A Cognitive Basis for Adaptive Utility

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Abstract

This paper provides a new approach to a utility function, by deriving it from deeper first principles. With a minimal set of cognitive tools - ordinal comparison and proportion (frequency-of-occurrence) estimation, an object can be evaluated by the perceived rank of its magnitude within a reference set. Such evaluation is isomorphic to a utility function, which adapts to the environmental context, and thus is evolutionarily advantageous. Consistent with empirical observations, a change in context leads to violations of reflexivity and transitivity. When the cognitive tools are perfect and environment is fixed, mistakes in binary choices are minimized. Cognitive imperfections tend to distort one's mental number line, leading to overvaluation of small and undervaluation of large objects. As the evaluation of consumption, income, and wealth is based on interpersonal comparisons, the relationship between welfare and income inequality is affected by cognitive imperfections. The proposed parsimonious model links recent developments in economics, psychology, and neuroscience.

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1 Introduction

How big is an apple? How high is a salary? How intense is an experience? How young is a person? How frequent is a name? How long is a piece of string? How can one tell?

The concept of a utility function is fundamental to economics and behavioral sciences, yet the process of utility acquisition is still not well understood. Recently a new evolutionary framework for decision-making has been advanced as an alternative to neoclassical utility theory. Rather than endowing living organisms with all necessary information, Nature provides the "tools" that enable one to extract information from one's environment and experience (Robson [2001, 2002], Samuelson [2004], Samuelson and Swinkels [2006]). These tools might have been further adapted to deal with more evolutionarily recent tasks (Cosmides and Tooby [1994]).

The present paper undertakes a novel approach to a concept of a utility function. Simply put, before one can construct a utility function $U(x)$, one needs to *measure* the size of x. I thus explore the process and tools by which a given magnitude x can be evaluated. I identify two well-documented domain-independent cognitive tools and, using a parsimonious mathematical model, show that one could evaluate a magnitude of an item entirely using *ordinal comparisons* which, by means of a *frequency* (proportion) tool, are keyed into a universal cardinal scale. This is done by calculating how frequently a given object "wins" a pairwise ordinal "tournament" against all other objects in the reference set. The resultant adaptive evaluation, which is the expected outcome of pairwise comparisons, is a non-decreasing function of a magnitude, and rationalizes "relatively more is better" preferences defined on a reference set. Thus, a possible candidate for an adaptive utility $U(x)$ is the adaptive evaluation of magnitude x relatively to a reference set by means of these two simple cognitive tools. Once the reference set changes, so does the adaptive utility. Specifically, a given magnitude is evaluated higher if the reference set is positively rather than negatively skewed, leading to a possibility of non-reflexivity, non-transitivity, and preference reversal.

I employ a convolution technique whereby an adaptive evaluation can be expressed as either a sum or a product of a reference magnitude and a pairwise comparison tournament. When the two cognitive tools are perfect, the adaptive magnitude evaluation is equal to the "veridical" (true) rank in the distribution of reference magnitudes. The convolution model allows one to explore of the effects of imperfections in cognitive tools on magnitude evaluation. An imperfect magnitude evaluation is only partially affected by the veridical rank, thus obscuring the empirical relationship between the environment and one's evaluation function. For example, if the veridical distribution is uniform (implying constant marginal utility), ordinal imperfections may lead to increasing marginal utility for smaller magnitudes and decreasing marginal utility for larger magnitudes, with constant marginal utility in-between. In general, smaller magnitudes tend to be overvalued, and larger tend to be undervalued, with such distortions being more severe if, in addition, small frequencies are overweighted and large are underweighted. Furthermore, I show that a (neoclassical) context-independent utility function may arise when one evaluates a magnitude relatively to a remembered sample and one has a long memory. In contrast, if one's memory is bounded, context effects may arise.

Since all economically relevant magnitudes (such as income, consumption, etc.) are attributable to individuals who possess them, the relevant magnitude evaluations necessarily involve interpersonal comparisons. I show that the properties of the ordinal comparison tool are important for utilitarian welfare, because economic processes such as economic growth and income redistribution may surprisingly be welfare neutral.[1] As welfare neutrality is an artefact of a restricted set of ordinal comparison tools, I further explore the effect of various ordinal comparison tools on welfare and show that welfare may or may not increase with economic growth and greater equality of income.

In contrast to the existing literature surveyed below, the present paper (i) explicitly models the process which might be involved in an evaluation of a magnitude; (ii) suggests the minimum set of domain-independent cognitive tools which are required for such evaluations; (iii) provides a parsimonious mathematical model of a magnitude evaluation; (iv) shows that this evaluation can represent "more is better" choices; (v) provides the conditions for the utility function to be equal to an empirical rank; (vi) provides the conditions for the utility function to be independent of the current context; (vii) explicitly incorporates the cognitive imperfections into evaluation; (viii) shows that cognitive imperfections obscure the empirical relationship between environment and utility; (ix) shows that the features of ordinal comparisons are important for societal welfare.

2 Related Literature

The parsimonious mathematical model presented here links a number of developments in economics, evolutionary and cognitive sciences.

2.1 Evolutionary Basis of Behavior

Robson [2001] advanced an argument in support of a utility function which adapts to the environment. [2] Rather than endowing living organisms with all necessary information, Nature provides the "tools" that enable one to extract information from one's environment and experience (Robson [2001, 2002], Samuelson [2004], Samuelson and Swinkels [2006]), which might have been further adapted to deal with more evolutionarily recent

[1] The welfare neutrality result is reminiscent of the famous paradox of happiness due to Easterlin [1974], whereby the average self-reported happiness is almost identical across time and nations despite individual happiness is increasing in one's own income and aggregate incomes vary widely.

[2] See a survey on hedonic adaptation by Frederick and Loewenstein [1999].

tasks (Cosmides and Tooby [1994], Gigerenzer [1998]). The quest for evolutionary basis of decision-making primitives has been expanded by Robson and Samuelson [2010] to decision and experienced utilities, while Rayo and Becker [2007] further suggest that Nature may endow humans with happiness as a context-dependent measurement tool which allow one to choose among the alternatives. Furthermore, Herold and Netzer [2011] suggested that the non-linear probability weighting function evolved as an evolutionary second-best to complement an adaptive S-shaped value function.

2.2 Domain-Independent Cognitive Tools

This paper argues that only two primitive tools are sufficient to "measure" a magnitude against a reference set. One such primitive tool - ordinal comparisons - enables one to tell whether a magnitude is larger, smaller, or equal to another magnitude. Brannon [2002] found that 9-11 months old human infants are able to discriminate between small arrays of objects, while Feigenson, Carey and Spelke [2002] suggested that human infants develop abilities to discriminate continuous variables (such as areas, sizes, densities) even earlier, while Xu [2003] reported that large numerosities are discriminated differently from the small ones. Furthermore Walsh [2003] suggested that time, space and numbers involve similar neural mechanisms. The other primitive tool is frequency-of-occurrence. Cognitive and evolutionary psychologists report that humans update frequencies easily and constantly (Hintzman and Stern [1978]), as well as automatically and accurately (Hasher and Zacks [1979, 1984]). Since human children and some animals are equipped with a mental system for counting (Gallistel and Gelman [1992]), natural frequencies can be stored in a numerical format (Jonides and Jones [1992]).

The issue of magnitude evaluation and measurement imperfections dates back to the 19th century, and is closely associated with the law of diminishing marginal utility. According to the Weber-Fechner law of just noticeable differences (see, for example, Link [1992]), humans fail to notice the difference between two magnitudes if this difference is less than a certain threshold, and such minimum noticeable difference is proportional to the stimulus level. For example, most people would notice the difference between having one dollar bill and two dollar bills in one's wallet, but would hardly notice the difference between having 67 and 68 dollar bills. This phenomenon was recently found on the brain level, as Nieder and Miller [2003] found that the representation of numerosities is "compressed" in a monkey brain, prompting Dehaene [2003] to advance an argument in support of a logarithmic mental number line to represent the Weber-Fechner law.[3]

The research on frequency (proportion) imperfections is still in its infancy. As Kahneman and Tversky [1979] pointed out, humans tend to overweight small probabilities and underweight large ones, and this probability distortion may happen at the encoding level

[3]Dehaene, Izard, Spelke, and Pica [2008] found that education has a "linearizing" effect on the mental number scale, as in Amazonian indigenous cultures the mental number line is found to be logarithmic, while in the Western societies it was found to be closer to the linear.

in the brain, as Hsu, Krajbich, Zhao and Camerer [2009] found that the neural activity in the striatum is nonlinear in stimulus probabilities. Psychologists have accumulated further evidence that humans tend to evaluate proportions (such as probabilities, proportions of graphical elements, numerosities, an so on) in a non-linear fashion, involving accurate evaluations at the "intermediate" proportions, overestimation of low and underestimate of large proportions (see Spence [1990] and Hollands and Dyre [2000]).

2.3 The Empirical Significance of Stimuli vs. Their Physical Properties

Robson [2001] proposed that the optimal hedonimeter is an empirical distribution of the environmental stimuli. Suppose one faces a choice between two alternatives, which are non-negative numbers independently drawn from a continuous distribution F. In terms of hedonic utility, the organism can only observe whether an alternative is above or below each of the N threshold values. That is, hedonic values (e.g. "High", "Low") are assigned following a series of ordinal comparisons to thresholds. When both draws fall within the same region, they are of the same hedonic value. Thus incorrect choices may occur. When both draws fall within the same region, as determined by adjacent threshold values, they are assigned the same hedonic value, despite being distinct. Robson shows that, with no complexity costs, the overall probability of error is minimized when the threshold values c_i satisfy $F(c_i) = \frac{i}{N+1}$ (e.g. if $N = 99$, each threshold level is a percentile). That is, the optimal hedonic utility function is the distribution function F.

This result was further examined by Netzer [2009][4] who further pointed out that the hedonic discriminability is particularly important for stimuli which are particularly environmentally relevant - that is, the utility function should be more sensitive for the stimuli which are particularly frequent.

Independently, cognitive scientists Yang and Purves [2004] suggested that it is the empirical significance of stimuli, rather than their physical properties, which is important for survival of a living organism. They note that the same visual aspect of an environment (such as luminance) in different contexts results in different perception of that aspect, and propose that the visual system does not record the physical aspects of stimuli, but instead their statistical relevance as represented by their relative rank in the distribution of stimuli.

[4] Netzer [2009], in addition, proposed an alternative fitness-maximizing utility function which also depends on the environment

2.4 Utility as Rank in a Distribution

Among the pioneers of context-dependent evaluation, psychologist Parducci [1963, 1965] developed his range-frequency theory to explain qualitative, but not quantitative, similarities between subjects' magnitude evaluations and veridical reference distributions. As the curvature of the observed magnitude evaluations were "flatter", less pronounced, than that of the veridical distribution, he modeled the magnitude evaluation as a weighted sum of a veridical rank and a rank in a uniform distribution with the same support.[5]

Independently, the isomorphism of a utility function and a cumulative density function was first formally explored by Van Praag [1968], who developed a formal model of consumer behavior where a choice over a large number of consumption goods resulted in the money utility being isomorphic to a normal distribution. This idea was later extended by Kapteyn [1985] (and references therein) who suggested that money utility is c.d.f of an income distribution.

Castagnoli and LiCalzi [1996] suggested to compare lotteries by the expectation of one lottery outperforming the other in the presence of a given prior.[6]

The case-based decision framework pioneered by Gilboa and Schmeidler [1995] utilizes a database of past experiences to make future decisions. This framework was extended in Gilboa and Schmeidler [2003] to show, as a special case, that one can use empirical frequencies to rank an alternative to be "at least as likely" as another.

Clark, Masclet and Villeval [2006] report the experimental and survey data suggest that one's rank in the income distribution is a strong determinant of effort. Brown, Gardner, Oswald and Qian [2008] found that one's rank in salary distribution is a strong determinant of job satisfaction and can predict job separation.

2.5 Ranking by Pairwise Comparisons

Rubinstein [1980] considered ranking participants in a tournament by counting the number of times a given player beats another player in a tournament. Landau [1951] used trichotomous outcomes of pairwise tournaments to describe societies' dominance structures.

Psychologists Stewart, Chater and Brown [2006] suggested that, using a series of binary, ordinal comparisons, individuals evaluate a magnitude by its rank in a sample drawn from one's memory.[7] They show that this "decision by sampling" procedure may

[5] The "flattened" evaluations observed by Parducci [1963, 1965] were among the motivating factors behind the present imperfect adaptive evaluation model.

[6] See also Kacelnik and Abreu [1998] for a similar model of risky choice with perceptual imperfections.

[7] Stewart, Chater and Brown [2006] refer to the formal model of magnitude evaluation independently

account for a number of observed regularities involving money, time, and probability, including concave utility functions, losses looming larger than gains, hyperbolic time discounting, and the overestimation of small probabilities and the underestimation of large probabilities.

A number of studies further support the decision-by-sampling theory. Olivola and Sagara [2009] argue against context-independent utility functions, and show that empirical distribution of death tolls affects experimental subjects' decisions involving human fatalities in hypothetical situations, and moreover, consistent with cross-country differences in environmental contexts, subjects' choices vary across countries. Wood, Brown and Maltby [2011] found that gratitude is affected by the magnitude distribution of help others provide. Stewart [2009] and Ungemach, Stewart and Reimers [2011] found that evaluations by experimental subjects depend on the relative rank of magnitudes in the reference sets. Stewart, Reimers, and Harris [2011] further argue that since utility, probability weighting and temporal discounting functions are context-dependent, they cannot be stable primitives of economic decision making.

2.6 Relative Comparisons

There is plenty of evidence that people compare their possessions relatively to what others have (see, for example, Frank [1985], Clark and Oswald [1996], Solnick and Hemenway (1998), and Neumark and Postlewaite [1998]). Nor are moral judgements absolute - as students were found to be appalled by the idea of poisoning a neighbour's dog, yet judged it as a petty crime when poisoning the neighbour herself was on the list (Parducci [1968]). Even evaluation of painful experience is not absolute as it varies with different sequences of pain (Redelmeier and Kahneman [1996]). And so are the judgements of size, weight, or numerousness (Parducci [1963, 1965]), as well as visual perception (Yang and Purves [2004]). Furthermore, the brain studies suggest that the anticipated reward magnitudes are coded in relative, and not absolute, terms (Seymour and McClure [2008] and references therein).

Thus, nearly everything that enters one's mental "in-box" is evaluated against a reference set, or its specific elements. This observation led economists and psychologists to develop a number of reference-dependent theories, including the "keeping up with the Joneses" theory of Duesenberry [1949], the adaptation-level theory of Helson [1948], the range-frequency theory of Parducci [1963, 1965], neo-cardinal theories of Van Praag [1968] and Kapteyn [1985 and references therein], the social rank concerns of Frank [1985], the prospect theory of Kahneman and Tversky [1979], the endogenous reference points of Koszegi and Rabin [2006], and the decision by sampling of Stewart, Chater and Brown [2006].

developed in Kornienko [2004].

3 The Adaptive Model of Magnitude Evaluation

Consider Robinson Crusoe on a desert island. There is a single coconut tree on the island, and Crusoe can see coconuts in a variety of sizes on that tree. One day, a coconut falls off the tree. How could Crusoe evaluate the size of this coconut? Clearly, if the coconut is the smallest one among the ones he can see, it is the least valuable. If instead it is the largest one, it is the most valuable. If it is neither smallest nor largest, its evaluation is in-between. I argue that Crusoe can construct an evaluation function using only two cognitive tools: first, by conducting *pairwise comparison* "tournaments" between the *target* coconut and each of the coconuts which he can see on the tree, and second, by evaluating the target magnitude by the *frequency*, or *proportion* of pairwise comparison tournaments the target coconut "wins".

As the adaptive evaluation procedure is context-dependent, Robinson Crusoe's evaluation is based on the size distribution of coconuts on his island, and will change if he were on another island. For example, Crusoe would be less happier with a 3 inch coconut on an island A where a typical coconut is 5 inch radius rather than on another island B where a typical coconut is 2 inch radius - simply because coconuts smaller than 3 inch are more frequent on island B. Thus it is possible to find a banana such that Crusoe will be happy to trade the 3-inch coconut for this banana on island A, but on island B he would be happy to trade this banana back for the 3-inch coconut - an apparent preference reversal.[8]

More formally, consider an *environment* S consisting of magnitudes of (potentially) observable objects (e.g. sizes of physical goods such as coconuts, houses, cars, incomes, and so on, or magnitudes of a characteristic such as height, beauty, intelligence, etc.), with veridical (true) distribution $F : S \to [0, 1]$. An individual observes a *sample* $\breve{S}_{N+1}$ of $N + 1$ observations which are independently and identically drawn from set S with distribution F. To evaluate the target magnitude $x \in \breve{S}_{N+1}$ he uses the remaining magnitudes y in the *reference set* $S_N = \breve{S}_{N+1} \setminus \{x\}$ using the following algorithm.

Adaptive Evaluation Algorithm *Suppose an individual faces a reference set S_N and is endowed with ordinal comparison and frequency (proportion) processing tools. Then the evaluation of the target magnitude x can be constructed as follows:*

Step 1 *Using the ordinal comparison tool, compare the "target" magnitude x to every magnitude y in the reference set S_N - that is, judge whether each is larger or smaller than x. Whenever y appears to be similar/equal to x (i.e. $y \sim x$), y is judged to be smaller with a probability p, and larger with probability $1 - p$.*

[8] If Robinson Crusoe is destined to have the same coconut (or height, intelligence, and so on) for life, he would choose the island where his coconut wins the pairwise tournaments most frequently. Yet, the adaptive evaluation model is silent regarding his choice over islands when other coconuts (income, consumption) could also be available.

Step 2 *Using the frequency (proportion) processing tool, estimate how often every $y \in S_N$ is judged to be smaller, or $\frac{N_{y<x}+\hat{N}^+_{y\sim x}}{N}$, where $\hat{N}^+_{y\sim x}$ is the realized count of cases when a reference magnitude y appears to be similar/equal to x but is judged to be smaller.*

Step 3 *Evaluate the magnitude x relatively to the reference set S_N by the proportion (or frequency of occurrence) of reference magnitudes y which are judged to be smaller:*

$$\hat{I}_{S_N}(x) = \frac{N_{y<x} + \hat{N}^+_{y\sim x}}{N} \tag{1}$$

The *realized evaluation* $\hat{I}_{S_N}(x)$ is a *perceived rank* of magnitude x in reference set S_N. It is stochastic whenever the reference set S_N contains elements which appear to have similar magnitudes. If the individual makes many evaluations of magnitude x using the above algorithm, on average the proportion of similar reference magnitudes which are judged to be smaller is p. Let us define the *adaptive evaluation* of magnitude x to be the expected outcome of pairwise comparison, or the expected realized evaluation of this magnitude:[9]

$$I_{S_N}(x) = E\hat{I}_{S_N}(x) = \frac{N_{y<x}}{N} + p\frac{N_{y\sim x}}{N} \tag{2}$$

Note that one can write the realized evaluation as $\hat{I}_{S_N} = I_{S_N} + \epsilon$, where ϵ is some random variable. Following the random utility models pioneered by Thurstone [1927], the rest of the paper will concentrate on the adaptive evaluation I_{S_N} as a candidate for a utility function.

Example 1 *Let $A = \{10, 11, 12, 12, 98, 98, 98, 98, 98, 99, 100\}$ and suppose any $y = x$ has an equal chance to be judged smaller or larger than y. Then*

$$\hat{I}_A(10) = \frac{0}{10} \quad \hat{I}_A(11) = \tfrac{1}{10} \quad \hat{I}_A(12) \in \left\{\frac{2}{10}; \frac{3}{10}\right\}$$
$$\hat{I}_A(98) \in \left\{\frac{4}{10}; \frac{5}{10}; \frac{6}{10}; \frac{7}{10}; \frac{8}{10}\right\} \quad \hat{I}_A(99) = \tfrac{9}{10} \quad \hat{I}_A(100) = \frac{10}{10} = 1$$

and the adaptive evaluations of each magnitude are

$$I_A(10) = 0; I_A(11) = 0.1; I_A(12) = 0.25; I_A(98) = 0.6; I_A(99) = 0.9; I_A(100) = 1$$

Evidently, the adaptive evaluation $I_{S_N}(x)$ on the reference set S_N is increasing in the magnitude x. Following Tversky and Kahneman [1991], consider complete and transitive "relatively more is better" *preference structure* $\succeq_{S_N}$ where set S_N is a *reference state.*

[9] The adaptive evaluation (2) is isomorphic to a *frequency accumulator* representation $I_{S_N}(x) = 1 \cdot \frac{\hat{N}_{y<x}}{N} + p \cdot \frac{\hat{N}_{y\sim x}}{N} + 0 \cdot \frac{\hat{N}_{y>x}}{N}$. where $p \in [0, 1]$ is the value of a similarity "tie".

This preference structure $\succeq_{S_N}$ is rationalizable by the adaptive evaluation I_{S_N}. As Tversky and Kahneman [1991] pointed out, "reference shift", or change in a reference state, often results in apparent preference anomalies. Thus, when the environment S changes, so does the reference set S_N, followed by a change in the adaptive evaluation $I_{S_N}(x)$, exhibiting non-reflexivity, non-transitivity and preference reversals when the choices are compared across different reference states.

Example 2 *Let* $B = \{10, 11, 12, 12, 12, 12, 12, 98, 98, 99, 100\}$ *and suppose any* $y = x$ *has an equal chance to be judged smaller or larger than* y. *Then*

$$\hat{I}_B(10) = \frac{0}{10} \quad \hat{I}_B(11) = \tfrac{1}{10} \quad \hat{I}_B(12) \in \left\{\frac{2}{10}; \frac{3}{10}; \frac{4}{10}; \frac{5}{10}; \frac{6}{10}\right\}$$
$$\hat{I}_B(98) \in \left\{\frac{7}{10}; \frac{8}{10}\right\} \quad \hat{I}_B(99) = \tfrac{9}{10} \quad \hat{I}_B(100) = \frac{10}{10}$$

Compare the adaptive evaluations I_B *of each magnitude to* I_A *from Example 1:*

$$I_B(10) = 0 = I_A(10) \quad I_B(11) = 0.1 = I_A(11) \quad I_B(12) = 0.3 > I_A(12)$$
$$I_B(98) = 0.75 < I_A(98) \quad I_B(99) = 0.9 = I_A(99) \quad I_B(100) = 1 = I_A(100)$$

That is, the same absolute magnitude may have different adaptive evaluation in different contexts.

The adaptive evaluation $I_{S_N}(x)$ and its realization $\hat{I}_{S_N}(x)$ depend on the composition of the reference set S_N and on the treatment of similar magnitudes p. The next assumption on the ordinal comparison tool, if holds, results in the realized evaluation $\hat{I}_{S_N}$ and the adaptive evaluation I_{S_N} being equal.

Assumption [A1]: *All reference magnitudes* y *which are similar to* x *are judged to be smaller with certainty, i.e.* $p = 1$ *for all* x *and* y.

The next assumption on the sample $\breve{S}_{N+1}$ is satisfied if the environment S is a continuum with a continuous veridical distribution $F(x)$.

Assumption [A2]: *Set* $\breve{S}_{N+1}$ *has no two elements* x *and* y *with equal magnitudes, i.e.* $Pr(y = x) = 0$ *in set* $\breve{S}_{N+1}$ *for any* N.

If at least one of the above assumptions hold, the following important result arises.

Proposition 1 *Suppose either A1 or A2 or both hold, and that only equal magnitudes are perceived to be similar, i.e.* $y \sim x \Leftrightarrow y = x$. *Then* $\hat{I}_{S_N} = I_{S_N}$, *and the adaptive evaluation* $I_{S_N}(x)$ *is isomorphic to the empirical distribution function* $F_N(x)$. *As the reference set* S_N *becomes large, it approaches in the limit to the veridical set* $S = \lim_{N\to\infty} S_N$ *and the evaluation* $I_S(x) = \lim_{N\to\infty} I_{S_N}(x)$ *converges uniformly to the veridical (true) distribution* $F(x)$. *Moreover,* $I_S(x)$ *is optimal in the sense of Robson [2001], i.e. a chance of mistaken choice of a smaller magnitude in a binary choice is minimized.*

Proof: By definition, the ordinal rank of x in S_N is given by the *empirical distribution function* $F_N(x) = \frac{N_{y\leq x}}{N} = \frac{N_{y<x}+N_{y=x}}{N}$, or, $F_N(x)$ is the frequency of magnitudes in S_N that do not exceed x. If A1 holds, then by assumption $p = 1$ for all x and y whenever $y \sim x \Leftrightarrow y = x$, so that $\hat{N}^{+}_{y\sim x} = N_{y=x}$. If A2 holds, $N_{y\sim x} = N_{y=x} = 0$ for any p. In either case, $I_{S_N}(x) = \hat{I}_{S_N}(x) = \frac{N_{y<x}+N_{y=x}}{N} = \frac{N_{y\leq x}}{N} = F_N(x)$. As all observations in $S_N \subset \breve{S}_{N+1}$ are i.i.d. draws from $F(x)$, the rest follows from Glivenko-Cantelli Theorem (e.g. Durrett [1996]). As Robson [2001] and Netzer [2009] showed, a mistake of choosing a smaller magnitude from a set of two draws from an environment $F(x)$ is minimized when the utility function equals to $F(x)$. ∎

That is, if either the ordinal comparison tool satisfies assumption A1, or if all reference magnitudes in the set S_N are distinct, the adaptive evaluation I_{S_N} is isomorphic to an empirical distribution, or cumulative density function. This isomorphism of a utility function and a cumulative density function was first noticed by Van Praag [1968], and further explored by Kapteyn [1985 and references therein]. Gilboa and Schmeidler [2003] extended the case-based decision theory to represent "at least as likely" binary relation with ranking alternatives by their empirical frequencies. Stewart, Chater and Brown [2006] suggested informally that one can evaluate attribute values by their ordinal rank in a distribution of a sample from memory - whether the attribute is money amounts, time, or probability.

While the logic behind the model presented here works for both discrete and continuous reference distributions, the continuous case is expositionally simpler. Let the environment be $S = (a, b) \subseteq R$, and the veridical distribution be a continuously differentiable distribution function F on S, with $F' > 0$ on S. Consider a large sample $\breve{S}_{N+1}$ such that $\lim_{N\to\infty} \breve{S}_N = \lim_{N\to\infty} S_N = S$, so that, by Proposition 1, $\lim_{N\to\infty} F_N = F(x)$. It is easy to see that (cumulative density) function $F(x)$ is a *frequency tool* as $F(x) = Pr(y \leq x)$, or equal to the frequency with which y is less or equal than x. To define the *ordinal comparison tool*, let an expected outcome of pairwise comparison of a fixed magnitude x with any magnitude y be represented by the *pairwise comparison function* $\mathcal{D}(x, y) : S \to [0, 1]$, which is non-decreasing in x and non-increasing in y, and is discontinuous at a countable number of points. Then, for any magnitude $x \in S$, the adaptive evaluation $I_S(x)$ of x against reference set S is parsimoniously constructed with two basic cognitive tools as the expected perceived outcome of pairwise comparison tournament of x against every element y in reference set S, or the perceived rank of x in S:[10]

$$I_S(x) = \int_S \underbrace{\mathcal{D}(x, y)}_{ordinal\ tool} \underbrace{dF(y)}_{frequency\ tool} \tag{3}$$

Here, the ordinal comparison tool is the integrand (kernel of integration), and the frequency (or proportion) tool is the variable of integration. The next result follows from the assumptions on the pairwise comparison function $\mathcal{D}$.

[10]Formally, the adaptive evaluation model (9) is closely related to Castagnoli and LiCalzi [1996]'s (expected) probability model.

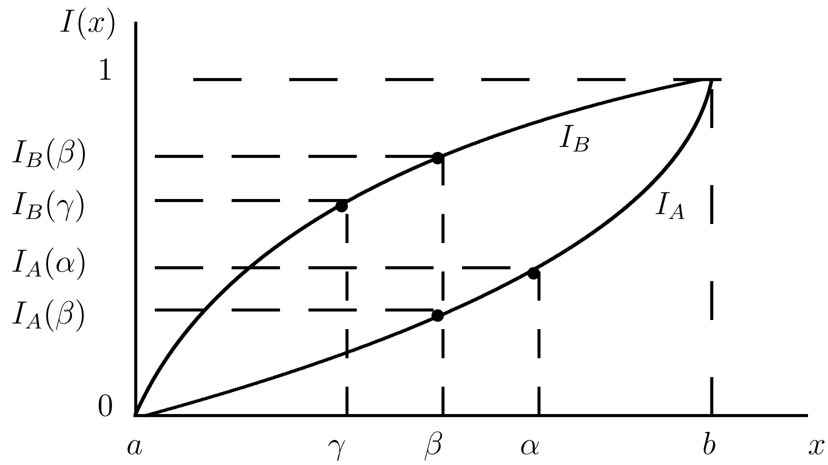


Figure 1: Violations of reflexivity and transitivity across different reference sets for $F_A \succeq_{FOSD} F_B$.

Proposition 2 *Suppose an individual is endowed with a frequency (proportion) tool F and a pairwise comparison tool $\mathcal{D}(x, y)$, which is non-decreasing in x, non-increasing in y, and has a finite number of discontinuities. Then*

(i) $I_S(x)$ is a non-decreasing continuous function on set S, and thus rationalizes continuous "more is better" preferences.

(ii) if the reference set A stochastically dominates set B, i.e. $F_A \succeq_{FOSD} F_B$, then $I_B(x) \geq I_A(x)$ for any $x \in X$.

Proof: (i) Since $\mathcal{D}$ is increasing in x and is discontinuous at finitely many points, $I_S(x)$ is non-decreasing and continuous. The preference rationalizability is a standard result.

(ii) By definition of first order stochastic dominance, for any non-decreasing function $U(t)$, $\int_A U(t)dF_A(t) \geq \int_B U(t)dF_B(t)$. But since $\mathcal{D}$ is non-increasing in y, we have that

$$I_A(x) = \int_A \mathcal{D}(x, y)dF_A(y) \leq \int_B \mathcal{D}(x, y)dF_B(y) = I_B(x) \quad \blacksquare$$

Thus, the adaptive evaluation $I_S(x)$ can rationalize continuous more is better preferences on S. Further, once the context changes, so does the adaptive evaluation. As Figure 1 shows, both reflexivity and transitivity may be violated. That is, one can find some $\beta \in A \cap B$, such that, first, $I_A(\beta) \neq I_B(\beta)$, and, second, it is possible to find $\alpha \in A$ and $\gamma \in B$ such that $I_A(\alpha) > I_A(\beta)$ and $I_B(\beta) > I_B(\gamma)$, but $I_A(\alpha) < I_B(\gamma)$. This has an important consequence. Suppose there exist an "outside" object with exogenously determined valuation $I_Y(\delta)$ where $(A \cup B) \cap Y = \emptyset$ such that $I_A(\beta) < I_Y(\delta) < I_B(\beta)$. Then, an individual would trade β for δ when reference set is A, but once reference set changes to B, he would trade δ back for β - an apparent preference reversal.

This systematic effect of a change in the reference set on subjects' evaluations has been documented in a number of psychological studies, including Parducci [1963, 1965], Stewart [2009], Olivola and Sagara [2009], Wood, Brown, and Maltby [2011], Ungemach, Stewart, and Reimers [2011], Stewart, Reimers, and Harris [2011].

4 Adaptive Evaluation with Ordinal Imperfections

The present model allows one to clarify the role of cognitive imperfections on magnitude evaluation. As psychologists discovered in the XIX century, humans tend not to notice the difference between two relatively similar magnitudes. The widely known Weber-Fechner psychophysical law (e.g. Laming [1973]) is the psychologist's counterpart to the law of diminishing marginal utility, and it states that the minimum amount by which stimulus intensity must be changed in order to produce a noticeable variation increases with the stimulus level.

Such findings open up a possibility that the ordinal comparison tool may be imperfect in the sense that any two magnitudes which belong to an interval of *doubt* are *perceived* to be similar. Such ordinal imperfections may lead to smaller magnitudes having higher realized evaluations than larger magnitudes. Thus, due to the stochastic nature of ordinal imperfections, apparent preference anomalies can occur even for the same reference set.

Example 3 *Consider the environments specified in Examples 1 and 2. Suppose the individual perceives* $10 \sim 11 \sim 12$ *and* $98 \sim 99 \sim 100$ *and suppose any* $y = x$ *has an equal chance to be judged smaller or larger than* y*. Then the sample* A *is perceived to be* $A' = \{11, 11, 11, 11, 99, 99, 99, 99, 99, 99, 99\}$ *and thus*

$$\hat{I}_{A'}(x) \in \left\{\frac{0}{10}; \frac{1}{10}; \frac{2}{10}; \frac{3}{10}\right\} \quad \textit{for} \quad x \in \{10, 11, 12\}$$
$$\hat{I}_{A'}(x) \in \left\{\frac{4}{10}; \frac{5}{10}; \frac{6}{10}; \frac{7}{10}; \frac{8}{10}; \frac{9}{10}; \frac{10}{10}\right\} \quad \textit{for} \quad x \in \{98, 99, 100\}$$

Thus the realized evaluation can be non-monotone in magnitude, e.g. $\hat{I}_{A'}(12) < \hat{I}_{A'}(11) < \hat{I}_{A'}(10) < \hat{I}_{A'}(100) = \hat{I}_{A'}(99) = \hat{I}_{A'}(98)$*. However, the adaptive evaluations are non-decreasing in magnitude:*

$$I_{A'}(10) = I_{A'}(11) = I_{A'}(12) = 0.15$$
$$I_{A'}(98) = I_{A'}(99) = I_{A'}(100) = 0.7$$

Similarly, B is perceived to be $B' = \{11, 11, 11, 11, 11, 11, 11, 99, 99, 99, 99\}$ and thus

$$\hat{I}_{B'}(x) \in \left\{ \frac{0}{10}; \frac{1}{10}; \frac{2}{10}; \frac{3}{10}; \frac{4}{10}; \frac{5}{10}; \frac{6}{10} \right\} \quad \textit{for} \quad x \in \{10, 11, 12\}$$
$$\hat{I}_{B'}(x) \in \left\{ \frac{7}{10}; \frac{8}{10}; \frac{9}{10}; \frac{10}{10} \right\} \quad \textit{for} \quad x \in \{98, 99, 100\}$$

so that the adaptive evaluations are:

$$I_{B'}(10) = I_{B'}(11) = I_{B'}(12) = 0.3$$
$$I_{B'}(98) = I_{B'}(99) = I_{B'}(100) = 0.85$$

Here, relatively to the "true" evaluation in Examples 1 and 2, the imperfect ordinal comparison tool leads to overvaluing of "small" magnitudes (10 and 11), undervaluing of "large" ones (99 and 100). Since large magnitudes are more frequent in set A' than in set B', all magnitudes are more valuable in set B'.

To understand the complex interactions between ordinal imperfections and the reference set, consider again a large sample from a continuous distribution $F(x)$, and suppose that an individual is endowed with a *perfect* frequency tool, perceiving the distribution of the reference magnitudes to be F. Suppose that the outcome of the pairwise comparison between x and y is determined with certainty whenever x and y are sufficiently far apart, but the ordinal discrimination is stochastic whenever x and y are sufficiently close. Let us suppose that the ordinal discrimination depends the ordinal discriminability variable $z = Z(x, y)$ with $Z_1 > 0, Z_2 < 0$ (e.g. $z = x + y$ or $z = \frac{x}{y}$). Let $G(z)$ be the probability that x is perceived to be no smaller when compared to y whenever z falls into the interval of doubt $K = [k_1, k_2]$. Then, the outcome of the imperfect ordinal comparison tournament is:

$$\mathcal{D}^Z(x, y) = \begin{cases} 0 & \text{whenever} \quad z < k_1 \\ G(z) & \text{whenever} \quad z \in [k_1, k_2] \\ 1 & \text{whenever} \quad z > k_2 \end{cases} \tag{4}$$

and the imperfect adaptive evaluation $I_S(x)$ can be written as the expected outcome of pairwise comparison:

$$I_S^Z(x) = \int_S \mathcal{D}^Z(x, y) dF(y) = \int_S G(Z(x, y)) dF(y) \tag{5}$$

The imperfect adaptive evaluation $I_S^Z(x)$ depends on the form of discriminability $z = Z(x, y)$ and will be explored below.

4.1 Adaptive Evaluation with Perfect Cognitive Tools

Adaptive evaluation with perfect cognitive tools provides a useful benchmark for the subsequent analysis. Suppose an individual is endowed with a *perfect* frequency (or proportion) tool, so that he correctly perceives the veridical distribution of magnitudes F. Suppose further he is endowed with a *perfect* ordinal comparison tool, i.e. he can tell a bigger from a smaller magnitude even if the magnitudes are only slightly different. In this case, an expected outcome of a pairwise comparison can be written as:[11]

$$\mathcal{D}^P(x,y) = \begin{cases} 0 & \text{whenever} \quad x < y \\ c & \text{whenever} \quad x = y \\ 1 & \text{whenever} \quad x > y \end{cases} \tag{6}$$

where $c \in [0,1]$. This perfect ordinal comparison tool permits two isomorphic representations as a degenerate discriminability z. First, perfect ordinal tool $\mathcal{D}^P$ can be represented in terms of a difference $z = x - y$, degenerate at 0, with $G(z) = H(z) = H(x-y)$, where $H(\cdot)$ is Heaviside (step) function.[12] Thus,

$$\mathcal{D}^P(x,y) = H(x-y) \tag{7}$$

Second, $\mathcal{D}^P$ can be represented in terms of a ratio $z = \frac{x}{y}$, degenerate at 1, with $G(z) = H(z-1) = H\left(\frac{x}{y} - 1\right)$, so that

$$\mathcal{D}^P(x,y) = H\left(\frac{x}{y} - 1\right) \tag{8}$$

Since $H\left(\frac{x}{y} - 1\right) = H(x-y)$, the formulations (7) and (8) represent the same degenerate random variable. This allows us to derive the following benchmark result for a continuous reference distribution.

Proposition 3 *Suppose an individual is endowed with a perfect ordinal comparison tool (6) and with a perfect frequency (proportion) tool F. Then his adaptive evaluation of magnitude x is given by the cumulative density function of the veridical (true) magnitude distribution $F(x)$:*

$$I_S^P(x) = F(x) \tag{9}$$

Moreover, the perfect adaptive evaluation is optimal in the sense of Robson [2001].

Proof: Using representations (7) and (8), get

$$I_S^P(x) = \int_S \mathcal{D}^P(x,y)dF(y) = \int_S H\left(x-y\right)dF(y) = \int_S H\left(\frac{x}{y} - 1\right)dF(y) = F(x)$$

The optimality follows from Robson [2001] and Netzer [2009]. ∎

[11] This perfect ordinal tool is similar to the optimal happiness function of Rayo and Becker [2007].

[12] A degenerate random variable z at c has a cumulative distribution equal to Heaviside (step) function $H(z-c)$, with $H(z-c) = 0$ for $z < c$, $H(z-c) = 1$ for $z > c$, and $H(c) \in [0,1]$. It has Dirak delta density function $\delta(x-c)$, and domain $(-\infty, \infty)$. For an arbitrary distribution F, $\int_{-\infty}^{\infty} H(c-z)dF(z) = F(c)$.

That is, if both ordinal comparison and frequency (proportion) tools are perfectly accurate, an individual's evaluation $I_S^P(x)$ of magnitude x is isomorphic to the veridical cumulative density function (veridical rank) $F(x)$, and is equal to the frequency of a magnitude x ordinarily "outperforming" other elements in the reference set S. Moreover, as it was shown by Robson [2001] and Netzer [2009], if the adaptive utility is $F(x)$, the probability of mistake in a binary choice in an environment S is minimized.

When cognitive tools are perfect, the shape of the magnitude distribution entirely determines the shape of the utility function. For example, if the veridical magnitude distribution is of the power function form, individual's adaptive evaluation is consistent with a constant relative risk aversion (CRRA) utility function; and if the veridical distribution is exponential, the evaluation exhibits constant absolute risk aversion (CARA). [13]

As the rest of the paper will show, cognitive imperfections result in an adaptive evaluation which is functionally distinct from the veridical distribution F, and thus suboptimal in the sense of Robson [2001].

4.2 Additively Imperfect Ordinal Comparison Tool

Let the ordinal discriminability z be in terms of differences, i.e. $z = x - y \in [k_1, k_2]$, with $k_1 \leq 0 \leq k_2$. For technical simplicity, assume that the interval of doubt is "small", i.e. $k_2 - k_1 < b - a$. Whenever $z > 0$, a reference magnitude y "looms large", with the ordinal tournament assessment of magnitude x being biased downwards. And vice versa, whenever $z < 0$, a reference magnitude y "looms small", and the evaluation of x is boosted upwards. The resulting adaptive evaluation $I_S(x)$ is isomorphic to a *convolution* of a reference variable y and an ordinal discriminability variable z:

$$I_S^A(x) = \int_S \mathcal{D}^A(x, y) dF(y) = \int_S G\left(x - y\right) dF(y) \tag{10}$$

Obviously, the perfect comparison tool (7) is a special case of the above.

Proposition 4 *Suppose the expectation of ordinal discriminability z is zero, i.e. $Ez = 0$. Then, the additively imperfect adaptive evaluation of $I^A(x)$ cross the veridical adaptive evaluation $I^P(x)$ once, and from above.*

Proof: The reference magnitude Y second order stochastically dominates the additively imperfect reference magnitude $Y + Z$ (or $Y \geq_{SOSD} Y + Z$), and the result is straightforward (see Shaked and Shantikumar [2007], Theorems 3.A.5 and 3.A.34). ∎

[13] See Castagnoli and Li Calzi [1996] for these relationships between distribution functions and the shape of utility function.

In other words, whenever ordinal discriminability z has zero mean, the low values of x are adaptively overvalued, and the high values are undervalued. For example, if the veridical (true) reference distribution is normal $N(\mu_x, \sigma_x^2)$, and the ordinal discriminability variable z is also normal $N(0, \sigma_z^2)$, then the additively imperfect adaptive evaluation of x is the rank in the convolution distribution, which is normal $N(\mu_x, \sigma_x^2 + \sigma_z^2)$, having "fatter" tails than the veridical (true) distribution. This "fatter tail" result holds for most reference distributions.

Proposition 5 *Suppose an individual is endowed with a perfect frequency (proportion) tool F, and an additively imperfect ordinal comparison tool D^A (10) with the ordinal discriminability variable z distributed with G on $[k_1, k_2]$ with $k_1 < 0 < k_2$ and $k_2 - k_1 < b - a$. Then $I_S^A(a) > 0$ and $I_S^A(b) < 1$ as long as $g = G' > 0$ on $[k_1, k_2]$ and $a > -\infty$ and $b < \infty$.*

Proof: For small interval of doubt $k_2 - k_1 < b - a$, an additively imperfect evaluation $I_S^A(x)$ (10) is:

$$I_S^A(x) = \begin{cases} \int_a^x \int_a^{t-k_1} g(t-y)dF(y)dt & \text{if } \ a \leq x \leq a + k_2 \\ \int_{a+k_2}^x \int_{t-k_2}^{t-k_1} g(t-y)dF(y)dt \quad + I_S^A(a+k_2) & \text{if } \ a + k_2 < x \leq b + k_1 \\ \int_{b+k_1}^x \int_{t-k_2}^b g(t-y)dF(y)dt \quad + I_S^A(b+k_1) & \text{if } \ b + k_1 < x \leq b \end{cases} \tag{11}$$

Thus, $I_S^A(a) = \int_{a+k_1}^a \int_a^{t-k_1} g(t-y)dF(y)dt$ is strictly positive as long as $a > -\infty$, and $I_S^A(b) = 1 - \int_b^{b+k_2} \int_{t-k_2}^b g(t-y)dF(y)dt$ is strictly less than one as long as $b < \infty$. ∎

Thus, ordinal imperfection alter the shape of magnitude evaluation, obscuring the relationship between context and evaluation, as an imperfect additively imperfect evaluation may not be a distribution function. Ordinal imperfections tend to result in overvaluation of low magnitides and undervaluation of high magnitudes because, when evaluating small objects, the individual incorrectly perceives the existence of objects that are smaller than the smallest object in the set S, and, similarly, he imagines non-existing large objects when evaluating large magnitudes.

Example 4 *Suppose the reference distribution F is uniform on $[\mu - \sigma; \mu + \sigma]$, and distribution of ordinal discriminability $G(z)$ is also uniform on $[k_1, k_2]$ with $k_2 - k_1 < 2\sigma$. Then*

$$I_S^{AU}(x) = \begin{cases} \frac{(\mu-\sigma+k_1-x)^2}{4\sigma(k_2-k_1)} & \textit{if } \ \mu - \sigma \leq x \leq \mu - \sigma + k_2 \\ \frac{x-\mu+\sigma-\frac{k_1+k_2}{2}}{2\sigma} & \textit{if } \ \mu - \sigma + k_2 < x \leq \mu + \sigma + k_1 \\ 1 - \frac{(\mu+\sigma+k_2-x)^2}{4\sigma(k_2-k_1)} & \textit{if } \ \mu + \sigma + k_1 < x \leq \mu + \sigma \end{cases}$$

Thus, in addition to overvaluing small and undervaluing large magnitudes, the evaluation exhibits increasing marginal utility for small magnitudes, decreasing marginal utility for large magnitudes, and constant marginal utility in-between.

4.3 Multiplicatively Imperfect Ordinal Comparison Tool

Let us consider a generalization of Rubinstein's [1988] similarity model (which can be seen as a form of psychophysical Weber-Fechner law). Let $a \geq 0$ and suppose the ordinal discriminability z be in terms of ratios, i.e. $z = \frac{x}{y} \in [k_1, k_2]$, with $0 < k_1 \leq 1 \leq k_2$. That is, the individual can tell apart two distinct magnitudes x and y with certainty only if the ratio of magnitudes $z = \frac{x}{y}$ is outside of the interval of doubt $[k_1, k_2]$, but within this interval an individual can only tell that x is greater than y with some probability $G(z)$. For technical simplicity, assume that the interval of doubt is "small", i.e. $\frac{k_2}{k_1} < \frac{b}{a}$. Whenever $z > 1$, a reference magnitude y "looms large", with the ordinal judgment of magnitude x being biased downwards. And vice versa, whenever $z < 1$, a reference magnitude y "looms small", and the judgment of x is boosted upwards.

The resulting adaptive evaluation $I_S(x)$ is isomorphic to a *scale mixture* of a reference variable y and an ordinal discriminability variable z:

$$I_S^M(x) = \int_S \mathcal{D}^M(x, y) dF(y) = \int_S G\left(\frac{x}{y}\right) dF(y) \qquad (12)$$

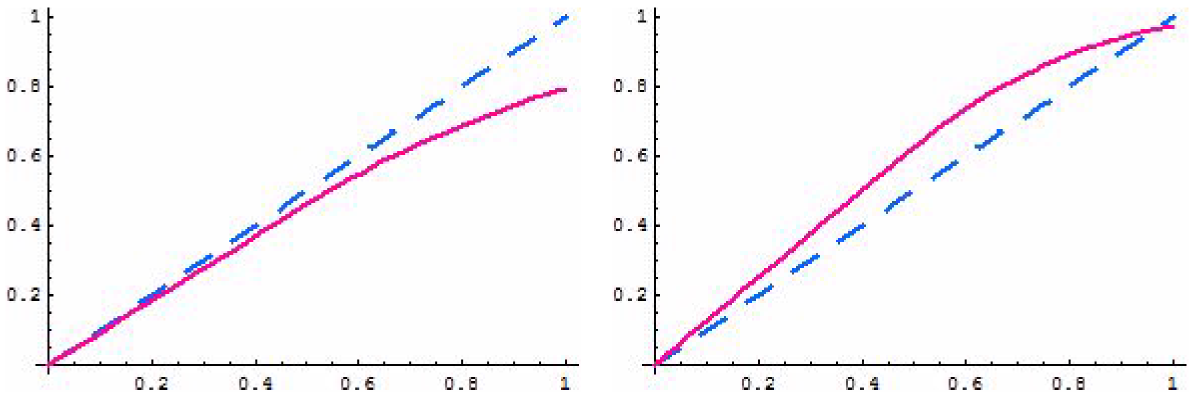
Obviously, the perfect comparison tool (8) as a special case of the above. This leads us to the following interesting result.

Proposition 6 *Suppose the veridical distribution F is uniform on $(0,1)$. Then, the multiplicatively imperfect ordinal tool results in a concave magnitude evaluation.*

Proof: This follows from the converse to Khinchine's representation for unimodal distributions and the definition of unimodal distributions (see Dharmadhikari and Joag-dev [1988], Theorem 1.3). ∎

That is, with an *arbitrary* multiplicative discriminability, one's adaptive evaluation with the uniform reference distribution on $(0,1)$ will tend to exhibit non-increasing marginal utility. For example, as Figure 2 shows, evaluation relatively to the uniform reference distribution on $(0,1)$ with a uniform multiplicative tool, will exhibit constant marginal utility for the lower range of veridical magnitudes and decreasing marginal utility for the higher range. The following statement shows that the "fatter tail" result holds for multiplicatively imperfect ordinal tool as well.

Proposition 7 *Suppose an individual is endowed with a perfect frequency (proportion) tool F, and a multiplicatively imperfect ordinal comparison tool D^M (12) with ordinal discriminability z distributed with G on $[k_1, k_2]$ with $0 < k_1 < 1 < k_2$ and $\frac{k_2}{k_1} < \frac{b}{a}$. Then $I_S^M(a) > 0$ and $I_S^M(b) < 1$ as long as $g = G' > 0$ on $[k_1, k_2]$ and $a > -\infty$ and $b < \infty$.*



a) $x \sim U(0,1)$, $z \sim U[0.5, 2.0]$ b) $x \sim U(0,1)$, $z \sim U[0.5, 1.2]$

Figure 2: Relatively to the uniform veridical distribution on $(0,1)$ (dashed lines) the multiplicatively imperfect evaluation (solid curves) exhibits non-increasing diminishing marginal utility.

Proof: For small interval of doubt $\frac{k_2}{k_1} < \frac{b}{a}$, adaptive evaluation $I_S^M(x)$ is:

$$I_S^M(x) = \begin{cases} \int_{k_1 a}^{x} \int_a^{\frac{t}{k_1}} \frac{1}{y} g\left(\frac{t}{y}\right) dF(y)dt & \text{if } \ a \le x \le k_2 a \\ \int_{k_2 a}^{x} \int_{\frac{x}{k_2}}^{\frac{x}{k_1}} \frac{1}{y} g\left(\frac{t}{y}\right) dF(y)dt \quad +I_S^M(k_2 a) & \text{if } \ k_2 a < x \le k_1 b \\ \int_{k_1 b}^{x} \int_{\frac{t}{k_2}}^{b} \frac{1}{y} g\left(\frac{t}{y}\right) dF(y)dt \quad +I_S^M(k_1 b) & \text{if } \ k_1 b < x \le b \end{cases}$$

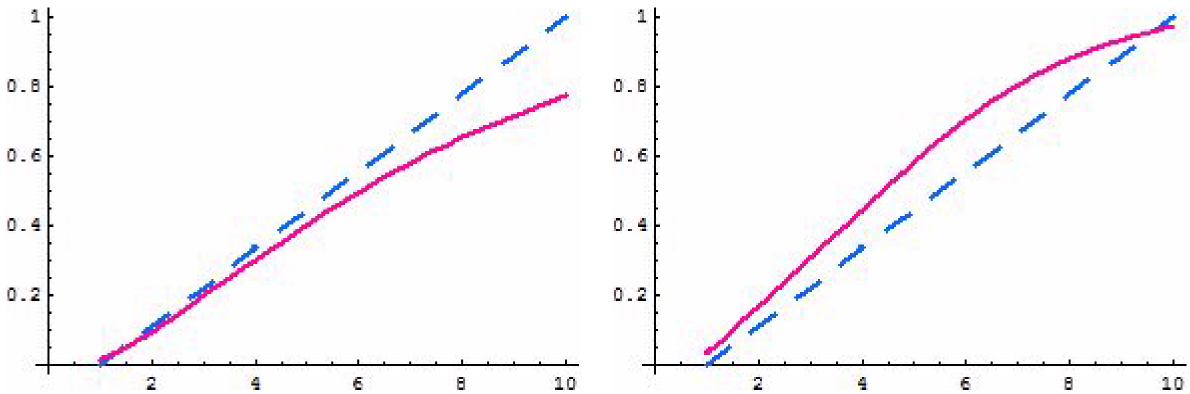
Thus, $I^M(a) = \int_{k_1 a}^{a} \int_a^{\frac{t}{k_1}} \frac{1}{y} g\left(\frac{t}{y}\right) dF(y)dt$ is positive as long as $a > 0$, and $I^M(b) = 1 - \int_b^{k_2 b} \int_{\frac{t}{k_2}}^{b} \frac{1}{y} g\left(\frac{t}{y}\right) dF(y)dt$ is strictly less than one as long as $b < \infty$. ∎

That is, whether imperfect ordinal tool is additive or multiplicative, in general, the imperfect adaptive evaluation is not a distribution function. This is because the ordinal imperfections lead to the individual incorrectly perceiving the existence of magnitudes which are outside of the reference set, leading to overvaluation of small magnitudes and undervaluation of large ones.

Example 5 *Suppose the reference distribution F is uniform on $[\mu-\sigma; \mu+\sigma]$, and distribution of multiplicative ordinal discriminability $G(z)$ is also uniform on $[k_1, k_2]$ with $\frac{k_2}{k_1} < \frac{\mu+\sigma}{\mu-\sigma}$. Then*

$$I_S^{MU}(x) = \begin{cases} \frac{1}{2\sigma(k_2-k_1)}\left(k_1(\mu-\sigma) - x + x\ln\frac{x}{k_1(\mu-\sigma)}\right) & \textit{if } \ \mu-\sigma < x < k_2(\mu-\sigma) \\ \frac{1}{2\sigma(k_2-k_1)}\left(-(k_2-k_1)(\mu-\sigma) + x\ln\frac{k_2}{k_1}\right) & \textit{if } \ k_2(\mu-\sigma) \le x \le k_1(\mu+\sigma) \\ 1 - \frac{1}{2\sigma(k_2-k_1)}\left(k_2(\mu+\sigma) - x + x\ln\frac{x}{k_2(\mu+\sigma)}\right) & \textit{if } \ k_1(\mu+\sigma) < x < \mu+\sigma \end{cases}$$

Thus, in addition to overvaluing small and undervaluing large magnitudes, the evaluation exhibits increasing marginal utility for small magnitudes, decreasing marginal utility for large magnitudes, and constant marginal utility in-between (see Figure 3).



a) $x \sim U(1,10)$, $z \sim U[0.5, 2.0]$ b) $x \sim U(1,10)$, $z \sim U[0.5, 1.2]$

Figure 3: Relatively to the uniform veridical distribution on $(1, 10)$ (dashed lines) the multiplicatively imperfect evaluation (solid curves) exhibits overvaluation for lower magnitudes and undervaluation for high magnitides.

As Figure 3a shows, if an individual faces similar ordinal imperfections for upward and downward comparisons, it is possible that undervaluation may occur for most of the veridical range. In contrast, as Figure 3b shows, if an individual is better at distinguishing magnitudes which are less rather than those which are greater than the target magnitude, most of the veridical magnitudes might be overvalued. That is, the overconfidence patterns found by Burks, Carpenter, Goette and Rustichini [2010] might be due to human difficulties in making upward ordinal comparisons.

Multiplicatively and additively imperfect evaluations have some features in common. This is at least in part because one can use a transformation $G\left(\frac{x}{y}\right) = \tilde{G}\left(\ln\frac{x}{y}\right) = \tilde{G}(\tilde{x} - \tilde{y})$, where $\tilde{x} = \ln x$ and $\tilde{y} = \ln y$. While computationally more complex, multiplicative discriminability is more conducive to cognitive non-linearities and has a more solid grounding in cognitive and brain research.

5 Adding Frequency Imperfections

In addition to the well-documented Weber-Fechner law suggesting that human ordinal comparison abilities are imperfect, is it possible that humans may also distort frequencies, albeit the form of frequency imperfections might differ from the form of ordinal imperfections. As Kahneman and Tversky [1979] point out, humans assign non-linear weights to probabilities, overweighing small probabilities and underweighing large ones.[14] Spence [1990] and Hollands and Dyre [2000] found even a more complex phenomenon for proportion judgments over a variety of stimuli, including continuous sensory modalities

[14]See Hsu, Krajbich, Zhao and Camerer [2009] for a comprehensive list of probability weighting models.

such as fullness of the glass, etc. Thus, it is plausible that the frequency tool is also imperfect, further distorting the relationship between the veridical magnitude distribution and adaptive evaluation, and making it more difficult to observe empirically.

Hsu, Krajbich, Zhao and Camerer [2009] suggest that the probability distortions arise at the level of coding in the brain, while Stewart, Chater and Brown [2006] argue that the distortions align with the empirical distribution of frequency-of-occurrence. As the research on probability/proportion imperfections is still in infancy, and the cognitive mechanism behind frequency distortions is not clear. Interestingly, in Dehaene, Izard, Spelke, and Pica [2008], subjects appear to lose count, undercounting large numerosities (particularly for sets of dots and sequences of tones). This opens up a possibility that frequency imperfections arise because of *counting errors*, as in large reference sets, the total number of elements N might be undercounted, as well as the number of smaller items $N_{y<x}$ when magnitude x is large (as it "beats" many reference magnitudes). Such imperfections in numerosity judgments may lead to overestimation of cumulative frequency for small magnitudes and underestimation for large ones.[15]

Example 6 *Consider the environments specified in Examples 1 and 2, with any $y = x$ has an equal chance to be judged smaller or larger than y. For illustrative purposes, suppose numerosities greater than* 6 *are undercounted by* 1. *Then for the sample A,*

$$\hat{I}_A^F(10) = \frac{0}{10-1} \qquad \hat{I}_A^F(11) = \frac{1}{10-1} \qquad \hat{I}_A^F(12) \in \left\{\frac{2}{10-1}; \frac{3}{10-1}\right\}$$
$$\hat{I}_A^F(98) \in \left\{\frac{4}{10-1}; \frac{5}{10-1}; \frac{6}{10-1}; \frac{7-1}{10-1}; \frac{8-1}{10-1}\right\} \qquad \hat{I}_A^F(99) = \frac{9-1}{10-1} \qquad \hat{I}_A^F(100) = \frac{10-1}{10-1}$$

and the adaptive evaluations are:

$$I_A^F(10) = 0 = I_A(10) \quad I_A^F(11) = 0.11 > I_A(11) \quad I_A^F(12) = 0.28 > I_A(12)$$
$$I_A^F(98) = 0.6 > I_A(98) \quad I_A^F(99) = 0.89 < I_A(99) \quad I_A^F(100) = 1 = I_A(100)$$

For the sample B

$$\hat{I}_B^F(10) = \frac{0}{10-1} \qquad \hat{I}_B^F(11) = \frac{1}{10-1} \qquad \hat{I}_B^F(12) \in \left\{\frac{2}{10-1}; \frac{3}{10-1}; \frac{4}{10-1}; \frac{5}{10-1}; \frac{6}{10-1}\right\}$$
$$\hat{I}_B^F(98) \in \left\{\frac{7-1}{10-1}; \frac{8-1}{10-1}\right\} \qquad \hat{I}_B^F(99) = \frac{9-1}{10-1} \qquad \hat{I}_B^F(100) = \frac{10-1}{10-1}$$

and the adaptive evaluations are:

$$I_B^F(10) = 0 = I_B(10) \quad I_B^F(11) = 0.11 > I_B(11) \quad I_B^F(12) = 0.44 > I_B(12)$$
$$I_B^F(98) = 0.72 > I_B(98) \quad I_B^F(99) = 0.89 < I_B(99) \quad I_B^F(100) = 1 = I_B(100)$$

[15]Furthermore, in Dehaene, Izard, Spelke, and Pica [2008] subjects from Amazonian indigenous societies also appear to overcount small numerosities, which might further lead to overcounting the number of smaller items $N_{y<x}$ when magnitude x is small (as it "beats" only a few reference magnitudes).

Combining frequency imperfections with the ordinal imperfections of Example 3 for for the sample A:

$$\hat{I}^F_{A'}(x) \in \left\{ \frac{0}{10-1}; \frac{1}{10-1}; \frac{2}{10-1}; \frac{3}{10-1} \right\} \quad \textit{for} \quad x \in \{10, 11, 12\}$$

$$\hat{I}^F_{A'}(x) \in \left\{ \frac{4}{10-1}; \frac{5}{10-1}; \frac{6}{10-1}; \frac{7-1}{10-1}; \frac{8-1}{10-1}; \frac{9-1}{10-1}; \frac{10-1}{10-1} \right\} \quad \textit{for} \quad x \in \{98, 99, 100\}$$

and the adaptive evaluations with both imperfections are

$$I^F_{A'}(10) = I^F_{A'}(11) = I^F_{A'}(12) = 0.17$$
$$I^F_{A'}(98) = I^F_{A'}(99) = I^F_{A'}(100) = 0.71$$

For the sample B:

$$\hat{I}^F_{B'}(x) \in \left\{ \frac{0}{10-1}; \frac{1}{10-1}; \frac{2}{10-1}; \frac{3}{10-1}; \frac{4}{10-1}; \frac{5}{10-1}; \frac{6}{10-1} \right\} \quad \textit{for} \quad x \in \{10, 11, 12\}$$

$$\hat{I}^F_{B'}(x) \in \left\{ \frac{7-1}{10-1}; \frac{8-1}{10-1}; \frac{9-1}{10-1}; \frac{10-1}{10-1} \right\} \quad \textit{for} \quad x \in \{98, 99, 100\}$$

and the adaptive evaluations are:

$$I^F_{B'}(10) = I^F_{B'}(11) = I^F_{B'}(12) = 0.33$$
$$I^F_{B'}(98) = I^F_{B'}(99) = I^F_{B'}(100) = 0.83$$

Here, the ordinal and frequency imperfections combined follow the pattern of ordinal distortion in Example 3.

More formally, let us assume an imperfect frequency tool $w(F)$ as in Quiggin [1982], with $w' > 0$. Then the adaptive evaluation of magnitude x is

$$\hat{I}_S(x) = \int_S G(Z(x, y))dw(F(y)) \tag{13}$$

That is, the adaptive evaluation is determined by three components: by the context of evaluation F, by the ordinal imperfection G and by the frequency (proportion) weighting $w(F)$, resulting in adaptive evaluation very different from the reference distribution. The next result is straightforward.

Proposition 8 *Suppose an individual's magnitude evaluation is given by (13). With a perfect ordinal comparison tool, his adaptive evaluation of magnitude x is entirely determined by its rank in the weighted magnitude distribution:*

$$I^{FP}_S(x) = w(F(x)) \tag{14}$$

so that $I^{FP}_S(x) \lesseqqgtr I^P_S(x)$ whenever $w(F(x)) \lesseqqgtr F(x)$. Moreover, an adaptive evaluation with imperfect frequency tool is suboptimal in the sense of Robson [2001].

Proof: Using representations (7) and (8), get

$$I_S^{FP}(x) = \int_S \mathcal{D}^P(x,y)dw(F(y)) = \int_S H(x-y)\,dw(F(y)) = \int_S H\left(\frac{x}{y}-1\right)dw(F(y)) = w(F(x)) \quad \blacksquare$$

With perfect ordinal tool, if the frequency imperfections follow the pattern found by Kahneman and Tversky [1979], frequency imperfections tend to boost the adaptive evaluations for relatively small magnitudes and depress evaluations of relatively large magnitudes, further aggravating the distortion caused by the ordinal imperfections.

6 Interaction Between Context and Memory

As Stewart, Chater and Brown [2006] hypothesize, an individual uses a reference set stored in one's memory. It thus may be only partially affected by the current environment, but it can also be affected by the past observations (which might be misremembered). In other words, Robinson Crusoe's evaluation of a target coconut may be based not only on the set of coconuts which are currently observable to him, but also on his memory of coconut sizes seen previously, or even imagined.

More formally, suppose that at time t_0 the individual's history of observations includes a collection of $T+1$ environments $S_{t_0}, S_{t_0-1}, \ldots, S_{t_0-t}, \ldots, S_{t_0-T}$ with magnitude distributions $F_{t_0}, F_{t_0-1}, \ldots, F_{t_0-t}, \ldots, F_{t_0-T}$. Suppose the individual remembers (samples) observations from period $t_0 - t$ uniformly with the memory rate $\delta_t \in [0,1]$, i.e. any two reference magnitudes $y, y' \in S_{t_0-t}$ have equal probabilities of being included in the reference set S. Using the general discounting model of Rubinstein [2003] (which permits hyperbolic discounting), write the remembered adaptive evaluation $I_S(x)$ of a magnitude x as a mixture of $T+1$ probability distributions:

$$I_S(x) = \frac{\delta_0}{\Delta}F_{t_0}(x) + \frac{1}{\Delta}\sum_{t=1,\ldots,T}\left(\Pi_{s=1,\ldots,t}\delta_s\right)F_{t_0-t}(x) \tag{15}$$

where $\Delta = \delta_0 + \sum_{t=1,\ldots,T}\left(\Pi_{s=1,\ldots,t}\delta_s\right)$ is a normalization constant.

Since past history may include imaginary environments, the above general formulation permits for saliency both in specific past histories and specific values. Specifically, a salient magnitude value $\tilde{x}$ might enter the reference set S as a past environment S_l containing a degenerate random variable at $\tilde{x}$, i.e. $F_l(x) = H(x-\tilde{x})$; while a salient environment at time k might enter S with $\delta_k > \max\{\delta_0, \ldots, \ldots, \delta_t, \ldots, \delta_T\}$ for all $t \neq k$.

Regardless of the structure of the reference set S, one can decompose the reference set S into a subset S_0 sampled from observations from the current period and a subset S_T sampled from remembered observations in the previous T periods, i.e. $S = S_0 \cup S_T$. Thus, the evaluation (15) can be written as a weighted sum of current F_{t_0} and remembered past $\mathcal{F}_T$ environment:

$$I_S(x) = \frac{\delta_0}{\Delta} F_{t_0}(x) + \left(1 - \frac{\delta_0}{\Delta}\right) F_T(x) \tag{16}$$

The following statement is obvious.

Proposition 9 *Suppose the reference set S includes observations beyond the current environment, i.e. $\frac{\delta_0}{\Delta} < 1$. Then the adaptive evaluation I_S is suboptimal in the sense of Robson [2001].*

That is, when an individual evaluates prospects based on a past memory, his mistakes in a binary choice in the present environment S_0 are not minimized. Furthermore, long memory may make the present environment to have a negligible effect on one's evaluation, leading to context-independent choices.

Proposition 10 *Suppose $\delta_t > 0$ for all t and consider an evaluation with long memory. Then $I(x) = \lim_{T\to\infty} I_S(x)$ is independent of the current context.*

Proof: As $\lim_{T\to\infty} \Delta = \infty$, the weight of the current context is negligible: $\lim_{T\to\infty} \frac{\delta_0}{\Delta} = 0$ and thus $I(x) = \lim_{T\to\infty} I_S(x) = F_T(x)$. ∎

Thus, with long memory, the individual behaves as if he possesses a neoclassical utility function which is determined by the distribution of remembered past observations. Such evaluation, however, is not optimal in the sense of Robson [2001]. Conversely, if one's memory is bounded, one's evaluations will exhibit dependence on the current context. The next result is straightforward.

Proposition 11 *Suppose $\delta_0 > 0$ and $T < \infty$, so that $\frac{\delta_0}{\Delta} > 0$. Suppose an individual faces either of the two environments, G_A or G_B. Then $I_{S_T \cup A}(x) \lesseqqgtr I_{S_T \cup B}(x)$ whenever $G_A(x) \lesseqqgtr G_B(x)$.*

In other words, when an individual is presented with variable context, her magnitude evaluation will exhibit qualitative, but not quantitative, correspondence with the present context. For example, magnitude judgments of subjects presented with positively skewed distributions will be higher than those presented with negatively skewed distributions as described in Section 3.

7 Interpersonal Comparisons and Social Welfare

To evaluate a coconut, Robinson Crusoe compares it against other coconuts present - whether these other coconuts are on a tree, on a store display, or in hands of other people. From the cognitive point of view, Mrs. Brown evaluates her house the same way as Crusoe evaluates a coconut, - using two primitive cognitive tools over a reference set consisting of other houses, - yet these reference houses tend to belong to other people. In a private ownership economy most (if not all) ordinal comparisons are necessarily done interpersonally, particularly for individual characteristics such as height, intelligence, beauty, as well as economically relevant variables such as consumption, income, and wealth. A change in Mr. Jones' possessions changes how Mrs. Brown evaluates what she has, appearing as a desire to "keep up with the Joneses".[16]

Such interpersonal comparisons have important economic consequences. To see that, consider a private ownership economy with a continuum of individuals, each endowed with an object of magnitude x (e.g. height, intelligence, income, happiness, and so on). Suppose that these magnitudes of individual endowments are observable and thus constitute the reference set S with the veridical magnitude distribution F. Assume that, being atomistic, all individuals have the same reference set S, have perfect frequency tool F, and the same ordinal tool $\mathcal{D}$. Importantly, during an encounter between two individuals possessing magnitude x and y, the result of ordinal comparison tournament for x is given by $\mathcal{D}(x, y)$, while the corresponding result for y is given by $\mathcal{D}(y, x)$. Define the *surplus* function $\Psi(x \perp y)$ from an encounter between x and y to be the sum of the results of both ordinal comparison tournaments:

$$\Psi(x \perp y) = \mathcal{D}(x, y) + \mathcal{D}(y, x) \tag{17}$$

Clearly, $0 \leq \Psi(x \perp y) \leq 2$. The surplus function $\Psi(x \perp y)$ plays an important role for the societal welfare. As the following Proposition suggests, if the surplus function Ψ has a constant sum property, the distributions of economically important goods can be welfare-neutral.

Proposition 12 *Consider a private ownership economy where all individuals have the same reference set S consisting of magnitudes of objects belonging to all other individuals in the economy with veridical (true) magnitude distribution F. Suppose all individuals have the perfect frequency (proportion) tool F, and identical ordinal comparison tool subject to a constant-sum property $\Psi(x \perp y) = \mathcal{D}(x, y) + \mathcal{D}(y, x) = C \in [0, 2]$ everywhere except at a countable number of encounters x, y. Then the utilitarian measure of the total welfare $W = \int_S I(x) dF(x)$ equals to $\frac{C}{2}$ independently of the magnitude distribution $F(x)$.*

[16] While an adaptive evaluation is sentiment-free, in private ownership economies it may lead to a concern with social status discussed by Veblen [1899], Frank [1995], and many others.

Proof: Rewrite the total welfare as:

$$W = \int_S I_S(x)dF(x) = \int_S \int_S \mathcal{D}(x,y)dF(x)dF(y)$$

Since adaptive evaluation is determined for any x and y, one can also write W as:

$$\begin{aligned} W &= \textstyle\int_S \int_S \mathcal{D}(y,x)dF(y)dF(x) &= \int_S \int_S (\Psi(x \perp y) - \mathcal{D}(x,y))dF(x)dF(y) = \\ & &= \int_S \int_S \Psi(x \perp y)dF(x)dF(y) - W \end{aligned}$$

Since $\Psi(x \perp y) = C$, we have that

$$W = \frac{1}{2}\int_S \int_S \Psi(x \perp y)dF(x)dF(y) = \frac{C}{2} \quad \blacksquare \tag{18}$$

The economic processes, such as economic growth, and public policies, such as redistributive taxation, tend to change the distribution of individual possessions. As individuals adapt their evaluations of their possessions to the new social environment, the cognitive processes involved in interpersonal comparisons might be important for societal welfare (and thus the success of economic policies). Whenever a pairwise comparison tournament entails opposite outcomes for any two individuals, welfare neutrality of economic processes may arise.[17]

Clearly, with frequency imperfections the welfare neutrality does not hold. With perfect frequency tool, if the ordinal tool allows for both individuals to "win" the ordinal comparison tournament when the magnitudes x and y are sufficiently similar, it is possible that more equal societal distributions increase welfare. Conversely, equality may decrease welfare if sufficiently similar magnitudes lead to "losses' for both individuals. The next example demonstrates this possibility.

Example 7 *Suppose individuals have additively imperfect evaluations, with the magnitude and ordinal discriminability distributions uniform as in Example 4. Then welfare is given by*

$$W = \frac{2k_1^3 - (k_1 - 2\sigma)^3 + (k_2 - 2\sigma)^3}{24\sigma^2(k_2 - k_1)}$$

Note first that the welfare is independent of mean magnitude μ. Yet if ordinal discriminability $z \sim U[-k,k]$ with $k < \sigma$, the surplus function $\Psi = 1$ for all x and y, and welfare $W = 0.5$ independently of σ. In contrast, if ordinal tool is subject to "downward inequality aversion", so that ordinal discriminability $z \sim U[-k,0]$ with $k < 2\sigma$, the surplus function Ψ has maximum at $x = y$ and welfare $W = \frac{1}{2} + \frac{k(6\sigma - k)}{24\sigma^2} > 0.5$. Instead, if

[17] Compare this welfare neutrality result to the "happiness paradox" of Easterlin [1974] who pointed out that average self-reported happiness in USA stayed practically unchanged in the post-war period despite real incomes nearly doubled.

ordinal tool is subject to "upward inequality aversion", so that ordinal discriminability $z \sim U[0,k]$ with $k < 2\sigma$, the surplus function Ψ has minimum at $x = y$ and welfare $W = \frac{1}{2} - \frac{k(6\sigma - k)}{24\sigma^2} < 0.5$.

The case of multiplicatively imperfect evaluations is more computationally more complex. To understand the interaction between the ordinal imperfections and inequality, consider first the following example for discrete distributions.

Example 8 *Consider an economy where proportion α of population have income x_L and proportion $1-\alpha$ have incomes $x_H > x_L$. Suppose all agents have perfect ordinal tool with $\mathcal{D}(x,x) = p$, so that $I^p(x_L) = \alpha p$ and $I^P(x_H) = \alpha + p(1-\alpha)$, and welfare $W^P = p + (1-2p)\alpha(1-\alpha)$. Now suppose that agents have a multiplicatively imperfect ordinal tool with the interval of doubt $[\lambda^{-1}, \lambda]$, where $\lambda > 1$. If income distribution F_A is sufficiently unequal with $x_H > \lambda x_L$, the imperfect evaluation $I_A^M = I_A^P$, so that $W_A^M = p + (1-2p)\alpha(1-\alpha)$. If instead, income distribution F_B is sufficiently equal with $x_H \leq \lambda x_L$, the imperfect evaluation $I_B^M(x_L) = I_B^M(x_H) = p$, so that $W_B^M = p$. As $W_A^M - W_B^M = (1-2p)\alpha(1-\alpha)$, welfare is higher in more equal society B as long as encounters with people of similar incomes tend to be seen favorably (i.e. $p > 0.5$). And vice versa, welfare is higher in more unequal society A if, instead, there is bias against encounters with people of similar incomes (i.e. $p < 0.5$). Again, welfare neutrality holds for either case whenever $p = 0.5$.*

The importance of the ordinal tournament between similar magnitudes is further highlighted by the following example.

Example 9 *Suppose individuals have multiplicatively imperfect evaluations, with the magnitude and ordinal discriminability distributions uniform as in Example 5. The resulting welfare W is a complex quadratic function of the ratio $\frac{\mu}{\sigma}$. Again, there is a relationship between the behavior of the surplus function $\Psi(x \perp y) = \frac{(x/y + y/x) - 2k_1}{k_2 - k_1}$ and welfare W. Specifically, for $k < \frac{\mu+\sigma}{\mu-\sigma}$, if ordinal tool is subject to "downward inequality aversion", so that ordinal discriminability $z \sim U[1/k, 1]$, the surplus function Ψ has maximum at $x = y$ and welfare W increases with the ratio $\frac{\mu}{\sigma}$. That is, for a given level of inequality σ, economic growth is socially desirable, while for a given level of aggregate income, inequality is not. Instead, if ordinal tool is subject to "upward inequality aversion", so that ordinal discriminability $z \sim U[1,k]$, the surplus function Ψ has minimum at $x = y$ and welfare $W =$ decreases with the ratio $\frac{\mu}{\sigma}$.*

Multiplicatively imperfect ordinal tools which satisfy constant sum property are rare, but they also result in welfare neutrality.

Example 10 *Suppose veridical income distribution is uniform on* $[\mu - \sigma, \mu + \sigma]$ *and suppose all agents a multiplicatively imperfect ordinal tool on* $[\lambda^{-1}, \lambda]$, *with* $1 < \lambda^2 < \frac{\mu+\sigma}{\mu-\sigma}$ *with ordinal discriminability* $z = \frac{x}{y}$ *being distributed with* $F^{\nu}(z) = 0.5 + 0.75\frac{\ln z}{\ln \lambda} - 0.25\left(\frac{\ln z}{\ln \lambda}\right)^3$. *One can verify that as* $F^{\nu}(z) + F^{\nu}(z^{-1}) = 1$, *the welfare neutrality holds.*

One can speculate that social aspects of interpersonal comparisons alter the shape of the ordinal comparison function. However, regardless of whether the ordinal comparison is entirely cognitive or has any social component to it, the relationship between welfare and income distribution will be determined by the behavior of the social surplus Ψ function, particularly around the point of parity $x = y$. Specifically, if the surplus is higher at parity, equality is likely to be desirable, and vice versa.

8 Conclusions

This paper provides a conceptual framework which allows one to combine the insights from economics, behavioral, evolutionary and cognitive sciences into one parsimonious mathematical model. Undoubtedly, this model has limitations. First, it concentrates on magnitude evaluation, rather than choice. As Rayo and Becker [2007] pointed out, to make a choice among the alternatives, one needs to compare adaptive evaluations across the alternatives, which may involve further perceptual limitations. Second, the model here describes the evaluation of objects that differ only in one dimension, while organisms are likely to use additional tools to evaluate multidimensional objects and bundles, and thus the similarity function in Gilboa and Schmeidler [1995, 2003] could be useful here. Third, it remains to be determined whether there are additional cognitive tools involved in processing of uncertainty, as Andreoni and Sprenger [2010] report on a qualitative difference between certain and uncertain utility. Yet, despite its shortcomings, the present paper highlights the importance of recent advances in cognitive and brain research for understanding of human economic decision making.

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