^1) \quad = \quad 0, \quad q_i^2(\theta, s^1) = \omega + \frac{b_i^1}{p^1} - q_i^1,$$

$$\text{bear } j \quad : \quad b_j^1(\theta) = q_j^1(\theta) = 0,$$

$$b_j^2(\theta, s^1) \quad = \quad (q_j^1 + \omega - \frac{b_j^1}{p^1})p^{nr}, \quad q_j^2(\theta, s^1) = 0,$$

$$\text{consumer } h \text{ (if } x_h^{nr}(p^{nr}) \quad \geq \quad \omega_h^x): \; b_h^1 = [x_h^{nr}(p^{nr}) - \omega_h^x]Ap^{nr}, \quad q_h^1 = 0,$$

$$b_h^2(p^1, b_h^1, q_h^1) \quad = \quad \left[x_h^{nr}(p^{nr}) - \omega_h^x + q_h^1 - \frac{b_h^1}{p^1} \right] p^{nr}, \quad q_h^2(p^1, b_h^1, q_h^1) = 0.$$

$$\text{consumer } h \text{ (if } x_h^{nr}(p^{nr}) \quad < \quad \omega_h^x): \; b_h^1 = 0, \quad q_h^1 = [\omega_h^x - x_h^{nr}(p^{nr})]A,$$

$$b_h^2(p^1, b_h^1, q_h^1) \quad = \quad 0, \quad q_h^2(p^1, b_h^1, q_h^1) = \omega_h^x - x_h^{nr}(p^{nr}) + \frac{b_h^1}{p^1} - q_h^1.$$

Along the equilibrium path, prices are given by

$$p^1(\theta) = p^2(\theta) = p^{nr}.$$

In the pooling equilibrium constructed in Proposition 4, the price in both periods equals the non-revealing REE price, and there are no arbitrage gains or losses. The equilibrium is supported by the beliefs that p^1 reveals nothing about the state. As a result, bulls and bears cannot manipulate the beliefs of consumers. Because the period 2 price equals p^{nr} on and off the equilibrium path, bulls and bears strictly prefer to make zero bids and offers in period 1, because any bid would push p^1 above p^{nr} and lead to arbitrage losses, and any offer would push p^1 below p^{nr} and lead to arbitrage losses. It is clear, therefore, that the pooling equilibrium constructed in Proposition 4 remains an equilibrium of the game without short sale restrictions.

We now construct a partially revealing REE, and use it to construct a pooling WPBE of the game with short sale restrictions. First, it is clear from the above analysis that for any $\widetilde{\theta} \in (\underline{\theta}, \overline{\theta})$, there is a non-revealing REE for the hypothetical competitive economy with the set of states equal to $[\underline{\theta}, \widetilde{\theta}]$ and the c.d.f. given by the conditional distribution $G(\theta | \theta \leq \widetilde{\theta})$. Denote the corresponding price by $\widetilde{f}(\widetilde{\theta})$

and the corresponding consumption of good x by consumer h by $\widetilde{x}_h(\widetilde{\theta})$. Then the following is a partially revealing REE for the actual economy: the price is $\widetilde{f}(\widetilde{\theta})$ for $\theta \leq \widetilde{\theta}$, and $f(\theta)$ for $\theta > \widetilde{\theta}$; the allocation is characterized by bulls and bears consuming zero units of good x in all states, and consumer h consuming $\widetilde{x}_h(\widetilde{\theta})$ for $\theta \leq \widetilde{\theta}$, and $\widehat{x}_h(\theta)$ for $\theta > \widetilde{\theta}$.[22]

From the partially revealing REE, Proposition 5 constructs a partially pooling WPBE. In all states, bulls make arbitrage losses, and bears and consumers make arbitrage gains. The beliefs assign the distribution $G(\theta | \theta \leq \widetilde{\theta})$ when $p^1 = \frac{\overline{x}^1 + \omega}{\overline{x}^1} \widetilde{f}(\widetilde{\theta})$ holds, and the beliefs assign a degenerate one-point distribution otherwise. However, beliefs are well-behaved in the sense that beliefs about p^2 are continuous in p^1 on and off the equilibrium path. On the equilibrium path, p^2 is the partially revealing REE price. If we let n approach infinity, then both prices and the allocation approach their REE values.

Proposition 5: *The following strategy profile[23], and beliefs constitute a type-*

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[22] Our regularity assumptions, including the single-crossing property, ensure that $\widetilde{f}(\widetilde{\theta}) < f(\theta)$ holds for $\theta > \widetilde{\theta}$, so the price is revealing for these states.

[23] See Remark 1.

symmetric, partially-revealing WPBE

$$\text{bull } i \quad : \quad b_i^1(\theta) = \frac{f(\theta)(\overline{x}^1 + \omega)}{n}, \qquad q_i^1(\theta) = 0, \ \text{ if } \theta > \widetilde{\theta}$$

$$b_i^1(\theta) = \frac{\widetilde{f}(\widetilde{\theta})(\overline{x}^1 + \omega)}{n}, \qquad q_i^1(\theta) = 0, \ \text{ if } \theta \leq \widetilde{\theta}$$

$$b_i^2(\theta, s^1) = 0, \quad q_i^2(\theta, s^1) = \omega + \frac{b_i^1}{p^1} - q_i^1,$$

$$\text{bear } j \quad : \quad b_j^1(\theta) = 0, \qquad q_j^1(\theta) = \overline{\omega} - \omega,$$

$$b_j^2(\theta, s^1) = (q_j^1 + \omega - \frac{b_j^1}{p^1}) f(\theta^e(p^1)), \qquad q_j^2(\theta, s^1) = 0, \ \text{ if } p^1 \neq \frac{\overline{x}^1 + \omega}{\overline{x}^1} \widetilde{f}(\widetilde{\theta})$$

$$b_j^2(\theta, s^1) = (q_j^1 + \omega - \frac{b_j^1}{p^1}) \widetilde{f}(\widetilde{\theta}), \qquad q_j^2(\theta, s^1) = 0, \ \text{ if } p^1 = \frac{\overline{x}^1 + \omega}{\overline{x}^1} \widetilde{f}(\widetilde{\theta})$$

$$\text{consumer } h \quad : \quad b_h^1 = 0, \qquad q_h^1 = \omega_h^x,$$

$$b_h^2(p^1, b_h^1, q_h^1) = \left[\widehat{x}_h(\theta^e(p^1)) + q_h^1 - \omega_h^x - \frac{b_h^1}{p^1} \right] f(\theta^e(p^1)), \ \text{ if } p^1 \neq \frac{\overline{x}^1 + \omega}{\overline{x}^1} \widetilde{f}(\widetilde{\theta})$$

$$b_h^2(p^1, b_h^1, q_h^1) = \left[\widetilde{x}_h(\widetilde{\theta}) + q_h^1 - \omega_h^x - \frac{b_h^1}{p^1} \right] \widetilde{f}(\widetilde{\theta}), \ \text{ if } p^1 = \frac{\overline{x}^1 + \omega}{\overline{x}^1} \widetilde{f}(\widetilde{\theta})$$

$$q_h^2(p^1, b_h^1, q_h^1) = 0.$$

$$\text{Beliefs:} \qquad \text{c.d.f. } G(\theta | \theta \leq \widetilde{\theta}) \ \text{ if } \ p^1 = \frac{\overline{x}^1 + \omega}{\overline{x}^1} \widetilde{f}(\widetilde{\theta})$$

$$\theta^e(p^1) = \overline{\theta} \ \text{ if } \ p^1 > \frac{\overline{x}^1 + \omega}{\overline{x}^1} f(\overline{\theta})$$

$$\theta^e(p^1) = f^{-1}(\frac{p^1 \overline{x}^1}{\overline{x}^1 + \omega}) \ \text{ otherwise.}$$

6. Concluding Remarks

For the model of Section 3, a bull is indifferent as to how much to bid in period 1. If information acquisition is costly, one must complicate the model in order to generate an incentive to gather information about θ. If bulls and bears have

initial holdings of good x that depend on θ, these players will have to monitor the state in order to determine their optimal action in period 2. Another possibility is to change the information structure so that privately informed players are not informationally small (see McLean and Postlewaite (2002)), perhaps with the introduction of risk averse informed traders.

Bulls and bears could have nonlinear utility over numeraire consumption, without affecting any of the results, because they are fully informed and face no uncertainty after any history. If consumer utility functions were not quasi-linear, then activity in period 1 would generate income effects, and the price in period 2 would no longer equal the REE price for the original economy, $f(\theta)$. However, I would conjecture that, under suitable regularity conditions, (i) the result about the unconstrained informed traders being at a disadvantage persists, and (ii) income effects become small as the economy is replicated, so that the equilibrium converges to the REE.

7. Appendix: Proofs.

Proof of Lemma 1. Given period 1 actions, p^1, (3.6), (3.8), (3.10), and (3.11), the price in period 2 is given by

$$p^2(\theta, \sigma; s^1) = \frac{\sum_{j \in A_-} (q_j^1 - \frac{b_j^1}{p^1}) + n\omega + \int_{h \in C} \left[\widehat{x}_h(\theta^e(p^1)) + q_h^1 - \omega_h^x - \frac{b_h^1}{p^1} \right] dh}{\sum_{i \in A_+} (\frac{b_i^1}{p^1} - q_i^1) + n\omega} f(\theta^e(p^1)).$$

(7.1)

Dividing both sides of (7.1) by $f(\theta^e(p^1))$, multiplying numerator and denominator on the right side by p^1, and using the market clearing property of REE, that $\int_{h \in C} \left[\widehat{x}_h(\theta^e(p^1)) - \omega_h^x \right] = 0$ holds, we have

$$\frac{p^2(\theta, \sigma; s^1)}{f(\theta^e(p^1))} = \frac{p^1 \left[\sum_{j \in A_-} q_j^1 + n\omega + \int_{h \in C} q_h^1 dh \right] - \sum_{j \in A_-} b_j^1 - \int_{h \in C} b_h^1 dh}{p^1(n\omega - \sum_{i \in A_+} q_i^1) + \sum_{i \in A_+} b_i^1}. \quad (7.2)$$

Denoting the left side of (7.2) as K (suppressing the dependence on the state and the period 1 actions), cross multiplying, and manipulating the expression, we have

$$(K-1)\sum_{i\in A_+} b_i^1 + \left[\sum_{i\in A_+} b_i^1 + \sum_{j\in A_-} b_j^1 + \int_{h\in C} b_h^1 dh\right]$$

$$= p^1\left[\sum_{j\in A_-} q_j^1 + \sum_{i\in A_+} q_i^1 + \int_{h\in C} q_h^1 dh\right] + (1-K)p^1(n\omega - \sum_{i\in A_+} q_i^1).$$

From the definition of p^1, the second term on the left side of the above equation cancels the first term on the right side, leaving

$$(K-1)\sum_{i\in A_+} b_i^1 = (1-K)p^1(n\omega - \sum_{i\in A_+} q_i^1). \tag{7.3}$$

For each bull i, we have $\omega + \frac{b_i^1}{p^1} - q_i^1 > 0$, so summing over $i \in A_+$ yields the conclusion from (7.3) that $K = 1$ must hold. ∎

Proof of Proposition 1. Equation (3.13) follows immediately from substituting (3.5), (3.7), and (3.9) into (3.1). To derive equation (3.14), we see from (3.11) and (3.1) that consumers' beliefs assign probability one to the correct state, $\theta^e(p^1(\theta)) = \theta$ for all θ. Therefore, along the equilibrium path, in period 2 and state θ, each bear bids $\bar{\omega}f(\theta)$ and for each h, consumer h bids $\widehat{x}_h(\theta)f(\theta)$. Each bull offers

$$\omega + \frac{f(\theta)(\overline{x}^1 + \omega)}{np^1(\theta)} = \omega + \frac{f(\theta)(\overline{x}^1 + \omega)}{n\left(\frac{\overline{x}^1+\omega}{\overline{x}^1}\right)f(\theta)} = \omega + \frac{\overline{x}^1}{n}.$$

Therefore, along the equilibrium path, we have

$$p^2(\theta) = \frac{n\bar{\omega}f(\theta) + f(\theta)\int_{h\in C}\widehat{x}_h(\theta)dh}{n\omega + \overline{x}^1} = f(\theta)\left[\frac{n\bar{\omega} + \int_{h\in C}\widehat{x}_h(\theta)dh}{n\bar{\omega} + \int_{h\in C}\omega_h^x dh}\right]. \tag{7.4}$$

Because bulls and bears do not demand any of good x in the REE, market clearing requires

$$\int_{h\in C}\widehat{x}_h(\theta)dh = n\omega + n(-\omega) + \int_{h\in C}\omega_h^x dh,$$

so (7.4) simplifies to $p^2(\theta) = f(\theta)$.

Now let us verify that (3.5)-(3.12) is a WPBE.

Beliefs are consistent. To see this, notice that all values of p^1 between 0 and $\left(\frac{\bar{x}^1+\omega}{\bar{x}^1}\right) f(\bar{\theta})$ are on the equilibrium path, and since $f'(\theta) > 0$ we can invert $p^1(\theta)$ to infer the correct state. Bayes' rule yields (3.11). For $p^1 > \left(\frac{\bar{x}^1+\omega}{\bar{x}^1}\right) f(\bar{\theta})$, the observed price is too high to be inconsistent with the strategy profile. Consumers believe that the state is $\bar{\theta}$ as specified in (3.12); since Bayes' rule does not apply, these beliefs are consistent.

Sequential rationality is satisfied in period 2 for each agent and each information set. For consumer h observing (p^1, b_h^1, q_h^1) and with beliefs $\theta^e(p^1)$, whose actions have a negligible effect on the period 2 price, $b_h^2(p^1, b_h^1, q_h^1)$ and $q_h^2(p^1, b_h^1, q_h^1)$ must satisfy

$$
\max_{b_h^2, q_h^2} u_h(x_h, \theta^e) + y_h
$$
$$
\text{subject to} \tag{7.5}
$$
$$
x_h = \omega_h^x + \frac{b_h^1}{p^1} - q_h^1 + \frac{b_h^2}{f(\theta^e)} - q_h^2,
$$
$$
y_h = \omega_h^y - b_h^1 + q_h^1 p^1 - b_h^2 + q_h^2 f(\theta^e),
$$
$$
0 \leq q_h^2 \leq \omega_h^x + \frac{b_h^1}{p^1} - q_h^1,
$$
$$
0 \leq b_h^2 \leq \omega_h^y - b_h^1 + q_h^1 p^1,
$$
$$
b_h^2 q_h^2 = 0
$$

Due to the quasi-linearity of the utility function, a necessary and sufficient condition for feasible bids and offers to solve (7.5) is that consumption of good x satisfy

$$
\frac{\partial u_h(x_h, \theta^e)}{\partial x_h} = f(\theta^e). \tag{7.6}
$$

However, condition (7.6) is also the necessary and sufficient condition characterizing utility maximization in the REE in state θ^e, with solution $x_h = \widehat{x}_h(\theta^e)$. Since (3.10) gives rise to $x_h = \widehat{x}_h(\theta^e)$ in (7.5), period 2 consumer behavior is sequentially rational.

Clearly, it is sequentially rational for a bull to bid zero and offer his entire holdings of good x in period 2, as specified in (3.6).

We now show that (3.8) is sequentially rational. From Lemma 1, by bidding according to (3.8) in period 2, bear j induces the price $p^2(\theta, \sigma; s^1) = f(\theta^e(p^1))$. Her

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net position of good x entering period 2 is $-(q_j^1 + \omega - \frac{b_j^1}{p^1})$, and her net purchases in period 2 are

$$\frac{(q_j^1 + \omega - \frac{b_j^1}{p^1})f(\theta^e(p^1))}{p^2(\theta, \sigma; s^1)},$$

which equals $(q_j^1 + \omega - \frac{b_j^1}{p^1})$. Thus, bear j exactly closes out her position of good x, so sequential rationality is satisfied at all information sets in period 2.

Sequential rationality is satisfied in period 1. Sequential rationality for each consumer follows from the fact that consumers have a negligible influence on prices, and from the fact that $p^1(\theta) \geq p^2(\theta)$ holds for each θ. Consumption of good x in state θ is $\widehat{x}_h(\theta)$ independent of consumer h's action in period 1, but consumption of good y is highest when the consumer adopts (3.9).

To verify sequential rationality by bulls in period 1, express the profit of bull i, given his continuation strategy and the equilibrium strategies of other agents, as a function of θ and arbitrary period 1 actions (b_i^1, q_i^1), as

$$\pi_i^{bull}(\theta, b_i^1, q_i^1) = -b_i^1 + q_i^1 p^{1,bull}(\theta, b_i^1, q_i^1) + \left[\omega + \frac{b_i^1}{p^{1,bull}(\theta, b_i^1, q_i^1)} - q_i^1\right] f(\theta^e(p^{1,bull}(\theta, b_i^1, q_i^1))). \tag{7.7}$$

Equation (7.7) uses Lemma 1, capturing the effect that bull i has on p^1, and the resulting effect on consumer behavior in period 2. The term $p^{1,bull}(\theta, b_i^1, q_i^1)$ is the price in period 1 when the state is θ and all agents other than bull i are choosing the strategies specified in (3.5)-(3.12), given by

$$p^{1,bull}(\theta, b_i^1, q_i^1) = \frac{\frac{(n-1)}{n}f(\theta)(\overline{x}^1 + \omega) + b_i^1}{\overline{x}^1 + q_i^1}. \tag{7.8}$$

Substituting (3.11) and (7.8) into (7.7) and simplifying, we have

$$\pi_i^{bull}(\theta, b_i^1, q_i^1) = \frac{n-1}{n}\omega f(\theta). \tag{7.9}$$

It follows that bull i receives profit that is independent of (b_i^1, q_i^1), so the specified strategy satisfies sequential rationality.

To verify sequential rationality by bears in period 1, the objective function for bear j, given her continuation strategy and the equilibrium strategies of other agents, as a function of θ and arbitrary period 1 actions (b_j^1, q_j^1), is given by:

$$\pi_j^{bear}(\theta, b_j^1, q_j^1) = -b_j^1 + q_j^1 p^{1,bear}(\theta, b_j^1, q_j^1) + \left[-\omega + \frac{b_j^1}{p^{1,bear}(\theta, b_j^1, q_j^1)} - q_j^1\right] f(\theta^e(p^{1,bear}(\theta, b_j^1, q_j^1))), \tag{7.10}$$

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where $p^{1,bear}(\theta, b_j^1, q_j^1)$ is the price in period 1 when the state is θ and all agents other than bear j are choosing the strategies specified in (3.5)-(3.12), given by

$$p^{1,bear}(\theta, b_j^1, q_j^1) = \frac{f(\theta)(\overline{x}^1 + \omega) + b_j^1}{\overline{x}^1 + q_j^1 - \overline{\omega}}.$$

Since (3.11) implies

$$f(\theta^e(p^{1,bear}(\theta, b_j^1, q_j^1))) = \frac{p^{1,bear}(\theta, b_j^1, q_j^1)\overline{x}^1}{\overline{x}^1 + \omega},$$

expression (7.10) can be simplified to

$$\pi_j^{bear}(\theta, b_j^1, q_j^1) = \left[\frac{\omega}{\overline{x}^1 + \omega}\right]\left[-b_j^1 + (q_j^1 - \overline{x}^1)p^{1,bear}(\theta, b_j^1, q_j^1)\right]. \qquad (7.11)$$

It is easy to see from (7.11) that $\pi_j^{bear}(\theta, b_j^1, q_j^1)$ is strictly decreasing in b_j^1 and strictly increasing in q_j^1. Therefore, it is sequentially rational to bid zero and offer up to one's short-sale limit. ∎

Proof of Proposition 2. We have an equilibrium for the replicated economy by substituting rn for n and $r\overline{x}^1$ for $\overline{x}^1$ in the construction given in Proposition 1.[24] If $\omega = 0$ holds, then we have $p^1(\theta) = p^2(\theta) = f(\theta)$ for all θ, and the result follows immediately. If $\omega > 0$ holds, we have

$$\lim_{r \to \infty} p^1(\theta; r) = \lim_{r \to \infty} \left(\frac{r\overline{x}^1 + \omega}{r\overline{x}^1}\right) f(\theta) = f(\theta).$$

Consumer i receives consumption of good x equal to her REE level, $\widehat{x}_h(\theta)$, based on the argument given in the proof of Proposition 1. Because the augmentation of her income, by selling her endowment in period 1 and buying it back in period 2, is converging to zero, her consumption of good y converges to the REE level, $\widehat{y}_h(\theta)$. It is easy to see that the bulls and bears receive the consumption that they receive under the REE. ∎

Proof of Lemma 2. Since the WPBE is fully revealing, consistency requires consumers observing p^1 to have beliefs assigning probability one to the state solving

---

[24]Consumers are assumed to be replicated as well, but this is not essential for the result.

$p^1 = p^1(\theta)$. Furthermore, consumers assign probability one to the continuation strategies of the other players in state θ following the price $p^1(\theta)$, and thus the period 2 price $p^2(\theta^e(p^1))$. Because consumers are negligible, sequential rationality for consumer h requires that (b_h^2, q_h^2) solve the following optimization problem

$$
\max_{b_h^2, q_h^2} u_h(x_h, \theta^e(p^1)) + y_h
$$
$$
\text{subject to} \tag{7.12}
$$
$$
\begin{aligned}
x_h &= \omega_h^x + \frac{b_h^1}{p^1} - q_h^1 + \frac{b_h^2}{p^2(\theta^e(p^1))} - q_h^2, \\
y_h &= \omega_h^y - b_h^1 + q_h^1 p^1 - b_h^2 + q_h^2 p^2(\theta^e(p^1)), \\
0 &\leq q_h^2, \\
0 &\leq b_h^2, \\
b_h^2 q_h^2 &= 0
\end{aligned}
$$

Due to the quasi-linearity of the utility function and the fact that consumer deviations have a negligible effect on prices, a necessary and sufficient condition for feasible bids and offers to solve (7.12) is that anticipated consumption of good x satisfies

$$
\frac{\partial u_h(x_h, \theta^e(p^1))}{\partial x_h} = p^2(\theta^e(p^1)). \tag{7.13}
$$

This is equal to the consumption along the equilibrium path in state $\theta^e(p^1)$, which we denote by $x_h(\theta^e(p^1))$. Therefore, we have

$$
\int_{h \in C} x_h(\theta^e(p^1)) dh = \int_{h \in C} \omega_h^x dh. \tag{7.14}
$$

Any bull or bear with a positive net holding of good x at the beginning of period 2 will offer exactly that quantity. For a bull or bear with a negative net holding of good x at the beginning of period 2, sequential rationality requires a bid in period 2 such that final holdings of good x are zero (given the profile of actions in period 2). Therefore, the actual consumptions of consumers must satisfy

$$
\int_{h \in C} x_h(\theta, \sigma; s^1) dh = \int_{h \in C} \omega_h^x dh. \tag{7.15}
$$

It follows from (7.14) and (7.15) that $p^2(\theta, \sigma; s^1) = p^2(\theta^e(p^1))$ holds. To see this, if $p^2(\theta, \sigma; s^1) > p^2(\theta^e(p^1))$ were to hold, then $x_h(\theta, \sigma; s^1) \leq x_h(\theta^e(p^1))$ would

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hold for all h, with strict inequality for any consumer h such that $b_h^2 > 0$. This would imply $\int_{h \in C} x_h(\theta^e(p^1)) dh > \int_{h \in C} x_h(\theta, \sigma; s^1) dh$, a contradiction. An identical argument applies to the case, $p^2(\theta, \sigma; s^1) < p^2(\theta^e(p^1))$. Thus, we have $p^2(\theta, \sigma; s^1) = p^2(\theta^e(p^1)) = f(\theta^e(p^1))$. ∎

Proof of Proposition 3. From Lemma 2, in any fully revealing type-symmetric WPBE, the equilibrium price in period 2, following any period 1 action profile yielding the price p^1, is $f(\theta^e(p^1))$. Sequential rationality requires bull i to choose b_i^1 to maximize the objective function given by

$$\pi_i(b_i^1, \theta) = -b_i^1 + (\omega + \frac{b_i^1}{p^1(b_i^1, \theta)}) f(\theta^e(p^1(b_i^1, \theta))).$$

Differentiating with respect to b_i^1, we have the necessary first order condition,

$$
\begin{aligned}
0 \;=\; & -1 + (\omega + \frac{b_i^1}{p^1(b_i^1, \theta)}) f'(\theta^e(p^1(b_i^1, \theta))) \frac{\partial \theta^e(p^1(b_i^1, \theta))}{\partial p^1} \frac{\partial p^1(b_i^1, \theta)}{\partial b_i^1} \quad (7.16) \\
& + f(\theta^e(p^1(b_i^1, \theta))) \frac{\partial}{\partial b_i^1}(\frac{b_i^1}{p^1(b_i^1, \theta)}).
\end{aligned}
$$

Evaluating at $b_i^1 = b^1(\theta)$ and $\theta^e(p^1(b_i^1, \theta)) = \theta$, (7.16) becomes

$$
\begin{aligned}
1 \;=\; & (\omega + \frac{b_i^1}{p^1(\theta)}) f'(\theta) \frac{\partial \theta^e(p^1(\theta))}{\partial p^1} \frac{1}{nq^1(\theta) + Q^1} \\
& + \frac{f(\theta)}{p^1(\theta)}(\frac{(n-1)b^1(\theta) + B^1}{nb^1(\theta) + B^1}),
\end{aligned}
$$

which, when multiplied by $(nq^1(\theta) + Q^1)p^1(\theta)$ simplifies to

$$
\begin{aligned}
nb^1(\theta) + B^1 \;=\; & (\omega p^1(\theta) + b_i^1) f'(\theta) \frac{\partial \theta^e(p^1(\theta))}{\partial p^1} \quad (7.17) \\
& + \frac{f(\theta)}{p^1(\theta)}((n-1)b^1(\theta) + B^1).
\end{aligned}
$$

Similarly, the objective function for bear j is given by

$$\pi_j(q_j^1, \theta) = q_j^1 p^1(q_j^1, \theta) - (q_j^1 + \omega) f(\theta^e(p^1(q_j^1, \theta))).$$

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The first order condition is

$$0 = p^1(q_j^1, \theta) + q_j^1 \frac{\partial p^1(q_j^1, \theta)}{\partial q_j^1} - f(\theta^e(p^1(q_j^1, \theta))) \tag{7.18}$$

$$-(q_j^1 + \omega)f'(\theta^e(p^1(q_j^1, \theta)))\frac{\partial \theta^e(p^1(q_j^1, \theta))}{\partial p^1}\frac{\partial p^1(q_j^1, \theta)}{\partial q_j^1}.$$

Evaluating at $q_j^1 = q^1(\theta)$ and $\theta^e(p^1(q_j^1, \theta)) = \theta$, (7.18) becomes

$$0 = p^1(\theta) + q^1(\theta)\left[-\frac{p^1(\theta)}{nq^1(\theta) + Q^1}\right] - f(\theta) + (q^1(\theta) + \omega)f'(\theta)\frac{\partial \theta^e(p^1(\theta))}{\partial p^1}\left[\frac{p^1(\theta)}{nq^1(\theta) + Q^1}\right],$$

which, when multiplied by $(nq^1(\theta) + Q^1)$, simplifies to

$$0 = nb^1(\theta) + B^1 - q^1(\theta)p^1(\theta) - f(\theta)(nq^1(\theta) + Q^1) \tag{7.19}$$

$$+(q^1(\theta) + \omega)f'(\theta)\frac{\partial \theta^e(p^1(\theta))}{\partial p^1}\left[p^1(\theta)\right].$$

Suppose that the realized state is such that $p^1(\theta) > f(\theta)$ holds. Then from (7.19), we have

$$(q^1(\theta) + \omega)f'(\theta)\frac{\partial \theta^e(p^1(\theta))}{\partial p^1}\left[p^1(\theta)\right]$$
$$= f(\theta)(nq^1(\theta) + Q^1) + q^1(\theta)p^1(\theta) - (nb^1(\theta) + B^1)$$
$$= f(\theta)(nq^1(\theta) + Q^1) + p^1(\theta)[q^1(\theta) - (nq^1(\theta) + Q^1)]$$
$$< p^1(\theta)q^1(\theta),$$

which implies

$$f'(\theta)\frac{\partial \theta^e(p^1(\theta))}{\partial p^1} < \frac{q^1(\theta)}{q^1(\theta) + \omega}. \tag{7.20}$$

From (7.17), we have

$$(\omega p^1(\theta) + b_i^1)f'(\theta)\frac{\partial \theta^e(p^1(\theta))}{\partial p^1}$$

$$= nb^1(\theta) + B^1 - \frac{f(\theta)}{p^1(\theta)}((n-1)b^1(\theta) + B^1)$$
$$> nb^1(\theta) + B^1 - ((n-1)b^1(\theta) + B^1)$$
$$= b^1(\theta),$$

which implies

$$f'(\theta)\frac{\partial \theta^e(p^1(\theta))}{\partial p^1} > \frac{b^1(\theta)}{\omega p^1(\theta) + b_i^1}. \tag{7.21}$$

Combining (7.20) and (7.21), we have

$$\frac{q^1(\theta)}{q^1(\theta) + \omega} > \frac{b^1(\theta)}{\omega p^1(\theta) + b_i^1},$$

which simplifies to

$$p^1(\theta) > \frac{b^1(\theta)}{q^1(\theta)},$$

implying

$$p^1(\theta) < \frac{B^1}{Q^1}. \tag{7.22}$$

Inequality (7.22) implies that in the aggregate, consumers are net buyers of good x in period 1.

Now suppose that the realized state is such that $p^1(\theta) < f(\theta)$ holds. Repeating the previous steps with the opposite inequality, we find

$$\frac{b^1(\theta)}{\omega p^1(\theta) + b_i^1} > f'(\theta)\frac{\partial \theta^e(p^1(\theta))}{\partial p^1} > \frac{q^1(\theta)}{q^1(\theta) + \omega},$$

from which we conclude

$$p^1(\theta) > \frac{B^1}{Q^1}. \tag{7.23}$$

Inequality (7.23) implies that in the aggregate, consumers are net sellers of good x in period 1.

Claim: $p^1(\theta) = f(\theta)$ holds for almost all θ.

Proof of Claim: Lemma 2 establishes that along the equilibrium path, $p^2(\theta) = f(\theta)$ holds for all θ. From the proof of Lemma 2, consumers receive the same consumption of good x that they receive in the REE: $x_h(\theta) = \widehat{x}_h(\theta)$ for all h, θ. Suppose the Claim is false. Then for a positive measure of states, consumers in the aggregate are making arbitrage losses as compared to transacting only in period 2, either buying in period 1 at a price above $f(\theta)$ or selling in period 1 at a price below $f(\theta)$. Therefore, we have

$$\int_\theta \int_{h \in C} y_h(\theta) dh dG(\theta) < \int_\theta \int_{h \in C} \widehat{y}_h(\theta) dh dG(\theta).$$

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It follows that there is a positive measure of consumers h who would receive higher utility by deviating to $b_h^1 = q_h^1 = 0$; since prices would not be affected by a unilateral consumer deviation, consumption of good x would not be affected and expected consumption of good y would be higher.

Since $p^1(\theta) = f(\theta)$ holds for almost all θ, and since both are continuous functions, it follows that the two functions are identical. Therefore, we have

$$\frac{\partial \theta^e(p^1(\theta))}{\partial p^1} = \frac{1}{\frac{\partial p^1(\theta)}{\partial \theta}} = \frac{1}{f'(\theta)}.$$

Then (7.19) can be simplified to

$$1 = \frac{q^1(\theta)}{q^1(\theta) + \omega}.$$

Whenever $\omega > 0$ holds, we have a contradiction to the supposition that there can be a WPBE of this form.

The argument given above also allows us to conclude that there cannot be an open-market, type-symmetric WPBE in which the bears are choosing the bid function $b^1(\theta)$ and bulls are choosing the offer function $q^1(\theta)$ in period 1. Simply change ω to $-\omega$ in the expressions for $\pi_i(b_i^1, \theta)$ (which now applies to bears) and $\pi_j(q_j^1, \theta)$ (which now applies to bulls), and in subsequent calculations. ∎

Proof of Proposition 4. Since only one price is observed on the equilibrium path, clearly the consumer beliefs are consistent. First we prove the analog of Lemma 1. Let C_+ denote $\{h : x_h^{nr}(p^{nr}) \geq \omega_h^x\}$ and let C_- denote $\{h : x_h^{nr}(p^{nr}) < \omega_h^x\}$. Given the period 1 action profile s^1 and the specified continuation strategies, we have

$$\frac{p^2(\theta, \sigma; s^1)}{p^{nr}} = \frac{\sum_{j \in A_-} (q_j^1 - \frac{b_j^1}{p^1}) + n\omega + \int_{h \in C_+} \left[x_h^{nr}(p^{nr}) + q_h^1 - \omega_h^x - \frac{b_h^1}{p^1} \right] dh}{\sum_{i \in A_+} (\frac{b_i^1}{p^1} - q_i^1) + n\omega + \int_{h \in C_-} \left[\omega_h^x - x_h^{nr}(p^{nr}) + \frac{b_h^1}{p^1} - q_h^1 \right] dh}.$$

(7.24)

Denoting the left side of (7.24) by K, multiplying numerator and denominator of

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the right side by p^1, then cross-multiplying, we have

$$
\begin{aligned}
K \sum_{i \in A_+} b_i^1 &- K p^1 \sum_{i \in A_+} q_i^1 + K p^1 n \omega + K p^1 \int_{h \in C_-} \left[\omega_h^x - x_h^{nr}(p^{nr}) \right] dh \\
&+ K \int_{h \in C_-} b_h^1 - K p^1 \int_{h \in C_-} q_h^1 dh \\
= \ p^1 \sum_{j \in A_-} q_j^1 &- \sum_{j \in A_-} b_j^1 + p^1 n \omega + p^1 \int_{h \in C_+} \left[x_h^{nr}(p^{nr}) - \omega_h^x \right] dh \\
&+ p^1 \int_{h \in C_+} q_h^1 dh - \int_{h \in C_+} b_h^1 dh,
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
K &\left[\sum_{i \in A_+} b_i^1 + \int_{h \in C_-} b_h^1 + \sum_{j \in A_-} b_j^1 + \int_{h \in C_+} b_h^1 dh \right] \\
&+ K p^1 \left[\int_{h \in C_-} \left[\omega_h^x - x_h^{nr}(p^{nr}) \right] dh - \int_{h \in C_+} \left[x_h^{nr}(p^{nr}) - \omega_h^x \right] dh \right] \qquad (7.25) \\
= \ K p^1 &\left[\sum_{i \in A_+} q_i^1 + \int_{h \in C_-} q_h^1 dh + \sum_{j \in A_-} q_j^1 + \int_{h \in C_+} q_h^1 dh \right] \\
&+ (K - 1) \left[-p^1 n \omega + \sum_{j \in A_-} b_j^1 - p^1 \sum_{j \in A_-} q_j^1 \right] \\
&+ (K - 1) \left[\int_{h \in C_+} b_h^1 dh - p^1 \int_{h \in C_+} q_h^1 dh - p^1 \int_{h \in C_+} \left[x_h^{nr}(p^{nr}) - \omega_h^x \right] dh \right].
\end{aligned}
$$

By the definition of p^1, the first expressions on the left and right of (7.25) cancel, and since the allocation is a non-revealing REE, market clearing implies the second expression on the left side of (7.25) is zero. Thus, we have

$$
\begin{aligned}
0 \ = \ (K - 1) &\left[\sum_{j \in A_-} b_j^1 - p^1 \sum_{j \in A_-} q_j^1 - p^1 n \omega \right] \\
&+ (K - 1) \left[\int_{h \in C_+} b_h^1 dh - p^1 \int_{h \in C_+} q_h^1 dh - p^1 \int_{h \in C_+} \left[x_h^{nr}(p^{nr}) - \omega_h^x \right] dh \right].
\end{aligned}
$$

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Because we are considering deviations that are not large enough to put agents on the opposite side of the market in period 2, both terms in brackets are negative, which requires $K = 1$. Thus, we have $p^2(\theta, \sigma; s^1) = p^{nr}$ in period 2 after all histories.

Because the price is p^{nr} in period 1 and in all period 2 subgames, it is immediate from the definition of a non-revealing REE that sequential rationality is satisfied for all consumers. Bulls and bears are exactly closing out their positions in period 2, so sequential rationality is satisfied in period 2 for each bull and bear.

Consider the period 1 decision for bull i in state θ. Taking into account the continuation strategies, bull i's payoff function is

$$\pi_i^{bull}(\theta, b_i^1, q_i^1) = -b_i^1 + q_i^1 p^{1,bull}(\theta, b_i^1, q_i^1) + \left[\omega + \frac{b_i^1}{p^{1,bull}(\theta, b_i^1, q_i^1)} - q_i^1\right] p^{nr}, \quad (7.26)$$

where (writing the aggregate consumer bids as B^1 and the aggregate consumer offers as Q^1) we have

$$p^{1,bull}(\theta, b_i^1, q_i^1) = \frac{B^1 + b_i^1}{Q^1 + q_i^1}. \quad (7.27)$$

To consider the effect of making a positive bid, substitute (7.27) into (7.26), set $q_i^1 = 0$, and differentiate with respect to b_i^1, yielding

$$-1 + p^{nr}\frac{B^1 Q^1}{(B^1 + b_i^1)^2}$$
$$= -1 + \frac{(B^1)^2}{(B^1 + b_i^1)^2},$$

where the last step follows from the fact that $p^{nr} = B^1/Q^1$. Since the expression is negative, it follows that submitting a positive bid strictly lowers bull i's payoff.

Similarly, to consider the effect of making a positive offer, substitute (7.27) into (7.26), set $b_i^1 = 0$, differentiate with respect to q_i^1, and substitute $p^{nr} = B^1/Q^1$, yielding

$$\left[\frac{(Q^1)^2}{(Q^1 + q_i^1)^2} - 1\right]\frac{B^1}{Q^1} < 0.$$

It follows that submitting a positive offer strictly lowers bull i's payoff, so sequential rationality is satisfied. To show that sequential rationality is satisfied for bear j, repeat the above steps with an endowment of $-\omega$. ∎

Proof of Proposition 5. Due to the similarity to the proofs of Propositions 1 and 4, a sketch will be provided. The logic of the proof of Lemma 1 and the first

part of the proof of Proposition 4 establishes that, after all histories, the price in period 2 is the competitive market clearing price given consumer beliefs. That is, we have $p^2(\theta, \sigma; s^1) = \widetilde{f}(\widetilde{\theta})$ if $p^1 = \frac{\overline{x}^1 + \omega}{\overline{x}^1} \widetilde{f}(\widetilde{\theta})$ holds, and $p^2(\theta, \sigma; s^1) = f(\theta^e(p^1))$ if $p^1 = \frac{\overline{x}^1 + \omega}{\overline{x}^1} \widetilde{f}(\widetilde{\theta})$ holds. It follows that, after all histories with period 1 price p^1, we have $p^2(\theta, \sigma; s^1) = \frac{p^1 \overline{x}^1}{\overline{x}^1 + \omega}$.

Given consumer beliefs and the period 2 price following any history, it follows from the structure of the fully revealing REE and the non-revealing REE of the "hypothetical" economy that sequential rationality is satisfied for consumers in period 2. Sequential rationality is satisfied in period 2 by bulls and bears, because they are exactly closing out their positions.

It is easy to see that consumers are satisfying sequential rationality in period 1, because $p^1 > p^2$ holds. To verify that beliefs are consistent following any p^1, notice that $p^1 > \frac{\overline{x}^1 + \omega}{\overline{x}^1} f(\overline{\theta})$ and $p^1 < \frac{\overline{x}^1 + \omega}{\overline{x}^1} \widetilde{f}(\widetilde{\theta})$ are off the equilibrium path, so Bayes' rule does not apply. Given the strategy profile, we have $p^1 = \frac{\overline{x}^1 + \omega}{\overline{x}^1} \widetilde{f}(\widetilde{\theta})$ if and only if $\theta \leq \widetilde{\theta}$ holds. This is because, for $\theta > \widetilde{\theta}$, we have $p^1 = \frac{\overline{x}^1 + \omega}{\overline{x}^1} f(\theta)$, and we know from our consumer regularity assumptions that $f(\theta) > \widetilde{f}(\widetilde{\theta})$. Thus, beliefs are consistent when we have $p^1 = \frac{\overline{x}^1 + \omega}{\overline{x}^1} \widetilde{f}(\widetilde{\theta})$. Finally, when we have $\frac{\overline{x}^1 + \omega}{\overline{x}^1} \widetilde{f}(\widetilde{\theta}) < p^1 \leq \frac{\overline{x}^1 + \omega}{\overline{x}^1} f(\overline{\theta})$, there is a unique $\theta \in (\widetilde{\theta}, \overline{\theta}]$ for which that price occurs on the equilibrium path, satisfying $f(\theta) = \frac{p^1 \overline{x}^1}{\overline{x}^1 + \omega}$. Therefore, the degenerate distribution given by $\theta^e(p^1)$ is consistent.

When $\theta \leq \widetilde{\theta}$ holds, the profits for bull i associated with bid b_i^1, given the continuation strategies, is given by

$$\pi_i^{bull} = -b_i^1 + (\omega + \frac{b_i^1}{p^{1,bull}}) \frac{p^{1,bull} \overline{x}^1}{\overline{x}^1 + \omega},$$

$$\text{where } p^{1,bull} = \frac{(\frac{n-1}{n}) \widetilde{f}(\widetilde{\theta})(\overline{x}^1 + \omega) + b_i^1}{\overline{x}^1}.$$

Then we can substitute and simplify the profit expression to $\pi_i^{bull} = \omega(\frac{n-1}{n}) \widetilde{f}(\widetilde{\theta})$; therefore, bull i is indifferent as to his bid, so sequential rationality is satisfied. As in the proof of Proposition 1, bull i receives the same profits by submitting an offer rather than a bid. When $\theta > \widetilde{\theta}$ holds, the profit expression for bull i associated with bid b_i^1 is unchanged, but we have

$$p^{1,bull} = \frac{(\frac{n-1}{n}) f(\theta)(\overline{x}^1 + \omega) + b_i^1}{\overline{x}^1}.$$

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Substituting and simplifying yields $\pi_i^{bull} = \omega(\frac{n-1}{n})f(\theta)$, so again bull i is indifferent and sequential rationality holds.

When $\theta \leq \widetilde{\theta}$ holds, the profits for bear j associated with offer q_j^1, given the continuation strategies, is given by

$$\pi_j^{bear} = q_j^1 p^{1,bear} - (\omega + q_j^1)\frac{p^{1,bear}\overline{x}^1}{\overline{x}^1 + \omega},$$

$$\text{where } p^{1,bear} = \frac{\widetilde{f}(\widetilde{\theta})(\overline{x}^1 + \omega)}{\overline{x}^1 + q_j^1 - \overline{\omega}}.$$

Then we can substitute and simplify the profit expression to $\pi_j^{bear} = \omega \widetilde{f}(\widetilde{\theta})(\frac{q_j^1 - \overline{x}^1}{\overline{x}^1 + q_j^1 - \overline{\omega}})$. One can verify that π_j^{bear} is strictly increasing in q_j^1, and that considering deviations to bid rather than offer also lead to lower profits. Therefore, sequential rationality is satisfied. When $\theta > \widetilde{\theta}$ holds, the profit expression for bear j associated with offer q_j^1 is unchanged, but we have

$$p^{1,bear} = \frac{f(\theta)(\overline{x}^1 + \omega)}{\overline{x}^1 + q_j^1 - \overline{\omega}}.$$

Substituting and simplifying yields $\pi_j^{bear} = \omega f(\theta)(\frac{q_j^1 - \overline{x}^1}{\overline{x}^1 + q_j^1 - \overline{\omega}})$, so again sequential rationality holds. ∎

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