Non-Supermodular Price-Setting Games

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Abstract

It is well known that the existence and uniqueness of Cournot equilibrium would extend to environments where firms prefer to be not active. However, we show that differentiated Bertrand oligopolies with constant unit costs and continuous best replies do not need to satisfy supermodularity (Topkis (1979)) or single crossing property (Milgrom and Shannon (1994)). Moreover, best replies might be negatively sloped and there are infinitely many undominated Bertrand-Nash equilibria on a wide range of parameter values when the number of firms is more than two. These results are very different from the existing literature on Bertrand models, where uniqueness, supermodularity, and single crossing property usually hold under a linear market demand assumption and best reply functions slope upwards. We fully characterize the set of undominated equilibria. We provide an iterative algorithm to find the set of players that are active in any equilibrium, and show that this set is the same in all undominated equilibria.

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1 Introduction

In several markets some firms may not able to actively participate, and many decide to shut down. The exit incentives of firms might be induced by the aggressive pricing strategies of their rivals by charging below their average costs. This practice, known as predatory pricing, is considered to be illegal by the main anti-trust authorities like Federal Trade Commission (FTC) and U.S. Department of Justice (DOJ). However, the exit behaviour of firms might also be efficiency based driven in highly competitive markets. A cost reducing innovation by competitors, the inability to adapt to changing market conditions, a cost-efficient merger among rival firms, or an increase in fixed costs may induce a firm to exit or to remain idle temporarily. These kinds of restructuring of the markets might force relatively more efficient firms to practice *efficiency based limit pricing* in order to induce exit of relatively less efficient rivals that have already entered. Efficiency based limit pricing strategies drive entirely from the non-cooperative nature of firm interactions. Relatedly, it cannot be considered as an illegal activity as all active firms charge above their average costs without taking into account possible future profits following the exit of their rivals.

In this paper, we study traditional static price-setting games among firms that have different levels of cost-efficiencies. The differences between these levels might due to one of the above factors. Our main aim is to identify the set of active and inactive firms in any equilibrium and to provide a full characterization of the equilibrium behaviour of active firms that potentially involves efficiency based limit pricing strategies. Such a characterization in static quantity-setting games is trivial because relatively efficient firms cannot be forced to increase their productions to induce exit of relatively inefficient firms. In particular, standard existence and uniqueness results for the Cournot equilibrium extend to environments where firms prefer to be not active (Novshek (1985) and Gaudet and Salant (1991)). However, the equilibrium behavior of firms constrained by non-negative production levels in Bertrand models has not been extensively studied. In contrast to the quantity-setting games, we argue that allowing pricing at marginal cost level (i.e., producing zero) sharply changes the set of equilibria in price setting games. We show that differentiated linear Bertrand oligopolies with constant unit costs and continuous best replies need not satisfy supermodularity (Topkis

(1979)) or single crossing property (Milgrom and Shannon (1994)). Consequently, existence and uniqueness results for games satisfying supermodularity or single crossing-property do not apply in our framework. In particular, Bertrand-best replies might be negatively sloped and there are infinitely many undominated efficiency based limit pricing equilibria for a wide range of parameter values when the number of firms is more than two. These results are different from the existing literature on Bertrand models, where uniqueness, supermodularity, and single crossing property hold under a linear market demand assumption and best reply functions slope upwards.

To explain our results, consider a symmetric three-firm differentiated product Bertrand-oligopoly where marginal cost levels are $c_i = \xi$ for $i = 1, 2, 3$. All firms are active, that is their equilibrium production levels are all strictly positive. Suppose now that there is a process innovation available for firms one and two. Accordingly, their cost levels reduce to $\widehat{\xi} = \widehat{c}_1 = \widehat{c}_2 < \widehat{c}_3 = \xi$. If the initial cost level ξ is high enough, then there are two cutoff levels for $\widehat{\xi}$, say $\widehat{\xi}_1$ and $\widehat{\xi}_2$ with $0 < \widehat{\xi}_1 < \widehat{\xi}_2$, such that equilibrium strategies of firms are qualitatively different when $\widehat{\xi}$ lies in regions $[0, \widehat{\xi}_1]$, $[\widehat{\xi}_1, \widehat{\xi}_2]$, or $[\widehat{\xi}_2, \xi)$. More specifically, if $\widehat{\xi} \in [\widehat{\xi}_2, \xi)$, then the level of innovation is not too high and all three firms continue to be active in the market. On the other extreme, if $\widehat{\xi} \in [0, \widehat{\xi}_1]$, then firm three becomes very inefficient compared to firm one and two and leaves the market. Accordingly firms one and two charge unconstrained duopoly prices. The most interesting region is the intermediate region here $\widehat{\xi} \in [\widehat{\xi}_1, \widehat{\xi}_2)$. This region involves efficiency based limit pricing induced by firms one and two to keep firm three out of the market. If they ignored firm three and charged unconstrained duopoly prices, then firm three would continue to be active in the market.

In the case of linear demand, limit pricing takes a particularly simple form. Consider any price combination of firms one and two such that $p_1 + p_2 = M$ where M is uniquely determined by the parameters of the model. If either firm one or firm two charges a higher price, then firm three would start to produce and the market becomes a triopoly market. On the other hand, when either of them decreases its price, the market is a duopoly market. For this reason, the profit functions of firms one and two exhibit kinks at price combinations where $p_1 + p_2 = M$. Moreover, the fact that demand is more sensitive to a change in the price that a firm sets in the region where all three firms are active[1] implies that the right hand derivative of the profit of firm 1 with respect to p_1 is more negative (or less positive) than the left hand derivative if $p_1 + p_2 = M$ as the demand drop is accelerated for prices where the third firm is active. At such price combinations, optimality conditions for firm 1 require the left hand derivative of the profit function to be positive, and the right hand derivative to be negative, which can be satisfied by multiple combinations of p_1 and p_2 satisfying $p_1+p_2 = M$. As a result there is a host of equilibria in our price setting game. Relatedly, the kink implies that the best reply for firm 1 when firm 2 sets p_2 satisfies $p_1 = M-p_2$, so firm one's and two's price choices are strategic substitutes at such a point.

The usefulness of analyzing efficiency based limit pricing strategies has already been pointed out in Bertrand-duopoly games. Muto (1993) considers a Bertrand-type duopoly model with differentiated goods under a linear demand and constant but different marginal costs. There is a cost-reducing process innovation by an external patentee and the patentee's payoff is compared under three licensing policies (the auction, the fee, and the royalty). A sufficient level of cost reduction might lead the efficient firm to exercise limit pricing, which induces the relatively inefficient firm to exit the market. Muto argues that a royalty may be the most beneficial for a patentee. Our results show that when there are more than two firms his comparison may drastically change when one considers the possibility of multiple equilibria. Zanchettin (2006) also studies a Bertrand duopoly model and considers one cost efficient and one inefficient firm. He mainly compares Cournot and Bertrand equilibrium prices, quantities, and profits of the efficient firm. Unlike Singh and Vives (1984), he considers the possibility of firms not producing as in this paper. He shows that both the efficient firm's and industry profits can be higher under Bertrand competition when the firm asymmetry is low enough. This reverses Singh and Vives' ranking. Our results show that the uniqueness of equilibrium in the two-firm case will disappear when there are more than two firms. It is clear from these arguments that the possibility of limit pricing and multiple equilibria might give rise to unexpected results in various contexts.

Our paper also contributes to the literature on equilibrium existence in price-

[1]The reason is that when firm 1 changes its price in the duopoly region (i.e. where $p_1 + p_2 < M$) then its quantity responds relatively mildly since there is only one other firm (firm 2), where customers divert to. In the region where $p_1 + p_2 \geq M$ any increase of p_1 makes customers divert to both firms 2 and 3.

setting games. Roberts and Sonnenschein (1977), Friedman (1983) and Vives (2001) provide examples of non-existence of equilibrium in non-supermodular price setting oligopolies. However, unlike our results, their examples depend on discontinuities in the best replies.[2] Topkis (1979) shows that if the goods are substitutes with linear demand and costs and if the players' strategies are prices constrained to lie in an interval $[0, p]$, then the game is supermodular.[3] Building on Topkis (1979), Milgrom and Roberts (1990b) show that there is a unique pure strategy Bertrand equilibrium with linear, CES, logit, and translog demand functions and constant marginal costs. In both Topkis (1979) and Milgrom and Roberts (1990b), demand function is assumed to be twice continuously differentiable. We argue that for standard demand functions, this assumption is only satisfied when all firms have positive production. Otherwise, demand functions might have kinks as we show it in our simple linear model.[4] More recently, Ledvina and Sircar (2011, 2012) study static entry price setting games, where some firms may not produce in equilibrium. They show that there is a unique pure strategy Bertrand equilibrium in a model that covers our set-up. However, our result establishes that it is necessary to assume positive production by all firms in order to assure supermodularity and single crossing, and thereby assure the uniqueness of pure strategy Bertrand equilibrium.

In Section 2 we describe the model and provide preliminary analysis. Section 3 provides the main theoretical analysis. Section 4 discusses implications for limit pricing models, and how firms may keep out rivals jointly in real world markets.

2 Model and Equilibrium Analysis

Let $N = \{1, 2, ..., n\}$ be a finite set of firms. Each firm $i \in N$ produces an imperfect substitutable product i at constant marginal cost c_i without incurring fixed costs.

[2] Friedman (1977) shows that when best response functions are contractions, costs are nondecreasing, and all firms actively produce imperfectly substitutable goods, then there is a unique Bertrand equilibrium.

[3] Later, Vives (1990) extends the result to the case of convex costs.

[4] Vives (1990, footnote 13) argues that we might alternatively impose positive productions by all firms rather than allowing zero production to ensure supermodularity. However, sufficient conditions for positive production levels are hard to provide.

Without loss of generality assume that $c_1 < c_2 < .... < c_n$.[5] Each firm $i \in N$ sets his price p_i simultaneously, knowing all the cost and demand parameters of the game.

Next we describe the demand side of the economy. The demand for product i is denoted by $q_i = D_i^N(\mathbf{p}_N)$, where $\mathbf{p}_N = (p_1, p_2, ..., p_n)$. The representative consumer has an exogenous income I and maximizes utility:

$$U = \sum_{k \in N} A q_k - \frac{1}{2} \sum_{k \in N} q_k^2 - \theta \sum_{k \in N} \sum_{k \in N, j > k} q_k q_j + (I - \sum_{k \in N} q_k q_k), \tag{1}$$

where A is the common demand parameter and $\theta \in (0, 1)$ is an inverse measure of product differentiation. Cumbul (2012) shows that U is concave. To obtain non-trivial results, we assume that for each $k \in N$, it holds that $A > c_k$.

This demand function is symmetric across firms, so it is clear that the consumer will consume a strictly positive amount of the cheapest $s(\mathbf{p}_S)$ products, which is offered by firms in set $S(\mathbf{p}) \subset N$.[6] The first order condition of the consumer's problem yields that for all products that are consumed in positive quantity, that is for all $i \in S(\mathbf{p}_S)$ it holds that

$$p_i = A - q_i - \theta \sum_{j \in S \setminus i} q_j. \tag{2}$$

Solving (2) for quantities yields

$$q_i = D_i^S(\mathbf{p}_S) = a_s - b_s p_i + d_s \sum_{j \in S \setminus i} p_j \tag{3}$$

where $a_s = \frac{A}{(1+\theta(s-1))}$, $b_s = \frac{1+\theta(s-2)}{(1-\theta)(1+\theta(s-1))}$, and $d_s = \frac{\theta}{(1-\theta)(1+\theta(s-1))}$.

Given a price vector $\mathbf{p}_N = (p_1, p_2, ..., p_n)$, one can calculate the profit of each firm $i \in N$ as follows. The profit of firm j, $\pi_j(\mathbf{p}_S)$ is equal to 0 if $j \in N \setminus S(\mathbf{p}_S)$. The profit of $i \in S(\mathbf{p}_S)$ can be written as

[5] All our results are valid for the case where some of the costs are equal, as we assume in some examples, but the notation becomes much more burdensome, so we do not cover this case formally.

[6] A full characterization of S for any price vector is not necessary at this point. We use the relevant properties of S when we proceed with our analysis.

$$\pi_i(\mathbf{p}_S) = (p_i - c_i)(a_{s(\mathbf{p}_S)} - b_{s(\mathbf{p}_S)} p_i + d_{s(\mathbf{p})} \sum_{j \in S(\mathbf{p}_S) \setminus i} p_j) \tag{4}$$

An *equilibrium* of the Bertrand-game requires that for all $k \in N$ it holds that $p_k \in \arg\max_x \pi_k(x, \mathbf{p}_{-k})$ where we let $\mathbf{p}_{-k}$ be the vector of prices set by all firms other than k. An *undominated equilibrium* is such that $p_l \geq c_l$ for all $l \in N$.

We show that π_i is globally quasi-concave, and thus a pure strategy equilibrium exists. First, the derivative of π_i with respect to p_i is $q_i - b_s(p_i - c_i)$ at all prices where the set of active firms does not change. As p_i increases, a new firm k may enter the market. At such a point the left hand derivative is still $q_i - b_s(p_i - c_i)$, while the right hand derivative becomes $q_i - b_{s+1}(p_i - c_i)$. Given that $b_{s+1} > b_s$, the right hand derivative is strictly lower than the left hand derivative. Therefore, as p_i increases, the derivative either exists and is decreasing or the right hand derivative is less than the left hand derivative. Therefore, the profit function is globally quasi-concave in p_i. This argument shows that the profit function exhibits a *kink* at a point where a new firm becomes active because at such a point the demand of i becomes more sensitive to changes in i's price (that is, $b_{s+1} > b_s$) due to the fact that the consumer may divert to more firms than before.

Lemma 1. *The profit function π_i is globally quasi-concave in p_i when $p_i \geq c_i$ and $q_i > 0$. Consequently, there exists a pure strategy undominated Bertrand-Nash equilibrium where each $i \in N$ charges $p_i \in [c_i, A]$.*

Proof: See the Appendix.

To find an equilibrium, one needs to check all possible combinations of firms that may be active. To facilitate the analysis, we first study a simpler game ignoring the non-negativity constraint for the output levels. In effect, we use (3) to calculate the demand even if $q_i < 0$ for some $i \in S$. We find the equilibrium of this modified game, which we call a naive equilibrium. In the next step, we impose the non-negativity constraints to find necessary conditions for equilibria of the original game. Then in Section 3, we propose an iterative algorithm to find the firms that are active in the equilibrium of the original game. Finally, we characterize equilibrium prices and quantities.

To provide a definition of a *naive-equilibrium* we use (3). In the S-firm market, a price vector $\mathbf{p}_S^*(S) = (p_i^*(S))_{i \in S} \geq 0$ is a naive Bertrand-Nash equilibrium if for each $i \in S$ it holds that

$$p_i = \arg\max_x (x - c_i)(a_s - b_s x + d_s \sum_{j \in S \setminus i} p_j). \tag{5}$$

Given our linearity assumptions, there is a unique such equilibrium, which can be found by differentiating (5) with respect to x and setting the derivative to zero. The best response of firm $i \in S$ is then given as

$$BR_i^S : \mathbb{R}^{s-1} \to \mathbb{R} \quad \text{s.t.} \quad BR_i^S(\mathbf{p}_{S \setminus i}) = \frac{a_s + d_s \sum_{S \setminus i} p_j + b_s c_i}{2b_s} \tag{6}$$

where $\mathbf{p}_{S \setminus i} = (p_j)_{j \in S \setminus \{i\}}$ is the price vector that does not contain the i^{th} dimension [7]. The graph of the related Bertrand best response function, i.e, $Gr(BR_i^S)$ can be defined as follows

$$Gr(BR_i^S) = \{\mathbf{p} \in \mathbb{R}^{|S|} : p_i = BR_i^S(\mathbf{p}_{S \setminus i})\} \tag{7}$$

Letting $c_T(S) = \sum\limits_S c_i$ and assuming that all firms best respond, we obtain the naive equilibrium price and quantity levels:

$$p_i^*(S) = \frac{a_s(2b_s + d_s) + b_s(2b_s - d_s(s-1))c_i + b_s d_s c_T(S)}{(2b_s + d_s)(2b_s - d_s(s-1))}, \tag{8}$$

and

$$q_i^*(S) = b_s(p_i^*(S) - c_i). \tag{9}$$

We now impose the constraint that the output of each firm is non-negative. Take any set of firms $S' \subset N$ with the cardinality of S' being s'. Let $h = \arg\min_{j \in N \setminus S'} c_j$ and $S'' = S' \cup \{h\}$. First, we derive a condition that ensures that if the set of active firms in the market is S', then firm h does not want to enter. Our starting point is that when firm h is inactive, any firm $g \in N \setminus S'$ that is less efficient than firm h can be ignored for the analysis as those firms are also inactive. Consequently, the demand that firm h faces when it sets $p_h = c_h$ and takes $\mathbf{p}_{S'}$ as given follows from (3):

[7] Throughout the paper, the bold letters show that the considered variable is written in the vectorial form.

$$D_h^{S''}(\mathbf{p}_{S'}, p_h = c_h) = a_{s'+1} - b_{s'+1}c_h + d_{s'+1}\sum_{j \in S'} p_j. \tag{10}$$

It is clear that firm h can profitably enter (produce $q_h > 0$) if and only if $D_h^{S''}(\mathbf{p}_{S'}, p_h = c_h) > 0$ because otherwise even if firm h charges his break even price c_h, it faces a non-positive demand.

Let us derive necessary conditions for an equilibrium where only firms in S_1 are active. Theorem 1 below shows that there are three possible types of equilibria of which exactly one type occurs for any parameter values. First, a **knife-edge equilibrium** may occur if firm h exactly produces zero when it interacts with firms in S_1 in the naive equilibrium (i.e., $q_h^*(S'') = 0$)[8]. In this knife-edge case, all firms in S_1 are active when they charge their naive equilibrium prices in the market of firms in S_1 and h, that is for all $i \in S^{'}$, $p_i^*(S'')$[9]. However, in an **unconstrained equilibrium** where only firms in S' active, each such firm charges the naive equilibrium prices when the set of firms on the market is just S' (i.e., $p_i^*(S')$), ignoring the presence of other firms completely. Moreover, no firm $j \in N \setminus S'$ has an incentive to disrupt the equilibrium, that is for all $j \in N \setminus S'$ it holds that $D_j^{S'}(p_{S'}^*(S')) \leq 0$. Lastly, a **constrained (limit pricing) equilibrium** is such that the active firms are constrained by the presence of some inactive firm h when they set their equilibrium prices. In this case $D_h^{S''}(\mathbf{p}_{S'}^*(S'), p_h = c_h) > 0$. While firms in $N \setminus S'$ have too high marginal costs to be able to enter in equilibrium, but some of them would enter if firms in S' charged the naive equilibrium price vector $\mathbf{p}_{S'}^*(S')$.

Theorem 1. *Take any set of firms $S' \subset N$ and let $h = \arg\min_{j \in N \setminus S'} c_j$ and $S'' = S' \cup h$. Let $(\widehat{p}_i, \widehat{q}_i)$ denote an undominated equilibrium price quantity vector of firm $i \in N$. If the set of active firms is S' (that is, $q_i > 0$ if and only if $i \in S'$) in an undominated equilibrium, then one of i), ii), or iii) holds:*

i) (knife-edge equilibrium) If $D_h^{S''}(\mathbf{p}_{S''}^*(S'')) = 0$ and for all $i \in S''$, then $\widehat{q}_i = q_i^*(S'')$, $\widehat{p}_i = p_i^*(S'')$;

[8]It can be shown from (9) that for each $i \in S^{'}$, $q_i^*(S^{'} \cup h) \neq q_i^*(S^{'})$, which explains the reason why we consider this knife-edge case as a possibility.

[9]Consider any $l \in N \setminus \{S^{''}\}$ and let $S''' = S'' \cup \{l\}$. We claim that firm l does not have an incentive to produce, that is $D_l^{S'''}(\mathbf{p}_{S'''}^*(S''')) \leq 0$. This requirement is trivially satisfied from (10) as $c_l > c_h$ and $q_h^*(S^{''}) = 0$, which implies that $D_h^{S^{''}}(\mathbf{p}_{S^{''}}^*(S^{''})) = 0$.

ii) (unconstrained equilibrium) If $D_h^{S''}(\mathbf{p}^*_{S''}(S'')) < 0$ *and* $D_h^{S''}(\mathbf{p}^*_{S'}(S'), p_h = c_h) \leq 0$ *then for all* $i \in S''$, $\widehat{q}_i = q_i^*(S')$, $\widehat{p}_i = p_i^*(S')$;

iii) (constrained equilibrium) If $D_h^{S''}(\mathbf{p}^*_{S''}(S'')) < 0$, $D_h^{S''}(\mathbf{p}^*_{S'}(S'), p_h = c_h) > 0$, *then* $D_h^{S''}(\widehat{\mathbf{p}}_{S'}(S'), p_h = c_h) = 0$

Proof: See the Appendix.

Theorem 1 investigates the set of possible undominated equilibria in case only firms in S' are active. It states that if such an equilibrium exists, then there are two possibilities. It might be the case that the most efficient inactive firm (firm h) would face a positive demand if any of the active firms increase its price at $\widehat{\mathbf{p}}_{S'}$ that satisfy $D_h^{S''}(\widehat{\mathbf{p}}_{S'}, p_h = c_h) = 0$ (part iii)). Otherwise, the active firms charge the prices they would if no firms other than the active firms existed in the market, i.e., $\widehat{p}_i = p_i^*(S')$ (parts i) and ii)). The result is intuitive because if firm h was not at the margin of entering but out of the market, then the active firms would not be constrained by firms not in S' when considering small deviations. In this case the first-order conditions of the unconstrained equilibrium would apply, pinning down the equilibrium prices at the unconstrained equilibrium levels.

Example: A simple numerical example helps fixing ideas. There are three firms with marginal costs $c_1 = c_2 = 15$, and $c_3 = 23.2$. The demand is $p_i = 30 - q_i - 0.5(q_j + q_l)$ where i, j, l are three different firms. Thus $\theta = 0.5$, $a_3 = 15, b_3 = 1.5$, $d_3 = 0.5$ and $a_2 = 20, b_2 = 1.33$, $d_2 = 0.67$. Since the first two firms are identical and they are the most efficient, they both produce in equilibrium. If firm three was also active then the naive three-firm equilibrium would apply. But using (8) and (9), a three-firm equilibrium calls for production $q_3 = -0.09 < 0$, so firm three is not active in equilibrium. Is there an unconstrained duopoly equilibrium, where only firms one and two are active? Using (8) and (9), such an equilibrium would call for $p_1 = p_2 = 20$, and $q_1 = q_2 = 20/3$. Then $p_3 = 23.33 - q_3$, and firm three is able to enter the market by charging slightly above its marginal cost of 23.2.

Therefore, if it exits, such an equilibrium (denote it by $(\widehat{p}_1, \widehat{p}_2, \widehat{p}_3)$) must be a constrained equilibrium, where $\widehat{p}_3 = c_3$ and $D_3(\widehat{p}_1, \widehat{p}_2, \widehat{p}_3 = c_3) = a_3 - b_3 c_3 + d_3(\widehat{p}_1 + \widehat{p}_2) = 0$ by iii) in Theorem 1. Then using $a_3 = 15, b_3 = 1.5$, $d_3 = 0.5$, we find that $\widehat{p}_1 + \widehat{p}_2 = 39.6$. By construction, firm three cannot profitably enter if $\widehat{p}_1 + \widehat{p}_2 = 39.6$ because firm three obtains a zero demand when he charges c_3. We now show that there exists a constrained equilibrium in which firms one and two are active and those equilibria are of the form [10]

$$19.69 \leq \widehat{p}_1, \widehat{p}_2 \leq 19.91 \text{ and } \widehat{p}_1 + \widehat{p}_2 = 39.6.$$

First, firm one should not have an incentive to increase its price and let firm three in[11] (this implies $p_1 \leq 19.69$), nor should it have the incentive to reduce his price[12] and steal market from firm 2 (this implies $p_1 \geq 19.91$). To explain equilibrium incentives, take the pair $(19.8, 19.8)$, which constitutes an equilibrium. By construction, the left hand derivative of the profit function π_1 with respect to p_1 is positive, while the right hand derivative is negative. It is not worth charging a price lower than 19.8 because then only customers from firm two can be attracted. It is not worth charging a higher price either because then customers may defect to *both* firms two and three. This kink in the profit function is the key property that makes *multiple equilibria* possible.

It is useful to vary c_3 and trace out the resulting equilibrium types. First, if $c_3 \geq 23.33$, then the unconstrained duopoly equilibrium of firms one and two ($p_1 = p_2 = 20$) constitutes an equilibrium. This holds because firm three faces a demand of $q_3 = 23.33 - p_3$, so firm three is not able to upset such an equilibrium when $c_3 > 23.33$. Second, we can show that firm three is active in equilibrium if and only if $c_3 < 23.08$ holds.[13] In the intermediate region where $c_3 \in (23.08, 23.33)$ the (multiple) equilibria are constrained, and in all such equilibria $D_3^N(\widehat{p}_1, \widehat{p}_2, \widehat{p}_3 = c_3) = a_3 - b_3c_3 + d_3(\widehat{p}_1 + \widehat{p}_2) = 0$.

2.1 Equilibrium Analysis

We provide an algorithm that constructively finds the set of active firms in equilibrium.

Bertrand Iteration Algorithm (B.I.A.): Let $N_i = \{1, 2, ..., i\}$.

Step 1: If $q_2^*(N_2) < 0$, then $N^* = N_1$. Otherwise proceed to the next step.

[10] There are only constrained equilibria in this case.

[11] This condition boils down to $q_1 - b_3(p_1 - c_1) \leq 0$, see the argument before Lemma 1.

[12] This condition boils down to $q_1 - b_2(p_1 - c_1) \geq 0$.

[13] This is shown by calculating the naive equilibria for all three firms, and setting $q_3 > 0$.

Step 2: If $q_3^*(N_3) < 0$, then $N^* = N_2$. Otherwise proceed to the next step.

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Step k: If $q_{k+1}^*(N_{k+1}) < 0$, then $N^* = N_k$. Otherwise proceed to the next step.

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Step n-1: If $q_n^*(N) < 0$, then $N^* = N_{n-1}$. Otherwise $N^* = N$.

The algorithm eventually stops and the set N^* is determined. Our next result characterizes the set of firms that are active in equilibrium.

Theorem 2. *Let $N^* = \{1, 2, ..., n^*\}$ be the set identified by the algorithm. Apart from the knife-edge equilibrium, the set of active firms is N^* in any undominated equilibrium*[14]*.*

Proof: See the Appendix.

The algorithm explicitly assumes that the most efficient firms are active, a necessary condition for any equilibrium. Suppose that the algorithm selects the first k firms for set N^*. This means that there is a naive-equilibrium in the market with k firms such that they all enter, but there is no such equilibrium in the market with the first $k+1$ firms. If the first k firms can play their unconstrained equilibrium without firm $k+1$ having an incentive to enter, then the result is immediate as all the other inactive firms can be safely ignored. If firm $k+1$ is not too inefficient, then it would enter if the first k firms played their unconstrained (naive) equilibrium strategies. In this case it seems reasonable, and is suggested by our numerical example, that there is an equilibrium where the first k firms all decrease their prices just to keep firm $k+1$ out.

The above argument provides an intuition for why an equilibrium exists, in which the first k firms enter. It is more difficult to rule out equilibria where a

[14] $q_{n^*}^* = 0$ is the knife-edge case. In such a case, the set of active firms is $N^* \setminus n^*$.

different set of firms are active. First, it is clear that there cannot be two unconstrained equilibria with different set of firms active. This follows from comparing naive equilibria with different number of firms. The novelty is to prove that there cannot be multiple constrained equilibria, or one unconstrained and at least one constrained equilibrium with different number of firms active. We cannot use supermodularity to argue this (see Theorem 5), but we can show that equilibria with more active firms feature lower prices on aggregate. This property is sufficient to pin the set of active firms down.

Next we prove conditions under which an unconstrained equilibrium exists, and provide full characterization for this case:

Theorem 3. *Let $\tilde{N} = N^* \cup \{n^*+1\}$. An unconstrained equilibrium exists if and only if*

$$D^{\tilde{N}}_{n^*+1}(p^*_1(N^*), p^*_2(N^*), ..., p^*_{n^*}(N^*), p^*_{n^*+1}(N^*) = c_{n^*+1}) \leq 0. \tag{11}$$

In such an unconstrained equilibrium each firm $i \in N^*$ charges price $p^*_i(N^*)$ and firm $j \in N \setminus \tilde{N}$ charges $p_j \geq c_j$ and $q_j = 0$ for all $j \in S \backslash N \setminus \tilde{N}$.

Proof: The result is a straightforward consequence of Theorems 1 and 2.

We turn to the more interesting case where the equilibrium is constrained. By Theorem 1, if an equilibrium is not unconstrained or not at the knife-edge, it can only be a constrained one. By Theorem 2, the set of active firms can be N^* in such a constrained equilibrium. A necessity condition for a constrained equilibrium to occur is that $\widehat{p}_{n^*+1} = c_{n^*+1}$ and

$$\textbf{Condition 1:} \qquad D^{\tilde{N}}_{n^*+1}(\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_{n^*}, \widehat{p}_{n^*+1} = c_{n^*+1}) = 0, \tag{12}$$

which means that the most efficient inactive firm n^*+1 is indifferent between entering or not. Recall that (12) implies that if firm $i \in N^*$ decreases its price, then firm n^*+1 does not produce, but if i slightly increases its price then the production of firm n^*+1 becomes positive. This implies that the profit function of firm $i \in N^*$ exhibits a kink in p_i at the candidate equilibrium price vector as we discussed before Lemma 1. Unfortunately, Condition 1 is not sufficient for a constrained equilibrium to exit. For $(\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_{n^*})$ to be an constrained equilibrium price vector, we further need that the right hand derivative of firm i's profit with respect to p_i must be negative, and the left hand derivative must be positive. Formally, for each $i \in N^*$,

$$\textbf{Condition 2:} \qquad \frac{\partial \pi_i^R(\tilde{N})}{\partial p_i} \Big|_{\widehat{\mathbf{p}}_{N^*}, p_{n^*+1}=c_{n^*+1}} < 0 \quad \text{and} \quad \frac{\partial \pi_i^L(N^*)}{\partial p_i} \Big|_{\widehat{\mathbf{p}}_{N^*}} > 0. \tag{13}$$

The first derivative in Condition 2 translates to $p_i \leq \overline{p}_i$, while the second translates to $p_i \geq \underline{p}_i$ where

$$\overline{p}_i = \frac{(b_{n^*} + d_{n^*})c_{n^*+1} + b_{n^*}c_i}{2b_{n^*} + d_{n^*}}, \tag{14}$$

and

$$\underline{p}_i = \frac{(b_{n^*+1} + d_{n^*+1})c_{n^*+1} + b_{n^*+1}c_i}{2b_{n^*+1} + d_{n^*+1}} \tag{15}$$

as we show it in the proof. We further provide the following two critical cutoff values for c_{n^*+1}, which allow the constrained equilibrium to exist. These levels are

$$\underline{c}_{n^*+1} = \frac{A(1-\theta)(2+\theta(2n^*-1))+\theta(1+\theta(n^*-1))c_T(N^*)}{(1+\theta n^*)(2+\theta(n^*-3))+\theta^2} \tag{16}$$

and

$$\bar{c}_{n^*+1} = \frac{A(1-\theta)(2+\theta(2n^*-3))+\theta(1+\theta(n^*-2))c_T(N^*)}{(1+\theta(n^*-1))(2+\theta(n^*-3))} \tag{17}$$

Our first main result is stated in Theorem 4. We provide two characterizations of constrained equilibrium price vectors of any given game. In the first one, we show that Conditions 1 and 2 are necessary and sufficient for a constrained equilibrium to exist. In the second characterization, we show that there exists an open interval $I = (\underline{c}_{n^*+1}, \bar{c}_{n^*+1})$ such that $c_{n^*+1} \in I$ and $c_l > c_{n^*+1}$ with $l \in N \setminus \tilde{N}$ imply that the equilibrium is constrained.

Theorem 4. *A price vector $(\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_{n^*})$ is a constrained equilibrium price vector for active firms if and only if it satisfies (12) and $\widehat{p}_i \in [\underline{p}_i, \overline{p}_i]$ for all $i \leq n^*$. Moreover, a constrained equilibrium exists if and only if $c_{n^*+1} \in (\bar{c}_{n^*+1}, \underline{c}_{n^*+1})$ for $c_l > c_{n^*+1}$ with $l \in N \setminus \tilde{N}$.*

Proof: See the Appendix.

The second characterization proves that constrained equilibria occur for a large set of parameter configurations. For example, it occurs if $c_3 \in (23.08, 23.33)$ in the example at the end of Section 2. In those situations, firm c_{n^*+1} is strong enough to upset the unconstrained equilibrium of the firms in N^* but not strong enough to enter himself.

It is also useful to study the sensitivity of limit pricing strategies to the degree of substitution (θ). Our results are summarized in the following lemma:

Lemma 2. *i)* $\frac{\partial \bar{p}_i}{\partial \theta} > 0$ *and* $\frac{\partial \underline{p}_i}{\partial \theta} > 0$.

ii) $\frac{\partial \underline{c}_{n^*+1}}{\partial \theta} < 0$ *and* $\frac{\partial \bar{c}_{n^*+1}}{\partial \theta} < 0$.

iii) $\underline{c}_{n^*+1} = \overline{c}_{n^*+1}$ *when* $\theta \to 1$ *or* $\theta = 0$*,*

Proof: See the Appendix.

An increase in θ generates two effects. First, the competition among the active firms in N^* are now higher, which will lead the slope of best response functions less steep (i.e., $\frac{\partial^2 BR_i}{\partial p_j \partial \theta} < 0$, $i \neq j$, by (6)). That would lead these firms to have more bundle of limit pricing strategies to keep out firm $n^* + 1$. Essentially, both $\frac{\partial \bar{p}_i}{\partial \theta}$ and $\frac{\partial \underline{p}_i}{\partial \theta}$ are positive by part i. However, there is also a second effect. Following an increase in θ, a higher competition makes it more difficult for firm n^*+1, which is at the verge of entering, to not produce when firms in N^* limit their prices. Therefore, limit pricing region would occur for less possible values of the marginal cost level of firm $n^* + 1$. In particular, when $\theta \to 1$, this region disappears and $\overline{c}_{n^*+1} = \underline{c}_{n^*+1}$. On the other hand, when $\theta = 0$, each firm produces a monopoly product and therefore there cannot be limit pricing by firms. Also in this case, $\overline{c}_{n^*+1} = \underline{c}_{n^*+1}$. The bottom line is that the limit pricing region can only occur when firms produce imperfectly substitutable products.

3 Limit pricing and equilibrium multiplicity

3.1 Multiplicity

Our second main result is the existence of efficiency based multiple limit-pricing (constrained) equilibria when there is more than one active firm in equilibrium.

Theorem 5. *Assume that a constrained equilibrium exists. There are a continuum of constrained equilibrium price vectors* $(\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_{n^*})$ *when* $n^* > 1$. *The constrained equilibrium price vector* $(\widehat{p}_1, \widehat{p}_2, ..., \widehat{p}_{n^*})$ *is unique when* $n^* = 1$.

Proof: See the Appendix.

Multiple equilibria occur for a large set of parameter configurations. For example, in the example in Section 2.1 when $c_3 \in (23.08, 23.33)$ there are multiple (constrained) equilibria.[15] As we discussed before, multiple equilibria is induced by the kink that the profit function exhibits at equilibrium points in constrained equilibria. The existence of multiple equilibria in simple linear Bertrand-models is in sharp contrast with the previous literature, which found a unique Bertrand-equilibrium for a large class of demand functions. The literature assumed that all firms are active in equilibrium, and showed uniqueness by establishing that the Bertrand-game is supermodular. We show that supermodularity no longer holds if some firms may not be active in equilibrium, and best responses may not be positively sloped (or monotone).

Theorem 6. *A linear Bertrand-model with continuous best replies may not be supermodular or satisfy the single-crossing property. Moreover, best replies may be non-monotone.*

Proof of Theorem 6: We prove it by giving an example. Our proof is based on a geometrical construction, which is illustrated in Figure 1. We first provide the multi-equilibria result. In the end, we show that our constructed example satisfies neither supermodularity nor single crossing property.

Step 1: (Multiple equilibria) Let us revisit our example from Section 2.1. Let $N = \{1, 2, 3\}$ and $S = \{1, 2\}$. Let $p_i = 30 - q_i - 0.5 \sum_{j \in N \setminus i} q_j$, and $\mathbf{c} =$

[15] In all such equilibria $D_3^N(p_1, p_2, p_3 = c_3) = a_3 - b_3c_3 + d_3(p_1 + p_2) = 0$ or $p_1 + p_2 = 3c_3 - 30$. Further restriction are given in Theorem 4 in the form of $p_i \in [\underline{p}_i, \overline{p}_i]$ for $i = 1, 2$.

$(15, 15, 23.2)$. Note that $\mathbf{q}^*_N(N) = (6.94, 6.94, -0.09)$ by (9), which is not feasible. By BIA, $N^* = \{1, 2\}$. In the absence of firm three, $\mathbf{p}^*_{N^*}(N^*) = (20, 20)$ (Point O in Figure 1) and $\mathbf{q}^*_{N^*}(N^*) = (6.67, 6.67)$. However, firm three deviates to produce $DC^N_3(\mathbf{p}^*_{N^*}(N^*), p_3 = c_3) = 0.2 > 0$ and firm three deviates to produce. In order for firm three to not produce, equilibrium prices of firms one and two should satisfy $DC^N_3(p^{**}_1, p^{**}_2, p_3 = c_3) = 0$. That is $p^{**}_1 + p^{**}_2 = \frac{b_3c_3 - a_3}{d_3} = 39.6$ from (9). Accordingly, $q_3 = 0$ and $p_3 = c_3$ by (8).

Now define one-dimensional simplex as $\triangle^1 = \{\mathbf{p} \in \mathbb{R}^2 : p_1 + p_2 = 39.6\}$ (see Figure 1). For each $i \in \{1, 2\}$, we create their two-firm and projected three firm best response graphs, i.e, $Gr(BR^S_i)$ and $Gr^{proj}(BR^N_i)$, where in the latter, we let $p_3 = c_3$ in the three firm best response and project it onto $p_1 - p_2$ quadrant as follows:

$$Gr^{proj}(BR^N_i) = \{\mathbf{p} \in \mathbb{R}^2 : p_i = BR^N_i(p_{S\setminus i}, p_3 = c_3)\} \tag{18}$$

Related substitutions using (6) yield $Gr(BR^S_1) = \{\mathbf{p} \in \mathbb{R}^2 : p_1 = (60 + p_2)/4\}$ and $Gr^{proj}(BR^N_1) = \{\mathbf{p} \in \mathbb{R}^2 : p_1 = (49.1 + 0.5p_2)/3\}$. Since firms one and two are symmetric, so do their best responses. $\triangle^1$ is a border for firm three's production. If either firm one or firm two charges a slightly higher price on this simplex given other's prices, some consumers of the higher priced firm would prefer firm three's product. Hence, firm three starts to produce and projected best response graphs become valid. Therefore, whereas the projected triopoly-market best response graphs are valid above the simplex, the duopoly ones are valid below it. We also have the individual rationality constraints stating that $p_1 \geq c_1$ and $p_2 \geq c_2$. The coordinates of the critical intersections of the simplex with best responses and individual rationality constraints are shown in Figure 1 as $A = (15, 24.6)$, $B = (19.68, 19.92)$, $C = (19.69, 19.91)$, $D = (19.91, 19.69)$, $E = (19.92, 19.68)$, and $F = (24.6, 15)$. Let $seg[CD] = \{\mathbf{p} \in \mathbb{R}^2 : 19.69 \leq p_1 \leq 19.91 \text{ and } p_1 + p_2 = 39.6\}$. We have already shown in the example of Section 2.1 that the following claim holds.

Claim: Each $\hat{\mathbf{p}} \in X = \{\mathbf{p} \in \mathbb{R}^3 : (p_1, p_2) \in seg[CD] \text{ and } p_3 = c_3\}$ is an undominated pure-strategy equilibria of the game played among N.

Step 2: (Supermodularity and single-crossing property) We claim that the above example neither satisfies super-modularity nor single-crossing. By construction and from the above proof, when $p_3 = c_3$, the best response graph of firm $i \in S$ is non-monotone and continuous and given by

$$Gr(BR_i) = \begin{cases} Gr(BR_i^S) & \text{if } p_{S\setminus i} < 19.68 \\ \triangle^1 & \text{if } 19.68 \leq p_{S\setminus i} \leq 19.91 \\ Gr^{proj}(BR_i^N) & otherwise \end{cases}$$

which is drawn in Figure 1.

A game with one-dimensional strategy choices is supermodular if and only if it possesses the property of increasing differences. Letting $p_1'' > p_1'$, $p_2'' > p_2'$ this property boils down to the following requirement in our game, fixing the action of firm 3:

$$\pi_1(p_1'', p_2'', p_3) - \pi_1(p_1'', p_2', p_3) \geq \pi_1(p_1', p_2'', p_3) - \pi_1(p_1', p_2', p_3).$$

The milder single crossing condition only requires that

$$\pi_1(p_1', p_2'', p_3) \geq \pi_1(p_1', p_2', p_3) \Rightarrow \pi_1(p_1'', p_2'', p_3) \geq \pi_1(p_1'', p_2', p_3).$$

Both conditions require that if the opponent has chosen a higher action, then I optimally respond with a higher action as well. Let $p_2' = p_1'' = 19.8$, $p_1' = 19.7$ and $p_2'' = 19.9$. In this range firm 1's best reply meets with the simplex $\triangle^1$ and therefore it is given by $39.6 - p_2$. Thus, $\pi_1(p_1', p_2'', p_3) > \pi_1(p_1', p_2', p_3)$ and $\pi_1(p_1'', p_2'', p_3) < \pi_1(p_1'', p_2', p_3)$ and both supermodularity and the single-crossing conditions fail.

□

3.2 Market performance

For welfare analysis, it is important to know how equilibrium multiplicity affects consumer welfare. Take our original example with $c_3 = 23.2$ and let us study which equilibria yield the highest welfare for the consumer. Let $V(p_1, p_2)$ denote the indirect utility function of the representative consumer. It is well-known from the consumer choice literature that V is quasi-convex, and it is easy to show that in our model V is strictly quasi-convex. Therefore, it follows that the consumers like extreme price combinations better than balanced prices. In the context of our set of equilibria, this means the consumers prefer the equilibrium, where $p_1 = 19.69$ and $p_2 = 19.91$ or vice versa over any other equilibrium.

This result has interesting consequences. First, if there is multiple equilibria when the incumbents keep a potential entrant out then it is better for consumer welfare if they choose different prices. For example if the incumbents alternate over time in terms of choosing price combinations so that other firms are kept out, this behavior enhances consumer welfare. Second, as c_3 changes in the range where the equilibria are constrained, there are equilibria for a higher value of c_3 which make the consumers better off than some equilibria that occur when c_3 is lower.

It is straightforward to show that each active firm prefers to be the one with the lower price. However, the sum of equilibrium profits is maximized at the symmetric equilibrium where $p_1 = p_2 = 19.8$. Therefore, if the active firms are able to coordinate over how they wish to keep out potential rivals, then they would choose the equilibrium with equal prices, which incidentally yields the lowest consumer welfare. Finally, total surplus is also highest when the two active firms charge the same price. Therefore, we have an interesting case where maximizing consumer surplus and total surplus over a range of equilibrium alternatives yield polar opposite recommendations.[16]

3.3 Implications for markets with limit pricing

{Do two things here: 1. Revisit the limit pricing literature; 2. Talk about real world examples and how incumbents may need to coordinate to keep out entrants}

4 Conclusion

Price-setting games are an important class of games that have been extensively studied in the literature. Most of the existing literature assumes that all firms are active, and shows uniqueness of equilibrium. However, firms might prefer not to be active in real-life situations. For instance, a cost-reducing innovation or cost-efficient mergers might induce firms to exit the market. In such settings,

[16] If the active firms have asymmetric costs then welfare calculations become more complicated but the main insights still hold as long as the marginal costs are not very different.

Vives (1990) argues that it is natural to expect that the game is supermodular. However, our analysis shows that when the number of firms is greater than two, the game need not satisfy supermodularity or even the single crossing property. Therefore previous existence and uniqueness of equilibrium theorems regarding supermodular games do not apply in our framework. We argue that Bertrand best replies might be negatively sloped and there are (infinite) many undominated Bertrand-Nash equilibria on a wide range of parameter values with more than two firms. As far as we know, our paper is the first that studies price-setting games in the context of potential entrants in a comprehensive way. As compared to quantity-setting games, we discovered that price-setting games provide a rich set of comparative statics that can help us finding links between economic theory and real life problems.

5 Appendix

Proof of Lemma 1: Note that the derivative of π_i with respect to p_i exists at all points where the set of active firms $S(p_i, \mathbf{p}_{-i})$ does not change in a neighborhood of p_i. In this case the derivative is $q_i + (p_i - c_i)\frac{\partial q_i}{\partial p_i} = q_i - b_s(p_i - c_i)$, which is strictly decreasing in p_i. As p_i increases, a new firm k enters the market when $p_i = \widetilde{p}_i$. At that point the left hand derivative of π_i with respect to p_i is still $q_i - b_s(p_i - c_i)$, while the right hand derivative becomes $q_i - b_{s+1}(p_i - c_i)$. Given that $b_{s+1} > b_s$ and $p_i - c_i \geq 0$, we obtain that $q_i - b_{s+1}(p_i - c_i) < q_i - b_s(p_i - c_i)$ so the right hand derivative is strictly lower than the left hand derivative. Therefore, as p_i increases and more and more inactive firms may become active, the derivative of π_i with respect to p_i either exists and is decreasing in a neighborhood or it does not exist, but one sided derivatives always exist, and the right hand derivative is always less than the left hand derivative. Therefore, as long as firm i remains active as it increases its price its profit function is strictly concave in p_i. However, at a point where firm i becomes inactive its profit becomes zero, and the profit stays zero for any p_i higher than that. Therefore, the profit function is quasi-concave in p_i.

Existence of equilibrium follows from standard results. In particular, note that $p_i > A$ and (2) together with the non-negativity of quantities implies that firm i cannot be active. Therefore, charging $p_i > A$ yields a zero profit, so such strategies can be ignored because a zero profit can be also achieved by charging c_i. So, the best reply of firm i always intersects with set $[c_i, A]$, and we can restrict the strategy space of firm i to $[c_i, A]$ without loss. Then we have a quasi-concave, continuous objective functions and convex, compact action spaces, so a pure strategy equilibrium exists.[17] □

Proof of Lemma 2:

i) We note that $\frac{\partial \bar{p}_i}{\partial \theta} = \frac{\partial \underline{p}_i}{\partial \theta} = \frac{(c_{n^*+1} - c)}{2+\theta(2n^*-3)} > 0$ from (14) and (15).

ii) and iii) Consider (16) and (17). We have $\frac{\partial \underline{c}_{n^*+1}}{\partial \theta} = \frac{-(An-c_T)(2+4\theta(n-1)+(2+n(2n-3))\theta^2}{(2+\theta(-3+\theta+n(3+\theta(n-3))))^2} < 0$ and $\frac{\partial \bar{c}_{n^*+1}}{\partial \theta} = \frac{-(An-c_T)(2+\theta(-8+7\theta+n(4+\theta(2n-7))}{(2+\theta(n-3))^2(1+\theta(n-1))^2} < 0$. Moreover, $\underline{c}_{n^*+1} = \bar{c}_{n^*+1}$ when $\theta \to 1$ or $\theta = 0$.

□

[17] See for example Theorem 2.2 of Reny (2008).

Proof of Theorem 1:

i) and **ii)** follow from the text.

iii) Take any $S' \subset N$ and let $c_h = \min_{j \in N \setminus S'} c_h$ and $S'' = S' \cup h$. Let $M^{'} = \sum_{i \in S'} p_i^*(S')$. Suppose both that $D_h^{S''}(\mathbf{p}_S^*(S), p_h = c_h) > 0$ and there exists a constrained equilibrium in which only firms in S' are active. We claim that any equilibrium price vector of firms in S', say $\widehat{\mathbf{p}}_{S'}$, satisfy $D_h^{S''}(\widehat{\mathbf{p}}_{S'}, p_h = c_h) = 0$. Let $M = \sum_{i \in S'} p_i^*(S')$. It is clear it cannot be the case that $D_h^{S''}(\widehat{\mathbf{p}}_{S'}, p_h = c_h) > 0$ for firm h to be inactive. Therefore suppose for a contradiction there exists an equilibrium price vector $\tilde{\mathbf{p}}_{S'}$ such that $D_h^{S''}(\tilde{\mathbf{p}}_{S'}, p_h = c_h) < 0$. That implies that $\sum_{i \in S'} \tilde{p}_i = M^{''} < M < M^{'}$ by (3). Now take any $j \in S'$. Since $M^{''} < M$, then for sufficiently small ϵ, any price deviation in the $\epsilon-$neighbourhood of $\tilde{p}_j$ given $\tilde{\mathbf{p}}_{S \setminus j}$ is still associated with a market where only firms in S actively produce. Hence S'-firm best responses, i.e., $BR_k^{S'}$, $k \in S'$, are still valid. But since $\bigcap_{l \in S'} Gr(BR_l^{S'}) = \mathbf{p}_{S'}^*(S')$ by definition of unconstrained equilibrium and $M^{''} < M^{'}$, then $\tilde{\mathbf{p}}_{S'}$ cannot be an equilibrium price vector trivially, which is a contradiction. $\square$

Proof of Theorem 2:

Let $N^* = \{1, 2, ..., n^*\}$ be the set of firms found by B.I.A.. Define

$$\upsilon_L = \frac{a_l(2b_l + d_l) + b_l d_l c_T(L)}{(b_l + d_l)(2b_l - d_l(l-1))} \tag{19}$$

for some $L \subset N$ with $\mid L \mid = l$. Throughout the proof let (p_k^*, q_k^*) denote a generic naive equilibrium price quantity pair for firm $k \in N$. Let (p_k^{**}, q_k^{**}) denote a generic unconstrained, and let (p_k^{***}, q_k^{***}) denote a generic constrained equilibrium price and quantity pair for firm $k \in N$.

We prove the result in three steps.

Step 1: In this step we prove that the set of active firms is such that if j is active and $c_i < c_j$, then i is active as well. Suppose not. Then we claim that firm i could charge a price equal to the equilibrium price of j, p_j and achieve a positive profit. Since $p_j < c_j < c_i$, we only need to prove that firm i obtains a positive demand when he charges p_j. Firm i's demand is equal to firm j's demand by symmetry, so we need to only show that firm j's demand remains positive when firm i switches to price p_j from its previous price that was above p_j. If this was not the case, then firm j's demand would change when firm i lowered his price without firm i being able to enter. But this is impossible because if firm i cannot enter even with the lower price p_j, then the price change of i does not change the demand of the firms active before i's price change, and in particular the demand of firm j, a contradiction.

Step 2: Let $1 \leq \widehat{n} \leq n^*$, and let $Y = \{1, 2, ..., \widehat{n}\}$ and $Z = Y \cup \{\widehat{n}+1\}$. We claim that there cannot be an equilibrium where only firms in Y are active. Assume by contradiction that there is.

2-i) Unconstrained Equilibrium: We have $(p_k^{**}, q_k^{**})_{k\in Y} = (p_k^*, q_k^*)_{k\in Y}$, that is unconstrained equilibrium prices are by definition the same as the naive equilibrium prices. Further, note that $q^*_{\widehat{n}+1}(Z) > 0$ by B. I. A. Hence using (8) and (9), we get

$$c_{\widehat{n}+1} < \upsilon_Z \tag{20}$$

Also remark that,

$$DC^Y_{\widehat{n}+1}(\mathbf{p}^*_Y(Y)) = a_{\widehat{n}+1} - b_{\widehat{n}+1}c_{\widehat{n}+1} + d_{\widehat{n}+1}\sum_{k=1}^{\widehat{n}} p_k. \tag{21}$$

from (10). Substitute $p_k^*(Y)$, $k \in Y$ into (21) from (8) and then subtract $b_{\widehat{n}+1}(\upsilon_Z - c_{\widehat{n}+1})$ from (21) to have

$$DC^Y_{\widehat{n}+1}(\mathbf{p}^*_Y(Y)) - b_{\widehat{n}+1}(\upsilon_Z - c_{\widehat{n}+1}) = \tag{22}$$

$$= \frac{\theta^3(A\widehat{n} - c_T(Y))}{(1+\theta\widehat{n})(1+\theta(\widehat{n}-1))(2+\theta(3((\widehat{n}-1)+\theta(1+\widehat{n}(\widehat{n}-1)))))} > 0.$$

Comparing (20) and (22) gives that $DC^Y_{\widehat{n}+1}(\mathbf{p}^*_Y(Y)) > 0$ by (10) and thus $q^*_{\widehat{n}+1} > 0$, which is a contradiction.

2-ii) Constrained (Limit Pricing) Equilibrium: To ensure that firm $\widehat{n}+1$ is priced out of the market, constrained equilibrium prices of firms in Y satisfy $\sum_{i\in Y} p_i^{***} = \frac{b_{\widehat{n}+1}c_{\widehat{n}+1} - a_{\widehat{n}+1}}{d_{\widehat{n}+1}}$ and we have $DC^Y_{\widehat{n}+1}(\mathbf{p}^{***}_Y) = 0$ and $q^{***}_{\widehat{n}+1} = 0$ from (3).

We claim that there exists a firm $j \in Y$ such that $p_j^{***} < \frac{(b_{\widehat{n}+1}+d_{\widehat{n}+1})c_{\widehat{n}+1} + b_{\widehat{n}+1}c_j}{2b_{\widehat{n}+1}+d_{\widehat{n}+1}}$.

Otherwise, summing up equilibrium prices of all firms in Y yields

$$\sum_{i\in Y} p_i^{***} = \frac{b_{\widehat{n}+1}c_{\widehat{n}+1} - a_{\widehat{n}+1}}{d_{\widehat{n}+1}} \geq \frac{(b_{\widehat{n}+1} + d_{\widehat{n}+1})\widehat{n}c_{\widehat{n}+1} + b_{\widehat{n}+1}c_T(Y)}{2b_{\widehat{n}+1} + d_{\widehat{n}+1}}. \tag{23}$$

But (23) can be rewritten as $c_{\widehat{n}+1} \geq \upsilon_Z$, which is a direct contradiction to (20).

Using that there exists a firm $j \in Y$ such that $p_j^{***} < \frac{(b_{\widehat{n}+1}+d_{\widehat{n}+1})c_{\widehat{n}+1}+b_{\widehat{n}+1}c_j}{2b_{\widehat{n}+1}+d_{\widehat{n}+1}}$ it is straightforward to show that firm j has an incentive to increase its price. Hence there cannot be any constrained equilibrium where only firms in Y are active either.

Step 3: Suppose $n^* < n$, and let $1 < n^* < t$, with $T = \{1, 2, ..., t\}$. To finish the proof, we claim that there cannot be any equilibrium where only firms in T are active. Assume by contradiction that there is and we consider constrained or unconstrained equilibria again.

3-i) Unconstrained Equilibrium: By B.I.A., $q_{n^*+1}(N^* \cup (n^* + 1)) \leq 0$. It is sufficient to show that if $q_x^*(X) \leq 0$ for some $X = \{1, 2, ..., x\}$ then for all $x' > x$, $X' = \{1, 2, ..., x'\}$ it holds that $q_{x'}^*(X') < 0$. We prove the claim by induction. Assume that for some $X \subset N$, we have $q_x^*(X) \leq 0$. We need to show that $q_{x+1}^*(X \cup \{x + 1\}) < 0$ holds. Using (8) and (9) and $q_x^*(X) \leq 0$, we obtain

$$c_x > \overline{c} = \frac{a_x(2b_x + d_x) + b_x d_x c_T(X\backslash x)}{(b_x + d_x)(2b_x - d_x(x - 1)) - b_x d_x}. \tag{24}$$

By construction, when $c_x = \overline{c}$, it holds that $q_x^*(X) = 0$.

Let $c_{x+1} = c_x + \xi$ for some $\xi > 0$. Using (8) and (9), we obtain

$$\frac{\partial q_{x+1}^*(X \cup \{x + 1\})}{\partial c_x} = -\frac{2 + \theta(3x - 4 + \theta(2 + x(x - 4)))}{(2 - 2\theta + x\theta)(2 - \theta + 2x\theta)} < 0, \tag{25}$$

and

$$\frac{\partial q_{x+1}^*(X \cup \{x + 1\})}{\partial \xi} = -\frac{2 + \theta(\theta - 3 + x(3 + \theta(x - 3)))}{(2 - 2\theta + x\theta)(2 - \theta + 2x\theta)} < 0. \tag{26}$$

Hence $q_{x+1}^*(X\cup\{x+1\})$ takes its maximum when c_x and ξ take their lowest possible values allowed (25) and (26). Those lowest possible values correspond to $c_x = \overline{c}$ and $\xi = 0$. Substituting those values, and using (9) yields upon simplifications that

$$q^*_{x+1}(X\cup\{x+1\}) \leq -\frac{b_{x+1}\theta^3(1-\theta)(Ax-A-C_T(X\backslash x))}{(2+\theta(x-2))(2+\theta(2x-1))(2+\theta(5\theta-6+x(3+(x-5)\theta)))},$$

which is negative if $A > c_i$, $\theta \in (0,1)$ and $x \geq 2$ desired. This completes the inductive step.

3-ii) Constrained (Limit Pricing) Equilibrium: Let $X'' = X \cup \{x+1\}$ assuming that there is a constrained equilibrium where the set of active firms is $X = \{1, 2, ..., x\}$ with $x > n^*$. In a constrained equilibrium, equilibrium prices of firm in X satisfy $\sum\limits_{i\in X} p_i^{***} = \frac{b_{x+1}c_{x+1}-a_{x+1}}{d_{x+1}}$, that is $DC^X_{x+1}(\mathbf{p}^{***}_X) = 0$ and we also have $q^{***}_{x+1} = 0$ by (3). Using (8) and (9), and Step 3-i, $q^*_x(X) < 0$ simplifies to $c_x > \upsilon_X$ and thus

$$c_{x+1} > \upsilon_X. \tag{27}$$

We claim that there exists a firm $k \in X$ such that $p_k^{***} > \frac{(b_x+d_x)c_x+b_xc_k}{2b_x+d_x}$. Otherwise, summing up equilibrium prices of all firms in X and reorganizing terms yields a contradiction to (27). The firm k for whom $p_k^{***} > \frac{(b_x+d_x)c_x+b_xc_k}{2b_x+d_x}$ has an incentive to decrease its price by following the same arguments as above. Hence there is no constrained equilibrium in which the set of active firms is X. Since x was an arbitrary integer such that $x > n^*$, our proof is now complete. ■

Proofs of Theorems 4 and 5:

Let $N^* = \{1, 2..., n^*\}$ be found by B.I.A. and denote $\tilde{N} = N^* \cup \{n^*+1\}$. Let $(\widehat{p}_k, \widehat{q}_k)$ denote equilibrium price and quantity pair of any firm $k \in N$. If $N^* = N$, then $(\widehat{p}_k, \widehat{q}_k) = (p^*_k(N), q^*_k(N))$ trivially. Therefore assume $N^* \subset N$. We consider three cases.

Case 1: Knife-edge Equilibrium If $q^*_{n^*}(N^*) = 0$, then we have $D^{N^*}_{n^*}(\mathbf{p}^*_{N^*}(N^*)) = 0$ by (3). Consider any $i \in N$ such that $i > n^*$. But by definition $D^{N^*}_i(\mathbf{p}^*_{N^*\backslash n^*}(N^*)) = D^{N^*}_{n^*}(\mathbf{p}^*_{N^*}(N^*)) + b_{n^*}(c_{n^*} - c_i) < 0$ for $\theta \in (0,1)$. Therefore $q^{**}_i = 0$ and for each $j \in N^*$, $(p^{**}_j, q^{**}_j) = (p^*_j(N^*), q^*_j(N^*))$.

Case 2: Unconstrained Equilibrium Suppose that $q^*_{n^*}(N^*) > 0$ and $D^{\tilde{N}}_{n^*+1}(\mathbf{p}^*_{N^*}(N^*), p_{n^*+1} = c_{n^*+1}) \leq 0$. Take any $l \in N$ such that $l > n^*+1$. But as in Case 1, $D^{\tilde{N}}_l(\mathbf{p}^*_{N^*}(N^*)) = D^{\tilde{N}}_{n^*+1}(\mathbf{p}^*_{N^*}(N^*)) + b_{n^*+1}(c_{n^*+1} - c_l) < 0$ for $\theta \in (0,1)$ under our supposition. Accordingly, $\hat{q}_l = 0$ and for each $j \in N^*$, $(\hat{p}_j, \hat{q}_j) = (p^*_j(N^*), q^*_j(N^*))$.

Case 3: Constrained Equilibria Finally remark that $q^*_{n^*+1}(\tilde{N}) < 0$ by the B.I.A. Suppose now that $D^{\tilde{N}}_{n^*+1}(\mathbf{p}^*_{N^*}(N^*), p_{n^*+1} = c_{n^*+1}) \leq 0$. Let $E^C_{N^*}(c_{n^*+1})$ denotes the set of all constrained equilibrium price vectors where only firms in N^* are active. In this case, we claim that $E^C_{N^*}(c_{n^*+1}) = E(c_{n^*+1})$, where

$$E = \{\mathbf{p}_{N^*} \in \mathbb{R}^{n^*}_+ : \mathbf{p}_{N^*} \in \triangle^{n^*-1} \text{ and for each } i \in N^*, \underline{p}_i \leq p_i \leq \bar{p}_i\} \tag{28}$$

where

$$\begin{aligned} \bar{p}_i &= \tfrac{(b_{n^*}+d_{n^*})c_{n^*+1}+b_{n^*}c_i}{2b_{n^*}+d_{n^*}} \\ \underline{p}_i &= \tfrac{(b_{n^*+1}+d_{n^*+1})c_{n^*+1}+b_{n^*+1}c_i}{2b_{n^*+1}+d_{n^*+1}} \end{aligned} \tag{29}$$

and the simplex is constructed from (12) and (3) as

$$\triangle^{n^*-1} = \{\mathbf{p}_{N^*} \in \mathbb{R}^{n^*}_+ : \sum_{i \in N^*} p_i = \frac{b_{n^*+1}c_{n^*+1} - a_{n^*+1}}{d_{n^*+1}}\} \tag{30}$$

We also define two vectors $\mathbf{A} = (\bar{p}_i(c_{n^*+1}))_{(i \in N^*)}$ and $\mathbf{B} = (\underline{p}_i(c_{n^*+1}))_{(i \in N^*)}$. Let $W = conhull[A, B]$. Note that $E(c_{n^*+1}) = W \cap \triangle^{n^*-1}$ from (28). To prove all the claims, we proceed in four steps. In Step 1, we show that the marginal cost level of the firm n^*+1 lies on the interval $I = (\underline{c}_{n^*+1}, \bar{c}_{n^*+1})$. Now consider any $c_{n^*+1} \in I$. In Step 2, we show that the critical price vectors are at the firm rational region, i.e., $\bar{p}_i(c_{n^*+1}) > \underline{p}_i(c_{n^*+1}) > c_i$. In Step 3, we derive that $E(c_{n^*+1})$ is well-defined. Moreover, this set is singleton when $n^* = 1$ and it is multiple when $n^* > 1$. But then, the set $E(c_{n^*+1})$ coincidences with the set $E^C_{N^*}(c_{n^*+1})$ by Step 4, as desired.

Step 1: For each $c_l > c_{n^*+1}$, $l \in N \setminus \tilde{N}$, we have $\underline{c}_{n^*+1} < c_{n^*+1} < \bar{c}_{n^*+1}$, where $\underline{c}_{n^*+1}$ and $\bar{c}_{n^*+1}$ are respectively defined in (16) and (17).

Proof of Step 1: Let $\underline{c}_{n^*+1}$ be such that $q^*_{n^*+1}(\tilde{N}) = 0$ if we had $c_{n^*+1} = \underline{c}_{n^*+1}$ ceteris paribus. But then $p^*_{n^*+1}(\tilde{N}) = \underline{c}_{n^*+1}$ and solving for $\underline{c}_{n^*+1}$ by using (8) yields (16). Moreover, since $q^*_{n^*+1}(\tilde{N}) < 0$ by B.I.A., then $c_{n^*+1} > \underline{c}_{n^*+1}$ from (9). Next let $\bar{c}_{n^*+1}$ be such that if we had $c_{n^*+1} = \bar{c}_{n^*+1}$, then $D^{\tilde{N}}_{n^*+1}(\mathbf{p}^*_{N^*}(N^*)) = 0$. Accordingly, $\bar{c}_{n^*+1}$ solves $D^{\tilde{N}}_{n^*+1}(\mathbf{p}^*_{N^*}(N^*), p_{n^*+1} = \bar{c}_{n^*+1}) = 0$ in (3), which is given by (17). But note that $\partial D^{\tilde{N}}_{n^*+1}(.)/\partial c_{n^*+1} = -b_{n^*+1} < 0$ and $D^{\tilde{N}}_{n^*+1}(\mathbf{p}^*_{N^*}(N^*), p_{n^*+1} = c_{n^*+1}) > 0$ by the initial supposition. Therefore, $c_{n^*+1} < \bar{c}_{n^*+1}$. Altogether, $\underline{c}_{n^*+1} < c_{n^*+1} < \bar{c}_{n^*+1}$.

Step 2 (Firm Rationality Constraint): For each $i \in N^*$, $\bar{p}_i(c_{n^*+1}) > \underline{p}_i(c_{n^*+1}) > c_i$.

Proof of Step 2: Take any $i \in N^*$. First note that $c_i < c_{n^*+1}$ by B.I.A.. Hence $\underline{p}_i(c_{n^*+1}) > c_i$ from (29). Next, define $Z_i = \bar{p}_i(c_{n^*+1}) - \underline{p}_i(c_{n^*+1})$. By using (29), we have $\frac{\partial Z_i}{\partial c_i} = -\frac{\theta^2}{(2+\theta(2n^*-1))(2+\theta(2n^*-3))} < 0$, which is negative for $\theta \in (0,1)$. Additionally it can be shown that $Z_i = 0$ if we had $c_i = c_{n^*+1}$. But since $c_i < c_{n^*+1}$, then $Z_i > 0$ by the above derivative calculation as desired. Altogether, the claim is proven.

Step 3: (Non-emptiness) $E = W \cap \vartriangle^{n^*-1}$ is non-empty. Moreover, if $n^* = 1$, then E is singleton. Otherwise, E is multiple.

Proof of Step 3:

To see that E is non-empty, it is sufficient to show that $\sum_{i \in N^*} \bar{p}_i(c_{n^*+1}) > M = \frac{b_{n^*+1}c_{n^*+1} - a_{n^*+1}}{d_{n^*+1}} > \sum_{i \in N^*} \underline{p}_i(c_{n^*+1})$. First let $T(c_{n^*+1}) = \sum_{i \in N^*} \bar{p}_i(c_{n^*+1}) - M$, where for $i \in N^*$, $\bar{p}_i(c_{n^*+1})$ is given by (29). Note that $\frac{\partial T(.)}{\partial c_{n^*+1}} = -\frac{(2+\theta(n^*-3))(1+\theta(n^*-1))}{\theta(2+\theta(2n^*-3))} < 0$ for $\theta \in (0,1)$. Moreover, $T(\bar{c}_{n^*+1}) = 0$ where $\bar{c}_{n^*+1}$ is provided in (17). But since $c_{n^*+1} < \bar{c}_{n^*+1}$ and $\frac{\partial T(.)}{\partial c_{n^*+1}} < 0$, then $T(c_{n^*+1}) > 0$ as desired.

Similarly define $V(c_{n^*+1}) = M - \sum_{i \in N^*} \underline{p}_i(c_{n^*+1})$ where for $i \in N^*$, $\underline{p}_i(c_{n^*+1})$ is given by (29). It can be shown that $\frac{\partial V(.)}{\partial c_{n^*+1}} = \frac{2+3\theta(n^*-1)+\theta^2(n^*(n^*-3)+1)}{\theta(2+\theta(2n^*-1))} > 0$ for $\theta \in (0,1)$. Additionally $T(\underline{c}_{n^*+1}) = 0$ where $\underline{c}_{n^*+1}$ is provided in (16). However, since $c_{n^*+1} > \underline{c}_{n^*+1}$ and $\frac{\partial V(.)}{\partial c_{n^*+1}} > 0$, then $V(c_{n^*+1}) > 0$ as desired.

Finally note that if $n^* = 1$, $\bar{p}_1(c_{n^*+1}) > \underline{p}_1(c_{n^*+1})$ and both $\vartriangle^0$ and $E(c_{n^*+1})$ are clearly singleton. Moreover, if $n^* \geq 2$, then for each $i \in N^*$, $\bar{p}_i(c_{n^*+1}) > \underline{p}_i(c_{n^*+1})$ by Step 2. Therefore, $E(c_{n^*+1}) = W \cap \vartriangle^{n^*-1}$ is multiple by construction.

Step 4: (Multiple Constrained Equilibria) $E^C_{N^*}(c_{n^*+1}) = E(c_{n^*+1})$.

Proof of Step 4: We prove the claim in two steps.

Step 4.1: $\acute{\mathbf{p}}_{N^*} \in E(c_{n^*+1})$ if and only if for each $i \in N^*$, $\frac{\partial \pi_i^R(\tilde{N})}{\partial p_i} |_{\mathbf{p}_{N^*}=\acute{\mathbf{p}}_{N^*}} < 0$ and $\frac{\partial \pi_i^L(N^*)}{\partial p_i} |_{\mathbf{p}_{N^*}=\acute{\mathbf{p}}_{N^*}} > 0$.

Proof of Step 4.1:

($\Rightarrow$) Take any firm $i \in N^*$ and consider any price vector $\acute{\mathbf{p}}_{N^*} \in E(c_{n^*+1})$. Since $\acute{\mathbf{p}}_{N^*} \in \Delta^{n^*-1}$ also, then $p_{n^*+1} = c_{n^*+1}$ by construction.

We first claim that $\frac{\partial \pi_i^R(\tilde{N})}{\partial p_i} |_{\mathbf{p}_{N^*}=\acute{\mathbf{p}}_{N^*}} < 0$. Note that $\frac{\partial \pi_i^R(\tilde{N})}{\partial p_i} |_{\mathbf{p}_{N^*}=\acute{\mathbf{p}}_{N^*}} = a_{n^*+1} + d_{n^*+1}(\sum_{l\in N^*} \acute{p}_l - \acute{p}_i + c_{n^*+1}) + b_{n^*+1}c_i - 2b_{n^*+1}\acute{p}_i$ by derivating (4) w.r.t. p_i and calculating it at $\mathbf{p}_{N^*} = \acute{\mathbf{p}}_{N^*}$ and $p_{n^*+1} = c_{n^*+1}$. But since $\acute{\mathbf{p}}_{N^*} \in E(c_{n^*+1})$, then substituting $\sum_{l\in N^*} \acute{p}_l = \frac{b_{n^*+1}c_{n^*+1}-a_{n^*+1}}{d_{n^*+1}}$ into this last equality gives

$$\frac{\partial \pi_i^R(\tilde{N})}{\partial p_i} |_{\mathbf{p}_{N^*}=\check{\mathbf{p}}_{N^*}} = (b_{n^*+1} + d_{n^*+1})c_{n^*+1} + b_{n^*+1}c_i - (2b_{n^*+1} + d_{n^*+1})\acute{p}_i \tag{31}$$

Remark also that $\acute{\mathbf{p}}_{N^*} \in E(c_{n^*+1})$ implies that $\acute{p}_i > \bar{p}_i(c_{n^*+1}) = \frac{(b_{n^*+1}+d_{n^*+1})(c_{n^*+1})+b_{n^*+1}c_i}{2b_{n^*+1}+d_{n^*+1}}$ by definition. Therefore $\frac{\partial \pi_i^R(\tilde{N})}{\partial p_i} |_{\mathbf{p}_{N^*}=\acute{\mathbf{p}}_{N^*}} < 0$ from (31), which proves the first part of the claim.

We next claim that $\frac{\partial \pi_i^L(N^*)}{\partial p_i} |_{\mathbf{p}_{N^*}=\acute{\mathbf{p}}_{N^*}} > 0$ as well. Noting that $\frac{\partial \pi_i^L(N^*)}{\partial p_i} |_{\mathbf{p}_{N^*}=\acute{\mathbf{p}}_{N^*}} = a_{n^*} + d_{n^*}(\sum_{l\in N^*} \underline{p}_l(c_{n^*+1}) - \acute{p}_i) + b_{n^*}c_i - 2b_{n^*}\acute{p}_i$ and substituting $\sum_{l\in N^*} \acute{p}_l = \frac{b_{n^*+1}c_{n^*+1}-a_{n^*+1}}{d_{n^*+1}}$ by using the fact that $\acute{\mathbf{p}}_{N^*} \in E(c_{n^*+1})$ yields

$$\frac{\partial \pi_i^L(N^*)}{\partial p_i} |_{\mathbf{p}_{N^*}=\hat{\mathbf{p}}_{N^*}} = \frac{b_{n^*+1}d_{n^*}c_{n^*+1} + b_{n^*}d_{n^*+1}c_i}{d_{n^*+1}} - (2b_{n^*} + d_{n^*})\hat{p}_i \tag{32}$$

But we have $i \in N^*$, we have $\acute{p}_i < \underline{p}_i = \frac{d_{n^*}b_{n^*+1}c_{n^*+1}+b_{n^*}d_{n^*+1}c_i}{d_{n^*+1}(2b_{n^*}+d_{n^*})}$ from Step 2. Hence (32) is positive as claimed.

($\Leftarrow$) Take any $\tilde{\mathbf{p}}_{N^*} \in \Delta^{n^*-1} \setminus E(c_{n^*+1})$. By definition of set $E(c_{n^*+1})$, there exists a firm $k \in N^*$ such that either $\tilde{p}_k \leq \bar{p}_k$ or $\tilde{p}_j \leq \underline{p}_k(c_{n^*+1})$. If $\tilde{p}_k \leq \bar{p}_k$, then a symmetric argument to the first part of ($\Rightarrow$) shows that $\frac{\partial \pi_k^L(\tilde{N})}{\partial p_k} |_{\mathbf{p}_{N^*}=\tilde{\mathbf{p}}_{N^*}} \geq 0$. However if $\tilde{p}_k \leq \underline{p}_k(c_{n^*+1})$, then a symmetric argument to the second part of ($\Rightarrow$) shows that $\frac{\partial \pi_k^R(N^*)}{\partial p_k} |_{\mathbf{p}_{N^*}=\tilde{\mathbf{p}}_{N^*}} \leq 0$ as claimed.

Step 4.2: $E^C_{N^*}(c_{n^*+1}) = E(c_{n^*+1})$.

Proof of Step 4.2:

By Theorems 1 and 2, there can only be a constrained equilibrium and the set of active firms is N^* in this case. Such an equilibrium would require that each equilibrium price vector is an element of the simplex, i.e., $\triangle^{n^*-1}$. But among those price vectors, only the set $E(c_{n^*+1})$ constitutes the equilibrium by Step 4.1. That is $E^C_{N^*}(c_{n^*+1}) = E(c_{n^*+1})$.

$\square$

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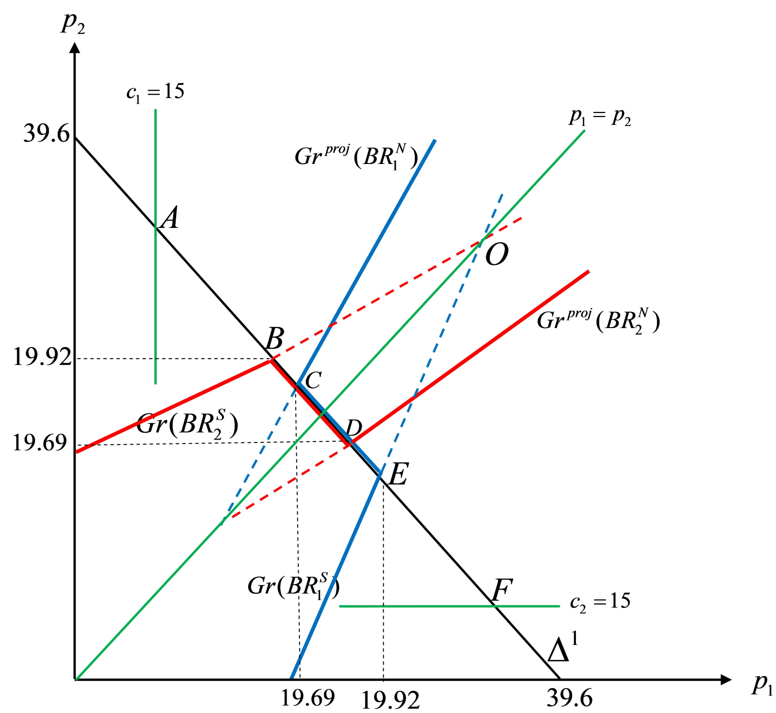

Figure 1: **The Sketch of the Proof of Theorem 6:** Let $N = \{1, 2, 3\}$ and $S = \{1, 2\}$. Let $p_i = 30 - q_i - 0.5 \sum_{j \in N \setminus i} q_j$, and $\mathbf{c} = (15, 15, 23.2)$. We draw best responses of firms one and two when $p_3 = c_3$, which are piecewise linear as shown in the figure. Moreover, they intersect at multiple points showing that each $\hat{\mathbf{p}} \in X = \{\mathbf{p} \in \mathbb{R}^3 : (p_1, p_2) \in seg[CD] \text{ and } p_3 = c_3\}$ is an undominated pure-strategy Bertrand equilibria.