A Model of Two-Party Representative Democracy: Endogenous Party Formation *

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Abstract

This paper proposes a model of two-party representative democracy on a single-dimensional political space, in which voters choose their parties in order to influence the parties' choices of representative. After two candidates are selected as the median of each party's support group, Nature determines the candidates' competence levels. Based on the candidates' political positions and competence levels, voters vote for the preferable candidate without being tied to their party's choice. We show that (1) there exists a nontrivial equilibrium under some conditions, and that (2) dependent on voter distribution over their political positions, the equilibrium party line and the ex ante probabilities of the two-party candidates winning are biased. In particular, we show that if a party has a strong subgroup with extreme positions, then the party tends to alienate its moderate subgroup, and its probability of winning the final election is reduced.

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1 Introduction

In a two-party electoral system, office-motivated parties set their policy platforms to attract the majority of voters in order to get elected. Downs (1957) and Black (1958) have shown that if the policy space is one-dimensional then both parties choose the median voter's "bliss point" as their party platform. Although this theoretical result is a nice justification for a two-party system, we do not observe this outcome in US politics. Why is this the case? First, in the real world, the policy space is not one-dimensional. However, given a two-party system, a similar result can occur when candidates are office-motivated even with a multidimensional policy space, if there is an equilibrium. In contrast, if candidates are policy-motivated, then we have policy divergence as Wittman (1983), Calvert (1985), and Roemer (2001) show in models with uncertainty in voting outcomes.

Although the result that candidates' levels of policy-orientation determine the level of equilibrium policy divergence is quite reasonable, one problem still remains. How did these policy-oriented candidates get elected as party candidates? In the real world, it appears that on many occasion candidates who are quite far from the median voter can be elected in party primaries in many occasions. In 2004, moderate Republican senator Arlen Specter faced a tough challenge from the right in the Republican primary election; but once Specter defeated the challenge with a narrow margin, he was comfortably reelected in the general election with great support from independents. During his reelection bid in 2006, moderate Democratic senator Joe Lieberman lost the Democratic Party primary election but won reelection in the general election as a third-party candidate. Electing extreme candidates in party primaries can have serious consequences: it can, for example, destabilize one party's domination over the other. A recent experience in California is vividly described by Fiorina, Abraham, and Pope (2011, pages 210-211). In the 1994 election, the California Republican Party won its governorship in a landslide, won four of the six other statewide races for state office, and Republicans defeated four Democratic House incumbents. However, thereafter, the California Republican Party was taken over by its extreme social conservative elements, nominating hard-core conservatives with limited in primary elections; and in 2002 Democrats won all the statewide races for the first time in California history. In less than a decade, California changed its hue from dark red to dark blue, and California is still known as a Democratic state (for now).

From these examples, we can make a few observations. First, primary election results can be quite biased, and the candidates who make it to the final election may not be close to the median voter's bliss point. Second, if the candidates elected in primary elections are too extreme, then the party has a high chance of losing the general election by alienating moderate central voters. Third, one party's domination is fragile and unstable, and it can be upset easily. Fourth, a dramatic change in a party's power may last for long time despite the fragility of a party's domination. In this paper, we will propose a simple two-party voting model that is consistent with these observations, and we will ask the following two questions: Does the presence of an extreme intraparty group within its political position (say, the social conservative subgroup in the Republican Party) have implications for the party line? Although it is under debate among researchers, some argue that US voters are becoming more and more politically divided. If this is true, what are the implications for equilibria and political stability?

Fiorina, Abraham, and Pope (2011) argue that each party's elite activists tend to have rather extreme views, and they influence primary election outcomes, resulting in more policy polarization. Levendusky (2009) stresses the role that party elites play in sorting voters into the two parties by clarifying each party's political positions.[1] Sorting of voters can aggravate the polarization of the party candidates even further. These authors investigate how we reached the current US political landscape over time with a dynamic analysis that checks the causality of events. Levendusky (2009) provides series of empirical evidences that support his hypothesis. However, it is very hard to construct a formal game-theoretical model with many players (party elites, voters, party candidates, etc.) that describes the dynamic evolution of party policies and voter sorting, since we need to specify our model precisely through specific assumptions on how rational party elites and voters are and what information they possess when they choose their actions. Although it would be elegant if we could construct such a dynamic model, it is extremely difficult to make the model robust to a specific setup and assumptions.

In this paper, we will take a simple alternative approach. We will provide a "static" model using equilibrium analysis without analyzing the causality of events. That is, we will focus on self-sustainable (internally consistent) allocations. This approach may appear to be a regression from the dynamic approach proposed by Levendusky (2009), which can explain how things evolved, but we believe that static analysis could complement dynamic analysis. By ignoring dynamics, we can draw simple conclusions from the analysis — for example, if there are multiple equilibria, then we can say that some sufficiently large shock can upset an equilibrium and shift it to a very different equilibrium. And the equilibrium is self-sustainable, so the existence of multiple equilibria can explain the third and the fourth observations mentioned. Our model predicts voter sorting in equilibrium, and it trivially causes more polarized party primary results by adopting our simplification assumption that the outcome of a party primary is the median voter's bliss point within the party.

[1] In the 60s, voters were not sorted to Democrats and Republicans by their political positions (Southern states were the stronghold of conservative democrats), but by the 80s the conservatives sorted to the Republicans while the liberals sorted to the Democrats. Levendusky (2009) asserts that party elites clarified party/ideology mapping, resulting in voter sorting.

Our main idea is described by introducing uncertainty in voting outcomes following Wittman (1983). Specifically, we assume that each candidate has a chance to win due to the uncertainty of the election. That is, our model involves common shocks rather than idiosyncratic shocks among voters in the final election. As is often seen in the real world, the candidates' campaign and debate performances can change the voting outcome.[2] Some voters may prefer the candidate from the opposite party even if her political position is very far from the candidate's position.[3] With such uncertainty in the voting outcome, even if an extreme candidate is selected in a primary, she may win the final election if she happens to be judged much more competent than the moderate candidate, although such an event would occur only with very low probability. Suppose that an extreme candidate is chosen in a party by the influence of a strong, extreme subgroup in that party. Then, moderate potential supporters of the party are alienated if the party does not reflect their voice in choosing the party candidate. If they participate in the other party which has more diverse support groups, they may be able to play a more significant role in choosing that party's candidate. As a result, the party line shifts accordingly, and the more diverse party selects a more moderate candidate, while the party supported by an extreme group selects a more extreme candidate. This is a self-sustaining outcome—an equilibrium. Obviously, the diverse party's candidate's political position is closer to the median voter's position, and she has a higher probability of getting elected.

To determine the party line in the equilibrium, we assume that voters are strategic in choosing their parties, foreseeing their influence on the choice of candidates, and we assume away all other strategic behaviors by voters and candidates.[4] We simply assume that a party candidate (or a party policy platform) is the median voter within the party support group.[5] In order to describe voters' strategic behavior in their party choice, we will not simply adopt Nash behavior, since we assume that voters are atomless. In this framework, unilateral deviations cannot affect the parties' candidate-selection processes, so any partition of voters can be a Nash equilibrium. To avoid this difficulty, we consider small coalitional deviations and define a "political equilibrium" as a partition of voters

[2]For example, we can recall the loss of the incumbent George Allen, a Republican, in the 2006 Virginia senator race and the victory by Scott Brown, a Republican, in the 2010 Massachusetts senator race to replace late Edward M. "Ted" Kennedy, a Democrat who had been the senator for more than 40 years. These shocks were clearly not idiosyncratic: the shocks can be quite dramatic and devastating.

[3]Persson and Tabelini (2000), Roemer (2001), and Bernhadt, Krasa, and Polborn (2008) use voting models with common shocks.

[4]We assume that the elected candidate cannot misrepresent her political position, following the citizen candidate literature (see Osborne and Slivinsky 1996; Besley and Coate 1997).

[5]Although this may sound like a strong assumption, we can generate similar results in a simplified model even if voters select their party's candidates strategically (see a companion paper: Kobayashi and Konishi 2012; draft). We will explain this companion paper in the Conclusion section.

from which any arbitrarily small coalitional deviations are unprofitable.[6]

Our game goes as follows. In stage 1, voters choose their parties by calculating their expected utilities from the final election from joining each party (with small groups of other voters). By the voters' party choice, the two party candidates are selected as the median voter of each party. In stage 2, Nature plays, and the two candidates' competence (relative attractiveness) is determined randomly. In stage 3, voters cast their ballots for the preferable candidate given the two candidates' positions and competence (relative attractiveness). A voter's party affiliation does not bind her voting behavior, and she votes sincerely. The final voting outcome is the equilibrium outcome of this voting game. Our solution concept, political equilibrium, is a subgame perfect equilibrium, except that we allow for small coalitional deviations instead of each voter's unilateral deviation in the party-choice stage (stage 1).[7]

We will first characterize our political equilibrium, and find that our equilibrium is consistent with voters' party sorting. Using this property, we provide sufficient conditions for the existence of a political equilibrium (Theorems 1 and 2). Then, we move on to investigate how the party line is affected by the distribution of voters over policy space: in particular, we show the relation of the median voter's position with the equilibrium party line (Proposition 4). Other things being equal, if a party's support group (in terms of its policy spectrum) becomes more extreme, then the party tends to lose support, making its candidate more extreme and the opponent party's candidate more moderate. Example 2 assumes a density function with a step where discontinuity occurs at the median voter, with higher density to the left of the median and lower density to the right. We find that the party line will move left, i.e., the right party expands by making it easier for voters to have their voice. In an example with a tri-peaked symmetric voter distribution (Example 3: a step function with peaks at the left extreme, the center, and the right extreme), we conduct a comparative static exercise to analyze what happens when voters are more politically divided. When the voter distribution is uniform, there is a unique symmetric equilibrium. However, as the population of the moderate left and right decreases gradually, two other asymmetric equilibria suddenly appear. Such an asymmetric equilibrium has

[6] This definition of equilibrium has superficial similarity to "ϵ-club's deviations" of Osborne and Tourky (2008). However, the uses of small coalitional deviations in these two equilibrium concepts are very different (see footnote 7 below).

[7] Our solution concept superficially resembles a part of Osborne and Tourky's (2007), small club Nash equilibrium. They consider deviations by positive measures of atomless voters to determine each subgame's voting outcome. However, this is a response to the insensitivity to the electoral outcome to a single voter's action (they assume that voting is costly), given a continuum of voters. In contrast, in our model, a small coalition's deviation has more significance. By a small coalitional deviation, candidates' political positions change, and voters' utilities are affected by that. Thus, in our paper, small coalitions play more significant role than in Osborne and Tourky (2008).

the feature of having one party composed mostly of extreme voters and the other party composed of the rest of the voters, including the centrist group. This equilibrium has two self-sustaining parties: the former party chooses an extreme candidate who has a low probability of winning (but there is still a chance to win if common shock is strongly in favor of him), while the latter chooses a moderately oppositely biased candidate with a high chance to win. Voters who are happy with the extreme candidate despite her low chance of winning continue to support the extremist party. However, there is another oppositely biased equilibrium that is self-sustainable. Thus, if voters are deeply divided politically, then there will be multiple equilibria and the equilibrium allocation can jump from one extreme to the other with a substantial shock in the environment.

Two articles are most closely related to our paper. Feddersen (1992) constructs a model in which voters choose political positions and calls a group of voters who choose the same political position a party. In the sense that voters choose their party strategically, our model is closest to Feddersen (1992), since voters are assumed to be strategic players in his model as well as ours. However, there are also a number of differences between the two approaches. Feddersen's model is deterministic, allows an arbitrary number of parties, and allows a multidimensional policy space. In contrast, uncertainty plays an essential role in our model, while we restrict our attention to the two-party case on a single-issue space. In our model, a party's political position (the candidate's position) is determined by aggregating the party supporters' political positions (via the party's median voter's policy). Extending the Wittman model (1983), Roemer (2001, Chapter 5) endogeneizes the party line through assuming that voters sort into parties by comparing their (deterministic) utility levels from two candidates' policies. In our model, voters compare the *expected* utility levels of joining each party. In this sense, voters in our model are more farsighted and strategic. Our Example 3 will bring out the difference between these two approaches.

In section 2, we present our model. In section 3, we define political equilibrium and investigate its properties. Using these properties, we provide some insights into how the party line is affected by the distribution of voters over their political positions. In section 4, we provide a sequence of examples that show when the equilibrium is biased and there are multiple equilibria. The main observations from the examples are: if one party has a stronger extreme subgroup, then the party loses some of centrist supporters; and if the voters are more polarized then there tend to be asymmetric equilibria in which one party consists of mostly extremists while the other party has both centrists and extremists as its supporters. In section 5, we conclude with a brief discussion of how relaxing our assumptions will affect our results.

2 The model

2.1 The overview of the model and the game

There is a one-dimensional policy space, and a continuum of atomless citizens, namely atomless voters, is distributed over the interval $[0, 1]$. There are two parties. The party names themselves do not matter, but for convenience, we call one party with more supporters from the left side the L party and the other with more supporters from the right side the R party. These parties are formed by the voters. Each party selects a candidate who represents the party, and each voter casts a vote for his or her most favorite candidate. Following the citizen-candidate models by Osborne and Slivinski (1996) and Besley and Coate (1997), we assume that the winner becomes the policy maker who implements her own preferred policy, which means that the policy maker elected by voters has complete authority, and we assume that candidates' political positions are common knowledge and that candidates cannot commit to anything, so that they cannot be tied to their ex-ante policies. We also assume that candidates' "political competence," which is the ability to implement a policy successfully and in a favorable way for voters, is a random variable that is initially unknown to the voters but is revealed after the candidate starts the campaign.[8] As a result, a candidate with the higher political competence has a higher probability to be elected to the office. Note that this random shock is not an idiosyncratic shock across voters but is common to all voters, and thus affects the voting outcome.[9] Once the candidates' political competence is realized, voters' behavior depends on the candidates' political competence and positions. Thus, at the voting stage some voters can prefer the candidate of the opposite party. Since there are no restrictions on voting, she does not necessarily vote for the candidate of her party.

We consider the following dynamic two-party representative election game. In stage 1-a, voters choose their parties; in stage 1-b, in each party, a member of the party is chosen as the party representative, who will choose the policy once elected; in stage 2, Nature plays and the competence (attractiveness to voters) of each candidate is realized; and in stage 3, all voters vote freely for one of the two candidates, and a winner becomes the policy maker and implements her favorite policy. Basically, we analyze these stages in

[8] To be concrete, through the electoral campaigns where voters watch candidates' debates, campaign gaffes, scandals, and so on, voters can know which candidate has the superior ability to implement policy and which is a more charismatic policymaker, which can affect the voting outcome.

[9] It is well known that each candidate takes the median voter's position if there is no uncertainty following the median voter theorem. On the other hand, when "the candidates are uncertain of the distribution of citizens' ideal points," they may take different positions (Osborne 1995). In this paper, by introducing this political competence, a simple uncertainty, we can explain the political phenomenon we pointed out in the previous section.

reverse order. However, following Besley and Coate (2003), we greatly simplify stage 1-b: a median of the support group of each party is selected as the candidate who represents the party. We solve this game by backward induction, so that the equilibrium is basically the subgame perfect equilibrium. However, we need to modify the equilibrium slightly in stage 1, since each voter is atomless. We introduce an equilibrium notion that is immune to any small coalitional deviations, as mentioned in the previous section. Regarding small deviations, Osborne and Tourky (2008) also use a similar deviation named "ϵ-club" and define the "small club Nash equilibrium."[10] This small club is not a coalition that weakly improves each member's payoff in the club, but a club that improves the sum of the members' payoff in the club by deviating. While Osborne and Tourky do not consider redistribution among the members in the club, our model is more rigid in terms of the redistribution of each member's payoff in the coalition by considering coalitional deviations.

2.2 Voters

Each voter cares about the policy chosen by the elected representative and cares about her competence, which is the ability to implement her policy successfully and in a favorable way for voters. Each voter is atomless and has a type θ, which is distributed continuously on $[0, 1]$ with density function $g(\theta)$.[11] Type θ voters have the following von Neuman-Morgenstern, hereafter vNM, expected utility function:

$$u(p_k; \theta, \epsilon_k) = -|p_k - \theta| + \epsilon_k,$$

where $p_k \in [0, 1]$ and $\epsilon_k \in \mathbb{R}$ denote the policy implemented by the elected representative $k \in C$ as a policy maker and a realization of a random variable that describes her competence, respectively. C denotes a candidate set composed of candidates selected from each party. The random variable ϵ_k follows probability density function f_k with zero expectation ($E(\epsilon_k) = 0$) and symmetric distribution with respect to 0. A positive realization ϵ_k shows that the candidate is competent, while a negative realization denotes her incompetence.

2.3 Allocations and Party-Candidates

In this section, we explain how each candidate is selected in each party. In this model, who becomes a candidate depends on the structure of the party.

[10] They use the ϵ-clubs deviations not in the party-formation stage but in the voting stage. In their model, candidates and voters are separated, which is not the citizen candidate model.

[11] Thus, even if all voters of type θ form one group, it is still atomless.

Definition 1 *An **allocation** is a list of membership densities of L party and R party, $g_L : [0,1] \to \mathbb{R}_+$ and $g_R : [0,1] \to \mathbb{R}_+$, respectively, such that for all $\theta \in [0,1]$, $g_L(\theta) + g_R(\theta) = g(\theta)$ holds.*

We assume that supporters of each party elect a party representative who becomes a candidate running for the representative (policy maker). Each candidate is of the majority's preferred type, namely a median voter elected as a party representative, following Besley and Coate (2003), as we said earlier.[12] Let $x(g_L)$ and $y(g_R)$ be such that

$$\int_0^{x(g_L)} g_L(\theta)d\theta = \int_{x(g_L)}^1 g_L(\theta)d\theta \iff G_L(x(g_L)) = G_L(1) - G_L(x(g_L))$$

and

$$\int_0^{y(g_R)} g_R(\theta)d\theta = \int_{y(g_R)}^1 g_R(\theta)d\theta \iff G_R(y(g_R)) = G_R(1) - G_R(y(g_R)),$$

respectively. They denote the candidates of the L party and the R party, respectively. Obviously, each candidate x and y depends on the distribution of her supporters (each party's distribution), respectively. From now on, our main focus is on "sorting allocations," that is all supporters of the L party are on the left side of a threshold type and those of the R party are on the right side of the type:

Definition 2 *A **sorting allocation** is an allocation $g_R^{\tilde{\theta}}$ and $g_L^{\tilde{\theta}}$ with a threshold $\tilde{\theta} \in [0,1]$ which partitions $[0,1]$ into two intervals: $L = [0, \tilde{\theta})$ and $R = (\tilde{\theta}, 1]$ such that*

1. *$g_L^{\tilde{\theta}}(\theta) = g(\theta)$ and $g_R^{\tilde{\theta}}(\theta) = 0$ for all $\theta \in [0, \tilde{\theta})$, and*
2. *$g_L^{\tilde{\theta}}(\theta) = 0$ and $g_R^{\tilde{\theta}}(\theta) = g(\theta)$ for all $\theta \in (\tilde{\theta}, 1]$.*[13]

Throughout the paper, we will denote these sorting allocations with threshold values $\tilde{\theta}$ by $g_L^{\tilde{\theta}}$ and $g_R^{\tilde{\theta}}$, respectively. We will focus our attention on a sorting allocation in later sections. On the basis of these characteristics, we can determine the candidates in a sorting allocation with threshold type $\tilde{\theta}$. From the definition of a sorting allocation, x is determined by $G(x) = G(\tilde{\theta}) - G(x)$ and y is determined by $G(y) - G(\tilde{\theta}) = 1 - G(y)$. In a sorting allocation, each candidate also depends on the threshold $\tilde{\theta}$. Thus, we will also denote each candidate as a function of $\tilde{\theta}$: i.e. $x = x(\tilde{\theta})$ and $y = y(\tilde{\theta})$ in the following sections when we focus on a change in the threshold $\tilde{\theta}$.

[12] We can obtain similar results with a model where voters select their party's candidate strategically, as long as the density function f is sufficiently widely spread; see Kobayashi and Konishi (2012; in progress).

[13] Although the voters of type $\tilde{\theta}$ are not choosing any parties in this notation, the results in the following sections do not change even if they are choosing either L or R party since only one type of voters, also including $\tilde{\theta}$, are atomless. However, we only assume that there are atomless voters of type 0 (1) in L (R) even if $\tilde{\theta} = 0$ ($\tilde{\theta} = 1$) for convenience.

2.4 Realization of Competence of a Candidate

After candidates x and y are selected, their competence ϵ_x and ϵ_y is realized. In principle, we assume that both ϵ_x and ϵ_y are independent distributed random variables following f_x and f_y, respectively. *But for expository simplicity, we will assume throughout the paper that only the party L candidate x has random variable ϵ with a density function f, and y has no shock.*[14]

2.5 Voting

First, note that voters' behavior is not determined by the parties they belong to. There is absolutely no commitment: voters consider only the candidates' political positions and their competence when deciding whom to vote for. *We assume that all voters vote sincerely.* Let us consider a type θ voter. We define a function of type θ's relative evaluation of y to x, which is the difference of type θ's utilities from policies chosen by each candidate, as $h(x, y; \theta) \equiv -|y-\theta| + |x-\theta|$ when $\epsilon = 0$, or

$$h(x, y; \theta) = \begin{cases} -(y-\theta) + (x-\theta) = x - y \leq 0 & \text{if } \theta \leq x \\ -(y-\theta) + (\theta - x) = 2\theta - x - y & \text{if } x < \theta < y \\ -(\theta - y) + (\theta - x) = y - x \geq 0 & \text{if } y \leq \theta \end{cases}.$$

Clearly, $h(\theta)$ is a weakly increasing function of θ. Note that to slide type $\theta \in (x, y)$ to the right by one unit enlarges the relative evaluation of y by two units, which means that voters prefer y to x as their type gets larger. Then competence ϵ, which makes the median voters indifferent between both candidates, is dependent on x and y. Thus, we will denote competence as a function of x and y: i.e.

$$\epsilon(x, y) \equiv h(x, y; \theta_{med}) = -|y - \theta_{med}| + |x - \theta_{med}| = 2\theta_{med} - x - y \tag{1}$$

We assume that a candidate who receives a plurality of the vote, i.e. who receives more ballots than the other candidate, at the voting stage becomes the elected representative.[15] Then we have the following lemma (for the proof, see Appendix A):

Lemma 1 *If $\epsilon > \epsilon(x, y)$, then x is the winner. If $\epsilon < \epsilon(x, y)$, then y is the winner.*

Since ϵ is a random variable drawn from a probability distribution with density function f, once x and y are determined, $1 - F(\epsilon(x, y))$ and $F(\epsilon(x, y))$ are the winning probabilities of candidates x and y, respectively, from this lemma; those probabilities are defined

[14] We can calculate that each voter's expected utility in the case where candidate x and y have competence ϵ_x and ϵ_y, respectively, is the same as that in the case where only a candidate x has competence ϵ by assuming that both ϵ_x and ϵ_y are independent and that $f(\epsilon) = \int_{-\infty}^{+\infty} f_x(\epsilon + \epsilon_y) f_y(\epsilon_y) d\epsilon_y$.

[15] We assume that all voters vote for either x or y sincerely, so that no voters abstain. As a result, obtaining a plurality of the vote means obtaining a majority of the vote.

as $P_y(x,y) \equiv F(\epsilon(x,y))$ and $P_x(x,y) \equiv 1 - F(\epsilon(x,y))$. Taking these probabilities and the political positions of both candidates into account, voters choose their parties.

Before the next section, we will present a clear-cut corollary of the above lemma (see Figure 1):

Corollary 1 *If $\theta_{med} < \frac{x+y}{2}$, then x has a higher chance of winning ($F(\epsilon(x,y)) < 1 - F(\epsilon(x,y))$). If $\theta_{med} > \frac{x+y}{2}$, then x has a higher chance of winning ($F(\epsilon(x,y)) > 1 - F(\epsilon(x,y))$). If $\theta_{med} = \frac{x+y}{2}$, then $F(\epsilon(x,y)) = 1 - F(\epsilon(x,y))$.*

2.6 Party Choice by Voters

In stage 1, all voters choose either the L party or the R party. We assume that there is no option of joining no party.[16] Each voter chooses one party $i \in \{L, R\}$, where she can obtain a higher expected utility than the other through influencing the choice of the party's candidate as the party representative. *Note that since every voter is atomless, each voter's party choice has absolutely no impact on the party's representative selection.* The expected utility of a voter of type θ when two candidates are x and y is

$$\begin{aligned}
Eu(x,y;\theta) &= \int_{-\infty}^{\epsilon(x,y)} f(\epsilon)(-|y-\theta|)d\epsilon + \int_{\epsilon(x,y)}^{+\infty} f(\epsilon)(-|x-\theta|+\epsilon)d\epsilon \\
&= \int_{-\infty}^{\epsilon(x,y)} f(\epsilon)d\epsilon(-|y-\theta|) + \int_{\epsilon(x,y)}^{+\infty} f(\epsilon)d\epsilon(-|x-\theta|) + \int_{\epsilon(x,y)}^{+\infty} \epsilon f(\epsilon)d\epsilon \\
&= \underbrace{F(\epsilon(x,y))}_{\text{prob. } y \text{ winning}} \times \underbrace{(-|y-\theta|)}_{\text{utility from } y \text{ winning}} + \underbrace{(1-F(\epsilon(x,y)))}_{\text{prob. } x \text{ winning}} \times \underbrace{(-|x-\theta|)}_{\text{utility from } x \text{ winning}} \\
&\quad + \underbrace{\int_{\epsilon(x,y)}^{+\infty} \epsilon f(\epsilon)d\epsilon}_{\text{ave. of } \epsilon \text{ when } x \text{ wins}} .
\end{aligned} \tag{2}$$

We denote the expected utility of each voter of type θ in stage 1 when voters' distributions are g_L and g_R by EU:

$$EU(g_L, g_R; \theta) = Eu(x(g_L), y(g_R); \theta).$$

In sorting allocations, noting that $x = x(\tilde{\theta})$ and $y = y(\tilde{\theta})$, the expected utility of type θ is

$$EU(g_L^{\tilde{\theta}}, g_R^{\tilde{\theta}}; \theta) = Eu(x(\tilde{\theta}), y(\tilde{\theta}); \theta).$$

2.7 Political Equilibrium

We now define our equilibrium concept. On the one hand, if we allow large coalitional deviations, then it is hard to assure any kind of stable allocation. On the other hand, if we

[16] We assume that each voter cannot choose "nothing," i.e. there are no independent voters. We can also consider the case where voters can choose nothing strategically. However, that case requires a further research.

allow only unilateral deviations of one voter or one type of voter, any allocation can be a Nash equilibrium at the party choice stage, since every voter is atomless. Thus, we adopt an equilibrium concept that is immune to any small but positive measure coalition $\gamma : [0,1] \to \mathbb{R}_+$.[17] Are there incentives to deviate from an allocation for a small coalition that is far from the party line? If we allow such coalitional deviations, we need to generalize the definition of political equilibrium. Note that if there are only finite voters then each voter has some impact on the selection of the party candidate. Our small coalitional deviations can be regarded as individual voters' deviations in a finite model. Let $Supp(\gamma) = \{\theta \in [0,1] : \gamma(\theta) > 0\}$. Formally, our equilibrium concept is defined as follows:

Definition 3 *A* ***political equilibrium*** *is an allocation with $g_L : [0,1] \to \mathbb{R}_+$ and $g_R : [0,1] \to \mathbb{R}_+$ ($g_L(\theta) + g_R(\theta) = g(\theta)$ for all $\theta \in [0,1]$) such that there is a small positive measure $\bar{\Delta} > 0$ such that*[18]

1. *for all $\gamma \leq g_L$ with $\int \gamma d\theta \leq \bar{\Delta}$ for all $\theta \in Supp(\gamma)$, $EU(g_L - \gamma, g_R + \gamma; \theta) \leq EU(g_L, g_R; \theta)$,*

2. *for all $\gamma \leq g_R$ with $\int \gamma d\theta \leq \bar{\Delta}$ for all $\theta \in Supp(\gamma)$, $EU(g_L + \gamma, g_R - \gamma; \theta) \leq EU(g_L, g_R; \theta)$.*

Condition 1 in Definition 3 concerns deviations from supporters from the L party to the R party, while condition 2 concerns deviations from the R party to the L party. We will be particularly interested in the following sorting political equilibrium.

Definition 4 *A* ***sorting political equilibrium*** *is a sorting allocation with party line $\tilde{\theta} \in (0,1)$ such that there is $\bar{\Delta} > 0$ such that*

1. *for all $\gamma \leq g$ with $Supp(\gamma) \subseteq [0, \tilde{\theta}]$ and $\int_0^1 \gamma(\theta')d\theta' \leq \bar{\Delta}$, and all $\theta \in Supp(\gamma)$, $EU(g_L^{\tilde{\theta}} - \gamma, g_R^{\tilde{\theta}} + \gamma; \theta) \leq EU(g_L^{\tilde{\theta}}, g_R^{\tilde{\theta}}; \theta)$,*

2. *for all $\gamma \leq g$ with $Supp(\gamma) \subseteq [\tilde{\theta}, 1]$ and $\int_0^1 \gamma(\theta')d\theta' \leq \bar{\Delta}$, and all $\theta \in Supp(\gamma)$, $EU(g_L^{\tilde{\theta}} + \gamma, g_R^{\tilde{\theta}} - \gamma; \theta) \leq EU(g_L^{\tilde{\theta}}, g_R^{\tilde{\theta}}; \theta)$.*

Condition 1 in Definition 4 concerns deviations from supporters from the L party to the R party, while condition 2 concerns deviations from the R party to the L party. Both definitions say that any small coalitional deviations less than or equal to $\bar{\Delta}$ in measure do

[17] As we will show in the next section, small coalitions' deviation incentives differ in the intervals (x, y) and others $[0, x)$ and $(y, 1]$. Assuming that the deviations are allowed only near the party line, we will be focusing on (x, y). It turns out that such an equilibrium allocation is immune to deviation of small coalitions with a positive measure if a sort of voters' psychological costs are introduced (see Section 3).

[18] Function $g_L - \gamma : [0,1] \to \mathbb{R}_+$ is such that $(g_L - \gamma)(\theta) = g_L(\theta) - \gamma(\theta)$ for all $\theta \in [0,1]$.

not yield a greater payoff than not deviating to each member of those deviations in the equilibrium.

In the next section, we will discuss deviation incentives by small coalitions from different policy areas — from intervals $(x(\tilde{\theta}), y(\tilde{\theta}))$, $[0, x(\tilde{\theta}))$ and $(y(\tilde{\theta}), 1]$. We will show that we unfortunately need an additional assumption to assure immunity to coalitional deviations from intervals $[0, x(\tilde{\theta}))$ and $(y(\tilde{\theta}), 1]$. The next section will show that every political equilibrium is a sorting political equilibrium under a realistic condition.

3 Deviation Incentives for Small Coalitions

In this section, we provide a general analysis of deviation incentives for small coalitions from an allocation described by g_L and g_R. We will start with coalitions from central regions.

3.1 Deviations from Interval (x, y)

Let us partition the space of voter types into three intervals: $[0, x)$, (x, y), and $(y, 1]$.[19] Since we will consider coalitional deviations near the party line, let us start with a coalitional deviation with size $\delta > 0$ that belongs to the interval (x, y), moving from R to L.[20] In this case, the coalitional deviation reduces the population of party R and increases that of party L by δ. To avoid confusion, we denote δ in this case by $\delta^{R \to L}_{(x,y)} > 0$. We can easily construct such a deviation. Consider $\gamma_{(x,y)} : [0, 1] \to \mathbb{R}_+$ such that $\int_0^1 \gamma_{(x,y)}(\theta) d\theta = \int_x^y \gamma_{(x,y)}(\theta) d\theta = \delta^{R \to L}_{(x,y)}$ and $\gamma_{(x,y)}(\theta) \leq g_R(\theta)$ for all $\theta \in (x, y)$. After the deviation by $\delta^{R \to L}_{(x,y)}$, party L's population distribution is $g_L^{\tilde{\theta}} + \gamma_{(x,y)}$, while party R's population distribution is $g_R^{\tilde{\theta}} - \gamma_{(x,y)}$. That is, the new median voter type x' of party L is determined by

$$G(x') = G(\tilde{\theta}) + \delta^{R \to L}_{(x,y)} - G(x'),$$

and y' of party R is by

$$G(y') - G(\tilde{\theta}) - \delta^{R \to L}_{(x,y)} = G(1) - G(y').$$

[19] We do not consider the case where $x = y$. This is the case where both parties have the same party medians, which means that x and y are also the median voter, and we cannot even distinguish between parties L and R.

[20] If a coalitional deviation in the interval (x, y) involves groups who switch parties $R \to L$ and $L \to R$, then the effect of the deviation is simply reduced by canceling them out. So, we can concentrate on one-sided move: either $R \to L$ or $L \to R$.

Since we are considering a small coalitional deviation, we will take $\delta^{R\to L}_{(x,y)} \to 0$. By totally differentiating them,[21] we have

$$g(x)dx = d\delta^{R\to L}_{(x,y)} - g(x)dx, \tag{3}$$

or

$$\frac{dx}{d\delta^{R\to L}_{(x,y)}} = \frac{1}{2g(x)},$$

and similarly, we have

$$\frac{dy}{d\delta^{R\to L}_{(x,y)}} = \frac{1}{2g(y)}.$$

These derivatives represent that, by the small coalitional deviation $\delta^{R\to L}_{(x,y)} \to 0$, both x and y move to the right. Thus, type θ's expected payoff is affected by such a deviation through changes in x and y. Since we are investigating the incentive of a coalition member to join the deviation, we consider voters of R in $(\tilde{\theta}, y)$. Thus, for $\theta \in (\tilde{\theta}, y)$ we have

$$Eu(x,y;\theta) = -F(\epsilon(x,y))(y-\theta) - (1-F(\epsilon(x,y)))(\theta - x) + \int_{\epsilon(x,y)}^{+\infty} \epsilon f(\epsilon) d\epsilon.$$

Note that $\theta_{med} \in [x,y]$ holds. Suppose that $\theta_{med} < x < y$. Then, since x and y are the medians of parties L and R, we reach a contradiction. The case where $x < y < \theta_{med}$ follows the same logic. Thus, $\theta_{med} \in [x,y]$ must hold. This implies $\epsilon(x,y) = 2\theta_{med} - x - y$, and the impact of the coalitional deviation from the interval $(\tilde{\theta}, y)$ is written as

$$\begin{aligned}
&\frac{dEu(x,y;\theta)}{d\delta^{R\to L}_{(x,y)}} \\
&= \frac{1}{2}\Bigg[\underbrace{-\frac{F(\epsilon(x,y))}{g(y)} + \frac{1-F(\epsilon(x,y))}{g(x)}}_{\text{changes in candidates' positions}} \underbrace{- (2\theta - x - y - \epsilon(x,y)) f(\epsilon(x,y)) \left(\frac{1}{g(x)} + \frac{1}{g(y)}\right)}_{\text{changes in winning probabilities}}\Bigg] \\
&= \frac{1}{2}\left[-\frac{P_y(x,y)}{g(y)} + \frac{P_x(x,y)}{g(x)} - (h(x,y;\theta) - h(x,y;\theta_{med})) f(\epsilon(x,y)) \left(\frac{1}{g(x)} + \frac{1}{g(y)}\right)\right].
\end{aligned} \tag{4}$$

The first two terms in the brackets of (4) are changes in the expected utility that both candidates bring by moving to the right. The last term in the brackets is a change in the expected utility that is brought about by the change in the winning probability of y, namely $f(\epsilon(x,y))\frac{d\epsilon(x,y)}{d\delta^{R\to L}_{(x,y)}}$. Especially, $h(x,y;\theta) - h(x,y;\theta_{med})$ in the last term in the brackets denotes a difference between θ's and θ_{med}'s relative evaluations of y to x, which means θ's evaluations of both candidates based on θ_{med} since a candidate who receives the ballots of θ_{med} voters becomes the winner by majority rule in a sorting allocation.

[21] In this case, x' and y' are functions of the coalitional deviation size $\delta^{R\to L}_{(x,y)}$, so that taking the difference of this equation between before and after deviation $G(x') - G(x) = \delta^{R\to L}_{(x,y)} - (G(x') - G(x))$, dividing it by $\delta^{R\to L}_{(x,y)}$ and taking $\delta^{R\to L}_{(x,y)}$ to zero, we can obtain the same result.

As a result, a change in the winning probability of y is evaluated by θ's expected utility based on θ_{med}.

Note that θ shows up only in the third term in the brackets of (4) (the effect due to the changes in winning probabilities), which is a decreasing function in θ.[22] Noting that $h(x, y; \theta) - h(x, y; \theta_{med}) = 2(\theta - \theta_{med})$ for $\theta \in (x, y)$, suppose that $\frac{dEu(x,y;\tilde{\theta})}{d\delta^{R\to L}_{(x,y)}} = 0$ with the threshold $\tilde{\theta}$ that divides into L and R ; i.e. $\tilde{\theta}$ is satisfied with

$$-\frac{F(\epsilon(x, y))}{g(y)} + \frac{1 - F(\epsilon(x, y))}{g(x)} = 2(\tilde{\theta} - \theta_{med})f(\epsilon(x, y))\left(\frac{1}{g(x)} + \frac{1}{g(y)}\right).$$

(It will be shown in section 4.1 that (4) becomes zero with $\tilde{\theta}$.) Then, for all $\theta < \tilde{\theta}$, we have $\frac{dEu(x,y;\theta)}{d\delta^{R\to L}_{(x,y)}} > 0$, while for all $\theta > \tilde{\theta}$, we have $\frac{dEu(x,y;\theta)}{d\delta^{R\to L}_{(x,y)}} < 0$. This implies that coalitions do not want to move from R to L if they are composed of the types in $(\tilde{\theta}, y)$, while some small coalitions composed of the types in $(x, \tilde{\theta})$ want to move from R to L if there are some voters belonging to R in $(x, \tilde{\theta})$. However, since we are considering a sorting allocation, all voters of $\theta \in [0, \tilde{\theta})$ are in L, so there are no small coalitions that want to move from R to L. From the analysis above, it is easy to see that if we consider a coalitional deviation with size $\delta \to 0$ moving from L to R that belongs to the interval (x, y), the analysis is symmetrically reversed. This argument shows that if only voters in the interval (x, y) are allowed to move, then only a sorting allocation is consistent with the political equilibrium.

3.2 Psychological Costs for $[0, x)$ and $(y, 1]$

In the previous subsection, we showed that unless we can exclude voters in intervals $[0, x)$ and $(y, 1]$, it is hard to achieve an equilibrium distribution of voters between the parties. We provide a very simple example to illustrate this point. As we have seen, coalitional deviations from intervals $[0, x)$ and $(y, 1]$ are sufficient to upset the immunity to the coalitions; by combining voters in (x, y) and $(y, 1]$, we can create an even simpler and robust example.

Example 1 *Assume that g is uniform $g(\theta) = 1$ for all $\theta \in [0, 1]$, and that f is very widely spread (for example, $f(\epsilon) = \frac{1}{2a}$ for all $\epsilon \in [-a, a]$ with a large number a. In this case, whoever the two candidates x and y are, their chances of winning are always almost $\frac{1}{2}$ and $\frac{1}{2}$, respectively. Now, since everything is symmetric, a natural candidate for an equilibrium is a symmetric allocation $g_L(\theta) = g(\theta)$ for all $\theta < \frac{1}{2}$ and $g_R(\theta) = g(\theta)$ for all $\theta > \frac{1}{2}$. In this case, $x = \frac{1}{4}$ and $y = \frac{3}{4}$. Can this be immune to a coalitional deviation far from the party line? We denote a coalitional deviation as γ. Consider a deviation from*

[22] While the first and second terms are common for each type θ because of their linear utility, changes in winning probability mean different changes in expected utility because the evaluations of x and y are different for each type.

party R to L: $\gamma(\theta) = g(\theta)$ for all $\theta \in (\frac{3}{4} - \delta, \frac{3}{4} - \frac{1}{2}\delta) \cup (\frac{3}{4} + \frac{1}{2}\delta, \frac{3}{4} + \delta]$ where $\delta > 0$ is a small positive number. That is, after the deviation, there is no impact on the R party's candidate: $y' = \frac{3}{4}$. However, clearly x' is closer to θ_{med} after the deviation. Given a widespread f, the chances of x' and y' to win are still almost $\frac{1}{2}$ and $\frac{1}{2}$. Then, deviators in γ have a closer candidate from L who wins with probability $\frac{1}{2}$, so they are all better off. $\square$

Although this example may appear extreme, the force of the coalitions is robust in our model. However, in fact, voters with an extreme political position tend to have a strong, sometimes even fanatical, belief in their position. It seems unnatural for voters who are even more right than the median of the R party to move to the L party, while moderate voters around the party line do not move. To avoid this, we simply assume that if a voter's political position is more extreme than the median of a party that is closer to her position, then it is psychologically costly for her to join the other party.[23] That is, we can assume the presence of psychological costs — if a voter joins a party that is not supported by her political-position neighbors, she feels deeply distressed, as if she is sinning her convictions. From the standpoint of analysis, this psychological cost shows up only on special occasions to eliminate unlikely behavior by voters outside interval (x, y).

Definition 5 ***Psychological cost*** *is defined as the following function:*

$$\Phi(\theta, i; x, y) = \begin{cases} \Phi > 0 & \text{if} \begin{cases} i = R \text{ and } \theta < x < y, \text{ or} \\ i = L \text{ and } x < y < \theta \end{cases} \\ 0 & \text{otherwise} \end{cases},$$

and $\Phi(\theta, i) = 0$, otherwise.

Clearly, psychological costs will prohibit small coalitions outside interval (x, y) from deviating. *More precisely, for any $\Phi > 0$, there exists a $\bar{\Delta} > 0$ (in the definition of political equilibrium) such that every political equilibrium is a sorting allocation.*

4 The Main Analysis

4.1 Existence of Political Equilibrium

In this section, we provide sufficient conditions for the existence of a sorting political equilibrium. To do so, we need to characterize sorting political equilibria — a sorting allocation that is immune to any small coalitional deviations near the party line of a threshold $\tilde{\theta}$. Let us begin by considering how type $\tilde{\theta}$ voters' expected utility changes

[23] For example, a politically active liberal person may not enjoy going to a conservative party's convention.

when the threshold $\tilde{\theta}$ slides slightly to the right. Differentiating (2) with respect to $\tilde{\theta}$,[24] we have

$$\frac{dEu(x(\tilde{\theta}), y(\tilde{\theta}); \tilde{\theta})}{d\tilde{\theta}} = \frac{g(\tilde{\theta})}{2}\varphi(\tilde{\theta}),$$

where function $\varphi : [0, 1] \to \mathbb{R}$ is defined by

$$\begin{aligned}\varphi(\tilde{\theta}) \quad \equiv \quad & -\frac{P_y(x(\tilde{\theta}), y(\tilde{\theta}))}{g(y(\tilde{\theta}))} + \frac{P_x(x(\tilde{\theta}), y(\tilde{\theta}))}{g(x(\tilde{\theta}))} \\ & -2(\tilde{\theta} - \theta_{med}) f(\epsilon(x(\tilde{\theta}), y(\tilde{\theta}))) \left(\frac{1}{g(y(\tilde{\theta}))} + \frac{1}{g(x(\tilde{\theta}))}\right). \end{aligned} \tag{5}$$

This function φ is very useful in characterizing sorting political equilibria.

Note that $\varphi(\tilde{\theta})$ denotes the change of the border type $\tilde{\theta}$'s expected utility when the party line $\tilde{\theta}$ moves, but it is not the change of the expected utility of any particular type of voters. This is because the evaluating type $\tilde{\theta}$ itself is also changing as the party line $\tilde{\theta}$ changes. To evaluate the expected utility change of some type θ, we need to adjust the formula in order to use the φ function to evaluate the expected utility change of each player when the party line $\tilde{\theta}$ changes. Here, we consider small coalitional deviations from the R party to the L party and from L to R around $\tilde{\theta}$, which are $\int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} g(\theta)d\theta$ and $\int_{\tilde{\theta}-\Delta}^{\tilde{\theta}} g(\theta)d\theta$ for a small interval $\Delta > 0$, respectively. Those deviations can be expressed by sliding the party line $\tilde{\theta}$ by Δ. In the following lemma, we provide the difference in the expected utility of type θ when the threshold changes from $\tilde{\theta}$ to $\tilde{\theta} + \Delta$ or to $\tilde{\theta} - \Delta$.

Lemma 2 *Consider sorting allocations described by $\tilde{\theta}$ and $\tilde{\theta} + \Delta$ such that $\Delta > 0$ and that Δ is sufficiently small. Then, we have*

$$\begin{aligned}& Eu(x(\tilde{\theta} + \Delta), y(\tilde{\theta} + \Delta); \theta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \theta) \\ = \quad & \int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} \frac{g(\theta')}{2} \left[\varphi(\theta') - 2(\theta - \theta') f(\epsilon(x(\theta'), y(\theta'))) \left(\frac{1}{g(x(\theta'))} + \frac{1}{g(y(\theta'))}\right)\right] d\theta'.\end{aligned}$$

As a consequence, $Eu(x(\tilde{\theta} + \Delta), y(\tilde{\theta} + \Delta); \theta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \theta)$ is decreasing in θ for all $\theta \in (\tilde{\theta}, y(\tilde{\theta}))$. Similarly, consider sorting allocations described by $\tilde{\theta}$ and $\tilde{\theta} - \Delta$. Then, we have

$$\begin{aligned}& Eu(x(\tilde{\theta} - \Delta), y(\tilde{\theta} - \Delta); \theta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \theta) \\ = \quad & -\int_{\tilde{\theta}-\Delta}^{\tilde{\theta}} \frac{g(\theta')}{2} \left[\varphi(\theta') - 2(\theta - \theta') f(\epsilon(x(\theta'), y(\theta'))) \left(\frac{1}{g(x(\theta'))} + \frac{1}{g(y(\theta'))}\right)\right] d\theta'.\end{aligned}$$

As a consequence, $Eu(x(\tilde{\theta} - \Delta), y(\tilde{\theta} - \Delta); \theta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \theta)$ is increasing in θ for all $\theta \in (\tilde{\theta}, y(\tilde{\theta}))$.

[24] Slightly sliding $\tilde{\theta}$ means that the number $g(\tilde{\theta}) \cdot d\tilde{\theta}$ of voters moves from R to L, so that this move can be regarded as $d\delta^{R \to L}_{(x,y)} = g(\tilde{\theta})d\tilde{\theta}$ in (3), then we obtain $\frac{dx}{d\tilde{\theta}} = \frac{g(\tilde{\theta})}{2g(x)}$. Similarly, we can also obtain $\frac{dy}{d\tilde{\theta}} = \frac{g(\tilde{\theta})}{2g(y)}$.

The first term of the content of bracket $\frac{g(\theta')}{2}\varphi(\theta')$ in the right side is the change of the border type's expected utility. Thus, integrating $\frac{g(\theta')}{2}\varphi(\theta')$ from $\tilde{\theta}$ to $\tilde{\theta}+\Delta$, the voter's type that is evaluating is also moving from $\tilde{\theta}$ to $\tilde{\theta}+\Delta$. The second term is a correction term that appears when the voter's type that is evaluating is fixed at θ. Note that the utility that each type of voter obtains from candidate y and x is different; in other words, each type evaluates candidate y and x differently. Thus, when the evaluator changes from θ' to θ, this difference in evaluation has to be adjusted.[25] More concretely, since the difference in the utility of y and x is $-|y-\theta|+|x-\theta|-\epsilon(x,y) = 2(\theta-\theta_{med})$, the difference of this term between type θ' and type θ is $2(\theta-\theta_{med})-2(\theta'-\theta_{med}) = 2(\theta-\theta')$. Now, note that

$$\frac{dF(\epsilon(x(\theta'),y(\theta')))}{d\theta'} = \frac{dF(\epsilon(x,y))}{d\epsilon}\cdot\frac{d\epsilon(x,y)}{d\theta'} = f(\epsilon(x(\theta'),y(\theta')))\cdot\frac{g(\theta')}{2}\left(\frac{1}{g(x(\theta'))}+\frac{1}{g(y(\theta'))}\right)$$

Since $F(\epsilon(x(\theta'),y(\theta')))$ is the probability of y winning, this denotes the change in the expected utility of the border type θ' by changing the probability of y winning when the threshold slightly moves to the right. (Note that increasing y's winning probability means decreasing x's winning probability, and vice versa, so that this expression also means a change in the probability of x losing at the same time.) As a result, for this adjustment, we have to subtract the term of $2(\theta-\theta')$ multiplied by the change in the probability $f(\epsilon(x(\theta'),y(\theta')))\cdot\frac{g(\theta')}{2}(\frac{1}{g(x(\theta'))}+\frac{1}{g(y(\theta'))})$ from $\frac{g(\theta')}{2}\varphi(\theta')$. Thus, voters of the edge type $\tilde{\theta}+\Delta$ in the coalition of $[\tilde{\theta},\tilde{\theta}+\Delta]$ receive a utility improvement from the deviation that is smallest among the coalition members, since the formula in Lemma 2 is a decreasing function of θ.

When $\varphi(\tilde{\theta})=0$, by reducing Δ to zero, both equalities in Lemma 2 are proportional to $\frac{\varphi'(\tilde{\theta})}{2} - f(\epsilon(x(\tilde{\theta}),y(\tilde{\theta})))\left(\frac{1}{g(x(\tilde{\theta}))}+\frac{1}{g(y(\tilde{\theta}))}\right)$. Although it is more involved to prove the following proposition formally, we can intuitively interpret the following characterization of a sorting political equilibrium.

Proposition 1 *Suppose that f and g are differentiable. A sorting allocation with threshold $\tilde{\theta}$ is a political equilibrium if (i) $\varphi(\tilde{\theta})=0$ and (ii) $\frac{\varphi'(\tilde{\theta})}{2} - f(\epsilon(x(\tilde{\theta}),y(\tilde{\theta})))\left(\frac{1}{g(x(\tilde{\theta}))}+\frac{1}{g(y(\tilde{\theta}))}\right) < 0$. On the other hand, a sorting allocation with threshold $\tilde{\theta}$ is a political equilibrium only if (i) $\varphi(\tilde{\theta})=0$ and (ii') $\frac{\varphi'(\tilde{\theta})}{2} - f(\epsilon(x(\tilde{\theta}),y(\tilde{\theta})))\left(\frac{1}{g(x(\tilde{\theta}))}+\frac{1}{g(y(\tilde{\theta}))}\right) \leq 0$.*

Proposition 1 says that $\varphi(\tilde{\theta})=0$ is not sufficient but is a necessary condition and that we also need a slope condition of $\varphi(\tilde{\theta})$ for a sorting allocation to become a political equilibrium.

From the above proposition, we can easily find a sufficient condition for a sorting allocation to become a political equilibrium.

[25] On the other hand, the change in expected utility of the border type θ' by changing each candidate's position is the same as that of the evaluator type θ, since linear utility is assumed.

Corollary 2 *Suppose that f and g are differentiable. A sorting allocation with threshold $\tilde{\theta}$ is a political equilibrium if (i) $\varphi(\tilde{\theta}) = 0$ and (ii) $\varphi'(\tilde{\theta}) \leq 0$.*

We can also find sufficient conditions for the existence of a sorting political equilibrium by imposing $\varphi(0) > 0$ and $\varphi(1) < 0$ in the below theorem. [26]

Theorem 1 *Suppose that f and g are differentiable, $Supp(f) \supset [-1, 1]$, and $g(\theta) > 0$ for all $\theta \in (0, 1)$, and $\frac{g(0)}{g(\theta_{med})} \leq \frac{P_x(x(0),y(0))}{P_y(x(0),y(0))}$ and $\frac{g(1)}{g(\theta_{med})} \leq \frac{P_y(x(1),y(1))}{P_x(x(1),y(1))}$. Then, there exists a sorting political equilibrium with an interior threshold $\tilde{\theta} \in (0, 1)$.*

The condition basically says that it is enough that (i) the ratios $g(0)/g(\theta_{med})$ and $g(1)/g(\theta_{med})$ are small, which means most extreme voters in both parties have less influence than moderate voters relatively, and (ii) uncertainty in relative competence of the two candidates is large, which means that even an extreme candidate has a relatively large chance to win to guarantee the existence of equilibrium. Later, we consider a special case of uniformly distributed f, and we can dispense with the second requirement.

5 The Party Structure in a Political Equilibrium

So far, although we have characterized political sorting equilibria, it is not yet clear what the intrinsic factor is for the determination of the party line. In this section, we make this clear by presenting more analyses and examples.

5.1 General Case

In this subsection, we investigate how the distribution of voter types is important to determining the equilibrium party structure by using our characterization of a sorting political equilibrium, which includes Proposition 1 or Corollary 2. We start with comparing the equilibrium party line $\tilde{\theta}$ with the traditional "median voter" θ_{med} in the following two propositions. It is convenient to define the changes in candidates' positions weighted by their chance of winning when the party line $\tilde{\theta}$ increases (moves to the right) slightly:[27]

$$\xi(\tilde{\theta}) \equiv -\frac{P_y(x(\tilde{\theta}), y(\tilde{\theta}))}{g(y(\tilde{\theta}))} + \frac{P_x(x(\tilde{\theta}), y(\tilde{\theta}))}{g(x(\tilde{\theta}))}.$$

Proposition 2 *Suppose that the conditions in Theorem 1 are met. Then, unless $\xi(\theta_{med}) = 0$, θ_{med} cannot be a party line of a political equilibrium.*

[26] Note that we are assuming that there are always two parties even in the case of $\tilde{\theta} = 0$ or 1. Here, we are considering the case where the minority party is extremely small ($\tilde{\theta} = \epsilon$ or $\tilde{\theta} = 1 - \epsilon$ for ϵ very small). Taking the limit, we have $\lim_{\epsilon \to 0} \varphi(\epsilon) = \varphi(0)$ and $\lim_{\epsilon \to 0} \varphi(1 - \epsilon) = \varphi(1)$.

[27] Strictly speaking, $\xi(\tilde{\theta})$ is an effect when x and y move by $2 \cdot dx/d\delta^{R \to L}_{(x,y)} = 1/g(x)$ and $2 \cdot dy/d\delta^{R \to L}_{(x,y)} = 1/g(y)$ in (3).

This proposition tells us that a sorting political equilibrium does not always divide the voters into two parties at the median type. Next, we will consider the condition where the median type becomes the threshold of a two-party structure. It turns out that it is not sufficient to have symmetric g and f, though the additional condition is often satisfied.

Proposition 3 *Suppose that the conditions in Theorem 1 are met, and that g and f are symmetric. Then, there is a political equilibrium with $\tilde{\theta} = \theta_{med}$ if and only if*

$$\frac{g(\theta_{med})}{4g(x(\theta_{med}))^2}\left(4f(0) - \frac{g'(x(\theta_{med}))}{g(x(\theta_{med}))}\right) - \frac{4f(0)}{g(x(\theta_{med}))} \leq 0.$$

The above proposition tells us that, in general, $\tilde{\theta}$ and θ_{med} have no reason to coincide with each other. They can coincide, but only in very special situations. We can elaborate this observation using the ξ function:

Proposition 4 *Suppose that the conditions in Theorem 1 are met. Then there exists a sorting equilibrium with threshold $\tilde{\theta}^* > (<)\theta_{med}$ if $\xi(\theta_{med}) > (<)0$.*

The condition $\xi(\theta_{med}) > 0$ says that if a small group near θ_{med} switches its party from R to L, then x moves more to the right than y moves to the left in the sense of the expectation. That is, the gain of x coming closer is greater than the loss of y moving away from θ_{med}, and such a small group in R prefers to switch its party. Thus, the statement of Proposition 4 is intuitive.

Incidentally, although the above propositions provide explanations for an equilibrium of a party line far from the median of voters on a distribution $g(\theta)$ that includes a less irregular distribution, which is in favor of our theory, those explanations do not mention that there is only a biased equilibrium once a distribution $g(\tilde{\theta})$ is determined. In other words, we cannot exclude the possibility that there may be another equilibrium that has an opposite bias. In the next subsection, we will impose a simplifying example on competence distribution f.

In concluding this subsection, we can discuss how the distribution of voters g affects the equilibrium party lines. Using $\xi(\tilde{\theta})$ and the above consideration, we can talk about equilibrium predictions by focusing on the sign of $\xi(\theta_{med})$. We suppose that when $\xi(\theta_{med}) \geq 0$ (≤ 0), $\varphi(\theta) > 0$ holds for all $\theta \in [0, \theta_{med})$ ($\varphi(\theta) < 0$ holds for all $\theta \in (\theta_{med}, 1]$). If $g(y(\theta_{med})) = g(x(\theta_{med}))$, then what matters is $P_y(x(\theta_{med}), y(\theta_{med})) \gtreqless P_x(x(\theta_{med}), y(\theta_{med}))$. If $y(\theta_{med})$ is further away from θ_{med} than $x(\theta_{med})$, then $P_y(x(\theta_{med}), y(\theta_{med})) < P_x(x(\theta_{med}, y(\theta_{med})))$ holds since Corollary 1, and $\theta_{med} < \tilde{\theta}$ such that $\varphi(\tilde{\theta}) = 0$ occurs. This means that if the support group of party R shifts more to the extreme right in the policy spectrum, then moderate R supporters tend to switch from R to L. If $P_y(x(\theta_{med}), y(\theta_{med})) = P_x(x(\theta_{med}, y(\theta_{med})))$ that is $\theta_{med} = \frac{x+y}{2}$ while $g(y(\theta_{med})) > g(x(\theta_{med}))$, then a coalitional move from R to L does not change the position of R's candidate much, while the position of L's candidate is pulled toward the median. Thus, the moderate R supporters tend to switch parties. That is, other things being equal, if a party's support group (in terms of its policy spectrum) becomes more extreme, then moderate voters' strategic behavior of switching from R to L makes the party lose of some of its support, by making its candidate more extreme and the opponent party's candidate more moderate (see Figure 2). As a result, the probability of R's candidate winning is reduced. To sum up, the distribution of voters $g(\theta)$ determines the relation between each candidate and a party line, and the candidates' positions determine their chances of winning. Thus, the distribution of voters plays a deterministic role.

5.2 A Special Case: Uniform f function

In the previous subsection, we saw that the importance of the voters' distribution. In this subsection, we focus on $g(\theta)$. Here, we assume that the common random variable called relative competence is uniform (f is a uniform distribution). Since our interest in this section is to see how the distribution of voters g affects the party line and voting outcome, the distribution of f is not really essential to our analysis.

$$f(\epsilon) = \begin{cases} \frac{1}{2a} & \text{if } \epsilon \in [-a, a] \\ 0 & \text{otherwise} \end{cases}$$

Here, we assume $a \geq 1$, so that even an extreme candidate has a chance to win. Then, with this uniformly distributed f, our $\varphi : [0,1] \to \mathbb{R}$ can be written as follows using $\epsilon(x(\tilde{\theta}), y(\tilde{\theta})) = 2\theta_{med} - x(\tilde{\theta}) - y(\tilde{\theta})$:

$$\begin{aligned} \varphi(\tilde{\theta}) &\equiv \frac{1}{g(y(\tilde{\theta}))}\left[-F(\epsilon(x(\tilde{\theta}), y(\tilde{\theta}))) - 2(\tilde{\theta} - \theta_{med})f(\epsilon(x(\tilde{\theta}), y(\tilde{\theta})))\right] \\ &\quad + \frac{1}{g(x(\tilde{\theta}))}\left[\left(1 - F(\epsilon(x(\tilde{\theta}), y(\tilde{\theta})))\right) - 2(\tilde{\theta} - \theta_{med})f(\epsilon(x(\tilde{\theta}), y(\tilde{\theta})))\right] \\ &= \frac{1}{g(y(\tilde{\theta}))} \times \frac{1}{2a} \times \left[-a + x(\tilde{\theta}) + y(\tilde{\theta}) - 2\tilde{\theta}\right] \\ &\quad + \frac{1}{g(x(\tilde{\theta}))} \times \frac{1}{2a} \times \left[a + x(\tilde{\theta}) + y(\tilde{\theta}) - 2\tilde{\theta}\right] \end{aligned} \tag{6}$$

From the above formula (6), we obtain

$$2a \cdot \varphi(\tilde{\theta}) = \left(x(\tilde{\theta}) + y(\tilde{\theta}) - 2\tilde{\theta}\right)\left(\frac{1}{g(y(\tilde{\theta}))} + \frac{1}{g(x(\tilde{\theta}))}\right) + a \times \left(\frac{1}{g(x(\tilde{\theta}))} - \frac{1}{g(y(\tilde{\theta}))}\right). \tag{7}$$

Note that $x(0) = 0$, $y(0) = \theta_{med}$, $x(1) = \theta_{med}$, and $y(1) = 1$. Thus, we obtain

$$2a \cdot \varphi(0) = \frac{1}{g(\theta_{med})} \times (-a + \theta_{med}) + \frac{1}{g(0)} \times (a + \theta_{med})$$

and

$$2a \cdot \varphi(1) = \frac{1}{g(1)} \times (-a - (1 - \theta_{med})) + \frac{1}{g(\theta_{med})} \times (a - (1 - \theta_{med}))\,.$$

Thus, $g(\theta_{med}) \geq g(0)$ is sufficient for $\varphi(0) > 0$, and $g(\theta_{med}) \geq g(1)$ is sufficient (but not necessary) for $\varphi(1) < 0$. This implies that we can assure the existence of political equilibrium under weak sufficient conditions when uniform f function is assumed.

Theorem 2 *If g is continuous with $g(\theta) > 0$ for all $\theta \in [0,1]$ and f is a uniform distribution with support $[-a,a]$ where $a \geq 1$, then $g(0) \leq g(\theta_{med})$ and $g(1) \leq g(\theta_{med})$ are sufficient for the existence of political equilibrium with interior $\tilde{\theta}$.*

It is still in general hard to tell how the sign of $\varphi(\tilde{\theta})$ changes as $\tilde{\theta}$ goes up, but we can decompose the effects. Rewriting (7), it is easy to see that $\varphi(\tilde{\theta}) \gtreqless 0$ holds if and only if

$$\underbrace{\left(x(\tilde{\theta}) + y(\tilde{\theta}) - 2\tilde{\theta}\right)}_{A} + \ a \times \underbrace{\left(\frac{g(y(\tilde{\theta})) - g(x(\tilde{\theta}))}{g(x(\tilde{\theta})) + g(y(\tilde{\theta}))}\right)}_{B} \gtreqless 0. \tag{8}$$

When $\tilde{\theta}$ is small, $x(\tilde{\theta})$ and $\tilde{\theta}$ are close to each other while $y(\tilde{\theta})$ is far from $\tilde{\theta}$. As a result, term A is positive. Similarly, when $\tilde{\theta}$ is large, $x(\tilde{\theta})$ is far from $\tilde{\theta}$ while $\tilde{\theta}$ and $y(\tilde{\theta})$ are close to each other, implying that term A is negative. It is easy to see that term A tends to be decreasing in $\tilde{\theta}$, although the sign of term A can change multiple times in the middle depending on the subtle shape of the density function g. However, term B has large first-order effects of changes in the relative sizes of $g(x(\tilde{\theta}))$ and $g(y(\tilde{\theta}))$. Clearly, the sign and the value of term B can be volatile as $\tilde{\theta}$ increases. However, in simple cases, we can more or less see the shape of the φ function.

Proposition 5 *Suppose that the conditions of Theorem 2 are met. Then, there is a unique equilibrium if we have*

1. $\dfrac{g(\tilde{\theta})}{g(x(\tilde{\theta}))} + \dfrac{g(\tilde{\theta})}{g(y(\tilde{\theta}))} \leq 4$, *and*

2. $g'(y(\tilde{\theta}))\dfrac{g(x(\tilde{\theta}))}{g(y(\tilde{\theta}))} \leq g'(x(\tilde{\theta}))\dfrac{g(y(\tilde{\theta}))}{g(x(\tilde{\theta}))}$ *for all* $\tilde{\theta} \in [0,1]$.

Condition 1 corresponds to $dA/d\tilde{\theta} \leq 0$, and condition 2 corresponds to $dB/d\tilde{\theta} \leq 0$. It is easy to see that the conditions are satisfied as long as g does not fluctuate much (a flat distribution: for example, if $\max_\theta g(\theta) \leq 2 \times \min_\theta g(\theta)$, then condition 1 is surely satisfied). Regarding Condition 2, suppose that the distribution g has a single-peak at $\theta^p \in [0,1]$. Clearly, if $x < \theta^p < y$, then $g'(x) > 0 > g'(y)$ the condition is satisfied. If $x < y < \theta^p$, then $g(y) > g(x)$, $g'(x) > 0$ and $g'(y) > 0$ hold, and the condition is still satisfied unless $g'(y)$ is much larger than $g'(x)$. Similarly, if $\theta^p < x < y$, then $g(x) > g(y)$, $g'(x) < 0$, and $g'(y) < 0$ hold, and the condition is again satisfied unless the absolute value of $g'(x)$ is much larger than $g'(y)$. Thus, if g is single-peaked and $g'(\theta)$ does not change much within interval $[0, \theta^p)$ and interval $(\theta^p, 1]$, then condition 2 is satisfied. Summarizing the above, we have the following corollary:

Corollary 3 *Suppose that the conditions of Theorem 2 are met. Then, there is unique equilibrium if (i)* $\max_\theta g(\theta) \leq 2 \times \min_\theta g(\theta)$ *and (ii) g is single-peaked.*

By the above analysis, we can see that if the voters' distribution is single-peaked and the slope on either side does not change much, then both A and B in (8) are decreasing; thus $\varphi(\tilde{\theta}) = 0$ can occur once and for all, implying unique equilibrium. Now, let us go back to Proposition 4. The proposition assures that if the conditions of Theorem 1 (alternatively 2) are met, and if $\xi(\theta_{med}) \gtreqless 0$, then there is a political equilibrium of which party line satisfies $\tilde{\theta}^* \gtreqless \theta_{med}$. Note that $\xi(\theta_{med}) = \varphi(\theta_{med})$. Thus, under the conditions in Proposition 5, we have a unique political equilibrium with $\tilde{\theta}^* \gtreqless \theta_{med}$ if and only if

$$\underbrace{(x(\theta_{med}) + y(\theta_{med}) - 2\theta_{med})}_{\text{winning probability effect}} + \; a \times \underbrace{\left(\frac{g(y(\theta_{med})) - g(x(\theta_{med}))}{g(x(\theta_{med})) + g(y(\theta_{med}))}\right)}_{\text{policy responsiveness effect}} \gtreqless 0.$$

That is, the L party has an advantage for attracting supporters over the R party if the winning probability effect is positive, which means x is closer to the median voter than y, and if the policy responsiveness effect is positive, which means L's policy platform is more responsive to the new centrists' participation than R's policy platform. Now, let's say $\theta^p < \theta_{med}$, which implies that the L party has a more extreme group. In this case, the winning probability effect tends to be positive while the policy responsiveness effect is negative (Figure 3: a figure of a single-peaked voter distribution with a biased peak to the left, so that x is closer to the median than y). Then, what is the sign of $\xi(\theta_{med})$? It is ambiguous. However, if a is large enough, which means the uncertainty in competence level is large enough, i.e., even an extreme candidate has a chance to win the election; then the policy responsiveness effect dominates the winning probability effect and $\xi(\theta_{med}) < 0$ and $\tilde{\theta}^* < \theta_{med}$ hold. A large a is required, since the L party's choosing an extreme candidate is justifiable only when that candidate has a good enough chance to win the election.

In the following example, we consider the case where the voters' distribution in the left area is more dense than the right area and the median is in the left side. For simplicity, we will allow discontinuity of g and analyze a minimally asymmetric voter distribution: density function g is a step function, and discontinuity occurs only at the median. We show that the L party has a denser voter distribution and a shorter tail, and that there is unique equilibrium in which the L party loses some of its moderate supporters.

Example 2. Consider the case where $g(\theta)$ is a step function and $f(\epsilon)$ is uniform:

$$g(\theta) = \begin{cases} \frac{1}{2\theta_{med}} & \text{if } \theta \leq \theta_{med} \\ \frac{1}{2(1-\theta_{med})} & \text{if } \theta > \theta_{med} \end{cases}$$

where $a > \frac{1}{2}$ to assure that no probabilities of x and y winning become zero ($1 - F(x) > 0$ and $F(y) > 0$). *Without loss of generality, we assume* $\theta_{med} \leq \frac{1}{2}$ *so that* $\frac{1}{2\theta_{med}} \geq \frac{1}{2(1-\theta_{med})}$. In this case, we have a unique political equilibrium with

$$\tilde{\theta} = \frac{2\theta_{med}((2a+1)\theta_{med} - a)}{4\theta_{med} - 1} < \theta_{med}.$$

Thus, the party line is unambiguously biased. The details of the figure and the calculations of $g(\theta)$ and $\varphi(\tilde{\theta})$ are in Appendix B and Figure 4. □

Example 2 shows that as long as g is relatively flat, the equilibrium is unique even if the voter distribution is asymmetric though the party line is biased.

On the other hand, we can also show a case of multiple equilibria. In the following, we consider a symmetric voter distribution g to show that there can be multiple equilibria if g goes up and down. There are three core groups in the example: Extreme Left (EL), Center (C), and Extreme Right (ER). We assume that Moderate Left (ML) and Moderate Right (MR) are distributed in a wider political range and are less concentrated than EL, C, and ER. This distribution expresses a political situation where major voters are divided in three different and narrow ranges and their opinions are conflicting. This political conflict brings multiple equilibria. With significant ups and downs, we can have multiple equilibria even if voter distribution is asymmetric.

Example 3. Let us consider the following symmetric voter distribution described by a step function ($0 < b \leq 1$: $b = 1$ corresponds to uniform g).

$$g(\theta) = \begin{cases} 3 - 2b & \text{for all } \theta \in \underbrace{[0, \frac{1}{9}]}_{EL} \cup \underbrace{[\frac{4}{9}, \frac{5}{9}]}_{C} \cup \underbrace{[\frac{8}{9}, 1]}_{ER} \\ b & \text{for all } \theta \in \underbrace{(\frac{1}{9}, \frac{4}{9})}_{ML} \cup \underbrace{(\frac{5}{9}, \frac{8}{9})}_{MR} \end{cases}$$

When $b = 1$, this example degenerates to uniformly distributed g. As b decreases from unity, the voters' distribution becomes more and more politically divided although we assume that there are still plenty in the centrist group. We can regard b's getting lower as a change from a situation where voters have scattered political opinions to a situation where voters' political opinions are becoming integrated around three positions and their conflict is getting more severe. See the Figures 5 - 8. Due to the discontinuity of the g function, the φ function becomes discontinuous, but we can easily approximate it by a continuous function by using the standard procedure. Note that $g(0) = g(1) = g(\theta_{med})$ holds ($\theta_{med} = \frac{1}{2}$), and the conditions of Theorem 2 are all satisfied after an approximation of g. Since everything is symmetric, we can focus on the cases of $\tilde{\theta} \in [0, \frac{1}{2}]$. We will investigate what will happen on $x(\tilde{\theta})$ and $y(\tilde{\theta})$ (thus including $\varphi(\tilde{\theta})$), as $\tilde{\theta}$ increases from 0 to $\frac{1}{2}$. Clearly, $x(0) = 0$ and $y(0) = \theta_{med} = \frac{1}{2}$. It is also easy to see $x(\frac{1}{9}) = \frac{1}{18}$ and

$y(\frac{1}{9}) = \frac{5}{9}$, and from $\tilde{\theta} = \frac{1}{9}$, $y(\tilde{\theta})$ enters into interval $(\frac{5}{9}, \frac{8}{9})$ with lower density while $x(\tilde{\theta})$ remains in interval $[0, \frac{1}{9}]$ with high density. From these calculations, we have $\varphi(\tilde{\theta}) > 0$ for $\tilde{\theta} \in [0, \frac{1}{9}]$. Since g and f are symmetric, $\varphi(\tilde{\theta}) < 0$ for $\tilde{\theta} \in [\frac{8}{9}, 1]$ is also obtained. Thus, Proposition 3 is also applicable in this example after the approximation, although the sufficient conditions of Theorem 1 are not met. In this example, the condition in the proposition becomes quite simple, that is, $g(\theta_{med}) \leq 4g(x(\theta_{med}))$ for any a. Since $g(x(\theta_{med}))$ is either equal to or less than $g(\theta_{med})$ from the definition of g, we have $b \geq \frac{1}{2}$ when $g(x(\theta_{med})) < g(\theta_{med})$. We can use the condition for checking whether θ_{med} is a symmetric equilibrium. However, from then on, we need to classify a few subcases. The following tables summarize the relevant information. We have three cases:

Table 1. x, y and $2 \cdot \varphi$ where $g(\theta)$ is the step function.

Case 1. $b \le \frac{3}{8}$ ($x(\tilde\theta)$ does not enter onto interval $(\frac{1}{9}, \frac{4}{9})$ implying $y(\frac{1}{2}) \in [\frac{8}{9}, 1]$ by symmetry)

	$x(\tilde\theta)$	$y(\tilde\theta)$	$2a \cdot \varphi(\tilde\theta)$
$0 \le \tilde\theta \le \frac{1}{9}$	$\frac{\tilde\theta}{2}$	$\frac{1}{2} + \frac{\tilde\theta}{2}$	$\frac{2}{3-2b}\left(\frac{1}{2} - \tilde\theta\right)$
$\frac{1}{9} < \tilde\theta < \frac{4}{9}$	$\frac{1-b}{6(3-2b)} + \frac{b}{2(3-2b)}\tilde\theta$	$\frac{1}{2} + \frac{\tilde\theta}{2}$	$\frac{3-b}{b(3-2b)}\left(\frac{10-7b}{6(3-2b)} - \frac{9-7b}{2(3-2b)}\tilde\theta\right) - \frac{3(1-b)}{b(3-2b)}a$
$\frac{4}{9} \le \tilde\theta < \frac{4}{3(3-2b)} - \frac{5b}{9(3-2b)}$	$\frac{-1+b}{2(3-2b)} + \frac{\tilde\theta}{2}$	$\frac{7}{6} - \frac{2}{3b} + \frac{3-2b}{2b}\tilde\theta$	$\frac{3-b}{b(3-2b)}\left(\frac{18-11b}{6(3-2b)} - \frac{2}{3b} + \frac{3-5b}{2b}\tilde\theta\right) - \frac{3(1-b)}{b(3-2b)}a$
$\frac{4}{3(3-2b)} - \frac{5b}{9(3-2b)} \le \tilde\theta \le \frac{1}{2}$	$\frac{-1+b}{2(3-2b)} + \frac{\tilde\theta}{2}$	$\frac{4-3b}{2(3-2b)} + \frac{\tilde\theta}{2}$	$\frac{2}{3-2b}\left(\frac{1}{2} - \tilde\theta\right)$

Case 2. $\frac{3}{8} < b \le \frac{3}{5}$ ($x(\tilde\theta)$ enters into interval $(\frac{1}{9}, \frac{4}{9})$ after $\tilde\theta$ enters into interval $(\frac{4}{9}, \frac{1}{2})$, implying $y(\frac{1}{2}) \in (\frac{5}{9}, \frac{8}{9})$)

	$x(\tilde\theta)$	$y(\tilde\theta)$	$2a \cdot \varphi(\tilde\theta)$
$0 \le \tilde\theta \le \frac{1}{9}$	$\frac{\tilde\theta}{2}$	$\frac{1}{2} + \frac{\tilde\theta}{2}$	$\frac{2}{3-2b}\left(\frac{1}{2} - \tilde\theta\right)$
$\frac{1}{9} < \tilde\theta < \frac{4}{9}$	$\frac{1-b}{6(3-2b)} + \frac{b}{2(3-2b)}\tilde\theta$	$\frac{1}{2} + \frac{\tilde\theta}{2}$	$\frac{3-b}{b(3-2b)}\left(\frac{10-7b}{6(3-2b)} - \frac{9-7b}{2(3-2b)}\tilde\theta\right) - \frac{3(1-b)}{b(3-2b)}a$
$\frac{4}{9} \le \tilde\theta \le \frac{5}{3(3-2b)} - \frac{13b}{9(3-2b)}$	$\frac{-1+b}{2(3-2b)} + \frac{\tilde\theta}{2}$	$\frac{7}{6} - \frac{2}{3b} + \frac{3-2b}{2b}\tilde\theta$	$\frac{3-b}{b(3-2b)}\left(\frac{18-11b}{6(3-2b)} - \frac{2}{3b} + \frac{3-5b}{2b}\tilde\theta\right) - \frac{3(1-b)}{b(3-2b)}a$
$\frac{5}{3(3-2b)} - \frac{13b}{9(3-2b)} < \tilde\theta \le \frac{1}{2}$	$\frac{5}{6} - \frac{5}{6b} + \frac{3-2b}{2b}\tilde\theta$	$\frac{7}{6} - \frac{2}{3b} + \frac{3-2b}{2b}\tilde\theta$	$\frac{2}{b}\left(2 - \frac{3}{2b} + \frac{3-4b}{b}\tilde\theta\right)$

Case 3. $\frac{3}{5} < b \le 1$ ($x(\tilde\theta)$ enters into interval $(\frac{1}{9}, \frac{4}{9})$ before $\tilde\theta$ enters into interval $(\frac{4}{9}, \frac{1}{2})$, implying $y(\frac{1}{2}) \in (\frac{5}{9}, \frac{8}{9})$)

	$x(\tilde\theta)$	$y(\tilde\theta)$	$2a \cdot \varphi(\tilde\theta)$
$0 \le \tilde\theta \le \frac{1}{9}$	$\frac{\tilde\theta}{2}$	$\frac{1}{2} + \frac{\tilde\theta}{2}$	$\frac{2}{3-2b}\left(\frac{1}{2} - \tilde\theta\right)$
$\frac{1}{9} < \tilde\theta < \frac{1}{3b} - \frac{1}{9}$	$\frac{1-b}{6(3-2b)} + \frac{b}{2(3-2b)}\tilde\theta$	$\frac{1}{2} + \frac{\tilde\theta}{2}$	$\frac{3-b}{b(3-2b)}\left(\frac{10-7b}{6(3-2b)} - \frac{9-7b}{2(3-2b)}\tilde\theta\right) - \frac{3(1-b)}{b(3-2b)}a$
$\frac{1}{3b} - \frac{1}{9} \le \tilde\theta < \frac{4}{9}$	$\frac{1}{6} - \frac{1}{6b} + \frac{\tilde\theta}{2}$	$\frac{1}{2} + \frac{\tilde\theta}{2}$	$\frac{2}{b}\left(\frac{2}{3} - \frac{1}{6b} - \tilde\theta\right)$
$\frac{4}{9} \le \tilde\theta \le \frac{1}{2}$	$\frac{5}{6} - \frac{5}{6b} + \frac{3-2b}{2b}\tilde\theta$	$\frac{7}{6} - \frac{2}{3b} + \frac{3-2b}{2b}\tilde\theta$	$\frac{2}{b}\left(2 - \frac{3}{2b} + \frac{3-4b}{b}\tilde\theta\right)$

As we mentioned above, when $b = 1$, $g(\theta)$ is a uniform distribution. Then the equilibrium is unique and symmetric, $\tilde{\theta} = \frac{1}{2}$. Even if b becomes only a little smaller than 1, $\varphi(\tilde{\theta})$ becomes discontinuous[28] at $\tilde{\theta} = \frac{1}{9}$ in Case 3 of Table 1 and at $\frac{8}{9}$ from the symmetry, and shifts below because of the discontinuity of $g(\theta)$ at $\frac{1}{9}$ and $\frac{8}{9}$; see Figure 5. As b gets smaller, this shift gets larger; then two asymmetric equilibria appear in $(\frac{1}{9}, \frac{4}{9})$ and $(\frac{5}{9}, \frac{8}{9})$ in addition to the symmetric equilibrium;[29] see Figure 6.[30] As b gets increasingly smaller, these asymmetric equilibria approach $\frac{1}{9}$ and $\frac{8}{9}$, respectively, and finally stick to them. In $b < \frac{1}{2}$ where b is in Case 2, the symmetric equilibrium disappears although $\varphi(\frac{1}{2}) = 0$ since the condition of Proposition 3 cannot be met, so that there are only two asymmetric equilibria; see Figure 7. With the case of deeply divided voters, in each asymmetric equilibrium, one party will be formed by all extremists ($\{EL\}$ and $\{ER\}$, respectively) and few moderates, while the other party will be formed by the rest ($\{L, C, R, ER\}$ and $\{EL, L, C, R\}$, respectively), and their candidates are extremist and moderately biased centrist. As a result, in one equilibrium $x(\tilde{\theta})$ and $y(\tilde{\theta})$ are around $\frac{1}{18}$ and $\frac{5}{9}$, and in the other those are around $\frac{4}{9}$ and $\frac{17}{18}$), respectively. When b goes on being even smaller and gets into Case 1, $x(\frac{1}{2})$ is in $(0, \frac{1}{9})$ at $\tilde{\theta} = \frac{1}{2}$. Then, although $b < \frac{1}{2}$, the condition of Proposition 3 is met again because of $g(x(\theta_{med}) = g(\theta_{med})$, so that the symmetric equilibrium appears again. Since the asymmetric equilibria still exist, there are again three equilibria; see Figure 8.□

This example shows that if there are core extreme groups (if voters are divided politically), political equilibria can be significantly biased and a political party may represent an extreme core group by alienating the center ground voters even if the voters' distribution is symmetric.[31] The existence of multiple equilibria means that even if political environments, $g(\theta)$ and $f(\theta)$, do not change, the political outcome can be different. That is, when voters are politically divided, if some large enough exogenous shock occurs, then the party supporters' allocation can jump from one political equilibrium to another. The episode of transforming from a deep red to a deep blue state in California (see introduction) can

[28] Function $\varphi(\tilde{\theta})$ becomes discontinuous at the point where x or y stride over a step. In Example 2, $\varphi(\tilde{\theta})$ is not discontinuous but only kinks at θ_{med} with a step because of $x \leq \theta_{med} \leq y$.

[29] When $a = 1$, multiple equilibria show up for a relatively high value of b: for example, we obtain asymmetric equilibria when $b = 0.8$ that makes the densities of EL, C and ER are 1.75 times larger than ML and MR.

[30] The fact that there can be multiple equilibria for medium size b also shows the importance of single-peakedness condition for the uniqueness of political equilibrium in Corollary 3.

[31] This example starkly contrasts our political equilibrium notion with Roemer's equilibrium notion of an endogenous party line (Roemer 2001 Chapter 5). In our model, the voters' party choice is determined by a comparison of expected utilities from joining the L and R parties, but Roemer assumes that $\tilde{\theta}$ is determined by a comparison of two parties' *policies*. For example, if $x = \frac{1}{18}$ and $y = \frac{5}{9}$, the party line is the middle point of the two: $\tilde{\theta} = \frac{11}{36}$. Thus, our biased equilibrium cannot be supported as an equilibrium. In fact, in this example, the Roemer equilibrium must be symmetric $\tilde{\theta} = \frac{1}{2}$.

be explained as a consequence of such a case of politically divided voters.

6 Conclusion

In this paper, we considered a two-party representative democracy and investigated how the distribution of voters' policy positions on a one-dimensional issue space affects the party line and the probability of each party's winning. We introduced a common shock that affects each voter's utility, instead of the standard idiosyncratic shocks in the probabilistic voting model. We also introduced a new equilibrium concept *political equilibrium*, which is immune to any small coalitional deviations near the party line, in contrast with Nash equilibrium and strong equilibrium. This notion simplifies the characterization of the equilibrium by focusing on a sorting allocation case. We showed that voters' distribution intrinsically affects the party line in the political equilibrium. In addition, we showed that, in Example 3 where voters are divided into three political positions, multiple equilibria appear as the division grows deeper. Especially, if voters are deeply divided, symmetric equilibrium disappears even though the distribution is symmetric. In each asymmetric equilibrium, the minority candidate becomes more extreme, and the other becomes more moderate. Those multiple equilibria appearing from deeply divided voters can be interpreted as bringing us to political instability that results in elections swinging extremely between left and right.

In future research, we may consider three extensions, although they may be difficult. First, we may try to generalize the functional form of the voters' utility function. To be concrete, we may consider a strictly convex utility (Osborne 1995) case. In this case, voters are more sensitive to candidates' positions who are close to their own positions. The convex utility function means that voters who are farther away from candidates do not take much interest in them. We redefine the utility function of the voter as

$$u(p_k; \theta, \epsilon) = -v(|p_k - \theta|) + \epsilon,$$

where $v'(\cdot) < 0$ and $v''(\cdot) > 0$ and $k \in C$ is a winner. This type of utility function is discussed in Osborne (1995). With such a utility function, one may think that extreme left or right voters — voters far to the left (right) of the median of party L (R) — have no incentive to switch parties, and we may be able to drop the assumption of psychological costs. It is perhaps true that such a convex cost function reduces the incentive to switch parties, but it would not totally resolve the problem, since a voter with an extreme position may be made better off by her party's candidate becoming more moderate and gaining a higher chance of winning even if the voter does not care about the other party's candidate's position. It all depends on the relative magnitudes of two effects: dissatisfaction with her party's candidate's position becoming more moderate and satisfaction with the candidate's increase in winning probability.

Second, it would be interesting to think about how to make each party's supporters select their candidate strategically in the original Besley-Coate model (Besley and Coate 1997). One way is to assume that given the party line, each voter tries to find her ideal candidate for the party (depending on her policy position and her candidate's chances of winning). It may be possible for us to drop our simple median voter assumption in order to show the existence of equilibrium. When f is uniformly distributed, Kobayashi and Konishi (2012; draft) assume that in primary elections, each voter announces her ideal policy position (taking winning probabilities and her true bliss point) given the other party's candidate position, and the median of announced positions becomes the party's candidate position. With this party decision rule, the candidates' position profile is determined as a Nash equilibrium. We show that the best response curve of each party is more moderate than the party median (bounded above by the party's true median position), and the equilibrium outcome is weakly more moderate than the naive primary elections we considered in this paper. However, the characterization of equilibrium under general assumptions can be very difficult.

Third, we used a static model in this paper. Although static approach has its own advantages, it also has drawbacks — we need to treat both candidates symmetrically, and we cannot introduce incumbents and challengers into the model. However, we can accommodate incumbents in our model if we assume static expectation dynamics (Kramer 1977; Ferejohn, Fiorina, and Packel 1980; Ferejohn, McKelvey, and Packel 1984; Kollman, Miller, and Page 1992; and, in particular, Bender, Diermeier, Siegel, and Ting 2011). Voters know the incumbent's competence level, while they do not know how competent a challenger is going to be compared with whoever wins in the other party's primary election. The incumbent's policy and competence level are intact, and the party line is determined by the previous election. Suppose that a party occupies the office and there is an incumbent candidate. The challenging party chooses its candidate in the way of previous paragraph. It may be interesting to see how the challenging party reacts against competent and incompetent incumbents, and how the dynamics of candidate profiles emerge.

Appendix A: Proofs

Proof of Lemma 1 Each candidate is a median type of each party, $x \leq \theta_{med} \leq y$. Assume that ϵ makes type $\hat{\theta} \in [x, y]$ being indifferent between x and y. Then, $\forall \theta \in [x, y)$

such that $\theta < \hat{\theta}$, and $\forall \bar{\theta} \in [0, x)$,

$$\begin{aligned}
0 = h(x, y; \hat{\theta}) - \epsilon &= -(y - \hat{\theta}) + (\hat{\theta} - x) - \epsilon = 2\hat{\theta} - x - y - \epsilon \\
&> 2\theta - x - y - \epsilon = h(x, y; \theta) - \epsilon \\
&\geq 2x - x - y - \epsilon \\
&= x - y - \epsilon = -(y - \bar{\theta}) + (x - \bar{\theta}) - \epsilon = h(x, y; \bar{\theta}) - \epsilon.
\end{aligned}$$

Thus, all voters of $\theta \in [0, \hat{\theta})$ prefer x to y, since $h(x, y; \theta)$, which is the relative evaluation of y to x is negative; that is all voters of $\theta < \hat{\theta}$ type vote for x when ϵ. Here, if $\epsilon > \epsilon(x, y)$, then, from

$$\begin{aligned}
0 = h(x, y; \hat{\theta}) - \epsilon &= 2\hat{\theta} - x - y - \epsilon \\
&< 2\hat{\theta} - x - y - \epsilon(x, y) \\
&= 2\hat{\theta} - x - y - h(x, y; \theta_{med}) = 2(\hat{\theta} - \theta_{med}),
\end{aligned}$$

we have $\hat{\theta} > \theta_{med}$. Hence, x gets a majority and wins when $\epsilon > \epsilon(x, y)$.

Similarly, if $\epsilon < \epsilon(x, y)$, $\hat{\theta} < \theta_{med}$ and every type $\theta > \hat{\theta}$ vote for y, and y wins. □

Proof of of Corollary 1 We consider the $h(x, y; \theta)$ such that $\theta = \frac{x+y}{2}$:

$$h(x, y; \frac{x+y}{2}) = -(y - \frac{x+y}{2}) + (\frac{x+y}{2} - x) = 0$$

This means that type $\frac{x+y}{2}$ is indifferent between x and y when $\epsilon = 0$. Thus, when $\theta_{med} < \theta = \frac{x+y}{2}$, we have the below inequality:

$$0 = h(x, y; \frac{x+y}{2}) > h(x, y; \theta_{med}) = \epsilon(x, y)$$

From the assumptions $E(\epsilon) = 0$ and the symmetry of the distribution of ϵ, $F(\epsilon(x, y)) < \frac{1}{2} < 1 - F(\epsilon(x, y))$ can be obtained. The other case is shown as well as the above. □

Proof of Lemma 2 Differentiating $Eu(x(\tilde{\theta}), y(\tilde{\theta}); \theta)$ with respect to $\tilde{\theta}$, we obtain:

$$\begin{aligned}
&\frac{dEu(x(\tilde{\theta}), y(\tilde{\theta}); \theta)}{d\tilde{\theta}} \\
&= \frac{g(\tilde{\theta})}{2} \left[-\frac{F(\epsilon(x, y))}{g(y)} + \frac{1 - F(\epsilon(x, y))}{g(x)} - 2(\theta - \theta_{med}) f(\epsilon(x, y)) \left(\frac{1}{g(x)} + \frac{1}{g(y)} \right) \right] \\
&= \frac{g(\tilde{\theta})}{2} \left[-\frac{F(\epsilon(x, y))}{g(y)} + \frac{1 - F(\epsilon(x, y))}{g(x)} - 2(\tilde{\theta} - \theta_{med}) f(\epsilon(x, y)) \left(\frac{1}{g(x)} + \frac{1}{g(y)} \right) \right. \\
&\qquad \left. - 2(\theta - \tilde{\theta}) f(\epsilon(x, y)) \left(\frac{1}{g(x)} + \frac{1}{g(y)} \right) \right] \\
&= \frac{g(\tilde{\theta})}{2} \left[\varphi(\tilde{\theta}) - 2(\theta - \tilde{\theta}) f(\epsilon(x, y)) \left(\frac{1}{g(x)} + \frac{1}{g(y)} \right) \right].
\end{aligned}$$

This implies that for small $\Delta > 0$, we have

$$
\begin{aligned}
& Eu(x(\tilde{\theta}+\Delta), y(\tilde{\theta}+\Delta); \theta) \\
= \; & Eu(x(\tilde{\theta}), y(\tilde{\theta}); \theta) + \int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} \frac{dEu(x(\theta'), y(\theta'); \theta)}{d\theta'} d\theta' \\
= \; & Eu(x(\tilde{\theta}), y(\tilde{\theta}); \theta) \\
& + \int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} \frac{g(\theta')}{2} \left[\varphi(\theta') - 2(\theta - \theta') f(\epsilon(x(\theta'), y(\theta'))) \left(\frac{1}{g(x(\theta'))} + \frac{1}{g(y(\theta'))} \right) \right] d\theta'.
\end{aligned}
$$

θ appears only in the brackets as $-2(\theta - \theta')$, so that the second term in this expression is decreasing in θ. Hence, $Eu(x(\tilde{\theta}+\Delta), y(\tilde{\theta}+\Delta); \theta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \theta)$ is decreasing in θ. The latter half of the statement in lemma 2 can be shown by a symmetric argument. $\square$

Proof of Proposition 1 Proof of Proposition 1 is provided by using the following four lemmas together with Lemma 2. The next lemma shows that there is a coalitional deviation around $\tilde{\theta}$ which has the same effect as small coalitions deviating from $(\tilde{\theta}, y)$ or $(x, \tilde{\theta})$ to the other party.

Lemma 3 *Consider an improving coalitional deviation γ from a sorting allocation with $\tilde{\theta}$ such that $Supp(\gamma) \subset [\tilde{\theta}, y(\tilde{\theta})]$. Then, there is another improving coalitional deviation γ_R^Δ such that (i) $\gamma_R^\Delta(\theta) = g(\theta)$ for all $\theta \in (\tilde{\theta}, \tilde{\theta}+\Delta)$ and $\gamma_R^\Delta(\theta) = 0$, otherwise; and (ii) $\int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} \gamma_R^\Delta(\theta) d\theta = \int_{\tilde{\theta}}^{y(\tilde{\theta})} \gamma(\theta) d\theta$. Similarly, consider an improving coalitional deviation γ from a sorting allocation with $\tilde{\theta}$ such that $Supp(\gamma) \subset [x(\tilde{\theta}+\Delta), \tilde{\theta}]$. Then, there is another improving coalitional deviation γ_L^Δ such that (i) $\gamma_L^\Delta(\theta) = g(\theta)$ for all $\theta \in (\tilde{\theta}-\Delta, \tilde{\theta})$ and $\gamma_L^\Delta(\theta) = 0$, otherwise; and (ii) $\int_{\tilde{\theta}-\Delta}^{\tilde{\theta}} \gamma_L^\Delta(\theta) d\theta = \int_{x(\tilde{\theta})}^{\tilde{\theta}} \gamma(\theta) d\theta$.*

Proof of Lemma 3 Note that as long as $Supp(\gamma) \subset [\tilde{\theta}, y(\tilde{\theta})]$, the effects of γ switching party from R to L on x and y are the same as those of γ_R^Δ switching party from R to L on x and y. Moreover, from Lemma 2, we know that $Eu(x(\tilde{\theta}+\Delta), y(\tilde{\theta}+\Delta); \theta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \theta)$ is decreasing in θ for $\theta \in (\tilde{\theta}, y(\tilde{\theta}))$. Thus, if there is an incentive to join the coalition for $\theta > \tilde{\theta}+\Delta$; i.e., $Eu(x(\tilde{\theta}+\Delta), y(\tilde{\theta}+\Delta); \theta) > Eu(x(\tilde{\theta}), y(\tilde{\theta}); \theta)$, then all $\theta' \leq \tilde{\theta}+\Delta$ have incentive to join the deviation. A symmetric argument proves the latter half of the statement. $\square$

$\square$

The following lemma is a direct consequence of the above two lemmas 2 and 3.

Proof of

Lemma 4 *Consider a sorting allocation with threshold $\tilde{\theta}$. This allocation is immune to a coalitional deviation γ with $Supp(\gamma) \subset (\tilde{\theta}, y(\tilde{\theta}))$ if and only if $Eu(x(\tilde{\theta}+\Delta), y(\tilde{\theta}+\Delta); \tilde{\theta}+\Delta) \leq Eu(x(\tilde{\theta}), y(\tilde{\theta}); \tilde{\theta}+\Delta)$ holds for Δ defined by γ^Δ. Similarly, this allocation is immune* *to a coalitional deviation γ with $Supp(\gamma) \subset (x(\tilde{\theta}+\Delta), \tilde{\theta})$ if and only if $Eu(x(\tilde{\theta}-\Delta), y(\tilde{\theta}-\Delta); \tilde{\theta}-\Delta) \leq Eu(x(\tilde{\theta}), y(\tilde{\theta}); \tilde{\theta}-\Delta)$ holds for Δ defined by γ^{Δ}.*

$\square$

As a result, in order to check whether a sorting allocation is a political equilibrium, this lemma tells us to confirm whether the type that is the furthest from $\tilde{\theta}$ in every coalition has an incentive for taking part in the coalition. More precisely, if there exists $\bar{\Delta} > 0$ such that (i) $Eu(x(\tilde{\theta}+\Delta), y(\tilde{\theta}+\Delta); \tilde{\theta}+\Delta) \leq Eu(x(\tilde{\theta}), y(\tilde{\theta}); \tilde{\theta}+\Delta)$ holds for all $\Delta \in (0, \bar{\Delta})$, and (ii) $Eu(x(\tilde{\theta}-\Delta), y(\tilde{\theta}-\Delta); \tilde{\theta}-\Delta) \leq Eu(x(\tilde{\theta}), y(\tilde{\theta}); \tilde{\theta}-\Delta)$ holds for all $\Delta \in (0, \bar{\Delta})$, then a sorting allocation with threshold $\tilde{\theta}$ is a political equilibrium.

We will simplify the above conditions by using the φ function. First, we provide a simple necessary condition to be a political equilibrium.

Proof of

Lemma 5 *Suppose that $\varphi(\theta)$ is continuous. Then, a sorting allocation with threshold $\tilde{\theta}$ is a political equilibrium only if $\varphi(\tilde{\theta}) = 0$.*

Proof of Lemma 5 Suppose that $\varphi(\tilde{\theta}) > 0$. Since φ is continuous, there exists $\tilde{\Delta} > 0$ such that $\varphi(\theta) > 0$ for all $\theta \in [\tilde{\theta}, \tilde{\theta}+\tilde{\Delta}]$. Then, from lemma 2, we have

$$\begin{aligned}
& Eu(x(\tilde{\theta}+\Delta), y(\tilde{\theta}+\Delta); \tilde{\theta}+\Delta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \tilde{\theta}+\Delta) \\
= & \int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} \frac{g(\theta')}{2} \left[\varphi(\theta') - 2((\tilde{\theta}+\Delta) - \theta') f(\epsilon(x(\theta'), y(\theta'))) \left(\frac{1}{g(x(\theta'))} + \frac{1}{g(y(\theta'))} \right) \right] d\theta'.
\end{aligned}$$

By choosing a small enough Δ (smaller than $\tilde{\Delta}$), the absolute value of the second term in the brackets becomes smaller than $\varphi(\theta')$, so that we can find an improving coalitional deviation γ_R^{Δ}. Similarly, if $\varphi(\tilde{\theta}) < 0$, then there is an improving coalitional deviation γ_L^{Δ}. Hence a sorting allocation with $\tilde{\theta}$ is a political equilibrium only if $\varphi(\tilde{\theta}) = 0$. $\square$

Thus, we will assume $\varphi(\tilde{\theta}) = 0$ in order to characterize political equilibrium. By applying the first-order Taylor expansion, we can approximate the utility change of the critical coalition member's utility in the below lemma when a coalition γ_R^{Δ} deviates.

Lemma 6 *Suppose that $\varphi(\tilde{\theta}) = 0$ and that f and g are differentiable functions. Then, for sufficiently small $\Delta > 0$, $Eu(x(\tilde{\theta}+\Delta), y(\tilde{\theta}+\Delta); \tilde{\theta}+\Delta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \tilde{\theta}+\Delta)$ is approximated as*

$$\begin{aligned}
& Eu(x(\tilde{\theta}+\Delta), y(\tilde{\theta}+\Delta); \tilde{\theta}+\Delta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \tilde{\theta}+\Delta) \\
= & \int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} \frac{g(\theta')}{2} \left[\varphi(\theta') - 2(\tilde{\theta}+\Delta - \theta') f(\epsilon(x(\theta''))) \left(\frac{1}{g(x(\theta'))} + \frac{1}{g(y(\theta'))} \right) \right] d\theta' \\
\simeq & \frac{\Delta^2 g(\tilde{\theta})}{2} \left[\frac{\varphi'(\tilde{\theta})}{2} - f(\epsilon(x(\tilde{\theta}), y(\tilde{\theta}))) \left(\frac{1}{g(x(\tilde{\theta}))} + \frac{1}{g(y(\tilde{\theta}))} \right) \right].
\end{aligned}$$

Proof of Lemma 6 First, we will approximate $Eu(x(\tilde{\theta}+\Delta), y(\tilde{\theta}+\Delta); \tilde{\theta}+\Delta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \tilde{\theta}+\Delta)$ by using the first-order Taylor expansion.

$$
\begin{aligned}
& Eu(x(\tilde{\theta}+\Delta), y(\tilde{\theta}+\Delta); \tilde{\theta}+\Delta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \tilde{\theta}+\Delta) \\
= & \int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} \frac{g(\theta')}{2} \left[\varphi(\theta') - 2(\tilde{\theta}+\Delta-\theta') f(\epsilon(x(\theta'), y(\theta'))) \left(\frac{1}{g(x(\theta'))} + \frac{1}{g(y(\theta'))} \right) \right] d\theta' \\
= & \frac{1}{2} \int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} \varphi(\theta') g(\theta') d\theta' \\
& + \int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} 2(\tilde{\theta}+\Delta-\theta') f(\epsilon(x(\theta'), y(\theta'))) \left(-\frac{g(\theta')}{2} \right) \left(\frac{1}{g(x(\theta'))} + \frac{1}{g(y(\theta'))} \right) d\theta'.
\end{aligned}
$$

Noting $\varphi(\tilde{\theta}) = 0$, the first term is approximated as

$$
\begin{aligned}
\frac{1}{2} \int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} \varphi(\theta') g(\theta') d\theta' & \simeq \frac{1}{2} \int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} (\varphi(\tilde{\theta}) g(\tilde{\theta}) + (\varphi'(\tilde{\theta}) g(\tilde{\theta}) + \varphi(\tilde{\theta}) g'(\tilde{\theta}))(\theta' - \tilde{\theta})) d\theta' \\
& = \frac{1}{2} \int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} \varphi'(\tilde{\theta}) g(\tilde{\theta}) (\theta' - \tilde{\theta}) d\theta' \\
& = \frac{1}{2} \varphi'(\tilde{\theta}) g(\tilde{\theta}) \left[\frac{(\theta' - \tilde{\theta})^2}{2} \right]_{\tilde{\theta}}^{\tilde{\theta}+\Delta} \\
& = \frac{\Delta^2}{4} \varphi'(\tilde{\theta}) g(\tilde{\theta}).
\end{aligned}
$$

In order to calculate the second term, first note that

$$
\frac{d}{d\theta'} F(\epsilon(x(\theta'), y(\theta'))) = f(\epsilon(x(\theta'), y(\theta'))) \left(-\frac{g(\theta')}{2} \right) \left(\frac{1}{g(x(\theta'))} + \frac{1}{g(y(\theta'))} \right).
$$

Thus, partially integrating the second term, we obtain

$$
\begin{aligned}
& \int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} 2\left(\tilde{\theta}+\Delta-\theta'\right) f(\epsilon(x(\theta'), y(\theta'))) \left(-\frac{g(\theta')}{2} \right) \left(\frac{1}{g(x(\theta'))} + \frac{1}{g(y(\theta'))} \right) d\theta' \\
= & \int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} 2\left(\tilde{\theta}+\Delta-\theta'\right) \frac{d}{d\theta'} F(\epsilon(x(\theta'), y(\theta'))) d\theta' \\
= & \underbrace{\left[2\left(\tilde{\theta}+\Delta-\theta'\right) F(\epsilon(x(\theta'), y(\theta')) \right]_{\tilde{\theta}}^{\tilde{\theta}+\Delta}}_{A} + \underbrace{\int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} 2F(\epsilon(x(\theta'), y(\theta'))) d\theta'}_{B}.
\end{aligned}
$$

Now, term A is rewritten as

$$
\begin{aligned}
& \left[2\left(\tilde{\theta}+\Delta-\theta'\right) F(\epsilon(x(\theta'), y(\theta')) \right]_{\tilde{\theta}}^{\tilde{\theta}+\Delta} \\
= & 2\left[\left(\tilde{\theta}+\Delta - \left(\tilde{\theta}+\Delta\right)\right) F(\epsilon(x(\tilde{\theta}+\Delta), y(\tilde{\theta}+\Delta)) - \left(\tilde{\theta}+\Delta-\tilde{\theta}\right) F(\epsilon(x(\tilde{\theta}), y(\tilde{\theta})) \right] \\
= & -2F(\epsilon(x(\tilde{\theta}), y(\tilde{\theta}))\Delta.
\end{aligned}
$$

Since $F(\epsilon(x(\theta'), y(\theta'))) \simeq F(\epsilon(x(\tilde{\theta}), y(\tilde{\theta})) + f(\epsilon(x(\tilde{\theta}), y(\tilde{\theta}))\epsilon'(x(\tilde{\theta}), y(\tilde{\theta})) \left(\theta' - \tilde{\theta}\right)$, by substituting $\epsilon'(x(\tilde{\theta}), y(\tilde{\theta})) = f(\epsilon(x(\tilde{\theta}), y(\tilde{\theta})))(-\frac{g(\tilde{\theta})}{2}) \left(\frac{1}{g(x(\tilde{\theta}))} + \frac{1}{g(y(\tilde{\theta}))} \right)$ into this approximation, term B can be approximated as

$$\begin{aligned}
& \int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} 2F(\epsilon(x(\theta'), y(\theta')))d\theta' \\
\simeq\ & 2F(\epsilon(x(\tilde{\theta}), y(\tilde{\theta})) \int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} d\theta' + 2f(\epsilon(x(\tilde{\theta}), y(\tilde{\theta}))\epsilon'(x(\tilde{\theta}), y(\tilde{\theta})) \int_{\tilde{\theta}}^{\tilde{\theta}+\Delta} (\theta' - \tilde{\theta})d\theta' \\
=\ & 2F(\epsilon(x(\tilde{\theta}), y(\tilde{\theta}))\Delta + f(\epsilon(x(\tilde{\theta}), y(\tilde{\theta})) \left(-\frac{g(\tilde{\theta})}{2}\right)\left(\frac{1}{g(x(\tilde{\theta}))} + \frac{1}{g(y(\tilde{\theta}))}\right)\Delta^2.
\end{aligned}$$

Thus, the second term is $A + B = f(\epsilon(x(\tilde{\theta}), y(\tilde{\theta}))(-\frac{g(\tilde{\theta})}{2})\left(\frac{1}{g(x(\tilde{\theta}))} + \frac{1}{g(y(\tilde{\theta}))}\right)\Delta^2$. Hence, we have the approximation formula:

$$\begin{aligned}
& Eu(x(\theta'), y(\theta'); \tilde{\theta} + \Delta) - Eu(x(\tilde{\theta}), y(\tilde{\theta}); \tilde{\theta} + \Delta) \\
\simeq\ & \frac{\Delta^2}{4}\varphi'(\tilde{\theta})g(\tilde{\theta}) + f(\epsilon(x(\tilde{\theta}), y(\tilde{\theta})) \left(-\frac{g(\tilde{\theta})}{2}\right)\left(\frac{1}{g(x(\tilde{\theta}))} + \frac{1}{g(y(\tilde{\theta}))}\right)\Delta^2 \\
=\ & \frac{\Delta^2 g(\tilde{\theta})}{2}\left[\frac{\varphi'(\tilde{\theta})}{2} - f(\epsilon(x(\tilde{\theta}), y(\tilde{\theta}))) \left(\frac{1}{g(x(\tilde{\theta}))} + \frac{1}{g(y(\tilde{\theta}))}\right)\right].
\end{aligned}$$

We have completed the proof. □

Proposition 1 directly follows from Lemma 6. □

Proof of Corollary 2 Since f and g are density functions, their values are nonnegative. Thus, from Lemma 6, we get the conclusion directly. □

Proof of Theorem 1 When $\tilde{\theta} = 0$, candidates of L and R are $x = 0$ and $y = \theta_{med}$, respectively. Then, we have

$$\varphi(0) = -\frac{F(\theta_{med})}{g(\theta_{med})} + \frac{1 - F(\theta_{med})}{g(0)} + 2\theta_{med} f(\theta_{med})\left(\frac{1}{g(\theta_{med})} + \frac{1}{g(0)}\right).$$

Thus, if $-\frac{F(\theta_{med})}{g(\theta_{med})} + \frac{1-F(\theta_{med})}{g(0)} \geq 0$ or $\frac{g(0)}{g(\theta_{med})} \leq \frac{P_x(x(0),y(0))}{P_y(x(0),y(0))}$ then it is sufficient for $\varphi(0) > 0$.

When $\tilde{\theta} = 1$, candidates of L and R are $x = \theta_{med}$ and $y = 1$, respectively. Then, we have

$$\varphi(1) = -\frac{F(\theta_{med} - 1)}{g(1)} + \frac{1 - F(\theta_{med} - 1)}{g(\theta_{med})} - 2(1 - \theta_{med})f(\theta_{med} - 1)\left(\frac{1}{g(1)} + \frac{1}{g(\theta_{med})}\right).$$

Thus, if $-\frac{F(\theta_{med}-1)}{g(1)} + \frac{1-F(\theta_{med}-1)}{g(\theta_{med})} \leq 0$ or $\frac{g(1)}{g(\theta_{med})} \leq \frac{P_y(x(1),y(1))}{P_x(x(1),y(1))}$ then it is sufficient for $\varphi(1) < 0$.

Since g and f are continuous in θ, $\varphi(\theta)$ is continuous. Thus, there exists at least a $\tilde{\theta} \in (0, 1)$ such that $\varphi(\tilde{\theta}) = 0$ and $\varphi'(\tilde{\theta}) \leq 0$. □

Proof of Proposition 2 From the necessary and sufficient conditions of the sorting political equilibrium in Theorem 1, if $\tilde{\theta}^*$ is a sorting political equilibrium, then

$$\varphi(\tilde{\theta}^*) = -\frac{F(\epsilon(x,y))}{g(y)} + \frac{1-F(\epsilon(x,y))}{g(x)} - (\tilde{\theta} - \theta_{med})f(\epsilon(x,y))\left(\frac{1}{g(y)} + \frac{1}{g(x)}\right) = 0.$$

In addition, if $Supp(f) \supset [-1,1]$ and $g(\theta) > 0$ for all $\theta \in (0,1)$ and $g(0) = g(1) = 0$, then a sorting political equilibrium becomes an interior solution from Proposition 1. Thus, with an equilibrium, $f(\epsilon(x,y))(\frac{1}{g(y)} + \frac{1}{g(x)}) > 0$. From these facts, if $-\frac{F(\epsilon(x,y))}{g(y)} + \frac{1-F(\epsilon(x,y))}{g(x)} \neq 0$ in $\varphi(\tilde{\theta})$, then $\tilde{\theta} - \theta_{med} \neq 0$. Hence θ_{med} is not an allocation of a sorting political equilibrium. $\square$

Proof of Proposition 3 Let $\tilde{\theta} = \theta_{med} = \frac{1}{2}$. Then, by symmetry of g, we have $\theta_{med} - x(\theta_{med}) = y(\theta_{med}) - \theta_{med}$, $g(x(\theta_{med})) = g(y(\theta_{med}))$ and $g'(x) = -g'(y)$. Thus, $\epsilon(x(\theta_{med}), y(\theta_{med})) = h(x, y; \theta_{med}) = 0$ is obtained. Since f is symmetric, $1 - F(0) = F(0) = \frac{1}{2}$. Then,

$$-\frac{F(\epsilon(x,y))}{g(y)} + \frac{1-F(\epsilon(x,y))}{g(x)} = 0.$$

Thus, when $\tilde{\theta} = \theta_{med}$, $\varphi(\theta_{med}) = 0$. In addition to this, by using the above facts, we have

$$\varphi'(\tilde{\theta}) = \frac{g(\theta_{med})}{2g(x)^2}\left[4f(0) - \frac{g'(x)}{g(x)}\right] - \frac{4f(0)}{g(x)}.$$

Moreover, the necessary and sufficient condition in Proposition 1,

$$\frac{\varphi'(\tilde{\theta})}{2} - f(0)\left(\frac{1}{g(x(\tilde{\theta}))} + \frac{1}{g(y(\tilde{\theta}))}\right) < 0$$

is equivalent to

$$\frac{g(\theta_{med})}{4g(x)^2}\left(4f(0) - \frac{g'(x)}{g(x)}\right) - \frac{4f(0)}{g(x)} < 0.$$

Hence, if this condition is satisfied, there is a political equilibrium with $\tilde{\theta} = \theta_{med}$. $\square$

Proof of Proposition 4 Note that $\xi(\theta_{med}) = \varphi(\theta_{med})$. The assumptions guarantee $\varphi(0) > 0$ and $\varphi(1) < 0$. From Theorem 1 (continuity of φ) and Proposition 1, we know that there exists $\tilde{\theta}$ such that $\varphi(\tilde{\theta}) = 0$ with $\varphi'(\tilde{\theta}) \leq 0$ under the assumption of this proposition. This proves the statement of the proposition. $\square$

Appendix B Uniform Distribution

Proof of Proposition 5. If both A and B are non-increasing, then we have $\varphi'(\tilde{\theta}) \leq$ for all $\tilde{\theta}$. First analyze term A. Since $x(\tilde{\theta})$ and $y(\tilde{\theta})$ are the solutions of

$$2G(x(\tilde{\theta})) = G(\tilde{\theta}),$$

and

$$1 - 2G(y(\tilde{\theta})) = 1 - G(\tilde{\theta}),$$

respectively, we obtain

$$\begin{aligned} \frac{dA}{d\tilde{\theta}} &= \frac{g(\tilde{\theta})}{2g(x(\tilde{\theta}))} + \frac{g(\tilde{\theta})}{2g(y(\tilde{\theta}))} - 2 \\ &= \frac{1}{2}\left[\frac{g(\tilde{\theta})}{g(x(\tilde{\theta}))} + \frac{g(\tilde{\theta})}{g(y(\tilde{\theta}))} - 4\right]. \end{aligned}$$

It is easy to see that the sign of $\frac{dA}{d\tilde{\theta}}$ tends to be negative as long as the voter density function does not go up or down in its magnitude too much. □

Second, let's analyze the behavior of B. The sign of B is clearly determined by $g(y) \gtreqless g(x)$. If g function is single-peaked at $p \in (0,1)$, then $g(y(\tilde{\theta})) - g(x(\tilde{\theta}))$ changes its sign only once as $\tilde{\theta}$ increases. Differentiating B with respect to $\tilde{\theta}$ we obtain

$$\begin{aligned} \frac{dB}{d\tilde{\theta}} &= \frac{\left(g'(y)\frac{dy}{d\tilde{\theta}} - g'(x)\frac{dx}{d\tilde{\theta}}\right)(g(x)+g(y)) - (g(y)-g(x))\left(g'(x)\frac{dx}{d\tilde{\theta}} + g'(y)\frac{dy}{d\tilde{\theta}}\right)}{(g(x)+g(y))^2} \\ &= \frac{\left(g'(y)\frac{g(\tilde{\theta})}{g(y)} - g'(x)\frac{g(\tilde{\theta})}{g(x)}\right)(g(x)+g(y)) - (g(y)-g(x))\left(g'(x)\frac{g(\tilde{\theta})}{g(x)} + g'(y)\frac{g(\tilde{\theta})}{g(y)}\right)}{(g(x)+g(y))^2} \\ &= \frac{2g'(y)\frac{g(\tilde{\theta})}{g(y)}g(x) - 2g'(x)\frac{g(\tilde{\theta})}{g(x)}g(y)}{(g(x)+g(y))^2} \\ &= \frac{2g(\tilde{\theta})}{(g(x)+g(y))^2}\left[g'(y)\frac{g(x)}{g(y)} - g'(x)\frac{g(y)}{g(x)}\right]. \end{aligned}$$

□

Proof of Example 2. We can explicitly calculate the φ function and under the population distribution. Since φ is a step function and is discontinuous at θ_{med}, we have two cases to calculate: (I) the case of $\tilde{\theta} \leq \theta_{med}$ and (II) the case of $\tilde{\theta} > \theta_{med}$. Noting that each candidate satisfies $x \leq \theta_{med} \leq y$ under any sorting political equilibria, the two cases are given below.

(I) The case of $\tilde{\theta} \leq \theta_{med}$. Two candidates are

$$x(\tilde{\theta}) = \frac{\tilde{\theta}}{2} \quad \text{and} \quad y(\tilde{\theta}) = \theta_{med} + \frac{1-\theta_{med}}{2\theta_{med}}\tilde{\theta}.$$

In this case, calculating (7), $\varphi \gtreqless 0$ holds if and only if

$$2a \cdot \varphi(\tilde{\theta}) = 2\left(\theta_{med} + \frac{1-4\theta_{med}}{2\theta_{med}}\tilde{\theta}\right) + 2a\left(2\theta_{med} - 1\right) \gtreqless 0.$$

If $\tilde{\theta}$ is a threshold of a sorting political equilibrium, it satisfies $\varphi(\tilde{\theta}) = 0$ from Lemma 5, namely, at a sorting political equilibrium,

$$\tilde{\theta}^* = \frac{2\theta_{med}((2a+1)\theta_{med} - a)}{4\theta_{med} - 1}$$

holds. We need to check if $\tilde{\theta}^* \leq \theta_{med}$ holds or not. In order to satisfy $\tilde{\theta}^* \leq \theta_{med}$, we must have $\theta_{med} > \frac{1}{4}$ since

$$\theta_{med} - \tilde{\theta}^* = \frac{(2a-1)(1-2\theta_{med})\theta_{med}}{4\theta_{med} - 1}$$

and $\theta_{med} \leq \frac{1}{2}$ and $a > \frac{1}{2}$. Indeed, if $\theta_{med} > \frac{1}{4}$ then the sufficient condition of the sorting political equilibrium at $\tilde{\theta}^*$ is satisfied (Corollary 2), since we have

$$2a \cdot \varphi'(\tilde{\theta}) = \frac{1 - 4\theta_{med}}{\theta_{med}} < 0.$$

We also need a condition on a, $a \leq \frac{\theta_{med}}{1-2\theta_{med}}$, since $\varphi(0) > 0$ must hold in order to have $\varphi(\tilde{\theta}^*) = 0$.

(II) The case of $\tilde{\theta} > \theta_{med}$. As well as (I), two candidates are

$$x(\tilde{\theta}) = \frac{\theta_{med}(1-2\theta_{med})}{2(1-\theta_{med})} + \frac{\theta_{med}}{2(1-\theta_{med})}\tilde{\theta} \quad \text{and} \quad y(\tilde{\theta}) = \frac{1+\tilde{\theta}}{2}.$$

In this case, calculating (7), $\varphi(\tilde{\theta}) \gtreqless 0$ holds if and only if

$$2a \cdot \varphi(\tilde{\theta}) = 2\left(\frac{1}{2} + \frac{\theta_{med}(1-2\theta_{med})}{2(1-\theta_{med})} + \frac{4\theta_{med} - 3}{2(1-\theta_{med})}\tilde{\theta}\right) + 2a(2\theta_{med} - 1) \gtreqless 0.$$

Noting that $\varphi(\tilde{\theta})$ function is not discontinuous at θ_{med} but just kinks because of $x \leq \theta_{med} \leq y$.[32] See Figure 4.

If $\tilde{\theta}^{**}$ is a threshold of a sorting political equilibrium in this range, i.e., $\varphi(\tilde{\theta}^{**}) = 0$ as well as the case (I), namely,

$$\tilde{\theta}^{**} = \frac{-(2+4a)\theta_{med}^2 + 6a\theta_{med} - 2a + 1}{3 - 4\theta_{med}}$$

must hold. However, calculating $\tilde{\theta}^{**} - \theta_{med}$, we have

$$\tilde{\theta}^{**} - \theta_{med} = \frac{(2a-1)\left[-(1-\theta_{med})^2 - \theta_{med}\right]}{3 - 4\theta_{med}} < 0.$$

Thus, there is no party line $\tilde{\theta}^{**}$ satisfying $\varphi'(\tilde{\theta}^{**}) = 0$ in this range, which implies that there is no political equilibrium in this range.

In conclusion, with the voter distribution g as in Example 2, there is a unique equilibrium if any, and the equilibrium party line satisfies $\tilde{\theta}^* < \theta_{med}$. This implies that the party with the shorter tail (or higher density: here the L party) loses some of its moderate supporters in any political equilibrium. □

[32] On the other hand, on the function $g(\theta)$ in Example 3, x and y stride over some steps. Thus $\varphi(\tilde{\theta})$ is discontinuous at several points.

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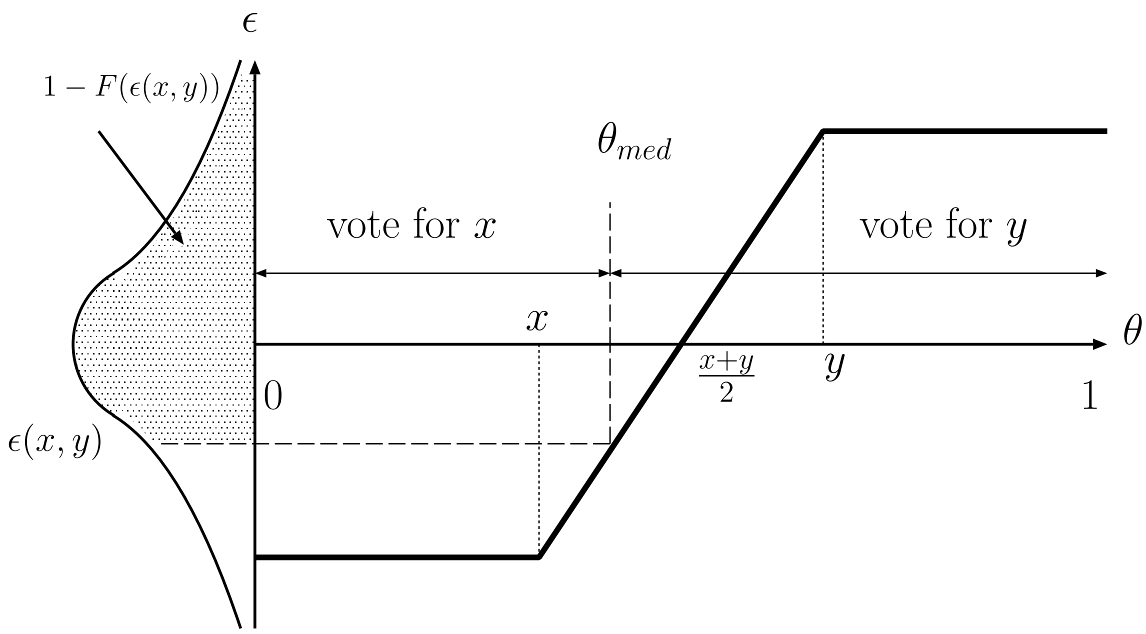

Figure 1: Voting at $\epsilon = \epsilon(x, y)$

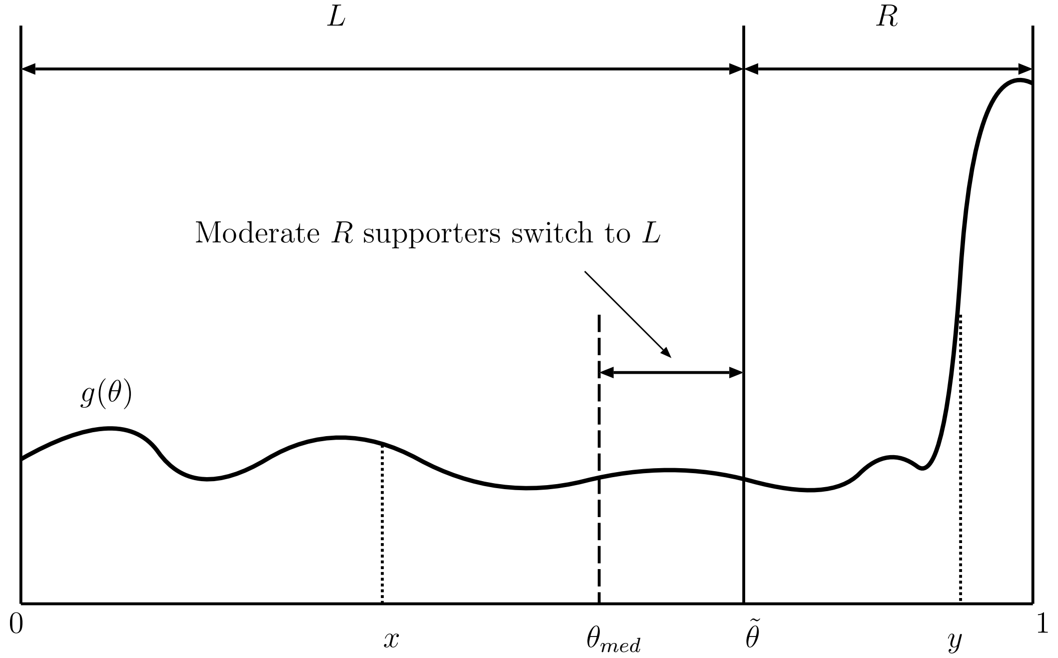

Figure 2: Extreme supporters in R get moderate supporters switch from R to L.

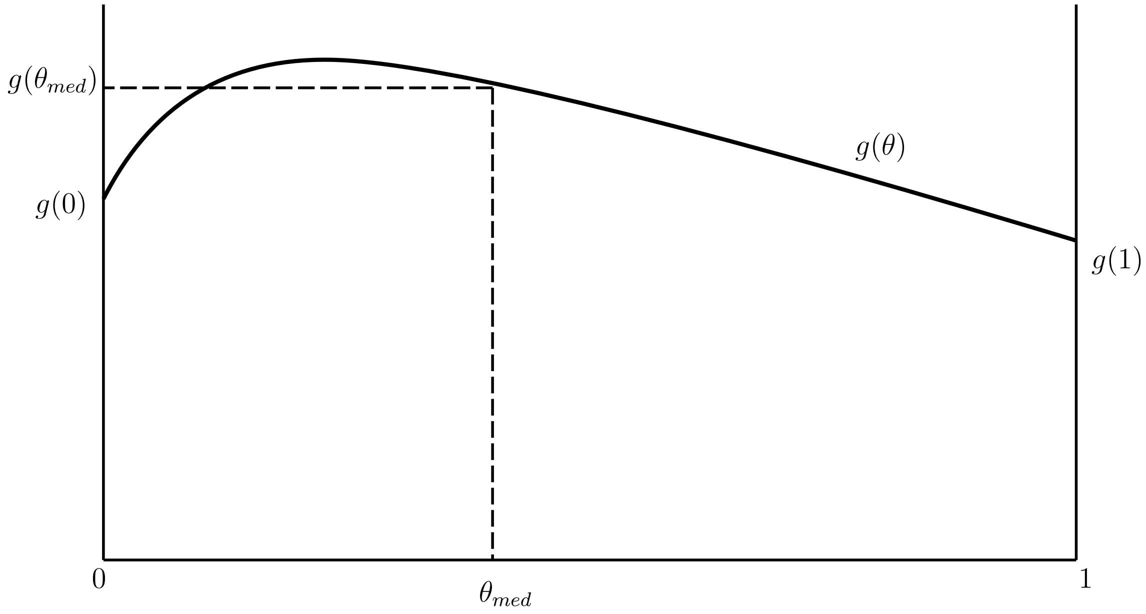

Figure 3: Sufficient conditions for unique equilibrium: single peaked and flat $g(\theta)$

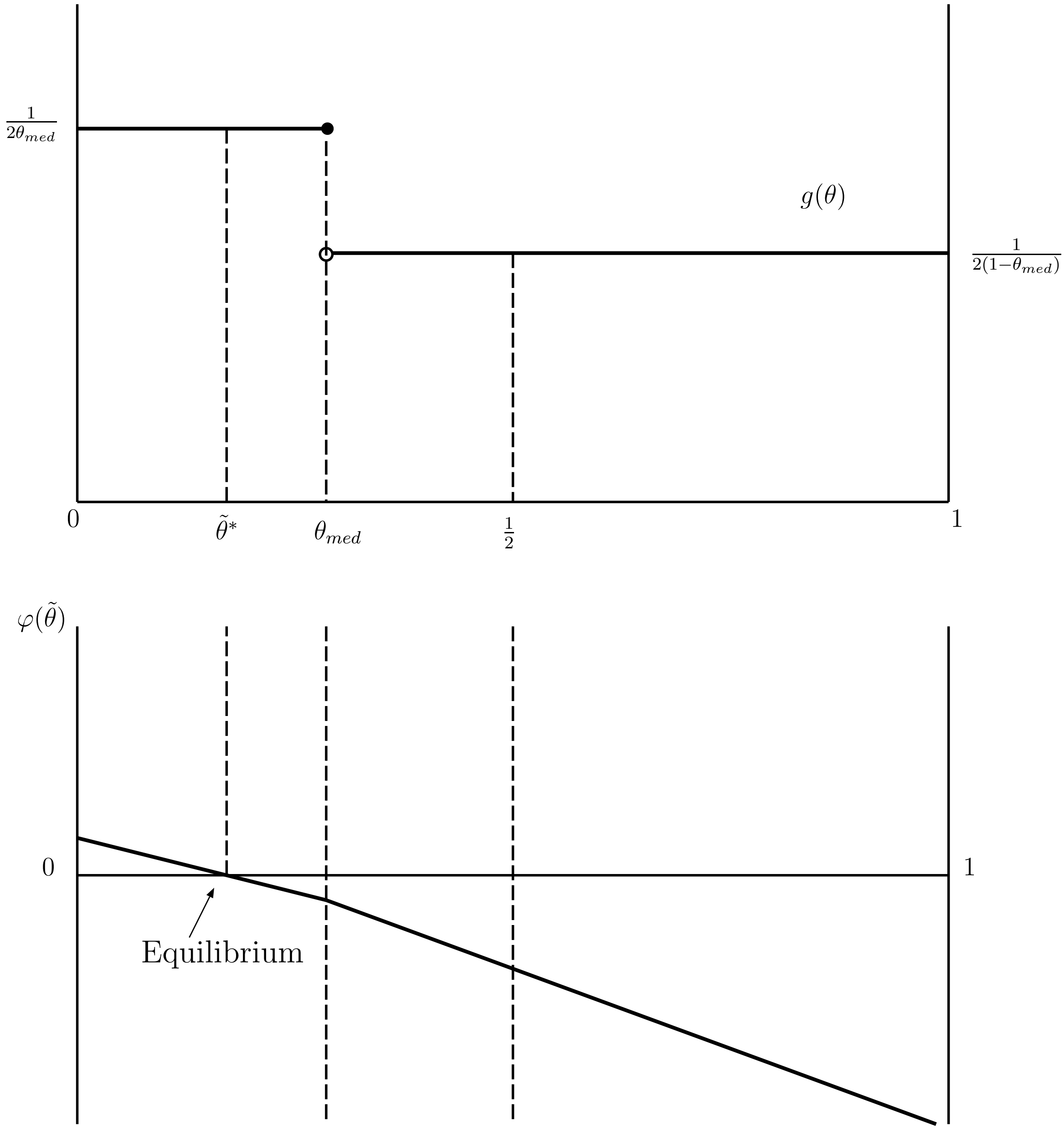

Figure 4: Example 2

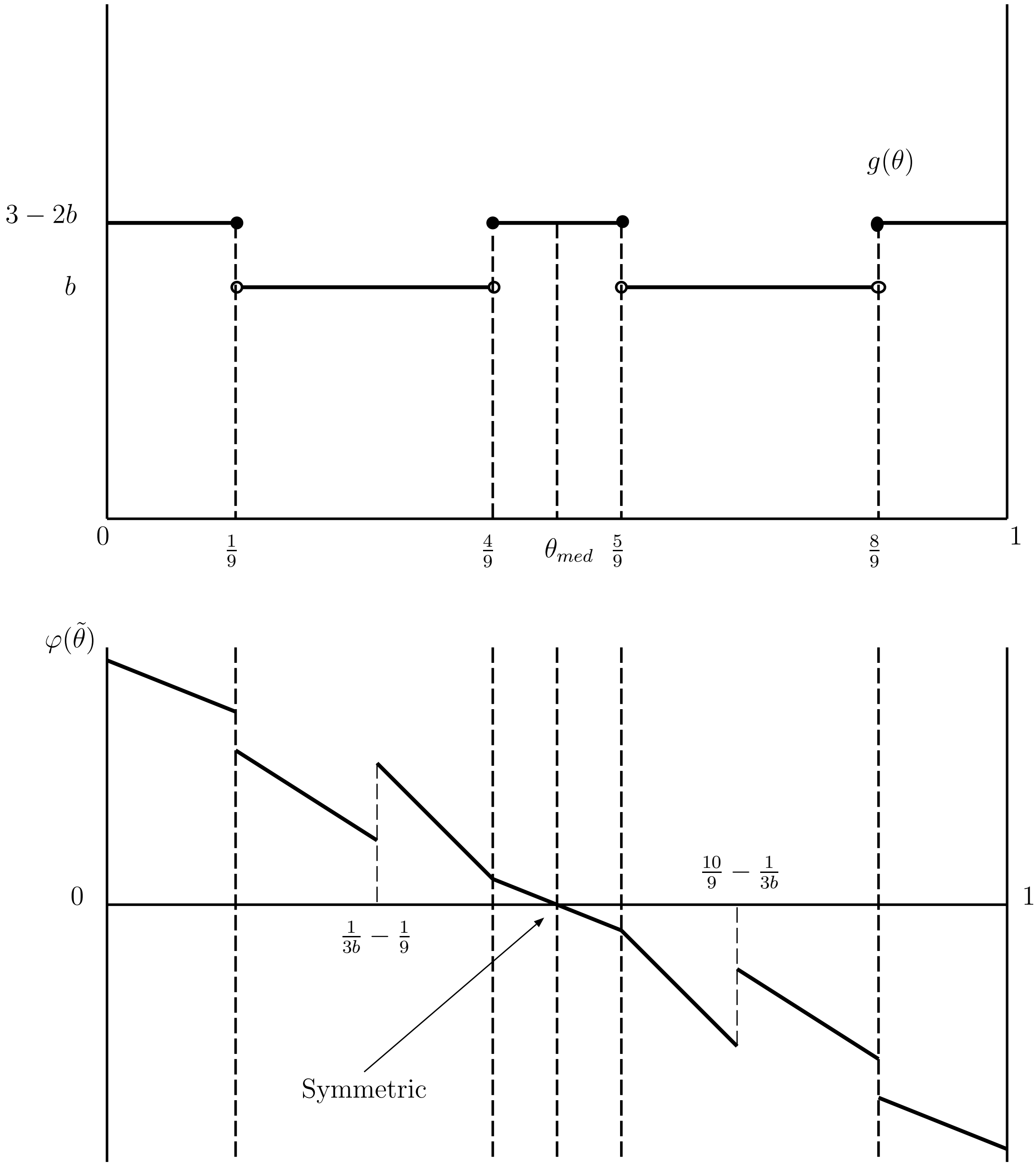

Figure 5: large b in Case 3 of Example 3

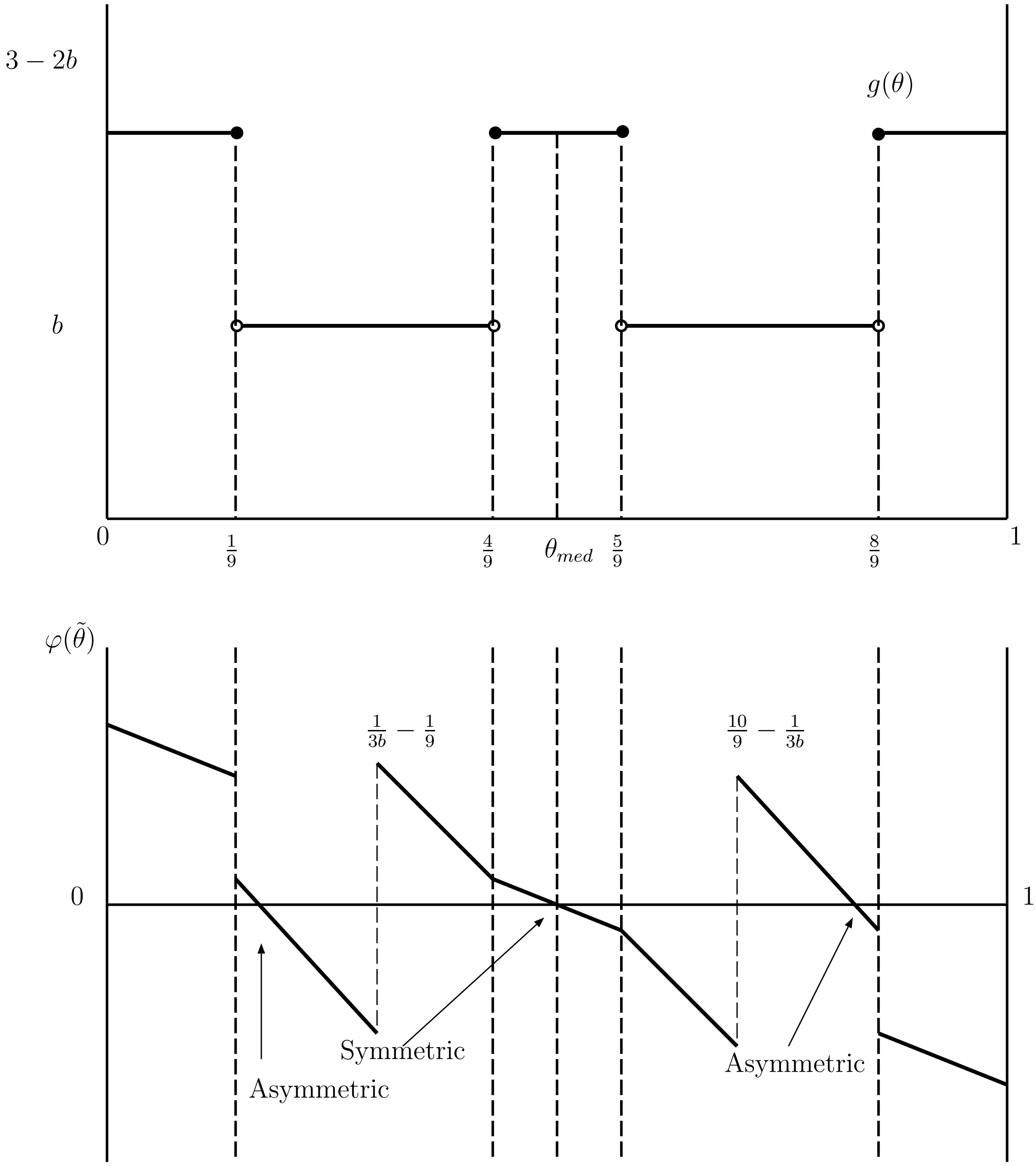

Figure 6: $b \geq \frac{1}{2}$ and not so large in Case 3 of Example 3

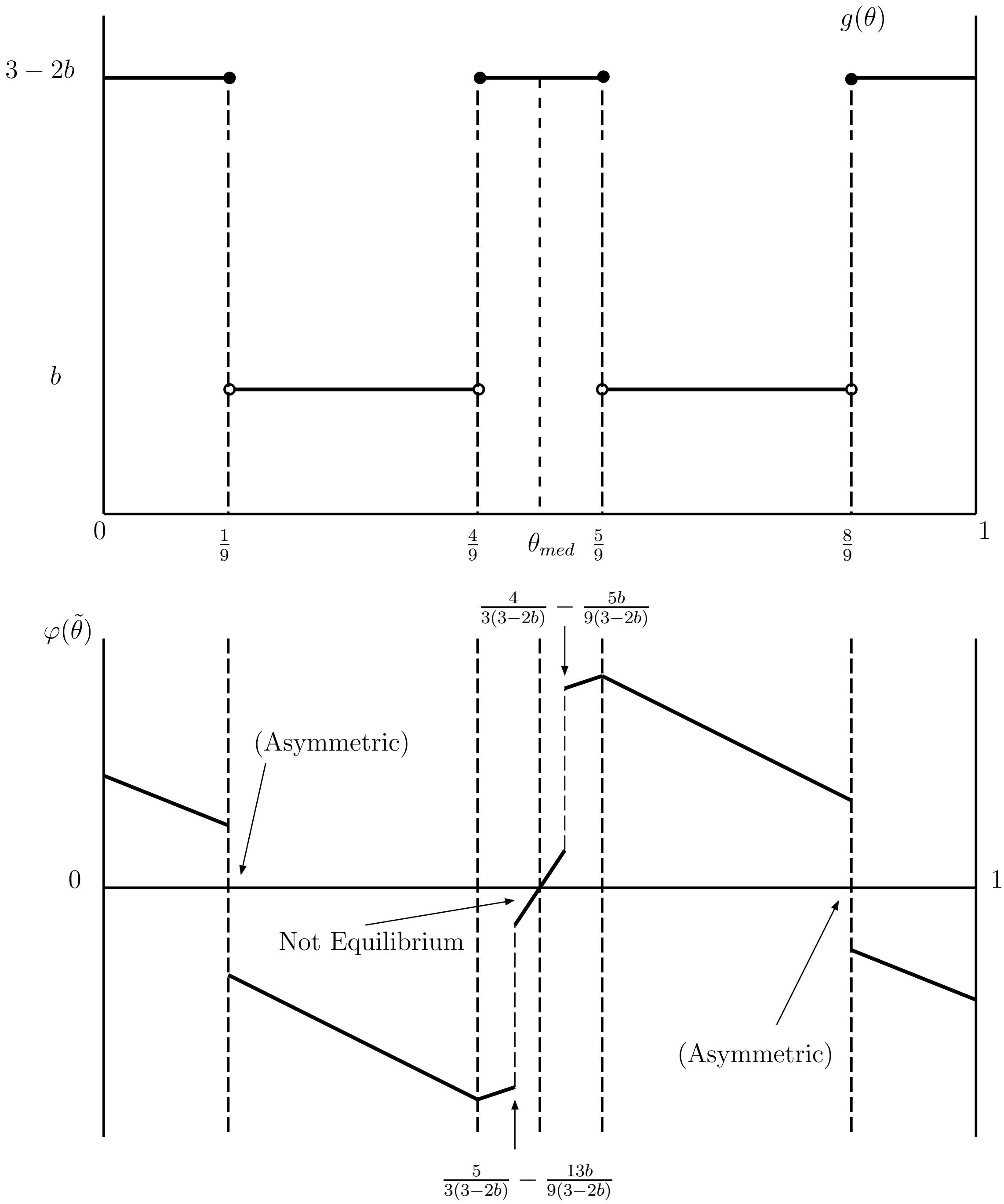

Figure 7: $\frac{3}{8} < b < \frac{1}{2}$ in Case 2 of Example 3

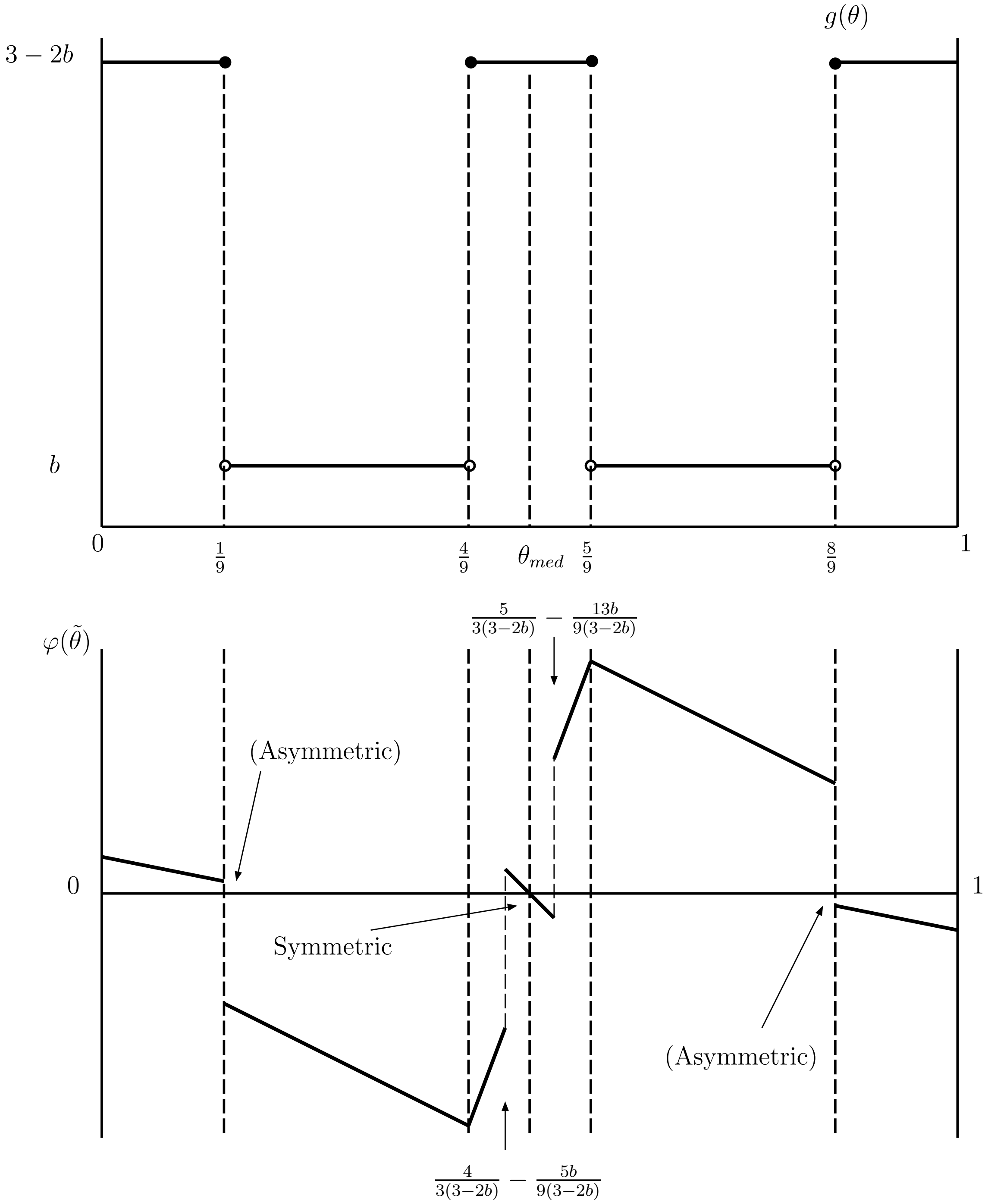

Figure 8: $b \leq \frac{3}{8}$ in Case 1 of Example 3