Entry Deterrence in Dynamic Auctions*

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Abstract

We examine a dynamic second-price auction with sequential and costly entry. We show that several types of equilibria exist, which differ in their allocations and prices. In one class of equilibrium, placing a low early bid can have a signaling effect that deters entry by subsequent bidders in the auction. As a result, fewer bidders enter on expectation, and the bidders who do enter obtain a higher expected payoff in equilibrium, compared to the benchmark equilibrium where all bidders submit their true values. A special case of this equilibrium is one with incremental bidding. In this equilibrium, after having submitted low opening bids, buyers raise their bids by a small incremental amount each period.

Keywords: English auctions, second price auctions, online auctions, bidder collusion, entry deterrence, early bidding, late bidding, incremental bidding.

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1 Introduction

Internet auctions provide a rich platform to observe competitive bidding in real life and to test wether observed bidding behavior is consistent with the predictions of auction theory.[1] Three types of bidding behavior, in particular, have been documented by the empirical literature on internet auctions: *Early bidding*, *incremental bidding*, and *late bidding*. Early bidding occurs when bids are placed shortly after opening of an internet auction (which typically runs for several days), while late bidding occurs when bids are placed in the final seconds. Incremental bidding occurs when a bidder places multiple bids over the course of the auction, with most of these bids being equal to the minimum required increment.[2] Together, these three patterns represent a large fraction of submitted bids in online auctions.[3]

Yet, these observations are puzzling in light of standard auction theory. Virtually all internet auctions are fundamentally second-price auctions. If valuations are private and all bidders are rational, then any profile of bidding strategies in which bidders bid their true valuations before the end of the auction is a Bayesian equilibrium.[4] According to this prediction, one should not observe bunching of bids early or late in the auction, nor is it clear why bidders should be submitting multiple bids in small increments. (With common values, bidders may strategically delay their bids in an effort to learn from other bids. Similarly, in the presence of naive "adaptive" bidders, late bidding may become a best response by rational bidders. However, early and incremental bidding are still difficult to explain in these cases.)

[1]See Bajari and Hortaçsu (2004), Hasker and Sickles (2010), and Levin (2011) for surveys of research in internet auction markets.

[2]Incremental bidding is also referred to as "multiple bidding," and late bidding is also referred to as "last-minute bidding" or "sniping."

[3]See Roth and Ockenfels (2002), Bajari and Hortaçsu (2003), Ockenfels and Roth (2006). Many other empirical studies confirm these findings. By examining several thousand eBay auctions for video gaming consoles, Shah *et al.* (2002) show that early, late, and incremental bidding make up 28%, 38%, and 34% of bids, respectively. Similarly, Bapna *et al.* (2003) report that 23% of bidders place early bids, 40% submit late bids, and 37% bid incrementally, in a sample of internet auctions. Che and Katayama (2013) provide a more detailed account of these bidding strategies in eBay auctions, reporting that 30–40% of bidders submit bids twice or more, and at least 70% of incremental bids equal the minimum required increments. Further, they observe that significant portions of bidders place bids in the last few seconds of the auction, i.e., 2–11% of bids are submitted in the last 15 second of the auction time, representing 4–17% of bidders. See also Shmueli *et al.* (2004), Anwar *et al.* (2004), Hossain (2008), Wintr (2008), Ely and Hossain (2009), Engelberg and Williams (2009), and Elfenbein and McManus (2010a,b).

[4]Note that, unlike in static second-price auctions, there generally exist no weakly dominant strategies in dynamic second-price auctions (see Ockenfels and Roth 2006).

In this paper, we provide a common, strategic explanation for all three bidding patterns. We examine a dynamic second-price auction with independent private values and risk neutral bidders. Potential bidders arrive to the auction in sequence. Upon arrival, each potential bidder observes the current auction price and then decides whether to enter the auction. If the bidder enters, he incurs a non-refundable entry cost and learns his valuation. He is then free to submit bid any number of bids at any time, until the auction closes at a predetermined time period.

We show that two basic classes of equilibrium exist in this environment, which differ in the bidders' participation decisions, the number of bids submitted by each participating bidder, and the final allocation and prices.

1. In the first, "immediate revelation" equilibrium, a bidder enters if and only if the current price is below some cutoff price p^*. Then, if the bidder's value exceeds the current auction price, he submits exactly one bid, equal to his valuation, immediately after entry. Once the auction price reaches p^*, entry ceases. This will be the case after two bidders with valuations above $p^{*}*$ have entered; the one with the higher valuation then wins and pays a price equal to the next highest valuation.

2. In the second, "delayed revelation" equilibrium, a bidder enters if and only if the current price is below a different cutoff price, p^{**}. If the entering bidder's value exceeds p^{**}, he submits a bid equal to p^{**} immediately after entry. Once the auction price reaches p^{**}, all further entry is deterred; this will be the case after two bidders with valuations above p^{**} have entered. These bidders will submit an additional pair of truthful late bids just prior to closing. In addition, they may periodically increase their bids between the time they enter and the final period. The bidder with the higher valuation wins and pays the next highest valuation.

The crucial result is that $p^{**} < p^*$. Thus, fewer bidders will enter in the second type equilibrium on average, and entry will cease earlier, than in the first equilibrium. By delaying the revelation of their true valuations until the final period, bidders in the delayed revelation equilibrium in effect collude to deter entry by potential rival bidders. The colluding bidders then compete against one another in a single Vickrey auction in the final period. The valuation of the winning bidder, and the price the winning bidder pays, are both lower (on expectation) in the second equilibrium. However, the bidders who do enter in the second equilibrium obtain higher expected surpluses than they would in the first equilibrium.

How does "collusion by bidding low" work? Conditional on having entered the auction, each bidder will eventually bid his true value—either immediately after entry, or in the final period. In this regard, our model is not different from other second-price auctions. Things are slightly more complicated for the bidders' entry decisions. Whether entry is worthwhile depends on the expectation a bidder holds about the valuations of competing bidders when observing the current auction price. This expectation, in turn, depends on the *particular* bidding strategies used by competing bidders, whence the multiplicity of equilibria. To see how expectations matter in the equilibria described above, note that each potential entrant cares about the distribution of the highest among his competitors' valuations. In the first (immediate revelation) equilibrium, the auction price provides a lower bound for this variable, in that exactly one of the current participants must have a valuation higher than the current price. In the second (delayed revelation) equilibrium, the auction price also provides a lower bound—but because the two highest bidders pool their bids, there are now *two* current competitors with valuations above the auction price. The price that makes a potential indifferent between entering and not entering the auction is hence lower in the second equilibrium.[5]

We also show that collusive equilibria exist in which entry is deterred by bidders adopting more sophisticated incremental bidding strategies. We argue that such incremental bidding equilibria are appealing, in the sense that bidders may more easily be able coordinate on such an equilibrium, compared to the simple delayed revelation equilibrium described earlier.

The remainder of the paper proceeds as follows. Section 2 reviews the literature related to bidding behavior in internet auctions as well as entry in auctions. In Section 3 we introduce our auction model. In Section 4 we characterize the immediate revelation equilibrium of our model. In Section 5 we explore the strategy of entry deterrence via delayed revelation, and in Section 6 we construct an equilibrium in which entry is deterred via a more sophisticated strategy of delayed revelation coupled with incremental bidding. Section 7 compares the expected buyer payoffs and seller revenues across the different types of equilibria. Section 8 concludes with a discussion of our results. Most proofs are in the Appendix.

[5]Even though bidders care about the distribution of their opponents' valuations, and learn about this distribution from previously submitted bids, we remark that our results do not rely on bidders' risk aversion, or on an assumption of correlated or common values. We assume risk neutrality and independent private values throughout.

2 Related Literature

[To be added.]

3 Sequential Second-Price Auctions with Entry

A single indivisible object is sold to $T > 2$ risk neutral potential bidders. All potential bidders are ex ante symmetric. Bidder $i \in \{1, \ldots, T\}$ has private value v_i for the object. All v_i are independent draws from a common atomless distribution F over support $[\underline{v}, \overline{v}]$, with $0 \leq \underline{v} < \overline{v}$. Initially, a bidder does not know his own private value, but knows only the distribution F. Bidders will be able to learn their valuations during the course of the auction.

3.1 Auction format

The auction format is a sequential second-price auction, or English auction, that is open over T periods. The auction price at the end of period t is denoted $p^t \geq 0$; the final ending price is p^T. The initial price at the beginning of the auction is $p^0 = 0$.

The bidders arrive to the auction in sequence, with bidder $i \in \{1, \ldots, T\}$ arriving in period i. Upon arrival, bidder i observes the current price p^{i-1} and decides whether to enter the auction. We denote this decision by $e_i \in \{0, 1\}$, where $e_i = 1$ means "entry" and $e_i = 0$ means "no entry." If i enters, he pays an entry cost $c > 0$ (which is the same for all bidders), learns his private value v_i, and is then free to bid in any period $t \in \{i, \ldots, T\}$. If he does not enter, he leaves the auction.[6] At the onset of the auction, the pool of participating bidders is $B^0 = \emptyset$. After potential entry in period $t \geq 1$, the pool of participating bidders becomes $B^t = \{i \leq t : e_i = 1\}$, so that $B^0 \subseteq B^1 \subseteq \ldots \subseteq B^T$.

In each period t, after potential entry, there is one round of simultaneous bidding during which all bidders in B_t submit simultaneous bids. We denote by $b_i^t \in [0, \infty)$ bidder i's bid in period t. We interpret a bid of zero as "no bid." For $t \geq 0$ and $i \notin B_t$, we automatically set $b_i^t = 0$. For $i \in B_t$, we require that $b_i^t \geq b_i^{t-1}$ for all $t \geq 1$. That is, bidders cannot revise previous bids downward during the auction. We further require that $b_i^t > p^{t-1}$ if $b_i^t > b_i^{t-1}$. That is, if a bidder revises his bid upward, he must bid more than the previous period's price.

[6] This assumption is not crucial; even if i were to remain in the pool of potential bidders he would not enter subsequently in our equilibria.

Following submission of period-t bids, the auction price p^t will be set to the second-second-highest bid among $b_1^t, \ldots, b_T^t$. (If there is more than one highest bid, the second-highest bid is equal to the highest bid.) Since $b_i^t \geq b_i^{t-1}$ $\forall i, t$, we have $p^0 \leq p^1 \leq \ldots \leq p^T$. Figure 1 depicts the timing of events.

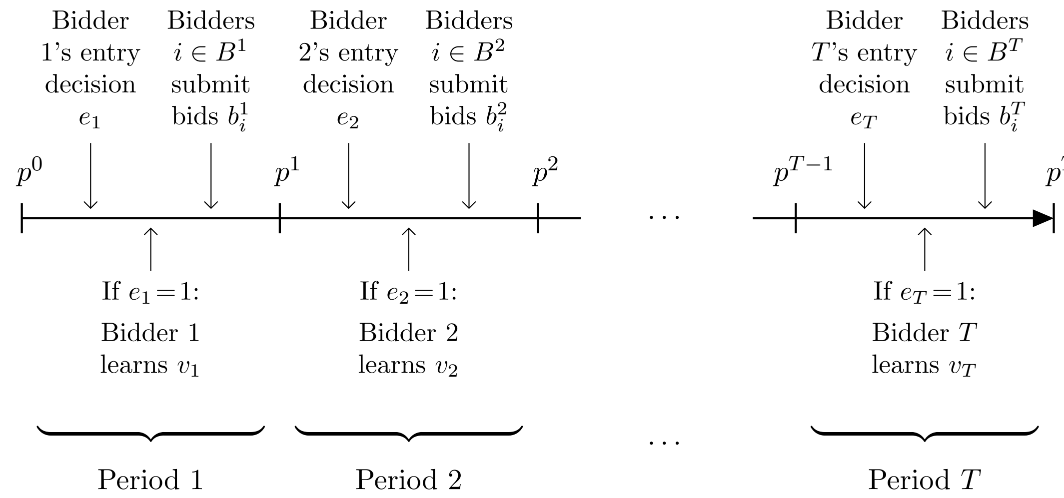

Figure 1: Timing

At the end of the final period T, if $B^T = \emptyset$ (i.e., if no bidders entered during the auction) the seller retains the object. If $B^T \neq \emptyset$, the bidder who submitted the highest bid wins the object and pays p^T. If two or more bidder submitted the highest bid, the object is awarded to the bidder who submitted the highest bid first. If there are two or more bidders who submitted the highest bid first, one of them is selected as winner by a random draw. Any bidder who does not win pays zero.

We assume that the auction rules, the entry cost c, the distribution F of values, and the arrival sequence of bidders are common knowledge. We also assume that $c < \int_{\underline{v}}^{\overline{v}} (1 - F(v))F(v)dv$. (This assumption ensures that at least two bidders can enter and obtain a positive expected surplus by bidding their valuations.)

3.2 Remarks

The entry cost c in our model has several possible interpretations. It could simply be the mental cost of introspection to determine one's willingness to pay for an item. Alternatively, c may represent the opportunity cost of the time and effort a potential bidder must spend to read and process the item description on an auction platform, in order to determine his willingness to pay. The assumption of valuations are independent and private then implies that the result of a bidder's introspection or research effort is idiosyncratic.

This interpretation is quite natural for objects such as collectibles, artwork, clothes, furniture, and the like. However, even for more "standardized" items such as electronics, some features (e.g., color) may be valued independently across buyers, or shipping costs may depend on a buyer's location (which is independent of the other location of others). Thus, the variation of the v_i should be interpreted to reflect such idiosyncratic differences. On the other hand, v_i may also contain a common value component, reflected in the level of v_i (i.e., the expectation of F). An implicit assumption in our model is that this common value component, if present, can be observed costlessly.

4 Equilibrium: Basics

In this Section, we introduce our notation for the bidders' decision rules and beliefs, and describe our immediate revelation equilibrium.

4.1 Strategies and beliefs

A bidder must make two decisions: Whether to enter the auction or not, and conditional on having entered, whether and how much to bid after entry and in each subsequent period.

An entry strategy for bidder i is, in general, a mapping from the current auction price p^{i-1} to entry decisions (either 0 or 1). Note that entry never depends on the bidder's valuation, since the bidder learns his valuation only after having entered the auction. In this section, we focus on entry strategies that are simple threshold strategies, prescribing entry if and only if the current price is low enough:

$$e_i(p^{i-1}) \;=\; \begin{cases} 1 & \text{if } p^{i-1} < p^*, \\ 0 & \text{if } p^{i-1} \geq p^*. \end{cases}$$

The entry threshold p^* will be the same across buyers in equilibrium, and therefore describes every bidder's entry strategy in a given equilibrium. However, entry thresholds will differ *across* equilibria.

Once a bidder has entered the auction, he can submit a bid in the present bidding round and in any round thereafter. Thus, a bidding strategy for bidder i

prescribes, for each period $t = i, \ldots, T$, a bid b_i^t as a function of i's information in period t. This information set includes i valuation v_i, the sequence of prices $p^1, \ldots, p^{t-1}$, i's previous bids $b_i^i, \ldots, b_i^{t-1}$, and on whether i is the high bidder at the beginning of round t. In general, i's bidding strategy can depend on all of these variables. In this section, we focus on bidding strategies that depend only on the round, a bidder's valuation, and the current price and bid:

$$b_i^t : [\underline{v}, \overline{v}] \times [0, \infty) \times [0, \infty) \to [0, \infty).$$

Here, $b_i^t(v_i, p^{t-1}, b_i^{t-1})$ is bidder i's bid in period t if his valuation is v_i, the current price is p^{t-1}, and i's bid in the previous period was b_i^{t-1}, and satisfies the restrictions on bids imposed in Section 3.1.[7]

A bidder will also entertain beliefs about the distribution of opponents' valuations, conditional on observed information. The belief that will be relevant in our equilibria is a potential bidder's belief about the highest valuation among the currently participating bidders, conditional on the last observed price. For $t \leq T$, let $w^t \equiv \max_{i \in B^t} v_i$ be the highest valuation among the bidders who have entered up to period t. In period t, the entering bidder's belief about w^{t-1} is then a conditional distribution

$$G(w^{t-1}|p^{t-1}) : [\underline{v}, \overline{v}] \to [0, 1].$$

Our solution concept will be a version of sequential equilibrium (Kreps and Wilson 1982). Specifically, we say that a profile of entry strategies $(e_i)_{i=1,\ldots,T}$, bidding strategies $(b_i)_{i=1,\ldots,T}$, and beliefs $G(\cdot)$, constitutes an *equilibrium* of the auction game if the following conditions hold for all $i = 1, \ldots, T$:

(i) Bidder i's bidding strategy b_i are optimal given $(b_j)_{j \neq i}$ and $(e_j)_{j \neq i}$;

(ii) bidder i's entry strategy e_i are sequentially rational given beliefs $G(\cdot|p^{i-1})$ for all $p^{i-1} \in [0, \overline{v}]$;

(iii) there exists a sequence of perturbed strategy profiles $(\widetilde{b}_i(\eta), \widetilde{e}_i(\eta)) \to (b_i, e_i)$ as $\eta \to 0$ such that any weakly increasing price sequence is possible under $(\widetilde{b}_i(\eta), \widetilde{e}_i(\eta))$, and for every $p^{t-1} \in [0, \overline{v}]$ the belief $G(\cdot|p^{t-t})$ is the limit of conditional distributions derived from Bayes' Rule under the perturbed strategies, as $\eta \to 0$.

[7]That is, $b_i^t(\cdot) \geq b_i^{t-1}(\cdot)$ $\forall t$ and $b_i^t(\cdot) > b_i^{t-1}(\cdot) \Rightarrow b_i^t(\cdot) > p^{t-1}$. These conditions will be satisfied in all strategies we examine.

4.2 Immediate revelation equilibrium

We call a bidding strategy an *immediate revelation strategy* if, immediately after entry, a bidder submits a bid equal to his private valuation if it exceeds the current price, and never revises his bid thereafter:

$$b_i^t(v_i, p^{t-1}, b_i^{t-1}) = \begin{cases} 0 & \text{if } \left[t = i \text{ and } v_i \leq p^{t-1}\right], \\ v_i & \text{if } \left[t = i \text{ and } v_i > p^{t-1}\right], \\ b_i^{t-1} & \text{otherwise.} \end{cases} \tag{1}$$

We will show that an equilibrium exists where all bidders follow the immediate revelation bidding strategy.

The optimality of the bidding strategy (1) is readily established. Conditional on all other bidders following strategy (1), and conditional on a fixed set of participants, a single bidder who enters the auction clearly cannot do better than bid his true valuation at some point before the end of the auction. But since potential bidders adopt a threshold entry strategy (as will be shown below), it is optimal for every bidder who has already entered the auction to bid his valuation immediately after entry: This results in a weakly higher price path than strategies that delay a truthful bid, and thus reduces the likelihood of entry by competitors.

Let us therefore assume a profile of immediate revelation bidding strategies, in order to fully characterize the bidders' entry decisions under this hypothesis. Consider bidder T, who observes price p^{T-1} before deciding whether to enter in period T. The payoff relevant variables for this bidder are his own valuation, v_T, and the highest valuation among participating rival bidders, which is w^{T-1}. Since v_T is itself a draw from F, bidder T's expected surplus after entry (if he bids according to (1)) is given by

$$U_T(p^{T-1}) = \int_{p^{T-1}}^{\overline{v}} \int_{p^{T-1}}^{\overline{v}} (v_T - w^{T-1})\, dG(w^{T-1}|p^{T-1})\, dF(v_T). \tag{2}$$

Bidder T enters if and only if $U_T(p^{T-1}) > c$.

Given any current price p^t, and assuming that bids are generated by the immediate revelation strategy (1), the conditional distribution of w^t is

$$G(w^t|p^t) = \frac{F(w^t) - F(p^t)}{1 - F(p^t)}. \tag{3}$$

Now use (3) to express (2) as

$$\begin{aligned}
U_T(p^{T-1}) &= \int_{p^{T-1}}^{\overline{v}} \left[\int_{p^{T-1}}^{v_T} (v_T - w^{T-1}) \frac{1}{1-F(p^{T-1})} dF(w^{T-1}) \right] dF(v_T) \\
&= \int_{p^{T-1}}^{\overline{v}} \frac{1}{1-F(p^{T-1})} \left[-F(p^{T-1})(v_T - p^{T-1}) \right. \\
&\qquad \left. + \int_{p^{T-1}}^{v_T} F(w^{T-1}) dw^{T-1} \right] dF(v_T) \\
&= \int_{p^{T-1}}^{\overline{v}} \frac{1}{1-F(p^{T-1})} \left[\int_{p^{T-1}}^{v_T} F(w^{T-1}) - F(p^{T-1}) dw^{T-1} \right] dF(v_T) \\
&= \int_{p^{T-1}}^{\overline{v}} \frac{1}{1-F(p^{T-1})} \left[\int_{w^{T-1}}^{\overline{v}} dF(v_T) \right] (F(w^{T-1}) - F(p^{T-1})) dw^{T-1} \\
&= \int_{p^{T-1}}^{\overline{v}} \frac{F(w^{T-1}) - F(p^{T-1})}{1-F(p^{T-1})} (1 - F(w^{T-1})) dw^{T-1}. \qquad (4)
\end{aligned}$$

(The second line is by integration by parts, and the fourth line by reversing the order of integration.) Note that (4) is strictly decreasing in p^{T-1}, larger than c at $p^{T-1} = \underline{v}$ (by our assumption on c), and zero at $p^{T-1} = \overline{v}$. Thus a unique price $p^* \in (\underline{v}, \overline{v})$ exists for which bidder T becomes indifferent between entering and not entering the auction. This price is implicitly defined by the condition

$$\int_{p^*}^{\overline{v}} \frac{F(v) - F(p^*)}{1 - F(p^*)} (1 - F(v)) dv \;=\; c. \qquad (5)$$

Bidder T hence enters in period T if $p^{T-1} < p^*$, and stays out if $p^{T-1} \geq p^*$.

We show in the Appendix that, given a profile of immediate revelation strategies, p^* is the entry threshold adopted by *all* potential bidders, regardless of their position in the arrival queue. We thus obtain the following result:

Proposition 1. ***(Immediate Revelation Equilibrium)*** *There exists an equilibrium of the auction game in which the following holds for all* $i = 1, \ldots, T$

(i) Upon arrival, bidder i's ($i > 1$) *belief about the distribution of the highest value among bidders* $1, \ldots, i-1$ *is* $G(w^{i-1}|p^{i-1})$*, given by* (3)*;*

(ii) bidder i enters if and only if $p^{i-1} < p^*$*, where* p^* *is implicitly defined by* (5)*;*

(iii) bidder i's bidding strategy (conditional on entry) is an immediate revelation strategy; that is, bidder i adopts the strategy (1).

The equilibrium characterized in Proposition 1 is interim efficient, in that the participating bidder with the highest valuation wins (and pays a price equal to the second-highest value among the participants). However, the outcome is not necessarily ex-post efficient. Once the auction price reaches p^*, entry ceases. Since only participating bidders learn their valuation, it is possible for a non-participating bidder to have a higher valuation than the winning bidder.

5 Delayed Revelation and Entry Deterrence

In this section, we explore how early entrants in the auction can deter entry by later potential participants via a strategy of delayed revelation. By this, we mean a strategy of bidding below one's true value after entry and revising this bid upward later.

5.1 Preliminaries

Delayed revelation will impact bidders' beliefs about the distribution of their opponents' valuations, and thereby affect entry in the auction. We now establish a preliminary result connecting bidders' beliefs and entry decisions. This result will then be used to construct various delayed revelation equilibria.

If in period T there are exactly two bidders participating in the auction whose values are larger than p^t, the conditional distribution of w^t is

$$H(w^t|p^t) \;=\; \left[\frac{F(w^t)-F(p^t)}{1-F(p^t)}\right]^2. \tag{6}$$

Now suppose that potential entrant T, after observing p^{T-1}, believes that two bidders in B^{T-1} have valuations larger than p^{T-1}, and that these two bidders will truthfully bid their valuations in period T. Bidder T's expected surplus from entering the auction and then bidding his own valuation in period T is then given by

$$U_T(p^{T-1}) \;=\; \int_{p^{T-1}}^{\overline{v}} \int_{p^{T-1}}^{v_T} (v_T - w^{T-1})\, dH(w^{T-1}|p^{I-1})\, dF(v_I) \;-\; c.$$

This is the same expression as (2) in Section 4.2, with H replacing G. Similar steps as those in Section 4.2 then show that a unique price $p^{**} \in (0,\overline{v})$ exists that makes bidder T indifferent between entering and not entering the auction. This indifference price is implicitly defined by the condition

$$\int_{p^{**}}^{\overline{v}} \left[\frac{F(v) - F(p^{**})}{1 - F(p^{**})} \right]^2 (1 - F(v)) dv \ = \ c. \tag{7}$$

If $p^{T-1} < p^{**}$ bidder T enters in period T, and if $p^{T-1} \geq p^{**}$ he does not enter.

As was the case for p^* in the immediate revelation equilibrium, it can be shown that p^{**} is also the entry threshold for every bidder $i < T$ in the arrival queue. More specifically, we can show the following result:

Lemma 2. *Suppose that every bidder i adopts a bidding strategy (conditional on entry) such that (i) if $p^{i-1} < p^{**}$ and $v_i > p^{i-1}$ then $b_i^i = \min\{v_i, p^{**}\}$; and (ii) if $v_i > p^{T-1}$ then $b_i^T = v_i$. Then in any equilibrium the profile of entry strategies satisfies the following for all i: If in period i bidder i believes that exactly two bidders in B^{i-1} have valuations above p^{i-1}, then i enters the auction in period i if and only if $p^{i-1} < p^{**}$, as defined in* (7)*. Moreover, $p^{**} < p^*$, as defined in* (5)*.*

The proof of Lemma 2 is in the Appendix; however, the intuition why $p^{**} < p^*$ is straightforward: At any price p, the larger the number of participating bidders whose valuation exceeds p, the lower is the expected surplus for an additional bidder who enters at price p. Thus, there exist values of p for which entering the auction is worthwhile if only one existing bidder's valuation exceeds p, and is not worthwhile if two existing bidders' valuations exceed p. This effect can be exploited by early bidders to deter entry by later bidders. We will show how in the following section.

5.2 A simple delayed revelation equilibrium

We call a bidding strategy a *delayed revelation strategy* if, after entry, some bidder submits a bid below his valuation, and then revises this bids to reflect his true valuation in the final bidding round. In particular, we will focus on the following bidding strategy (conditional on the bidder entering the auction):

$$b_i^t(v_i, p^{t-1}, b_i^{t-1}) \;=\; \begin{cases} 0 & \text{if } \big[t = i \text{ and } v_i \le p^{t-1}\big] \\ & \text{or } \big[t = i < T \text{ and } p^{t-1} \ge p^{**}\big], \\ v_i & \text{if } \big[t = i < T \text{ and } p^{t-1} < v_i \le p^{**}\big] \\ & \text{or } \big[t = T \text{ and } v_i > p^{T-1}\big], \\ p^{**} & \text{if } \big[t = i < T \text{ and } v_i > p^{**} > p^{t-1}\big], \\ b_i^{t-1} & \text{otherwise,} \end{cases} \tag{8}$$

where p^{**} is the price defined in (7).

Strategy (8) is identical to the immediate revelation strategy (1), with two exceptions: First, a bidder whose valuation is above the threshold p^{**} does not bid his valuation upon entry if the current price is below p^{**}. Instead, this bidder submits p^{**} after entry, but revises his bid to his true valuation in the final period. Second, a bidder who enters at price p^{**} or above does not bid until the final period, at which time he bids his valuation.

We will show that an equilibrium exists where all bidders follow the delayed revelation bidding strategy (8). This strategy is easily seen to constitute a mutual best response, *for a given set of participating bidders*: Note that all participating bidders submit their true valuation by the final period (unless the price at some point already exceeds a bidder's valuation). Against this profile, any strategy is optimal for which a bidder submits his true valuation in the final period (again, unless the price already exceeds the bidder's valuation). This is precisely what strategy (8) prescribes. Thus, given a fixed set of participating bidders a profile of delayed revelation strategies is an equilibrium of the auction. We therefore assume a profile of delayed revelation bidding strategies, and characterize the equilibrium entry decisions under this hypothesis.

Our equilibrium entry strategy is for bidder i to enter in period i if and only if $p^{i-1} < p^{**}$. This strategy is indeed optimal, assuming bidding proceeds as in (8): In any period t, assuming that all previous bids were generated by the delayed revelation strategy (8), the distribution of w^t conditional on prices p^{t-1} that are consistent with this strategy is given by

$$G(w^t|p^t) \;\text{ if }\; p^t < p^{**}, \qquad H(w^t|p^t) \;\text{ if }\; p^t = p^{**}. \tag{9}$$

This is so because the only possibility that a price of p^{**} is observed—under the presumed bidding strategy—is for exactly two bidders to have submitted a bid of

p^{**}. In this case, there will be exactly two bidders with valuations above p^{**} in B^t. By Lemma 2, therefore, it is optimal for bidder t not to enter in any period t if the previous price is $p^{t-1} = p^{**}$. If the participating bidders continue to follow bidding strategy (8), the price will stay at p^{**} until the final round of bidding, which means that entry will be deterred in all subsequent periods as well. On the other hand, if $p^{t-1} < p^{**}$, then exactly one bidder has a valuation above p^{t-1}, given the presumed bidding strategy. Since $p^{t-1} < p^{**} < p^*$, the analysis in Section 4.2 implies that bidder t should enter in period t.

Unlike in the immediate revelation equilibrium, there are now out-of-equilibrium beliefs to be specified. (Note that, given the entry and bidding strategies described above, a price $p^t > p^{**}$ cannot be observed for any $t < T$. Yet, the equilibrium entry strategy must be sequentially rational given beliefs at such prices as well.) This will be done in the Appendix. We thus have the following result:

Proposition 3. ***(Delayed Revelation Equilibrium)*** *There exists an equilibrium of the auction game in which the following holds for all $i = 1, \ldots, T$:*

(i) Upon arrival, bidder i's ($i > 1$) belief along the equilibrium price path about the distribution of the highest value among bidders $1, \ldots, i-1$ is given by (9)*;*

(ii) bidder i enters if and only if $p^{i-1} < p^{**}$, where p^{**} is implicitly defined by (7)*;*

(iii) bidder i's bidding strategy (conditional on entry) is a delayed revelation strategy; that is, bidder i adopts the strategy (8).

Just like in the immediate revelation equilibrium of Section 4.2, the object will get awarded to the bidder with the highest valuation among the participating bidders, and this bidder pays the second-highest valuation among the participants. However, the pool of participants will be different across the two equilibria. In particular, in the delayed revelation equilibrium entry ceases once the auction price is p^{**} or above. As shown in Lemma 2, the entry threshold p^{**} is less than the threshold p^* in the immediate revelation equilibrium. Thus, it has a positive probability that the participants with the highest and second-highest valuations in the immediate revelation equilibrium do not enter in the delayed revelation equilibrium. In this case, the final allocation and price will be different across the two equilibria.

6 Incremental Bidding

6.1 Collusion and coordination

By delaying the revelation of their true valuations until the final period, the first and second bidder to arrive who have valuations above p^{**} in effect collude to deter entry by potential rival bidders. The two colluding bidders then compete against one another in a single Vickrey auction in the final period. Thus, the delayed revelation equilibrium described in Proposition 3 leads to larger expected surpluses for these bidders than the immediate revelation equilibrium described in Proposition 1.[8]

The outcome of this collusive effort depends on whether two coordination attempts succeed. First, the equilibrium calls on the first two first bidders whose valuations exceeds p^{**} to delay truthful revelation and bid p^{**} at first. Suppose that only one of these bidders were to delay a truthful bid, and the second bidder submitted some other bid upon entry—e.g., his valuation, if this bidder is under the impression that the immediate revelation equilibrium is being played instead. In this event, the price would still not rise above p^{**}. Thus, a single bidder who wants to collude can attempt to do so safely, even if the bidder who he is colluding with is not aware of this attempt, as long as all future bidders interpret the price p^{**} as they should in the delayed revelation equilibrium.

Second, potential entrants who see a price of p^{**} must interpret this price to mean that two bidders participate in the auction whose valuations exceed p^{**}. However, the same price can also occur in the immediate revelation equilibrium—namely, if the second highest bidder in some period happens to have a valuation equal to p^{**}—and in this case entry would not cease at p^{**}. Thus, the entry deterring effect of bidding p^{**} in the delayed revelation equilibrium rests on all other bidders believing that this equilibrium, and not the immediate revelation equilibrium, is being played.

Unlike coordination among the two colluding bidders, coordination among the many potential entrants may be harder to achieve. Is there a way for colluding bidders to signal to potential entrants "more strongly" that the price they observe was generated by a delayed revelation strategy? In other words, does a collusive bidding strategy exist that induces in a price path which would be even less likely to

[8]More precisely, the distribution of surpluses received by the two colluding bidders in the delayed revelation equilibrium first-order stochastically dominates the distribution of surpluses these bidders obtain in the immediate revelation equilibrium.

occur under truthful bidding, compared to a path where the price simply remains stuck at p^{**}?

Consider the following bidding strategy. Upon entry, i bids 0 if $v_i \leq p^{i-1}$, v_i if $p^{i-1} < v_i < p^{**}$, and p^{**} if $v_i \geq p^{**} = p^{i-1}$. Once two bidders i, j with values $v_i, v_j \geq p^{**}$ have entered in period t, we have $p^t = p^{**}$. In all subsequent periods $t < t' < T$, bidders i and j submit bids $b_i^{t'} = b_i^{t'-1} + \kappa$ and $b_j^{t'} = b_j^{t'-1} + \kappa$, where $\kappa > 0$ is some small increment. (For example, if the auction format features a set minimum bid increment, κ can be the required amount.) If $b_i^{t'-1} + \kappa > v_i$ for some t', i stops raising his bids, and similarly for j. In the final period both bidders reveal their valuations, that is, $b_i^T = v_i$ and $b_j^T = v_j$.

If no other bidders enter after t, this strategy will induce a slowly rising price path, along which

$$p^{t+1} = p^t + \kappa.$$

In the immediate revelation equilibrium, this sequence of prices is infinitely less likely to be observed than the price sequence

$$p^t = p^{t+1} = \ldots = p^{**}.$$

The first sequence would require that in every period a bidder enters whose valuation is exceeds the previous entrant's valuation exactly by the amount κ. On the other hand, the second sequence only requires that one bidder enters with a valuation exactly equal to p^{**}, one bidder enters with a valuation larger than p^{**}, and all other participants have valuations below p^{**}. Thus, observing the former sequence virtually guarantees that it was generated by two bidders submitting incremental bids below their true valuations. But since $p^t \geq p^{**}$ and no bidder bids above his valuation, an incremental price path signals that two bidders have valuations above p^{**}. Hence, as long as the price is slowly rising by the increment κ in every period, no new bidders will enter.

6.2 Incremental bidding equilibria

To fully formalize the ideas introduced above, let us introduce a state variable $\theta^t \in \{0, 1\}$ defined as follows:

$$\theta^t = \begin{cases} 1 & \text{if } t \geq 3 \text{ and } \Big[\big[p^{t-1} = p^{t-2} + \kappa > p^{**} \big] \text{ or } \\ & \qquad\qquad\quad\; \big[p^{t-1} = p^{**} \text{ and } p^{t-2} < p^{**} \big] \Big], \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

If $\theta^t = 1$, the auction is in a state of *incremental bidding* in period t. This is the case after the price equals p^{**} for the first time, and in every period thereafter in which the last observed price is a κ-increment above the second-last price. The increment $\kappa > 0$ will be endogenous to the equilibrium; however, its value will be arbitrary.

We now assume that a potential entrant observes the last two prices, and can hence condition his entry and bidding strategies in period t on the state variable θ^t. In particular, we consider the entry strategy

$$e_i(p^{i-1}, \theta^t) = \begin{cases} 0 & \text{if } \theta^t = 1 \text{ or } p^{t-1} \geq p^*, \\ 1 & \text{otherwise.} \end{cases} \tag{11}$$

Under this strategy, a potential bidder enters the auction unless the auction is in the incremental bidding state, or the price has reached the entry threshold p^*. After entry, bidder i's plays the following bidding strategy:

$$b_i^t(v_i, p^{t-1}, b_i^{t-1}, \theta^t) = \begin{cases} 0 & \text{if } \left[t = i \text{ and } v_i \leq p^{t-1}\right] \\ & \text{or } \left[t = i < T \text{ and } p^{t-1} \geq p^*\right] \\ & \text{or } \left[t = i < T \text{ and } \theta^t = 1 \right. \\ & \qquad\qquad \left. \text{and } v_i < p^{t-1} + \kappa\right], \\ v_i & \text{if } \left[t = i < T \text{ and } p^{t-1} < v_i \leq p^{**}\right] \\ & \text{or } \left[t = T \text{ and } v_i > p^{T-1}\right], \\ p^{**} & \text{if } \left[t = i < T \text{ and } v_i > p^{**} > p^{t-1}\right], \\ p^{t-1} + \kappa & \text{if } \left[\theta^t = 1 \text{ and } v_i \geq p^{t-1} + \kappa\right] \\ & \text{or } \left[t = i < T \text{ and } \theta^t = 0 \right. \\ & \qquad \left. \text{and } v_i - \kappa \geq p^{t-1} \geq p^{**}\right], \\ b_i^{t-1} & \text{otherwise.} \end{cases} \tag{12}$$

This *k-incremental bidding strategy* is identical to the delayed revelation strategy (8), with two exceptions: First, bidders with values above the current price plus κ submit incremental bids, if the auction is in the incremental bidding state. Second, should a bidder enter at price p^{**} or higher and the auction is *not* in the incremental bidding state, the entering bidder will attempt to restart an incremental bidding phase by submitting an incremental bid.[9]

[9]This case will occur if the auction was in the incremental bidding state but has left that state because the price increased to the valuation of one of the bidders who were submitting incremental bids.

Now consider the beliefs of potential entrants. In any period t, assuming that all previous bids were generated by the k- incremental bidding strategy (12), the distribution of w^{t-1} conditional on observed prices p^{t-1} that are consistent with this strategy is given by

$$G(w^{t-1}|p^{t-1}) \text{ if } \theta^t = 0, \quad H(w^{t-1}|p^{t-1}) \text{ if } \theta^t = 1. \tag{13}$$

Under these beliefs, the arguments made in Section 4 imply that bidder t should enter the auction if $\theta^t = 0$ (as long as $p^{t-1} < p^*$), and the arguments made in Section 5 imply that bidder t should not enter the auction if $\theta^t = 1$. Entry strategy (11) is therefore sequentially rational given Bayesian beliefs under the presumed bidding strategy.

The optimality of the bidding strategy given the entry strategy still needs to be established, and out-of-equilibrium beliefs taken care of. Again, this will be done in the Appendix. We then have:

Proposition 4. ***(κ-Incremental Bidding Equilibrium)*** *Let $\kappa > 0$. There exists an equilibrium of the auction game in which the following holds for all $i = 1, \ldots, T$:*

(i) Upon arrival, bidder i's ($i > 1$) belief along the equilibrium price path about the distribution of the highest value among bidders $1, \ldots, i-1$ is given by (13);

(ii) bidder i enters if and only if $p^{i-1} < p^*$ and $\theta^t = 0$, where p^* is implicitly defined by (5) and θ^t is the state variable given by (10);

(iii) bidder i's bidding strategy (conditional on entry) is the κ-incremental bidding strategy; that is, bidder i adopts the strategy (12) using κ as the bid increment.

Note that many incremental bidding equilibria exist, which differ by the value of the increment κ. The smaller is κ, the more closely will the equilibrium price sequence under κ-incremental bidding resemble the price sequence in the delayed revelation equilibrium characterized in Proposition 3.

Furthermore, note that in an incremental bidding equilibrium, entry may resume after a phase of entry deterrence. This will be the case whenever the price increases to the valuation of one of the colluding bidders, who then stops incrementing his bid and thereby halts the progression of increasing prices. If, at this moment, the price is still below the entry threshold p^* (i.e., the threshold in the immediate revelation equilibrium), entry resumes until a bidder enters whose valuation exceeds the current price sufficiently to reinitiate the incremental bidding phase. Therefore, the final allocation and price in an incremental bidding equilibrium will differ from that in the delayed revelation equilibrium with positive probability. The smaller κ, however, the smaller is the probability that an incremental bidding phase is interrupted, and the larger is the probability that the final allocation and price in the incremental bidding equilibrium coincides with that in the delayed revelation equilibrium.

7 Expected Surplus Comparison

[To be added.]

8 Discussion

[To be added.]

Appendix

Proof of Proposition 1

Part (iii) of the result was established in the text. Part (i) is straightforward: For $t \geq 3$, all $p^{t-1} \in [0, \overline{v}]$ can occur in equilibrium. In period 2, the second-price auction format implies that bidder 2 observes first-period price $p^1 = 0$ regardless of actions taken by the previous bidder. Thus, for all $t \geq 2$, the belief $G(w^{t-1}|p^{t-1})$ is given by (3) and computed from the equilibrium strategies using Bayes' Rule for all feasible p^{t-1}.

To establish part (ii) of the result, we need to prove that p^* is the entry threshold not only for bidder T (which was already shown in the text), but also for all bidders $t < T$. We split the argument into two steps.

Step 1. We show that $p^{t-1} \geq p^*$ implies that bidder t does not enter. This will be done by induction. Suppose $p^{T-2} \geq p^*$; then $p^{T-1} \geq p^*$ and bidder T will not enter in period T. Knowing that bidder T will not enter, bidder $T-1$ competes against the highest bidder in B^{T-2}, whose valuation is distributed by $G(w^{T-2}|p^{T-2})$. This is the problem examined in the main text in Section 4.2, and we know that, since $p^{T-2} \geq p^*$, it is optimal for bidder $T-1$ not to enter in period $T-1$. Now suppose that $p^{T-3} \geq p^*$. Then $p^{T-1} \geq p^{T-2} \geq p^*$, so that bidders T and $T-1$ will not enter. Bidder $T-2$ therefore competes against the highest valuation bidder in B^{T-3}, whose valuation is distributed by $G(w^{T-3}|p^{T-3})$. Because $p^{T-3} \geq p^*$, it is optimal for bidder $T-2$ not to enter. Continuing in this fashion, we conclude that bidder $t \in \{1, \ldots, T\}$ does not enter in period t if $p^{t-1} \geq p^*$.

Step 2. We show that $p^{t-1} < p^*$ implies that bidder t enters. Let z^{t+1} be the highest bids submitted by bidders who will enter after period t, and let $Z(z^{t+1}|p^t)$ be the distribution of z^{t+1} conditional on period-t price p^t. Note that, if bidder t enters in period t and bids v_t, then $p^t = \min\{v_t, w^{t-1}\}$, where w^{t-1} is the highest valuation of bidders $j \in B^{t-1}$. Under the bidding strategies (1), the continuation payoff (not including the entry cost c) from entering at price $p^{t-1} < p^*$ to bidder t is thus given by

$$U_t(p^{t-1}) = \int_{p^{t-1}}^{\overline{v}} \int_{p^{t-1}}^{v_t} \int_0^{v_t} (v_t - \max\{w^{t-1}, z^{t+1}\}) \, dZ(z^{t+1}| \min\{v_t, w^{t-1}\}) \, dG(w^{t-1}|p^{t-1}) \, dF(v_t).$$

Define

$$A(v_t) = \int_{p^{t-1}}^{p^*} \int_0^{v_t} (v_t - \max\{w^{t-1}, z^{t+1}\}) \, dZ(z^{t+1}| \min\{v_t, w^{t-1}\}) \, dG(w^{t-1}|p^{t-1}),$$

$$B(v_t) = \int_{p^*}^{v_t} \int_0^{v_t} (v_t - \max\{w^{t-1}, z^{t+1}\}) \, dZ(z^{t+1}| \min\{v_t, w^{t-1}\}) \, dG(w^{t-1}|p^{t-1}),$$

and express bidder t's payoff from entering as follows:

$$U_t(p^{t-1}) = \int_{p^{t-1}}^{\overline{v}} [A(v_t) + B(v_t)] \, dF(v_t) > \int_{p^*}^{\overline{v}} [A(v_t) + B(v_t)] \, dF(v_t). \quad (14)$$

Now consider two cases.

1. First, suppose $v_t \geq p^*$ and $w^{t-1} \geq p^*$. Then the price at the end of period t will be $p^t = \min\{v_t, w^{t-1}\} \geq p^*$, and no entry will occur after period t as shown in Step 1. Thus, conditional on $v_t \geq p^*$ and $w^{t-1} \geq p^*$ we have $z^{t+1} = 0$, which allows us to write

$$B(v_t) = \int_{p^*}^{v_t} (v_t - w^{t-1}) dG(w^{t-1}|p^{t-1})$$
$$= (1 - G(p^*|p^{t-1})) \int_{p^*}^{v_t} (v_t - w^{t-1}) dG(w^{t-1}|p^*). \quad (15)$$

2. Second, suppose $v_t \geq p^*$ and $w^{t-1} \leq p^*$. If during some period $s > t$ a bidder enters with $v_s \geq p^*$, the price at the end of period s will be $p^s = \min\{v_t, v_s\} \geq p^*$, and no further entry will occur after period s, as shown in Step 1. In this event, $z^{t+1} = v_s > p^*$ with distribution $G(z^{t+1}|p^*)$. If no bidder with $v_s \geq p^*$ enters during any period $s > t$, we have $z^{t+1} < p^*$. Since bidder t's payoff will be lower in the first event than in the second, we can write

$$A(v_t) > \int_{p^{t-1}}^{p^*} \int_{p^*}^{v_t} (v_t - z^{t+1}) dG(z^{t+1}|p^*) dG(w^{t-1}|p^{t-1})$$
$$= G(p^*|p^{t-1}) \int_{p^*}^{v_t} (v_t - z^{t+1}) dG(z^{t+1}|p^*). \quad (16)$$

Combining (14)–(16), we have

$$U_t(p^{t-1}) > \int_{p^*}^{\overline{v}} \int_{p^*}^{v_t} (v_t - v) dG(v|p^*) = c$$

as shown in the main text in Section 4.2. Thus, when $p^{t-1} < p^*$, the expected surplus for bidder t from entering the auction in period t exceeds the entry cost c, so bidder t enters. □

Proof of Lemma 2

We need to prove that, under the assumptions of Lemma 2, p^{**} is the entry threshold not only for bidder T (which was already shown in Section 5.1), but also for all bidders $t < T$. The argument is parallel to the one we made to prove Proposition 1 (ii), and proceeds in two steps. We then prove in a third step that $p^{**} < p^*$.

Step 1. We show that $p^{t-1} \geq p^{**}$ implies that bidder t does not enter. Suppose $p^{T-2} \geq p^{**}$; then $p^{T-1} \geq p^{**}$ and bidder T will not enter in period T. Knowing that bidder T will not enter, bidder $T-1$ competes against the highest valuation among bidders in B^{T-2}, w^{T-2}. Furthermore, if i believes that exactly two bidders in B^{T-2} have valuations above p^{T-2}, w^{T-2} follows distribution $H(w^{T-2}|p^{T-2})$. The same argument we made for bidder T in Section 5.1 then implies that it is optimal for bidder $T-1$ not to enter in period $T-1$. Continuing inductively, we conclude that bidder $t \in \{1, \ldots, T\}$ does not enter in period t if $p^{t-1} \geq p^{**}$.

Step 2. We show that $p^{t-1} < p^{**}$ implies that bidder t enters. Define z^{t+1} and $Z(z^{t+1}|p^t)$ as in the proof of Proposition 1. Since, by assumption, all bidders $j \in B^T$ with $v_j > p^{T-1}$ will bid their valuations in period T, the continuation payoff (not including the entry cost c) from entering at price $p^{t-1} < p^*$ to bidder t is thus given by

$$U_t(p^{t-1}) = \int_{p^{t-1}}^{\overline{v}} \int_{p^{t-1}}^{v_t} \int_0^{v_t} (v_t - \max\{w^{t-1}, z^{t+1}\}) \, dZ(z^{t+1}| \min\{v_t, w^{t-1}\}) \, dH(w^{t-1}|p^{t-1}) \, dF(v_t).$$

By setting

$$A(v_t) = \int_{p^{t-1}}^{p^{**}} \int_0^{v_t} (v_t - \max\{w^{t-1}, z^{t+1}\}) \, dZ(z^{t+1}| \min\{v_t, w^{t-1}\}) \, dH(w^{t-1}|p^{t-1}),$$

$$B(v_t) = \int_{p^{**}}^{v_t} \int_0^{v_t} (v_t - \max\{w^{t-1}, z^{t+1}\}) \, dZ(z^{t+1}| \min\{v_t, w^{t-1}\}) \, dH(w^{t-1}|p^{t-1}),$$

we can express bidder t's payoff from entering as follows:

$$U_t(p^{t-1}) = \int_{p^{t-1}}^{\overline{v}} [A(v_t) + B(v_t)] \, dF(v_t) > \int_{p^{**}}^{\overline{v}} [A(v_t) + B(v_t)] \, dF(v_t).$$

Mirroring our proof of Proposition 1 (ii), we consider two cases.

1. First, suppose $v_t \geq p^{**}$ and $w^{t-1} \geq p^{**}$. Under the assumed bidding strategies, the price at the end of period t will be $p^t = p^{**}$ and no entry will occur after period t (as shown in Step 1), so we can write

$$B(v_t) = \int_{p^{**}}^{v_t} (v_t - w^{t-1}) dG(w^{t-1}|p^{t-1}) = (1 - G(p^{**}|p^{t-1})) \int_{p^{**}}^{v_t} (v_t - w^{t-1}) dG(w^{t-1}|p^{**}).$$

2. Second, suppose $v_t \geq p^{**}$ and $w^{t-1} \leq p^{**}$. If during some period $s > t$ a bidder enters with $v_s \geq p^{**}$, under the assumed bidding strategies the price

at the end of period s will be $p^s = p^{**}$ and no further entry will occur after period s (as shown in Step 1). In this event, $z^{t+1} = v_s > p^{**}$ with distribution $G(z^{t+1}|p^{**})$. If no bidder with $v_s \geq p^{**}$ enters during any period $s > t$, we have $z^{t+1} < p^{**}$. Since bidder t's payoff will be lower in the first event than in the second, we can write

$$A(v_t) \; > \; \int_{p^{t-1}}^{p^{**}} \int_{p^{**}}^{v_t} (v_t - z^{t+1}) dG(z^{t+1}|p^{**}) dG(w^{t-1}|p^{t-1})$$
$$= \; G(p^{**}|p^{t-1}) \int_{p^{**}}^{v_t} (v_t - z^{t+1}) dG(z^{t+1}|p^{**}).$$

Combining the last three equations, we have

$$U_t(p^{t-1}) \; > \; \int_{p^{**}}^{\overline{v}} \int_{p^{**}}^{v_t} (v_t - v) dH(v|p^*) \; = \; c$$

as shown in Section 5.1. Thus, when $p^{t-1} < p^{**}$, the expected surplus for bidder t from entering the auction in period t exceeds the entry cost c, so bidder t enters. □

Step3. Finally, we need to show that $p^{**} < p^*$. To do so, let

$$L_k(p) \; \equiv \; \int_p^{\overline{v}} \left[\frac{F(v) - F(p)}{1 - F(p)} \right]^k (1 - F(v)) dv.$$

Note that $L_1(p^*) = L_2(p^{**}) = c$ and $L_1(p) < L_2(p) \; \forall p < \overline{v}$. Therefore, $L_2(p^*) < c$, and since L_2 is strictly decreasing we conclude that $p^* > p^{**}$. □

Proof of Proposition 3

Most of the result was shown already in the text in Section 5.2. What is left is to establish the optimality of the equilibrium bidding strategy given the equilibrium entry strategy (Step 1), and the optimality of the entry strategy following off-equilibrium prices; that is, prices that exceed p^{**} (Step 2).

Step 1. Clearly, in the final period a truthful bid $b_i^T = v_i$ is optimal for every i with value $v_i > \min\{b_i^{T-1}, p^{T-1}\}$. Let us therefore consider bidding in periods $t < T$. We need to consider three types of deviations.

1. Suppose bidder i with valuation $v_i > p^{**}$ deviates from the equilibrium strategy by bidding $b_i^t \neq p^{**}$ in period $t < T$. If this deviation does not change the price in periods $s \geq t$, the deviation has no effect on the final allocation and price. If the deviation changes the price in some period $s \geq t$

from $p^s = p^{**}$ to $p^s \neq p^{**}$, then additional bidders will enter with positive probability, reducing the payoff to existing bidder i.

2. Suppose bidder i with valuation $v_i \leq p^{**}$ deviates from the equilibrium strategy by bidding $b_i^t < v_i$ in period $t < T$. This deviation has no effect on the final allocation and price, and hence on payoffs.

3. Suppose bidder i with valuation $v_i \leq p^{**}$ deviates by bidding $b_i^t > v_i$ in period $t < T$. At best, this deviation reduces the number of bidders who enter in periods $s > t$, namely if i submits bid $b_i^t = p^{**}$ that results in an entry-deterring price p^{**} earlier than what would otherwise have been the case. However, this requires one other bid $b_j^s = p^{**}$, submitted by bidder $j \neq i$ in some period s. Under our bidding strategy this only happens if $v_j \geq p^{**}$. If such a bidder participates, he will outbid i in the final period, leaving the outcome for i unchanged. If such a bidder does not participate, then i will have the highest final bid and win. In this case, i either pays the same price as before (if i had won without the deviation), or pays price $p^T > v_i$ (if i had lost without the deviation).

We therefore conclude that no entering bidder has an incentive to deviate from the equilibrium bidding strategies (8).

Step 2. Next, we consider the bidders' entry decisions. Recall that our entry strategy asks potential bidders to stay out of the auction if they see a price larger than p^{**}; yet, in equilibrium, no prices larger than p^{**} are observed (except for the final price p^T). For sequential equilibrium we need to find a sequence of strategy profiles, converging to the equilibrium strategies, such that prices above p^{**} are possible along the sequence and the equilibrium entry strategies are sequentially rational under the limit of Bayesian beliefs generated by the sequence of perturbed strategies.

To this end, consider the following perturbed strategy for every player i:

$\widetilde{e}_i$: In period i, enter with probability $(1-\eta)e_i(p^{i-1}) + \eta(1 - e_i(p^{i-1}))$, where $e_i(\cdot)$ is the equilibrium entry strategy.

$\widetilde{b}_i$: Conditional on having entered, bid as follows: If $v_i \geq p^{**}$, then with probability $1-\eta$ play the equilibrium bidding strategy (8), and with probability η submit a random sequences of bids b_i^t ($i \leq t \leq T$) drawn from cumulative distribution function

$$\Gamma(b_i^t | b_i^{t-1}) \;=\; \frac{1}{2} + \frac{1}{2}\frac{b_i^t - b_i^{t-1}}{\overline{v} - b_i^{t-1}}. \tag{17}$$

(The precise functional form of $\Gamma(\,\cdot\,|b_i^{t-1})$ is unimportant, as long as it has support $[b_i^{t-1}, \overline{v}]$ and mass at b_i^{t-1}.) If $v_i < p^{**}$ follow the equilibrium bidding strategy (8).

Note that any weakly increasing sequence of prices $p^1 \leq p^2 \leq \ldots$ can occur under this strategy profile, as long as $\eta > 0$. Furthermore, as $\eta \to 0$ the profile converges to the equilibrium strategies.

Now suppose some potential bidder t observes a price $p^{t-1} > p^{**}$. This can only happen if at least two participating bidders did not play their equilibrium strategies, and submitted bids larger than p^{**}. Given the perturbed profile, the probability that both of these deviating players have valuations of at least p^{**} is

$$\frac{\eta^2}{\eta^2 + \alpha(\eta)} \xrightarrow{\eta \to 0} 1,$$

where $\alpha(\eta) = o(\eta^2)$. Thus, the resulting limit belief about w^{t-1}, conditional on $p^{t-1} > p^{**}$, is

$$\widetilde{H}(w|p^{t-1}) \equiv Pr[w^{t-1} < w | p^{t-1} > p^{**}] = H(w|p^{**})$$

where H is defined in (6). By Lemma 2, it is optimal for a bidder to stay out of the auction at price p^{**} if the bidder believes that two participating bidders have valuations above p^{**}. Since $p^{t-1} > p^{**}$, it is optimal for bidder t to stay out after observing p^{t-1}, given beliefs $\widetilde{H}(w^{t-1}|p^{t-1})$. □

Proof of Proposition 4

We need to establish the optimality of the equilibrium bidding strategy given the equilibrium entry strategy (Step 1), and the optimality of the entry strategy following off-equilibrium prices (Step 2).

Step 1. This step is almost identical to that in the proof of Proposition 3. By not following strategy (12) when all rivals follow (12), the best bidder i can hope for is to retard the entry process in periods when there would otherwise be entry. In parallel to our arguments above, this will entail i bidding above v_i in some period, and at least one other bidder j submitting the same bid in the same period. Since j is still following the equilibrium strategy, his valuation v_j will exceed the collusive bid and therefore exceed v_i. This means that j will outbid i in period T, guaranteeing a loss for i. On the other hand, if no such bidder j exists, then i will either lose, or win but pay a price above v_i. In all cases, i is no better off than he would be had he followed strategy (12).

Step 2. For all $2 \leq t \leq T$, prices p^{t-1} above p^{**} that are not κ-increments over p^{t-2} cannot arise under the incremental bidding strategy. The equilibrium prescribes entry in all periods t where this is the case, unless $p^{t-1} \geq p^*$. To show that, in this case, entry is sequentially rational under limit Bayesian beliefs, consider the following perturbed strategy for every player i:

$\widetilde{e}_i$: In period i, enter with probability $(1-\eta)e_i(p^{i-1}) + \eta(1 - e_i(p^{i-1}))$, where $e_i(\cdot)$ is the equilibrium entry strategy.

$\widetilde{b}_i$: Conditional on having entered, bid as follows: If $v_i \geq p^{**}$, then with probability $1-\eta$ play the equilibrium bidding strategy (12). With probability η, play a strategy that is identical to (12) up to a randomly and uniformly selected period $t^* \geq i$. If $p^{t-1} < v_i$, then in every period $t \geq t^*$ bid $b_i^t = v_i$. If $v_i < p^{**}$, play the equilibrium strategy (12).

Note that any weakly increasing sequence of prices $p^1 \leq p^2 \leq \ldots$ can occur under this strategy profile, as long as $\eta > 0$. Furthermore, as $\eta \to 0$ the profile converges to the equilibrium strategies.

Now suppose some potential bidder t observes a price $p^{t-1} > p^{**}$ which is not a κ-increment over p^{t-1}. Given the perturbed strategies, this means that at least one bidder submitted a truthful bid when he should not have done so in the equilibrium. Furthermore, p^{t-1} will then be equal the valuation of one of the bidders who trembled. Consider the following cases.

1. $\theta^{t-1} = 1$ and $p^{t-1} < p^{t-2} + \kappa$. Let i and j denote the two participating bidders who submitted incremental bids in period $t-2$. Under the perturbed profile, the most likely explanation for out-of-equilibrium price p^{t-1} is that exactly one of bidders i, j did not submit an incremental bid in period $t-1$ (and instead submitted a truthful bid), while all other bidders followed their equilibrium bidding and entry strategies. This event has conditional probability $2\eta(1-\eta)/[2\eta(1-\eta) + \alpha(\eta)] \to 1$ as $\eta \to 0$, where $\alpha(\eta) = o(\eta)$.

2. $\theta^{t-1} = 1$ and $p^{t-1} > p^{t-2} + \kappa$. Let i and j denote the two participating bidders who submitted incremental bids in period $t-2$. Under the perturbed profile, the most likely explanation for out-of-equilibrium price p^{t-1} is that both i, j did not submit incremental bids in period $t-1$ (and instead submitted truthful bids), while all other bidders followed their equilibrium strategies.

This event has conditional probability $\eta^2/[\eta^2+\alpha(\eta)] \to 1$ as $\eta \to 0$, where $\alpha(\eta) = o(\eta^2)$.

3. $\theta^{t-1} = 0$ and $p^{t-2} < p^{**}$. Under the perturbed profile, the most likely explanation for out-of-equilibrium price p^{t-1} is that in some period $i \le t-2$ a bidder entered whose valuation is $v_i \ge p^{t-1}$ and who bid $b_i^i = v_i$; and in period $t-1$ a bidder entered whose valuation is $v_{t-1} \ge p_{t-1}$ and who bid $b_{t-1}^{t-1} = v_{t-1}$; and $p^{t-1} = \min\{v_i, v_{t-1}\}$. (For both bidders i and $t-1$, entry is not a mistake but bidding truthfully is.) This event has conditional probability $\eta^2/[\eta^2+\alpha(\eta)] \to 1$ as $\eta \to 0$, where $\alpha(\eta) = o(\eta^2)$.

4. $\theta^{t-1} = \theta^{t-2} = 0$ and $p^{t-1} \ge p^{**}$. Under the perturbed profile, the most likely explanation for out-of-equilibrium price p^{t-1} is that in some period $i \le t-2$ such that $\theta^i = 0$ and $p^i \ge p^{**}$ a bidder entered whose valuation is $v_i \ge p^{t-1}$ and who bid $b_i^i = v_i$; and in period $t-1$ a bidder entered whose valuation is $v_{t-1} \ge p_{t-1}$ and who bid $b_{t-1}^{t-1} = v_{t-1}$; and $p^{t-1} = \min\{v_i, v_{t-1}\}$. (For both bidders i and $t-1$, entry is not a mistake but bidding truthfully is.) This event has conditional probability $\eta^2/[\eta^2+\alpha(\eta)] \to 1$ as $\eta \to 0$, where $\alpha(\eta) = o(\eta^2)$.

5. $\theta^{t-1} = 0$, $\theta^{t-2} = 1$. (Note that $\theta^{t-2} = 1$ implies $p^{t-2} \ge p^{**}$). Let i and j be the colluding bidders in period $t-2$ with the highest and second-highest valuation, respectively. Under the perturbed profile, the most likely explanation for out-of-equilibrium price p^{t-1} is that, in period $t-1$, bidder i bid v_i (a mistake), bidder j bid p^{t-2} because $v_j < p^{t-2}+\kappa$ (not a mistake), and bidder $t-1$ entered in period $t-1$ and bid valuation $v_{t-1} > p^{t-2}$ (entering is not a mistake, but bidding truthfully is). This event has conditional probability $\eta^2/]\eta^2+\alpha(\eta)] \to 1$ as $\eta \to 0$, where $\alpha(\eta) = o(\eta^2)$.

In all three cases, potential bidder t will be certain that there is exactly one participating bidder in B^{t-1} whose valuation exceeds p^{t-1}. The resulting limit belief about w^{t-1}, conditional on observing out-of-equilibrium price p^{t-1}, is therefore

$$\widetilde{H}(w|p^{t-1}) \equiv Pr[w^{t-1}<w|p^{t-1}] = G(w|p^{t-1}),$$

where G is defined in (3). As shown in Section 4.2, it is then optimal for bidder t to enter the auction as long as $p^{t-1} \le p^*$. □

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