Sense and Suspense: Implementation by Gradual Revelation*

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Abstract

We investigate the feasibility of implementing an allocation rule with a *gradual revelation mechanism*, where the agents reveal their private information over time (rather than all at once). With independently distributed types, private values, and transferable utilities satisfying a single crossing property, an *ex-post monotonicity* condition is sufficient for *budget-balanced* implementation of any *incentive compatible* allocation rule with any gradual revelation scheme. When we extend the single crossing property over the set of *randomized* allocations, a weaker monotonicity condition is both necessary and sufficient for budget-balanced implementation by gradual revelation.

KEYWORDS: Mechanism Design; Sequential Revelations; Budget-balanced Implementation

1 Introduction

In her celebrated novel, *Sense and Sensibility*, Jane Austen writes about the "life and loves" of the Dashwood sisters in the 18th century England. Elinor Dashwood falls in love with Edward Ferrars. The attraction is mutual but Edward is already secretly engaged to another girl, who turns out to be more interested in Edward's family fortune than in Edward himself. At the end of the novel, Edward loses his rights to the family money but ends up marrying Elinor. Obviously, not all love stories are meant to have such happy endings. In the same novel, Elinor's sister Marianne is romantically involved with John Willoughby. But unlike Edward, Willoughby prefers a comfortable lifestyle to the pursuit of true love. Therefore he gets married to a young lady with a large fortune instead of Marianne.

From a cold blooded mechanism design perspective, it is easy to come up with a simple incentive scheme which would generate the above *allocation rule* where idealist Edward would lose his fortune but succeed in love and materialist Willoughby would secure a financially comfortable life but be deprived

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of romance. A *direct revelation mechanism* would ask each of the gentlemen if he is the more idealist or the more materialist type and then assign him to the appropriate spouse.

Although the *revelation principle* is an essential tool in the analysis of design problems, it does not provide a good recipe for writing a literary masterpiece. In Sense and Sensibility, it takes three volumes with more than 300 pages for Elinor and Edward to unite. First, Edward starts acting detached and Elinor comes to the conclusion that he has lost interest in her. Then she learns of Edward's prior engagement, which he feels compelled to honor. Edward's mother finds out about the engagement as well and disinherits him in disapproval. Once Edward loses the connection to his wealthy family, Edward's fiancé breaks up with him and marries his brother, who is now the sole inheritor of the family estate. Elinor hears about the marriage in Edward's family and assumes that it was Edward who got married. It is almost at the end of the novel that Edward shows up unexpectedly and announces to Elinor that he is not burdened with his earlier engagement or with his obligations to the family any more.

To sustain the suspense throughout the novel, Jane Austen does not reveal the traits of her characters at once. Instead, the revelations are supported by a sequence of decisions made by the characters. These decisions *make sense* from the characters' perspective. Each decision reflects the preferences of the decisionmaker given what he / she knows about the other characters at the time that the decision is made. The decisions not only serve to reveal the qualities of the characters to the audience (and to the other characters in the book) but they also lead to the outcome to be faced at the end of the novel.[1]

In this paper, we investigate the feasibility of implementing an allocation rule with *gradual revelation* schemes such as the one that serves as the foundation of the story line in Sense and Sensibility. We study this question in the context of an economy with independently distributed types, private values, and transferable utilities satisfying a single crossing property. A well known condition which is indispensable – both for implementation by simultaneous revelation as prescribed by the revelation principle and for implementation by gradual revelation that we try to motivate in this paper – is *incentive compatibility*: An implementable allocation rule should provide each agent with the incentive to report his type truthfully under his prior beliefs on the types of the other agents. The results of this paper identify other conditions that guarantee implementation of an allocation rule with a gradual revelation mechanism. These results are presented in three propositions with closely related proofs.

In a recent paper, Ely, Frankel, and Kamenica (2013) discuss the idea that gradual revelation of information has an entertainment value for a Bayesian audience. They give many examples from real life auctions, political primaries, mystery novels, sports events, gambling, and reality TV supporting the contention that receivers of information have preferences over the evolution of their beliefs in time. The model we develop in Section 2 benefits a great deal from their formalization of sequential revelations

[1] For many other examples of strategic thinking in Austen's work, see the intriguing book by Chwe (2013).

as a *stochastic path of beliefs*. Ely, Frankel, and Kamenica identify the revelation sequences that a designer would commit to in order to maximize the "suspense" or the "surprise" that the audience would experience. The current paper complements their work by detailing how such a revelation sequence can be supported by the behavior of self-interested agents instead of the actions of a central designer with a commitment power.

Sustaining suspense and surprise is not the only economic motivation for pursuing gradual revelation. Some settings require the participation in the mechanism to be a voluntary decision for the agents. By rejecting to participate in a mechanism, an agent can resume a predetermined strategic interaction with the other agents. For instance, a firm can revert to a state of competition by refusing to take part in a cartel with its rivals. If this firm's payoff in the competition game is not convex in the information on its rivals, the participation payoff threshold of this firm can be reduced by revealing some preliminary information during the cartel negotiations (Celik and Peters, 2011, 2013).

Another justification for gradual revelation is the agents' ability to *partially verify* the state by using *hard evidence* (Bull and Watson, 2007; Deneckere and Severinov, 2008). For instance, a worker may have access to two pieces of evidence, the first one ruling out that the worker is unskilled for the job, and the second one establishing that she is not lazy. However, due to the time and attention span constraints, this worker may have the ability to successfully report only a single piece of evidence to her superiors. In this case, in order to evaluate whether this worker is the right person for the job, the management should follow a two step procedure where the co-workers reveal the nature of their complaints first and then the worker makes her defense with the help of the appropriate hard evidence. A similar logic applies under the presence of *communication costs* (Van Zandt, 2007; Fadel and Segal, 2009; Mookherjee and Tsumagari, 2013). For instance, if the cost structure constrains an agent to a binary message space, an efficient use of this space would require conditioning the exact meaning of this agent's message to the messages chosen by the other agents earlier in the mechanism.

There is one class of allocation rules which can be trivially implemented through gradual revelation schemes. When an allocation rule is *dominant strategy* incentive compatible, truthful revelation remains as the optimal strategy of an agent regardless of what he learns about the types of the other agents. Mookherjee and Reichelstein (1992) establish that an incentive compatible allocation rule can be transformed into a dominant strategy incentive compatible one as long as the original allocation rule is *ex-post monotone*. The resulting allocation rule from this transformation is identical to the original one up to a transfer function which yields the same interim expected payoff for the agents as in the original allocation rule. Mookherjee and Reichelstein prove this result in a continuous type environment by invoking the *revenue equivalence theorem*. In Section 3, we extend this transformation to our discrete type model (Proposition 1).

In Section 4, we turn our attention to the analysis of the *budget-balanced* mechanisms, which remain as the main focus for the rest of the paper. It is well known that dominant strategy implementation is generally incompatible with balancing the budget. Therefore the transformation discussed above requires some outside intervention in order to provide the agents with the appropriate transfers. However, with Proposition 2, we show that the ex-post monotonicity condition, which is sufficient for dominant strategy implementation of an incentive compatible allocation rule, is also sufficient for its budget-balanced implementation under any possible sequence that the private information may be revealed.

Unlike the dominant strategy transfers, which depend only on the final type reports of the agents, the balanced budget transfers depend on the entire path of revelations made by them. With these transfers, the agents find it optimal to report their types truthfully not only at the start of the game under their prior beliefs but also at any information set they may reach during the process of revelations. Moreover, these transfers make the agents indifferent between any revelation they may make prior to their final reporting of the exact types. Thanks to this indifference condition, it is an equilibrium behavior for each type of each agent to randomize between different revelations with the appropriate frequencies that would generate the aimed sequence of information disclosures.

We construct these balanced budget transfers by following the *expected externality* method introduced by Arrow (1979) and d'Aspremont and Gerard-Varet (1979). The crucial point in supporting a balanced budget is making sure that each agent's transfer reflects the expected level of the externality he imposes on the other agents at each step of the revelation procedure. However, achieving this in our dynamic setting is a more involved exercise than what is required in a simultaneous revelation game, since the expectations in our case are based on the endogenous probability distributions which are determined by the earlier revelations made by the same set of agents.

Ex-post monotonicity is a *sufficient* condition to reconcile gradual revelation with a balanced budget, but it is not *necessary*. In Section 5, we introduce a weaker monotonicity condition which is defined with respect to the intended sequence of revelations. When the single crossing property is extended over the stochastic allocations, this weaker monotonicity condition is necessary and sufficient for budget-balanced implementation by gradual revelation (Proposition 3). Mookherjee and Tsumagari (2013) refer to a similar monotonicity condition to characterize the implementable *output functions* in the context of communication costs. We discuss the relation between these monotonicity conditions in Section 5.

One complication in proving the sufficiency part of Proposition 3 is the identification of the optimal continuation behavior of an agent when this agent makes an off-the-equilibrium-path revelation which is inconsistent with his type. We resolve this complication by designating a "type to imitate" for each deviation that a particular type can follow. We show that a gradual revelation mechanism can provide the incentives for a deviating type to imitate only these designated types once he makes the decision to deviate from the equilibrium behavior. This mechanism ensures that a type of an agent is indifferent between the revelations he makes with a positive probability (but not necessarily *all* possible revelations).

2 The Model

In an environment with incomplete information, I is the set of agents. We refer to the private information of an agent as his type. The agents' types are drawn independently from finite sets of real numbers. Formally, each agent $i \in I$ has a finite type set $\Theta_i \subset \mathbb{R}$ with generic element θ_i. Agent i's type is distributed with respect to the prior distribution $\mu_i^0 \in \Delta\Theta_i$, independently of the other agents' types. In standard notation, the cross product space $\Theta = \times_{i\in I}\Theta_i$ is the set of type profiles with generic element $\theta = \{\theta_i\}_{i\in I}$. We define $\mu^0 = \{\mu_i^0\}_{i\in I}$ as the collection of the priors on the agents' types. Similarly, Θ_{-i} stands for $\times_{j\in I-\{i\}}\Theta_j$ with generic element $\theta_{-i} = \{\theta_j\}_{j\in I-\{i\}}$ and μ_{-i}^0 represents $\{\mu_j^0\}_{j\in I-\{i\}}$.

Y is the finite set of economic alternatives. An agent's preferences over the economic alternatives depend on his own type but not on the types of the other agents. Accordingly, preferences of agent i are represented by the utility function $u_i : Y \times \Theta_i \to \mathbb{R}$. The continuous type model where the agents' types are drawn from a continuous distribution and their utility functions are continuously differentiable can be seen as a limit of our discrete type model. We will refer to this limit case to better illustrate some of the points we make. We assume that the preferences satisfy the following **single crossing property**:

Assumption (SC) *For any agent i and any two economic alternatives y and $\hat{y} \in Y$, the utility difference $u_i(y,\theta_i) - u_i(\hat{y},\theta_i)$ is either weakly increasing or weakly decreasing in θ_i.*

Notice that this assumption is trivially satisfied when agents have at most two types.

The single crossing property implies an order on the set of economic alternatives Y for each agent. That is, for each agent i, there exists a function $h_i : Y \to \mathbb{R}$ such that for any two economic alternatives y and $\hat{y} \in Y$,

$$h_i(y) \geq h_i(\hat{y}) \text{ if and only if } u_i(y,\theta_i) - u_i(\hat{y},\theta_i) \text{ is weakly increasing in } \theta_i. \tag{1}$$

If function $h_i(\cdot)$ satisfies condition (1), any positive monotone transformation of it also satisfies this condition. Non-uniqueness of function $h_i(\cdot)$ will not be a problem for our analysis, since we will be exclusively concerned with its monotonicity properties, which are robust under such a monotone transformation. For many settings, where preferences satisfy a one dimensional condensation condition (Mookherjee and Reichelstein, 1992), function $h_i(\cdot)$ has a natural interpretation such as the probability of receiving an object, the level of a public good, or the amount of production.

In addition to the economic alternative and his type, an agent's payoff is also affected by the monetary transfer he receives. We assume that the payoff functions are quasilinear in transfers and can be written as

$$u_i(y,\theta_i) + x_i \tag{2}$$

for agent i, where y is the economic alternative, θ_i is agent i's type, and $x_i \in \mathbb{R}$ is his monetary transfer.

Following the earlier literature, we define a **decision rule** as a mapping from type profiles into economic alternatives $y : \Theta \to Y$ and a **transfer rule** as a mapping from type profiles into agents' transfers $x : \Theta \to \mathbb{R}^{|I|}$, with x_i yielding the relevant dimension of the transfer rule for agent i. We refer to $(y(\cdot), x(\cdot))$ as an **allocation rule**. $(y(\cdot), x(\cdot))$ is **incentive compatible** under belief μ^0 if the following constraint is satisfied for all $i \in I$ and all $\left(\theta_i, \hat{\theta}_i\right) \in \Theta_i^2$ pairs:

$$\mathbb{E}_{\theta_{-i}|\mu_{-i}^0} \left\{u_i\left(y\left(\theta_i, \theta_{-i}\right), \theta_i\right) + x_i\left(\theta_i, \theta_{-i}\right)\right\} \geq \mathbb{E}_{\theta_{-i}|\mu_{-i}^0} \left\{u_i\left(y\left(\hat{\theta}_i, \theta_{-i}\right), \theta_i\right) + x_i\left(\hat{\theta}_i, \theta_{-i}\right)\right\}. \quad (3)$$

If $(y(\cdot), x(\cdot))$ satisfies the incentive compatibility constraints above, we know from the proof of the revelation principle that there exists a direct revelation mechanism where the agents reveal their types all at once, knowing that the economic alternative and their transfers will be chosen according to this allocation rule. Of course, the revelation principle does not rule out the possibility of implementing the same allocation rule in alternative ways.

What we study in this paper is these alternative ways of implementation. In particular, we investigate if we can implement an allocation rule – or a close variant of it – while delaying the full disclosure of private information of the agents by making them reveal their types *gradually*, presumably to cater to the demand for suspense.[2]

Our model is a discrete type model but perhaps this point is better explained by referring to a well known example where types are distributed on a continuum. Consider a *first price sealed bid auction* such that the type (the private value for the auctioned object) of each bidder is uniformly distributed on the interval $[0, 1]$. This auction has an equilibrium where each bidder bids $\frac{|I|-1}{|I|}$ times his value, so that the types of the bidders are revealed simultaneously at the observation of their bids. The *dutch auction* (the descending price auction where the asking price of the object lowers in time) is another mechanism which implements the same allocation rule with a gradual revelation scheme. The dutch auction has an equilibrium where each bidder waits until the asking price declines to the $\frac{|I|-1}{|I|}$ times his value and accepts to buy the object at that price if no other bidder chose to buy it earlier at a higher price. As time passes and the asking price declines, the bidders in the dutch auction (and the outside observers) keep updating their beliefs about their rivals until some bidder buys the object and reveals his type completely. This makes the dutch auction more exciting than the first price sealed bid auction, even though both auction formats generate the same economic allocation where the bidder with the highest value receives the object and pays $\frac{|I|-1}{|I|}$ times his value.

The idea that an audience may demand suspense (and surprise) in the revelation of information is recently studied by Ely, Frankel, and Kamenica (2013). We refer to their model in order to integrate

[2] In what follows, we assume that suspense is demanded by an audience which is not a direct participant to the mechanism. Our model also applies to a setting where the participating agents have preferences over the evolution of their beliefs, but these preferences are separable from their preferences over the economic alternatives.

the notion of gradual revelation in the mechanism design framework. Ely, Frankel, and Kamenica are concerned with information disclosures by a single informed party, but their approach can be extended to accommodate revelations made by multiple agents. In this extended model, $t \in \{0, 1, ..., T\}$ denotes the period. In each period, each agent chooses a signal from a finite[3] set. The audience and all the agents observe the signals sent and update their beliefs on the sending agents. The resulting period t belief on the type of agent i is denoted by $\mu_i^t \in \Delta\Theta_i$, and $\mu^t = \{\mu_i^t\}_{i \in I}$ is a collection of these beliefs. Technically, an agent's **information policy** is a function that maps the current period t, the agent's type θ_i, the current belief μ^t into a distribution over the signals.[4] The agents' information policies together generate a stochastic path of beliefs on the agents' types. Let $\tilde{\mu}_i^t \in \Delta(\Delta\Theta_i)$ be a distribution on agent i's type distributions and define $\tilde{\mu}^t = \{\tilde{\mu}_i^t\}_{i \in I}$ as a collection of these distributions. Following the terminology of Ely, Frankel, and Kamenica, a **belief martingale** is a sequence $\{\tilde{\mu}^t\}_{t=0}^T$ such that

i) $\tilde{\mu}^0$ is degenerate at prior μ^0, and

ii) $\mathbb{E}\left[\mu^t | \mu^0, ..., \mu^{t-1}\right] = \mu^{t-1}$ for all $t = 1, ..., T$.

A realization of a belief martingale is the belief path $\{\mu^t\}_{t=0}^T$.

We also add a final period, period $T+1$, to Ely, Frankel, and Kamenica's timing, where each agent can send a last signal. In our formulation, this will be the period where the agents will be given the incentive to fully reveal their private information if they have not done so in earlier periods. A **gradual revelation mechanism** determines the economic alternative and the agents' transfers as functions of all the signals sent in $T+1$ periods.

For an illustration of how a gradual revelation mechanism would work, we reconsider the independent private values auction setting. The bidders in the example start with the prior belief that the types of their rivals are uniformly distributed on the interval $[0, 1]$. This corresponds to the period 0 belief μ^0 in our model. Then, in period 1, the gradual revelation mechanism may ask each bidder to send either a "high" signal or a "low" signal. After observing the period 1 signal of a bidder, his rivals update their belief on the bidder's value to a triangular distribution with the cumulative distribution function θ_i^2 if the signal is high or with the cumulative distribution function $2\theta_i - \theta_i^2$ if the signal is low.[5] This corresponds to the period 1 belief μ_i^1, which happens to have two possible realizations in this example. In period 2, depending on the period 1 revelations, some of the bidders may be asked to send a second signal revealing their type completely.[6] Finally in period 3, all the remaining bidders reveal their types and

[3] We concentrate on finite message sets in order to use sequential equilibrium as our solution concept.

[4] Following Ely, Frankel, and Kamenica, we assume that the information policy is *memoryless*, i.e., the signal depends on the current period and the current belief instead of the full history.

[5] These cumulative distribution functions are derived from the Bayes rule with the assumption that each bidder sends the high signal with a probability equal to his value $\theta_i \in [0, 1]$.

[6] For instance, bidder i may reveal his type if and only if he is the only bidder who happened to send the "high" signal

the mechanism determines the economic alternative and the transfers by processing the data generated by the bidders in all the three time periods.

Any allocation rule implemented by a gradual revelation mechanism would still respect the incentive compatibility conditions: Starting with the first period of the mechanism, type θ_i of agent i can choose to imitate type $\hat{\theta}_i$ by following the equilibrium strategy of this latter type. For this strategy not be a profitable deviation, condition (3) must hold in expectation. However, a gradual revelation scheme as in the preceding paragraph introduces many additional conditions to be satisfied in the construction of an equilibrium. First, consider an agent who waits until the last period to reveal his type. Such an agent will have superior information about the other agents than what he knew in period zero. Therefore his truthtelling constraints in period $T+1$ will be more stringent than the interim version of the incentive compatibility constraints in (3). Moreover, a gradual revelation mechanism should also provide the incentive to send the accurate signals in periods earlier than $T+1$. For instance, if an agent is supposed to randomize between different signals (as is the case in period 1 in the above example), his expected continuation payoff from these signals must be equal to each other and weakly larger than the continuation payoff from any other signal that he is not supposed to send in equilibrium.

3 Dominant Strategy Incentive Compatibility

There is one class of allocation rules which are easily shown to be implementable with gradual revelation. Suppose $(y(\cdot), x(\cdot))$ is **dominant strategy incentive compatible**, i.e., it satisfies the incentive constraints in (3) for all $\theta_{-i} \in \Theta_{-i}$ instead of satisfying them in expectation only. In this case, we can construct a gradual revelation mechanism where the chosen allocation depends only on the type reports at time $T+1$ but not on the sent signals or the updated beliefs in earlier periods. Regardless of what an agent learns on the types of the other agents in these earlier periods, he would prefer to report his type truthfully at the end of the procedure. Moreover, since the payoffs do not depend on the revelations made in periods 1 to T, all types of all agents would be indifferent between all the information policies available to them. Accordingly, sending their signals in a type dependent manner to generate any given martingale would be an equilibrium behavior for the agents.[7]

Under the single crossing property, it is well known that dominant strategy incentive compatibility demands a monotonic relation between agent i's type and function h_i. If decision rule $y(\cdot)$ is dom-

in period 1.

[7] These strategy profiles constitute an *ex-post* equilibrium, where each agent's strategy is optimal whatever the types of the other agents are. As noted by Fadel and Segal (2009) in their Proposition 6, such strategy profiles can be supported as equilibria under any prior beliefs. In fact, this observation extends to the interdependent values case, where an agent's payoff may depend on another agent's type, as long as the allocation rule satisfies the ex-post version of the incentive compatibility constraints (see Van Zandt, 2007).

inant strategy incentive compatible with some transfers, then it must be **ex-post monotone**, i.e., $h_i[y(\theta_i, \theta_{-i})]$ must be weakly increasing in θ_i for all $\theta_{-i} \in \Theta_{-i}$ and all $i \in I$. For instance, for private value auctions, ex-post monotonicity corresponds to the requirement that the probability of assigning the object to any given agent increases in the private value of this agent, regardless of the values of all the other agents.

Mookherjee and Reichelstein (1992) argue that there is a sense in which the ex-post monotonicity condition is also sufficient for dominant strategy incentive compatibility. In the context of a model with a continuum of types and continuously differentiable utility functions, they show that if allocation rule $(y(\cdot), x(\cdot))$ is incentive compatible and decision rule $y(\cdot)$ is ex-post monotone, then transfer rule $x(\cdot)$ can be transformed into another transfer rule $x^{DS}(\cdot)$, which constitutes a dominant strategy incentive compatible allocation rule together with $y(\cdot)$ and which yields the same interim transfers as $x(\cdot)$. Mookherjee and Reichelstein construct $x^{DS}(\cdot)$ by invoking the revenue equivalence theorem in their continuous type environment. We show that this result extends to our model with discrete types.

Proposition 1 *Suppose that the single crossing condition in Assumption SC holds and* $(y(\cdot), x(\cdot))$ *is an incentive compatible allocation rule under belief* μ^0. *There exists a transfer rule* $x^{DS}(\cdot)$ *such that*

i) allocation rule $\left(y(\cdot), x^{DS}(\cdot)\right)$ *is dominant strategy incentive compatible and*

ii) $\mathbb{E}_{\theta_{-i}|\mu^0_{-i}} x_i^{DS}(\theta_i, \theta_{-i}) = \mathbb{E}_{\theta_{-i}|\mu^0_{-i}} x_i(\theta_i, \theta_{-i})$ *for all* $\theta_i \in \Theta_i$, *all* $i \in I$,

if and only if decision rule $y(\cdot)$ *is ex-post monotone.*

Proof. The "only if" part is a standard result from screening theory, which can be proved by adding the two dominant strategy incentive compatibility constraints between any two types under allocation rule $\left(y(\cdot), x^{DS}(\cdot)\right)$.

As a first step to proving the "if" part, define $\Delta_i\left(\theta_i, \hat{\theta}_i|\theta_{-i}\right)$ as the **payoff premium** of type θ_i for revealing his type truthfully instead of imitating an "adjacent" type $\hat{\theta}_i$ when the other agents' types are given as θ_{-i}:

$$\Delta_i\left(\theta_i, \hat{\theta}_i|\theta_{-i}\right) = u_i(y(\theta_i, \theta_{-i}), \theta_i) + x_i(\theta_i, \theta_{-i}) - u_i\left(y\left(\hat{\theta}_i, \theta_{-i}\right), \theta_i\right) - x_i\left(\hat{\theta}_i, \theta_{-i}\right). \tag{4}$$

Notice that function $\Delta_i\left(\theta_i, \hat{\theta}_i|\theta_{-i}\right)$ can take positive or negative values depending on θ_{-i} but (3) implies that $\mathbb{E}_{\theta_{-i}|\mu^0_{-i}}\Delta_i\left(\theta_i, \hat{\theta}_i|\theta_{-i}\right)$ is non-negative.

The next step is defining function $g_i(\theta_i, \theta_{-i})$. The rate of change of this function between any two adjacent types θ_i and $\hat{\theta}_i$ is given as:

$$g_i(\theta_i, \theta_{-i}) - g_i\left(\hat{\theta}_i, \theta_{-i}\right) = \frac{\Delta_i\left(\hat{\theta}_i, \theta_i|\theta_{-i}\right)\mathbb{E}_{\tilde{\theta}_{-i}|\mu^0_{-i}}\Delta_i\left(\theta_i, \hat{\theta}_i|\tilde{\theta}_{-i}\right) - \Delta_i\left(\theta_i, \hat{\theta}_i|\theta_{-i}\right)\mathbb{E}_{\tilde{\theta}_{-i}|\mu^0_{-i}}\Delta_i\left(\hat{\theta}_i, \theta_i|\tilde{\theta}_{-i}\right)}{\mathbb{E}_{\tilde{\theta}_{-i}|\mu^0_{-i}}\Delta_i\left(\theta_i, \hat{\theta}_i|\tilde{\theta}_{-i}\right) + \mathbb{E}_{\tilde{\theta}_{-i}|\mu^0_{-i}}\Delta_i\left(\hat{\theta}_i, \theta_i|\tilde{\theta}_{-i}\right)} \tag{5}$$

for all θ_{-i}. This equation determines $g_i(\cdot, \theta_{-i})$ up to a constant[8] for any θ_{-i}. The following equation yields this constant term and thus pins down the function:

$$\mathbb{E}_{\theta_i|\mu_i^0}\{g_i(\theta_i, \theta_{-i})\} = 0 \tag{6}$$

for all θ_{-i}.

When function g_i is defined as above, its expectation with respect to its second argument θ_{-i} is zero as well:

$$\mathbb{E}_{\theta_{-i}|\mu_{-i}^0}\{g_i(\theta_i, \theta_{-i})\} = 0 \tag{7}$$

for all θ_i. To see this last point observe that the expectation of the right hand side of equation (5) over θ_{-i} given μ_{-i}^0 is equal to zero. Hence $\mathbb{E}_{\theta_{-i}|\mu_{-i}^0}\{g_i(\theta_i, \theta_{-i})\}$ is a constant function of θ_i with the expected value of zero (equation (6)).

We define $x^{DS}(\cdot)$ with the equation $x_i^{DS}(\theta_i, \theta_{-i}) = x_i(\theta_i, \theta_{-i}) + g_i(\theta_i, \theta_{-i})$. Part (ii) of the proposition follows from (7). Under condition SC and the ex-post monotonicity of $y(\cdot)$, the dominant strategy incentive compatibility constraints between the adjacent types will be sufficient for all the other dominant strategy incentive compatibility constraints. Therefore, to prove part (i), it is sufficient to show that the updated payoff premium to revealing the type as θ_i rather than imitating an adjacent type $\hat{\theta}_i$ under the allocation rule $\left(y(\cdot), x^{DS}(\cdot)\right)$ is non-negative for all i and all θ_{-i}:

$$\begin{aligned} & u_i(y(\theta_i, \theta_{-i}), \theta_i) + x_i^{DS}(\theta_i, \theta_{-i}) - u_i\left(y\left(\hat{\theta}_i, \theta_{-i}\right), \theta_i\right) - x_i^{DS}\left(\hat{\theta}_i, \theta_{-i}\right) \\ = \; & \Delta_i\left(\theta_i, \hat{\theta}_i|\theta_{-i}\right) + g_i(\theta_i, \theta_{-i}) - g_i\left(\hat{\theta}_i, \theta_{-i}\right) & (8) \\ = \; & \frac{\Delta_i\left(\hat{\theta}_i, \theta_i|\theta_{-i}\right) + \Delta_i\left(\theta_i, \hat{\theta}_i|\theta_{-i}\right)}{\mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^0}\Delta_i\left(\theta_i, \hat{\theta}_i|\tilde{\theta}_{-i}\right) + \mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^0}\Delta_i\left(\hat{\theta}_i, \theta_i|\tilde{\theta}_{-i}\right)}\mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^0}\Delta_i\left(\theta_i, \hat{\theta}_i|\tilde{\theta}_{-i}\right) & (9) \end{aligned}$$

The terms in expectations are all non-negative thanks to the incentive compatibility of $(y(\cdot), x(\cdot))$. Hence the sign of

$$\begin{aligned} \Delta_i\left(\hat{\theta}_i, \theta_i|\theta_{-i}\right) + \Delta_i\left(\theta_i, \hat{\theta}_i|\theta_{-i}\right) \;=\; & \left[u_i(y(\theta_i, \theta_{-i}), \theta_i) - u_i\left(y\left(\hat{\theta}_i, \theta_{-i}\right), \theta_i\right)\right] \\ & - \left[u_i\left(y(\theta_i, \theta_{-i}), \hat{\theta}_i\right) - u_i\left(y\left(\hat{\theta}_i, \theta_{-i}\right), \hat{\theta}_i\right)\right] \end{aligned} \tag{10}$$

determines the sign of the updated payoff premium. This sum is a non-negative number due to the single crossing and the ex-post monotonicity conditions. (Either $\theta_i \geq \hat{\theta}_i$ and therefore $u_i\left(y(\theta_i, \theta_{-i}), \tilde{\theta}_i\right) -$

[8]If the incentive constraints between types θ_i and $\hat{\theta}_i$ are binding in both directions (which would indeed be the case if the allocation rule is *pooling* these types together), then both $\mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^0}\Delta_i\left(\theta_i, \hat{\theta}_i|\tilde{\theta}_{-i}\right)$ and $\mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^0}\Delta_i\left(\hat{\theta}_i, \theta_i|\tilde{\theta}_{-i}\right)$ will be equal to zero. In this case, we adopt the convention that these terms cancel each other out so that the right hand side of (5) is equal to $\frac{\Delta_i\left(\hat{\theta}_i, \theta_i|\theta_{-i}\right) - \Delta_i\left(\theta_i, \hat{\theta}_i|\theta_{-i}\right)}{2}$. We follow the same practice in defining similar functions in the proofs of our later results.

$u_i\left(y\left(\hat{\theta}_i, \theta_{-i}\right), \tilde{\theta}_i\right)$ is weakly increasing in $\tilde{\theta}_i$, or $\theta_i \leq \hat{\theta}_i$ and therefore $u_i\left(y\left(\theta_i, \theta_{-i}\right), \tilde{\theta}_i\right) - u_i\left(y\left(\hat{\theta}_i, \theta_{-i}\right), \tilde{\theta}_i\right)$ is weakly decreasing in $\tilde{\theta}_i$.) ■

In the proof of the proposition, transfer rule $x^{DS}(\cdot)$ is constructed by the equation

$$x_i^{DS}(\theta_i, \theta_{-i}) = x_i(\theta_i, \theta_{-i}) + g_i(\theta_i, \theta_{-i}), \tag{11}$$

where $g_i(\cdot)$ is a function which gives agent i the incentive to reveal his true type θ_i after he observes θ_{-i}. Another crucial property of function $g_i(\cdot)$ is that it has an expected value of zero when the expectation is taken either with respect to θ_i given any θ_{-i} or with respect to θ_{-i} given any θ_i. In the continuous type limit of our model, $x^{DS}(\cdot)$ defined in (11) corresponds to the unique (up to a constant depending on θ_{-i}) transfers that the revenue equivalence theorem yields for the *equivalent* implementation of $(y(\cdot), x(\cdot))$ in dominant strategies (see Mookherjee and Reichelstein, 1992, Proposition 1). In the special case of our model where the utility function $u_i(y, \theta_i)$ is linear in θ_i for all i, $x^{DS}(\cdot)$ is the same as the transfer rule derived by Gershkov et al. (2013) in their Theorem 2. Gershkov et al. use this transfer rule in the construction of a dominant strategy incentive compatible allocation rule which delivers the same interim expected payoff as an incentive compatible (but possibly ex-post non-monotonic) allocation rule.

If we apply the transformation described in the proof of Proposition 1 to the independent private value auction example of the previous section, we end up with converting the allocation rule of the first price auction into the allocation rule of the second price auction, where the highest value bidder receives the object and pays a monetary amount equal to the second highest value. This allocation rule has the advantage of being implementable through a gradual revelation mechanism. Whatever the bidders learn about their rivals in the earlier stages of this mechanism, they still find it optimal to report their types truthfully at the final stage. Furthermore, since the transfers will depend only on these final period type reports, the bidders would be indifferent between any of the signals they can submit in the earlier periods. This last point makes it possible to construct an equilibrium where the bidders' randomizations over the signals would generate any desired belief martingale.

Both the first price and the second price auction allocation rules yield the same decision regarding the identity of the winning bidder, the same expected transfers from the bidders at the interim stage, and the same expected revenue for the seller at the ex-ante stage. However, the realized level of the revenue has different distributions under these two auctions. As first remarked by Vickrey (1961, Appendix 1), the second price auction revenue has a larger support and a higher variance than the first price one, implying that a risk averse seller would strictly prefer the first price auction. In other words, even though the risk neutral agents are indifferent between the original and the modified allocation rules, these two rules may not necessarily describe equivalent outcomes for a principal with more elaborate risk preferences. Perhaps more importantly, in many design settings (such as bilateral trade and provision of public goods), there does not exist a principal (like the seller in the auction setting) who can cover the differences between the monetary transfers to be made under the original allocation rule and the dominant strategy incentive compatible variant of this allocation rule. These observations point to the fact that balancing the budget is a desirable property for gradual revelation mechanisms as it is for the direct revelation mechanisms.

In general, balancing the budget and attaining dominant strategy incentive compatibility are two objectives which cannot be achieved together. In the following section, we will abandon the latter objective and concentrate on sustaining gradual revelation without insisting on dominant strategy implementation. We will see that it is possible to balance the budget and at the same time to provide the incentives for gradual revelation.

4 Gradual Revelation with Budget Balance: A Sufficient Condition

In the previous section, we argued that ex-post monotonicity of the decision rule is a sufficient condition to transform an incentive compatible allocation rule into a dominant strategy incentive compatible one, without considering the budget balance. In this section, we will see that ex-post monotonicity is also sufficient for the construction of a gradual revelation mechanism adhering to the balanced budget requirement.

Unlike the transfers identified in the previous section, the transfers ensuring a balanced budget will not give us dominant strategy incentive compatibility. Nevertheless, these transfers will provide the agents with the incentive to make accurate revelations in all periods of the constructed mechanism, including the last period where they will fully reveal their types.[9]

Proposition 2 *Suppose that the single crossing condition in Assumption SC holds, $\left\{\tilde{\mu}^t\right\}_{t=0}^{T}$ is an arbitrary belief martingale, and $(y(\cdot), x(\cdot))$ is an incentive compatible allocation rule under belief μ^0. There exists a gradual revelation mechanism and a sequential equilibrium of this mechanism where*

i) types are gradually revealed according to martingale $\left\{\tilde{\mu}^t\right\}_{t=0}^{T}$ and decision rule $y(\cdot)$ is implemented,

ii) the interim expected payoff for type θ_i of agent i is $\mathbb{E}_{\theta_{-i}|\mu^0_{-i}}\left\{u_i\left(y\left(\theta_i, \theta_{-i}\right)\right) + x_i\left(\theta_i, \theta_{-i}\right)\right\}$, and

iii) transfers are budget-balanced (they add up to $\sum_{i \in I} x_i(\theta)$ regardless of the path of revelation),

if decision rule $y(\cdot)$ is ex-post monotone.

[9] A gradual revelation mechanism induces a dynamic optimization problem for each type of each agent. For the mechanism we construct, we will show that following an accurate revelation path is a solution to this dynamic optimization problem. Solving this problem involves identifying an optimal action in each round, contingent on the history of the earlier signals. The complication is that, as Fadel and Segal (2009) observe, these histories must include the histories which are not supposed to be reached on the equilibrium path given the type of the agent.

The proof of the proposition will follow from the lemma below. When the agent types are independently distributed, the typical method to attain budget-balanced implementation – while keeping the Bayesian incentives intact – is outlined by the *expected externality* approach of Arrow (1979) and d'Aspremont and Gerard-Varet (1979).[10] In our context, applying the expected externality method directly would involve replacing function $g_i(\cdot)$ that we used in the construction of $x_i^{DS}(\cdot)$ in (11) with the summation of $|I|$ different terms, each of which depends on the type report of each of the $|I|$ agents:

$$\mathbb{E}_{\tilde{\theta}_{-i}|\mu^0_{-i}} g_i\left(\theta_i, \tilde{\theta}_{-i}\right) - \sum_{j \neq i} \frac{1}{|I|-1} \mathbb{E}_{\tilde{\theta}_{-j}|\mu^0_{-j}} g_j\left(\theta_j, \tilde{\theta}_{-j}\right). \tag{12}$$

Here, agent i's incentive to report truthfully is provided by the first term, and the remaining terms are there just to balance the budget. However, since the expected level of $g_i(\cdot)$ is zero in both θ_i and $\tilde{\theta}_{-i}$ in our setting, this construction amounts to reverting to the original transfer function $x_i(\cdot)$, which assures *simultaneous* revelation of the types but which is not necessarily suitable for gradual revelation.

We break this tension between the provision of the right incentives and the budget balance by letting the transfers depend on the signals sent by the agents in the first T periods as well as the type reports submitted in period $T+1$. Construction of these transfers will benefit from the expected externality approach discussed above. We will still refer to an analogous expression to (12) to determine the evolution of the transfers in each time period. However, the relevant probability distribution in the calculation of the expectations in this expression will not be the prior belief μ^0. Instead, the gradual revelation mechanism transfers will depend on the expected values of the functions $g_i\left(\cdot, \tilde{\theta}_{-i}\right)$ conditional on the updated belief of the period in question. Since the beliefs evolve on the path of play according to the signals sent by the agents in our setting, the analogous expression to (12) will not be additively separable in the agents' signals. Agent i will have an effect not only in the value of its first term through his final type report but also in the values of the remaining $|I|-1$ terms through his information policy. Nevertheless, we will be able to provide a proof for the lemma below by invoking the fact that the last $|I|-1$ terms in this expression will be equal to zero in expectation for agent i regardless of the signal he chooses to submit.

Lemma 1 *Suppose that the single crossing property in Assumption SC holds, decision rule $y(\cdot)$ is ex-post monotone, allocation rule $\left(y(\cdot), x^{\tau-1}(\cdot)\right)$ is incentive compatible under belief $\mu^{\tau-1}$, and $\mathbb{E}\left[\mu^\tau | \mu^{\tau-1}\right] = \mu^{\tau-1}$. There exists a belief dependent transfer rule $x^\tau(\cdot, \mu^\tau)$ such that*

a) allocation rule $(y(\cdot), x^\tau(\cdot, \mu^\tau))$ is incentive compatible under belief μ^τ,

b) for all θ_i, all μ_i^τ, and all i, $\mathbb{E}_{\mu^\tau_{-i}|\mu^{\tau-1}_{-i}} \mathbb{E}_{\theta_{-i}|\mu^\tau_{-i}} x_i^\tau\left(\theta_i, \theta_{-i}, \mu_i^\tau, \mu_{-i}^\tau\right) = \mathbb{E}_{\theta_{-i}|\mu^{\tau-1}_{-i}} x_i^{\tau-1}(\theta_i, \theta_{-i})$, and

[10] See d'Aspremont, Cremer, and Gerard-Varet (2004) and Kosenok and Severinov (2008) for an extension of this method to correlated types. See Borgers and Norman (2009) for an extension to interdependent values. See Eso and Futo (1999) for how to smooth the transfers when the principal is risk averse, ambiguity averse (Bose, Ozdenoren, Pape, 2006), or is expecting the agents to collude (Che and Kim, 2006, 2009, and Pavlov, 2008).

c) $x^{\tau}(\cdot,\mu^{\tau})$ is budget-balanced: $\sum_i x_i^{\tau}(\theta,\mu^{\tau}) = \sum_i x_i^{\tau-1}(\theta)$ for all θ and all μ^{τ}.[11]

Proof. The proof of the lemma follows similar steps as in the proof of the "if" part of Proposition 1. We first define the payoff premium of type θ_i for revealing his type truthfully instead of imitating an adjacent type $\hat{\theta}_i$ when the other agents' types are given as θ_{-i}:

$$\Delta_i^{\tau}\left(\theta_i,\hat{\theta}_i|\theta_{-i}\right) = u_i\left(y\left(\theta_i,\theta_{-i}\right),\theta_i\right) + x_i^{\tau-1}\left(\theta_i,\theta_{-i}\right) - u_i\left(y\left(\hat{\theta}_i,\theta_{-i}\right),\theta_i\right) - x_i^{\tau-1}\left(\hat{\theta}_i,\theta_{-i}\right). \tag{13}$$

Then we define function $g_i^{\tau}(\theta_i,\theta_{-i})$ with equations

$$g_i^{\tau}(\theta_i,\theta_{-i}) - g_i^{\tau}\left(\hat{\theta}_i,\theta_{-i}\right) = \frac{\Delta_i^{\tau}\left(\hat{\theta}_i,\theta_i|\theta_{-i}\right)\mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^{\tau-1}}\Delta_i^{\tau}\left(\theta_i,\hat{\theta}_i|\tilde{\theta}_{-i}\right) - \Delta_i^{\tau}\left(\theta_i,\hat{\theta}_i|\theta_{-i}\right)\mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^{\tau-1}}\Delta_i^{\tau}\left(\hat{\theta}_i,\theta_i|\tilde{\theta}_{-i}\right)}{\mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^{\tau-1}}\Delta_i^{\tau}\left(\theta_i,\hat{\theta}_i|\tilde{\theta}_{-i}\right) + \mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^{\tau-1}}\Delta_i^{\tau}\left(\hat{\theta}_i,\theta_i|\tilde{\theta}_{-i}\right)} \tag{14}$$

where θ_i and $\hat{\theta}_i$ are two adjacent types and

$$\mathbb{E}_{\theta_i|\mu_i^{\tau-1}}\left\{g_i^{\tau}(\theta_i,\theta_{-i})\right\} = 0 \tag{15}$$

for all θ_{-i}. As in the proof of Proposition 1, this definition implies that

$$\mathbb{E}_{\theta_{-i}|\mu_{-i}^{\tau-1}}\left\{g_i^{\tau}(\theta_i,\theta_{-i})\right\} = 0 \tag{16}$$

for all θ_i. Finally we define the period τ transfers as

$$x_i^{\tau}(\theta,\mu^{\tau}) = x_i^{\tau-1}(\theta) + \mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^{\tau}}g_i^{\tau}\left(\theta_i,\tilde{\theta}_{-i}\right) - \frac{1}{|I|-1}\mathbb{E}_{\tilde{\theta}_i|\mu_i^{\tau}}\sum_{j\neq i}\mathbb{E}_{\tilde{\theta}_{-i-j}|\mu_{-i-j}^{\tau}}g_j^{\tau}\left(\theta_j,\tilde{\theta}_i,\tilde{\theta}_{-i-j}\right). \tag{17}$$

Budget balancedness in part (c) of the lemma holds by construction. Part (b) follows from (15) and (16). Under Assumption *SC* and ex-post monotonicity of $y(\cdot)$, the incentive compatibility constraints in (3) between the adjacent types are sufficient for all the other incentive compatibility constraints. The expected value of the updated payoff premium to revealing the type as θ_i rather than imitating an adjacent type $\hat{\theta}_i$ is

$$\begin{aligned} &\mathbb{E}_{\theta_{-i}|\mu_{-i}^{\tau}}\left\{\Delta_i^{\tau}\left(\theta_i,\hat{\theta}_i|\theta_{-i}\right) + g_i^{\tau}(\theta_i,\theta_{-i}) - g_i^{\tau}\left(\hat{\theta}_i,\theta_{-i}\right)\right\} \\ = &\frac{\mathbb{E}_{\theta_{-i}|\mu_{-i}^{\tau}}\left\{\Delta_i^{\tau}\left(\hat{\theta}_i,\theta_i|\theta_{-i}\right) + \Delta_i^{\tau}\left(\theta_i,\hat{\theta}_i|\theta_{-i}\right)\right\}}{\mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^{\tau-1}}\Delta_i^{\tau}\left(\theta_i,\hat{\theta}_i|\tilde{\theta}_{-i}\right) + \mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^{\tau-1}}\Delta_i^{\tau}\left(\hat{\theta}_i,\theta_i|\tilde{\theta}_{-i}\right)}\mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^{\tau-1}}\Delta_i^{\tau}\left(\theta_i,\hat{\theta}_i|\tilde{\theta}_{-i}\right) \end{aligned} \tag{18}$$

under belief μ_{-i}^{τ}. To prove part (a) of the lemma, it is sufficient to show that this payoff premium is non-negative. The terms $\mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^{\tau-1}}\Delta_i^{\tau}\left(\theta_i,\hat{\theta}_i|\tilde{\theta}_{-i}\right)$ and $\mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^{\tau-1}}\Delta_i^{\tau}\left(\hat{\theta}_i,\theta_i|\tilde{\theta}_{-i}\right)$ are both non-negative since

[11] Notice that we still need to identify $x_i^{\tau}(\theta,\mu^{\tau})$ when the type profile θ is not in the support of belief μ^{τ}. In the equilibrium we will construct to prove Proposition 2, this would correspond to an off-the-equilibrium-path event that an agent first sends a signal, and later reports a type which is not in the support of the equilibrium belief generated by the earlier signal. In this case, our budget balance requirement still demands the sum of the transfers to be equal to $\sum_i x_i(\theta)$, where θ will be determined by the final type reports.

$(y(\cdot), x^{\tau-1}(\cdot))$ is incentive compatible under belief $\mu^{\tau-1}$. As in the proof of Proposition 1, Assumption SC and ex-post monotonicity of $y(\cdot)$ imply that $\Delta_i^\tau\left(\hat{\theta}_i, \theta_i|\theta_{-i}\right) + \Delta_i^\tau\left(\theta_i, \hat{\theta}_i|\theta_{-i}\right)$ is non-negative for all θ_{-i} as well. ■

This lemma establishes the following: Start with a transfer rule which makes a decision rule incentive compatible under a certain belief. If this belief is updated in a Bayesian fashion, it is possible to modify the transfer rule to make the original decision rule incentive compatible under the updated belief. Moreover, the resulting *belief dependent* allocation rules yield the same interim expected payoff as the initial allocation rule. This ensures that the agents will be indifferent between all the belief updates they may generate about their own types. In Appendix A, we provide a numerical example to this construction.

To see how Proposition 2 follows from Lemma 1, first consider the case where $T = 1$. In this case, the agents are given an opportunity to send some signals simultaneously in period 1 and then they report their types in period 2. The gradual revelation mechanism maps the reported types into the economic alternative prescribed by the decision rule $y(\cdot)$. The transfers depend on both the signals sent in period 1 and the types reported in period 2. We relabel the signals available to each agent i in period 1 as the beliefs in the support of $\tilde{\mu}_i^1$. The gradual revelation mechanism maps the sent signals and the reported types into transfers by using the belief dependent transfer rule $x^1\left(\cdot, \mu^1\right)$ described in Lemma 1 (setting $x^0(\cdot)$ in the lemma equal to $x(\cdot)$ in the proposition).

This gradual revelation mechanism has an equilibrium where each agent reports his type truthfully in period 2 and randomizes between the signals in period 1 so that the resulting updated belief after sending a signal is identical to the label of the signal. Optimality of truthful reporting under the updated beliefs follows from part (a) of the lemma. Moreover, part (b) implies that agents are indifferent between all the signals available to them in period 1, proving that the randomizations prescribed by $\tilde{\mu}^1$ constitute an equilibrium behavior. Another implication of this indifference condition is that the equilibrium yields the interim payoff $\mathbb{E}_{\theta_{-i}|\mu_{-i}^0}\left\{u_i\left(y\left(\theta_i, \theta_{-i}\right)\right) + x_i\left(\theta_i, \theta_{-i}\right)\right\}$ for agent i with type θ_i. Finally, budget balancedness follows from part (c) of the lemma.

The gradual revelation mechanism above can be extended to longer horizons ($T > 1$), by relabeling signals at each period t as the beliefs in the support of the distribution $\tilde{\mu}_i^t$ and then letting the transfers depend on signals sent in all periods and the types reported in period $T + 1$. These transfers are determined iteratively by using the evolution of the equilibrium beliefs and function $x^T\left(\cdot, \mu^T\right)$ described in Lemma 1 (set function $x^{T-1}(\cdot)$ equal to $x^{T-1}\left(\cdot, \mu^{T-1}\right)$, function $x^{T-2}(\cdot)$ equal to $x^{T-2}\left(\cdot, \mu^{T-2}\right)$, etc.). This extended gradual revelation mechanism has an equilibrium where the agents' randomizations on the signals respect martingale $\left\{\tilde{\mu}^t\right\}_{t=0}^T$ and where they report their true types in period $T + 1$.

Notice that Proposition 2 does not put a restriction on the belief martingales. As long as the decision rule is ex-post monotone, we can construct the transfers that would induce the martingale we choose.

Unlike the transfer rule in Proposition 1, however, the transfers we construct here are contingent on the belief martingale that we intend to support.

How we make use of Arrow (1979) and d'Aspremont and Gerard-Varet's (1979) expected externality approach to balance the budget here is similar to the way that Athey and Segal (2013) calculate the transfers for their *balanced team mechanism* in an infinite horizon design setting.[12] In each period of their dynamic setting, agents acquire additional private information and decide on a new economic alternative. The distribution of the private information is affected by both the past information and the past decisions. Athey and Segal are interested in the implementation of the *efficient* decision rule. This requires the design of a mechanism with a *full revelation* property, such that the agents would reveal all the private information they hold at each period. In the process of identifying this mechanism, Athey and Segal show that, for any mechanism satisfying the full revelation property, there exists another full revelation mechanism which implements the same decision rule with a balanced budget. The analysis in this section suggests that Athey and Segal's result extends to mechanisms where the agents are induced to follow more general information policies than fully revealing their information at every opportunity.

The results derived in this section refer to an ex-post monotonicity condition, which is satisfied by many allocation rules of particular interest. For instance, Mookherjee and Reichelstein (1992) show that the incentive compatible allocation rule which maximizes the objective function of a *principal* (ex-ante expected value of the gross benefit from the chosen economic alternative minus the transfers to the agents) is ex-post monotone. In the context of linear utility functions (such as the bidders' value functions in auctions), Manelli and Vincent (2010) and Gershkov et al. (2013) argue that for any incentive compatible allocation rule, there exists an ex-post monotone and incentive compatible rule which generates the same interim expected payoffs (but not necessarily the same economic alternatives and the same interim transfers).

We close this section by remarking that the ex-post monotonicity condition is sufficient for constructing a gradual revelation mechanism but it is not necessary. Characterization of a necessary and sufficient condition will be the subject of the following section.

5 Gradual Revelation with Budget Balance: A Necessary and Sufficient Condition

The single crossing property in Assumption *SC* is concerned with the agents' preferences only over the *deterministic* economic alternatives. As demonstrated by Strausz (2006), this assumption does not

[12] In fact, Fadel and Segal (2009) refer to Athey and Segal's work (their footnote 18) to suggest that the budget-unbalanced mechanism in their Proposition 6 can be transformed into a budget-balanced one.

imply a regularity on how different types of an agent would evaluate the *randomizations* on the economic alternatives. We start this section by extending the single crossing property over these randomized alternatives.

Assumption $(\overline{\mathbf{SC}})$ *For any agent i and any two (possibly randomized) economic alternatives q and $\hat{q} \in \Delta Y$, the expected utility difference $E_{y|q}\{u_i(y,\theta_i)\} - E_{y|\hat{q}}\{u_i(y,\theta_i)\}$ is either weakly increasing or weakly decreasing in θ_i.*

Notice that this stronger version of the single crossing property is also trivially satisfied when agents have at most two types.

The same way that Assumption SC implies an order on set Y, the extended single crossing property in Assumption $\overline{SC}$ implies an order on set ΔY for each agent i. Moreover, since the expected utility is linear in utility levels from deterministic economic alternatives, this order satisfies the independence condition. Accordingly, for each agent i, there exists a function $\bar{h}_i : Y \to \mathbb{R}$ such that for any two randomized economic alternatives q and $\hat{q} \in \Delta Y$,

$$\mathbb{E}_{y|q}\left\{\bar{h}_i(y)\right\} \geq \mathbb{E}_{y|\hat{q}}\left\{\bar{h}_i(y)\right\} \text{ if and only if } \mathbb{E}_{y|q}\{u_i(y,\theta_i)\} - \mathbb{E}_{y|\hat{q}}\{u_i(y,\theta_i)\} \text{ is weakly increasing in } \theta_i.$$

In Section 3, we remarked that the ex-post monotonicity of the decision rule $y(\cdot)$ is a necessary condition for dominant strategy incentive compatibility under Assumption SC. Below, we identify a similar monotonicity property on $y(\cdot)$ which would be a necessary condition for sustaining gradual revelation under Assumption $\overline{SC}$.

Consider a gradual revelation mechanism where the agents reveal their private information with respect to the belief martingale $\left\{\tilde{\mu}^t\right\}_{t=0}^T$. Take an arbitrary period t and a belief μ^t in the support of $\tilde{\mu}^t$ such that μ_i^t assigns positive probabilities to types θ_i and $\hat{\theta}_i$ of agent i. Starting with this period, a possible deviation for type θ_i is following the equilibrium strategy of type $\hat{\theta}_i$ in the continuation of the mechanism. For this deviation not to be profitable, $\mathbb{E}_{\theta_{-i}|\mu_{-i}^t}\left\{u_i(y(\theta_i,\theta_{-i}),\theta_i) - u_i\left(y\left(\hat{\theta}_i,\theta_{-i}\right),\theta_i\right)\right\}$ must be at least as large as the difference between the expected equilibrium transfers to type $\hat{\theta}_i$ and type θ_i, where the expectation is taken under period t belief μ_{-i}^t. Similarly, the same expected transfer difference must be at least as large as $\mathbb{E}_{\theta_{-i}|\mu_{-i}^t}\left\{u_i\left(y(\theta_i,\theta_{-i}),\hat{\theta}_i\right) - u_i\left(y\left(\hat{\theta}_i,\theta_{-i}\right),\hat{\theta}_i\right)\right\}$, for type $\hat{\theta}_i$ not to find it optimal to start imitating type θ_i in period t. Supposing that θ_i is larger than $\hat{\theta}_i$, these two inequalities imply that $\mathbb{E}_{\theta_{-i}|\mu_{-i}^t}\left\{\bar{h}_i[y(\theta_i,\theta_{-i})]\right\} \geq \mathbb{E}_{\theta_{-i}|\mu_{-i}^t}\left\{\bar{h}_i\left[y\left(\hat{\theta}_i,\theta_{-i}\right)\right]\right\}$ under Assumption $\overline{SC}$. This discussion yields the following condition on the decision rule as a necessary condition for gradual revelation: Decision rule $y(\cdot)$ is **monotone with respect to martingale** $\left\{\tilde{\mu}^t\right\}_{t=0}^T$ if for all periods $t \leq T$, all μ^t in the support of $\tilde{\mu}^t$, and all $i \in I$, $\mathbb{E}_{\theta_{-i}|\mu_{-i}^t}\bar{h}_i[y(\theta_i,\theta_{-i})]$ is weakly increasing in θ_i *when the domain of θ_i is restricted to the support of μ_i^t.*

For an illustration of this monotonicity requirement, consider a private value auction with two bidders labelled as bidders A and B. Each bidder's private value for the auctioned object can take one of three equally likely values: $\underline{\theta} < \hat{\theta} < \bar{\theta}$. A natural choice for function $\bar{h}_i$ here is letting it be equal to bidder i's probability of receiving the object. Consider the symmetric decision rule which assigns the object to a type $\bar{\theta}$ bidder with probability 1 if and only if the rival's type is $\hat{\theta}$, and to a type $\hat{\theta}$ bidder with probability 1/2 if and only if the rival's type is $\underline{\theta}$. In all the remaining cases, no bidder receives the object. Accordingly, $\bar{h}_A$ and $\bar{h}_B$ are given as below:

$\bar{h}_A$	$\underline{\theta}_A$	$\hat{\theta}_A$	$\bar{\theta}_A$
$\underline{\theta}_B$	0	1/2	0
$\hat{\theta}_B$	0	0	1
$\bar{\theta}_B$	0	0	0

$\bar{h}_B$	$\underline{\theta}_A$	$\hat{\theta}_A$	$\bar{\theta}_A$
$\underline{\theta}_B$	0	0	0
$\hat{\theta}_B$	1/2	0	0
$\bar{\theta}_B$	0	1	0

(19)

Suppose that we want to implement this decision rule gradually with a belief martingale $\left\{\tilde{\mu}^t\right\}_{t=0}^{1}$ where $T = 1$. Since the types are assumed to be equally likely, μ_i^0 is uniform. Moreover, we construct the belief martingale such that bidder B's type is fully revealed in period 1, i.e., $\tilde{\mu}_B^1$ consists only of the three degenerate beliefs.

The decision rule is monotone for both bidders in period 0 since $\mathbb{E}_{\theta_j|\mu_j^0}\bar{h}_i\left[y\left(\theta_i, \theta_j\right)\right]$ is weakly increasing in θ_i. This requirement is trivially satisfied for bidder B in period 1 as well, since the supports of the beliefs on his types are singletons. We now discuss the restrictions on distribution $\tilde{\mu}_A^1$ of period 1 beliefs on bidder A that would ensure monotonicity for this bidder in period 1.

First, notice that this decision rule cannot be implemented in dominant strategies since $\bar{h}_A\left[y\left(\theta_A, \underline{\theta}_B\right)\right]$ is not monotone in θ_A and therefore ex-post monotonicity fails. This implies that $\tilde{\mu}_A^1$ cannot assign full weight on the prior belief μ_A^0: Bidder A cannot wait until after hearing the type of bidder B to make his first revelation. However, monotonicity of $\bar{h}_A$ would be restored if the domain of θ_A is restricted either to $\left\{\underline{\theta}_A, \hat{\theta}_A\right\}$ or to $\left\{\underline{\theta}_A, \bar{\theta}_A\right\}$ in period 1. This observation implies that the decision rule is monotone with respect to $\left\{\tilde{\mu}^t\right\}_{t=0}^{T}$ if the support of $\tilde{\mu}_A^1$ consists only of beliefs assigning positive probabilities either only to types $\underline{\theta}_A$ and $\hat{\theta}_A$ or only to types $\underline{\theta}_A$ and $\bar{\theta}_A$. In other words, the monotonicity condition demands that bidder A sends a signal fully separating his types $\hat{\theta}_A$ and $\bar{\theta}_A$ before he hears the type of bidder B, but it allows for type $\underline{\theta}_A$ to stay mixed with either one of the other two types.

With the following proposition, we establish that this monotonicity condition is also sufficient for gradual revelation.

Proposition 3 *Suppose that the single crossing property in Assumption $\overline{SC}$ holds, $\left\{\tilde{\mu}^t\right\}_{t=0}^{T}$ is a belief martingale, and $\left(y\left(\cdot\right), x\left(\cdot\right)\right)$ is an incentive compatible allocation rule under belief μ^0. There exist a gradual revelation mechanism and a sequential equilibrium of this mechanism where*

i) types are gradually revealed according to martingale $\left\{\tilde{\mu}^t\right\}_{t=0}^{T}$ and decision rule $y\left(\cdot\right)$ is implemented, *ii) the interim expected payoff for type* θ_i *of agent* i *is* $\mathbb{E}_{\theta_{-i}|\mu^0_{-i}}\left\{u_i\left(y\left(\theta_i,\theta_{-i}\right),\theta_i\right)+x_i\left(\theta_i,\theta_{-i}\right)\right\}$,

iii) transfers are budget-balanced (they add up to $\sum_{i\in I}x_i\left(\theta\right)$ *regardless of the path of revelation),*[13]

if and only if decision rule $y\left(\cdot\right)$ *is monotone with respect to martingale* $\left\{\tilde{\mu}^t\right\}_{t=0}^T$.

The "only if" part of the proposition is already discussed above. The "if" part will follow from the lemma below:

Lemma 2 *Suppose that the single crossing property in Assumption* $\overline{SC}$ *holds,* τ *is a time period such that* $0<\tau\leq T$*, decision rule* $y^{\tau-1}\left(\cdot\right)$ *is monotone with respect to martingale* $\left\{\tilde{\mu}^t\right\}_{t=0}^T$*, and allocation rule* $\left(y^{\tau-1}\left(\cdot\right),x^{\tau-1}\left(\cdot\right)\right)$ *is incentive compatible under belief* $\mu^{\tau-1}$*. There exist a decision rule* $y^{\tau}\left(\cdot,\mu^{\tau}\right)$ *and a transfer rule* $x^{\tau}\left(\cdot,\mu^{\tau}\right)$ *for all* μ^{τ} *in the support of* $\tilde{\mu}^{\tau}$ *such that*

a) allocation rule $\left(y^{\tau}\left(\cdot,\mu^{\tau}\right),x^{\tau}\left(\cdot,\mu^{\tau}\right)\right)$ *is incentive compatible under belief* μ^{τ},

b) for all θ_i *and all* i,

$$
\begin{aligned}
&\mathbb{E}_{\mu^{\tau}_{-i}|\mu^{\tau-1}_{-i}}\mathbb{E}_{\theta_{-i}|\mu^{\tau}_{-i}}\left\{u_i\left(y^{\tau}\left(\theta_i,\theta_{-i},\mu^{\tau}_i,\mu^{\tau}_{-i}\right),\theta_i\right)+x^{\tau}_i\left(\theta_i,\theta_{-i},\mu^{\tau}_i,\mu^{\tau}_{-i}\right)\right\}\\
\leq\;&\mathbb{E}_{\theta_{-i}|\mu^{\tau-1}_{-i}}\left\{u_i\left(y^{\tau-1}\left(\theta_i,\theta_{-i}\right),\theta_i\right)+x^{\tau-1}_i\left(\theta_i,\theta_{-i}\right)\right\},
\end{aligned}
\tag{20}
$$

with equality when θ_i *is in the support of* μ^{τ}_i,

c) $x^{\tau}\left(\cdot,\mu^{\tau}\right)$ *is budget-balanced:* $\sum_i x^{\tau}_i\left(\theta,\mu^{\tau}\right)=\sum_i x^{\tau-1}_i\left(\theta\right)$ *for all* $\left(\theta,\mu^{\tau}\right)$,

d) $y^{\tau}\left(\theta,\mu^{\tau}\right)=y^{\tau-1}\left(\theta\right)$ *when* θ_i *is in the support of* μ^{τ}_i *for all* i*, and*

e) $y^{\tau}\left(\theta,\mu^{\tau}\right)$ *is monotone with respect to* $\left\{\tilde{\mu}^t\right\}_{t=0}^T$.

Proof. Given belief μ^{τ}_i, we define function $\Phi^{\mu^{\tau}_i}:\Theta_i\rightarrow\Theta_i$ as follows. Suppose that under belief $\mu^{\tau-1}_{-i}$, type θ_i of agent i has to choose a type to imitate within the support of belief μ^{τ}_i. If θ_i is in this support, incentive compatibility of $\left(y^{\tau-1}\left(\cdot\right),x^{\tau-1}\left(\cdot\right)\right)$ implies that he will choose his own type. On the other hand, if μ^{τ}_i does not assign a positive probability to θ_i, he chooses the type which would minimize his payoff loss. If there is more than one such type, we define $\Phi^{\mu^{\tau}_i}\left(\theta_i\right)$ as the highest of these types. Formally,

$$
\Phi^{\mu^{\tau}_i}\left(\theta_i\right)=\begin{cases}\theta_i & \text{if } \theta_i\in supp\left(\mu^{\tau}_i\right)\\ \max\left\{\arg\max_{\hat{\theta}_i\in supp\left(\mu^{\tau}_i\right)}\mathbb{E}_{\theta_{-i}|\mu^{\tau-1}_{-i}}\left\{u_i\left(y^{\tau-1}\left(\hat{\theta}_i,\theta_{-i}\right),\theta_i\right)+x^{\tau-1}_i\left(\hat{\theta}_i,\theta_{-i}\right)\right\}\right\} & \text{if } \theta_i\notin supp\left(\mu^{\tau}_i\right)\end{cases}
$$

[13] The equilibrium is consistent with the intended allocation rule in the sense that the agents expect the same interim payoffs, the equilibrium economic alternative is $y\left(\theta\right)$, and the sum of the equilibrium transfers is $\sum_{i\in I}x_i\left(\theta\right)$. Off the equilibrium path (for instance, when an agent first sends a signal and later reports an inconsistent type with this signal), the mechanism will determine an economic alternative such as $y\left(\hat{\theta}\right)$ in the range of the original decision rule. The balanced budget constraint requires the sum of the transfers to be equal to $\sum_{i\in I}x_i\left(\theta\right)$ in this case (where θ is determined by the final type reports of the agents). The proposition is still valid if we impose an alternative budget constraint and ask this sum to be equal to $\sum_{i\in I}x_i\left(\hat{\theta}\right)$.

where $supp\left(\mu_i^\tau\right)$ is the support of belief μ_i^τ.

Single crossing condition $\overline{SC}$ and incentive compatibility of $\left(y^{\tau-1}(\cdot), x^{\tau-1}(\cdot)\right)$ under $\mu_{-i}^{\tau-1}$ imply that $\Phi^{\mu_i^\tau}(\theta_i)$ is weakly increasing in θ_i: It cannot be that a higher type than θ_i chooses to imitate a lower type than $\Phi^{\mu_i^\tau}(\theta_i)$. We define decision rule $y^\tau(\cdot, \mu^\tau)$ by setting $y^\tau(\theta, \mu^\tau) = y^{\tau-1}\left(\left\{\Phi^{\mu_i^\tau}(\theta_i)\right\}_{i\in I}\right)$. This proves parts (d) and (e) of the lemma.

As a first step in the construction of transfer rule $x^\tau(\cdot, \mu^\tau)$, we define the payoff premium between two adjacent types θ_i and $\hat{\theta}_i$ of agent i under the condition that both types have to imitate types within the support of belief μ_i^τ:

$$\begin{aligned}\Delta_i^{\mu_i^\tau}\left(\theta_i, \hat{\theta}_i|\theta_{-i}\right) &= \left[u_i\left(y^{\tau-1}\left(\Phi^{\mu_i^\tau}(\theta_i), \theta_{-i}\right), \theta_i\right) + x_i^{\tau-1}\left(\Phi^{\mu_i^\tau}(\theta_i), \theta_{-i}\right)\right] \\ &\quad - \left[u_i\left(y^{\tau-1}\left(\Phi^{\mu_i^\tau}\left(\hat{\theta}_i\right), \theta_{-i}\right), \theta_i\right) + x_i^{\tau-1}\left(\Phi^{\mu_i^\tau}\left(\hat{\theta}_i\right), \theta_{-i}\right)\right]. \end{aligned} \tag{21}$$

It follows from the definition of $\Phi^{\mu_i^\tau}$ that $\mathbb{E}_{\theta_{-i}|\mu_{-i}^{\tau-1}}\Delta_i^{\mu_i^\tau}\left(\theta_i, \hat{\theta}_i|\theta_{-i}\right)$ is non-negative. Similar to our earlier proofs, we define function $g_i^{\mu_i^\tau}(\theta_i, \theta_{-i})$ for all agents i and all beliefs μ_i^τ with equations

$$\begin{aligned}& g_i^{\mu_i^\tau}(\theta_i, \theta_{-i}) - g_i^{\mu_i^\tau}\left(\hat{\theta}_i, \theta_{-i}\right) \\ =\; & \frac{\Delta_i^{\mu_i^\tau}\left(\hat{\theta}_i, \theta_i|\theta_{-i}\right)\mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^{\tau-1}}\Delta_i^{\mu_i^\tau}\left(\theta_i, \hat{\theta}_i|\tilde{\theta}_{-i}\right) - \Delta_i^{\mu_i^\tau}\left(\theta_i, \hat{\theta}_i|\theta_{-i}\right)\mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^{\tau-1}}\Delta_i^{\mu_i^\tau}\left(\hat{\theta}_i, \theta_i|\tilde{\theta}_{-i}\right)}{\mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^{\tau-1}}\Delta_i^{\mu_i^\tau}\left(\theta_i, \hat{\theta}_i|\tilde{\theta}_{-i}\right) + \mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^{\tau-1}}\Delta_i^{\mu_i^\tau}\left(\hat{\theta}_i, \theta_i|\tilde{\theta}_{-i}\right)}\end{aligned} \tag{22}$$

where θ_i and $\hat{\theta}_i$ are two adjacent types, and

$$\mathbb{E}_{\theta_i|\mu_i^\tau}\left\{g_i^{\mu_i^\tau}(\theta_i, \theta_{-i})\right\} = 0 \tag{23}$$

for all θ_{-i}. This definition implies that

$$\mathbb{E}_{\theta_{-i}|\mu_{-i}^{\tau-1}}\left\{g_i^{\mu_i^\tau}(\theta_i, \theta_{-i})\right\} = 0 \tag{24}$$

for all θ_i and all μ_i^τ. To see this last point take the expectation of both sides of equations (22) and (23).

Finally, we define the period τ transfers with the following equation:

$$\begin{aligned}x_i^\tau(\theta, \mu^\tau) &= x_i^{\tau-1}\left(\Phi^{\mu_i^\tau}(\theta_i), \theta_{-i}\right) + \mathbb{E}_{\tilde{\theta}_{-i}|\mu_{-i}^\tau}g_i^{\mu_i^\tau}\left(\theta_i, \tilde{\theta}_{-i}\right) - \frac{1}{|I|-1}\mathbb{E}_{\tilde{\theta}_i|\mu_i^\tau}\sum_{j\neq i}\mathbb{E}_{\tilde{\theta}_{-i-j}|\mu_{-i-j}^\tau}g_j^{\mu_j^\tau}\left(\theta_j, \tilde{\theta}_i, \tilde{\theta}_{-i-j}\right) \\ &\quad + \frac{1}{|I|-1}\sum_{j\neq i}\left[x_j^{\tau-1}(\theta_j, \theta_{-j}) - x_j^{\tau-1}\left(\Phi^{\mu_j^\tau}(\theta_j), \theta_{-j}\right)\right]\end{aligned} \tag{25}$$

Budget balancedness in part (c) of the lemma holds by construction. To see the proof for part (b), notice that equations (23), (24), and $\Phi^{\mu_j^\tau}(\theta_j) = \theta_j$ for $\theta_j \in supp\left(\mu_j^\tau\right)$ imply that

$$\mathbb{E}_{\mu_{-i}^\tau|\mu_{-i}^{\tau-1}}\mathbb{E}_{\theta_{-i}|\mu_{-i}^\tau}\left\{x_i^\tau\left(\theta_i, \theta_{-i}, \mu_i^\tau, \mu_{-i}^\tau\right)\right\} = \mathbb{E}_{\theta_{-i}|\mu_{-i}^{\tau-1}}\left\{x_i^{\tau-1}\left(\Phi^{\mu_i^\tau}(\theta_i), \theta_{-i}\right)\right\}, \tag{26}$$

and

$$y^\tau\left(\theta_i, \theta_{-i}, \mu_i^\tau, \mu_{-i}^\tau\right) = y^{\tau-1}\left(\Phi^{\mu_i^\tau}(\theta_i), \theta_{-i}\right) \tag{27}$$

for all θ_{-i} in the support of μ^{τ}_{-i}. It follows from the last two equations that the left hand side of (20) equals to

$$\mathbb{E}_{\theta_{-i}|\mu^{\tau-1}_{-i}}\left\{u_i\left(y^{\tau-1}\left(\Phi^{\mu^{\tau}_i}(\theta_i),\theta_{-i}\right),\theta_i\right)+x_i^{\tau-1}\left(\Phi^{\mu^{\tau}_i}(\theta_i),\theta_{-i}\right)\right\}. \tag{28}$$

Incentive compatibility of $\left(y^{\tau-1}(\cdot),x^{\tau-1}(\cdot)\right)$ under belief μ^{τ}_{-i} implies that this last expression is weakly smaller than the right hand side of (20) and exactly equal to it for $\theta_i \in supp\left(\mu^{\tau}_i\right)$, proving part (b) of the lemma.

Assumption $\overline{SC}$ and monotonicity of function $\Phi^{\mu^{\tau}_i}$ imply that the incentive compatibility constraints in (3) between the adjacent types are sufficient for all the other incentive compatibility constraints. The expected value of the updated payoff premium to revealing the type as θ_i rather than imitating an adjacent type $\hat{\theta}_i$ (for allocation rule $\left(y^{\tau}(\cdot,\mu^{\tau}),x^{\tau}(\cdot,\mu^{\tau})\right)$) is

$$\begin{aligned}&\mathbb{E}_{\theta_{-i}|\mu^{\tau}_{-i}}\left\{\Delta_i^{\mu^{\tau}_i}\left(\theta_i,\hat{\theta}_i|\theta_{-i}\right)+g_i^{\mu^{\tau}_i}(\theta_i,\theta_{-i})-g_i^{\mu^{\tau}_i}\left(\hat{\theta}_i,\theta_{-i}\right)\right\}\\ =\;&\frac{\mathbb{E}_{\theta_{-i}|\mu^{\tau}_{-i}}\left\{\Delta_i^{\mu^{\tau}_i}\left(\hat{\theta}_i,\theta_i|\theta_{-i}\right)+\Delta_i^{\mu^{\tau}_i}\left(\theta_i,\hat{\theta}_i|\theta_{-i}\right)\right\}}{\mathbb{E}_{\tilde{\theta}_{-i}|\mu^{\tau-1}_{-i}}\Delta_i^{\mu^{\tau}_i}\left(\theta_i,\hat{\theta}_i|\tilde{\theta}_{-i}\right)+\mathbb{E}_{\tilde{\theta}_{-i}|\mu^{\tau-1}_{-i}}\Delta_i^{\mu^{\tau}_i}\left(\hat{\theta}_i,\theta_i|\tilde{\theta}_{-i}\right)}\mathbb{E}_{\tilde{\theta}_{-i}|\mu^{\tau-1}_{-i}}\Delta_i^{\mu^{\tau}_i}\left(\theta_i,\hat{\theta}_i|\tilde{\theta}_{-i}\right)\end{aligned} \tag{29}$$

under belief μ^{τ}_{-i}. To prove part (a) of the lemma, it is sufficient to show that this payoff premium is non-negative. We have already seen that the terms $\mathbb{E}_{\tilde{\theta}_{-i}|\mu^{\tau-1}_{-i}}\Delta_i^{\mu^{\tau}_i}\left(\theta_i,\hat{\theta}_i|\tilde{\theta}_{-i}\right)$ and $\mathbb{E}_{\tilde{\theta}_{-i}|\mu^{\tau-1}_{-i}}\Delta_i^{\mu^{\tau}_i}\left(\hat{\theta}_i,\theta_i|\tilde{\theta}_{-i}\right)$ are both non-negative. Moreover,

$$\begin{aligned}&\mathbb{E}_{\theta_{-i}|\mu^{\tau}_{-i}}\left\{\Delta_i^{\mu^{\tau}_i}\left(\hat{\theta}_i,\theta_i|\theta_{-i}\right)+\Delta_i^{\mu^{\tau}_i}\left(\theta_i,\hat{\theta}_i|\theta_{-i}\right)\right\}\\ =\;&\mathbb{E}_{\theta_{-i}|\mu^{\tau}_{-i}}\left\{\begin{array}{c}\left[u_i\left(y\left(\Phi^{\mu^{\tau}_i}(\theta_i),\theta_{-i}\right),\theta_i\right)-u_i\left(y\left(\Phi^{\mu^{\tau}_i}\left(\hat{\theta}_i\right),\theta_{-i}\right),\theta_i\right)\right]\\ -\left[u_i\left(y\left(\Phi^{\mu^{\tau}_i}(\theta_i),\theta_{-i}\right),\hat{\theta}_i\right)-u_i\left(y\left(\Phi^{\mu^{\tau}_i}\left(\hat{\theta}_i\right),\theta_{-i}\right),\hat{\theta}_i\right)\right]\end{array}\right\}\end{aligned} \tag{30}$$

is non-negative as well because of Assumption $\overline{SC}$ and monotonicity of function $\Phi^{\mu^{\tau}_i}$. ■

Like Lemma 1, the current lemma is also concerned with transforming an incentive compatible allocation rule under belief $\mu^{\tau-1}$ into a family of allocation rules which are all incentive compatible under the corresponding updated beliefs μ^{τ}. According to part (b) of the lemma, the resulting allocation rules yield the same expected payoff as the initial allocation rule only for the types which are in the support of the updated belief μ^{τ}_i. The types which are not in the support of this belief may end up with a strictly lower expected payoff. Hence, under allocation rule $\left(y^{\tau}(\cdot,\mu^{\tau}),x^{\tau}(\cdot,\mu^{\tau})\right)$, whenever an agent is given the opportunity to send a signal, he will be indifferent between all the signals which lead to beliefs assigning a positive weight to his realized type. Moreover, he will (weakly) prefer these signals to any other signal which would generate an inconsistent belief with his type.

In the proof of Lemma 2, we deal with the possibility of the inconsistency of the beliefs and the type reports as follows: For each type θ_i which is not assigned a positive probability by the belief μ^{τ}_i, we designate a "best type to imitate" within the support of this belief. In the event that agent

i sends a signal leading to belief μ_i^τ and then reports his type as θ_i, the constructed allocation rule $(y^\tau(\cdot,\mu^\tau), x^\tau(\cdot,\mu^\tau))$ would treat this agent as if he were the designated type in the support of μ_i^τ.[14] An implication of this construction is that function $g_i(\cdot)$, which plays a crucial role in the proof of this lemma as well, depends on the belief μ_i^τ unlike in the proofs of our earlier results. In Appendix B, we provide a numerical example to the construction of these modified allocation rules.

In order to see how Lemma 2 implies the "if" part of Proposition 3, we start with a revelation scheme with $T=1$ as we have done when discussing Proposition 2. In this case, the gradual revelation mechanism would ask the agents to send some signals in period 1 and to report their types in period 2. We relabel the signals available to agent i as the beliefs μ_i^1 in the support of $\tilde{\mu}_i^1$. We set $y^0(\cdot)$ and $x^0(\cdot)$ in Lemma 2 equal to $y(\cdot)$ and $x(\cdot)$ in Proposition 3. The mechanism would determine the economic alternative and the transfers by using the sent signals and the reported types as arguments of the functions $y^1(\cdot,\cdot)$ and $x^1(\cdot,\cdot)$ mentioned in Lemma 2.

The gradual revelation mechanism introduced above has an equilibrium where each agent i reports his type truthfully in period 2 and follows a type dependent randomization over his period 1 signals so that $\tilde{\mu}_i^1$ is the distribution over the equilibrium beliefs on his types. Part (a) of the lemma implies the sequential rationality of truthful reporting in period 2, regardless of the signals sent in the earlier period. Optimality of the signaling behavior in period 1 follows from part (b). Notice that, when the agents follow their equilibrium path of play, the implemented economic alternative is $y(\theta)$ (part d), and the agents receive the type dependent interim payoff $\mathbb{E}_{\theta_{-i}|\mu_{-i}^0}\{u_i(y(\theta_i,\theta_{-i}),\theta_i)+x_i(\theta_i,\theta_{-i})\}$ (part b). Finally, budget balancedness follows directly from part (c).

The gradual revelation mechanism above can be extended to deal with longer horizons where $T>1$. The extended version of the mechanism conditions the decision rule and the transfers to the signals sent in all periods $t=1,...,T$ and the reported types in period $T+1$ through the functions $y^T(\cdot,\mu^T)$ and $x^T(\cdot,\mu^T)$ identified by Lemma 2. These functions are iteratively determined by using the evolution of the equilibrium beliefs and setting functions $x^{T-1}(\cdot)$ and $y^{T-1}(\cdot)$ equal to $x^{T-1}(\cdot,\mu^{T-1})$ and $y^{T-1}(\cdot,\mu^{T-1})$, functions $x^{T-2}(\cdot)$ and $y^{T-2}(\cdot)$ equal to $x^{T-2}(\cdot,\mu^{T-2})$ and $y^{T-2}(\cdot,\mu^{T-2})$, and so on. This extended gradual revelation mechanism has an equilibrium where the agents' randomizations on the signals respect martingale $\{\tilde{\mu}^t\}_{t=0}^T$ and where their type reports are truthful in period $T+1$.

As mentioned in the Introduction, Mookherjee and Tsumagari (2013) consider a setting where gradual revelation is an implication of communication costs. The economic decision in question is the production level for each of the two productive agents. These agents have linear utility functions (in their respective

[14] A crucial step here is ruling out potential deviations where an agent deviates from his equilibrium behavior and conditions his type report on the revelations of the other agents. The *static* incentive compatibility constraints do not exclude the profitability of these *dynamic* deviations. Under the allocation rule we construct, these deviations are dominated by imitating the designated type regardless of the other agents' revelations.

production levels and types) and their types (production costs) are drawn from a continuum. The agents use deterministic communication strategies so that each round of communication can be represented as a partition of their type spaces. Mookherjee and Tsumagari are mainly interested in finding the optimal output function which maximizes a principal's objective subject to the incentive and communication constraints. They make important contributions to the debate on organizations by comparing the performance of centralized versus decentralized production decisions and simultaneous versus sequential communication protocols from this principal's perspective under different communication cost structures. Since the principal is risk neutral in monetary transfers, balancing the budget is not an issue for their analysis. As a preliminary result, Mookherjee and Tsumagari identify a monotonicity condition which is necessary and sufficient for implementability of an output function.

To see how our Proposition 3 relates to Mookherjee and Tsumagari's characterization, consider a belief martingale which is generated by deterministic information policies, such that each agent's type dependent strategy specifies a single signal at each period rather than a non-degenerate distribution over signals. For this belief martingale, the monotonicity condition we introduce in this section boils down to the monotonicity condition in Mookherjee and Tsumagari's paper. In this sense, Proposition 3 complements Mookherjee and Tsumagari's analysis by extending their characterization to environments where the agents have discrete types, communication strategies are stochastic, the economic decisions are not necessarily one dimensional, and the monetary transfers are budget-balanced.[15]

6 Appendix A

In this Appendix, we illustrate the construction of the budget-balanced transfers described by Proposition 2 and Lemma 1 with the help of a numerical example based on an independent private values auction. The two bidders are labelled as A and B. As in the continuous type example mentioned in the text, the private values of the bidders are uniformly distributed on the interval $[0, 1]$. We are interested in the first price auction allocation rule, where the auctioned object is efficiently allocated to the highest value

[15] The main difference between the proofs of the Mookherjee-Tsumagari result and our Proposition 3 is as follows. In order to implement decision rule $y(\cdot)$, we construct a gradual revelation mechanism which uses only the economic alternatives in the range of $y(\cdot)$. Even if an agent deviates from his equilibrium behavior and sends inconsistent signals in different stages of this mechanism, he will still be facing an economic alternative within the range of the original decision rule. By contrast, Mookherjee and Tsumagari make use of the single dimensional nature of their economic alternatives and generate an *auxiliary decision rule* for off-the-equilibrium-path events. The resulting mechanism makes the agents in their model indifferent between all the signals that are available to them, even though each type of each agent chooses a single signal (per period) with probability one in equilibrium. In our case, the gradual revelation mechanism ensures that each type of each agent is indifferent between only the signals he would send with positive probability on the equilibrium path, but this indifference does not necessarily extend to the signals which are not supposed to be sent by this particular type.

bidder and this bidder pays half of his value as the price of the object.[16] The resulting transfer rule for bidder A is described as

$$x_i(\theta_i,\theta_j) = \begin{cases} -\theta_i/2 & \text{if } \theta_i > \theta_j \\ 0 & \text{otherwise} \end{cases}. \tag{31}$$

We want to implement this allocation rule or a budget-balanced close variant of it with a gradual implementation mechanism, which would generate the symmetric belief martingale $\{\tilde{\mu}^t\}_{t=0}^{1}$. Distribution $\tilde{\mu}_i^0$ is degenerate at the uniform prior μ_i^0 and distribution $\tilde{\mu}_i^1$ has two equally likely beliefs, μ_i^{high} and μ_i^{low}, in its support. Beliefs μ_i^{high} and μ_i^{low} are represented by cumulative distribution functions θ_i^2 and $2\theta_i - \theta_i^2$ respectively on the full support $[0,1]$. These beliefs can be sustained by an information policy where bidder i sends a "high" signal with probability θ_i and a "low" signal with probability $1-\theta_i$.

We now concentrate on the derivation of the gradual revelation transfers for bidder A. The analogous transfers for bidder B can be constructed identically. The proof of Lemma 1 refers to function g_A^1 which can transform an implementable transfer rule into a dominant strategy implementable one. When the type space is discrete, this function can be constructed by referring to the payoff premium Δ_A^1 defined in the same proof. The continuous type assumption in this example allows us to follow a more direct approach. The only transfers that would achieve incentive compatibility of the efficient allocation with dominant strategies in a continuous type setting are the *Vickrey Clarke Groves* transfers below:

$$x_A^{DS}(\theta_A,\theta_B) = \begin{cases} -\theta_B + k(\theta_B) & \text{if } \theta_A > \theta_B \\ k(\theta_B) & \text{otherwise} \end{cases}, \tag{32}$$

where $k(\cdot)$ gives a constant term which does not depend on the type of bidder A. (Notice that, when $k(\cdot)$ is set to be zero, these transfers can be implemented by the second price auction.) Accordingly, function $g_A^1(\theta_A,\theta_B)$ will have the form

$$g_A^1(\theta_A,\theta_B) = x_A^{DS}(\theta_A,\theta_B) - x_A(\theta_A,\theta_B) = \begin{cases} \theta_A/2 - \theta_B + k(\theta_B) & \text{if } \theta_A > \theta_B \\ k(\theta_B) & \text{otherwise} \end{cases}. \tag{33}$$

Equation (15) in the proof of the lemma yields the constant $k(\theta_B)$ and therefore pins down function g_A^1:

$$g_A^1(\theta_A,\theta_B) = \begin{cases} \frac{\theta_A}{2} - \frac{3}{4}\theta_B^2 - \frac{1}{4} & \text{if } \theta_A > \theta_B \\ \theta_B - \frac{3}{4}\theta_B^2 - \frac{1}{4} & \text{otherwise} \end{cases}. \tag{34}$$

The next step involves taking the expectation of this function under the equilibrium beliefs that bidder A may have about the type of bidder B:

$$\begin{aligned} \mathbb{E}_{\tilde{\theta}_B|\mu_B^{high}}\left\{g_A^1\left(\theta_A,\tilde{\theta}_B\right)\right\} &= \int_0^1 g_A^1\left(\theta_A,\tilde{\theta}_B\right)\left[2\tilde{\theta}_B\right]d\tilde{\theta}_B = -\frac{1}{6}\theta_A^3 + \frac{1}{24}, \\ \mathbb{E}_{\tilde{\theta}_B|\mu_B^{low}}\left\{g_A^1\left(\theta_A,\tilde{\theta}_B\right)\right\} &= \int_0^1 g_A^1\left(\theta_A,\tilde{\theta}_B\right)\left[2-2\tilde{\theta}_B\right]d\tilde{\theta}_B = \frac{1}{6}\theta_A^3 - \frac{1}{24}, \end{aligned} \tag{35}$$

[16] To simplify the exposition, in the zero probability event that both bidders have the same value, we assume that the object is not allocated to either bidder.

where the terms in the square brackets are the density functions derived from the cumulative distribution functions.

Now we are ready to give the belief dependent transfer rule $x_A^1\left(\theta_A,\theta_B,\mu_A^1,\mu_B^1\right)$ by using equation (17) in the proof of Lemma 1:

$$x_A^1\left(\theta_A,\theta_B,\mu_A^{high},\mu_B^{high}\right) = x_A\left(\theta_A,\theta_B\right)-\frac{1}{6}\theta_A^3+\frac{1}{24}+\frac{1}{6}\theta_B^3-\frac{1}{24}=\begin{cases} -\frac{\theta_A}{2}-\frac{1}{6}\theta_A^3+\frac{1}{6}\theta_B^3 & \text{if } \theta_A>\theta_B \\ -\frac{1}{6}\theta_A^3+\frac{1}{6}\theta_B^3 & \text{otherwise}\end{cases},$$

$$x_A^1\left(\theta_A,\theta_B,\mu_A^{low},\mu_B^{high}\right) = x_A\left(\theta_A,\theta_B\right)-\frac{1}{6}\theta_A^3+\frac{1}{24}-\frac{1}{6}\theta_B^3+\frac{1}{24}=\begin{cases} -\frac{\theta_A}{2}-\frac{1}{6}\theta_A^3-\frac{1}{6}\theta_B^3+\frac{1}{12} & \text{if } \theta_A>\theta_B \\ -\frac{1}{6}\theta_A^3-\frac{1}{6}\theta_B^3+\frac{1}{12} & \text{otherwise}\end{cases},$$

$$x_A^1\left(\theta_A,\theta_B,\mu_A^{high},\mu_B^{low}\right) = x_A\left(\theta_A,\theta_B\right)+\frac{1}{6}\theta_A^3-\frac{1}{24}+\frac{1}{6}\theta_B^3-\frac{1}{24}=\begin{cases} -\frac{\theta_A}{2}+\frac{1}{6}\theta_A^3+\frac{1}{6}\theta_B^3-\frac{1}{12} & \text{if } \theta_A>\theta_B \\ +\frac{1}{6}\theta_A^3+\frac{1}{6}\theta_B^3-\frac{1}{12} & \text{otherwise}\end{cases},$$

$$x_A^1\left(\theta_A,\theta_B,\mu_A^{low},\mu_B^{low}\right) = x_A\left(\theta_A,\theta_B\right)+\frac{1}{6}\theta_A^3-\frac{1}{24}-\frac{1}{6}\theta_B^3+\frac{1}{24}=\begin{cases} -\frac{\theta_A}{2}+\frac{1}{6}\theta_A^3-\frac{1}{6}\theta_B^3 & \text{if } \theta_A>\theta_B \\ +\frac{1}{6}\theta_A^3-\frac{1}{6}\theta_B^3 & \text{otherwise}\end{cases}.$$

These transfers are budget-balanced and they make the efficient allocation of the object incentive compatible under the updated beliefs in period 1. Moreover, regardless of μ_A^1 equals to μ_A^{high} or μ_A^{low},

$$\mathbb{E}_{\mu_B^1|\mu_B^0}\mathbb{E}_{\theta_B|\mu_B^1}x_A^1\left(\theta_A,\theta_B,\mu_A^1,\mu_B^1\right)=\mathbb{E}_{\theta_B|\mu_B^0}x_A\left(\theta_A,\theta_B\right)=-\frac{\theta_A^2}{2}. \tag{36}$$

We can use function x_A^1 above to construct a gradual revelation mechanism, where the bidders send either the high signal or the low signal in period 1 and reveal their type in period 2. The object is allocated to the bidder with the highest reported type. The transfers for the bidders are determined by function x_A^1 where arguments θ_A and θ_B are the reported types in period 2 and beliefs μ_A^1 and μ_B^1 are determined by the signals in period 1. This gradual revelation mechanism has an equilibrium where each bidder i sends the high signal with probability θ_i in period 1 (since he is indifferent between the signals) and reveals his type truthfully in period 2.

7 Appendix B

In this Appendix, we illustrate the construction of the gradual revelation mechanism described by Proposition 3 and Lemma 2 with the help of a numerical example based on an independent private values auction. The two bidders are referred to as bidders A and B. As in the example worked out in the text just before the statement of the proposition, we assume that bidder i's private value for the auctioned object can take one of three equally likely values $\underline{\theta}_i$, $\hat{\theta}_i$, or $\bar{\theta}_i$. We assume further that $\underline{\theta}_i=\underline{\theta}$, $\hat{\theta}_i=2\underline{\theta}$, and $\bar{\theta}_i=3\underline{\theta}$ for both bidders. We want to implement the symmetric decision rule described in table (19) given in the text, with $\bar{h}_i$ representing the probability that agent i receives the auctioned object. The transfer for each bidder depends on his own value but not on the value of the other bidder, as described by the transfer rule below:

$$x_i(\theta_i, \theta_j) = \begin{cases} 0 & \text{if } \theta_i = \underline{\theta}_i \\ -\frac{1}{4}\underline{\theta} & \text{if } \theta_i = \hat{\theta}_i \\ -\frac{2}{3}\underline{\theta} & \text{if } \theta_i = \bar{\theta}_i \end{cases} \tag{37}$$

for all θ_j. Notice that these decision and transfer rules constitute an incentive compatible allocation rule. We want to implement this allocation rule or a budget-balanced close variant of it with a gradual revelation mechanism generating the belief martingale $\{\tilde{\mu}^t\}_{t=0}^1$. Distribution $\tilde{\mu}_i^0$ is degenerate at the uniform prior μ_i^0 and distribution $\tilde{\mu}_i^1$ has two equally likely beliefs in its support: $\bar{\mu}_i^1$ and $\hat{\mu}_i^1$. Belief $\bar{\mu}_i^1$ assigns probability 2/3 to type $\bar{\theta}_i$ and probability 1/3 to type $\underline{\theta}_i$. Belief $\hat{\mu}_i^1$ assigns probability 2/3 to type $\hat{\theta}_i$ and probability 1/3 to type $\underline{\theta}_i$. These beliefs can be supported with an information policy such that types $\bar{\theta}_i$ and $\hat{\theta}_i$ send separate signals in period 1 and type $\underline{\theta}_i$ randomizes between these signals with equal probabilities.

The first point we establish is that the decision rule in table (19) is monotone with respect to the belief martingale above. The following table gives the expected probability of receiving the object for bidder A under the equilibrium beliefs he may hold during the implementation:

$\bar{h}_A$	$\underline{\theta}_A$	$\hat{\theta}_A$	$\bar{\theta}_A$
μ_B^0	0	1/6	1/3
$\hat{\mu}_B^1$	0	1/6	2/3
$\bar{\mu}_B^1$	0	1/6	0

(38)

Notice that the acquisition probabilities are increasing in bidder A's type under beliefs μ_B^0 and $\hat{\mu}_B^1$ but not under belief $\bar{\mu}_B^1$. Yet, monotonicity with respect to the martingale in question is satisfied, since this martingale fully separates types $\hat{\theta}_A$ and $\bar{\theta}_A$ (the two types responsible for the non-monotonicity) in period 1.

In what follows, we will concentrate on the construction of the gradual revelation mechanism transfers for bidder A. The analogous transfers for bidder B can be constructed identically. Under the targeted allocation rule and the prior belief μ_B^0, type $\bar{\theta}_A$ prefers to imitate type $\hat{\theta}_A$ rather than type $\underline{\theta}_A$. Therefore, type $\bar{\theta}_A$ would mimic type $\hat{\theta}_A$ if he is restricted to choose a type in the support of $\hat{\mu}_A^1$. In the language of the proof of Lemma 2, this means that $\Phi^{\hat{\mu}_A^1}(\bar{\theta}_A) = \hat{\theta}_A$. Similarly, $\Phi^{\bar{\mu}_A^1}(\hat{\theta}_A) = \bar{\theta}_A$.[17] For all other (μ_A^1, θ_A) pairs, $\Phi^{\mu_A^1}(\theta_A)$ is equal to θ_A.

Following the proof of Lemma 2, we calculate the values of the payoff premium functions $\Delta_A^{\bar{\mu}_A^1}$ and

[17] To be accurate, type $\hat{\theta}_A$ is indifferent between imitating type $\bar{\theta}_A$ or type $\underline{\theta}_A$. The proof of the lemma sets $\Phi^{\bar{\mu}_A^1}(\hat{\theta}_A)$ as $\bar{\theta}_A$ since this function designates the *highest* of the "best types to imitate."

$\Delta_A^{\hat{\mu}_A^1}$ as defined in (21):

$$\Delta_A^{\bar{\mu}_A^1}\left(\theta_A, \theta'_A | \theta_B\right) = \begin{cases} 0 & \text{if } \left(\theta_A, \theta'_A\right) = \left(\bar{\theta}_A, \hat{\theta}_A\right) \text{ or } \left(\hat{\theta}_A, \bar{\theta}_A\right) \\ \frac{4}{3}\underline{\theta} & \text{if } \left(\theta_A, \theta'_A\right) = \left(\hat{\theta}_A, \underline{\theta}_A\right) \text{ and } \theta_B = \hat{\theta}_B \\ -\frac{2}{3}\underline{\theta} & \text{if } \left(\theta_A, \theta'_A\right) = \left(\hat{\theta}_A, \underline{\theta}_A\right) \text{ and } \theta_B \neq \hat{\theta}_B \\ -\frac{1}{3}\underline{\theta} & \text{if } \left(\theta_A, \theta'_A\right) = \left(\underline{\theta}_A, \hat{\theta}_A\right) \text{ and } \theta_B = \hat{\theta}_B \\ \frac{2}{3}\underline{\theta} & \text{if } \left(\theta_A, \theta'_A\right) = \left(\underline{\theta}_A, \hat{\theta}_A\right) \text{ and } \theta_B \neq \hat{\theta}_B \end{cases} \tag{39}$$

$$\Delta_A^{\hat{\mu}_A^1}\left(\theta_A, \theta'_A | \theta_B\right) = \begin{cases} 0 & \text{if } \left(\theta_A, \theta'_A\right) = \left(\bar{\theta}_A, \hat{\theta}_A\right) \text{ or } \left(\hat{\theta}_A, \bar{\theta}_A\right) \\ \frac{3}{4}\underline{\theta} & \text{if } \left(\theta_A, \theta'_A\right) = \left(\hat{\theta}_A, \underline{\theta}_A\right) \text{ and } \theta_B = \underline{\theta}_B \\ -\frac{1}{4}\underline{\theta} & \text{if } \left(\theta_A, \theta'_A\right) = \left(\hat{\theta}_A, \underline{\theta}_A\right) \text{ and } \theta_B \neq \underline{\theta}_B \\ -\frac{1}{4}\underline{\theta} & \text{if } \left(\theta_A, \theta'_A\right) = \left(\underline{\theta}_A, \hat{\theta}_A\right) \text{ and } \theta_B = \underline{\theta}_B \\ \frac{1}{4}\underline{\theta} & \text{if } \left(\theta_A, \theta'_A\right) = \left(\underline{\theta}_A, \hat{\theta}_A\right) \text{ and } \theta_B \neq \underline{\theta}_B \end{cases} \tag{40}$$

These payoff premium functions yield the following expected values:

$$\begin{aligned} \mathbb{E}_{\tilde{\theta}_B|\mu_B^0}\left\{\Delta_A^{\bar{\mu}_A^1}\left(\bar{\theta}_A, \hat{\theta}_A | \tilde{\theta}_B\right)\right\} &= \mathbb{E}_{\tilde{\theta}_B|\mu_B^0}\left\{\Delta_A^{\bar{\mu}_A^1}\left(\hat{\theta}_A, \bar{\theta}_A | \tilde{\theta}_B\right)\right\} = \mathbb{E}_{\tilde{\theta}_B|\mu_B^0}\left\{\Delta_A^{\bar{\mu}_A^1}\left(\hat{\theta}_A, \underline{\theta}_A | \tilde{\theta}_B\right)\right\} = 0 \\ \mathbb{E}_{\tilde{\theta}_B|\mu_B^0}\left\{\Delta_A^{\bar{\mu}_A^1}\left(\underline{\theta}_A, \hat{\theta}_A | \tilde{\theta}_B\right)\right\} &= \frac{1}{3} \\ \mathbb{E}_{\tilde{\theta}_B|\mu_B^0}\left\{\Delta_A^{\hat{\mu}_A^1}\left(\bar{\theta}_A, \hat{\theta}_A | \tilde{\theta}_B\right)\right\} &= \mathbb{E}_{\tilde{\theta}_B|\mu_B^0}\left\{\Delta_A^{\hat{\mu}_A^1}\left(\hat{\theta}_A, \bar{\theta}_A | \tilde{\theta}_B\right)\right\} = 0 \\ \mathbb{E}_{\tilde{\theta}_B|\mu_B^0}\left\{\Delta_A^{\hat{\mu}_A^1}\left(\hat{\theta}_A, \underline{\theta}_A | \tilde{\theta}_B\right)\right\} &= \mathbb{E}_{\tilde{\theta}_B|\mu_B^0}\left\{\Delta_A^{\hat{\mu}_A^1}\left(\underline{\theta}_A, \hat{\theta}_A | \tilde{\theta}_B\right)\right\} = \frac{1}{12} \end{aligned} \tag{41}$$

When we substitute the expressions above in (22) and use equation (23), we identify functions $g_A^{\bar{\mu}_A^1}$ and $g_A^{\hat{\mu}_A^1}$ as

$$g_A^{\bar{\mu}_A^1}\left(\theta_A, \theta_B\right) = \begin{cases} \frac{8}{9}\underline{\theta} & \text{if } \theta_A = \underline{\theta}_A \text{ and } \theta_B = \hat{\theta}_B \\ \frac{2}{9}\underline{\theta} & \text{if } \theta_A \neq \underline{\theta}_A \text{ and } \theta_B \neq \hat{\theta}_B \\ -\frac{4}{9}\underline{\theta} & \text{otherwise} \end{cases}, \tag{42}$$

$$g_A^{\hat{\mu}_A^1}\left(\theta_A, \theta_B\right) = \begin{cases} \frac{4}{12}\underline{\theta} & \text{if } \theta_A = \underline{\theta}_A \text{ and } \theta_B = \underline{\theta}_B \\ \frac{1}{12}\underline{\theta} & \text{if } \theta_A \neq \underline{\theta}_A \text{ and } \theta_B \neq \underline{\theta}_B \\ -\frac{2}{12}\underline{\theta} & \text{otherwise} \end{cases}. \tag{43}$$

The next step is finding the expected values of $g_A^{\bar{\mu}_A^1}$ and $g_A^{\hat{\mu}_A^1}$ under the equilibrium beliefs on bidder B's type:

$$\mathbb{E}_{\theta_B|\mu_B}\left\{g_A^{\bar{\mu}_A^1}\left(\theta_A, \theta_B\right)\right\} = \begin{cases} -\frac{4}{9}\underline{\theta} & \text{if } \theta_A = \underline{\theta}_A \text{ and } \mu_B = \bar{\mu}_B^1 \\ \frac{2}{9}\underline{\theta} & \text{if } \theta_A \neq \underline{\theta}_A \text{ and } \mu_B = \bar{\mu}_B^1 \\ \frac{4}{3}\underline{\theta} & \text{if } \theta_A = \underline{\theta}_A \text{ and } \mu_B = \hat{\mu}_B^1 \\ -\frac{2}{3}\underline{\theta} & \text{if } \theta_A \neq \underline{\theta}_A \text{ and } \mu_B = \hat{\mu}_B^1 \end{cases} \tag{44}$$

and

$$\mathbb{E}_{\theta_B|\mu_B}\left\{g_A^{\hat{\mu}_A^1}(\theta_A,\theta_B)\right\}=0 \text{ for all } \theta_A \text{ and } \mu_B=\hat{\mu}_B^1 \text{ or } \bar{\mu}_B^1. \tag{45}$$

Now we are ready to identify function $x_A^1(\theta_A,\theta_B,\mu_A^1,\mu_B^1)$ which yields the transfer to bidder A as a function of the signals and the type reports by using equation (25). The values of this function are reported in the tables below:

$x_A^1(\cdot,\bar{\mu}_A^1,\bar{\mu}_B^1)$	$\underline{\theta}_A$	$\hat{\theta}_A$	$\bar{\theta}_A$
$\underline{\theta}_B$	0	0	0
$\hat{\theta}_B$	$-\frac{1}{4}\underline{\theta}$	$-\frac{1}{4}\underline{\theta}$	$-\frac{1}{4}\underline{\theta}$
$\bar{\theta}_B$	$-\frac{2}{3}\underline{\theta}$	$-\frac{2}{3}\underline{\theta}$	$-\frac{2}{3}\underline{\theta}$

$x_A^1(\cdot,\hat{\mu}_A^1,\bar{\mu}_B^1)$	$\underline{\theta}_A$	$\hat{\theta}_A$	$\bar{\theta}_A$
$\underline{\theta}_B$	$-\frac{4}{3}\underline{\theta}$	$-\frac{19}{12}\underline{\theta}$	$-\frac{19}{12}\underline{\theta}$
$\hat{\theta}_B$	$\frac{13}{12}\underline{\theta}$	$\frac{5}{6}\underline{\theta}$	$\frac{5}{6}\underline{\theta}$
$\bar{\theta}_B$	$\frac{2}{3}\underline{\theta}$	$\frac{5}{12}\underline{\theta}$	$\frac{5}{12}\underline{\theta}$

$x_A^1(\cdot,\bar{\mu}_A^1,\hat{\mu}_B^1)$	$\underline{\theta}_A$	$\hat{\theta}_A$	$\bar{\theta}_A$
$\underline{\theta}_B$	$\frac{4}{3}\underline{\theta}$	$-\frac{4}{3}\underline{\theta}$	$-\frac{4}{3}\underline{\theta}$
$\hat{\theta}_B$	$\frac{4}{3}\underline{\theta}$	$-\frac{4}{3}\underline{\theta}$	$-\frac{4}{3}\underline{\theta}$
$\bar{\theta}_B$	$\frac{11}{12}\underline{\theta}$	$-\frac{7}{4}\underline{\theta}$	$-\frac{7}{4}\underline{\theta}$

$x_A^1(\cdot,\hat{\mu}_A^1,\hat{\mu}_B^1)$	$\underline{\theta}_A$	$\hat{\theta}_A$	$\bar{\theta}_A$
$\underline{\theta}_B$	0	$-\frac{1}{4}\underline{\theta}$	$-\frac{1}{4}\underline{\theta}$
$\hat{\theta}_B$	0	$-\frac{1}{4}\underline{\theta}$	$-\frac{1}{4}\underline{\theta}$
$\bar{\theta}_B$	$-\frac{5}{12}\underline{\theta}$	$-\frac{2}{3}\underline{\theta}$	$-\frac{2}{3}\underline{\theta}$

As an example to the calculation of this function, consider the off-the-equilibrium-path situation where both bidders send the signal leading to belief $\bar{\mu}_i^1$ and then reveal their types as $\hat{\theta}_i$. In this case, the transfer to bidder A is determined as

$$\begin{aligned}&x_A^1\left(\hat{\theta}_A,\hat{\theta}_B,\bar{\mu}_A^1,\bar{\mu}_B^1\right)\\=\;&\underbrace{x_A\left(\bar{\theta}_A,\hat{\theta}_B\right)}_{-\frac{2}{3}\underline{\theta}}+\underbrace{\mathbb{E}_{\theta_B|\bar{\mu}_B^1}\left\{g_A^{\bar{\mu}_A^1}\left(\hat{\theta}_A,\theta_B\right)\right\}}_{\frac{2}{9}\underline{\theta}}-\underbrace{\mathbb{E}_{\theta_A|\bar{\mu}_A^1}\left\{g_B^{\bar{\mu}_A^1}\left(\theta_A,\hat{\theta}_B\right)\right\}}_{\frac{2}{9}\underline{\theta}}+\underbrace{x_B\left(\hat{\theta}_A,\hat{\theta}_B\right)}_{-\frac{1}{4}\underline{\theta}}-\underbrace{x_B\left(\hat{\theta}_A,\bar{\theta}_B\right)}_{-\frac{2}{3}\underline{\theta}}=-\frac{1}{4}\underline{\theta}.\end{aligned}$$

This gives us the number in the middle of the upper left table above. All the other entries in these tables are calculated in the same way.

As mentioned above, transfer function x_B^1 can be constructed symmetrically to function x_A^1. The proof of Lemma 2 yields the belief dependent decision rule as $y^1(\theta_A,\theta_B,\mu_A^1,\mu_B^1)=y\left(\Phi^{\mu_A^1}(\theta_A),\Phi^{\mu_B^1}(\theta_B)\right)$. Allocation rule (x_A^1,x_B^1,y^1) satisfies conditions (a) to (e) listed in Lemma 2. We can use this allocation rule to construct a gradual revelation mechanism, where the bidders send either the high signal (leading to belief $\bar{\mu}_i^1$ in equilibrium) or the low signal (leading to $\hat{\mu}_i^1$) in period 1 and report their types in period 2. The allocation decision and the transfers are determined by the signals and the type reports through (x_A^1,x_B^1,y^1). This gradual revelation mechanism has an equilibrium where bidders report their true types in period 2, type $\bar{\theta}_i$ sends the high signal in period 1, type $\hat{\theta}_i$ sends the low signal in period 1, and type $\underline{\theta}_i$ randomizes between these two signals with equal probabilities.

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