SYMMETRIC AUCTIONS

RAHUL DEB[†] AND MALLESH M. PAI[‡]

ABSTRACT. Real-world auctions are often restricted to being symmetric (anonymous and nondiscriminatory) due to practical or legal constraints. We examine when this restriction prevents a seller from achieving his objectives. In an independent private value setting with heterogenous buyers, we characterize the set of incentive compatible and individually rational outcomes that can be implemented via a symmetric auction. Our characterization shows that symmetric auctions can yield a large variety of discriminatory outcomes such as revenue maximization and affirmative action. We also characterize the set of implementable outcomes when individual rationality holds in an ex-post rather than an interim sense. This additional requirement may prevent the seller from maximizing revenue.

[†]DEPARTMENT OF ECONOMICS, UNIVERSITY OF TORONTO
[‡]DEPARTMENT OF ECONOMICS, UNIVERSITY OF PENNSYLVANIA
E-mail addresses: rahul.deb@utoronto.ca, mallesh@econ.upenn.edu.
Date: May 8, 2013.
We would like to thank Dirk Bergemann, Amos Fiat, Johannes Hörner, René Kirkegaard, Kevin Lleyton-Brown, Steve Matthews, Maher Said, Xianwen Shi, Andrzej Skrzypacz, Jeroen Swinkels, Juuso Toikka, Gabor Virag and seminar audiences at Arizona State University, NTU Singapore, University of Western Ontario, Washington University at St. Louis for their insightful comments. We also thank Yunan Li for providing excellent research assistance. Mallesh Pai gratefully acknowledges the support of NSF Grant CCF-1101389.

An optimal auction extends the asymmetry of the buyer roles to the allocation rule itself. The assignment of the good and the appropriate buyer payment will depend not only on the list of offers, but also on the identities of the buyers who submit the bids. In short, an optimal auction under asymmetric conditions violates the principle of buyer anonymity.

J. Riley and W. Samuelson (1981). "Optimal Auctions," *American Economic Review.*

1. Introduction

The revelation principle is the foundation underlying theoretical mechanism design. It states that restricting attention to direct mechanisms is without loss of generality. However, in order to implement desired outcomes in practical applications, the focus is often on the design of indirect mechanisms which account for real-world constraints. One such important constraint is that auctions often have to be *symmetric*[1] (anonymous and nondiscriminatory)—payments and allocation rules cannot depend on the identities of the bidders either because they bid anonymously or due to practical and legal constraints. The requirement of symmetry may conflict with the objectives of the seller. An example is provided in the epigraph above. It is well known that when buyers are ex-ante heterogenous, the revenue optimal direct mechanism discriminates amongst bidders based on their identity (see also Myerson 1981). That said, the direct mechanism is just one possible implementation of the optimal auction.

In this paper, our aim is to characterize the set of allocation (and payment) rules for which there exists *a* symmetric auction implementation. We define a symmetric auction to be a sealed bid game in which buyers submit bids (real numbers), the highest bidder over the reservation bid wins and the transfers are determined via an anonymous function which maps bids to payments (and which can depend on the distributions of values). Importantly, in a symmetric auction, bidders remain anonymous both to the mechanism designer and to the other auction participants and hence outcomes depend only on the profile of bids and not on the identities of any of the auction participants. "Standard" sealed bid auction formats such as the first price, second price and all pay auctions are symmetric in this sense. We focus on the independent private value setting with ex-ante heterogenous buyers and our main characterization shows that, in a sense we make precise, a large variety of incentive compatible, individually rational outcomes are achievable using symmetric auctions. A strength of our analysis is that it requires very mild assumptions on the distributions of buyer valuations.

In particular, an important implication of our main result is that the revenue optimal outcome can always be achieved via a symmetric auction. This result counters what appears to be common intuition and received wisdom. Because its direct implementation is asymmetric, the optimal auction was believed to be nonanonymous in the earliest seminal work (see epigraph) and since then, there are numerous instances in the auction theory literature where similar beliefs are stated. Some argue that this observation justifies the removal of legal hurdles that prevent discrimination. In the context of international trade, McAfee & McMillan (1989) used the theory of optimal auctions to show that explicitly discriminating amongst suppliers can reduce the costs of procurement.

[1]The term 'symmetric auctions' has been used to describe such anonymous mechanisms as far back as (Maskin & Riley 1984) (if not earlier).

Their aim was to provide an argument against the 1981 Agreement on Government Procurement (in the General Agreement on Tariffs and Trade) which set out rules to ensure that domestic and international suppliers were treated equally.[2] Similarly, Cramton & Ayres (1996) suggest that in government license auctions, subsidizing minority owned or local businesses may actually result in more revenue to the government.[3] We show that in order to achieve such goals, explicit discrimination by the auctioneer is unnecessary.

That a symmetric auction can be used to achieve a broad class of different objectives has implications for government procurement auctions which often have distributional goals in addition to generating revenue. Governments often desire to favor certain bidders (small businesses, women, minorities, etc.) who are economically disadvantaged and hence may be unable to compete with stronger bidders unless the auction rules are skewed in their favor. However, such a preferential policy is often viewed as unfair. This policy was successfully challenged in the US Supreme Court case *Adarand Constructors, Inc. v. Pena* (1995), and states like California and Michigan have explicitly changed their laws (Proposition 209 and Proposal 2 respectively) to prohibit favored treatment on the basis of race, sex or ethnicity. In Europe, Article 87(1) of the European Commission Treaty prohibits "aid granted by a Member State or through State resources in any form whatsoever which distorts or threatens to distort competition by favoring certain undertakings..." Our results suggest that with careful auction design, it is potentially possible to achieve outcomes where particular classes of bidders are favored without having to resort to explicitly biasing the auction.

This implication can also be interpreted another way— symmetry of the auction does not imply fairness of the outcome. In a sense, this intuition is already well known, as ex-ante heterogenous buyers may have different equilibrium strategies even in a symmetric auction. For instance, Maskin & Riley (2000) show that stronger bidders often favor second price to first price auctions and that the latter format can yield higher revenues for the seller. The observation that a 'fair' and transparent auction can be constructed in a way to implement discriminatory outcomes is important in formulating policy which prevents favoritism.

In addition to symmetry, auction designers may want their indirect implementation to have other desirable properties. We consider one such desideratum: that the equilibrium of the auction be ex-post individually rational. An implication of ex-post individual rationality is that losers never have make payments and that winners do not pay more than their value for the object. This property is satisfied by first and second price auctions but not by all pay auctions, which is perhaps one of the reasons why the latter format is rarely used in practice. We provide necessary and sufficient conditions which an allocation rule must satisfy in order to be implementable with a symmetric auction that additionally satisfies this condition. These conditions impose nontrivial additional restrictions on the space of outcomes that can be implemented. In particular, the revenue optimal outcome may not have a symmetric ex-post individually rational implementation.

---

[2]Such an agreement is also currently present in the World Trade Oraganization which has replaced the GATT.

[3]Corns & Schotter (1999) test these arguments empirically by conducting a laboratory experiment.

There are a few distinct strands of literature that are related to this paper. This paper was in part motivated by the work focused on the revenue ranking of commonly utilized auction formats such as the first and second price auctions (Maskin & Riley 2000, Mares & Swinkels 2010, 2011, Kirkegaard 2012). One motivation for this research is the recognition that the direct implementation of optimal auction may be infeasible in practice and therefore it is important to examine the revenue properties of commonly utilized mechanisms. On the other hand, the recent papers of Athey et al. (2013) and Pai & Vohra (2012) study the optimal methods of bidder discrimination in settings where the auctioneer is free to explicitly treat bidders differently. Related also is the work which shows that the optimal auction can be implemented by mechanisms involving multiple rounds where bidders are allowed to resell the good to each other and their identities are not private (Zheng 2006, Caillaud & Robert 2005, Lebrun 2012).

A different way of framing our main result is that the requirement of symmetry alone is quite unrestrictive for an auctioneer. In this regard, our results are related to the recent work of Manelli & Vincent (2010) and Gershkov et al. (2013). These authors show that in the independent private values model, any incentive compatible and individually rational outcome that can be achieved in Bayes-Nash equilibrium can also be achieved (in expectation) in dominant strategies. Thus, the requirement of dominant strategy implementation is not restrictive in and of itself; differences in implementable outcomes between the two solution concepts arise only due to additional desiderata such as budget balance. Analogously, we show that the additional requirement of ex-post individual rationality imposes stricter restrictions on outcomes implementable by symmetric auctions. Related to this latter result is the literature on mechanism design with budget constraints (a recent work with a good overview of the literature is Pai & Vohra 2008), as both consider the design of auctions where the ex-post payments that buyers can make are bounded.

The paper is organized as follows. In Section 2, we describe the model and set up the notation. Section 3 presents an example which highlights our approach. The main results are presented in Section 4. We discuss ex-post individually rational implementation for an auction with two bidders in Section 5. Finally concluding remarks are provided in Section 6. The appendix contains proofs and some additional results.

2. The Model

We consider an independent private value auction setting. A set $N = \{1, 2, \ldots, n\}$ of risk neutral buyers or bidders (used interchangeably) compete for a single indivisible object.[4] Buyer $i \in N$ draws a value $v_i \in V_i \equiv [\underline{v}_i, \overline{v}_i]$ independently from a distribution F_i. We assume F_i is twice continuously differentiable with corresponding density f_i which is strictly positive throughout the support $[\underline{v}_i, \overline{v}_i]$. Note that both V_i and F_i can be different across i and hence we allow for ex-ante heterogenous bidders. We denote $\mathbf{V} \equiv \prod_{j \in N} V_j$ and $\mathbf{V}_{-i} \equiv \prod_{j \neq i} V_j$ with $\mathbf{v} \in \mathbf{V}$ and $\mathbf{v}_{-i} \in \mathbf{V}_{-i}$ denoting typical elements of these sets. As with values, we use notation $F \equiv \prod_{j \in N} F_j$ and $F_{-i} \equiv \prod_{j \neq i} F_j$. We will use similar notation for other vectors and vector-valued functions throughout the paper.

[4]Equivalently, our model could be considered to be one of procurement where a firm or government wants a single project to be completed and solicits quotes from contractors, each of whom has an independent private cost.

A *direct mechanism* asks bidders to report their values, and uses these reports to determine allocations and payments. Allocations are determined via an ordered list of functions

$$a^d = \left(a_1^d, \ldots, a_n^d\right) \quad \text{where} \quad a_i^d : \mathbf{V} \to [0,1] \text{ and } \sum_{i=1}^n a_i^d(\mathbf{v}) \leq 1. \tag{Direct Allocation}$$

Here, $a_i^d(\mathbf{v})$ is the probability that bidder i wins the auction when the profile of reported types is $\mathbf{v}$. The inequality above reflects the fact that the seller has a single unit to sell and hence the probability of allocating it cannot exceed one at any profile $\mathbf{v}$. Additionally, this allows for the possibility that the seller may choose to withhold the good. Similarly, payments are determined via an ordered list of functions

$$p^d = \left(p_1^d, \ldots, p_n^d\right) \quad \text{where} \quad p_i^d : \mathbf{V} \to \mathbb{R}. \tag{Direct Payment}$$

Here, $p_i^d(\mathbf{v})$ is the payment made by bidder i when the profile of reported types is $\mathbf{v}$. Note that, when it is positive, this is a transfer to the seller and, when it is negative, it is a subsidy from the seller. Also, the bidder may be required to make payments even when she doesn't receive the object.

Values are private, that is, buyers do not know the realized valuations of other bidders. Hence, each bidder's expected utility from participating in this mechanism is determined by their expected allocation and payment. For a given direct mechanism $\left(a^d, p^d\right)$, we define *interim* allocations and payments to be the expected allocations and payments conditioning on truthful reporting by all the bidders. Formally, these are given by

$$a_i^d(v_i) \equiv \int_{\mathbf{V}_{-i}} a_i^d(v_i, \mathbf{v}_{-i}) dF_{-i}(\mathbf{v}_{-i}), \tag{Interim Allocation}$$

$$p_i^d(v_i) \equiv \int_{\mathbf{V}_{-i}} p_i^d(v_i, \mathbf{v}_{-i}) dF_{-i}(\mathbf{v}_{-i}). \tag{Interim Payment}$$

For simplicity, we deliberately abuse notation by denoting interim allocations using the same symbol; the difference is determined by whether the argument is a single value or a value profile.

We make the additional standard assumption that the bidders are risk neutral and their utilities are quasilinear in the transfers. Conditional on truthful reporting by the other bidders, the interim expected utility for bidder i with value v_i who announces a value v_i' is simply

$$v_i\, a_i^d(v_i') - p_i^d(v_i'). \tag{Bidder Utility}$$

A mechanism $\left(a^d, p^d\right)$ is said to be (Bayesian) *incentive compatible* or simply IC if reporting truthfully is a Bayes-Nash equilibrium, i.e.

$$v_i\, a_i^d(v_i) - p_i^d(v_i) \geq v_i\, a_i^d(v_i') - p_i^d(v_i') \qquad \forall i \in N,\ \forall v_i, v_i' \in V_i. \tag{IC}$$

Myerson (1981) showed that incentive compatibility implies that the allocation rule a^d pins down the payments p^d up to constants $c_i \in \mathbb{R}$, that is,

$$p_i^d(v_i) = v_i a_i^d(v_i) - \int_{\underline{v}_i}^{v_i} a_i^d(w) dw + c_i. \tag{Payoff Equivalence}$$

Additionally, a mechanism is said to be *individually rational* or simply IR if truthful reporting leads to a nonnegative payoff or

$$v_i\, a_i^d(v_i) - p_i^d(v_i) \geq 0 \qquad \forall v_i \in V_i. \tag{IR}$$

2.1. *Symmetric Auctions*

We define a *symmetric auction* to be a game with three properties: (i) buyers simultaneously submit real numbers called bids; (ii) the winner is the highest bidder over a given reservation bid (ties are broken uniformly); and (iii) payments are determined via an anonymous payment function. This is an indirect sealed bid auction mechanism with the additional restriction that allocations and payments depend only on the profile of bids and not the identity of the bidders. Formally, in a symmetric auction, each bidder i chooses a bid $b_i \in \mathbb{R}$, allocations and payments are determined by functions $a^s : \mathbb{R}^n \to [0,1]$ and $p^s : \mathbb{R}^n \to \mathbb{R}$ respectively. Bidder i's allocation or simply her probability of winning the item is given by

$$a^s\left(b_i, \mathbf{b}_{-i}\right) = \begin{cases} \frac{1}{\#\{j \in N : b_j = b_i\}} & \text{when } b_i \geq \max\{\mathbf{b}_{-i}, r\}, \\ 0 & \text{otherwise.} \end{cases} \tag{Symmetric Auction Allocation}$$

where r is the reservation bid. As with values, we use $\mathbf{b}$ and $\mathbf{b}_{-i}$ to denote the vector of all bids and the vector of all bids except that of bidder i respectively.

Bidder i's payment is given by

$$p^s\left(b_i, \mathbf{b}_{-i}\right), \tag{Symmetric Auction Payment}$$

where p^s is invariant to permutations of $\mathbf{b}_{-i}$ but can depend on the underlying distribution of values $(F_1, \dots, F_n)$. Notice that since the allocation and payment rules do not depend on the identity of the bidders, we only need a single function, as opposed to lists of functions, to define these mechanisms. Most commonly used auction formats, such as the first price, second price and all pay auctions are symmetric in this sense.

In a symmetric auction, a pure strategy (henceforth referred to simply as a strategy) for a bidder i is a mapping

$$\sigma_i : V_i \to \mathbb{R}, \tag{Buyer Strategy}$$

which specifies the bid corresponding to each possible value. A profile of strategies $\sigma = (\sigma_1, \dots, \sigma_n)$ constitutes a (Bayesian Nash) equilibrium of the symmetric auction (a^s, p^s) if each buyer's strategy is a best response to the strategies of other buyers. Formally this requires that for all $i \in N$ and $v_i \in V_i$, we have

$$\sigma_i(v_i) \in \underset{b \in \mathbb{R}}{\operatorname{argmax}} \int_{\mathbf{V}_{-i}} \left[v_i\, a^s\left(b, \sigma_{-i}(\mathbf{v}_{-i})\right) - p^s\left(b, \sigma_{-i}(\mathbf{v}_{-i})\right)\right] dF_{-i}(\mathbf{v}_{-i}).$$

Symmetric auctions are useful in situations where the seller knows the underlying value distributions (perhaps from having conducted similar auctions in the past) but cannot condition the mechanism on bidder identity. As we argued in the introduction, one reason for this is that discrimination may be explicitly prohibited by the law. Alternatively, the seller could be conducting the auction in an environment where it is easy for bidders to conceal their identity (such as auctions conducted over the internet). An advantage of a symmetric auction format is that it maintains buyer privacy by ensuring that bids do not reveal identities. However, we require the buyers to know the underlying value distributions so that they can compute their equilibrium bid. Admittedly, this might be an unrealistic assumption in certain settings. That said, this requirement is imposed in almost all auction theory and, in particular, is necessary for buyers to calculate equilibrium bids even in standard first price auctions.

We say an IC and IR direct mechanism $\left(a^d, p^d\right)$ is *implemented by a symmetric auction* (a^s, p^s) if there is a pure strategy equilibrium in undominated strategies of the latter mechanism which yields the same allocation and expected payment as the former. Specifically, we say that a direct mechanism is implementable if there exists an undominated[5] equilibrium strategy profile σ such that for all $\mathbf{v} \in \mathbf{V}$

$$\begin{aligned} a_i^d(\mathbf{v}) &= a^s\left(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i})\right) \\ p_i^d(v_i) &= \int_{\mathbf{V}_{-i}} p^s\left(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i})\right) dF_{-i}(\mathbf{v}_{-i}). \end{aligned}$$

In this notion of implementability, we require the equilibrium allocation of the symmetric auction to be identical to the direct mechanism for each profile of values but the payments to be equal in expectation. This is a partial implementation criterion as we do not require the symmetric auction to have a unique equilibrium.[6] A weaker criterion would be to require the allocation along with the payment rule to be implemented in an expected sense for which we use the term *interim implementation*. The recent work on the equivalence of Bayesian and dominant strategy implementability (Manelli & Vincent 2010, Gershkov et al. 2013) uses this notion.

More generally, we say that an IC and IR direct mechanism $\left(a^d, p^d\right)$ is *implementable* by a symmetric auction (or simply implementable) if there exists a symmetric auction (a^s, p^s) which implements it. The main goal of this paper is to characterize the set of IC and IR direct mechanisms which are implementable by symmetric auctions.[7] For simplicity of exposition, we have deliberately defined implementation only in terms of pure strategies for the bidders. This restriction does not affect any of the results in the paper. We show in the appendix that allowing for mixed strategies does not expand the set of implementable or ex-post IR implementable (formally defined in Section 5) mechanisms .

It is important to mention at the outset that it is possible to implement all IC and IR outcomes using non-auction indirect mechanisms which are not anonymous but can be construed to be nondiscriminatory. Such mechanisms require bidders to reveal their identity either to the designer or to other bidders and so may not be able to overcome the legal hurdles that may be present in practical applications. Consider a simple example of such a symmetric mechanism in an environment where bidders are anonymous but the message space for the bidders is multidimensional–

[5]We use the additional restriction of undominated equilibrium strategies to ensure that our symmetric implementation is not based on 'implausible' buyer behavior.

[6]Multiple equilibria can result in standard auction formats like the second price auction and, with heterogenous buyers, in the first price auction.

[7]Since the additional requirement of IR only involves changing the payment rules by a constant, our characterization results can also be viewed as simply characterizing the set of IC direct mechanisms which are implementable.

they are asked to announce both their identity and their value for the good. Bidders can always be incentivized to reveal their identity truthfully as misreports can be detected in equilibrium (two bidders would have reported the same identity) and punished. Incentives to report values truthfully can be provided as usual. These mechanisms are nondiscriminatory since a permutation of messages would lead to the same permutation of outcomes. More generally, similar mechanisms can be used to get the agents to reveal any common knowledge they might have by asking them to simultaneously reveal their joint information and punishing everyone if they all fail to send the same information (see for instance Maskin 1999). For example, such mechanisms can be used for implementation in cases where the agents know the value distributions but the principle does not.

3. Example: Implementing the Optimal Auction with Two Buyers

In this section, we explain our approach by describing a symmetric implementation of the optimal auction when there are two buyers. For simplicity, we additionally assume that the distributions of both buyers satisfy the increasing virtual value property. Formally, this condition requires that for each buyer $i \in N$, the virtual value

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \qquad \text{(Virtual Value)}$$

is increasing in v_i. An implication is that ϕ_i^{-1} is well defined.

We denote the allocation and payment rule of the optimal auction by (a^*, p^*). Recall that in the optimal auction, bidders announce their values and the mechanism awards the good to the bidder who has the highest positive virtual value (ties can be broken equally). Hence, when bidders draw their values from different distributions, this direct mechanism is not symmetric as the allocation rule depends on the bidder-specific value distribution.

A natural way to attempt a symmetric implementation of the optimal auction is to construct a payment rule such that it is an equilibrium for both bidders to bid their virtual values. The auction could then allocate the good to the higher bid and have a reservation bid of 0. We denote the set of virtual values of bidder i by

$$B_i \equiv [\phi_i(\underline{v}_i), \phi_i(\overline{v}_i)].$$

The distribution F_i over V_i induces a distribution G_i over the set B_i of virtual values.

We claim that the optimal auction can be implemented if we can construct a payment rule p^s which satisfies

$$p_i^*(v_i) = \int_{B_j} p^s\left(\phi_i(v_i), b_j\right) dG_j(b_j) \qquad \text{for } i \neq j \text{ and all } v_i \in V_i.$$

This is simply a restatement of the implementability requirement where equilibrium strategies of bidding the virtual value have been substituted in. This claim is easy to see:

(1) If a bidder i with value v_i chooses to bid $b_i \in B_i$ but $b_i \neq \phi_i(v_i)$. This is equivalent to her reporting a value $\phi_i^{-1}(b_i) \neq v_i$ in the direct mechanism (a^*, p^*) which yields a lower payoff as the optimal auction is IC.

(2) If bidder i with value v_i bids $b_i \notin B_i$, this can be detected with positive probability by the auctioneer when the other bidder is bidding truthfully. This is because there will be a

positive measure of bids b_j such that $(b_i, b_j) \notin (B_1 \times B_2) \cup (B_2 \times B_1)$. The payment function can be chosen to be high enough at these off-equilibrium bids to discourage such behavior.

We now construct such a symmetric payment rule. Since it is easy to discourage bids that lie outside the support of the virtual values, the payment rule is deliberately defined only for equilibrium bid profiles $(b_i, b_j) \in (B_1 \times B_2) \cup (B_2 \times B_1)$. We separately construct the payment for bids that lie in the supports of only one and both virtual value distributions respectively. In equilibrium, bids $b_i \in B_i \backslash B_j$ are made only by buyer i. Hence, for such bids, we can simply define the payment rule to be the interim payment from the optimal auction or

$$p^s(b_i, b_j) = p_i^*(\varphi_i^{-1}(b_i)) \text{ when } b_i \in B_i \backslash B_j \text{ and } b_j \in B_j.$$

To construct the payments for bids $b_i \in B_1 \cap B_2$ that lie in the support of both virtual value distributions, we first observe that for asymmetric buyers ($F_1 \neq F_2$), there exists a $\hat{b} \in \mathbb{R}$ such that $G_1(\hat{b}) \neq G_2(\hat{b})$. In other words, different value distributions yield different virtual values distributions. Consider the payment rule

$$p^s(b_i, b_j) = \begin{cases} p^u(b_i) & \text{if } b_j \geq \hat{b} \text{ and } b_j \in B_1 \cup B_2, \\ p^l(b_i) & \text{if } b_j < \hat{b} \text{ and } b_j \in B_1 \cup B_2, \end{cases}$$

where

$$p^u(b_i) = \frac{p_1^*(\phi_1^{-1}(b_i))G_1(\hat{b}) - p_2^*(\phi_2^{-1}(b_i))G_2(\hat{b})}{G_1(\hat{b}) - G_2(\hat{b})},$$
$$p^l(b_i) = \frac{p_2^*(\phi_2^{-1}(b_i))[1 - G_2(\hat{b})] - p_1^*(\phi_1^{-1}(b_i))[1 - G_1(\hat{b})]}{G_1(\hat{b}) - G_2(\hat{b})}.$$

According to this payment rule, a bidder i who bids b_i pays an amount $p^u(b_i)$ when her opponent bids higher than $\hat{b}$ and an amount $p^l(b_i)$ when her opponent's bid is lower than $\hat{b}$. Hence, the expected payment of a bidder i who bids $b_i \in B_1 \cap B_2$ when bidder j bids $\phi_j(v_j)$ for all $v_j \in V_j$ is

$$p^u(b_i)[1 - G_j(\hat{b})] + p^l(b_i)G_j(\hat{b}) = p_i^*(\phi_i^{-1}(b_i)), \tag{1}$$

which is precisely the required payment for implementation.

Notice also that the above equation (1) can be used to derive the expressions for p^u and p^l. An equivalent matrix representation is the following system for $b_i \in B_1 \cap B_2$

$$\mathcal{M} \begin{bmatrix} p^u(b_i) \\ p^l(b_i) \end{bmatrix} = \begin{bmatrix} p_1^*\left(\phi_1^{-1}(b_i)\right) \\ p_2^*\left(\phi_2^{-1}(b_i)\right) \end{bmatrix} \quad \text{where} \quad \mathcal{M} = \begin{bmatrix} 1 - G_2(\hat{b}) & G_2(\hat{b}) \\ 1 - G_1(\hat{b}) & G_1(\hat{b}) \end{bmatrix}. \tag{2}$$

By definition, $G_1(\hat{b}) \neq G_2(\hat{b})$ implies that $\mathcal{M}$ is a full rank matrix. Therefore (2) has a solution for all $b_i \in B_1 \cap B_2$ and p^u, p^l can be obtained by inverting $\mathcal{M}$.

In summary, the symmetric payment rule which implements the optimal auction in this example is

$$p^s(b_i, b_j) = \begin{cases} p^u(b_i) & \text{if } b_i \in B_1 \cap B_2,\ b_j \geq \hat{b} \text{ and } b_j \in B_1 \cup B_2, \\ p^l(b_i) & \text{if } b_i \in B_1 \cap B_2,\ b_j < \hat{b} \text{ and } b_j \in B_1 \cup B_2, \\ p_1^*(\varphi_1^{-1}(b_i)) & \text{if } b_i \in B_1 \backslash B_2 \text{ and } b_j \in B_2, \\ p_2^*(\varphi_2^{-1}(b_i)) & \text{if } b_i \in B_2 \backslash B_1 \text{ and } b_j \in B_1. \end{cases}$$

The following numerical example illustrates this construction.

Example 1. Consider a setting with two buyers. Buyer 1 has a value that uniformly distributed over $[2, 4]$, while buyer 2's value is uniformly distributed over $[1, 2]$. The seller wants to conduct a symmetric implementation of the optimal auction. In this setting, the virtual value of buyer 1 is $\phi_1(v_1) = 2v_1 - 4$ and the virtual value of buyer 2 is $\phi_2(v_2) = 2v_2 - 2$. Therefore, buyer 1's virtual value (bid) is uniformly distributed over $B_1 \equiv [0, 4]$ while buyer 2's is uniformly distributed over $B_2 \equiv [0, 2]$.

We begin by deriving the interim payments. These can be determined using (Payoff Equivalence) as follows

$$\begin{aligned} p_1^*(v_1) &= v_1 a_1^*(v_1) - \int_2^{v_1} a_1^*(w)dw = v_1 \min\{v_1 - 2, 1\} - \int_2^{v_1} \min\{w - 2, 1\}dw \\ &= \begin{cases} \frac{v_1^2}{2} - 2 & \text{for } v_1 \in [2, 3] \\ \frac{5}{2} & \text{for } v_1 \in (3, 4] \end{cases} \end{aligned}$$

and

$$\begin{aligned} p_2^*(v_2) &= v_2 a_2^*(v_2) - \int_1^{v_2} a_2^h(w)dw \\ &= v_2 \left[\frac{v_2 - 1}{2}\right] - \int_1^{v_2} \left[\frac{w - 1}{2}\right] dw = \frac{v_2^2 - 1}{4} \qquad \text{for } v_2 \in [1, 2]. \end{aligned}$$

Interim payments expressed in terms of bids are then

$$\begin{aligned} p_1^*\left(\phi_1^{-1}(b_1)\right) &= \begin{cases} \frac{b_1^2}{8} + b_1 & \text{for } b_1 \in [0, 2], \\ \frac{5}{2} & \text{for } b_1 \in (2, 4], \end{cases} \\ p_2^*\left(\phi_1^{-1}(b_2)\right) &= \frac{b_2^2}{16} + \frac{b_2}{4} \qquad \text{for } b_2 \in [0, 2]. \end{aligned}$$

Consider now $\hat{b} = 1$ which implies $G_1(\hat{b}) = \frac{1}{4}$ and $G_2(\hat{b}) = \frac{1}{2}$. This choice of $\hat{b}$ yields

$$p^u(b_i) = -\frac{b_i}{2} \qquad \text{and} \qquad p^l(b_i) = \frac{5b_i}{2} + \frac{b_i^2}{4},$$

from which we can define the symmetric payment rule for equilibrium bids

$$p^s(b_i, b_j) = \begin{cases} -\frac{b_i}{2} & \text{if } b_i \in [0, 2] \text{ and } b_j \in [1, 4], \\ \frac{5b_i}{2} + \frac{b_i^2}{4} & \text{if } b_i \in [0, 2] \text{ and } b_j \in [0, 1), \\ \frac{5}{2} & \text{if } b_i \in (2, 4] \text{ and } b_j \in [0, 2]. \end{cases}$$

Our main result in the next section builds on the intuition in this example. The key difficulty in a symmetric implementation is that the same bid, when made by different bidders, must lead to the appropriate expected (interim) payments. In order for this to be the case, the payment rule needs to be designed in a way which utilizes the difference in the distribution of equilibrium bids of each bidder. In this example, we simply had to charge different amounts depending on whether the opponent's bid was above or below $\hat{b}$. The proof of the main result, generalizes this construction to n bidders.

4. Characterization of Mechanisms with Symmetric Implementations

In this section, we present and discuss the main result— a characterization of the IC and IR direct mechanisms that can be implemented by a symmetric auction. A constructive approach to determining whether a particular direct mechanism is implementable would require first the design of a symmetric auction and then a derivation of its equilibrium. However, deriving equilibria for a given symmetric auction can be a hard task. For instance, it is well known that deriving closed form solutions for equilibrium bids in the first price auction is difficult for arbitrary distributions. In order to simplify our task, we reformulate our problem in terms of hierarchical mechanisms.

We begin by defining *hierarchical allocation rules*.[8] These are generated by an ordered list $I = (I_1, \ldots, I_n)$ of nondecreasing *index functions* where $I_i : V_i \to \mathbb{R}$ for each bidder $i \in N$. A hierarchical allocation rule is generated from a given list of index functions I as follows

$$a_i^h(\mathbf{v}) = \begin{cases} \frac{1}{\#\{j \in N : I_j(v_j) = I_i(v_i)\}} & \text{when } I_i(v_i) \geq \max\{I_{-i}(\mathbf{v}_{-i}), 0\}, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{(Hierarchical Allocations)}$$

Each bidder's value is transformed into an index via the index function. The good is then allocated to the bidder with the highest positive index and ties are broken equally. Restricting allocations to buyers with positive indices is essentially equivalent to setting reservation bids. Choosing a reserve of 0 for the index functions is without loss of generality as they can always be moved up or down by a constant. Moreover, index functions can be chosen so that allocations occur above different reservation values across the buyers.

A *hierarchical mechanism* (I, p^h) is an IC and IR mechanism which consists of index functions I and payment functions p^h. The allocation a^h is determined as above from the index functions. For the results that follow, we find it convenient to denote a hierarchical mechanism in terms of the index functions I as opposed to the allocation rule a^h. If two lists of index functions I, I' generate the same allocation rule a^h, then it must be that one is a monotone transformation of the other. Formally, if I, I' generate the same allocation a^h, then there exists a strictly monotone function $\Gamma : \mathbb{R} \to \mathbb{R}$ such that $I_i(v_i) = \Gamma(I'_i(v_i))$ for all i and v_i. The particular choice of index functions that correspond to a given allocation a^h does not matter for the statement of our results.

Since the index functions are nondecreasing, having a higher value implies a weakly higher probability of winning. This implies that every hierarchical allocation rule a^h has associated IC transfers p^h (pinned down to constants) which yield a hierarchical mechanism. Most commonly utilized mechanisms fall within the class of hierarchical mechanisms. In the efficient Vickrey auction, values serve as indices or $I_i(v_i) = v_i$ and in the optimal auction (with increasing virtual values) the indices are given by the virtual values or $I_i(v_i) = \phi_i(v_i)$. When the virtual values are not increasing, the index functions are simply the 'ironed' virtual value functions (Myerson 1981). Alternatively, suppose an auctioneer with affirmative action concerns wants to 'subsidize' a historically disadvantaged bidder i over a bidder j where the latter has index $I_j(v_j) = v_j$. The index for bidder i could reflect either a flat subsidy $I_i(v_i) = v_i + s$ (where $s > 0$) or a percentage subsidy $I_i(v_i) = s\, v_i$ (where $s > 1$).

[8] This term was introduced by Border (1991).

As in analysis of the previous section, restricting attention to hierarchical mechanisms simplifies the implementation task. Since, the allocation rule of a symmetric auction that implements a hierarchical mechanism must allocate the good to the bidder with the highest index, a natural assumption is to make equilibrium bids correspond to the index values. Then constructing the symmetric implementation essentially boils down to finding a symmetric payment rule that yields the same interim payments. Given a hierarchical mechanism (I, p^h), the distribution F_i on the set of values V_i induces a distribution G_i on the set of indices or bids

$$B_i \equiv \{I_i(v_i) \mid v_i \in V_i\}. \tag{Bid Space}$$

At times, we will slightly abuse notation and use G_i both as a distribution and a measure. The meaning will be clear depending on whether the argument of G_i is a real or a set. The notation G_i deliberately suppresses the dependence on the index function I_i; the meaning will always be clear from the context. Since index functions I are not necessarily strictly increasing, the induced distributions G_i may have atoms. Additionally, notice that the set B_i need not be an interval as the index functions I may be discontinuous.

A hierarchical allocation mechanism (I, p^h) can be implemented if we can find a symmetric payment function p^s such that

$$p_i^h(v_i) = \int_{\mathbf{B}_{-i}} p^s\left(I_i(v_i), \mathbf{b}_{-i}\right) dG_{-i}(\mathbf{b}_{-i}) \qquad \text{for all } i \in N \text{ and } v_i \in V_i. \tag{$\star$}$$

If such a symmetric payment function exists, it follows that an equilibrium of the symmetric auction with this payment rule will involve each buyer i with value v_i bidding their index $I_i(v_i)$. By construction, such bids generate the required allocation.

The intuition is straightforward and is identical to that of the example. Suppose a bidder with value v_i makes a bid $b_i' \in B_i$ other than her index so $b_i' \neq I_i(v_i)$. Her corresponding allocation and payment would be identical to what she would get by reporting a value $v_i' \in I_i^{-1}(b_i')$, resulting in lower utility as the direct mechanism (I, p^h) is IC.[9] Off-equilibrium bids $b_i' \notin B_i$ which lie outside the bid space can be punished by requiring high expected payments at these bids.

We are now in a position to present our main result.

Theorem 1. *A hierarchical mechanism (I, p^h) has a symmetric implementation if and only if for every pair of distinct buyers $i, j \in N$, at least one of the two following conditions is satisfied:*

(i) The induced distributions G_i and G_j on the bids satisfy $G_i \neq G_j$, or

(ii) For all $v_i \in V_i$, $v_j \in V_j$ such that $I_i(v_i) = I_j(v_j)$, we have that $p_i^h(v_i) = p_j^h(v_j)$.

Part (i) of the theorem states that whenever bid distributions G_i differ across the buyers, it is possible to construct a payment rule so that ($\star$) is satisfied. When there are two bidders, a payment rule like the one in the previous section can be used to construct the symmetric auction implementation. The construction for more that two bidders is less straightforward and can be found in the appendix. Part (ii) states that when the two induced bid distributions are the same or $G_i = G_j$, then it must be that the interim payments are the same for any two values with the

[9]Here $I_i^{-1}(\cdot)$ is the correspondence defined by $I_i^{-1}(b_i) = \{v_i \in V_i \mid I_i(v_i) = b_i\}$.

same indices. This is because it is no longer possible to generate different equilibrium expected payments for distinct buyers who make the same bid.

We now present two examples of hierarchical mechanisms that cannot be implemented symmetrically, thus showing that the conditions of the above theorem are not vacuous. In the first example, the good is allocated randomly and in the second, the seller would like to subsidize one of the buyers.

Example 2. There are 2 buyers. Buyer 1 has a value uniformly distributed on $[0,1]$. Buyer 2 has a value uniformly distributed on $[0.5,1]$. The seller assigns the good at random (with equal probability) to each of the two buyers irrespective of their value. Buyer 1 is never asked to pay anything whereas buyer 2 is always asked to pay 0.25.

Notice that this mechanism is a hierarchical mechanism where each bidders' index function is a constant nonnegative function or $I_1(v_1) = I_2(v_2) \geq 0$ for all $v_1 \in [0,1]$ and $v_2 \in [.5,1]$. Here the bid space just consists of a single point and distributions G_1, G_2 are degenerate and therefore satisfy $G_1 = G_2$. Notice that this mechanism does not satisfy (i) in Theorem 1 and additionally since the payments differ, it does not satisfy (ii) either.

Example 3. Consider an environment where there are two buyers. Buyer 1 has a value v_1 which is uniformly distributed in $[0,1]$. Buyer 2 has a value v_2 which is uniformly distributed in $[1,2]$.

Suppose the seller would like to 'subsidize' the bid of buyer 1 by a dollar. Put differently, buyer 2 wins the good if and only if his value exceeds that of buyer 1 by 1. Therefore for any $v_1 \in [0,1]$, the interim allocation probabilities are given by

$$a_1^h(v_1) = a_2^h(1+v_1).$$

The IC and IR payments are chosen to be such that the lowest type of both buyers for whom there is no probability of winning neither make payments nor are paid. This is clearly a hierarchical mechanism with index functions $I_1(v_1) = I_2(v_1+1)$, where $I_1(\cdot)$ is strictly increasing on the interval $[0,1]$.

Observe that this implies that the distributions over the bid spaces are identical as G_1 and G_2 are both $\mathbb{U}[0,1]$. This violates condition (i) of Theorem 1. Moreover, IC pins down payments which satisfy

$$p_2^h(v_1+1) = p_1^h(v_1) + a_1^h(v_1).$$

For all values $v_1 \in (0,1]$ which have a strictly positive probability of winning, the above equation implies that

$$p_2^h(v_1+1) \neq p_1^h(v_1).$$

Since the interim payments differ for values that have the same index and the bid spaces have identical distributions, symmetric payments cannot be constructed to implement this mechanism.

However, note that this mechanism could have been implemented if buyer 2's value distribution was anything other than $\mathbb{U}[1,2]$ as this would imply that condition (i) of Theorem 1 would be satisfied.

The conditions in Theorem 1 were on the distributions of the bid space. The following Corollary qualitatively describes the types of hierarchical allocation rules that cannot be implemented.

Corollary 1. *Suppose a hierarchical mechanism (I, p^h) does not have a symmetric implementation. Then there must exist two distinct buyers j, j' such that their index functions can be written as*

$$\text{for } i = j, j' : \; I_i(v_i) = \Gamma(F_i(v_i)) \text{ for almost every } v_i \in V_i,$$

for some non-decreasing function $\Gamma(\cdot)$.

In words, the above corollary demonstrates that the only hierarchical mechanisms (with continuous index functions) that cannot be implemented are ones where the indices corresponding to each value depend solely on the 'statistical rank.' Thus only a very specific subclass of mechanisms are unimplementable; such mechanisms are clearly nongeneric as a small perturbation of the allocation rule would restore implementability.

It is worth repeating that this is one of the main insights of this paper and has two main implications. The first is that symmetry need not imply fairness– just because an auction treats the bids of different buyers similarly, this doesn't imply that the resulting outcomes are equal from an ex-ante perspective. The second is that careful auction design can allow the mechanism designer to achieve a wide variety of goals in environments where explicit favoritism is impractical or prohibited. For instance, the auction designer can choose formats which favor weaker bidders without explicitly biasing the mechanism. This can be helpful for governments striving to reach distributional goals (favoring small businesses, minorities etc.) without facing legal challenges over favoritism policies. Alternatively, this can be useful to encourage competition (and thereby enhance revenue) amongst asymmetric bidders in settings such as online auctions where the seller may have a good knowledge about value distributions (from previous auctions conducted) but where bids are placed anonymously. In fact, the following corollary points out that the revenue optimal auction can always be implemented. We feel that this is one of the most surprising results of the paper.[10]

Corollary 2. *The optimal auction can be implemented symmetrically.*

It is worth stressing that the above corollary requires no hazard rate assumptions on the value distributions. When the distributions satisfy the increasing virtual value property, it is easy to show that if the bidders are asymmetric, the distribution over virtual values must also be different. Here, condition (i) of Theorem 1 can be shown to be true. When the virtual values are not increasing then the proof of the Corollary shows that if the distributions over the 'ironed' virtual values are the same (i.e., condition (i) does not hold) then condition (ii) must be true.

We end this section with a discussion of the implementability of IC and IR mechanisms that are not hierarchical mechanisms; Theorem 1 does not apply to such mechanisms. A simple two buyer example of a mechanism that is not a hierarchical mechanism is one where irrespective of

[10]In an influential paper, Cantillon (2008) conjectured that bidder asymmetries hurt the auctioneer in any anonymous mechanism. In this same paper, she showed that asymmetries do not necessarily hurt the auctioneer in the optimal auction. Corollary 2 answers this conjecture in the negative by showing that the optimal auction can be implemented by an anonymous mechanism.

the values, buyer 1 gets the good 25% of the time and buyer 2 gets it 75%. Clearly this is not a hierarchical allocation since our definition of the latter requires the equal breaking of ties. Another example is a mechanism in which the seller randomly allocates the good 50% of the time and runs a second price auction the remaining 50%. A simple way in which Theorem 1 can be used to implement such non-hierarchical mechanisms is via randomization.

The principal can employ randomization by choosing amongst a set of mechanisms via a lottery. After choosing one such mechanism from the set, the principal can announce it to the buyer. For instance, the principal could toss a coin and choose between a first and second price auction. Having chosen, the buyer is informed of the auction format and the game proceeds. Such a randomization expands the set of outcomes that the principle can implement in an ex-ante sense. As we mentioned earlier, such an implementation concept is appropriate for a principal concerned about expected outcomes (see Manelli & Vincent 2010, Gershkov et al. 2013).

We now characterize the set of outcomes that are achievable via randomization. A mechanism (a^d, p^d) is defined to be a *randomization* over a set of mechanisms $\mathscr{M}$, if there is a measure ζ defined on $\mathscr{M}$ such that

$$a_i^d(v_i) = \int_{\mathscr{M}} a_i(v_i) d\zeta((a,p)) \qquad \text{and} \qquad p_i^d(v_i) = \int_{\mathscr{M}} p_i(v_i) d\zeta((a,p)).$$

The lemma below shows that all IC and IR direct mechanisms can be obtained as a randomization over hierarchical allocation rules. This lemma follows from results in Border (1991) and Mierendorff (2011).

Lemma 1. *Every IC and IR direct mechanism is a randomization over the set of hierarchical mechanisms.*

Clearly, the outcome from any mechanism that is a randomization over implementable hierarchical mechanisms can be achieved in an ex-ante sense. The auctioneer can just randomly choose (using measure ζ) from the symmetric auctions that correspond to the implementable hierarchical mechanisms. Notice, that strictly speaking this isn't interim implementation in the way we have defined it but for practical applications it serves the same purpose as randomization is done before the chosen symmetric auction is announced to the buyers. The next corollary summarizes this discussion and in it, we use the terminology *outcomes are achievable* to clarify the distinction from interim implementation.

Corollary 3. *The outcomes from an IC and IR direct mechanism are achievable if it is a randomization over implementable hierarchical mechanisms.*

Finally, we discuss the two examples and examine whether their outcomes can be achieved via randomization.

Example 2. (Continued) Recall that in this example, the seller assigns the good at random (with equal probability), buyer 1 is never asked to pay anything and buyer 2 is always asked to pay 0.25. The outcome from mechanism can be achieved by randomizing with equal probability over two implementable hierarchical mechanisms. In the first hierarchical mechanism, buyer 1 is awarded the good with probability 1 irrespective of value and is not asked to pay anything. In the second hierarchical mechanism, buyer 2 is awarded the good with probability 1 irrespective of value and is asked to pay 0.5.

Example 3. (Continued) Recall that in this example, buyer 2 wins the good if and only if her value exceeds that of buyer 1 by 1. The outcome of this mechanism cannot be achieved using randomization.

Consider the index function $I_1(v) = I_2(v+1) = v$. By observation, the allocation rule a^h corresponding to these index functions is the unique (almost everywhere) maximizer of

$$\int_{\mathbf{V}} \left(\sum_{j \in \{1,2\}} a_j^d(v_j, \mathbf{v}_{-j}) I_j(v_j) f_j(v_j) \right) d\mathbf{v},$$

amongst all IC direct allocations a^d.

Therefore, for any hierarchical allocation rule $\tilde{a}^h \neq a^h$ that differs from a^h at a positive measure subset of values, it must be that

$$\int_{\mathbf{V}} \left(\sum_{j \in \{1,2\}} \tilde{a}_j^h(v_j, \mathbf{v}_{-j}) I_j(v_j) f_j(v_j) \right) d\mathbf{v} < \int_{\mathbf{V}} \left(\sum_{j \in \{1,2\}} a_j^h(v_j, \mathbf{v}_{-j}) I_j(v_j) f_j(v_j) \right) d\mathbf{v}.$$

Moreover, any allocation rule that is equal to a^h almost everywhere is not implementable. Therefore, a^h is not a randomization over implementable hierarchical allocations and hence its outcome is not achievable.

5. Symmetric Ex-Post IR Implementation

Our main result from the previous section argued that a large class of mechanisms can be implemented symmetrically. However, symmetry is just one desideratum of a practical implementation. While the symmetric implementations we construct are by definition IR, they are IR in an interim sense. The equilibrium need not be IR in an ex-post sense however— certain bid profiles may result in losing bidders having to make payments or winners having to pay more than their valuation. This is unappealing and may result in certain bidders choosing not to participate. Perhaps more importantly, this may result in non-payment by budget constrained bidders. This is because a bidder's valuation may reflect their ability to pay for the good. Additionally, certain bidders who plan to pay by taking a loan may be unable to obtain credit upon losing the auction.[11] This might be one reason why all-pay auctions are rarely used in practice, whereas first and second price auctions (the equilibria of which are ex-post IR) are ubiquitous.

Formally, we say that a hierarchical mechanism (I, p^h) has a *symmetric ex-post IR implementation* if it has a symmetric implementation (a^s, p^s) with equilibrium strategies σ in which for each $\mathbf{v} \in \mathbf{V}$, the following holds for all $i \in N$

$$p^s\left(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i})\right) \leq v_i a^s\left(\sigma_i(v_i), \sigma_{-i}(\mathbf{v}_{-i})\right). \qquad \text{(Ex-post IR Implementation)}$$

In words, this states that at any bid profile which occurs in equilibrium, winning buyers are never charged more than their value and losers do not have to make payments (although they may receive subsidies). Notice that when there are ties, the above inequality implies that buyers only have to pay in the event that they win.

[11]If we were to take the procurement interpretation of our model, the ex-post IR requirement would ensure that firms can cover their costs and complete the project.

The principal objective of this section is to show that the additional ex-post IR requirement reduces the set of hierarchical mechanisms that can be implemented. To this end, the following simple example shows that the optimal auction may not always have a symmetric ex-post IR implementation even when the increasing virtual value condition holds.

Example 1. (Continued) Recall that buyer 1 has a value that uniformly distributed over $[2,4]$, while buyer 2's value is uniformly distributed over $[1,2]$ and the seller wants to maximize revenue. Virtual values are distributed $\mathbb{U}[0,4]$ and $\mathbb{U}[0,2]$ respectively.

Consider buyer 1 with value $v_1 = 3$. This buyer has a virtual value of $\phi_1(3) = 2$, always wins the good and makes $p_1^*(3) = \frac{5}{2}$. In order for there to be a symmetric ex-post IR implementation there must exist a symmetric payment p^s such that

$$\int_0^2 p^s(2, b_2) dG_2(b_2) = \frac{5}{2}$$

which in turn implies that there must exist at least one $b \in [0,2]$ such that

$$p^s(2,b) \geq \frac{5}{2}.$$

However, note that a buyer 2 with value $v_2 = 2$ also has virtual value $\phi_2(2) = 2$. Since there is a $b \in [0,2]$ such that $p^s(2,b) \geq \frac{5}{2}$, there will be a bid profile in the support of the equilibrium bids at which buyer 2 is paying more than her value. This violates the ex-post IR requirement.

We derive necessary and sufficient conditions for a hierarchical mechanism to admit a symmetric ex-post IR implementation. These conditions will be in terms of the index functions and the distributions on the bid space that they induce. Once again, we would like to point out that a given hierarchical allocation rule has a continuum of corresponding index functions and the characterization is unaffected by the particular choice of index rule. Due to the complexity of the characterization, we impose three simplifying restrictions which aid exposition. Firstly, we only consider the case of two bidders. Secondly, we restrict attention to hierarchical mechanisms (I, p^h) in which the index functions I are differentiable and strictly increasing—this ensures that the implied distribution over bids for any buyer has a density. Third, we further restrict attention to the case where the lower bounds of the supports of the bid space do not coincide, or $I_1(\underline{v}_1) \neq I_2(\underline{v}_2)$. The necessary and sufficient conditions for this case are simpler to state. In the appendix, we present the characterization for allocation rules in which $I_1(\underline{v}_1) = I_2(\underline{v}_2)$. The characterization for more than two bidders is beyond the scope of this paper.

Without loss of generality, we assume the Bidder 1's bid space has the lower support or

$$I_1(\underline{v}_1) = \underline{b}_1 < \underline{b}_2 = I_2(\underline{v}_2).$$

Additionally we define $I_i(\overline{v}_i) = \overline{b}_i$ for $i \in \{1,2\}$ and

$$v(b) \equiv \min\{I_1^{-1}(b), I_2^{-1}(b)\} \qquad \text{for } b \in B_1 \cap B_2,$$

as the lower of the values of the two buyers corresponding to a bid b which lies both bid spaces. Recall that since we have restricted attention to strictly increasing index functions, this inverse is well defined.

We can now state a simple first necessary condition that a hierarchical mechanism (I, p^h) must satisfy in order to have a symmetric ex-post IR implementation.

Condition C1: The distribution of values F_1 and F_2 induce distributions G_1 and G_2 such that

$$\forall b \in B_1 \cap B_2 : \quad v(b)G_2(b) \geq p_1^h\left(I_1^{-1}(b)\right). \tag{C1}$$

This is an intuitive necessary condition. $v(b)$ is the maximum amount that can be charged to a winning buyer who bids $b \in B_1 \cap B_2$ and whose opponent bids $b' \in B_1 \cap B_2$, $b' \leq b$. Since the auction is symmetric, such a profile of bids will not reveal the identity of the winning bidder and therefore the ex-post IR requirement restricts the payment to be lower than both possible values of the winning bidder. Hence, bidder 1's interim payment $p_1^h\left(I_1^{-1}(b)\right)$ cannot be higher than $v(b)G_2(b)$ for any bid $b \in B_1 \cap B_2$. Notice that the necessity of this condition does not hinge on the lower bounds of the supports of the bid spaces being different and C1 will continue to remain necessary when $\underline{b}_1 = \underline{b}_2$. We revisit Example 1 and show that it violates this condition.

Example 1. (Continued) Once again consider buyer 1 with value $v_1 = 3$ at which the interim payment is $p_1^*(3) = \frac{5}{2}$. We argue that at the bid $\phi_1(3) = 2$, Condition C1 is violated. This is because

$$v(2) = \min\{\phi_1^{-1}(2), \phi_2^{-1}(2)\} = \min\{3, 2\} = 2,$$

and hence,

$$v(2)G_2(2) = 2 < p_1^*\left(\phi_1^{-1}(2)\right) = \frac{5}{2}.$$

It remains to derive a similar condition for the interim payment of buyer 2 which accounts for the fact that the lower bounds of the supports of the bid distributions differ ($\underline{b}_1 < \underline{b}_2$). Suppose one buyer bids $b \in B_1 \cap B_2$ while the other bid is in $[\underline{b}_1, \underline{b}_2)$. Then it is clear that the buyer bidding b is buyer 2 and hence payments on this range of bids can be chosen to be up to her value $I_2^{-1}(b)$ which may be higher than $v(b)$. By contrast, when buyer 1 bids b, she can never be charged more than $v(b)$ even if her value $I_1^{-1}(b)$ is strictly greater. This argument yields an analogous necessary condition for buyer 2.

Condition C1': The distribution of values F_1 and F_2 induce distributions G_1 and G_2 such that

$$\forall b \in B_1 \cap B_2 : \quad v(b)\left(G_1(b) - G_1(\underline{b}_2)\right) + I_2^{-1}(b)G_1(\underline{b}_2) \geq p_2^h\left(I_2^{-1}(b)\right). \tag{C1'}$$

However, conditions C1 and C1′ together need not be sufficient. This is because ensuring the appropriate interim payment for buyer 1 places a bound on the amount that can be extracted from buyer 2 from bids that lie in the common support $B_1 \cap B_2$. Suppose that at a bid b, the interim payment $p_1^h\left(I_1^{-1}(b)\right)$ of buyer 1 is substantially lower than that of buyer 2 which is $p_2^h\left(I_2^{-1}(b)\right)$. This may prevent the seller from extracting the entire expected payment $v(b)[G_1(b) - G_1(\underline{b}_2)]$ from buyer 2 when buyer 1's bids lie in the range $[\underline{b}_2, b]$.

Hence, we need to derive the maximum payment $\eta(b) \leq v(b)[G_1(b) - G_1(\underline{b}_2)]$ that can be extracted symmetrically from buyer 2 when (i) she bids $b \in B_1 \cap B_2$, (ii) positive payments are only taken when b is the winning bid, i.e. the other buyer's bids are in the range $[\underline{b}_2, b]$ and (iii) buyer 1's expected payment from bid b is $p_1^h\left(I_1^{-1}(b)\right)$.

In words, we need to define payments for bids $b \in B_1 \cap B_2$ in a way that maximizes the amount extracted from buyer 2 while ensuring that buyer 1's expected payment remains $p_1^h\left(I_1^{-1}(b)\right)$. If this amount extracted is greater than the required payment $p_2^h\left(I_2^{-1}(b)\right)$ for buyer 2, subsidies can always be provided when buyer 1's bids lie in the range $[\underline{b}_1, \underline{b}_2)$ because, in equilibrium, such bids can only come from buyer 1.

We now need some additional notation. First we define the following function for $b \in B_2$ which depends on the ratios of the densities:

$$L(b) = \frac{g_1(b)}{g_2(b)},$$

that is, $L(\cdot)$ is the likelihood ratio of a buyer bidding b being buyer 1 versus buyer 2. Further, define

$$\underline{\ell} \equiv \min_{b \in B_2} \{ L(b) \}.$$

This is the lowest value of the likelihood ratio for bids in B_2. Since index functions are assumed to be differentiable and strictly increasing, densities g_1 and g_2 are well defined and continuous on B_1 and B_2 respectively. As a result, $\underline{\ell}$ is well defined and is positive when $\overline{b}_2 \leq \overline{b}_1$ and 0 when $\overline{b}_2 > \overline{b}_1$.

Additionally, we define the following sets.

$$\gamma(\ell) \equiv \left\{ b \in B_2 \,\middle|\, L(b) \leq \ell \right\}$$

is the set of bids less than b where the likelihood ratio is at most ℓ and

$$\bar{\bar{\gamma}}(\ell) \equiv \left\{ b \in B_2 \,\middle|\, L(b) = \ell \right\}$$

is similarly the set of bids less than b where the likelihood ratio is exactly ℓ. These sets will be useful to describe payment rules which derive $\eta(b)$. In order to obtain $\eta(b)$, we concentrate the maximum payment $v(b)$ on bids that are more likely to lie in the bid space of buyer 1 relative to that of buyer 2 and buyer 1's interim payment is then guaranteed by providing a subsidy at bids that are least likely.

When Condition C1 holds, that is when $v(b)G_2(b) \geq p_1^h\left(I_1^{-1}(b)\right)$, the following two cases are mutually exclusive and exhaustive for any $b \in B_1 \cap B_2$.[12]

$$G_2(\bar{\bar{\gamma}}(\underline{\ell})) > 0 \qquad \text{OR} \qquad v(b)G_2(b) = p_1^h\left(I_1^{-1}(b)\right). \tag{Case 1}$$

$$G_2(\bar{\bar{\gamma}}(\underline{\ell})) = 0 \qquad \text{AND} \qquad v(b)G_2(b) > p_1^h\left(I_1^{-1}(b)\right). \tag{Case 2}$$

$\eta(b)$ needs to be derived separately for each of these two cases and hence, we analyze them separately below.

Case 1. Let $\hat{B}$ be a subset of $\bar{\bar{\gamma}}(\underline{\ell})$ such that

$$v(b)G_2\left([\underline{b}_2, b] \backslash \hat{B}\right) \geq p_1^h\left(I_1^{-1}(b)\right).$$

[12] Recall that we use G_i to represent both a measure and a CDF.

If $v(b)G_2(b) = p_1^h\left(I_1^{-1}(b)\right)$, then $\hat{B}$ must be a G_2-null set, else consider any set $\hat{B}$ which satisfies the above inequality and has strictly positive measure.

We now define a payment rule

$$\hat{p}(b,b') = \begin{cases} v(b) & \text{for } b' \in [\underline{b}_2, b]\backslash\hat{B}, \\ s & \text{for } b' \in \hat{B}, \\ 0 & \text{for } b' \in B_2 \text{ and } b' \notin \left([\underline{b}_2, b] \cup \hat{B}\right). \end{cases} \tag{C2,P1}$$

where s is chosen to solve

$$v(b)G_2\left([\underline{b}_2, b]\backslash\hat{B}\right) + sG_2\left(\hat{B}\right) = p_1^h\left(I_1^{-1}(b)\right).$$

Notice that s here is a subsidy. We set

$$\eta(b) = \int_{\underline{b}_2}^{\overline{b}_2} \hat{p}(b,b')dG_1(b'). \tag{3}$$

Observe that $\eta(b)$ does not depend on the choice of $\hat{B}$. Also observe that when $\overline{b}_2 > \overline{b}_1$, then $\hat{B} \subset (\overline{b}_1, \overline{b}_2]$ and $\eta(b) = v(b)$.

Case 2. Since $G_2(\bar{\bar{\gamma}}(\underline{\ell})) = 0$, it must be that $\overline{b}_2 \leq \overline{b}_1$. Here, we define the payment rule $\hat{p}_\ell$ for $\ell > \underline{\ell}$ as follows:

$$\hat{p}_\ell(b,b') = \begin{cases} v(b) & \text{for } b' \in [\underline{b}_2, b]\backslash\gamma(\ell), \\ s & \text{for } b' \in \gamma(\ell), \\ 0 & \text{for } b' \in B_2 \text{ and } b' \notin \left([\underline{b}_2, b] \cup \hat{B}\right). \end{cases} \tag{C2,P2}$$

where s is chosen to solve

$$v(b)G_2\left([\underline{b}_2, b]\backslash\gamma(\ell)\right) + sG_2(\gamma(\ell)) = p_1^h\left(I_1^{-1}(b)\right).$$

Notice, that for ℓ close to $\underline{\ell}$, s is negative and therefore the payment rule $\hat{p}_\ell$ is ex-post IR. Define:

$$\eta_\ell(b) = \int_{\underline{b}_2}^{\overline{b}_2} \hat{p}_\ell(b,b')dG_1(b'), \tag{4}$$

and let

$$\eta(b) = \lim_{\ell \downarrow \underline{\ell}} \left[\eta_\ell(b)\right].$$

We can now define the second condition.

Definition 5.1 (Condition C2)**.** The distribution of values F_1 and F_2 induce distributions G_1 and G_2 such that

$$\forall b \in B_1 \cap B_2 : \quad \eta(b) + I_2^{-1}(b)G_1(\underline{b}_2) \geq p_2^h\left(I_2^{-1}(b)\right), \tag{C2}$$

with the inequality holding strictly for any b such that

$$G_2(\bar{\bar{\gamma}}(\underline{\ell})) = 0 \text{ and } v(b)G_2(b) > p_1^h\left(I_1^{-1}(b)\right).$$

The following proposition states that the two conditions C1 and C2 are necessary and sufficient for a symmetric ex-post IR implementation.

Proposition 1. *Suppose there are 2 buyers. Consider a hierarchical allocation mechanism* (I, p^h) *with differentiable and strictly increasing index functions such that the lower bounds of the supports of the bid distributions differ, that is,* $\underline{b}_1 < \underline{b}_2$*. Then Conditions C1 and C2 are necessary and sufficient for there to exist a symmetric, ex-post IR implementation of* (I, p^h)*.*

We end this section by observing that Proposition 1 can be adapted to accommodate entry fees. In many practical situations, auctions are often conducted in two steps– buyers first pay to participate following which the auction is conducted. Such entry fees can relax ex-post IR constraints of the auction itself as buyers are making a part of the payment before participating. In particular, if the seller could charge a high enough entry fee, he would not need the buyers to make payments in the auction and could offer rebates instead. Having sunk the entry cost, ex-post IR would then be obtained automatically. Conditions C1 and C2 can be appropriately weakened to accommodate a given entry fee; the construction in this section can simply be altered so that winning bidder never pays more than her value plus the fee and the loser never has to pay more than the fee.

6. Concluding Remarks

While auction theory is a mature field with an enormous body of work, simple auction formats continue to dominate in practical applications even when they are provably suboptimal for the auctioneer's goals. Perhaps this is because the majority of mechanism design focuses on design of direct mechanisms which are often not suitable for real world applications. This paper considers the requirement of symmetry which is often necessary for legal and practical reasons. We have shown that symmetry, in and of itself, does not prevent the auctioneer from achieving a wide variety of goals. In particular, the optimal auction can be implemented as can auctions in which certain bidders are subsidized over others. We have also shown requiring ex-post individual rationality in addition to symmetry, imposes stronger restrictions on the set of implementable outcomes. In particular, the optimal auction may not have a symmetric, ex-post individually rational implementation.

Given the ubiquitous use of standard auction formats in the real-world, an interesting avenue for future research is to isolate the properties of these auctions that make them suitable for practical applications and to try and design indirect mechanisms with these properties that achieve the seller's goals. Apart from symmetry, two other desiderata that come to mind are "simplicity" and distribution independence of the mechanism. In order for the behavior of the buyers to be predictable, the mechanism employed should have simple, transparent rules and buyers should be able to easily compute their equilibrium strategies. While the auctions we construct have comparatively complex payment rules, the equilibrium bids can easily be derived by buyers. By contrast, first price auctions have simple rules but equilibrium bids may be hard to compute. Needless to say, one of the challenges in designing simple mechanisms is the definition of simplicity itself.

Finally, the symmetric auctions we construct depend critically on the fact that the seller knows the underlying distribution of values. Moreover, equilibrium bidding requires the buyers to possess this knowledge as well. This is a widespread assumption in mechanism design. In fact, knowledge about value distributions is necessary even for revenue ranking first and second price auctions and the equilibrium bids in the former require bidders to know these distributions. However, such an assumption may be unsuitable for some practical applications. There is a relatively recent literature in computer science on “prior-free” mechanism design in which underlying distributions are unknown (see for example Hartline (2012) and the references therein). Here the aim is to design approximately optimal mechanisms. In reality, a seasoned auctioneer may not have exact knowledge about distributions but may know certain summary statistics. An important topic for future research is the design of auctions only using properties of the distribution which can be estimated from previous auction data.

APPENDIX A. PROOFS FROM SECTION 4

A.1. *Proof of Theorem 1*

Sufficiency. At a high level, we generalize the ideas in our construction of the two bidder example in Section 3. Recall that our goal is to construct a symmetric auction game which implements a hierarchical mechanism (I, p^h) with corresponding allocation rule a^h.

We construct a symmetric auction game which has a pure strategy Bayes-Nash equilibrium in which buyer i with value v_i reports $I_i(v_i)$. By construction therefore, the allocation of this mechanism equals a^h. We are left to show that:

i. As constructed, this auction game implements the desired payments p^h.
ii. For each buyer i, bidding according to $I_i(\cdot)$ constitutes a Bayes-Nash equilibrium of the construction auction.

Step 1: Preliminaries. Our goal is to show that we can construct a symmetric $p^s : \mathbb{R}^n \to \mathbb{R}$, such that

$$\forall i, \forall v_i \in V_i : \quad p_i^h(v_i) = \int_{\mathbf{B}_{-i}} p^s\left(I_i(v_i), \mathbf{b}_{-i}\right)\ \mathrm{d}G_{-i}(\mathbf{b}_{-i}). \tag{5}$$

In other words, we need to show that we can construct a p^s such that each buyer i's expected payment, expectation over candidate equilibrium bids of other buyers, equals $p_i^h(\cdot)$.

Step 2: Full Rank Events. We say that an event $E \subseteq \mathbb{R}^{n-1}$ is *symmetric* if

$$\text{for every permutation } \rho : \{1, 2, \ldots, n-1\} \to \{1, 2, \ldots, n-1\},$$
$$(b_1, b_2, \ldots b_{n-1}) \in E \implies (b_{\rho(1)}, b_{\rho(2)}, \ldots b_{\rho(n-1)}) \in E.$$

We start with a simple observation.

Observation 1. *Consider $k \leq n$ symmetric events $E_1, E_2 \ldots E_k \subseteq \mathbb{R}^{n-1}$ and define the $k \times k$ matrix*

$$\mathcal{M} \equiv [G_{-i}(E_j)]_{i,j=1}^k.$$

If matrix $\mathcal{M}$ is full rank, then, there exists a symmetric payment rule p^s such that

$$\forall i = 1, \ldots k, \forall v_i \in V_i : \quad p_i^h(v_i) = \int_{\mathbf{B}_{-i}} p^s(I_i(v_i), \mathbf{b}_{-i})\, dG_{-i}(\mathbf{b}_{-i}). \tag{6}$$

In particular, if $k = n$, then there exists a payment rule p^s that satisfies (5).

Proof. Define the payment rule this way: there are k numbers associated with each bid $b \in \mathbb{R}$, denote the j^{th} number $\pi_j(b)$. Suppose a bid b is made by a buyer, and other buyers make the profile of bids $\mathbf{b}_-$. For each event E_j that occurs among other buyers' bids, i.e. $\mathbf{b}_- \in E_j$, the buyer is asked to pay $\pi_j(b)$. Formally, the payment function is defined as

$$p^s(b, \mathbf{b}_-) = \sum_{j=1}^k \pi_j(b) \chi_{\{\mathbf{b}_- \in E_j\}}, \tag{7}$$

where χ is the characteristic function. Note that since each of the E_j's are symmetric (by assumption) the payment rule defined thus is symmetric as well.

Given this definition of p^s, the expected payment made by buyer i bidding $b_i \in B_i$ when all other buyers are bidding according to their candidate equilibrium strategies is

$$\begin{aligned}
&\int_{\mathbf{B}_{-i}} p^s(b_i, \mathbf{b}_{-i})\, \mathrm{d}G_{-i}(\mathbf{b}_{-i}),\\
&= \int_{\mathbf{B}_{-i}} \left(\sum_{j=1}^{k} \pi_j(b_i) \chi_{\{\mathbf{b}_{-i} \in E_j\}} \right) \mathrm{d}G_{-i}(\mathbf{b}_{-i}),\\
&= \sum_{j=1}^{k} \pi_j(b_i) G_{-i}(E_j).
\end{aligned}$$

By the full rank assumption, for any $b \in \mathbb{R}$, there exists a solution $\pi(b) \in \mathbb{R}^k$ to the system of equations:

$$\mathcal{M}\pi(b) = \tilde{p}(b), \tag{8}$$

$$\text{where } \tilde{p}(b) = [\tilde{p}_1(b), \ldots, \tilde{p}_k(b)]^T,$$

$$\tilde{p}_i(b) = \begin{cases} p_i^h(I_i^{-1}(b)) & \text{if } b \in B_i, \\ \max_{i \in N}\{\overline{v}_i\} & \text{otherwise.} \end{cases} \tag{9}$$

Therefore, the payment rule defined using $\pi(\cdot)$ that satisfies this system of equations satisfies (6). When $k = n$, the constructed system satisfies (5). $\square$

Theorem 2 shows that there always exist such events.

Theorem 2. *For any $n > 1$ and any $k \leq n$ such that $G_1, G_2, \ldots G_k$ are all pairwise distinct, there exist symmetric events $E_1, \ldots, E_k \subseteq \mathbb{R}^{n-1}$ such that the $(k \times k)$ matrix $\mathcal{M} = [G_{-i}(E_j)]_{i,j=1}^k$ has full rank.*

A proof of the theorem is deferred to Appendix C.

Step 3: Matching Payments. First consider Case 1, i.e. $G_i \neq G_{i'}$ for all $i \neq i'$. Then by Theorem 2 there exist symmetric events $E_1, E_2 \ldots E_n \subseteq \mathbb{R}^{n-1}$ such that the $n \times n$ matrix $\mathcal{M} = [G_{-i}(E_j)]$ is full rank. Therefore, by Observation 1, we can construct a symmetric payment rule p^s that matches the desired interim payment rule p^h when all buyers make their candidate equilibrium bids, i.e., satisfies (5).

Now to consider the other case, i.e. there exist i, i' such that $G_i = G_{i'}$. Note that if $G_i = G_{i'}$ for some $i \neq i'$, then $G_{-i} = G_{-i'}$.

We define N_U as the set of "distributionally unique buyers." Formally for any induced distribution over bids, G define $N_G = \{i \in N : G_i = G\}$. Now we can define $N_U = \cup_{i \in N}\{\min\{N_{G_i}\}\}$. In other words, N_U is the largest subset of N s.t. for any distinct $i, i' \in N_U$, $G_i \neq G_{i'}$. Renumber the buyers so that the first $|N_U|$ buyers are distributionally unique. By Theorem 2, we can construct full row rank events for these buyers. We are then done, because by assumption, if $G_i = G_{i'}$ we have that $p_i^h(I_i^{-1}(b)) = p_{i'}^h(I_{i'}^{-1}(b))$.

Step 4: Equilibrium. We have already shown that if each buyer followed the candidate equilibrium strategy, the desired payment rule p^h is implemented. We are left to show that following the candidate strategy (i.e. that buyer i with value v_i bids $I_i(v_i)$) is a Bayes-Nash equilibrium of the game.

Consider buyer i, with value v_i. His candidate equilibrium bid is $b_i = I_i(v_i)$. Let us divide possible deviations into two types:

(1) Buyer i bids $b_i' \in B_i$.
(2) Buyer i bids $b_i' \notin B_i$.

Since the original mechanism (a^h, p^h) is Bayes Incentive Compatible, it should be clear that deviations of type 1 cannot be profitable. If player i with value v_i deviates to some other $b_i' = I_i(v_i')$. Assuming all other players are playing their equilibrium strategies player i will win the good with probability $a_i^h(v_i')$ and make an expected payment of $p_i^h(v_i')$. Incentive compatibility of the original direct revelation mechanism guarantees that:

$$v_i a_i^h(v_i) - p_i^h(v_i) \geq v_i a_i^h(v_i') - p_i^h(v_i').$$

By construction (8,9), deviations of type 2 will require the buyer to make an expected payment of $\max_{i \in N}\{\overline{v}_i\}$ and hence, such deviations cannot be profitable.

Therefore our candidate equilibrium strategies constitute a Bayes-Nash equilibrium of the symmetric auction game we constructed, concluding our proof of sufficiency.

Necessity. We now show that our condition is necessary for there to exist a symmetric implementation. Let us consider a hierarchical allocation rule with index functions $I_1, \ldots, I_n$ such that for buyers 1 and 2, $G_1 = G_2$.

Firstly, note that any other index function I' that implements the same allocation rule must be a strictly monotone transform of I. Therefore the resulting distributions will be such that $G_1' = G_2'$. It is therefore without loss to only check whether there exists an implementation corresponding to the 'original' index rule I.

Pick v_1, v_2 such that $I_1(v_1) = I_2(v_2)$, and $p_1^h(v_1) \neq p_2^h(v_2)$. Note that $a_1^h(v_1) = a_2^h(v_2)$ since $G_1 = G_2 \implies G_{-1} = G_{-2}$.

Recall that a symmetric implementation in pure strategies is a symmetric payment rule p^s, such that for all buyers i and all valuations v_i in V_i,

$$p_i^h(v_i) = \int_{\mathbf{B}_{-i}} p^s(I_i(v_i), \mathbf{b}_{-i}) dG_{-i}(\mathbf{b}_{-i}).$$

Since $G_1 = G_2$, the product distributions G_{-1} and G_{-2} are also the same. Therefore for any b,

$$\int_{\mathbf{B}_{-1}} p^s(b, \mathbf{b}_{-1}) dG_{-1}(\mathbf{b}_{-1}) = \int_{\mathbf{B}_{-2}} p^s(b, \mathbf{b}_{-2}) dG_{-2}(\mathbf{b}_{-2}).$$

For $b = I_1^{-1}(v_1) \ \left(= I_2^{-1}(v_2)\right)$, we have the required contradiction. $\square$

Mixed Strategies. We now argue that allowing for mixed strategies does not expand the set of implementable mechanisms. Fix a symmetric auction game. A mixed strategy equilibrium in this setting is a mapping for each buyer i

$$\sigma_i : V_i \to \Delta\mathbb{R},$$

i.e. buyer i with value v_i randomizes over bids with probability measure $\sigma_i(v_i)$.

A little notation is useful. Given that buyer i's values are distributed according to F_i, and that when he has value v_i he randomizes over bids with measure $\sigma_i(v_i)$, let G_i^r denote the implied distribution over bids.

Let us denote by $\breve{a}_i(b_i)$ the interim winning probability of buyer i when he bids b_i, with associated interim payment $\breve{p}_i(b_i)$. Note that $\breve{a}_i(b_i)$ is non-decreasing in b_i for all buyers i.

The following observation shows that the bids over which different values of a given buyer randomize are disjoint and ordered.

Observation 2. *For any buyer i and values $v_i < v_i'$, the support of the distributions of bids by these two values is effectively disjoint, i.e.*

$$\max\{\breve{a}_i(b_i) : b_i \in supp(\sigma_i(v_i))\} \leq \min\{\breve{a}_i(b_i) : b_i \in supp(\sigma_i(v_i'))\}.$$

Proof. Firstly, note that if buyer i with value v_i mixes over bids $b_i < b_i'$ with $\breve{a}_i(b_i) < \breve{a}_i(b_i')$, then he must be indifferent between these bids. Therefore $\breve{p}_i(b_i') - \breve{p}_i(b_i) = v_i(\breve{a}_i(b_i') - \breve{a}_i(b_i))$, implying that v_i' cannot be indifferent between both these bids.

So now suppose buyer i with value $v_i' > v_i$ has an equilibrium bid b_i'' with $\breve{a}_i(b_i'') < \breve{a}_i(b_i')$. Combining the equilibrium constraints that v_i prefers to bid b_i' than b_i'' and that v_i' prefers to bid b_i'' than b_i', we have a contradiction. The observation follows. □

Observation 3. *For any buyer i, the set of values $v_i \in V_i$ such that*

$$\exists b_i, b_i' \in supp(\sigma_i(v_i)) : \breve{a}_i(b_i) \neq \breve{a}_i(b_i')$$

has measure 0*.*

Proof. By Observation 2 we have that the support of distribution of bids for a given buyer is effectively disjoint. Therefore, at most a countable number of values for any buyer can have two different interim probabilities of winning in their support, since the range of $\breve{a}_i(\cdot)$ is $[0,1]$— the reals can have an at most countable set of positive length intervals. Since F_i is differentiable, the measure of a countable set of values is 0. □

Therefore, for any buyer i, and almost any possible value v_i, any randomization by this buyer must be such that for any two distinct bids $b_i < b_i'$, such that $b_i, b_i' \in \text{supp}(\sigma_i(v_i))$, $\breve{a}_i(b_i) = \breve{a}_i(b_i')$. It follows that $\breve{a}_i(\cdot)$ is constant on $[b_i, b_i']$. This implies that there cannot be a positive measure of values of other buyers $-i$ that submit bids in the interval $[b_i, b_i']$.

It follows therefore that $G_1 = G_2$ implies $G_1^r = G_2^r$. To see this, note that there cannot be any interval of bids in which only one of the two buyers puts positive mass. As a result, $G_{-1}^r = G_{-2}^r$.

Suppose a hierarchical mechanism (I, p^h) cannot be implemented in pure strategies. Then without loss of generality, $G_1 = G_2$ and there are values v_1 and v_2 such that condition (ii) of Theorem 1 is violated. For interim implementation in mixed strategies we must have that

$$a_i^h(v_i) = \int_{B_i} \breve{a}_i(b_i) d\sigma_i(v_i)(b_i),$$

$$p_i^h(v_i) = \int_{B_i} \breve{p}_i(b_i) d\sigma_i(v_i)(b_i).$$

Now suppose (without loss of generality) that there is a mixed strategy symmetric implementation of the case where $G_1 = G_2$, $p_1^h(v_1) > p_2^h(v_2)$ and $I_1(v_1) = I_2(v_2)$. Then, buyer 1 with value v_1, strictly prefers the strategy $\sigma_2(v_2)$ over $\sigma_1(v_1)$ (since $a_1^h(v_1) = a_2^h(v_2)$ by assumption), contradicting the assumption that these strategies constitute an equilibrium.

A.2. *Proof of Corollary 1*

Without loss of generality, consider only buyers 1 and 2. Since the auction does not have a symmetric implementation, it must be the case that $G_1 = G_2$. First consider the case that index functions I_1 and I_2 are continuous.

Suppose v_1, v_2 are such that $F_1(v_1) = F_2(v_2)$ but $I_1(v_1) > I_2(v_2)$— if no such v_1, v_2 exists we are done. Define

$$v_1' = \max\{v \in V_1 : I_1(v) = I_2(v_2)\}.$$

By continuity of I_1, v_1' exists. By monotonicity of I_1, $v_1' < v_1$. By assumption, $G_1(I_1(v_1')) = G_2(I_2(v_2))$. Combining, we have that

$$F_1(v_1) > F_1(v_1') = G_1(I_1(v_1')) = G_2(I_2(v_2)) \geq F_2(v_2),$$

implying that $F_1(v_1) > F_2(v_2)$. This contradicts our assumption that $F_1(v_1) = F_2(v_2)$.

Now suppose I_1 and I_2 are not necessarily continuous. The common support must lie on an at most countable collection of intervals and at most countable atoms. For any point in the interior of any interval in the support of G_1, and any atom, the above argument shows that

$$\text{for } i = 1, 2: \;\; I_i(v_i) = \Gamma(F_i(v_i)),$$

for any v such that $I_1(v_1)$ is in the interior of an interval in the support of G_1 or an atom on G_1. This leaves only measure 0 end points of the intervals, of which there are an at most countable set. These correspond to discontinuities in the index rules, which are also at most countable. □

A.3. *Proof of Corollary 2*

Recall from Myerson (1981) that if the function $\varphi_i(v_i)$, defined as

$$\varphi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)},$$

is non-decreasing in v_i, then the allocation rule for the optimal auction is defined by the hierarchical allocation rule with the index rule φ_i for buyer i. If φ_i is not non-decreasing, then the optimal allocation rule is given by the "ironed" virtual value $\overline{\varphi}_i$. Let G_i be the distribution over bids of buyer i induced by $\overline{\varphi}_i$.

The following simple lemma shows if two buyers (without loss of generality 1 and 2) have the same distribution of (possibly ironed) virtual values, then the two buyers also have the same virtual value function. Therefore, the hierarchical allocation rule implementing the optimal auction satisfies either condition (i) of Theorem 1, or if not, then this Lemma shows it satisfies condition (ii). The Corollary follows.

Lemma 2. *Suppose two buyers are such that $G_1 = G_2$. Then $V_1 = V_2$ and $\overline{\varphi}_1 = \overline{\varphi}_2$.*

Proof. Define $v_i(b) = \overline{\varphi}_i^{-1}(b)$ for $b \in B_1$. Since $\overline{\varphi}_i(\cdot)$ need not be strictly increasing, it follows that $\overline{\varphi}_i^{-1}(\cdot)$ is a correspondence. Define $\underline{v}_i(b) = \inf \overline{\varphi}_i^{-1}(b)$ and $\overline{v}_i(b) = \sup \overline{\varphi}_i^{-1}(b)$.
Since $\overline{\varphi}_i$ non-decreasing, it follows that

$$G_i(b) = F_i(\overline{v}_i(b)).$$

There can be at most a countable number of pooling intervals in $\overline{\varphi}_i$ (see Myerson 1981, Section 6). Each of these pooling intervals correspond to an atom in G_i. We denote the set of atomic bids by $\mathcal{B}_i \subseteq B_i$, denote by β_{in} the bid that corresponds to the n^{th} atom in G_i, the size of the atom is denoted by

$$\varsigma_{in} = F_i(\overline{v}_i(\beta_{in})) - F_i(\underline{v}_i(\beta_{in})).$$

$\overline{\varphi}_i$ is differentiable everywhere else, therefore so is $v_i(\cdot)$ whenever it is a singleton. For any $b \in B_i \backslash \mathcal{B}_i$, differentiating we have that

$$g_i(b) = f_i(v_i(b))v_i'(b).$$

For any $b \in B_i \backslash \mathcal{B}_i$, we know that $\varphi_i(v_i(b)) = \overline{\varphi}_i(v_i(b))$, and therefore by the definition of φ_i

$$\begin{aligned} v_i(b) - \frac{1 - F_i(v_i(b))}{f_i(v_i(b))} &= b. \\ \Longrightarrow v_i(b) - \frac{1 - G_i(b)}{g_i(b)} v_i'(b) &= b. \end{aligned} \tag{10}$$

Observation 4. *Consider any interval $[\underline{b}, \overline{b}]$ in the support of G such that there are no atoms in this interval. Further, suppose $v_1(\overline{b}) = v_2(\overline{b})$. Then $v_1(b) = v_2(b)$ for every $b \in [\underline{b}, \overline{b}]$.*

Proof. From (10), we know that for each $b \in [\underline{b}, \overline{b}]$, and $i = 1, 2$

$$\begin{aligned} v_i(b) - \frac{1 - G_i(b)}{g_i(b)} v_i'(b) &= b, \\ \Longrightarrow v_1(b) \lesseqgtr v_2(b) &\iff v_1'(b) \lesseqgtr v_2'(b), \end{aligned}$$

where the implication follows from the fact that $G_1 = G_2$. Therefore if $v_1(b) \neq v_2(b)$ for some $b \in [\underline{b}, \overline{b}]$, it cannot be that $v_1(\overline{b}) = v_2(\overline{b})$. $\square$

For any $\beta_{in} \in \mathcal{B}_i$, the 'ironed' virtual value pools all buyers in $[\underline{v}_i(b), \overline{v}_i(b)]$. Therefore

$$\begin{aligned} \beta_{in} &= \frac{\int_{\underline{v}_i(\beta_{in})}^{\overline{v}_i(\beta_{in})} \varphi_i(v) f_i(v)\, \mathrm{d}v}{F_i(\overline{v}_i(\beta_{in})) - F_i(\underline{v}_i(\beta_{in}))}, \\ &= \underline{v}_i(\beta_{in}) - (\overline{v}_i(\beta_{in}) - \underline{v}_i(\beta_{in})) \frac{1 - F_i(\overline{v}_i(\beta_{in}))}{F_i(\overline{v}_i(\beta_{in})) - F_i(\underline{v}_i(\beta_{in}))}, \\ &= \underline{v}_i(\beta_{in}) - (\overline{v}_i(\beta_{in}) - \underline{v}_i(\beta_{in})) \frac{1 - G_i(\beta_{in})}{\varsigma_{in}}. \end{aligned} \tag{11}$$

Since $G_1 = G_2$, both have the same (at most countable set of) atoms— we denote the set of atoms $\mathcal{B}$ with generic element β_n of 'size' ς_n.

Observation 5. *Consider any atom $\beta_n \in \mathcal{B}$ of size ς_n, and suppose that $\overline{v}_1(\beta_n) = \overline{v}_2(\beta_n)$. Then we have that $\underline{v}_1(\beta_n) = \underline{v}_2(\beta_n)$, i.e. $v_1(\beta_n) = v_2(\beta_n)$.*

Proof. By (11), we have for $i = 1, 2$

$$\beta_n = \underline{v}_i(\beta_n) - (\overline{v}_i(\beta_n) - \underline{v}_i(\beta_n))\frac{1 - G_i(\beta_n)}{\varsigma n},$$
$$= \underline{v}_i(\beta_n)\left(1 + \frac{1 - G_i(\beta_n)}{\varsigma n}\right) - \overline{v}_i(\beta_n).$$

Therefore if $\overline{v}_1(\beta_n) = \overline{v}_2(\beta_n)$, then $\underline{v}_1(\beta_n) = \underline{v}_2(\beta_n)$. □

Finally, letting $\overline{b}$ be the upper bound of the support of $G_1 (= G_2)$, note that by definition:

$$v_1(\overline{b}) = v_2(\overline{b}) = \overline{b}.$$

The fact that $v_1(\cdot) = v_2(\cdot)$ now follows from this initial condition and Observations 4 and 5. Therefore $G_1 = G_2 \implies \overline{\varphi}_1 = \overline{\varphi}_2$. □

A.4. *Proof of Lemma 1*

Define the set of non-decreasing interim allocation rules achieved by some index rule as $\mathscr{H}_M$ the set of all feasible, non-decreasing interim allocation rules by $\mathscr{F}_M$ and the set of all feasible interim allocation rules by $\mathscr{F}$. By feasible, we mean that this interim allocation rule can result from some feasible ex-post allocation rule. The proof follows from two observations.

Observation 6. *$\mathscr{F}_M$ is as compact subset of L_2^n in the weak/ weak* topology $\sigma(L_2^n, L_2^n)$.*

Proof. Lemma 8 of Mierendorff (2011) shows that the set of feasible interim allocation rules $\mathscr{F}$ is a compact convex subset of L_2^n in this topology.

By observation, $\mathscr{F}_M$ is convex. We now argue that $\mathscr{F}_M$ is also compact in this topology. By the Eberlein-Smulian theorem (Theorem 6.34, Aliprantis & Border 2006), sequential compactness and compactness coincide in this topology. It is therefore enough to show that if for some sequence $\{a^n\}_{n=1}^{\infty} \subset \mathscr{F}_M$, $a^n \rightharpoonup a$, then $a \in \mathscr{F}_M$. Since each a^n is monotone, it is a function of bounded variation and therefore by Helly's selection theorem, there exists a subsequence which converges pointwise. Therefore a is also non-decreasing, and $a \in \mathscr{F}_M$, concluding our argument. □

Therefore, we have that the closure of the convex hull of $\mathscr{H}_M$ is a subset of $\mathscr{F}_M$ or

$$\overline{\text{conv}(\mathscr{H}_M)} \subseteq \mathscr{F}_M.$$

Observation 7. *For any index function $I : V \to \mathbb{R}^n$, the corresponding hierarchical allocation rule $a^h \in \mathscr{H}_M$ which solves*

$$\max_{a \in \mathscr{F}_M} \int_V \left(\sum_j a_j(v) I_j(v) f_j(v)\right) dv, \tag{I-OPT-M}$$

Proof. If I is non-decreasing, i.e. $I_j(v)$ is non-decreasing in v for each $j \in N$, then the solution to (I-OPT-M) is in $\mathscr{H}_M$. This follows easily from the definition of hierarchical allocation rule. Since at every profile of values, the good is allotted to the buyer with the higher index, the rule maximizes the 'index revenue' profile-by-profile. Therefore it solves the maximization problem (I-OPT-M).

So let us consider the solution to (I-OPT-M) for other index functions. We can re-write the problem as

$$\max_{a\in\mathscr{F}} \int_V \left(\sum_j a_j(v) I_j(v) f_j(v)\right) dv,$$
$$a \text{ is non-decreasing.}$$

In this case, we can 'relax' the non-decreasing constraint into the objective function. By the ironing procedure of Myerson (1981), there exists an 'ironed' non-decreasing index rule $\hat{I}$ such that the solution to the above problem is the same as

$$\max_{a\in\mathscr{F}} \int_V \left(\sum_j a_j(v) \hat{I}_j(v) f_j(v)\right) dv.$$

Note that the corresponding hierarchical rule for index rule $\hat{I}$ lies in $\mathscr{H}_M$. □

To conclude the proof, suppose by way of contradiction that

$$\overline{\text{conv}(\mathscr{H}_M)} \subsetneq \mathscr{F}_M.$$

Then there exists $a \in \mathscr{F}_M$ such that $a \notin \overline{\text{conv}(\mathscr{H}_M)}$. By Corollary 7.47 of Aliprantis & Border (2006) there exists an $I \in L_2^n$ such that

$$\langle a, I\rangle > \max_{a'\in\overline{\text{conv}(\mathscr{H}_M)}} \langle a', I\rangle,$$

where $\langle a, I\rangle$ is the standard inner product $\int_V \left(\sum_j a_j(v) I_j(v) f_j(v)\right) dv$.

By Observation 7, the hierarchical allocation rule corresponding to I solves (I-OPT-M), implying that

$$\langle a, I\rangle > \max_{a'\in\mathscr{F}_M} \langle a', I\rangle.$$

Since $a \in \mathscr{F}_M$, this is a contradiction. It follows that

$$\overline{\text{conv}(\mathscr{H}_M)} = \mathscr{F}_M.$$

The Lemma follows. □

Appendix B. Proofs from Section 5

B.1. *Proof of Proposition 1*

We will first prove the theorem as stated for differentiable and strictly increasing index rules (that is, no atoms in G_i). Later, we will extend to the more general case.

Proof. We first demonstrate sufficiency, and then argue necessity.

Sufficiency. For simplicity, we will only define payments for equilibrium bids; off-equilibrium bids can be discouraged in the same way as in the proof of Theorem 1. We consider the two cases of

Condition C2 separately. For (Case 1) we consider the following payment rule:

$$p^s(b,b') = \begin{cases} \frac{1}{G_1(\underline{b}_2)}\left[p_2^h(I_2^{-1}(b)) - \eta(b)\right] & \text{for } b' \in [\underline{b}_1, \underline{b}_2) \\ \hat{p}(b,b') & \text{for } b' \notin [\underline{b}_1, \underline{b}_2) \end{cases}$$

where $\hat{p}$ is given by (C2,P1), and $\eta(b)$ is given by (3).

Similarly, (Case 2) we consider the following payment rule:

$$p^s(b,b') = \begin{cases} \frac{1}{G_1(\underline{b}_2)}\left[p_2^h(I_2^{-1}(b)) - \eta_l(b)\right] & \text{for } b' \in [\underline{b}_1, \underline{b}_2) \\ \hat{p}_l(b,b') & \text{for } b' \notin [\underline{b}_1, \underline{b}_2) \end{cases}$$

where in this case $\hat{p}_\ell$ is given by (C2,P2) for a given $\ell > \underline{\ell}$, and $\eta_l(b)$ is as defined in (4). From Condition C2 and continuity there is a ℓ close enough to $\underline{\ell}$ for which $p^s(b,b') < v(b)$ for $b' \in [\underline{b}_1, \underline{b}_2)$.

By construction, for each buyer, his expected payment will equal his interim payment in the hierarchical mechanism. Condition (C2) guarantees that in the range $b' \in [\underline{b}_1, \underline{b}_2)$, the implementation still satisfies ex-post IR, $p^s(b,b') \leq I_2^{-1}(b)$. Indeed it might be the case that $p^s(b,b') < 0$.

Necessity. We first verify these conditions are necessary for implementation in a symmetric auction where bidding the actual index rule is each buyer's equilibrium strategy. We then show that the same conditions also rule out other implementations as well.

Let us verify that C1 is necessary. So suppose not, i.e. suppose: $v(b)G_2(b) < p_1^h(I_1^{-1}(b))$ for some $b \in B_1 \cap B_2$. Note that if a buyer bids b, and the other bidder bids $b' \in B_1 \cap B_2$, the maximum she can be asked to pay without violating ex-post IR is $v(b)$. But now, for bidder 1, it follows that the maximum expected payment that she can be asked to make is $v(b)G_2(b)$. If her required payment, $p_1^h(I_1^{-1}(b))$, exceeds this, then there cannot be a symmetric, ex-post IR implementation.

With buyer 2, there is a little more 'wriggle room.' When buyer 2 bids $b \in B_1 \cap B_2$, she could be a winner in some 'asymmetric' profiles; i.e. when buyer 1 bids in the range $[\underline{b}_1, \underline{b}_2)$. At these bid profiles, a potentially higher payment (up to $I_2^{-1}(b)$) can be extracted from buyer 2. Condition (C2) then guarantees that the required interim payment, $p_2^h(I_2^{-1}(b))$ can be extracted.

Note that in the construction of either $\hat{p}$ or $\hat{p}_\ell$, either the maximum permissible amount $v(b)$ is being paid by the *winning* buyer, or a rebate of s is being returned to the buyer who bids b. The rebates are being paid when the other buyer's bid b' has the lowest possible value of $L(b')$. This means that the rebates are worth the lowest possible in expectation to a winning buyer 2, because they occur where $L(\cdot)$ is minimized.

We begin by considering the following maximization problem for $b \in B_1 \cap B_2$ and any given $s \leq 0$:

$$m_s(b) = \max_{\varrho(\cdot)} \int_{\underline{b}_2}^{\bar{b}_1} \varrho(b')dG_1(b'), \tag{Max-P}$$

$$\text{s.t.} \int_{\underline{b}_2}^{\bar{b}_2} \varrho(b')dG_2(b') = p_1^h(I_1^{-1}(b)), \tag{λ}$$

$$s \leq \varrho(b') \leq v(b), \qquad \forall b' \in [\underline{b}_2, b], \tag{$\delta(b'),\kappa(b')$}$$

$$s \leq \varrho(b') \leq 0, \qquad \forall b' \in (b, \max\{\bar{b}_1, \bar{b}_2\}]. \tag{$\delta(b'),\kappa(b')$}$$

To understand this optimization program in words, fix a bid b. Think of $\varrho(\cdot)$ as the payment made by the buyer in this case as a function of the other buyer's bid. The program asks what the maximum expected payment that can be extracted from buyer 2 is subject to constraints we describe next. The first constraint requires that the expected payment of buyer 1 under $\varrho(\cdot)$ is his correct interim payment. The latter two constraints require that $\varrho(\cdot)$ is pointwise bounded below by s and bounded above by the maximum possible ex-post IR payment $v(b)$ when winning and 0 when losing. The terms in the parentheses to the right of the constraints denote the corresponding dual (co-state) variables.

We claim that $\lim_{s\downarrow-\infty} m_s(b) = \eta(b)$. When $v(b)G_2(b) = p_1^h(I_1^{-1}(b))$, then $\varrho(b') = v(b)$ for all $b' \in [\underline{b}_2, b]$ is the only feasible function, so this case is trivial. Hence, we focus on the case $v(b)G_2(b) > p_1^h(I_1^{-1}(b))$.

The Hamiltonian in this case is:

$$\begin{aligned} &g_1(b') - \lambda g_2(b') + \delta(b') - \kappa(b') = 0, \\ \Longrightarrow\ &\frac{g_1(b')}{g_2(b')} - \lambda + \frac{1}{g_2(b')}(\delta(b') - \kappa(b')) = 0 \end{aligned}$$

with complementary slackness conditions:

$$\begin{aligned} &\delta(b')(s - \varrho(b')) = 0, \\ \text{for } b' \in [\underline{b}_2, b], \quad &\kappa(b')(v(b) - \varrho(b')) = 0, \\ \text{for } b' \in (b, \max\{\bar{b}_1, \bar{b}_2\}], \quad &\kappa(b')\varrho(b') = 0, \\ \text{and} \quad &\delta(b'), \kappa(b') \geq 0. \end{aligned}$$

By observation, the solution to this for any s is 'bang bang', i.e.

$$\varrho(b') = \begin{cases} s & \text{if } L(b') \leq \lambda^\star, \\ v(b) & \text{if } L(b') > \lambda^\star, b' \in [\underline{b}_2, b], \\ 0 & \text{otherwise.} \end{cases}$$

with $\lambda^\star$ selected such that the corresponding primal equation binds for $\varrho(\cdot)$ selected thus. The corner case that needs care is when $G_2(\bar{\bar{\gamma}}(\underline{\ell})) > 0$. In this case, there is a positive measure of $b' \in [\underline{b}_2, \bar{b}_2]$ such that $L(b') = \underline{\ell}$. Here, the solution is bang bang, but possibly (depending on s), there is $\hat{B} \subseteq \bar{\bar{\gamma}}(\underline{\ell})$ such that

$$\varrho(b') = \begin{cases} s & \text{if } b' \in \hat{B} \subseteq \bar{\bar{\gamma}}(\underline{\ell}), \\ v(b) & \text{if } b' \in [\underline{b}_2, b] \backslash \hat{B}, \\ 0 & \text{otherwise.} \end{cases}$$

It follows by construction, therefore, that $\lim_{s\downarrow-\infty} m_s(b) = \eta(b)$. Therefore, subject to the payment rule extracting the appropriate interim payment $p_1^h(I_1^{-1}(b))$ when buyer 1 bids b, $\eta(b)$ is the maximum expected payment that can be extracted from buyer 2 when she bids b and buyer 1 makes a bid higher than $\underline{b}_2$. It follows therefore that if inequality (C2) is violated, there cannot be an implementation satisfying both symmetry and ex-post individual rationality.

Next, consider any other mechanism (I', p^h) with a differentiable index rule, that implements the same mechanism. Then, it must be that

$$I_i'(v_i) = \Gamma(I_i(v_i)).$$

for some differentiable and strictly increasing function $\Gamma : \mathbb{R} \to \mathbb{R}$. Note that the resulting distribution on bids, which we shall denote by G_i', is

$$G_i'(\Gamma(b)) = G_i(b).$$

Note that this implies that

$$g_i'(\Gamma(b))\Gamma'(b) = g_i(b).$$

Our previous arguments already imply that Conditions C1 and C2, written in terms of G_i''s are necessary for an implementation. By the equations above, we see that for $b \in B_1 \cap B_2$

$$v(b)G_2'(\Gamma(b)) \geq p_1^h(I_1^{-1}(b)) \implies v(b)G_2(b) \geq p_1^h(I_1^{-1}(b)).$$

Also, for any $b \in B_2$,

$$L(b) = \frac{g_1(b)}{g_2(b)} = \frac{g_1'(\Gamma(b))}{g_2'(\Gamma(b))}.$$

Therefore our conditions in terms of the original G_i's are necessary for any pure strategy implementation. □

Weakly increasing index rules. So far we have only considered strictly increasing index rules. If the index rules are not strictly increasing, the corresponding bid distributions will have atoms. Denote by $\mathcal{B}_i$ the atoms in G_i. For $b_i \in \mathcal{B}_i$, the size of the atom is $G_i(\{b_i\})$—recall that this is a measure and not a density. Further, $I_i^{-1}(\cdot)$ may be correspondence— $v(\cdot)$ may not be well defined. Redefine $v(b)$ as

$$v(b) = \inf\{v \in I_1^{-1}(b) \cup I_2^{-1}(b)\}.$$

Note that when $I_1^{-1}(b)$ and $I_2^{-1}(b)$ are singletons, this is the same as the old definition of $v(b)$. Now Condition C1 will be as before with this extended definition of $v(\cdot)$.

Next, note that Condition C2 depends on g_1/g_2, which again may not be well defined. We redefine $L(\cdot)$ as follows

$$L(b) = \begin{cases} \frac{g_1(b)}{g_2(b)} & b \in B_2 \text{ and } b \notin \mathcal{B}_1 \cup \mathcal{B}_2, \\ \frac{G_1(\{b\})}{G_2(\{b\})} & b \in \mathcal{B}_1 \cap \mathcal{B}_2, \\ 0 & b \in \mathcal{B}_2 \backslash \mathcal{B}_1. \end{cases}$$

We can now redefine $\eta(b)$ with this definition $L(b)$. It should be clear that Conditions C1 and C2 thus extended are necessary and sufficient.

Mixed Strategies. Observation 3 shows that in any mixed strategy implementation, an at most probability 0 set of values for any set of buyers can be mixing. It follows that to induce the same interim allocation rule, any mixed strategy implementation must induce the same distribution over bids as some pure strategy index rule implementing the allocation rule. Therefore our conditions are necessary for implementation in mixed strategies as well.

B.2. *Symmetric Ex-Post IR Implementation with Common Lower Bound of Bid Space Support*

We now use the previous intuition to derive axioms for the case where $\underline{b}_1 = \underline{b}_2$. This adds a little more complexity to our analysis. To see why, recall that our previous implementation 'heavily' used the fact that $\underline{b}_1 < \underline{b}_2$. In particular, profiles of the sort (b, b') where $b \in B_1 \cap B_2$ and $b' < \underline{b}_2$ were used as a sort of residual claimant. The payment of the winning buyer in profiles could be set as high v_2 to make up for any 'shortfall' in buyer 2's expected payment vis-a-vis her interim payment. Conversely, she can be given a rebate to make up for any surplus.

Since $G_1(\underline{b}_2) = 0$, Condition C2 rewritten in this case reflects the fact that there is no such region to make up for any shortfall:

Definition B.1 (Condition C2′)**.** Condition C2′ requires that for all b in $B_1 \cap B_2$, with $\underline{b}_1 = \underline{b}_2$

$$\eta(b) \geq p_2^h(I_2^{-1}(b)) \tag{12}$$

with the inequality holding strictly for any b such that:

$$G_2(\bar{\bar{\gamma}}(\underline{\ell})) = 0 \text{ and } v(b)G_2(b) > p_1^h(I_1^{-1}(b)).$$

Intuitively, Condition $C2'$ requires that the maximum expected payment $\eta(b)$ that can be extracted from buyer 2 when she bids b, among all payment rules that extract exactly $p_1^h(I_1^{-1}(b))$ from buyer 1 in expectation, is more than $p_2^h(I_2^{-1}(b))$. In the previous section this was enough, because any excess $\eta(b) - p_2^h(I_2^{-1}(b))$ can be rebated to buyer 2 when the other buyer bids in the range $[\underline{b}_1, \underline{b}_2)$. Now, this is no longer enough.

We need an additional condition to account for the fact that there is no lower region to 'rebate' any surplus to. We now write down the exact analog condition, i.e. that the minimum expected payment $\zeta(b)$ that can be extracted from buyer 2 when she bids b, among all payment rules that extract exactly $p_1^h(I_1^{-1}(b))$ from buyer 1 in expectation, is at most $p_2^h(I_2^{-1}(b))$.

If both conditions hold, there clearly exists a payment rule which will achieve the required implementation, since the set of all payment rules that extract exactly $p_1^h(I_1^{-1}(b))$ from buyer 1 in expectation is convex.

We consider two cases depending on the ordering of the upper bound of the possible bids, $\bar{b}_1$ and $\bar{b}_2$.

If $\bar{b}_1 > \bar{b}_2$, we can rebate money to buyer 2 similarly as before—in this case when the other bidder bids in the range $(\bar{b}_2, \bar{b}_1]$. In this case define $\zeta(b) = 0$ for all $b \in B_1 \cap B_2$.

Now let us consider the other case, i.e. that $\bar{b}_1 \leq \bar{b}_2$—in this case $B_1 \subseteq B_2$. We need some additional notation. First, we define

$$\bar{\ell} = \max_{b' \in [\underline{b}_2, \bar{b}_2]} L(b').$$

As before $\overline{\ell}$ is well defined. As before, there are two sub-cases. The first sub-case is when

$$G_2(\bar{\bar{\gamma}}(\overline{\ell})) > 0.$$

Let $\hat{B} \subset \bar{\bar{\gamma}}(\overline{\ell}(b)) > 0$ be a (potentially empty) subset such that:

$$v(b)G_2\left([\underline{b}, b]\backslash\hat{B}\right) \geq p_1^h(I_1^{-1}(b)).$$

We now define a payment rule

$$\hat{p}'(b, b') = \begin{cases} v(b) & \text{for } b' \in [\underline{b}_2, b]\backslash\hat{B}, \\ s & \text{for } b' \in \hat{B}, \\ 0 & \text{o.w.} \end{cases} \tag{C3,P1}$$

where s is chosen to solve

$$v(b)(G_2([\underline{b}, b]\backslash\hat{B})) + sG_2(\hat{B}) = p_1^h(I_1^{-1}(b)).$$

Notice that s here is a subsidy. We set:

$$\zeta(b) = \int_{\underline{b}_2}^{\overline{b}_2} \hat{p}'(b, b')dG_1(b').$$

The second sub-case is when

$$G_2(\bar{\bar{\gamma}}(\overline{\ell})) = 0,$$

we define the payment rule for $\ell < \overline{\ell}$

$$\hat{p}'_\ell(b, b') = \begin{cases} v(b) & \text{for } b' \in [\underline{b}, b],\ L(b') \leq \ell, \\ s & \text{for } L(b') > \ell, \\ 0 & \text{otherwise.} \end{cases} \tag{C3,P2}$$

where s is chosen to solve

$$v(b)G_2(\{b' : b' \in [\underline{b}, b], L(b') \leq \ell\}) + sG_2(\{b' : L(b') > \ell\}) = p_1^h(I_1^{-1}(b)).$$

Here we set

$$\zeta(b) = \lim_{\ell \uparrow \overline{\ell}} \left[\int_{\underline{b}_2}^{b} \hat{p}'_\ell(b, b')dG_1(b')\right].$$

Definition B.2 (Condition C3)**.** Condition C3 requires that

$$\zeta(b) \leq p_2^h(I_2^{-1}(b))$$

with the inequality holding strictly when:

$$G_2(\bar{\bar{\gamma}}(\overline{\ell})) = 0 \text{ and } v(b)G_2(b) > p_1^h(I_1^{-1}(b)).$$

We can now state the proposition

Proposition 2. *Suppose there are 2 buyers. Consider a hierarchical allocation mechanism* (I, p^h) *with differentiable and strictly increasing index functions such that the lower bounds of the supports of the bid* *distributions are the same, that is, $\underline{b}_1 = \underline{b}_2$. Then Conditions C1, C2′ and C3 are necessary and sufficient for there to exist a symmetric, ex-post IR implementation of (I, p^h).*

The proof follows from also considering the analogous minimization problem to (Max-P) and is omitted.

APPENDIX C. FULL RANK EVENTS

A little more notation will be useful. We say that an event $E \subseteq \mathbb{R}^{n-1}$ is of type l if there exists a $\beta \in \mathbb{R}$ such that E is the event "l randomly chosen buyers out of the $n-1$ have bids of β or less." For any number k, let $[k] \equiv \{1, 2, \ldots, k\}$. For any set K, $|K| = k$, $l \leq k$, define

$$\binom{K}{l} \equiv \{X : X \subseteq K, |X| = l\},$$

that is, the set of all subsets of K of cardinality exactly l.

By definition, if E is an event of type l with corresponding β, then

$$G_{-i}(E) = \frac{l!(n-1-l)!}{(n-1)!} \sum_{M \in \binom{N \setminus i}{l}} \prod_{j \in M} G_j(\beta). \tag{13}$$

We also allow for an event of type l to have a random cutoff $\tilde{\beta} \in \Delta\mathbb{R}$. This corresponds to the event that there are l randomly chosen buyers out of the $n-1$ and each of them has a bid less than an i.i.d. realization of the random variable $\tilde{\beta}$. Denote by $G_j(\tilde{\beta})$ the probability that a draw according to G_j is less than or equal to the random variable $\tilde{\beta}$.

Note that if we have an event E of type l with corresponding cutoff $\tilde{\beta}$,

$$G_{-i}(E) = \frac{l!(n-1-l)!}{(n-1)!} \sum_{M \in \binom{N \setminus i}{l}} \prod_{j \in M} G_j(\tilde{\beta}). \tag{14}$$

Recall the theorem:

Theorem 2. For any $n > 1$ and any $k \leq n$ such that $G_1, G_2, \ldots G_k$ are all pairwise distinct, there exist symmetric events $E_1, \ldots, E_k \subseteq \mathbb{R}^{n-1}$ such that the $(k \times k)$ matrix $\mathcal{M} = [G_{-i}(E_j)]_{i,j=1}^{k}$ has full rank.

Proof. Fix the number of buyers $n > 1$. We will prove the lemma by induction on k.
Base Case: $k = 2$. Since $G_1 \neq G_2$, pick $\beta^\star \in \mathbb{R}$ s.t. $G_1(\beta^\star) \neq G_2(\beta^\star)$. Now pick E_1 to be an event of type 1 with cutoff $\beta^\star$, and $E_2 = \mathbb{R}^{n-1} \setminus E_1$. The corresponding matrix $\mathcal{M}$ is

$$\mathcal{M} = \begin{bmatrix} \frac{1}{n-1}\sum_{i \neq 1} G_i(\beta^\star) & 1 - \frac{1}{n-1}\sum_{i \neq 1} G_i(\beta^\star) \\ \frac{1}{n-1}\sum_{i \neq 2} G_i(\beta^\star) & 1 - \frac{1}{n-1}\sum_{i \neq 2} G_i(\beta^\star) \end{bmatrix}.$$

By observation, this is full rank.
Inductive Hypothesis. Suppose this is true for all $k \leq \hat{k}$ for some $\hat{k} < n$.
Inductive step. We will show this true for $k = \hat{k} + 1$. By the inductive hypothesis we have events $E_1, \ldots, E_{\hat{k}} \subseteq \mathbb{R}^{n-1}$ such that

$$\mathcal{M} = [G_{-i}(E_j)]_{i,j=1}^{\hat{k}} \text{ is full rank.}$$

We need to show that we can find a $E_{\hat{k}+1}$ such that

$$\mathcal{M}' = [G_{-i}(E_j)]_{i,j=1}^{\hat{k}+1} \text{ is full rank.}$$

Note that since $\mathcal{M}$ is full rank, there exists a unique row-vector $\alpha \in \mathbb{R}^{\hat{k}}$ such that:

$$\alpha\mathcal{M} = \left[G_{-(\hat{k}+1)}(E_1), G_{-(\hat{k}+1)}(E_2), \ldots G_{-(\hat{k}+1)}(E_{\hat{k}})\right]$$

If it is not the case that

$$\sum_{i=1}^{\hat{k}} \alpha_i = 1,$$

then we are already done. To see this note that we can select $E_{\hat{k}+1} = \mathbb{R}^{n-1}$. With this selection, $\mathcal{M}'$ will be full rank, since $G_{-i}(\mathbb{R}^{n-1}) = 1$ for all i by definition, and therefore

$$\sum_{i=1}^{\hat{k}} \alpha_i G_{-i}(\mathbb{R}^{n-1}) \neq G_{-(\hat{k}+1)}(\mathbb{R}^{n-1}).$$

We will now proceed to prove that there exists an event such that M' is full rank. In particular, we will show that either $\mathbb{R}^{n-1}$ suffices or there must exist an event of type 1 to $\hat{k}$. So suppose that for any event $E_{\hat{k}+1}$ of type 1, the matrix $\mathcal{M}'$ is not full rank. For any event of type 1 with corresponding cutoff β, by (13)

$$G_{-i}(E_{\hat{k}+1}) = \frac{1}{n-1} \sum_{j=1, j\neq i}^{n} G_j(\beta).$$

Since by assumption no such event $E_{\hat{k}+1}$ results in a full rank matrix, we have that for all $E_{\hat{k}+1}$ of type 1 with corresponding β,

$$G_{-(\hat{k}+1)}(E_{\hat{k}+1}) = \sum_{i=1}^{\hat{k}} \alpha_i G_{-i}(E_{\hat{k}+1}),$$

$$\implies \forall \beta \in \mathbb{R},\ G_{\hat{k}+1}(\beta) = \sum_{i=1}^{\hat{k}} \alpha_i G_i(\beta).$$

As notational shorthand, we will write this as

$$G_{\hat{k}+1} = \sum_{i=1}^{\hat{k}} \alpha_i G_i.$$

Claim 1. Suppose $\hat{l} \leq \hat{k}$ is such that for all $l = 1 \ldots \hat{l}$, selecting $E_{\hat{k}+1}$ from events of types 1 to $\hat{l}$ cannot make $\mathcal{M}'$ full rank. Then for all $l = 1 \ldots \hat{l}$:

$$(G_{\hat{k}+1})^l = \sum_{i=1}^{\hat{k}} \alpha_i (G_i)^l, \tag{15}$$

$$\sum_{M \in \binom{[\hat{k}]}{l}} \left(1 - \sum_{i\in M} \alpha_i\right) \prod_{i\in M} G_i = \left(\sum_{M \in \binom{[\hat{k}]}{l}} \prod_{i \in M} G_i\right) - G_{\hat{k}+1} \sum_{M \in \binom{[\hat{k}]}{l-1}} \left(1 - \sum_{i \in M} \alpha_i\right) \prod_{i\in M} G_i. \tag{16}$$

Recall that (15) is notational shorthand for

$$\forall \tilde{\beta} \in \Delta\mathbb{R} : \ (G_{\hat{k}+1}(\tilde{\beta}))^l = \sum_{i=1}^{\hat{k}} \alpha_i (G_i(\tilde{\beta}))^l,$$

Proof of Claim 1. We prove this claim by induction on $\hat{l}$. While the base case $\hat{l} = 1$ is true by observation, to build intuition we will now prove it for the case of $\hat{l} = 2$. Since by assumption no event of type 2 produces a full rank matrix, it must be that for every event E of type 2,

$$G_{-(\hat{k}+1)}(E) = \sum_{i=1}^{\hat{k}} \alpha_i G_{-i}(E).$$

Substituting in from (13), and canceling terms, we have

$$\begin{aligned}
\sum_{M \in \binom{[k]}{2}} \prod_{i \in M} G_i &= \sum_{q=1}^{\hat{k}} \alpha_q \left(\sum_{M \in \binom{[\hat{k}+1] \setminus q}{2}} \prod_{i \in M} G_i \right), \\
&= \sum_{q=1}^{\hat{k}} \alpha_q \left(\sum_{M \in \binom{[\hat{k}] \setminus q}{2}} \prod_{i \in M} G_i + G_{\hat{k}+1} \sum_{i=1, i \neq q}^{\hat{k}} G_i \right),
\end{aligned}$$

since $\sum_{i=1}^{\hat{k}} \alpha_i = 1$, we have,

$$= \sum_{M \in \binom{[\hat{k}]}{2}} \left(1 - \sum_{i \in M} \alpha_i \right) \prod_{i \in M} G_i + G_{\hat{k}+1} \sum_{i=1}^{\hat{k}} (1 - \alpha_i) G_i.$$

By observation therefore we have (16) for $\hat{l} = 2$. Substituting in that $\sum_i \alpha_i G_i = G_{\hat{k}+1}$, we have

$$\begin{aligned}
\sum_{M \in \binom{[k]}{2}} \prod_{i \in M} G_i &= \sum_{M \in \binom{[\hat{k}]}{2}} \left(1 - \sum_{i \in M} \alpha_i \right) \prod_{i \in M} G_i + \left(\sum_{i=1}^{\hat{k}} \alpha_i G_i \right) \sum_{i=1}^{\hat{k}} G_i - (G_{\hat{k}+1})^2 \\
\implies 0 &= \sum_{M \in \binom{[\hat{k}]}{2}} \left(- \sum_{i \in M} \alpha_i \right) \prod_{i \in M} G_i + \left(\sum_{i=1}^{\hat{k}} \alpha_i G_i \right) \sum_{i=1}^{\hat{k}} G_i - (G_{\hat{k}+1})^2.
\end{aligned}$$

Canceling terms, we have

$$\begin{aligned}
0 &= \sum_{i=1}^{\hat{k}} \alpha_i (G_i)^2 - (G_{\hat{k}+1})^2, \\
\implies (G_{\hat{k}+1})^2 &= \sum_{i=1}^{\hat{k}} \alpha_i (G_i)^2,
\end{aligned}$$

as desired.

For our inductive hypothesis, assume that (15,16) are true for all $l \leq \hat{l} - 1$ and now suppose that no event of type $\hat{l}$ can make matrix $\mathcal{M}'$ full rank. We are therefore left to show (15,16) for $l = \hat{l}$. It therefore must be that for any event E of type $\hat{l}$,

$$G_{-(\hat{k}+1)}(E) = \sum_{i=1}^{\hat{k}} \alpha_i G_{-i}(E).$$

Substituting in from (13), and canceling terms, we have

$$\begin{aligned}
\sum_{M \in \binom{[\hat{k}]}{\hat{l}}} \prod_{i \in M} G_i &= \sum_{q=1}^{\hat{k}} \alpha_q \left(\sum_{M \in \binom{[\hat{k}+1] \setminus q}{\hat{l}}} \prod_{i \in M} G_i \right), \\
&= \sum_{q=1}^{\hat{k}} \alpha_q \left(\sum_{M \in \binom{[\hat{k}] \setminus q}{\hat{l}}} \prod_{i \in M} G_i + G_{\hat{k}+1} \sum_{M \in \binom{[\hat{k}] \setminus q}{\hat{l}-1}} \prod_{i \in M} G_i \right),
\end{aligned}$$

since $\sum_{i=1}^{\hat{k}} \alpha_i = 1$, we have,

$$= \sum_{M \in \binom{[\hat{k}]}{\hat{l}}} \left(1 - \sum_{i \in M} \alpha_i\right) \prod_{i \in M} G_i + G_{\hat{k}+1} \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1}} (1 - \sum_{i \in M} \alpha_i) \prod_{i \in M} G_i.$$

Therefore, we have (16) as desired for $\hat{l}$. Rearranging, we have

$$\sum_{M \in \binom{[\hat{k}]}{\hat{l}}} \left(\sum_{i \in M} \alpha_i\right) \prod_{i \in M} G_i = G_{\hat{k}+1} \left(\sum_{M \in \binom{[\hat{k}]}{\hat{l}-1}} (1 - \sum_{i \in M} \alpha_i) \prod_{i \in M} G_i \right).$$

Substituting the term in the parentheses on the right hand side from (16) for $\hat{l}-1$,

$$\sum_{M \in \binom{[\hat{k}]}{\hat{l}}} \left(\sum_{i \in M} \alpha_i\right) \prod_{i \in M} G_i = G_{\hat{k}+1} \left(\left(\sum_{M \in \binom{[\hat{k}]}{l-1}} \prod_{i \in M} G_i \right) - G_{\hat{k}+1} \sum_{M \in \binom{[\hat{k}]}{l-2}} \left(1 - \sum_{i \in M} \alpha_i\right) \prod_{i \in M} G_i. \right).$$

Proceeding inductively and collecting terms, we have

$$\begin{aligned}
&\sum_{M \in \binom{[\hat{k}]}{\hat{l}}} \left(\sum_{i \in M} \alpha_i\right) \prod_{i \in M} G_i = G_{\hat{k}+1} \left(\sum_{s=0}^{\hat{l}-2} (-1)^s (G_{\hat{k}+1})^s \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1-s}} \prod_{i \in M} G_i + (-1)^{\hat{l}-1} (G_{\hat{k}+1})^{\hat{l}-1} \right) \\
\implies &\sum_{M \in \binom{[\hat{k}]}{\hat{l}}} \left(\sum_{i \in M} \alpha_i\right) \prod_{i \in M} G_i = (-1)^{\hat{l}-1} (G_{\hat{k}+1})^{\hat{l}} + G_{\hat{k}+1} \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1}} \prod_{i \in M} G_i \\
&\qquad\qquad + G_{\hat{k}+1} \left(\sum_{s=1}^{\hat{l}-2} (-1)^s (G_{\hat{k}+1})^s \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1-s}} \prod_{i \in M} G_i \right). \\
\implies &0 = (-1)^{\hat{l}-1} (G_{\hat{k}+1})^{\hat{l}} + \sum_{M \in \binom{[\hat{k}]}{\hat{l}-1}} \left(\sum_{i \in M} \alpha_i G_i \right) \prod_{i \in M} G_i
\end{aligned}$$

$$+ G_{\hat{k}+1}\left(\sum_{s=1}^{\hat{l}-2}(-1)^s(G_{\hat{k}+1})^s \sum_{M\in\binom{[\hat{k}]}{\hat{l}-1-s}} \prod_{i\in M} G_i\right).$$

$$\Longrightarrow 0 = (-1)^{\hat{l}-1}(G_{\hat{k}+1})^{\hat{l}} + \sum_{M\in\binom{[\hat{k}]}{\hat{l}-1}}\left(\sum_{i\in M}\alpha_i G_i\right)\prod_{i\in M} G_i - (G_{\hat{k}+1})^2 \sum_{M\in\binom{[\hat{k}]}{\hat{l}-2}}\prod_{i\in M} G_i$$

$$+ G_{\hat{k}+1}\left(\sum_{s=2}^{\hat{l}-2}(-1)^s(G_{\hat{k}+1})^s \sum_{M\in\binom{[\hat{k}]}{\hat{l}-1-s}} \prod_{i\in M} G_i\right).$$

Substituting in from (15) for $l = 2$,

$$\Longrightarrow 0 = (-1)^{\hat{l}-1}(G_{\hat{k}+1})^{\hat{l}} + \sum_{M\in\binom{[\hat{k}]}{\hat{l}-1}}\left(\sum_{i\in M}\alpha_i G_i\right)\prod_{i\in M} G_i - \left(\sum_{i=1}^{\hat{k}}\alpha_i (G_i)^2\right) \sum_{M\in\binom{[\hat{k}]}{\hat{l}-2}}\prod_{i\in M} G_i$$

$$+ G_{\hat{k}+1}\left(\sum_{s=2}^{\hat{l}-2}(-1)^s(G_{\hat{k}+1})^s \sum_{M\in\binom{[\hat{k}]}{\hat{l}-1-s}} \prod_{i\in M} G_i\right).$$

Canceling terms

$$\Longrightarrow 0 = (-1)^{\hat{l}-1}(G_{\hat{k}+1})^{\hat{l}} - \sum_{M\in\binom{[\hat{k}]}{\hat{l}-2}}\left(\sum_{i\in M}\alpha_i (G_i)^2\right)\prod_{i\in M} G_i$$

$$+ G_{\hat{k}+1}\left(\sum_{s=2}^{\hat{l}-2}(-1)^s(G_{\hat{k}+1})^s \sum_{M\in\binom{[\hat{k}]}{\hat{l}-1-s}} \prod_{i\in M} G_i\right).$$

Continuing to open out the summation and cancel terms, we have, as desired,

$$(G_{\hat{k}+1})^{\hat{l}} = \sum_{i=1}^{\hat{k}}\alpha_i (G_i)^{\hat{l}}.$$

This concludes the proof of the claim. □

Having shown Claim 1, we now show that there exist an event of type 1 to $\hat{k}$ such that the matrix $\mathcal{M}'$ has full rank. To see this, assume the converse. Then, by (15) we have that

$$\forall l = 1 \dots \hat{k},\ (G_{\hat{k}+1})^l = \sum_{i=1}^{\hat{k}}\alpha_i (G_i)^l,$$

and further we know by our previous arguments that

$$1 = \sum_{i=1}^{\hat{k}}\alpha_i.$$

We can rewrite these together as

$$\forall l = 0 \dots \hat{k},\ (G_{\hat{k}+1})^l = \sum_{i=1}^{\hat{k}} \alpha_i (G_i)^l,$$

We now have $\hat{k}+1$ functional equations, but only $\hat{k}$ variables (α's). Since the distributions are different, it should be intutive that this system of equations cannot have a solution.

Claim 2. Suppose the distributions G_1 to $G_{\hat{k}+1}$ are pairwise different. Then,

$$\exists \tilde{\beta} \in \Delta\mathbb{R} \text{ s.t. } G_1(\tilde{\beta}) \text{ to } G_{\hat{k}+1}(\tilde{\beta}) \text{ are all different.}$$

Proof. Consider the subset of $\mathbb{R}^{\hat{k}+1}$ defined as

$$S \equiv \{(a_1, a_2, \dots, a_{\hat{k}+1}) : \exists \tilde{\beta} \in \Delta\mathbb{R} \text{ s.t. } a_j = G_j(\tilde{\beta}) \text{ for } j = 1, \dots, \hat{k}+1\}.$$

Further, for every j, j', define $X_{j,j'} \subseteq \mathbb{R}^{\hat{k}+1}$

$$X_{j,j'} \equiv \{(a_1, a_2, \dots, a_{\hat{k}+1}) : a_j = a_{j'}\}.$$

Note that each $X_{j,j'}$ is a $\hat{k}$ dimensional subspace of $\mathbb{R}^{\hat{k}+1}$.

By definition S is convex. Since the distributions are pairwise different, for every j, j' there exists $\beta \in \mathbb{R}$ such that $G_j(\beta) \neq G_{j'}(\beta)$. Therefore for each j, j', $S \not\subseteq X_{j,j'}$. Further note that $X \equiv \bigcup_{j \neq j'} X_{j,j'}$ is not convex, so $S \not\subseteq X$, and therefore we have our desired result. □

Note that by Claim 2, possibly by adding a little weight on a low β such that $G_j(\beta) = 0$ for all j, we have that there exists $\tilde{\beta} \in \Delta\mathbb{R}$ such that all $G_1(\tilde{\beta})$ to $G_{\hat{k}+1}(\tilde{\beta})$ are pairwise different, and also different from 1.

Therefore, for this $\tilde{\beta}$, there must exist a solution to:

$$\forall l = 0, \dots, \hat{k}, (G_{\hat{k}+1}(\tilde{\beta}))^l = \sum_{i=1}^{\hat{k}} \alpha_i (G_i(\tilde{\beta}))^l.$$

Taking the appropriate Farkas alternative, therefore, for the previous system to have a solution, here there must exist a non-zero solution $\nu \in \mathbb{R}^{\hat{k}+1}$ to:[13]

$$\forall i = 1, \dots \hat{k}+1 : \sum_{l=0}^{\hat{k}} \nu_l (G_i(\tilde{\beta}))^l = 0.$$

But note that this suggests there are $\hat{k}+1$ distinct roots of the $\hat{k}$ degree polynomial

$$\sum_{l=0}^{\hat{k}} \nu_l x^l,$$

[13] The Farkas Lemma states that either the system $Cx = d$ has a solution or $yC = 0$, $yd > 0$ has a solution but never both. For the latter system to have no solution, it must be that for every non-zero y such that $yC = 0$, it is the case that $yd = 0$. This is the version we stated.

which is impossible. Therefore there is no solution to

$$\forall l = 0, \ldots, \hat{k}, (G_{\hat{k}+1}(\tilde{\beta}))^l = \sum_{i=1}^{\hat{k}} \alpha_i (G_i(\tilde{\beta}))^l,$$

concluding our proof. □

References

Adarand Constructors, Inc. v. Pena (1995), Vol. 515 U.S. 200, Supreme Court.

Aliprantis, C. D. & Border, K. C. (2006), *Infinite Dimensional Analysis*, Springer-Verlag.

Athey, S., Coey, D. & Levin, J. (2013), 'Set-asides and subsidies in auctions', *American Economic Journal: Microeconomics* **5**(1), 1–27.

Border, K. C. (1991), 'Implementation of reduced form auctions: A geometric approach', *Econometrica* pp. 1175–1187.

Caillaud, B. & Robert, J. (2005), 'Implementation of the revenue-maximizing auction by an ignorant seller', *Review of Economic Design* **9**(2), 127–143.

Cantillon, E. (2008), 'The effect of bidders' asymmetries on expected revenue in auctions', *Games and Economic Behavior* **62**(1), 1–25.

Corns, A. & Schotter, A. (1999), 'Can affirmative action be cost effective? an experimental examination of price-preference auctions', *The American Economic Review* **89**(1), 291–305.

Cramton, P. & Ayres, I. (1996), 'Deficit reduction through diversity: How affirmative action at the fcc increased auction competition', *Stanford Law Review* **48**, 761–815.

Gershkov, A., Goeree, J., Kushnir, A., Moldovanu, B. & Shi, X. (2013), 'On the equivalence of bayesian and dominant strategy implementation', *Econometrica* **81**(1), 197–220.

Hartline, J. D. (2012), 'Approximation in mechanism design', *The American Economic Review* **102**(3), 330–336.

Kirkegaard, R. (2012), 'A mechanism design approach to ranking asymmetric auctions', *Econometrica* **80**(5), 2349–2364.

Lebrun, B. (2012), 'Optimality and the second-price auction with resale', *Games and Economic Behavior* **75**, 731–751.

Manelli, A. & Vincent, D. (2010), 'Bayesian and dominant-strategy implementation in the independent private-values model', *Econometrica* **78**(6), 1905–1938.

Mares, V. & Swinkels, J. M. (2010), On the analysis of asymmetric first price auctions, Technical report, Northwestern University.

Mares, V. & Swinkels, J. M. (2011), 'Near-optimality of second price mechanisms in a class of asymmetric auctions', *Games and Economic Behavior* **72**(1), 218–241.

Maskin, E. (1999), 'Nash equilibrium and welfare optimality', *Review of Economic Studies* **66**(1), 23–38.

Maskin, E. & Riley, J. (1984), 'Optimal auctions with risk averse buyers', *Econometrica* pp. 1473–1518.

Maskin, E. & Riley, J. (2000), 'Asymmetric auctions', *Review of Economic Studies* **67**(3), 413–438.

McAfee, R. & McMillan, J. (1989), 'Government procurement and international trade', *Journal of international economics* **26**(3), 291–308.

Mierendorff, K. (2011), 'Asymmetric reduced form auctions', *Economics Letters* **110**(1), 41–44.

Myerson, R. (1981), 'Optimal auction design', *Mathematics of operations research* **6**(1), 58–73.

Pai, M. & Vohra, R. (2012), Auction design with fairness concerns: Subsidies vs. set-asides, Technical report, Discussion Paper, Center for Mathematical Studies in Economics and Management Science.

Pai, M. & Vohra, R. V. (2008), Optimal auctions with financially constrained bidders, Technical report, Discussion paper, Center for Mathematical Studies in Economics and Management Science.

Zheng, C. (2006), 'Optimal auction with resale', *Econometrica* **70**(6), 2197–2224.