Sequential Auctions with Budget-Constrained Bidders*

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Abstract

Many important auctions—from spectrum rights to collectors' items on eBay—involve the same set of bidders repeated over time and bidders may face budget constraints in the course of auctions. We consider sequential auctions in which bidders have multi-unit demand, but are potentially budget constrained. For the case of two bidders with binary valuations and two auctions, we first derive the equilibrium to the second-price auction. In contrast to the standard static analogue, the auction involves mixed strategies in the first auction. The results are used as a departure point for deriving the optimal mechanism when the two auctions are independently conducted by separate designers. While the second auction mechanism is very sensitive to the amount of the bidders' remaining budgets a deterministic direct mechanism is employed. In contrast, the optimal mechanism in the first auction involves offering lotteries to the bidders that assure that with positive probability a least one bidders budget constraint binds in the second auction.

Keywords: sequential auctions, budget constraints, optimal auctions

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1 Introduction

In many sequential auctions—including large scale ones such a for spectrum rights and smaller ones such as collector's items on eBay—the same set of bidders frequently interact and compete against one-another repeatedly. While such sequential settings have been given much attention of late in the literature on auctions [cites here], an important aspect that often features in such settings has by and large not been discussed; namely that in the course of such sequential auctions bidders may have limited budgets available to them for the purposes of bidding and therefore my deplete their budget in the course of the sequence of sales. Indeed, especially in larger auctions, bidders may be capital constrained, implying that budgets may be depleted in the course of bidding (see, e.g., Cramton 1995 or Burguet and McAfee 2005). With the notable exception of Pitchik 2009 the implications of budget constraints in sequential auctions has not yet been extensively explored. The present manuscript contains some preliminary observations and results concerning bidding in such settings and the design of optimal mechanisms when auctioneers are independent, but the group of bidders remain the same across auctions. In contrast to the work by Pitchik, our model differs in that we consider stochastically equivalent goods whose values are drawn only just prior to commencement of the auction, whereas Pitchik considers heterogenous goods with *ex ante* privately known values. Moreover, her focus is then specifically on the implications of the order sales (an issue that does not arise in our framework); while we are interested in how limited budgets affect bidding and optimal mechanism especially when absent budget constraints the auctions are otherwise equivalent.

The implication of budget constraints in sequential auctions is three-fold. An immediate implication of potentially depleted budgets in latter auctions is that bidders' bids are capped at their remaining budget, affecting equilibrium bidding when their valuations exceed their budgets. Forward looking bidders in earlier auctions must anticipate this, which brings about two further considerations. First, a bidder must determine whether it is worth shielding her budget in order to remain more competitive in the subsequent auctions. But second, driving up first period bids has the effect of depleting the rival's budget in the first auction when one loses—which reduces competition in the subsequent auctions, as a rival's bidding ability is curtailed due to high first-auction prices.

After introducing the basic model in Section 2 we consider various aspects of bidding in different settings in order to draw out some of the salient features in sequential auctions with budget-constrained bidders. In particular, in Section 3 we consider the equilibrium to a sequence of second-price auctions. Here we assume that budgets are sufficiently generous to cover the first period equilibrium price with budgets remaining for the second auction; while the amount carried over into the second is insufficient to win the second auction. We show that bidders with low values in the first auction bid aggressively in the first auction as losing with high bids entails greater payoffs in the second auction. Indeed, in contrast to static settings (with or without budget constraints) bidders utilize mixed strategies as they trade-off higher probabilities of winning the first auction with higher expected payoffs in the second auction when a rival's budget is depleted.

In Section 4 we consider the optimal direct mechanism in the second of two auctions, allowing for all possible budget constellations. While the analysis is broadly separated into two cases—depending on bidders' possible values—each of these falls into several sub-cases depending on the budgets that are available to the bidders. This analysis is followed in Section 5 with a derivation of the optimal auction mechanism across the sequence of auctions, however, making the simplifying assumption that there is only one bidder. The insights from Sections 4 and 5 are used in Section 6 to comment on aspects of the optimal mechanism in the first auction of the two-bidder model. In particular, budget constraints generate bidder surpluses in the second auction mechanism when bidders may obtain goods below their value when their budgets are capped. The first auction mechanism designer is able to extract these potential surpluses in the first auction mechanism by introducing mean-preserving lotteries that place positive weight on binding budget constraints for the bidders.

2 The Basic Model

Consider two risk-neutral bidders $i = 1, 2$ who are in a sequence of two auctions for two (stochastically) identical indivisible goods. Bidders' values V_i are i.i.d. draws from a two-point distribution with $V_i \in \{\underline{v}, \overline{v}\}$ (where $\underline{v} > 0$)[1] with $\rho := \Pr\{V = \overline{v}\}$. Bidders' values are drawn at the beginning of each auction. All bids are made public at the conclusion of an auction. Thus, values for the second good are not known until after the first auction is completed with a revelation of the bids placed in the first auction.

Both bidders have a budget of $m \geq \overline{v}$ available to them for purposes of bidding, with remaining money being directly consumed so that a bidders' payoffs when not participating in an auction is given by m. Budgets are (for simplicity) assumed to be the same for both bidders and are common knowledge.

We first consider the equilibrium to a sequence of second-price auctions. Then we derive properties of the optimal direct mechanisms when the auctioneers are distinct and act independently of one-another.

3 A Sequence of Second-Price Auctions

Using backward induction we first consider the final auction, before deriving the equilibrium for the first auction.

3.1 The Second Auction

The second auction is essentially a standard static second-price sealed bid auction. However, with (potentially binding) budget constraints bidders may no longer be able to afford to pay their value, placing an upper bound on the bid placed in the second auction. In general, with budget constraints the bidder's optimal strategy in a second price auction is to bid $b_i = \min\{m_i^2, v_i\}$, where m_i^2 denotes the budget available to bidder i in the second auction.

As is demonstrated below, having assumed that $m \geq \overline{v}$, the bidder who loses the first auction is not budget-constrained in the second auction, whereas the winner of the first auction may be, depending on the price paid in the first auction. Letting p_1 denote the price paid in the first auction, the equilibrium in the second auction is thus summarized by,

[1] So that the potential for meaningful bid-shading below $\underline{v}$ exists.

Proposition 1 (Second Auction Bidding in the Second-Price Format) *Equilibrium bidding of the second auction in second-price sealed bid auctions is given by*

$$b_i^* = \begin{cases} \min\{m - p_1, v_i\} & \text{if } i \text{ won the first auction,} \\ v_i & \text{else.} \end{cases} \tag{1}$$

It follows from Proposition 1 that since equilibrium bids in the second auction are a function of the first auction price, expected payoffs in the second auction depend on the outcome of the first auction. As bidders are forward-looking in the first auction, they therefore must anticipate how their first-auction bids potentially impact their second auction payoffs. Thus, the equilibrium of the first auction must be solved while simultaneously accounting for the implications of bidding in the first auction on payoffs in the second auction. In general this can be done by (1) postulating possible equilibrium prices of the first auction, (2) using these to determine second-auction payoffs, (3) determining *expected* second-auction payoffs, as a function of first period strategies, and (4) confirming that equilibrium bidding in the first auction generates the price structure initially postulated, thus confirming the equilibrium.

In the present exposition we restrict attention to cases in which budgets m are sufficiently tight (i.e., small, relative to V) so that—in equilibrium—the winner of the first auction is surely unable to bid above $\underline{v}$ in the second auction.[2] In order to formalize this, let B denote the set of possible equilibrium bids in the first auction with $\underline{b} := \inf\{B\}$ then we suppose that $m - \underline{b} < \underline{v}$.[3]

Then expected payoffs for the second period auction are those given in Table 1.

[2]Cases with less severe budget constraints have similar features as the equilibrium we consider. In the interest of expediency we abstract from these cases, although these analyses are available from the authors upon request.

[3]As is derived later, this implies that $m < 2\underline{v} - \rho(1-\rho)\,(\overline{v} - \underline{v})$ (see Eq. 8). We maintain throughout that $m \geq \overline{v}$.

history	value	bid	payoff	expected payoff
Won 1st	$\overline{v}\vert w$	$m - p_1$	0	0
	$\underline{v}\vert w$	$m - p_1$	0	
Lost 1st	$\overline{v}\vert l$	$\overline{v}$	$\overline{v} - (m - p_1)$	$\rho\overline{v} + (1-\rho)\underline{v} - (m - p_1)$
	$\underline{v}\vert l$	$\underline{v}$	$\underline{v} - (m - p_1)$	

Table 1: Equilibrium in the Second Auction with Severe Budget Constraints

3.2 The First Auction

3.2.1 The High-Value Bidder's Strategy

Similar to a static second-price auction, in the first auction, the high valued bidder places the bid that makes him indifferent between winning and losing the first auction whenever he is called upon to actually pay the bid he placed. A high-value bidder who places the bid b_i and loses the first auction obtains an overall expected payoff of

$$0 + \left[\rho\overline{v} + \underline{v}(1-\rho) - (m - b_i)\right],$$

since in losing the first auction, his bid becomes the price paid by his rival in the first auction, i.e., $p_1 = b_i$.

If he wins, and has to pay his bid of b_i his expected payoff is

$$[\overline{v} - b_i] + 0.$$

The equilibrium bid, denoted by $\overline{b}$, equates these two payoffs. Hence,

$$\overline{b} = \frac{(1-\rho)\left(\overline{v} - \underline{v}\right) + m}{2}. \tag{2}$$

3.2.2 The Low-Value Bidder's Strategy

It is easy to demonstrate that no pure-strategy equilibrium bidding strategy exists for the low-type bidder. Instead, he uses a mixed strategy under which on the support of the strategy he trades-off lower probabilities of winning the first auction with higher expected second-auction payoffs by placing higher bids (which lowers the competition in the second auction, as the rivals' funds are more depleted).

We now derive the equilibrium bid distribution of the low-type bidder. Note first that in equilibrium he will never bid above the high-value bidder's equilibrium bid. Indeed, it is clear that the upper end of the support of the mixed strategy $\overline{b}$ of the low types must be equal to the bid placed by high types. It is seen by contradiction that it cannot be below this: if it were, then whenever called upon to place a bid near the upper end the bidder does not decrease the probability of winning by simply bidding $\overline{b}$, but should he then lose, he has managed to more severely tighten the rival's budget for the second auction, thereby increasing his own second auction expected payoff.

Consider first his expected payoff for the game if he bids b_i and wins the first auction. Since he then surely loses the second auction (by assumption of the severity of the budget constraint), his overall payoff is simply:

$$\underline{\Pi}_{\text{win 1}} = [(\underline{v} - E\left[b_{-i}|b_i > b_{-i}\right]) \Pr\{b_i > b_{-i}\}] + 0,$$

where $E\left[b_{-i}|b_i > b_{-i}\right]$ is the expected price to be paid (i.e., his rival's bid) when he wins with having placed the bid b_i, and $\Pr\{b_i > b_{-i}\}$ is the probability that he wins when placing the bid b_i.

Hence, letting $F(\cdot) : \left[\underline{b}, \overline{b}\right] \to [0,1]$ denote the mixed strategy distribution used by low-valued bidders in the first period (where $\overline{b}$ is given by (2) and $\underline{b}$ is yet to be determined),

$$\underline{\Pi}_{\text{win 1}} = \left(\underline{v} - \frac{\int_{\underline{b}}^{b_i} b_{-i} dF}{F(b_i)}\right)(1-\rho)F(b_i) = \left(\underline{v}F(b_i) - \int_{\underline{b}}^{b_i} b_{-i} dF\right)(1-\rho).$$

If, when placing the bid b_i he loses the first auction, his payoff from the auction is obviously zero, but his rival pays a price b_i in the first auction, and therefore the bidder's second auction payoff is as given in the table above with p_1 being replaced by b_i. Hence his expected payoff for this case is:

$$\underline{\Pi}_{\text{lose 1}} = 0 + \pi_{2|lost1st} \Pr\{b_i < b_{-i}\} = \left[\rho\overline{v} + (1-\rho)\underline{v} - (m - b_i)\right]\left[\rho + (1-\rho)(1 - F(b_i))\right].$$

Taking these two together we get the low-type's overall expected payoff at the beginning of the game as:

$$\underline{\Pi}_i = \left(\underline{v}F(b_i) - \int_{\underline{b}}^{b_i} b_{-i} dF\right)(1-\rho) + \left[\rho\overline{v} + (1-\rho)\underline{v} - (m - b_i)\right]\left[\rho + (1-\rho)(1 - F(b_i))\right].$$

Now recall that equilibrium requires that his payoff be constant for all bids on the support of the strategy, i.e.,

$$\underline{\Pi}_i(b) = \text{const.} \; \forall\, b \in \left[\underline{b}, \overline{b}\right] ;$$

and since the upper end of the support is given by $\overline{b}$ from above, we have $F(\overline{b}) = 1$. Therefore

$$\begin{aligned}\underline{\Pi}(b) = \underline{\Pi}(\overline{b}) &= \left(\underline{v} - \int_{\underline{b}}^{\overline{b}} b_{-i} dF\right)(1-\rho) + \left[\rho\overline{v} + (1-\rho)\underline{v} - (m - \overline{b})\right]\rho \\ &= \left(\underline{v} - \int_{\underline{b}}^{\overline{b}} b_{-i} dF\right)(1-\rho) + \frac{\rho\overline{v} + (1-\rho)\underline{v} + \overline{v} - m}{2}\rho\end{aligned}$$

And therefore $F(b)$ is implied by:

$$\begin{aligned}&\left(\underline{v}F(b) - \int_{\underline{b}}^{b} x dF\right)(1-\rho) + [\rho\overline{v} + (1-\rho)\underline{v} - (m-b)]\,[\rho + (1-\rho)(1-F(b))] = \\ &\qquad = \left(\underline{v} - \int_{\underline{b}}^{\overline{b}} x dF\right)(1-\rho) + \frac{\rho\overline{v} + (1-\rho)\underline{v} + \overline{v} - m}{2}\rho. \qquad (3)\end{aligned}$$

Differentiating the left-hand side of equation (3) with respect to b yields,

$$f(b)[2b + \rho(\overline{v} - \underline{v}) - m] - \frac{1}{1-\rho} + F(b) = 0. \qquad (4)$$

Equation (4) is a differential equation in $F(b)$. To solve it, rewrite it using $f(b) = \frac{dF(b)}{db}$ as follows:

$$\frac{dF(b)}{\frac{1}{1-\rho} - F(b)} = \frac{db}{[2b + \rho(\overline{v} - \underline{v}) - m]} \qquad (5)$$

For now assume that $(2b + \rho(\overline{v} - \underline{v}) - m)$ is positive—something that will be ascertained later. Solving (5), we obtain:

$$F(b) = \frac{1}{1-\rho} - \frac{C}{\sqrt{2b + \rho(\overline{v} - \underline{v}) - m}} \qquad (6)$$

where C is a positive constant.

Since $F(\overline{b}) = 1$, using $\overline{b}$ from Equation (2) we get,

$$\begin{aligned}F(\overline{b}) = 1 &= \frac{1}{1-\rho} - \frac{C}{\sqrt{2\overline{b} + \rho(\overline{v} - \underline{v}) - m}} \\ &= \frac{1}{1-\rho} - \frac{C}{\sqrt{(\overline{v} - \underline{v})}},\end{aligned}$$

therefore

$$C = \frac{\rho}{1-\rho}\sqrt{\overline{v}-\underline{v}},$$

and

$$F(b) = \frac{1}{1-\rho}\left[1 - \frac{\rho\sqrt{\overline{v}-\underline{v}}}{\sqrt{2b+\rho(\overline{v}-\underline{v})-m}}\right]. \tag{7}$$

And as $F(\underline{b}) = 0$, it follows that

$$\underline{b} = \frac{m-\rho(1-\rho)(\overline{v}-\underline{v})}{2};$$

with $\underline{b} > 0$, since $m > \overline{v}$.

Note that the expression under the square root in (6) and (7), $2b+\rho(\overline{v}-\underline{v})-m$, is positive for all $b \in [\underline{b}, \overline{b}]$, since $\underline{b} = \frac{m-\rho(1-\rho)(\overline{v}-\underline{v})}{2}$, confirming the postulated solution to the differential equation (4).

Finally, our initial conjecture of severe budget constraints (i.e., $m-b < \underline{v}\ \forall\, b \Rightarrow m-\underline{b} < \underline{v}$) holds whenever

$$m - \underline{b} = \frac{m+\rho(1-\rho)(\overline{v}-\underline{v})}{2} < \underline{v},$$

or

$$m < 2\underline{v} - \rho(1-\rho)\left(\overline{v}-\underline{v}\right). \tag{8}$$

The derivation is summarized in the following proposition:

Proposition 2 (Bidding in the First Auction in the Second-Price Format) *Suppose that $m \in \left(\overline{v}, 2\underline{v} - \rho(1-\rho)\left(\overline{v}-\underline{v}\right)\right)$, so that there are sever budget constraints, but bidders can afford to pay at least $\overline{v}$. Equilibrium bidding in the first of the two second-price sealed bid auctions is then given by*

$$b_i^* = \overline{b} = \frac{(1-\rho)\left(\overline{v}-\underline{v}\right)+m}{2},$$

if the bidder has a high value. If the bidder has a low value he uses the mixed strategy given by

$$F(b) = \frac{1}{1-\rho}\left[1 - \frac{\rho\sqrt{\overline{v}-\underline{v}}}{\sqrt{2b+\rho(\overline{v}-\underline{v})-m}}\right],$$

on $\left[\underline{b}, \overline{b}\right] = \left[\frac{m-\rho(1-\rho)(\overline{v}-\underline{v})}{2}, \frac{(1-\rho)(\overline{v}-\underline{v})+m}{2}\right]$.

3.3 Expected bid from low-type

Let $E_1\underline{b}$ stand for the expected bid by a low type $\underline{v}$ in the first period. That is, $E_1\underline{b} = \int_{\underline{b}}^{\bar{b}} bdF(b)$. Given the probability distribution $F(.)$ of the first-period bids of the low type, we can complete the RHS Equation (3) from above (assuming no masspoints). We have:

$$\begin{aligned}
E_1\underline{b} = \int_{\underline{b}}^{\bar{b}} bdF &= bF|_{\underline{b}}^{\bar{b}} - \int_{\underline{b}}^{\bar{b}} Fdb \\
&= \bar{b} - \frac{\bar{b}-\underline{b}}{1-\rho} + \frac{\rho}{1-\rho}\sqrt{\bar{v}-\underline{v}}\int_{\underline{b}}^{\bar{b}} (2b+\rho(\bar{v}-\underline{v})-m)^{-\frac{1}{2}}db \\
&= \frac{1}{1-\rho}\left[\underline{b}-\rho\bar{b}+\rho\sqrt{\bar{v}-\underline{v}}(2b+\rho(\bar{v}-\underline{v})-m)^{\frac{1}{2}}|_{\underline{b}}^{\bar{b}}\right] \\
&= \frac{1}{1-\rho}\left[\underline{b}-\rho\bar{b}+\rho\sqrt{\bar{v}-\underline{v}}\left((2\bar{b}+\rho(\bar{v}-\underline{v})-m)^{\frac{1}{2}}-(2\underline{b}+\rho(\bar{v}-\underline{v})-m)^{\frac{1}{2}}\right)\right] \\
&= \frac{1}{1-\rho}\left[\frac{m-\rho(1-\rho)(\bar{v}-\underline{v})}{2} - \rho\frac{(1-\rho)(\bar{v}-\underline{v})+m}{2} + \right. \\
&\qquad \left. +\rho\sqrt{\bar{v}-\underline{v}}\left(\sqrt{\bar{v}-\underline{v}}-\rho\sqrt{\bar{v}-\underline{v}}\right)\right] \\
&= \frac{1}{1-\rho}\left[(1-\rho)\frac{m}{2}-\rho(1-\rho)(\bar{v}-\underline{v})+\rho(1-\rho)(\bar{v}-\underline{v})\right] \\
&= \frac{m}{2}
\end{aligned}$$

3.4 Expected seller's payoff

Note that the sum of the seller's payoffs over the two periods is equal to m. Indeed, if bidder i wins in period 1, he pays b_j, the first-period bid of bidder j. Then in the second period bidder j wins by bidding, say $\underline{v}$ and pays $m - b_j$, the bid of bidder i in period 2.

Let $E_1\underline{p}$ stand for the expected payment in the first-period auction conditional on both bidders having type $\underline{v}$ in the first period. Note that $E_1\underline{p} = \int_{\underline{b}}^{\bar{b}} bd(F^2(b) + 2F(b)(1-F(b))$. Then the expected payment received by the seller in the first period is equal to:

$$E\pi_1^s = \rho^2\bar{b} + 2\rho(1-\rho)E_1\underline{b} + (1-\rho)^2 E_1\underline{p} \tag{9}$$

Let us compute the value of $E_1\underline{p}$. We have:

$$E_1\underline{p} = \int_{\underline{b}}^{\overline{b}} bd(F^2(b) + 2F(b)(1-F(b)) = 2\int_{\underline{b}}^{\overline{b}} bdF(b) - \int_{\underline{b}}^{\overline{b}} bdF^2(b) = m - \overline{b} + \int_{\underline{b}}^{\overline{b}} F^2(b)db$$

$$= \frac{m - (1-\rho)\left(\overline{v} - \underline{v}\right)}{2} + \int_{\underline{b}}^{\overline{b}} F^2(b)db \tag{10}$$

Let us compute $\int_{\underline{b}}^{\overline{b}} F^2(b)db$. We have:

$$\int_{\underline{b}}^{\overline{b}} F^2(b)db = \int_{\underline{b}}^{\overline{b}} \left(\frac{1}{1-\rho}\left[1 - \frac{\rho\sqrt{\overline{v}-\underline{v}}}{\sqrt{2b+\rho(\overline{v}-\underline{v})-m}}\right]\right)^2 db$$

$$= \frac{1}{(1-\rho)^2}\int_{\underline{b}}^{\overline{b}} \left[1 - 2\frac{\rho\sqrt{\overline{v}-\underline{v}}}{\sqrt{2b+\rho(\overline{v}-\underline{v})-m}} + \left(\frac{\rho^2(\overline{v}-\underline{v})}{(2b+\rho(\overline{v}-\underline{v})-m}\right)\right] db$$

$$= -\frac{\overline{b}-\underline{b}}{(1-\rho)^2} + \frac{2}{(1-\rho)^2}\int_{\underline{b}}^{\overline{b}} \left(1 - \frac{\rho\sqrt{\overline{v}-\underline{v}}}{\sqrt{2b+\rho(\overline{v}-\underline{v})-m}}\right) db$$

$$+ \frac{\rho^2(\overline{v}-\underline{v})}{(1-\rho)^2}\int_{\underline{b}}^{\overline{b}} \frac{1}{(2b+\rho(\overline{v}-\underline{v})-m)}db$$

$$= -\frac{\overline{b}-\underline{b}}{(1-\rho)^2} + \frac{2}{(1-\rho)}\int_{\underline{b}}^{\overline{b}} F(b)db + \frac{\rho^2(\overline{v}-\underline{v})}{(1-\rho)^2}|_{\underline{b}}^{\overline{b}} \log(2b+\rho(\overline{v}-\underline{v})-m)$$

$$= -\frac{\overline{b}-\underline{b}}{(1-\rho)^2} + \frac{2}{(1-\rho)}\left(\overline{b} - \frac{m}{2}\right)$$

$$+ \frac{\rho^2(\overline{v}-\underline{v})}{(1-\rho)^2}\left(\log(2\overline{b}+\rho(\overline{v}-\underline{v})-m) - \log(2\underline{b}+\rho(\overline{v}-\underline{v})-m)\right)$$

$$= -\frac{(1+\rho)\left(\overline{v}-\underline{v}\right)}{2(1-\rho)} + (\overline{v}-\underline{v}) + \frac{\rho^2(\overline{v}-\underline{v})}{(1-\rho)^2}\left(\log(\overline{v}-\underline{v}) - \log(\rho^2(\overline{v}-\underline{v}))\right)$$

$$= -\frac{(1+\rho)\left(\overline{v}-\underline{v}\right)}{2(1-\rho)} + (\overline{v}-\underline{v}) - \frac{2\rho^2(\overline{v}-\underline{v})}{(1-\rho)^2}\log(\rho) \tag{11}$$

Next, combine (11) with (10) to obtain:

$$E_1\underline{p} = \frac{m-(1-\rho)\left(\overline{v}-\underline{v}\right)}{2} - \frac{(1+\rho)\left(\overline{v}-\underline{v}\right)}{2(1-\rho)} + (\overline{v}-\underline{v}) - \frac{2\rho^2(\overline{v}-\underline{v})}{(1-\rho)^2}\log(\rho)$$

$$= \frac{m}{2} - \frac{\rho(1+\rho)(\overline{v}-\underline{v})}{2(1-\rho)} - \frac{2\rho^2(\overline{v}-\underline{v})}{(1-\rho)^2}\log(\rho) \tag{12}$$

Therefore, the expected seller's payoff in period 1 is equal to:

$$E\pi_1^s = \rho^2\left(\frac{(1-\rho)\left(\overline{v}-\underline{v}\right)+m}{2}\right)$$

$$+ \rho(1-\rho)m + (1-\rho)^2\left(\frac{m}{2} - \frac{\rho(1+\rho)(\overline{v}-\underline{v})}{2(1-\rho)} - \frac{2\rho^2(\overline{v}-\underline{v})}{(1-\rho)^2}\log(\rho)\right)$$

$$= \frac{1}{2}\left(m - \rho(1-\rho)(\overline{v}-\underline{v})\right) - 2\rho^2(\overline{v}-\underline{v})\log(\rho) = \underline{b} - 2\rho^2(\overline{v}-\underline{v})\log(\rho) \tag{13}$$

4 Optimal Direct Mechanism in the Second Auction

Let us investigate the optimal direct mechanism. We start with period 2. Let m_1^2 and m_2^2 denote the remaining budget of bidder 1 and bidder 2, respectively. Without loss of generality we let $m_1^2 \geq m_2^2$.

Assumption 1 *The second-period auction is offered after the buyers have learned their second-period values.*

Assumption 2 *The current owner's value of the object is normalized to 0.*

Also, let $\overline{p}_i$ and $\underline{p}_i$ denote the probabilities with which bidder i with announced valuation $\overline{v}$ and $\underline{v}$, respectively, trades in period 2. Let $\overline{t}_i$ and $\underline{t}_i$ denote the transfers which bidder i with announced valuation $\overline{v}$ and $\underline{v}$, respectively, makes to the mechanism designer.

The optimal mechanism in period 2 maximizes

$$\rho(\overline{t}_1 + \overline{t}_2) + (1-\rho)(\underline{t}_1 + \underline{t}_2)$$

subject to the following incentive constraints for $i \in \{1, 2\}$:

$$\begin{aligned} \overline{p}_i \overline{v} - \overline{t}_i &\geq \underline{p}_i \overline{v} - \underline{t}_i \\ \underline{p}_i \underline{v} - \underline{t}_i &\geq \overline{p}_i \underline{v} - \overline{t}_i \end{aligned} \quad (14)$$

We omit here the upwards incentive constraints. The latter require the probabilities to increase in value i.e. $\overline{p}_i \geq \underline{p}_i$ for $i \in \{1, 2\}$—it is straightforward to show that these are satisfied in the mechanism that we construct.

The mechanism also has to satisfy the following individual rationality constraint:

$$\underline{p}_i \underline{v} - \underline{t}_i \geq 0$$

and budget constraints:

$$\max\{\overline{t}_i, \underline{t}_i\} \leq m_i^2 \quad \text{for} \quad i \in \{1, 2\}$$

The probabilities of trade have to satisfy the following conditions: $1 \geq \overline{p}_i \geq 0$, $1 \geq \underline{p}_i \geq 0$, for $i \in \{1, 2\}$ and

$$\rho(\overline{p}_1 + \overline{p}_2) + (1-\rho)(\underline{p}_1 + \underline{p}_2) \leq 1 \quad (15)$$

Three additional restrictions follow from the fact that the good cannot be allocated with a probability exceeding 1:

$$\overline{p}_1 + \overline{p}_2 \leq 2 - \rho \tag{16}$$

$$\underline{p}_1 + \underline{p}_2 \leq 1 + \rho \tag{17}$$

$$(1-\rho)\underline{p}_i + \rho\overline{p}_j \leq 1 - \rho(1-\rho) \tag{18}$$

For (16), suppose that i always get's the good when he has a high value (i.e., $\overline{p}_i = 1$), then j can only be awarded the good when he has a high value if i does not have a high value (i.e., $\overline{p}_j = 1 - \rho$).

For (17), suppose that i only get's the good when both bidders have a low value (i.e., $\underline{p}_i = 1-\rho^2$), then the most j can expect is to get it the good when he has a low value and the rival has a high value (i.e., $\overline{p}_j = (1-\rho)\rho$). Together this yield $\underline{p}_1+\underline{p}_2 = 1-\rho^2+(1-\rho)\rho = 1+\rho$.

Our goal is to provide a closed form solution for the mechanism designer's profits and the bidders' payoffs. First, let us provide some preliminary results. If the mechanism is efficient then $\overline{p}_1 + \overline{p}_2 = 2 - \rho$ and $\underline{p}_1 + \underline{p}_2 = 1 - \rho$.

Further, in the absence of budget constraints it is optimal to set $\underline{p}_1 + \underline{p}_2 = 0$ if $\rho\overline{v} \geq \underline{v}$. Conversely, it is optimal to set $\underline{p}_1 + \underline{p}_2 = 1 - \rho$ if $\rho\overline{v} < \underline{v}$. We will use these observations to derive optimal mechanisms.

First, as in a standard situation without budget constraints, in an optimal mechanism we should have binding individual rationality constraints of low-valuation types and binding incentive constraints of high-value types: (note that there may be other mechanisms where these constraints do not bind, but there is an optimal mechanism where they bind since one can simply lower the associated probabilities of trade.[4]) Thus, we can assume:

$$\overline{p}_1\overline{v} - \overline{t}_1 = \underline{p}_1\overline{v} - \underline{t}_1$$
$$\overline{p}_2\overline{v} - \overline{t}_2 = \underline{p}_2\overline{v} - \underline{t}_2$$
$$\underline{p}_1\underline{v} - \underline{t}_1 = 0$$
$$\underline{p}_2\underline{v} - \underline{t}_2 = 0$$

Then, letting $\Delta := \overline{v} - \underline{v}$, the budget constraints can be rewritten as follows:

$$\overline{p}_i\overline{v} - \underline{p}_i\Delta \leq m_i^2 \quad \text{for} \quad i \in \{1, 2\}$$

Using the above equations, we can rewrite the principal's objective function as follows:

$$\max_{\overline{p}_1,\overline{p}_2,\underline{p}_1,\underline{p}_2} \rho(\overline{p}_1\overline{v} - \underline{p}_1\Delta + \overline{p}_2\overline{v} - \underline{p}_2\Delta) + (1-\rho)\underline{v}(\underline{p}_1 + \underline{p}_2) = \tag{19}$$

$$\max_{\overline{p}_1,\overline{p}_2,\underline{p}_1,\underline{p}_2} \rho\overline{v}(\overline{p}_1 + \overline{p}_2) + (\underline{v} - \overline{v}\rho)(\underline{p}_1 + \underline{p}_2) \tag{20}$$

subject to the following constraints:[5]

$$\overline{p}_i\overline{v} - \underline{p}_i\Delta \leq m_i^2 \quad \text{for} \quad i \in \{1, 2\} \tag{23}$$

$$\rho(\overline{p}_1 + \overline{p}_2) + (1-\rho)(\underline{p}_1 + \underline{p}_2) \leq 1 \tag{24}$$

$$\overline{p}_1 + \overline{p}_2 \leq 2 - \rho \tag{25}$$

$$1 \geq \overline{p}_i \geq \underline{p}_i \geq 0, \quad for \;\; i \in \{1, 2\} \tag{26}$$

[4]Note: Suppose that $\max\{m_1^2, m_2^2\} \leq .5\underline{v}$. Then—if in fact the auctioneer has no value for the object—taking both budgets and randomly (equally likely) allocating the good to either bidder is an optimal mechanism in which bidders have positive expected payoffs. We are suggesting that for this case we consider the (equivalently optimal) mechanism in which the auctioneer might hold on to the object, assigning only just a high enough probability to the bidders winning that they are willing to surrender their whole budgets (a similar story can be told for budget constrained high types). The tricky thing here is then when one considers the two-auction game this inefficiency in the second auction can have negative feedback into the first auction.

[5]The solution also has to satisfy the following constraints:

$$\underline{p}_1 + \underline{p}_2 \leq 1 + \rho \tag{21}$$
$$(1-\rho)\underline{p}_i + \rho\overline{p}_j \leq 1 - \rho(1-\rho) \tag{22}$$

However, we can show that both (21) and (22) are redundant. To see that (21) is redundant, note that by (26) $\overline{p}_i \geq \underline{p}_i$. So (21) follows from a combination of (24) and (26). To see that (22) is redundant, again use the fact that $\overline{p}_i \geq \underline{p}_i$. Then, if $\overline{p}_i \leq 1-\rho$, then $\underline{p}_i \leq 1-\rho$ and so (22) holds because $\overline{p}_j \leq 1$. On the other hand, if $\overline{p}_i > 1-\rho$, then (22) is implied by (24)

Bidder's Payoffs From the binding IC and IR constraints:

$$\underline{t}_i = \underline{p}_i \underline{v}$$
$$\overline{t}_i = \overline{p}_i \overline{v} - \underline{p}_i \Delta$$

So bidders' ex ante expected payoffs are

$$\pi_i^2 := m_i^2 + \rho\left[\overline{p}_i \overline{v} - \overline{t}_i\right] + (1-\rho)\left[\underline{p}_i \underline{v} - \underline{t}_i\right] = m_i^2 + \rho \underline{p}_i \Delta. \tag{27}$$

Also, the maximal total surplus (i.e., assuming an efficient allocation, viz. trade takes place with prob. 1) is given by

$$TS^{\max} := Rev + \sum_i \pi_i^2 = (1-\rho)^2 \underline{v} + \rho(2-\rho)\overline{v} + \sum_i m_i^2$$

Whereas the equilibrium expected surplus is

$$TS^{Eq} = \rho\overline{v}(\overline{p}_1 + \overline{p}_2) + (\underline{v} - \overline{v}\rho)(\underline{p}_1 + \underline{p}_2) + \sum_i m_i^2.$$

4.1 Case 0: Benchmark without budget constraints

As a benchmark, let us consider Problem (19) subject to (24)-(26), i.e. omit the budget constraints. Then the solution depends on the sign of $\rho\overline{v} - \underline{v}$.

If $\rho\overline{v} - \underline{v} \geq 0$, then the expected value of the high valuation sufficiently exceed the low valuation to make it desirable to attempt to extract surplus only from the high valued bidder. Thus, it is optimal to set $\overline{p}_1 + \overline{p}_2 = 2 - \rho$ and $\underline{p}_1 + \underline{p}_2 = 0$. The principal's expected profits are equal to $(1-(1-\rho)^2)\overline{v} = \rho(2-\rho)\overline{v}$. Bidders' expected payoffs are

$$\pi_i^2 = m_i^2.$$

If $\rho\overline{v} - \underline{v} < 0$, then it is optimal to set $\overline{p}_1 + \overline{p}_2 = 2 - \rho$ and $\underline{p}_1 + \underline{p}_2 = 1 - \rho$. The latter follows from the objective and the constraint (24). In particular, for instance, the principal can set $\overline{p}_i = 1, \underline{p}_i = 0, \overline{p}_j = 1 - \rho, \underline{p}_j = 1 - \rho$, i.e., bidder i is assigned the good *iff* he has a high value, otherwise the good is given to bidder j.

The principal's expected profits are equal to $\rho\overline{v} + (1-\rho)\underline{v}$. The bidders' expected payoffs for the chosen allocation probabilities are $\pi_i^2 = m_i^2$ and $\pi_j^2 = m_j^2 + \rho(1-\rho)\Delta$.

In general,

$$\pi_1^2 \in \left[m_1^2, m_1^2 + \rho(1-\rho)\Delta\right]$$
$$\pi_2^2 = m_2^2 + \rho(1-\rho)\Delta - \pi_1^2 + m_1^2$$
$$\in \left[m_2^2, m_2^2 + \rho(1-\rho)\Delta\right]$$

Remark 1 *Notice, thus, that the bidders' expected payoffs need not be identical even when they have the same budget and a bidder's payoff need not be uniquely defined by the optimal mechanism—only the bounds of individual payoffs and the sum of bidders' payoffs is pinned down.*

4.2 Case 1: $\rho\overline{v} \geq \underline{v}$

Returning now to the problem with budget constraint we first consider the case in which $\rho\overline{v} \geq \underline{v}$

Lemma 1 (Binding Budget Constraint) *Bidder i's budget constraint binds* iff $m_i^2 \leq \overline{v}$, *and when it does not bind it is optimal to set* $\underline{p}_i = 0$.

Proof of Lemma 1: Note first that that if $m_i^2 \geq \overline{v}, \forall i$ then trivially neither budget constraint binds and $\underline{p}_1 = \underline{p}_2 = 0$.

Second if $\max\{m_1^2, m_2^2\} = m_1^2 \geq \overline{v}$, then bidder 1's budget constraint does not bind so $\underline{p}_1 = 0$. If bidder 2's budget constraint is not binding, then $\underline{p}_2 = 0$ and by (25) $\overline{p}_2 \geq 1 - \rho$, since $\overline{p}_1 \leq 1$. But then bidder 2's budget constraint will fail whenever $m_2^2 < \overline{v}$.

Lastly, suppose that $\max\{m_1^2, m_2^2\} = m_1^2 \leq \overline{v}$. Then both budget constraints are binding. The proof is by contradiction.

At first, suppose that neither budget constraint in (23) is binding. Then $\overline{p}_i < 1$ since $\max\{m_1^2, m_2^2\} = m_1^2 < \overline{v}$, and $\underline{p}_i = 0$ for $i \in \{1, 2\}$ since the value of the objective is decreasing in $\underline{p}_1$ and $\underline{p}_2$. Since $m_1^2 + m_2^2 < \overline{v} + (1-\rho)\overline{v}$, we have $\overline{p}_1 + \overline{p}_2 < 2 - \rho$, i.e. (25) is non-binding. But then the value of the objective in (19) can be increased by raising $\overline{p}_1$ and $\overline{p}_2$, so the original mechanism cannot be optimal.

Next, suppose that only one budget constraint in (23) is binding. Suppose without loss of generality that the budget constraint of bidder 1 is non-binding, while that of bidder 2 is binding. Then $\underline{p}_2 = 0$, since the objective function is decreasing in $\underline{p}_2 = 0$, and $\overline{p}_2 < 1$, since $\max\{m_1^2, m_2^2\} = m_1^2 < \overline{v}$. If $\underline{p}_1 = 0$, then $\overline{p}_1 + \overline{p}_2 < 2 - \rho$ i.e. (24) and (25) are non-binding, since $m_1^2 + m_2^2 < \overline{v} + (1-\rho)\overline{v}$. Therefore, the value of the objective (19) can be increased by raising $\overline{p}_2$.

If $\underline{p}_1 > 0$, then the value of the objective (19) can be increased by increasing $\overline{p}_2$, decreasing $\overline{p}_1$ by the same amount, and also decreasing $\underline{p}_1$ so that the budget constraint of bidder 1 remains binding.

□

Remark 2 (Binding Budget Constraints) *When the budget constraint binds then (23) becomes* $\underline{p}_i = \frac{\overline{v}\overline{p}_i - m_i^2}{\Delta}$. *Since* $\underline{p} \geq 0$, *it follows that* $\overline{p} \geq \frac{m_i^2}{\overline{v}}$; *and since* $\underline{p} \leq 1$, *it follows that* $\overline{p}_i \leq \frac{\Delta + m_i^2}{\overline{v}}$; *and since* $\overline{p}_i = \frac{\underline{p}\Delta + m_i^2}{\overline{v}}$, *it follows that* $\underline{p} \leq \frac{\overline{v} - m_i^2}{\Delta}$. *Moreover, if* $m_i^2 \leq \underline{v}$ *then it is possible to extract all of the bidder's budget by setting* $\overline{p}_i = \underline{p}_i = \frac{m_i^2}{\underline{v}}$. *In sum then,*

$$\textit{Whenever } i\textit{'s budget constraint binds then} \begin{cases} \overline{p}_i \in \left[\frac{m_i^2}{\overline{v}}, \min\left\{1, \frac{m_i^2}{\underline{v}}, \frac{\Delta + m_1^2}{\overline{v}}\right\}\right] \\ \underline{p}_i \in \left[0, \min\left\{\frac{\overline{v} - m_i^2}{\Delta}, \frac{m_i^2}{\underline{v}}, 1\right\}\right]. \end{cases} \tag{28}$$

In order to determine the exact optimal (sum of) probabilities, one must distinguish between various constellations of available budgets relative to bidders' possible valuations. In order to do so, we refer to Figure 1 in which—without loss of generality—we postulate that the bidder with the larger budget is indexed by $i = 1$, i.e., $m_1^2 \geq m_2^2$.

Figure 1: *Case 1—Subcases*

Case 1.a $m_1^2 + m_2^2 \geq \overline{v} + (1-\rho)\overline{v} = (2-\rho)\overline{v}$, $\min\{m_1^2, m_2^2\} = m_2^2 \geq (1-\rho)\overline{v}$.

In this case, the principal can attain the expected payoff $(1-(1-\rho)^2)\overline{v}$—the same as without budget constraints by setting $\overline{p}_1 + \overline{p}_2 = 2 - \rho$, with $1 \geq \max\{\overline{p}_1, \overline{p}_2\}$, $\min\{\overline{p}_1, \overline{p}_2\} \geq 1 - \rho$, and $\underline{p}_1 + \underline{p}_2 = 0$.

Hence

$$\pi_i^2 = m_i^2, \quad \forall i.$$

Remark 3 *Since* $\underline{p}_i = 0$ *the bidders' payoffs follow readily. In order to determine equilibrium values for* $\overline{p}_1$ vs. $\overline{p}_2$ *we use the budget constraint. Specifically, with* $\min\{m_1^2, m_2^2\} = m_2^2$, *note that it must be the case that* $m_2^2 < \overline{v}$ *(otherwise both budgets are greater than* $\overline{v}$ *and we have Case 0—the case without budgets being an issue). Hence,*

$\overline{p}_2 \in [1-\rho, m_2^2/\overline{v}]$ *and* $\overline{p}_1 = 2 - \rho - \overline{p}_2$.

Case 1.b $\max\{m_1^2, m_2^2\} = m_1^2 \geq \overline{v}$, $\min\{m_1^2, m_2^2\} = m_2^2 < (1-\rho)\overline{v}$.

Since $m_1^2 \geq \overline{v}$, bidder 1's budget constraint is not binding and it is optimal to set $\underline{p}_1 = 0$. And bidder 2's budget constraint must be binding i.e. $\overline{p}_2\overline{v} - \underline{p}_2\Delta = m_2^2$. It follows that $\underline{p}_2 = \frac{\overline{p}_2\overline{v} - m_2^2}{\Delta}$. Substituting this into the objective (19) we can rewrite it as follows:

$$\max_{\overline{p}_1, \overline{p}_2, \underline{p}_1, \underline{p}_2} \rho\overline{v}\overline{p}_1 + (\underline{v} - \overline{v}\rho)\underline{p}_1 + \frac{\overline{p}_2\overline{v}}{\Delta}(\rho\Delta + (\underline{v} - \overline{v}\rho)) - \frac{m_2^2}{\Delta}(\underline{v} - \overline{v}\rho) \tag{29}$$

Note that $\rho\Delta + \underline{v} - \overline{v}\rho = \underline{v}(1-\rho) > 0$. So, the coefficient on $\overline{p}_2$ in (29) is positive. However, it is smaller than the coefficient on $\overline{p}_1$. So in the optimal mechanism we have: $\overline{p}_1 = 1$, $\underline{p}_1 = 0$, and $\overline{p}_2$ and $\underline{p}_2$ should be set to that $\overline{p}_2$ is maximized subject to $(1-\rho) \geq \overline{p}_2 \geq \underline{p}_2$ and $\underline{p}_2 = \frac{\overline{p}_2\overline{v} - m_2^2}{\Delta}$. The solution is to set $\overline{p}_2 = (1-\rho)$ when $m_2^2 \geq (1-\rho)\underline{v}$. Then $\underline{p}_2 = \frac{\overline{p}_2\overline{v} - m_2^2}{\Delta} \leq (1-\rho)$. When $m_2^2 < (1-\rho)\underline{v}$, we should set $\overline{p}_2 = \underline{p}_2$ so that $\overline{p}_2\underline{v} = m_2^2$.

Bidders' payoffs are

$$\pi_1^2 = m_1^2, \tag{30}$$

$$\pi_2^2 = \begin{cases} m_2^2 + \rho\frac{m_2^2}{\underline{v}}\Delta & \text{if } m_2^2 < (1-\rho)\underline{v}, \\ m_2^2 + \rho\frac{(1-\rho)\overline{v} - m_2^2}{\Delta}\Delta = m_2^2 + \rho\left[(1-\rho)\overline{v} - m_2^2\right] & \text{if } m_2^2 \geq (1-\rho)\underline{v}. \end{cases} \tag{31}$$

Cases 1.c, 1.d, 1.e, 1.f. $m_1^2 + m_2^2 < \overline{v} + (1-\rho)\overline{v}$, $\max\{m_1^2, m_2^2\} = m_1^2 < \overline{v}$.

From the binding budget constraints we have $\underline{p}_i = \frac{\overline{p}_i\overline{v} - m_i^2}{\Delta} \geq 0$, so $\overline{p}_i \geq \frac{m_i^2}{\overline{v}}, \forall i$. Also, given that both budget constraints in (23) are binding, the value of the objective becomes:

$$\max_{\overline{p}_1, \overline{p}_2, \underline{p}_1, \underline{p}_2} \frac{(\overline{p}_1 + \overline{p}_2)\overline{v}}{\Delta}(\rho\Delta + (\underline{v} - \overline{v}\rho)) - \frac{m_1^2 + m_2^2}{\Delta}(\underline{v} - \overline{v}\rho) \tag{32}$$

Note that (32) depends only on the sum $(\overline{p}_1 + \overline{p}_2)$ and is increasing in it. Therefore, it is optimal to set $(\overline{p}_1 + \overline{p}_2)$ as high as possible.

Since both budget constraints are binding, let us consider when this implies that both (24) and (25) are non-binding. First, suppose that $m_1^2 \geq \underline{v}$ (Recall that we also have $m_1^2 \leq \overline{v}$). Then $(\overline{p}_1 + \overline{p}_2)$ cannot reach its upper bound of $2-\rho$ i.e., (25) is non-binding if $m_2^2 < (1-\rho)\underline{v}$. In this case, with binding budget constraint of bidder 2, $\overline{p}_2$ reaches its maximal value when we set $\overline{p}_2 = \underline{p}_2 = \frac{m_2^2}{\underline{v}} < 1-\rho$. Therefore $\overline{p}_1 + \overline{p}_2 < 2-\rho$ since $\overline{p}_1 \leq 1$.

Then, given the binding budget constraint of bidder 1, $\overline{p}_1$ and $\underline{p}_1$ are maximal when we set $\overline{p}_1 = 1$ and $\underline{p}_1 = \frac{\overline{v}-m_1^2}{\Delta}$. Then (24) will be non-binding if

$$\rho\left(1+\frac{m_2^2}{\underline{v}}\right)+(1-\rho)\left(\frac{\overline{v}-m_1^2}{\Delta}+\frac{m_2^2}{\underline{v}}\right)<1.$$

which can be rearranged as:

$$\frac{m_2^2}{\underline{v}} < (1-\rho)\left(\frac{m_1^2-\underline{v}}{\Delta}\right). \tag{33}$$

Thus, (33) delivers a condition for both (24) and (25) to be non-binding in case $\overline{v} \geq m_1^2 \geq \underline{v}$

Now, suppose that $m_1^2 < \underline{v}$ (and so $m_2^2 < \underline{v}$). Then to maximize $\overline{p}_1 + \overline{p}_2$, we should set $\overline{p}_1 = \underline{p}_1 = \frac{m_1^2}{\underline{v}}$ and $\overline{p}_2 = \underline{p}_2 \leq \frac{m_2^2}{\underline{v}}$ and so (24) is non-binding if $\sum m_i < \underline{v}$.

Thus, if (33) fails or $\sum m_i \geq \underline{v}$, then either (24) and (25) will be binding **our Cases 1.c and 1.d** below. Let us derive the condition determining which of the constraints will be binding (24) and (25)

Substituting $\underline{p}_1 + \underline{p}_2 = \frac{(\overline{p}_1+\overline{p}_2)\overline{v}-m_1^2-m_2^2}{\Delta}$ into (24) we obtain:

$$(\overline{p}_1+\overline{p}_2)\left(\rho+\frac{(1-\rho)\overline{v}}{\Delta}\right)-\frac{(1-\rho)(m_1^2+m_2^2)}{\Delta}\leq 1 \tag{34}$$

If (34) holds with $\overline{p}_1 + \overline{p}_2 = 2-\rho$, then in the optimal mechanism (24) is non-binding, and (25) is binding. If (34) fails with $\overline{p}_1 + \overline{p}_2 = 2-\rho$, then in the optimal mechanism $\overline{p}_1$ and $\overline{p}_2$ are determined by (34) holding as equality.

Remark 4 (Binding constraints) *Notice that the condition that distinguishes whether 24 or 25 is binding is derived from the two constraints. Viz.*

$$\sum \overline{p}_i \leq \frac{\Delta+(1-\rho)\sum m_i^2}{\overline{v}-\rho\underline{v}}$$

$$\sum \overline{p}_i \leq 2-\rho.$$

Hence,

$$\frac{\Delta + (1-\rho)\sum m_i^2}{\overline{v} - \rho\underline{v}} \gtreqless 2-\rho$$

$$\sum m \gtreqless \overline{v} + (1-\rho)\underline{v}.$$

Therefore if $\underline{v} \leq \sum m < \overline{v} + (1-\rho)\underline{v}$ *(Case 1.d), then 24 binds. If* $\overline{v} + (1-\rho)\underline{v} \leq \sum m \leq \overline{v} + (1-\rho)\overline{v}$ *(Case 1.c).*

Case 1.c:

$\overline{v} + (1-\rho)\underline{v} \leq \sum m \leq \overline{v} + (1-\rho)\overline{v}$, $m_1^2 < \overline{v}$.

Note that in this case $\frac{m_2^2}{\underline{v}} \geq (1-\rho)\left(\frac{m_1^2 - \underline{v}}{\Delta}\right)$. So, as established above, in the optimal mechanism $\overline{p}_1 + \overline{p}_2 = 2-\rho$ (i.e., 25 binds).

Since $\overline{p}_2 = 2-\rho-\overline{p}_1$, $\overline{p}_2 \leq 2-\rho-\frac{m_1^2}{\overline{v}}$. Therefore $\underline{p}_2 = \frac{\overline{p}_2\overline{v} - m_2^2}{\Delta} \leq \frac{(2-\rho)\overline{v} - m_1^2 - m_2^2}{\Delta}$. Similarly, $\underline{p}_1 = \frac{\overline{p}_1\overline{v} - m_1^2}{\Delta} \leq \frac{(2-\rho)\overline{v} - m_1^2 - m_2^2}{\Delta}$.

Since $\overline{p}_1 + \overline{p}_2 = 2-\rho$, we have $\underline{p}_1 + \underline{p}_2 = 1-\rho$. Hence, $\underline{p}_i = \frac{\overline{p}_i\overline{v} - m_i^2}{\Delta} \leq 1-\rho$. for $i \in \{1,2\}$. So, for $i,j \in \{1,2\},\ i \neq j$,

$$\overline{p}_i \leq \frac{(1-\rho)\Delta + m_i^2}{\overline{v}}, \quad \overline{p}_j \geq 2-\rho-\frac{(1-\rho)\Delta + m_i^2}{\overline{v}} = \frac{\overline{v} + \underline{v}(1-\rho) - m_i^2}{\overline{v}}. \tag{35}$$

We also have, for $i,j \in \{1,2\},\ i \neq j$:

$$\overline{p}_i \geq 1-\rho \tag{36}$$

$$\overline{p}_i \geq \frac{m_i^2}{\overline{v}}, \quad \overline{p}_j \leq 2-\rho-\frac{m_i^2}{\overline{v}} = \frac{(2-\rho)\overline{v} - m_i^2}{\overline{v}}. \tag{37}$$

Combining (35), (36) and (37), we conclude that for $i,j \in \{1,2\}$ $i \neq j$ the following restrictions have to hold:

$$\max\left\{1-\rho, \frac{m_i^2}{\overline{v}}, \frac{\overline{v} + \underline{v}(1-\rho) - m_j^2}{\overline{v}}\right\} \leq \overline{p}_i \leq \min\left\{1, \frac{(2-\rho)\overline{v} - m_j^2}{\overline{v}}, \frac{(1-\rho)\Delta + m_i^2}{\overline{v}}\right\}. \tag{38}$$

Since $\overline{v} + \underline{v}(1-\rho) \leq m_1^2 + m_2^2$, (38) gives us

$$\max\left\{1-\rho, \frac{m_i^2}{\overline{v}}\right\} \leq \overline{p}_i \leq \min\left\{1, \frac{(2-\rho)\overline{v} - m_j^2}{\overline{v}}\right\}. \tag{39}$$

It follows that

$$\pi_i^2 \in \left[m_i^2 + \rho\max\left\{(1-\rho)\overline{v} - m_i^2, 0\right\}, m_i^2 + \rho\min\left\{\overline{v} - m_i^2, (2-\rho)\overline{v} - m_1^2 - m_2^2\right\}\right]$$

With

$$\sum_i \pi_i^2 = (1-\rho)\sum_i m_i^2 + \rho(2-\rho)\overline{v}.$$

Case 1.d:

$\underline{v} \leq \sum m < \overline{v} + (1-\rho)\underline{v}$, $m_1^2 < \overline{v}$.

$\frac{m_2^2}{\underline{v}} \geq (1-\rho)\left(\frac{m_1^2 - \underline{v}}{\Delta}\right)$.

As established above, $\overline{p}_1 + \overline{p}_2 = \frac{\Delta + (1-\rho)\sum m_i^2}{\overline{v} - \rho\underline{v}}$ (i.e., 24 binds). Since $\overline{p}_i \leq \min\left\{\frac{m_i^2}{\underline{v}}, 1\right\}$, it follows that $\overline{p}_j \geq \frac{\Delta + (1-\rho)\sum m_i^2}{\overline{v} - \rho\underline{v}} - \min\left\{\frac{m_i^2}{\underline{v}}, 1\right\} = \max\left\{\frac{\Delta\left(1 - \frac{m_i^2}{\underline{v}}\right) + (1-\rho)m_j^2}{\overline{v} - \rho\underline{v}}, \frac{(1-\rho)\left(\sum m_i^2 - \underline{v}\right)}{\overline{v} - \rho\underline{v}}\right\}$. for $i, j \in \{1, 2\}$, $i \neq j$.

Since $\overline{p}_i \geq \frac{m_i^2}{\overline{v}}$, it follows that $\overline{p}_j \leq \frac{\Delta + (1-\rho)\sum m_i^2}{\overline{v} - \rho\underline{v}} - \frac{m_i^2}{\overline{v}} = \frac{\Delta + (1-\rho)m_j^2 - \frac{\rho\Delta m_i^2}{\overline{v}}}{\overline{v} - \rho\underline{v}}$ for $i, j \in \{1, 2\}$, $i \neq j$

Thus, we have:

$$\max\left\{\frac{\Delta\left(1 - \frac{m_i^2}{\underline{v}}\right) + (1-\rho)m_j^2}{\overline{v} - \rho\underline{v}}, \frac{(1-\rho)\left(\sum m_i^2 - \underline{v}\right)}{\overline{v} - \rho\underline{v}}, \frac{m_i^2}{\overline{v}}\right\} \leq \overline{p}_i \tag{40}$$

$$\overline{p}_i \leq \min\left\{\frac{m_i^2}{\underline{v}}, 1, \frac{\Delta + (1-\rho)m_i^2 - \rho\frac{\Delta m_j^2}{\overline{v}}}{\overline{v} - \rho\underline{v}}\right\}. \tag{41}$$

Since $\underline{p}_i = \frac{\overline{p}_i\overline{v} - m_i^2}{\Delta}$, (40) implies that:

$$\max\left\{\frac{\overline{v}\Delta\left(1 + (1-\rho)m_j^2\right) - \left(\left(\frac{\overline{v}}{\underline{v}}\right)^2 - \rho\right)\underline{v}m_i^2}{(\overline{v} - \rho\underline{v})\Delta}, \frac{\overline{v}(1-\rho)(m_j^2 - \underline{v}) - \rho m_i^2\Delta}{(\overline{v} - \rho\underline{v})\Delta}, 0\right\} \leq \underline{p}_i \tag{42}$$

$$\underline{p}_i \leq \min\left\{\frac{m_i^2}{\underline{v}}, \frac{\overline{v} - m_i^2}{\Delta}, \frac{\overline{v} - \rho\sum m}{\overline{v} - \rho\underline{v}}\right\}. \tag{43}$$

Note that the first term in the max operator on the left-hand side of (40) is greater than the second term, 0, if

$$m_j^2 > \frac{\rho}{1-\rho}\frac{\Delta}{\overline{v}}m_i^2 + \underline{v} \tag{44}$$

Finally, (43) implies that

$$\pi_i^2 \in \left[m_i^2 + \rho \max \left\{ \frac{\overline{v}\Delta\left(1+(1-\rho)m_j^2\right) - \left(\left(\frac{\overline{v}}{\underline{v}}\right)^2 - \rho\right)\underline{v}m_i^2}{(\overline{v}-\rho\underline{v})\Delta}, \frac{\overline{v}(1-\rho)(m_j^2-\underline{v}) - \rho m_i^2\Delta}{(\overline{v}-\rho\underline{v})}, 0 \right\}, \right.$$
$$\left. m_i^2 + \rho \min\left\{\overline{v} - m_i^2, \Delta\frac{\overline{v}-\rho\sum m}{\overline{v}-\rho\underline{v}}\right\}\right]$$

With

$$\sum_i \pi_i^2 = \sum_i m_i^2 + \rho\frac{\overline{v}-\rho\sum_i m_i^2}{\overline{v}-\rho\underline{v}}\Delta$$

Case 1.e:

$m_1^2 < \overline{v}$.

$\frac{m_2^2}{\underline{v}} < (1-\rho)\left(\frac{m_1^2-\underline{v}}{\Delta}\right)$.

Note that the second condition implies that $m_1^2 \geq \underline{v}$ and $m_2^2 \leq (1-\rho)\underline{v}$. As established above, in this case neither (24) nor (25) are binding, and the optimal mechanism involves setting $\overline{p}_1 = 1$, $\underline{p}_1 = \frac{\overline{v}-m_1^2}{\Delta}$; $\overline{p}_2 = \underline{p}_2 = \frac{m_2^2}{\underline{v}}$.

Then the seller's revenue is equal to $\rho m_1^2 + (1-\rho)\frac{\overline{v}-m_1^2}{\Delta}\underline{v} + m_2^2$.

The bidders profits are:

$$\pi_1^2 = m_1^2 + \rho(\overline{v} - m_1^2)$$
$$\pi_2^2 = m_2^2 + \rho\frac{\Delta m_2^2}{\underline{v}} \tag{45}$$

Case 1.f: $\sum m \leq \underline{v}$.

In this case, neither (24) nor (25) is binding. We have $\overline{p}_i = \underline{p}_i = p_i = m_i/\underline{v}$.

Payoffs are $\sum_{i=1,2} m_i^2$ for the mechanism designer and bidder payoffs are

$$\pi_i^2 = m_i^2 + \rho\frac{m_i^2}{\underline{v}}\Delta.$$

Summary of bidder 1's payoff in case 1:

$$\pi_1^2(m_1^2,m_2^2):\begin{cases}\text{if } m_1^2>\overline{v}\begin{cases}\text{if } m_2^2>(1-\rho)\overline{v}: & 1a\\ \text{if } m_2^2<(1-\rho)\overline{v}: & 1b\end{cases}\\ \text{if } m_1^2\in[(1-.5\rho)\overline{v},\overline{v}]\begin{cases}\text{if } m_2^2>(2-\rho)\overline{v}-m_1^2: & 1a\\ \text{if } m_2^2\in[\overline{v}+(1-\rho)\underline{v}-m_1^2,(2-\rho)\overline{v}-m_1^2]: & 1c\\ \text{if } m_2^2\in\left[(1-\rho)\underline{v}\frac{m_1^2-v}{\Delta},\overline{v}+(1-\rho)\underline{v}-m_1^2\right]: & 1d\\ \text{if } m_2^2<(1-\rho)\underline{v}\frac{m_1^2-v}{\Delta}: & 1e\end{cases}\\ \text{if } m_1^2\in\left[\frac{1}{2}[\overline{v}+(1-\rho)\underline{v}],(1-.5\rho)\overline{v}\right]\begin{cases}\text{if } m_2^2>\overline{v}+(1-\rho)\underline{v}-m_1^2: & 1c\\ \text{if } m_2^2\in\left[(1-\rho)\underline{v}\frac{m_1^2-v}{\Delta},\overline{v}+(1-\rho)\underline{v}-m_1^2\right]: & 1d\\ \text{if } m_2^2<(1-\rho)\underline{v}\frac{m_1^2-v}{\Delta}: & 1e\end{cases}\\ \text{if } m_1^2\in\left[\underline{v},\frac{1}{2}[\overline{v}+(1-\rho)\underline{v}]\right]\begin{cases}\text{if } m_2^2\geq(1-\rho)\underline{v}\frac{m_1^2-v}{\Delta}: & 1d\\ \text{if } m_2^2<(1-\rho)\underline{v}\frac{m_1^2-v}{\Delta}: & 1e\end{cases}\\ \text{if } m_1^2\in\left[\frac{1}{2}\underline{v},\underline{v}\right]\begin{cases}\text{if } m_2^2\in[\underline{v}-m_1^2,m_1^2]: & 1d\\ \text{if } m_2^2\in[0,\underline{v}-m_1^2]: & 1f\end{cases}\\ \text{if } m_1^2\leq\frac{1}{2}\underline{v}: \quad 1f\end{cases}$$

Or,

$$\pi_1^2 \in \begin{cases} \{m_1^2\} & \text{if } m_1^2 > \overline{v} \\ \left.\begin{array}{ll} \{m_1^2\} & \text{if } m_2^2 > (2-\rho)\overline{v} - m_1^2 \\ \left[m_1^2, m_1^2 + \rho[(2-\rho)\overline{v} + \sum m]\right] & \text{if } m_2^2 \in [\overline{v} + (1-\rho)\underline{v} - m_1^2, (2-\rho)\overline{v} - m_1^2] \\ 1d & \text{if } m_2^2 \in \left[(1-\rho)\underline{v}\frac{m_1^2 - \underline{v}}{\Delta}, \overline{v} + (1-\rho)\underline{v} - m_1^2\right] \\ \{m_1^2 + \rho(\overline{v} - m_1^2)\} & \text{if } m_2^2 < (1-\rho)\underline{v}\frac{m_1^2 - \underline{v}}{\Delta} \end{array}\right\} & \text{if } m_1^2 \in [(1-.5\rho)\overline{v}, \overline{v}] \\ \left.\begin{array}{ll} \left[m_1^2, m_1^2 + \rho(\overline{v} - m_1^2)\right] & \text{if } m_2^2 > \overline{v} + (1-\rho)\underline{v} - m_1^2 \\ 1d & \text{if } m_2^2 \in \left[(1-\rho)\underline{v}\frac{m_1^2 - \underline{v}}{\Delta}, \overline{v} + (1-\rho)\underline{v} - m_1^2\right] \\ \{m_1^2 + \rho(\overline{v} - m_1^2)\} & \text{if } m_2^2 < (1-\rho)\underline{v}\frac{m_1^2 - \underline{v}}{\Delta} \end{array}\right\} & \text{if } m_1^2 \in \left[\frac{1}{2}[\overline{v} + (1-\rho)\underline{v}], (1-.5\rho)\overline{v}\right] \\ \left.\begin{array}{ll} 1d & \text{if } m_2^2 \geq (1-\rho)\underline{v}\frac{m_1^2 - \underline{v}}{\Delta} \\ \{m_1^2 + \rho(\overline{v} - m_1^2)\} & \text{if } m_2^2 < (1-\rho)\underline{v}\frac{m_1^2 - \underline{v}}{\Delta} \end{array}\right\} & \text{if } m_1^2 \in \left[\underline{v}, \frac{1}{2}[\overline{v} + (1-\rho)\underline{v}]\right] \\ \left.\begin{array}{ll} 1d & \text{if } m_2^2 \in \left[\underline{v} - m_1^2, m_1^2\right] \\ \{m_1^2 + \rho\frac{m_1^2}{\underline{v}}\Delta\} & \text{if } m_2^2 \in \left[0, \underline{v} - m_1^2\right] \end{array}\right\} & \text{if } m_1^2 \in \left[\frac{1}{2}\underline{v}, \underline{v}\right] \\ \{m_1^2 + \rho\frac{m_1^2}{\underline{v}}\Delta\} & \text{if } m_1^2 \leq \frac{1}{2}\underline{v} \end{cases}$$

We can break this down into cases, dependent on the size of bidder 2's (remaining) budget m_2^2:

Case A: $m_2^2 \in \left[0, \max\left\{\underline{v} - m_1^2, (1-\rho)\underline{v}\frac{m_1^2 - \underline{v}}{\Delta}\right\}\right]$

$$\pi_1^2(m_1^2) = \begin{cases} m_1^2 & \text{if } m_1^2 \geq \overline{v} \\ m_1^2 + \rho(\overline{v} - m_1^2) & \text{if } m_1^2 \in (\underline{v}, \overline{v}) \\ m_1^2 + \rho\frac{m_1^2}{\underline{v}}\Delta & \text{if } m_1^2 \leq \underline{v} \end{cases} \tag{46}$$

Case B: $m_2^2 \in \left[\max\left\{\underline{v} - m_1^2, (1-\rho)\underline{v}\frac{m_1^2 - \underline{v}}{\Delta}\right\}, \min\left\{\overline{v} + (1-\rho)\underline{v} - m_1^2, m_1^2\right\}\right]$

$$\pi_1^2(m_1^2) = \begin{cases} m_1^2 & \text{if } m_1^2 \geq \overline{v} \\ 1d & \text{if } m_1^2 \in (\underline{v}, \overline{v}) \\ m_1^2 + \rho\frac{m_1^2}{\underline{v}}\Delta & \text{if } m_1^2 \leq \underline{v} \end{cases} \tag{47}$$

Case C: $m_2^2 \in [\overline{v} + (1-\rho)\underline{v} - m_1^2, \min\{(2-\rho)\overline{v} - m_1^2, m_1^2\}]$

$$\pi_1^2(m_1^2) = \begin{cases} m_1^2 & \text{if } m_1^2 \geq \overline{v} \\ [m_1^2, m_1^2 + \rho[(2-\rho)\overline{v} + \sum m]] & m_1^2 \in [(1-.5\rho)\overline{v}, \overline{v}] \\ \ldots \end{cases} \tag{48}$$

We now consider the case of the expected high value being below the low value.

4.3 Case 2: $\rho\overline{v} < \underline{v}$

Case 2. $\rho\overline{v} < \underline{v}$.

Note that the objective function (19) is increasing in each of $\overline{p}_1, \overline{p}_2, \underline{p}_1, \underline{p}_2$. Therefore, either both budget constraints in (23) are binding or (24) is binding, or both. For suppose otherwise, i.e. (24) is not binding and at least one budget constraint (23), say, for bidder 1, is non-binding. Then we must have $\overline{p}_1 = \underline{p}_1 = 1$, for otherwise we can increase the value of the objective by raising either $\overline{p}_1$ or $\underline{p}_1$. But this implies that (24) is binding.

Further, if we ignore the budget constraints (23), then (19) subject to (23)-(26) attains its maximum when we set $\overline{p}_1 + \overline{p}_2 = 2 - \rho$ and $\underline{p}_1 + \underline{p}_2 = 1 - \rho$. Hence, we should set the probabilities in this way when budget constraints allows this. For this to be possible, it is necessary that $\min\{\overline{p}_1, \overline{p}_2\} \geq 1 - \rho$. But since $\min\{\overline{p}_1, \overline{p}_2\} \leq \frac{\min\{m_i^2, m_2^2\}}{\underline{v}}$, we must have $\min\{m_1^2, m_2^2\} = m_2^2 \geq (1-\rho)\underline{v}$. We must also have $m_1^2 + m_2^2 \geq \overline{v} + (1-\rho)\underline{v}$, since this is necessary for $\overline{p}_1 + \overline{p}_2$ to reach $2 - \rho$ when $\underline{p}_1 + \underline{p}_2 \leq 1 - \rho$. So, we have:

Case 2.a $\min\{m_1^2, m_2^2\} = m_2^2 \geq (1-\rho)\underline{v}$, $m_1^2 + m_2^2 \geq \overline{v} + (1-\rho)\underline{v}$.

Suppose without loss of generality that $m_1^2 \geq m_2^2$. Then set: $\underline{p}_1 = 0$, $\underline{p}_2 = 1 - \rho$, $\overline{p}_1 = \min\{1, \frac{m_1^2}{\overline{v}}\}$, $\overline{p}_2 = \min\{2 - \rho - \min\{1, \frac{m_1^2}{\overline{v}}\}, \frac{m_2^2 + (1-\rho)\Delta}{\overline{v}}\}$. Note that $\overline{p}_1 + \overline{p}_2 = 2 - \rho$ since $\min\{m_1^2, m_2^2\} = m_2^2 \geq (1-\rho)\underline{v}$ and $m_1^2 + m_2^2 \geq \overline{v} + (1-\rho)\underline{v}$.

In this case, the principal attains the maximal expected payoff $\rho\overline{v} + (1-\rho)\underline{v}$ that she can attain without budget constraints.

Note that when $\min\{m_1^2, m_2^2\} = m_2^2 \geq (1-\rho)\overline{v}$, $m_1^2 + m_2^2 \geq \overline{v} + (1-\rho)\overline{v} = (2-\rho)\overline{v}$, then $\underline{p}_2 = 0, \underline{p}_1 = 1 - \rho$ and $\overline{p}_2 = \frac{m_2^2}{\overline{v}}, \overline{p}_1 = 2 - \rho - \frac{m_2^2}{\overline{v}}$ is also feasible.

If $m_1^2 + m_2^2 \leq \overline{v} + (1-\rho)\overline{v} = (2-\rho)\overline{v}$ or $m_2^2 \leq (1-\rho)\overline{v}$ then $\overline{p}_1 = 1, \overline{p}_2 = 1-\rho$; and $\underline{p}_2 = \frac{(1-\rho)\overline{v} - m_2^2}{\Delta}, \underline{p}_1 = \frac{(1-\rho)\underline{v} + m_2^2}{\Delta}$.

In sum,

$$\pi_1^2 \in \left[m_1^2, m_1^2 + \rho(1-\rho)\Delta - \max\left\{0, \rho(1-\rho)\overline{v} - \rho m_2^2\right\}\right]$$
$$\pi_2^2 = \left[m_2^2 + \max\left\{0, \rho(1-\rho)\overline{v} - \rho m_2^2\right\}, m_2^2 + \rho(1-\rho)\Delta\right];$$

with

$$\sum \pi_i = \sum_i m_i^2 + \rho(1-\rho)\Delta.$$

Case 2.b $\rho\overline{v} < \underline{v}$, $\min\{m_1^2, m_2^2\} = m_2^2 < (1-\rho)\underline{v}$, $\max\{m_1^2, m_2^2\} = m_1^2 > \overline{v}$.

Suppose without loss of generality that $m_1^2 > m_2^2$. In this case, the budget constraint of bidder 1 cannot be binding so, as established above, (24) is binding. Using (24) to substitute $\underline{p}_1 + \underline{p}_2$ in (19), we can rewrite the objective as follows:

$$\max_{\overline{p}_1, \overline{p}_2, \underline{p}_1, \underline{p}_2} \rho\overline{v}(\overline{p}_1 + \overline{p}_2) + (\underline{v} - \overline{v}\rho)\left(\frac{1}{1-\rho} - \frac{\rho(\overline{p}_1 + \overline{p}_2)}{1-\rho}\right) \tag{49}$$

Since $\rho\overline{v} - \frac{\rho(\underline{v} - \overline{v}\rho)}{1-\rho} > 0$, in the optimal mechanism the value of $\overline{p}_1 + \overline{p}_2$ should be maximized.

Hence, we set $\overline{p}_2 = \underline{p}_2 = \frac{m_2^2}{\underline{v}} < 1-\rho$. We also set $\overline{p}_1 = 1$ and $\underline{p}_1 = 1 - \frac{m_2^2}{\underline{v}(1-\rho)}$.

The principal's profits are equal to:

$$\underline{v} + \frac{m_2^2\rho(\frac{\overline{v}}{\underline{v}} - 1)}{(1-\rho)} \tag{50}$$

And

$$\pi_1^2 = m_1^2 + \rho\left(1 - \frac{m_2^2}{\underline{v}(1-\rho)}\right)\Delta,$$
$$\pi_2^2 = m_2^2 + \rho\frac{m_2^2}{\underline{v}}\Delta;$$

with

$$\sum \pi_i = \sum_i m_i^2 + \rho\Delta\frac{(1-\rho)\underline{v} - \rho m_2^2}{(1-\rho)\underline{v}}.$$

Cases 2.c,d,e $\max\{m_1^2, m_2^2\} = m_1^2 < \overline{v}$, $m_1^2 + m_2^2 < \overline{v} + (1-\rho)\underline{v}$. In this case both budget constraints are binding. To see this note that if not, then we should set $\overline{p}_1 + \overline{p}_2 = 2-\rho$,

$\underline{p}_1 + \underline{p}_2 = 1 - \rho$. But in this case at least one budget constraint will fail since $m_1^2 + m_2^2 < \underline{v}(1-\rho) + \overline{v}$. So, in the optimal mechanism we must have $\overline{p}_1 + \overline{p}_2 < 2 - \rho$ and therefore (25) cannot be binding.

Let us now suppose that (24) is binding. And suppose that one budget constraint is binding, say that of bidder 2, and one budget constraint is non-binding, that of bidder 1. If $\underline{p}_1 > 0$, then we can increase $\overline{p}_1$ and lower $\underline{p}_1$ keeping (24) binding. Then the value of the objective will increase (see (49)). If $\underline{p}_1 = 0$, then $\underline{p}_2 > 0$. Then we can lower both $\overline{p}_2$ and $\underline{p}_2$ and raise $\overline{p}_1$ keeping (24) binding. Then $\overline{p}_1 + \overline{p}_2$ will increase and hence the value of the objective will also increase. We conclude that both budget constraints must be binding.

Finally, let's determine when (24) also is not binding:

Since the value of the objective (19) is increasing in $\overline{p}_1, \overline{p}_2, \underline{p}_1$ and $\underline{p}_2$ we desire to have: $\overline{p}_1 = \underline{p}_1 = \frac{m_1^2}{\underline{v}}$ and $\overline{p}_2 = \underline{p}_2 = \frac{m_2^2}{\underline{v}}$, which allows us to extract the entire budgets; but when this is feasible, then (24) is non-binding if $m_1^2 + m_2^2 \leq \underline{v}$ (Case 2e).

Moreover, suppose that $m_1^2 > \underline{v}$, then $\frac{m_1^2}{\underline{v}} > 1$, and therefore it is optimal to set $\overline{p}_1 = 1$, and $\underline{p}_1 = \frac{\overline{v} - m_1^2}{\Delta}$, then we find that (24) also fails when

$$\frac{m_2^2}{\underline{v}} < (1-\rho)\left(\frac{m_1^2 - \underline{v}}{\Delta}\right)$$

(see Equation 33), which is Case 2.d.

Case 2.c (this is equivalent to Case 1.d):

$\underline{v} \leq \sum m < \overline{v} + (1-\rho)\underline{v}$, $m_1^2 < \overline{v}$.

$\frac{m_2^2}{\underline{v}} \geq (1-\rho)\left(\frac{m_1^2 - \underline{v}}{\Delta}\right)$.

Hence

$$\pi_i^2 \in \left[m_i^2 + \rho \max\left\{ \frac{\overline{v}\Delta\left(1 + (1-\rho)m_j^2\right) - \left(\left(\frac{\overline{v}}{\underline{v}}\right)^2 - \rho\right)\underline{v} m_i^2}{(\overline{v} - \rho\underline{v})\Delta}, \frac{\overline{v}(1-\rho)(m_j^2 - \underline{v}) - \rho m_i^2 \Delta}{(\overline{v} - \rho\underline{v})}, 0 \right\}, \right.$$
$$\left. m_i^2 + \rho \min\left\{ \overline{v} - m_i^2, \Delta \frac{\overline{v} - \rho \sum m}{\overline{v} - \rho \underline{v}} \right\} \right]$$

With

$$\sum_i \pi_i^2 = \sum_i m_i^2 + \rho \frac{\overline{v} - \rho \sum_i m_i^2}{\overline{v} - \rho \underline{v}} \Delta$$

Case 2.d $m_1^2 < \overline{v}$.
$\frac{m_2^2}{\underline{v}} < (1-\rho)\left(\frac{m_1^2 - \underline{v}}{\Delta}\right)$.
Note that the second condition implies that $m_1^2 \geq \underline{v}$ and $m_2^2 \leq (1-\rho)\underline{v}$. This case is analogous to Case 1.e, and the optimal mechanism involves setting $\overline{p}_1 = 1$, $\underline{p}_1 = \frac{\overline{v} - m_1^2}{\Delta}$; $\overline{p}_2 = \underline{p}_2 = \frac{m_2^2}{\underline{v}}$.

The the seller's revenue is equal to $\rho m_1^2 + (1-\rho)\frac{\overline{v}-m_1^2}{\Delta}\underline{v} + m_2^2$

The bidders profits are:

$$\begin{aligned}\pi_1^2 &= m_1^2 + \rho(\overline{v} - m_1^2) \\ \pi_1^2 &= m_2^2 + \rho\frac{\Delta m_2^2}{\underline{v}}\end{aligned} \tag{51}$$

Case 2.e $\sum m \leq \underline{v}$.

In this case, neither (24) nor (25) is binding. We have $\overline{p}_i = \underline{p}_i = p_i = m_i/\underline{v}$.

Payoffs are $\sum_{i=1,2} m_i^2$ for the mechanism designer and bidder payoffs are

$$\pi_i^2 = m_i^2 + \rho\frac{m_i^2}{\underline{v}}\Delta.$$

5 Optimal Mechanism with One Bidder

To understand what goes on in the first-period auction, let us consider a single bidder case. Then, with $\rho\overline{v} > \underline{v}$. The second period optimal mechanism is as follows:

If $m^2 \geq \overline{v}$, then $\overline{p} = 1$, $\overline{t} = \overline{v}$, $\underline{p} = 0$, $\underline{t} = 0$.

If $\underline{v} \leq m^2 < \overline{v}$, then $\overline{p} = 1$, $\overline{t} = m^2$, $\underline{p} = \frac{\overline{v}-m^2}{\Delta}$, $\underline{t} = \underline{v}\frac{\overline{v}-m^2}{\Delta}$.

If $\underline{v} > m^2$, then $\overline{p} = \underline{p} = \frac{m^2}{\underline{v}}$, $\underline{t} = \overline{t} = m^2$.

Let $u_2(m^2)$ be the second period ex-ante payoff of the bidder as a function of the budget. We have $u_2(m^2) = m^2$ if $m^2 \geq \overline{v}$; $u_2(m^2) = m^2 + \rho(\overline{v} - m^2)$ if $\underline{v} \leq m^2 < \overline{v}$; $u_2(m^2) = m^2 + \rho\frac{m^2}{\underline{v}}\Delta$ if $\underline{v} > m^2$. Note that $u_2(m^2)$ is increasing in m^2. Or

$$u_2(m^2) = \pi^2(m^2) = \begin{cases} m^2 & \text{if } m^2 \geq \overline{v} \\ m^2 + \rho(\overline{v} - m^2) & \text{if } \underline{v} \leq m^2 < \overline{v} \\ m^2 + \rho\frac{m^2}{\underline{v}}\Delta & \text{if } m^2 < \underline{v} \end{cases} \tag{52}$$

This payoff is depicted in Figure 2.

Figure 2: *Second-Auction Payoff*

To simplify the analysis, we suppose that the first-period budget is $m^1 = \overline{v}$. And we let $\overline{q}$, $\underline{q}$, $\overline{s}$, $\underline{s}$, be the probabilities of trade and expected transfers.

The following constraints have to hold in the first-period mechanism.

$$\overline{vq} + u_2(m^1 - \overline{s}) \geq \overline{v}\underline{q} + u_2(m^1 - \underline{s}) \tag{53}$$

$$\underline{vq} + u_2(m^1 - \underline{s}) \geq m^1 \quad (= u_2(m^1)) \tag{54}$$

As standard, we will omit that the incentive constraint of the low type holds in the first period. We will confirm this later. Also by standard arguments, $\overline{q} \geq \underline{q}$, hence $\overline{s} \geq \underline{s}$. Also, (53) and (54) must be binding in the optimal mechanism. Hence binding (53) can be rewritten as follows:

$$\overline{vq} + u_2(m^1 - \overline{s}) = \Delta\underline{q} + m^1 \tag{55}$$

We need to consider the following three cases:

(i) $m^1 - \overline{s} \geq \underline{v}$; $m^1 - \underline{s} \geq \underline{v}$.

(ii) $m^1 - \overline{s} \leq \underline{v}$; $m^1 - \underline{s} \geq \underline{v}$.

(iii) $m^1 - \overline{s} \leq \underline{v}$; $m^1 - \underline{s} \leq \underline{v}$.

(Note that the fourth case, $m^1 - \overline{s} \geq \underline{v}$; $m^1 - \underline{s} < \underline{v}$, is impossible since $\overline{s} \geq \underline{s}$.)

Case (i). $m^1 - \overline{s} \geq \underline{v}$; $m^1 - \underline{s} \geq \underline{v}$.

Then binding (53) and (54) give us:

$$\underline{vq} + m^1 - \underline{s} + \rho(\overline{v} - m^1 + \underline{s}) = m^1$$

$$\overline{vq} + m^1 - \overline{s} + \rho(\overline{v} - m^1 + \overline{s}) = \Delta\underline{q} + m^1 \tag{56}$$

Since $m^1 = \overline{v}$, (56) simplifies to:

$$\overline{vq} - \Delta\underline{q} = (1 - \rho)\overline{s} \tag{57}$$

$$\underline{vq} = (1 - \rho)\underline{s} \tag{58}$$

Substituting these values into the profit function of the first-period auctioneer we obtain:

$$\rho\overline{s} + (1-\rho)\underline{s} = \rho\frac{\overline{vq} - \Delta\underline{q}}{1-\rho} + \underline{vq} = \rho\frac{\overline{vq}}{1-\rho} - \underline{q}\left(\frac{\rho\Delta}{1-\rho} - \underline{v}\right) \tag{59}$$

Note that $\frac{\rho\Delta}{1-\rho} - \underline{v} > 0$ since $\rho\overline{v} > \underline{v}$. So, $\underline{q}$ should be set as low as possible. That is, by (57) we should set it so that $m^1 - \overline{s} = \overline{v} - \overline{s} = \overline{v} - \frac{\overline{vq}-\Delta\underline{q}}{1-\rho} = \underline{v}$ (where the last equality stems from the restriction on case i, in conjunction with the minimization problem) which gives us:

$$\underline{q} = \frac{\overline{v}}{\Delta}\overline{q} - 1 + \rho$$

Substituting this into the objective (59), we obtain:

$$\begin{aligned} \rho\frac{\overline{vq}}{1-\rho} - \underline{q}\left(\frac{\rho\Delta}{1-\rho} - \underline{v}\right) &= \rho\frac{\overline{vq}}{1-\rho} - \frac{\overline{v}}{\Delta}\overline{q}\left(\frac{\rho\Delta}{1-\rho} - \underline{v}\right) + (1-\rho)\left(\frac{\rho\Delta}{1-\rho} - \underline{v}\right) \\ &= \frac{\overline{v}\underline{v}}{\Delta}\overline{q} + (1-\rho)\left(\frac{\rho\Delta}{1-\rho} - \underline{v}\right) \end{aligned} \tag{60}$$

Note that the coefficient on $\overline{q}$ is positive so we should set $\overline{q} = 1$, and hence $\underline{q}$ should solve:

$$\underline{q} = \frac{\overline{v}}{\Delta} - 1 + \rho$$

provided that $\underline{q} \leq 1$, which is the case when $\overline{v} \geq \underline{v}\frac{2-\rho}{1-\rho}$. Otherwise, i.e. if we have $\overline{v} < \underline{v}\frac{2-\rho}{1-\rho}$ we should set $\overline{q} = \underline{q} = \frac{\Delta(1-\rho)}{\underline{v}}$.

The seller's profits are:

$$\frac{\overline{v}\underline{v}}{\Delta} + (1-\rho)\left(\frac{\rho\Delta}{1-\rho} - \underline{v}\right) = \rho\overline{v} + \frac{\underline{v}^2}{\Delta} \quad if \quad \overline{v} \geq \underline{v}\frac{2-\rho}{1-\rho} \tag{61}$$

$$\overline{v}(1-\rho) + (1-\rho)\left(\frac{\rho\Delta}{1-\rho} - \underline{v}\right) = \Delta \quad if \quad \overline{v} < \underline{v}\frac{2-\rho}{1-\rho} \tag{62}$$

Case (ii). $m^1 - \overline{s} \leq \underline{v}$; $m^1 - \underline{s} \geq \underline{v}$.

In this case, binding (54) and (55) give us:

$$\begin{aligned} \underline{vq} + m^1 - \underline{s} + \rho(\overline{v} - m^1 + \underline{s}) &= m^1 \\ \overline{vq} + m^1 - \overline{s} + \rho\frac{m^1 - \overline{s}}{\underline{v}}\Delta &= \Delta\underline{q} + m^1 \end{aligned} \tag{63}$$

Since $m^1 = \overline{v}$, (63) simplifies to:

$$\begin{aligned} \underline{vq} &= (1-\rho)\underline{s} \\ \overline{vq} - \Delta\underline{q} + \frac{\rho\Delta\overline{v}}{\underline{v}} &= \overline{s}\left(1 + \frac{\rho\Delta}{\underline{v}}\right) \end{aligned} \tag{64}$$

Substituting these values into the profit function of the first-period auctioneer we obtain:

$$\rho\overline{s} + (1-\rho)\underline{s} = \frac{\rho}{\left(1+\frac{\rho\Delta}{\underline{v}}\right)}\left(\overline{v}\overline{q} + \frac{\rho\Delta\overline{v}}{\underline{v}} - \Delta\underline{q}\right) + \underline{q}\underline{v} = \frac{\rho\overline{v}}{\left(1+\frac{\rho\Delta}{\underline{v}}\right)}\left(\overline{q} + \frac{\rho\Delta}{\underline{v}} + \underline{q}\underline{v}\right) \tag{65}$$

Since in (65) the coefficients on $\overline{q}$ and $\underline{q}$ are positive, in the optimal mechanism we should set $\overline{q}$ and $\underline{q}$ as high as possible. In particular, we should set $\overline{q} = 1$. We should also choose $\underline{q}$ so that $m^1 - \overline{s} \leq \underline{v}$; $m^1 - \underline{s} \geq \underline{v}$, which changes (64) to

$$\begin{aligned}
&\frac{\underline{v}\underline{q}}{1-\rho} \leq \Delta \\
&\frac{\overline{v} - \Delta\underline{q} + \frac{\rho\Delta\overline{v}}{\underline{v}}}{1+\frac{\rho\Delta}{\underline{v}}} \geq \Delta
\end{aligned} \tag{66}$$

Note that (66) is equivalent to:

$$\underline{q} \leq \frac{\Delta(1-\rho)}{\underline{v}} \tag{67}$$

$$\underline{q} \leq \frac{\underline{v}}{\Delta} + \rho \tag{68}$$

Note that the right-hand side of (67) is less than or equal than the maximum of 1 and the right-hand side of (68) if and only if $\overline{v} \leq \underline{v}\frac{2-\rho}{1-\rho}$. The opposite is true if $\overline{v} \geq \underline{v}\frac{2-\rho}{1-\rho}$. I.e.,

$$\frac{\Delta(1-\rho)}{\underline{v}} \leq \max\left\{1, \frac{\underline{v}}{\Delta} + \rho\right\} = \frac{\underline{v}}{\Delta} + \rho \quad \Leftrightarrow \quad \overline{v} \leq \underline{v}\frac{2-\rho}{1-\rho}$$

$$\frac{\Delta(1-\rho)}{\underline{v}} \geq \max\left\{1, \frac{\underline{v}}{\Delta} + \rho\right\} = 1 \quad \Leftrightarrow \quad \overline{v} \geq \underline{v}\frac{2-\rho}{1-\rho}$$

Thus, if $\overline{v} \leq \underline{v}\frac{2-\rho}{1-\rho}$, the seller's profits are equal to:

$$\begin{aligned}
\rho\overline{s} + (1-\rho)\underline{s} &= \frac{\rho}{\left(1+\frac{\rho\Delta}{\underline{v}}\right)}\left(\overline{v} + \frac{\rho\Delta\overline{v}}{\underline{v}} - \Delta\frac{\Delta(1-\rho)}{\underline{v}}\right) + (1-\rho)\Delta \\
&\stackrel{?}{=} \rho\overline{v}\left(1 + \frac{(1-\rho)\underline{v}\Delta}{\underline{v}+\rho\Delta}\right)
\end{aligned} \tag{69}$$

If $\overline{v} \geq \underline{v}\frac{2-\rho}{1-\rho}$, the seller's profits are equal to:

$$\rho\overline{s} + (1-\rho)\underline{s} = \rho\Delta + \underline{v}\left(\frac{\underline{v}}{\Delta} + \rho\right) = \rho\overline{v} + \frac{\underline{v}^2}{\Delta} \tag{70}$$

Case (iii). $m^1 - \overline{s} \leq \underline{v}$; $m^1 - \underline{s} \leq \underline{v}$.

In this case, binding (53) and (54) give us:

$$\underline{v}\underline{q} + m^1 - \underline{s} + \rho\frac{m^1 - \underline{s}}{\underline{v}}\Delta = m^1$$
$$\overline{v}\overline{q} + m^1 - \overline{s} + \rho\frac{m^1 - \overline{s}}{\underline{v}}\Delta = \Delta\underline{q} + m^1 \tag{71}$$

Since $m^1 = \overline{v}$, (71) simplifies to:

$$\overline{v}\overline{q} - \Delta\underline{q} + \frac{\rho\Delta\overline{v}}{\underline{v}} = \overline{s}\left(1 + \frac{\Delta\rho}{\underline{v}}\right) \tag{72}$$
$$\underline{v}\underline{q} + \frac{\rho\Delta\overline{v}}{\underline{v}} = \underline{s}\left(1 + \frac{\rho\Delta}{\underline{v}}\right) \tag{73}$$

Substituting these values into the profit function of the first-period auctioneer we obtain:

$$\rho\overline{s} + (1-\rho)\underline{s} = \frac{1}{\left(1 + \frac{\rho\Delta}{\underline{v}}\right)}\left(\rho\overline{v}\overline{q} + \frac{\rho\Delta\overline{v}}{\underline{v}} - \underline{q}(\rho\overline{v} - \underline{v})\right) \tag{74}$$

Note that $\underline{q}$ should be set as low as possible since $\rho\overline{v} > \underline{v}$. So, we should set $\underline{q}$ so that $m^1 - \underline{s} = \underline{v}$, which is equivalent to: $\underline{s} = \Delta$. In this case, we have $\underline{q} = \frac{(1-\rho)\Delta}{\underline{v}}$. Since $\underline{q} \leq 1$, this is feasible iff $\overline{v} \leq \underline{v}\frac{2-\rho}{1-\rho}$, and the seller profits are equal to (69).

Otherwise, i.e., if $\overline{v} > \underline{v}\frac{2-\rho}{1-\rho}$, this case is not feasible.

5.1 Optimal Mechanism

Now we are in a position to summarize the results of the analysis and identify the optimal mechanism. Note that (69) is weakly greater than (62) and both these expressions apply when $\overline{v} \leq \underline{v}\frac{2-\rho}{1-\rho}$. Also, profits in case (iii) are equal to (69).

Also, (70) is the same as (61) and both these expressions apply when $\overline{v} \geq \underline{v}\frac{2-\rho}{1-\rho}$.

Hence, we can state the following result. If $\overline{v} \leq \underline{v}\frac{2-\rho}{1-\rho}$, then the optimal mechanism is given by the following: $\overline{q} = 1$, $\underline{q} = \frac{\Delta(1-\rho)}{\underline{v}}$, $\overline{s} = \frac{1}{\left(1+\frac{\rho\Delta}{\underline{v}}\right)}\left(\overline{v} + \frac{\rho\Delta\overline{v}}{\underline{v}} - \Delta\frac{\Delta(1-\rho)}{\underline{v}}\right) \geq m^1 - \underline{v} = \Delta$, $\underline{s} = m^1 - \underline{v} = \Delta$. The seller's expected profits are equal to (69).

If $\overline{v} \geq \underline{v}\frac{2-\rho}{1-\rho}$, then the optimal mechanism is given by the following: $\overline{q} = 1$, $\underline{q} = \frac{\underline{v}}{\Delta} + \rho$, $\overline{s} = m^1 - \underline{v} = \Delta$, $\underline{s} = \frac{\frac{\underline{v}^2}{\Delta} + \rho\underline{v}}{1-\rho} \leq m^1 - \underline{v} = \Delta$. The seller's expected profits are equal to (70)

6 Thoughts on the optimal mechanism with 2 bidders

Deriving the optimal mechanism with two bidders is considerably more involved than that with one bidder, since here the constellations of budgets and values to one anther also matter. However, the analysis of previous sections provides the key characteristic of the optimal mechanism in the first period, when the designer of the first period acts independently of the mechanism designer in the second period.

Note, in particular, that the mechanism designer of the second auction is able to either (a) keep bidders at utility levels that do not exceed their outside option, or (b) extract their entire budgets. Only in the latter case does a bidder receive a payoff that exceeds his initial budget. This happens when his expected payoff from the auction (i.e., the probability of obtaining a good of yet unknown value) exceeds the cost of participating when these costs are low due to a limited budget.

This affords the first-period mechanism designer the opportunity to extract potential bidder-surplus that is otherwise obtainable in the second auction. In particular, while second-auction payoffs are more elaborate under the optimal mechanism than those depicted in Figure 2 for the one-bidder case (note that in the two-bidder case payoffs depend on the values and budgets of both bidders and are not uniquely pinned-down), the figure can nonetheless be used to illustrate the principle of rent-extraction at play. Thus, consider Figure 3, and suppose that an arbitrary first period mechanism leaves the bidder with an expected budget of $\hat{m}^2$. His continuation utility associated with this budget is $u^2(\hat{m}^2)$. Now suppose that

Figure 3: *Payoff from Lottery before the Second Auction*

prior to the begin of the second auction, the first-auction mechanism designer offers the bidder the following lottery L in exchange for his second-auction budget:

$$m_L^2 = \begin{cases} \overline{M} & \text{with probability } \frac{\hat{m}^2 - \underline{v}}{\overline{M} - \underline{v}}, \\ \underline{v} & \text{with probability } \frac{\overline{M} - \hat{m}^2}{\overline{M} - \underline{v}}. \end{cases} \tag{75}$$

Note that such a lottery is costless to the mechanism designer, since its expected payout is equal to its price, i.e.,

$$E\left[m_L^2\right] = \overline{M}\frac{\hat{m}^2 - \underline{v}}{\overline{M} - \underline{v}} + \underline{v}\frac{\overline{M} - \hat{m}^2}{\overline{M} - \underline{v}} = \hat{m}^2. \tag{76}$$

However, the bidder is strictly better off with the lottery than with the certain budget, viz. $u^2(L) > u^2\left(\hat{m}^2\right)$ (see Figure 3). As a result, the first-auction mechanism designer is able to extract (virtually all of) this surplus and increase his revenue beyond those obtained from the arbitrary mechanism that initially left the bidder $\hat{m}^2$.

The implication of this is, of course, that the first auction mechanism designer is able to extract the bidders' potential surpluses from the second auction. Now note that (keeping in mind that the two-bidder case differs quantitatively, but not qualitatively) if the budgets are small, then the first-auction mechanism designer can directly extract the budgets, and if budgets are large, he can offer lotteries that generate additional surplus in the second auction that he can extract. Indeed, for the latter case, if his choice of $\overline{M}$ is unlimited, the added surplus that can be extracted from the second auction is as high as (i.e., when $m^2 > \overline{v}$)

$$\lim_{\overline{M}\to\infty} u^2\left(L|\overline{M}\right) - u^2(m^2) = \rho\Delta. \tag{77}$$

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