A revealed preference test for weakly separable preferences

John K.-H. Quah*

February 21, 2013

Abstract: Consider a finite data set of price vectors and consumption bundles; under what conditions will be there a weakly separable utility function that rationalizes the data? This paper shows that rationalization in this sense is possible if and only if there exists a preference order on some finite set of consumption bundles that is consistent with the observations and that is weakly separable. Since there can only be a finite number of preference orders on this set, the problem of rationalization with a weakly separable utility function is solvable.

Keywords: Afriat's Theorem, concave utility function, budget set, generalized axiom of revealed preference, preorder

JEL classification numbers: C14, C60, C61, D11, D12

1. Introduction

Suppose we observe a consumer making purchases from ℓ goods, with a typical observation t consisting of the bundle $x^t \in R^\ell_+$ chosen by the consumer and the price vector $p^t \in R^\ell_{++}$ that the consumer faces. A finite data set $\{(p^t, x^t)\}_{t \in T}$ is said to be *rationalized* by the utility function $U : R^\ell_+ \to R$ if, for all $t \in T$, the observed bundle x^t maximizes $U(x)$ in the budget set

$$B^t = \{x \in R^\ell_+ : p^t \cdot x \leqslant p^t \cdot x^t\}. \tag{1}$$

Afriat's Theorem gives us the precise condition under which a data set can be rationalized by a *well-behaved*, i.e., strongly monotone[1] and continuous, utility function U. It says that this is possible if and only if the data set obeys an intuitive property called the *generalized axiom of revealed preference* or GARP, for short (see Afriat (1967) and Varian (1982)). GARP also has the feature that it can be easily tested using a linear program, so that Afriat's Theorem has become the cornerstone of a large empirical literature on consumer demand.

* Email address: john.quah@economics.ox.ac.uk.

I am grateful to Matthew Polisson for useful conversations on this issue.

[1]This means that $U(x') > U(x)$ whenever $x' > x$.

It is common in empirical and theoretical work to impose additional conditions on the utility function (apart from requiring it to be well-behaved). A particularly common and convenient property is *weak separability* (which we shall often refer to in this paper simply as 'separability'). For this reason, it is useful to develop a characterization for data sets that could be rationalized by utility functions with this added feature. The objective of this paper is to solve this problem.

To state the issue formally, suppose that the goods available to the consumer can be partitioned into the subsets X and Z, consisting of ℓ_X goods and ℓ_Z goods respectively. At $t \in T$, the consumer faces prices $p^t = (p^t_X, p^t_Z) \in R^{\ell_X}_{++} \times R^{\ell_Z}_{++}$, where the subvector p^t_X (p^t_Z) gives the prices of the X-goods (Z-goods), and chooses to buy the bundle $q^t = (x^t, z^t) \in R^{\ell_X}_{+} \times R^{\ell_Z}_{+}$. We are interested in finding a necessary and sufficient condition on the data set $\{(q^t, p^t)\}_{t \in T}$ that permits rationalization by a function $G : R^{\ell_X} \times R^{\ell_Z} \to R$ of the form

$$G(x, y, z) = F(U(x), z), \tag{2}$$

where U and F well-behaved functions. In other words, at every observation t, $q^t = (x^t, z^t)$ maximizes $G(x, z)$ in the budget set

$$B^t_S = \{(x, z) \in R^{\ell_X} \times R^{\ell_Z} : (p^t_X, p^t_Z) \cdot (x, z) \leqslant (p^t_X, p^t_Z) \cdot (x^t, z^t)\}. \tag{3}$$

Notice that the preference represented by G exhibits weak separability on X-goods since the preference over these goods is represented by the utility function U and is independent of consumption levels of Z-goods.

To understand our approach to this issue, it is useful that we first re-visit Afriat's Theorem. Denote the set of observed consumption bundles by $\mathcal{X}$, i.e. a bundle x is in $\mathcal{X}$ if it is bought by the consumer at some observation. If the consumer is maximizing a well-behaved utility function, then the choices made by the consumer will convey information about the consumer's ranking on these bundles; for example, if some bundle x^s in $\mathcal{X}$ satisfies $p^t \cdot x^s \leqslant (<)\, p^t \cdot x^t$, then we know the consumer prefers (strictly prefers) x^t to x^s. In essence, GARP says that these relationships revealed by the data must be mutually consistent, i.e., that there are no revealed strict preference cycles. It is straightforward to show that GARP is equivalent to the existence of a preference $\succeq$ (i.e., a reflexive, transitive and complete preorder) defined on $\mathcal{X}$ that agrees with these revealed relationships. In other words, GARP allows for the completion (on $\mathcal{X}$) of the partial rankings revealed by the data. If such a preference $\succeq$

exists, then at every observation t, x^t must be the most preferred bundle according to $\geq$ in the budget set

$$\{x \in \mathcal{X} : p^t \cdot x \leqslant p^t \cdot x^t\}. \tag{4}$$

Of course the consumer's actual budget set is (1) and not (4), so we could understand Afriat's Theorem as a result that extends the domain of the rationalization: *if there exists a preference on $\mathcal{X}$ that rationalizes the data, then there exists a preference on R^ℓ_+, represented by a well-behaved utility function, that rationalizes the data.*

A result that is loosely analogous to this version of Afriat's Theorem can be formulated for separable preferences. Let $\mathcal{C} = \mathcal{X} \times \mathcal{Z}$, where $\mathcal{X}$ ($\mathcal{Z}$) is the set of bundles of X-goods (Z-goods) observed at some observation.[2] Suppose the consumer is maximizing a well-behaved utility function that is separable on X; then the observed purchases will reveal information about the consumer's rankings across certain elements of $\mathcal{C}$. Furthermore, these revealed (partial) rankings amongst bundles in $\mathcal{C}$ could be completed in a way that forms a separable preference $\geq_S$ on $\mathcal{C}$; by a separable preference we mean a preference with the added property that the induced preference on $\mathcal{X}$, holding fixed $z \in \mathcal{Z}$, is independent of z. This is clear since $\geq_S$ can always be chosen to be the preference on $\mathcal{C}$ induced by G. It follows that, at every observation t, $q^t = (x^t, z^t)$ must be the most preferred bundle according to $\geq_S$ in the budget set

$$\{(x, z) \in \mathcal{C} : (p^t_X, p^t_Z) \cdot (x, z) \leqslant (p^t_X, p^t_Z) \cdot (x^t, z^t)\} \tag{5}$$

The main result of this paper is the converse of this result. It could be loosely stated as follows: *if there exists a separable preference on $\mathcal{C}$ that rationalizes the data, then there exists a separable preference on $R^{\ell_X}_+ \times R^{\ell_Z}_+$ that rationalizes the data. This preference is representable by a utility function G of the form (2), where U and F are well-behaved.*[3] This theorem provides us with a test of the hypothesis that the consumer is maximizing a well-behaved utility function that is separable on X, since whether there is a separable preference on (the finite set) $\mathcal{C}$ that rationalizes the data is a finite problem.

To understand the nature of the difficulty involved in proving this result, notice that, if a

[2]Note that $\mathcal{C}$ is typically larger than the set of bundles purchased by the consumer at some observation; for example, if the consumer is observed to buy $q^t = (x^t, z^t)$ at observation t and $q^s = (x^s, z^s)$ at observation s, then q^t and q^s are in $\mathcal{C}$, and so is (x^t, z^s), but this last bundle may not be purchased at any observation.

[3]More generally, the test we develop could be used to test the hypothesis that the rationalizing utility function exhibits weak separability over several disjoint subsets of goods (rather than just the subset X).

data set $\{((p_X^t, p_Z^t), (x^t, z^t))\}_{t\in T}$ admits such a rationalization, then the 'restricted' data set $\{p_X^t, x^t)\}_{t\in T}$ must obey GARP and, by Afriat's Theorem, could be rationalized by a well-behaved utility function $\bar{U}$. To establish the result, however, we need to go further: it is necessary and sufficient that we find a well-behaved function F such that $(\bar{u}^t, z^t)$, where $\bar{u}^t = \bar{U}(x^t)$, maximizes $F(u, z)$ in the constraint set

$$\bar{B}_S^t = \{(u, z): \quad u = \bar{U}(x) \text{ and } (x, z) \in B_S^t \} \tag{6}$$

(with B_S^t given by (3). Of course, there are many possible sub-utility functions $\bar{U}$ that rationalize $\{(p_X^t, x^t)\}_{t\in T}$, so the issue is to construct the sub-utility function that admits such a function F. This task is facilitated by a recent result of Forges and Minelli (2009) which generalizes Afriat's Theorem to non-linear constraint sets. Translated to our context, their result says that F exists so long as the set $\{((u^t, z^t), \bar{B}_S^t)\}_{t\in T}$ obeys a generalized version of GARP. The nontrivial part of our proof consists precisely in showing that, whenever there exists a separable preference on $\mathcal{C}$ that rationalizes the data, then we can construct a sub-utility function $\bar{U}$ such that $\{((u^t, z^t), \bar{B}_S^t)\}_{t\in T}$ obeys generalized GARP.

To construct the correct sub-utility functions, we rely on a sharper version of Afriat's Theorem which is proved in Section 2. Typical formulations of Afriat's Theorem simply specify a well-behaved utility function that rationalizes the data, without paying too much attention to the ranking this utility function induces on the set of observed bundles $\mathcal{X}$. Our version of Afriat's Theorem says that *every* preference ordering on $\mathcal{X}$ that is consistent with the data could be extended to a well-behaved utility function on R_+^ℓ and provides an explicit way of constructing a well-behaved utility function with this property. In a corollary to this result, we also show how to construct a well-behaved utility function that simultaneously rationalizes the data and controls for levels of indirect utility at some finite set of price-income combinations *outside* the set of observed data. Apart from its use in proving our main result, this sharper version of Afriat's Theorem could be useful in other situations. For example, while the data itself may not reveal that some observed bundle x^t is preferred to another observed bundle x^s, the modeler may have some other information which suggests that that is true. In that case, he may like to construct a utility function rationalizing the data that has this added feature, and our theorem allows him to do precisely that.

It is well-known that the rationalizing utility function provided by Afriat's Theorem is not just well-behaved but also concave. In our main theorem, the rationalizing function we construct, G (as defined by (2)), is such that U is a concave function but F need not be a concave function. We give an example of a data set which can be rationalized by a well-behaved utility function that is separable on X, but in which U and F *cannot* be simultaneously concave. This example also highlights the difference between our main theorem and the revealed preference test of weak separability developed by Varian (1983). Varian provides necessary and sufficient conditions under which a data set is rationalized by a utility function G such that U and F are well-behaved and concave. The concavity assumption that he imposes on both functions makes for a relatively short proof of the validity of his conditions using concave programming.[4] The example we provide shows that rationalization in the sense of Varian is substantively different from the one considered in this paper.

The rest of the paper is organized as follows. Section 2 re-presents and extends Afriat's Theorem; Section 3 extends the Forges-Minelli Theorem along lines analogous to our extension Afriat's Theorem; using the results in Sections 2 and 3, Section 4 presents our characterization of data sets that admit rationalization with well-behaved separable utility functions.

2. An extension of Afriat's Theorem

Let $\mathcal{O} = \{(p^t, x^t)\}_{t \in T}$ be a finite set, where $p^t \in R^\ell_{++}$ and $x^t \in R^\ell_{+}$. We interpret $\mathcal{O}$ as a set of observations, where x^t is the observed bundle of ℓ goods chosen by the agent (the *demand bundle*) at the price vector p^t. Given a price vector $p \in R^\ell_{++}$ and income $w > 0$, the agent's *budget set* is defined as the set $B(p, w) = \{x \in R^\ell_{+} : p \cdot x \leqslant w\}$. A function $U : R^\ell_{+} \to R$ is said to *rationalize* the set $\mathcal{O}$ if, at all $t \in T$, $U(x^t) \geqslant U(x)$ for all $x > 0$ such that $p^t \cdot x \leqslant p^t \cdot x^t$. In other words, x^t is the bundle that maximizes the agent's utility function U within the budget set $B(p^t, w^t)$, where $w^t = p^t \cdot x^t$.

We are interested in finding conditions under which $\mathcal{O}$ is rationalizable by *well-behaved* utility function U; by this we mean that U is continuous and strongly monotone (i.e. $U(x') > U(x)$ whenever $x' > x$). For this purpose, it is useful to introduce a number of concepts. Denote the set of observed demands by $\mathcal{X}$, i.e., $\mathcal{X} = \{x^t\}_{t \in T}$. ($\mathcal{X}$ includes, if necessary, multiple copies of the same vector.) For x^t, $x^s \in \mathcal{X}$, we say that x^t is *directly revealed preferred to* x^s if $p_t \cdot x_s \leqslant p_t \cdot x_t$; when this inequality is strict, we say that x^t is *directly revealed strictly*

[4] However, there are serious computational challenges involved in implementing Varian's test; for a recent treatment see Cherchye, Demuynck, and de Rock (2011).

preferred to x^s. We denote these relations by $x^t \succeq^{**} x^s$ and $x^t \ggcurly^{**} x^s$ respectively. We say that x^t is *revealed preferred* to x^s (and denote it by $x^t \succeq^* x^s$) if there are observations $t = 1, 2, ..., n$ such that $x^t \succeq^{**} x^1$, $x^1 \succeq^{**} x^2$, ..., $x^{n-1} \succeq^{**} x^n$, and $x^n \succeq^{**} x_s$; x^t is said to be *revealed strictly preferred* to x^s (denoted by $x^t \ggcurly^* x^s$) if any of the direct preferences in this sequence is strict. If $x^t \succeq^* x^s$ and $x^s \succeq^* x^t$, then we say that x^t and x^s are *revealed indifferent* and denote it by $x^t \sim^* x^s$. We refer to $\succeq^*$, $\ggcurly^*$, and $\sim^*$ as the *revealed relations* (or *revealed preference relations*) *of* $\mathcal{O}$.

Let $\succeq$ be a *preorder* on $\mathcal{X}$, i.e., a relation that is reflexive, transitive, and complete. We write $x \succ y$ if $x \succeq y$ but $y \not\succeq x$ and $x \sim y$ if $x \succeq y$ and $y \succeq x$. A preorder on $\mathcal{X}$ is said to be *consistent with the revealed relations of* $\mathcal{O}$ if it has the following two properties: (i) $x \succeq y$ whenever $x \succeq^* y$ and (ii) $x \succ y$ whenever $x \ggcurly^* y$.[5] Note that (i) also implies that $x \sim y$ if $x \sim^* y$. It is clear that to check that (i) and (ii) holds, we need only check these properties for the direct relations $\succeq^{**}$ and $\ggcurly^{**}$.

The following result (whose simple proof we shall skip) gives the motivation for introducing the concept of a consistent preorder: it says that the existence of such a preorder on $\mathcal{X}$ is *necessary* for rationalizability by a well-behaved utility function.

Proposition 1 *Suppose that* $\mathcal{O}$ *is drawn from an agent who maximizes a locally non-satiated utility function* U.[6] *Then the preorder* $\succeq_U$ *on* $\mathcal{X}$ *induced by* U *(i.e.,* $x^t \succeq_U x^s$ *if* $U(x^t) \geqslant U(x^s)$*) is consistent with the revealed relations of* $\mathcal{O}$.

When does $\mathcal{O}$ admit a preorder on $\mathcal{X}$ that agrees with its revealed relations? It is quite easy to see that the *generalized axiom of revealed preference* (GARP) is a necessary and sufficient condition. The set $\mathcal{O}$ is said to obey GARP if whenever there are observations (p^t, x^t) (for $t = 1, 2, ..., n$) satisfying

$$p^1 \cdot x^2 \leqslant p^1 \cdot x^1;\ p^2 \cdot x^3 \leqslant p^2 \cdot x^2; \ldots ; p^{n-1} \cdot x^n \leqslant p^{n-1} \cdot x^{n-1}; \text{ and } p^n \cdot x^1 \leqslant p^n \cdot x^n$$

then all the inequalities have to be equalities. One could re-formulate GARP in terms of the revealed relations: the data set $\mathcal{O}$ obeys GARP if whenever there are observations (p^t, x^t)

[5]Note it does not say that $x \sim y$ implies that $x \sim^* y$ nor does it say that $x \succ y$ implies that $x \ggcurly^* y$.

[6]Local non-satiation means that, at any bundle $x \in R^\ell_+$ and any open neighborhood of x, there is x' in that neighborhood such that $U(x') > U(x)$. Note that any well-behaved utility function is strongly monotone and hence locally non-satiated.

(for $t = 1, 2, ..., n$) satisfying

$$x^1 \geq^{**} x^2,\ x^2 \geq^{**} x^3,\ ...,\ x^{n-1} \geq^{**} x^n, \text{ and } x^n \geq^{**} x^1, \tag{7}$$

then none of direct revealed preferences in this sequence can be replaced with the strict preference $\gg^{**}$.[7] The following result is proved in the Appendix.

PROPOSITION 2 *The set $\mathcal{O}$ admits a preorder that is consistent with its revealed relations if and only if it obeys GARP.*

The next result is the main result of this section and the converse of Proposition 4; it says that the existence of a consistent preorder on $\mathcal{X}$ is also *sufficient* for rationalizability.

THEOREM 1 *Suppose $\mathcal{O}$ admits a preorder $\succeq$ (on $\mathcal{X}$) that is consistent with its revealed relations. Then there exists a well-behaved and concave function $U : R^\ell_+ \to R$ with the following properties: (i) it rationalizes $\mathcal{O}$ and (ii) the preorder on $\mathcal{X}$ induced by U coincides with $\succeq$, i.e., $U(x^t) > (=)\, U(x^s)$ if and only if $x^t \succ (\sim)\, x^s$ for x^t, x^s in $\mathcal{X}$.*

Given Proposition 2, part (i) of Theorem 1 is equivalent to the following: *$\mathcal{O}$ can be rationalized by a strongly monotone, continuous and concave utility function if it obeys GARP.* This is, of course, well-known and corresponds to a part of Afriat's Theorem. Part (ii) of Theorem 1 appears to be new. It strengthens Afriat's Theorem by saying that *any* preorder on $\mathcal{X}$ which is consistent with the revealed relations could be extended into a preorder on R^ℓ_+ which is representable by a strongly monotone, continuous and concave utility function. In other words, no consistent preorder on $\mathcal{X}$ can be eliminated by rationality. Our proof of this theorem gives an explicit procedure for constructing a utility function extending (to the consumption space R^ℓ_+) any given consistent preorder on $\mathcal{X}$ in a way that also rationalizes $\mathcal{O}$.

This sharper version of Afriat's Theorem, and the even sharper version we develop in Corollary 1, play a crucial role in helping us develop a test for separable preferences. Theorem 1 may also be of interest in itself. For example, it may be the case that a modeler has some information on the agent's preference over $\mathcal{X}$, in addition to that revealed by the agent's demand at different price vectors; Theorem 1—or rather the algorithm for constructing the utility function given in the proof—gives a precise way of incorporating such information.

[7] To say the obvious, even if $x^t \geq^* x^s$ and $x^s \not\geq^* x^t$ we do not obtain $x^t \gg x^s$.

The proof of Theorem 1 requires the following lemma. This lemma is well-known but we include it here for completeness. The inequalities (8) and the form of U in this lemma were both introduced by Afriat.

LEMMA 1 *Given the data set $\mathcal{O}$, suppose there are numbers ϕ^t and $\lambda^t > 0$ (for every $t \in T$) that obey the* Afriat inequalities*, i.e.,*

$$\phi^t \leqslant \phi^k + \lambda^k p^k \cdot (x^t - x^k) \text{ for all } k \neq t. \tag{8}$$

Then the function $U : R^l_+ \to R$ given by

$$U(x) = \min_{(p_t, x_t) \in \mathcal{O}} \{\phi_t + \lambda_t p_t \cdot (x - x_t)\} \tag{9}$$

rationalizes $\mathcal{O}$ and satisfies $U(x^t) = \phi^t$. This function is strongly monotone and concave.

Proof: The fact that $U(x^t) = \phi^t$ follows immediately from the definition of U and the Afriat inequalities. Note that U is a strongly monotone utility function since $\lambda_t > 0$ for all t and it is concave because it is the minimum of a family of concave functions. To see that it generates the observations in $\mathcal{O}$, let x satisfy $p^s \cdot x = p^s \cdot x^s$. It follows from the definition of U that $U(x) \leqslant \phi_s$ and so $U(x) \leqslant U(x^s)$. Therefore, $x^s \in \arg\max_{x \in B(p^s, p^s \cdot x^s)} U(x)$. QED

Proof of Theorem 1: With no loss of generality, write $\mathcal{X} = \{x^1, x^2, ..., x^N\}$, where either $x^{n+1} \succ x^n$ or $x_{n+1} \sim x^n$, for $n = 1, 2, ...N - 1$. All we need to do is to find numbers ϕ^s and $\lambda^s > 0$ (for $s = 1, 2, ..., N$) that (a) obey the Afriat inequalities and (b) satisfy $\phi^{n+1} > (=)\phi^n$ if $x^{n+1} \succ (\sim)x^n$. Then Lemma 1 guarantees that U (as defined by (9)) rationalizes the data set and satisfies $U(x^n) = \phi^n$ for $n = 1, 2, ..., N$. Note that the latter property, together with (b), guarantee that (ii) holds, i.e., the restriction of U to $\mathcal{X}$ coincides with $\succeq$. We shall find ϕ^n and λ^n with a step-by-step approach, explicitly constructing the numbers ϕ^n and λ^n one at a time.

Denote $p^i \cdot (x^j - x^i)$ by a^{ij}. Choose ϕ^1 to be any number and λ^1 to be any positive number. Since $x^j \succeq x^1$ for all $j > 1$, we have $x^j \nprec x_1$ and so

$$a^{1j} = p^1 \cdot (x^j - x^1) \geqslant 0$$

(because if not, $x^1 \ggcurly^{**} x^j$ and $x^1 \succ x^J$ by the consistency of $\succeq$). Suppose $x^2 \succ x^1$; then $\min_{j>1} a_{1j} > 0$. This is because if $a^{1J} = 0$ for some $J > 1$, then $x^1 \succeq x^J$ (again, by the consistency of $\succeq$), which is impossible since $x^J \succ x^1$. So there is ϕ^2 such that

$$\phi^1 < \phi^2 < \min_{j>1}\{\phi^1 + \lambda^1 a^{1j}\}. \tag{10}$$

On the other hand, if $x^2 \sim x^1$, then we can choose ϕ^2 such that

$$\phi^1 = \phi^2 \leqslant \min_{j>1}\{\phi^1 + \lambda^1 a^{1j}\}. \tag{11}$$

Now choose $\lambda^2 > 0$ sufficiently small so that

$$\phi^1 \leqslant \phi^2 + \lambda^2 a^{21}.$$

Clearly this is possible if $a^{21} \geqslant 0$. If $a^{21} < 0$, then $x^2 \succ x^1$ (by consistency of $\succeq$), in which case $\phi^2 > \phi^1$, and the inequality is still possible for a λ^2 sufficiently small.

We now go on to choose ϕ^3 and λ^3. It follows from $x^j \succeq x^i$ for all $j > 2$ and $i = 1, 2$ that $x^j \nprec x^i$ and so

$$a^{ij} = p^i \cdot (x^j - x^i) \geqslant 0 \ \text{ for } i = 1, 2.$$

Once again we consider two cases: when $x^3 \succ x^2 \succeq x^1$ and when $x^3 \sim x^2 \succeq x^1$. In the case of the former, we know that $\min_{j>2} a^{2j} > 0$ since, if $a^{2J} = 0$ for some $J > 2$, then $x^2 \succeq x^J$ which contradicts $x^J \succ x^2$. Therefore,

$$\phi^2 < \min_{j>2}\{\phi^2 + \lambda^2 a^{2j}\}.$$

Similarly, $\min_{j>2} a^{1j} > 0$; if $a^{1J} = 0$ for some $J > 2$, then $x^1 \succeq x^J$ which contradicts $x^J \succ x^1$. Therefore,

$$\phi^2 < \min_{j>2}\{\phi^1 + \lambda^1 a^{1j}\};$$

this is the case because either ϕ^2 was chosen to satisfy (10) or $\phi^2 = \phi^1$. We conclude that there is ϕ^3 such that

$$\phi^2 < \phi^3 < \min\left\{\min_{j>2}\{\phi^1 + \lambda^1 a^{1j}\}, \min_{j>2}\{\phi^2 + \lambda^2 a^{2j}\}\right\}.$$

We turn to the case where $x^3 \sim x^2 \succeq x^1$. It follows from (10) and (11) that $\phi^2 \leqslant \min_{j>2}\{\phi^1 + \lambda^1 a^{1j}\}$. We also know that $a^{2j} \geqslant 0$ for all $j > 2$. Therefore, we can choose ϕ^3 such that

$$\phi^2 = \phi^3 \leqslant \min\left\{\min_{j>2}\{\phi^1 + \lambda^1 a^{1j}\}, \min_{j>2}\{\phi^2 + \lambda^2 a^{2j}\}\right\}.$$

Now choose $\lambda^3 > 0$ sufficiently small so that

$$\phi^i \leqslant \phi^3 + \lambda^3 a^{3i} \text{ for } i = 1, 2.$$

Clearly this is possible if $a^{3i} \geqslant 0$. If $a^{3i} < 0$, then $x^3 \succ x^i$, in which case $\phi^3 > \phi^i$, and the inequality is still possible for a λ^3 sufficiently small.

Repeating this argument, we can choose ϕ^k (for $k = 2, 3, ..., N$) such that if $x^k \succ x^{k-1}$ then

$$\phi^{k-1} < \phi^k < \min_{s \leqslant k-1} \left\{ \min_{j > k-1} \{\phi^s + \lambda^s a_{sj}\} \right\} \tag{12}$$

and if $x^k \sim x^{k-1}$ then

$$\phi^{k-1} = \phi^k \leqslant \min_{s \leqslant k-1} \left\{ \min_{j > k-1} \{\phi^s + \lambda^s a_{sj}\} \right\}; \tag{13}$$

and $\lambda_k > 0$ (for $k = 2, 3, ..., N$) such that

$$\phi^i \leqslant \phi^k + \lambda^k a^{ki} \text{ for } i \leqslant k - 1. \tag{14}$$

For any fixed m, (12) and (13) guarantee that $\phi^m \leqslant \phi^s + \lambda^s a^{sm}$ for $s < m$ (setting $k = m$ and letting $j = m$), while (14) guarantees that this inequality holds for $s > m$ (with $k = s$ and $i = m$). In other words, we have found λ^s and ϕ^s to obey the Afriat inequalities. QED

Our final objective in this section is to develop a sharper version of Theorem 1 that allows us to control utility levels at price-income combinations *outside* the set of observations. While this result may be independently interesting, our reason for proving it is to use it later in Section 4 to establish the validity of our revealed preference test for separability. The basic message of Corollary 1 is easy to explain. Given the utility function U, the *indirect utility* at $(p, w) \in R^\ell_{++} \times R_+$ is the highest utility achievable in the budget set $B(p, w)$, i.e., $I_U(p, w) = \max\{U(x) : x \in B(p, w)\}$. Suppose U rationalizes $\mathcal{O}$ and agrees with some consistent preorder $\succeq$. Clearly, $I_U(p, w) \geqslant U(x^t)$ for all $x^t \in B(p, w) \cap \mathcal{X}$. The corollary goes further by saying that we could always choose U such that, if x^s is ranked (by $\succeq$) above all bundles in $B(p, w) \cap \mathcal{X}$, then $I_U(p, w) \leqslant U(x^s)$. In other words, $I_U(p, w)$ could be chosen so that it will *not* be higher than the utility of any bundle which it is not 'required' (by $\succeq$) to be higher.

It is necessary to introduce a number of concepts formally before we state the result. For any $(p, w) \in R^\ell_{++} \times R_+$ such that $B(p, w) \cap \mathcal{X}$ is nonempty, we define

$$\beta(p, w) = \{x' \in B(p, w) \cap \mathcal{X} : x' \succeq x \; \forall \, x \in B(p, w) \cap \mathcal{X}\}. \tag{15}$$

This is the set of elements that are ranked by $\succeq$ (at least weakly) ahead of the other elements in $B(p,w) \cap \mathcal{X}$; clearly it is nonempty so long as $B(p,w) \cap \mathcal{X}$ is nonempty. If $B(p,w) \cap \mathcal{X}$ is empty, we let $\beta(p,w) = \{0\}$. We define

$$\bar{\beta}(p,w) = \{x'' \in \mathcal{X} : x'' \succ \beta(p,w) \text{ and if } y \in \mathcal{X} \text{ obeys } y \succ \beta(p,w) \text{ then } y \succeq x'' \}. \quad (16)$$

In other words, this is the set of elements in $\mathcal{X}$ ranked just above $\beta(p,w)$. Note that this set is empty if $B(p,w)$ contains a highest ranked element of $\mathcal{X}$ (according to $\succeq$).

Corollary 1 *Suppose $\mathcal{O}$ admits a preorder $\succeq$ on $\mathcal{X}$ that is consistent with its revealed relations. For any finite set $\{(p^m, w^m)\}_{m\in M} \subset R^\ell_{++} \times R_+$, there is a well-behaved and concave utility function $U : R^\ell_+ \to R$ satisfying (i) and (ii) in Theorem 1 and also the following property:*

$$U(\bar{\beta}(p^m,w^m)) > I_U(p^m,w^m) \geqslant U(\beta(p^m,w^m)) \text{ for all } m \in M \quad (17)$$

and $U(\beta(p^m,w^m)) = I_U(p^m,w^m)$ if $p^m \cdot \underline{x} = w^m$ for all $\underline{x} \in \beta(p^m,w^m)$.

Remark 1: Since *any* rationalization of $\mathcal{O}$ must satisfy $I_U(p^m,w^m) \geqslant U(\underline{X}(p^m,w^m))$, the nontrivial part of (17) lies in the claim that U can be chosen such that $I_U(p^m,w^m)$ is bounded above by $U(\bar{X}(p^m,w^m))$.

Remark 2: We prove this result in the Appendix. Like the proof of Theorem 1, this proof provides an explicit procedure for constructing U.

3. A revealed preference test for non-linear budget sets

There are a number of results that extend Afriat's Theorem to account for budget (more generally, constraint) sets that are nonlinear, including Matzkin (1991), Chavas and Cox (1993) and Forges and Minelli (2009). The last of these is most relevant for our purposes. Forges and Minelli consider a scenario where an observer has access to a set of observations, with each observation consisting of a (possibly) nonlinear constraint set and a choice from that set. There is a natural and obvious generalization of the GARP property for such a set of observations; Forges and Minelli pointed out that this generalized GARP property is necessary and sufficient for the observations to be rationalizable. The utility function they construct for the rationalization has a form similar to the classic Afriat-form (see (9)); in particular, it is the minimum of a finite family of functions, though that family is no longer linear in x. Therefore, this utility function need not be concave and indeed one could construct data sets where any rationalizing utility function *must not* be concave. In other words, Afriat's Theorem has a general extension to nonlinear constraint sets, so long as we do not require the concavity of the utility function rationalizing the data.

In this section we re-present and generalize the Forges-Minelli Theorem. This generalization is similar to our generalization of Afriat's Theorem in Theorem 1. Our version of the result emphasizes the flexibility of the utility function rationalizing the data set; in particular, it is possibe to construct a utility function that agrees with *any* consistent preorder on the observed choices. We will apply this result later in Section .

A set $K \subset R^\ell_+$ is said to be a *regular* if it has the following properties: (i) there is $x \gg 0$ such that $x \in K$; (ii) K is *monotone*, i.e., if $x \in K$ then any $x' \in R^\ell_+$ such that $x' \leqslant x$ is also in K; and (iii) K is compact (iv) if x is on the *upper boundary of* K, (i.e., if, for all $y \gg x$, $y \notin K$) then λx is not on the upper boundary of K for all $\lambda \in [0, 1)$; and (v) if x is on the upper boundary of K, then $y \notin K$ for all $y > x$. In our formulation of the Forges-Minelli Theorem, we shall be requiring the constraint sets to be regular. Clearly, the classical budget set $B(p, w)$ (for $p \gg 0$ and $w > 0$) is a regular set.[8]

It is straightforward to check that when K is regular, then for every nonzero $x \in R^\ell_+$, there is a unique $\mu > 0$ such that μx is on the upper boundary of K. Define $g : R^\ell_+ \to R$ by $g(x) = 1/\mu$ for $x > 0$ and $g(0) = 0$; we shall refer to g as K's *gauge function.* This function is continuous, 1-homogeneous and—because of (iv)—it is strongly monotone.[9] Lastly, the set K can be characterized by the gauge function, .e., $K = \{x \in R^\ell_+ : g(x) \leqslant 1\}$.

Let $\mathcal{O} = \{(K^t, x^t)\}_{t \in T}$ be a finite set, where $K^t \subset R^\ell_{++}$ is a regular set and x^t is on the upper

[8]Forges and Minelli imposed conditions (i) to (iv) on their constraint sets, but not (v). Notice, for example, that conditions (i) to (iv) will permit a constraint set of the form $[0, 1] \times [0, 1]$ but $[0, 1] \times [0, 1] \cup \{(0, r) : r \in [1, 2]\}$ is excluded by (iv). Neither is a regular set; in particular, the former is excluded by (v). We add assumption (v) here because it is convenient for our purposes (specifically in the application of Theorem 2 in the next section). This added assumption leads to a stronger conclusion: the rationalizing utility function in Theorem 2 is continuous and strongly monotone, while the rationalizing function in Forges and Minelli's version of this result (Proposition 3 in their paper) is continuous and *monotone*, i.e., if $x' \gg x$ then $U(x') > U(x)$. If we drop (v), then the same proof we give for Theorem 2 will still go through, except that it leads to a monotone (rather than strongly monotone) utility function.

[9]Suppose $x' > x$ but $g(x') = g(x) = 1/m$. This implies that mx' and mx are both on the upper boundary of K, which is excluded by (v) since $mx' > mx$.

boundary of K^t. We interpret $\mathcal{O}$ as a set of observations, where x^t is the observed bundle of ℓ goods chosen by the agent from the constraint set K^t. The function $U : R^\ell_+ \to R$ is said to *rationalize the set* $\mathcal{O}$ if $x^t \in \arg\max\{U(x) : x \in K^t\}$ for all $t \in T$.

We are interested in finding conditions under which $\mathcal{O}$ is rationalizable. For this purpose, it is useful to re-introduce the revealed preference relations in this more general setting. Denote the set of observed demands by $\mathcal{X}$, i.e., $\mathcal{X} = \{x^t\}_{t\in T}$. For x^t, $x^s \in \mathcal{X}$, we say that x^t is *directly revealed preferred to* x^s if $x^s \in K^t$; if there is $\lambda > 1$ such that $\lambda x^s \in K^t$ (so x^s is not on the upper boundary of K^t), we say that x^t is *directly revealed strictly preferred to* x^s. We denote these relations by $x^t \succeq^{**} x^s$ and $x^t \succ^{**} x^s$ respectively. From these we may construct the revealed preferred ($\succeq^*$), revealed strictly preferred ($\succ^*$), and revealed indifference ($\sim^*$) relations, in exactly the same way as in Section 2.

It is clear that Proposition 4 remains true in this setting, i.e., if some locally non-satiated utility function U rationalizes $\mathcal{O}$ then it induces a preorder on $\mathcal{X}$ that is consistent with the revealed relations of $\mathcal{O}$. And there is an analog to Proposition 2 as well, namely the existence of a consistent preorder is equivalent to GARP. In this context, $\mathcal{O}$ is said to obey GARP if the following holds: whenever there are observations (K^t, x^t) (for $t = 1, 2, ..., n$) satisfying $x^2 \in K^1$, $x^3 \in K^2$,..., $x^n \in K^{n-1}$, and $x^1 \in K^n$, then x^2 is on the upper boundary of K^1, x^3 is on the upper boundary of K^2,..., and x^1 is on the upper boundary of K^n. More succinctly, if $x^1 \succeq^{**} x^2$, $x^2 \succeq^{**} x^3$,..., $x^{n-1} \succeq^{**} x^n$, and $x^n \succeq^{**} x^1$, then none of the direct revealed preferences can be replaced with the strict preference $\succ^{**}$.

Last but not least there is an analog to Theorem 1 in this setting.

THEOREM 2 *Suppose $\mathcal{O} = \{(K^t, x^t)\}_{t\in T}$ admits a preorder $\succeq$ (on $\mathcal{X}$) that is consistent with its revealed relations. Then there exists a well-behaved utility function $U : R^\ell_+ \to R$ with the following properties: (i) it rationalizes $\mathcal{O}$ and (ii) the preorder on $\mathcal{X}$ induced by U coincides with $\succeq$, i.e., $U(x^t) > (=)\, U(x^s)$ if and only if $x^t \succ (\sim)\, x^s$ for x^t, x^s in $\mathcal{X}$. The utility function U can be chosen to take the form*

$$U(x) = \min_{t\in T} \left\{\phi^t + \lambda^t(g^t(x) - 1)\right\} \tag{18}$$

where $g^t : R^\ell_+ \to R_+$ is the gauge function of K^t and ϕ^t, $\lambda^t \in R$ with $\lambda^t > 0$. Furthermore,

$$\phi^t \leqslant \phi^k + \lambda^k(g^k(x^t) - 1) \ \ \textit{for all } k \neq t \tag{19}$$

so that $U(x^t) = \phi^t$.

REMARK: This result includes Theorem 1 as a special case, where $K^t = B(p^t, p^t \cdot x^t)$. With no loss of generality, we may normalize p^t so that $p^t \cdot x^t = 1$ for all $t \in T$; the gauge function of K^t is then simply $g^t(x) = p^t \cdot x$, so (18) has precisely the same form as (9) and (19) reduces to the Afriat inequalities (8).

Proof: We need to find ϕ^t and λ^t to satisfy (19) and with $\phi^t > (=)\phi^s$ if $x^t \succ (\sim) x^s$. Then it is clear that U, as defined by (18), rationalizes $\mathcal{O}$. (To see this, adapt the proof of Lemma 1 in the obvious way and use the fact that x^t is on the upper boundary of K^t so $g^t(x^t) = 1$.) Furthermore $U(x^t) = \phi^t$ for all $t \in T$, so the preorder on $\mathcal{X}$ induced by U agrees with $\succeq$. Since g^t (for all $t \in T$ are continuous and strongly monotone functions, the same is true of U.

With no loss of generality, write $\mathcal{X} = \{x^1, x^2, ..., x^N\}$, where either $x^{n+1} \succ x^n$ or $x_{n+1} \sim x^n$, for $n = 1, 2, ...N-1$. Denote $g^i(x^j) - 1$ by a^{ij}. Note that if $a^{ij} \leqslant 0$ then $x^i \succeq^{**} x^j$ and so (by the consistency of $\succeq$) $x^i \succeq x^j$. Similarly, if $a^{ij} < 0$ then $x^i \ggcurly^{**} x^j$ and so the consistency of $\succeq$ guarantees that $x^i \succ x^j$. Therefore, a^{ij} has precisely the same properties as a^{ij} defined in the proof of Theorem 1. The proof given for Theorem 1 works equally well as a proof for this result, once we substitute this more general formula for a^{ij}. Using that method, we can construct ϕ^1 and λ^1, ϕ^2 and λ^2, and so forth. QED

4. Weakly separable preferences

We assume that the agent chooses from a finite set of goods Q which can be divided into three non-overlapping subsets X, Y and Z, consisting of ℓ_X goods, ℓ_Y goods, and ℓ_Z goods respectively. We denote the agent's consumption bundle by $q = (x, y, z) \in R_+^{\ell_X} \times R_+^{\ell_Y} \times R_+^{\ell_Z}$. The agent faces the price vector $p = (p_X, p_Y, p_Z) \in R_{++}^{\ell_X} \times R_{++}^{\ell_Y} \times R_{++}^{\ell_Z}$, where the subvector p_X gives the prices of the X-goods, etc.

At price $p^t = (p_X^t, p_Y^t, p_Z^t)$, we observe the agent purchasing the bundle $q^t = (x^t, y^t, z^t)$. Let $\mathcal{O} = \{(p^t, q^t)\}_{t \in T}$ be a finite set of observations. The set of observed consumption bundles of goods is denoted by $\mathcal{Q}$, i.e., $\mathcal{Q} = \{q^t\}_{t \in T}$. The set of observed consumption bundles of X-goods is denoted by $\mathcal{X}$; i.e., $\mathcal{X} = \{x^t\}_{t \in T}$. In a similar way, we define $\mathcal{Y}$ and $\mathcal{Z}$. As in the previous section, we allow for multiple copies of the same vector.

We are interested in conditions under which the data set $\mathcal{O}$ can be *rationalized by a weakly* *separable utility function* (or, more briefly, by a *separable utility function*). By this we mean that there are strongly monotone and continuous functions $U : R_+^{\ell_X} \to R_+$ and $V : R_+^{\ell_Y} \to R_+$ and $G : R_+^{\ell_X} \times R_+^{\ell_Y} \times R_+^{\ell_Z}$ which rationalize (respectively) the sets $\mathcal{O}_X = \{(p_X^t, x^t)\}_{t\in T}$ and $\mathcal{O}_Y = \{(p_Y^t, y^t)\}_{t\in T}$ and $\mathcal{O} = \{(p^t, q^t)\}_{t\in T}$, where

$$G(x, y, z) = F(U(x), V(y), z) \tag{20}$$

for some strongly monotone and continuous function $F : R_+^3 \to R$. Clearly there is no loss of generality in assuming that $U(0) = V(0) = 0$ (so that $U(x) > 0$ and $V(y) > 0$ all non-zero bundles x and y) and we shall be imposing this convenient condition throughout this section.[10]

The special structure of G makes it possible for the agent's decision to be decomposed into a decision on the consumption of X-goods, a decision on the consumption of Y-goods and an overall decision on the allocation of income over the three good categories. To be precise, we define, at price $p = (p_X, p_Y, p_Z) \gg 0$ and income $w > 0$, the sets

$$K(p, w) = \{(u, v, z) \in R_+^2 \times R_+^{\ell_Z} : u = U(x),\ v = V(y), \text{ and } (x, y, z) \in B(p, w)\} \tag{21}$$

$$L(p, w) = \{(w_X, w_Y, z) \in R_+^2 \times R_+^{\ell_Z} : w_X + w_Y + p_Z \cdot z \leqslant w\} \tag{22}$$

$K(p, w)$ gives the possibilities open to the agent at (p, w) in terms of the bundle of Z-goods and the utility derived from the X-goods and Y-goods, while $L(p, w)$ identifies the budget possibilities for the agent in terms of the expenditure devoted to X-goods and Y-goods, w_X and w_Y respectively, and the bundle of Z-goods. The next result is well-known and has a straightforward proof which we shall omit.

PROPOSITION 3 *Suppose that G has the form given by (20) where U, V, and F (and thus G) are well-behaved functions. Then the following are equivalent:*
[1] $(\bar{x}, \bar{y}, \bar{z}) \in \arg\max\{G(q) : q \in B(p, w)\}$
[2] $\bar{x} \in \arg\max\{U(x) : x \in B(p_X, p_X \cdot \bar{x})\}$, $\bar{y} \in \arg\max\{V(y) : y \in B(p_Y, p_Y \cdot \bar{y})\}$, *and*

[10] Note that while we allow for the separation of two subsets of goods, X and Y, the same arguments go through if there is just one subset of goods over which there is separable utility (which is the case discussed in the Introduction) or if there are more than two disjoint subsets of goods with separable utility. Our treatment of the case with two separable sets, X and Y, together with a residual set of goods Z, covers all the mathematically substantive issues. Note also that the analysis allows for Z to be empty. In other words, we also provide a test for rationalization with utility functions of the form $F(U(x), V(y))$.

$(U(\bar{x}), U(\bar{y}), z) \in \arg\max\{F(u,v,z) : (u,v,z) \in K(p,w)\}$;
[3] $\bar{x} \in \arg\max\{U(x) : x \in B(p_X, p_X \cdot \bar{x})\}$, $\bar{y} \in \arg\max\{V(y) : y \in B(p_Y, p_Y \cdot \bar{y})\}$, *and* $(p_X \cdot \bar{x}, p_Y \cdot \bar{y}, \bar{z}) \in \arg\max\{F(I_U(p_X, w_X), I_V(p_Y, w_Y), z) : (w_X, w_Y, z) \in L(p,w)\}$.

Statement [3] in Proposition 3 says that the decision making of this agent can be thought of as consisting of two parts. First, the agent's choice of X-goods must maximize U at that level of expenditure on X-goods; similarly, the bundle of Y-goods is chosen to maximizes V at that level of expenditure on Y-goods. Second, the expenditure on X and Y-goods (w_X and w_Y respectively) are determined by maximizing F, after taking into account the indirect utilities $I_U(p_X, w_X)$ and $I_V(p_Y, w_Y)$. The formulation [2] is closely related to [3], but imagines the agent as choosing from the constraint set $K(p,w)$, having first worked out the sub-utilities derived from the X and Y-goods; this formulation of the agent's choice problem turns out to be useful for our purposes.

Our objective is to provide necessary and sufficient conditions on $\mathcal{O}$ to guarantee that the observations are rationalizable by a separable utility function. We would like a result that has a similar structure to Theorem 1, though the conditions we require of $\mathcal{O}$ must be stronger since the restrictions on the utility function are more stringent. As in the last section, we begin with a discussion of the revealed preference relations.

Let $\mathcal{C} = \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}$; in other words, a typical element q in $\mathcal{C}$ may be written as (x^r, y^s, z^t) where x^r is the consumption bundle of X-goods at the observation $r \in T$, y^s the consumption bundle of Y-goods at the observation $s \in T$ (possibly different from r), etc. Note that $\mathcal{Q} \subset \mathcal{C}$.

Given bundles $q = (x,y,z)$ and $q' = (x', y', z')$ in $\mathcal{C}$, we say that q *is directly revealed preferred to* q' *under separability* (and denote this relation by $q \succeq^{**} q'$) if any one of the conditions hold:
(S1) $q = (x^t, y^t, z^t)$ for some $t \in T$ (in other words, $q \in \mathcal{Q}$)) and $p^t \cdot q' \leqslant p^t \cdot q$ or
(S2) (a) $z = z'$, $y = y'$ and $x \succeq^{**}_X x'$ where $\succeq^{**}_X$ is the direct revealed preference relation (on $\mathcal{X}$) induced by $\mathcal{O}_X = \{(p^t_X, x^t)\}_{t \in T}$.
(S2) (b) $z = z'$, $x = x'$ and $y \succeq^{**}_X y'$ where $\succeq^{**}_Y$ is the direct revealed preference relation (on $\mathcal{Y}$) induced by $\mathcal{O}_Y = \{(p^t_Y, y^t)\}_{t \in T}$.
We say that q is *directly strongly revealed preferred to* q' *under separability* (denoted $q \succ\!\!\succ^{**} q'$) if (S1) is satisfied with a strong inequality or (S2)(a) is satisfied with $\succ\!\!\succ^{**}_X$ in place of $\succeq^{**}_X$ or

(S2)(b) is satisfied with $\gg_Y^{**}$ in place of $\succeq_Y^{**}$. The bundle q is said to be *revealed preferred to q' under separability* if there are bundles q^1, q^2,...,q^n in $\mathcal{C}$ such that $q \succeq^{**} q^1$, $q^1 \succeq^{**} q^2$,..., $q^{n-1} \succeq^{**} q^n$, and $q^n \succeq^{**} q'$. If any of the revealed preferences in this (finite) sequence are strong, then we say that q is *strongly revealed preferred to q' under separability.* The revealed preference and revealed strong preference of q over q' are denoted by $q \succeq^* q'$ and $q \gg^* q'$ respectively. The bundles q and q' are said to be *revealed indifferent under separability* if $q \succeq^* q'$ and $q' \succeq^* q$; this relation is denoted by $q \sim^* q'$. We shall refer to $\succeq^*$, $\gg^*$ and $\sim^*$ as the *revealed separability relations.*

A relation $\succeq$ on $\mathcal{C}$ is said to be *separable* if it obeys the following properties:
(a) it is transitive and reflexive;
(b) for every $y' \in \mathcal{Y}$ and $z' \in \mathcal{Z}$, the relation on $x \in \mathcal{X}$ given by the restriction of $\succeq$ to the elements (x, y', z') is a preorder (i.e., it is transitive, reflexive and complete) and this preorder is independent of y' and z';
(c) for every $x' \in \mathcal{X}$ and $z' \in \mathcal{Z}$, the relation on $y \in \mathcal{Y}$ given by the restriction of $\succeq$ to the elements of the form (x', y, z') is a preorder and independent of x' and z';
(d) $\succeq$ is a preorder when restricted to the set of observed consumption bundles $\mathcal{Q}$.
We denote the preorder induced by $\succeq$ on X and Y by $\succeq_X$ and $\succeq_Y$ respectively. Note that (a), (b), and (c) imply that if $x \succeq_X x'$ and $y \succeq_Y y'$ then

$$(x, y, z) \succeq (x', y, z) \succeq (x', y', z) \tag{23}$$

and if, in addition, either $x \succ_X x'$ or $y \succ_Y y'$ then $(x, y, z) \succ (x', y', z)$. The separable relation $\succeq$ on $\mathcal{Q}$ is said to be *consistent with the revealed separability relations* or simply *consistent with revealed separability* if $q \succeq q'$ whenever $q \succeq^* q'$ and $q \succ q'$ whenever $q \gg^* q'$. The former property also guarantees that $q \sim q'$ whenever $q \sim^* q'$. Clearly, if $\succeq$ is a consistent separable relation then $\succeq_X$ ($\succeq_Y$) is also consistent with the revealed relations of $\mathcal{O}_X$ ($\mathcal{O}_Y$).

Note that if the revealed separability relations ($\succeq^*$, $\gg^*$, and $\sim^*$) are such that it admits a separable and consistent relation $\succeq$, then $\succeq$ can be chosen to *minimal*; by this we mean that whenever $q \succeq q'$, there exists q^i ($i = 1, 2, ...(n-1)$) such that $q = q^0 \succeq q^1$, $q^1 \succeq q^2$, $q^1 \succeq q^2$,..., $q^{n-1} \succeq q^n = q'$, such that either both q^i and q^{i+1} are in $\mathcal{Q}$, the vectors q^i and q^{i+1} differ *only* in their X-subvectors or Y-subvectors, or $q^i \succeq^{**} q^{i+1}$. This is clear since if $q \succeq q'$ but no such property holds then this particular relationship between q and q' can simply be removed. The 'remaining' relation will still be separable and consistent with revealed separability. It is also possible for us to move in the other direction: if $\succeq$ is consistent with revealed separability then $\succeq$ can be chosen to be a preorder on $\mathcal{C}$. This is clear because we can always complete $\succeq$ in a way that preserves reflexivity and transitivity, and any such completion will be separable and consistent with revealed separability.

Example 1: Consider a data set with the following two observations, drawn from an agent who is choosing a consumption bundle out of two X-goods and two Z-goods.
$p^1_X = (1, 3/2)$, $p^1_Z = (2, 1)$, $x^1 = (1, 0)$, $z^1 = (2, 1)$, $w^1 = 6$
$p^2_X = (3/2, 1)$, $p^2_Z = (2, 1)$, $x^2 = (0, 1)$, $z^2 = (1, 2)$, $w^2 = 5$.
In this case, $\mathcal{X} = \{x^1, x^2\}$, $\mathcal{C} = \{(x^1, z^1), (x^1, z^2), (x^2, z^1), (x^2, z^2)\}$, and $\mathcal{Q} = \{(x^1, z^1), (x^2, z^2)\}$. The non-trivial direct revealed separability relations on $\mathcal{C}$ are precisely the following: $(x^1, z^1) \ggcurly^{**} (x^1, z^2)$ and $(x^1, z^1) \ggcurly^{**} (x^2, z^2)$. Therefore, any separable relation on $\mathcal{C}$ consistent with revealed separability must satisfy $(x^1, z^1) \succ (x^1, z^2)$ and $(x^1, z^1) \succ (x^2, z^2)$. To complete the specification of $\succeq$ as a separable relation, we need to specify the relation between x^1 and x^2. It is straightforward to check the following specifies a minimal and consistent separable relation $\succeq$ on $\mathcal{C}$: each element is related to itself, $(x^1, z^1) \succ (x^1, z^2)$, $(x^1, z^1) \succ (x^2, z^2)$, $(x^1, z^1) \succ (x^2, z^1)$, and $(x^1, z^2) \succ (x^2, z^2)$.

The next result says that if a data set is drawn from an agent who maximizes a separable utility function, then the data set will admit a separable relation which is consistent with the revealed separability relations. This is not hard to show. The result following that is more substantial and is the main result of the paper. It says that the admissibility of a consistent separable relation is also sufficient for rationalization with a separable utility function.

PROPOSITION 4 *Suppose that $\mathcal{O}$ is drawn from an agent who maximizes a utility function G of the form (20), where U and V are locally non-satiated and F is strongly monotone. Then the preorder $\succeq$ on $\mathcal{C}$ induced by G is a separable relation and it is consistent with the revealed separability relations.*

Proof: It is clear that $\succeq$ is a separable relation (in fact, it is a preorder on $\mathcal{C}$). So only consistency needs to be checked. If $q \succeq^{**} q'$ because (S1) holds, then q is the bundle chosen at some observation t (when price is p^t) and clearly $G(q) \geqslant G(q')$, because otherwise the agent is better off choosing q'. Now suppose $q \succeq^{**} q'$ because (S2)(a) holds, i.e., $x \succeq^{**} x'$. Suppose

$x = x^s$ for some observation s. We claim that $U(x) \geqslant U(x')$. If not, $F(U(x'), V(y^s), z^s) > F(U(x^s), V(y^s), z^s)$ (because F is strongly monotone) and so the bundle (x', y^s, x^s) does not cost more than $q^s = (x^s, y^s, z^s)$ at the price vector p^s and gives higher utility, which contradicts the optimality of q^s at observation s. So $U(x) \geqslant U(x')$ and thus $G(q) \geqslant G(q')$ (again by the strong monotonicity of F). A similar argument can be applied if $q \succeq^{**} q'$ because (S2)(b) holds.

Suppose $q \ggcurly^{**} q'$ because (S1) holds with a strict inequality, i.e., $p^t \cdot q' < p^t \cdot q$, assuming that $q = q^t = (x^t, y^t, z^t)$ is the bundle chosen at observation t. We need to show that $G(q) > G(q')$. Since U is locally non-satiated, there is some bundle $\hat{x}$ such that $U(\hat{x}) > U(x')$ (where $q' = (x', y', z')$) and $\hat{q} = (\hat{x}, y', z')$ satisfies $p^t \cdot \hat{q} \leqslant p^t \cdot q$. So $G(\hat{q}) > G(q')$ since F is strongly monotone and if $G(q') \geqslant G(q)$, we obtain $G(\hat{q}) > G(q)$ which contradicts the optimality of q at t. Now suppose $q \ggcurly^{**} q'$ because (S2)(a) holds with a strict inequality, i.e., $x \ggcurly^{**}_X x'$. Write $q = (x, y, z)$ and suppose $x = x^s$ for some observation s. We claim that $U(x) > U(x')$; if not, by the local non-satiation of U, there is some bundle x'' such that $p^s_X \cdot x'' \leqslant p^s_X \cdot x$ such that $U(x'') > U(x') \geqslant U(x)$. Since F is strongly monotone, $F(U(x''), V(y^s), z^s) > F(U(x^s), V(y^s), z^s)$ and the bundle (x'', y^s, z^s) costs no more than (x^s, y^s, z^s) at price p^s, which contradicts the optimality of q^s. A similar argument can be applied if $q \ggcurly^{**} q'$ because (S2)(b) holds with a strict inequality. QED

Our main result is the converse of Proposition 4.

THEOREM 3 *Suppose there exists a separable relation $\succeq$ on $\mathcal{C}$ that is consistent with revealed separability. Then there is a well-behaved function F, and well-behaved and concave functions U and V such that G (as defined by (20)) rationalizes $\mathcal{O}$. The preorder induced by U on $\mathcal{X}$ coincides with $\succeq_X$, the preorder induced by V on $\mathcal{Y}$ coincides with $\succeq_Y$, and the preorder induced by G on $\mathcal{Q}$ coincides with the restriction of $\succeq$ to $\mathcal{Q}$. Furthermore, if $\succeq$ is minimal then the preorder induced by G on $\mathcal{C}$ coincides with the restriction of $\succeq$ to $\mathcal{C}$.*

Proposition 3 guarantees that in any rationalization with a separable utility function, U and V must also rationalize (respectively) $\mathcal{O}_X = \{(p^t_X, x^t)\}_{t \in T}$ and $\mathcal{O}_Y = \{(p^t_Y, y^t)\}_{t \in T}$. Theorem 3 assumes that these data sets obey GARP (because the preorders $\succeq_X$ and $\succeq_Y$ induced by $\succeq$ are consistent with the revealed relations of $\mathcal{O}_X$ and $\mathcal{O}_Y$ respectively), so the existence of well-behaved and concave functions U and V rationalizing $\mathcal{O}_X$ and $\mathcal{O}_Y$ is guaranteed by

Theorem 1. The delicate part of the proof of Theorem 3 consists of a careful construction of U and V so that, *in addition to these rationalizations*, we also have

$$\widehat{\mathcal{O}} = \{(K(p^t, p^t \cdot q^t), (u^t, v^t, z^t))\}_{t \in T} \tag{24}$$

obeying GARP (where $u^t = U(x^t)$ and $v^t = V(y^t)$ and K is defined by (21)). Theorem 2 then guarantees the existence of F that rationalizes $\widehat{\mathcal{O}}$. That the resulting G rationalizes $\mathcal{O}$ then follows from Proposition 3.

$\widehat{\mathcal{O}}$ obeys GARP if and only if there exists a preorder $\widehat{\gtrsim}$ on the set $\widehat{\mathcal{Q}} = \{(u^t, v^t, z^t)\}_{t \in T}$ that is consistent with the revealed relations of $\widehat{\mathcal{O}}$. An obvious candidate for $\widehat{\gtrsim}$ is the following:

$$(u^t, v^t, z^t) \widehat{\gtrsim} (u^s, v^s, z^s) \text{ if } (x^t, y^t, z^t) \geq (x^s, y^s, z^s). \tag{25}$$

Note that this relation is a well-defined preorder on $\widehat{\mathcal{Q}} = \{(u^t, v^t, z^t)\}_{t \in T}$. Given than $\geq$ is a preorder on $\mathcal{Q}$ (property (d) in our definition of $\geq$), it suffices to check the following: if there are t, $\bar{t} \in T$ such that $(u^t, v^t, z^t) = (u^{\bar{t}}, v^{\bar{t}}, z^{\bar{t}})$, then

$$(x^t, y^t, z^t) \sim (x^{\bar{t}}, y^{\bar{t}}, z^{\bar{t}}). \tag{26}$$

Since U rationalizes $\{(p_X^t, x^t)\}_{t \in T}$, if $u^t = u^{\bar{t}}$ then $x^t \sim x^{\bar{t}}$; similarly, $y^t \sim y^{\bar{t}}$ and so (26) follows from (23). What is less clear is the consistency of $\widehat{\gtrsim}$ with $\widehat{\mathcal{O}}$'s revealed relations. Given the definition (26), and denoting the direct revealed preference and revealed strong preference relations of $\widehat{\mathcal{O}}$ by $\widehat{\gtrsim}^{**}$ and $\widehat{\succ\!\!\succ}^{**}$, consistency holds if and only if

$$\textit{for all } \hat{q}^t,\, \hat{q}^s \in \widehat{\mathcal{Q}},\ \textit{ if } \hat{q}^t \,\widehat{\gtrsim}^{**} (\widehat{\succ\!\!\succ}^{**})\, \hat{q}^s \textit{ then } (u^t, v^t, z^t) \geq (>) \,(u^s, v^s, z^s). \tag{27}$$

This property does *not* hold for every well-behaved U and V rationalizing $\mathcal{O}_X$ and $\mathcal{O}_Y$ and requires a more careful construction of U and V. The following example brings out some of the subtleties involved in the construction of U and V.

Example 2: Consider a data set with the following two observations, drawn from an agent who is choosing a consumption bundle out of two X-goods and two Z-goods.
$p_X^1 = (2, 1)$, $p_Z^1 = (1, 3/2)$, $x^1 = (0, 1)$, $z^1 = (1, 2)$, $w^1 = 5$
$p_X^2 = (1, 2)$, $p_Z^2 = (3/2, 1)$, $x^2 = (1, 0)$, $z^2 = (2, 1)$, $w^2 = 5$.

We claim that this data set can be rationalized by a separable utility function. It is easy to check that $(x^1, z^1) \succ\!\!\succ^{**} (x^1, z^2)$ but $(x^1, z^1) \not\gtrsim^{**} (x^2, z^2)$ and $(x^2, z^2) \succ\!\!\succ^{**} (x^2, z^1)$ but

$(x^2, z^2) \not\succeq^{**} (x^1, z^1)$. Any consistent separable relation must satisfy $(x^1, z^1) \succ (x^1, z^2)$ and $(x^2, z^2) \succ (x^2, z^1)$. The separable relation must also relate x^1 and x^2. Note that we are not completely free to choose this relation. In particular, it is not possible to specify $x^1 \sim_X x^2$, because this creates the following cycle:

$$(x^1, z^1) \succ (x^1, z^2) \sim (x^2, z^2) \succ (x^2, z^1) \sim (x^1, z^1).$$

So we are left with either $x^1 \succ_X x^2$ or $x^2 \succ_X x^2$. Given the symmetry in this example, we could confine our analysis to the former; the latter case will be analogous.

Assuming that $x^1 \succ_X x^2$, this leads to

$$(x^1, z^1) \succ (x^1, z^2) \succ (x^2, z^2)$$

and we obtain the following minimal and consistent separable relation on $\mathcal{C}$: every element is related to itself, $(x^1, z^1) \succ (x^1, z^2)$, $(x^1, z^2) \succ (x^2, z^2)$, $(x^1, z^1) \succ (x^2, z^2)$, $(x^2, z^2) \succ (x^2, z^1)$, $(x^1, z^2) \succ (x^2, z^1)$, and $(x^1, z^1) \succ (x^2, z^1)$.

We could check directly that $\mathcal{O}$ can be rationalized with a separable utility function whose restriction to $\mathcal{C}$ agrees with $\succeq$. To do this we need to find U that rationalizes $\mathcal{O}_X$ such that

$$\widehat{\mathcal{O}} = \left\{ \left(K(p^1, w^1), (U(x^1), z^1)\right), \left(K(p^2, w^2), (U(x^2), z^2)\right) \right\}$$

obeys GARP. It is then clear (given that $\widehat{\mathcal{O}}$ consists of just two elements), that there is a strongly monotone and continuous function F that rationalizes $\widehat{\mathcal{O}}$.

More formally, we require U to satisfy several properties. We require (i) x^1 to be optimal in $B(p^1_X, 1)$ and x^2 to be optimal in $B(p^2_X, 1)$ (under U); we also require (ii) $U(x^1) > U(x^2)$. Since $(x^1, z^1) \ggcurly^{**} (x^1, z^2)$, we have $(U(x^1), z^2) \in K(p^1, w^1)$, which implies that $(U(x^2), z^2) \in K(p^1, w^1)$ and not on its upper boundary (in other words, $(U(x^1), z^1) \ggcurly^{**} (U(x^2), z^2)$). Thus, for $\widehat{\mathcal{O}}$ to obey GARP, it is necessary and sufficient that $(U(x^1), z^1) \notin K(p^2, w^2)$. This means that at (p^2, w^2), if the agent chooses to buy z^1, then the money left for buying X-goods, which is $w^2 - p^2 \cdot z^1 = 1.5$, must not give him a utility greater than the bundle x^1. In other words, it is necessary and sufficient that (iii) $U(x^1) > I_U(p^2_X, 1.5)$. It is not hard to see that one could draw in convex indifference curves representing U on R^2_+ such that (i), (ii), and (iii) are satisfied.

The next result (Lemma 2) spells out the properties on U and V that are sufficient for our purpose and guarantees that U and V with those properties exist. The proof is an application of Corollary 1 so in fact it provides a way of explicitly constructing U and V. Following that, Lemma 3 says that when U and V have the properties listed in Lemma 2, then $\widehat{\gtrsim}$, as defined by (26) is a preorder consistent with the revealed preference relations of $\widehat{\mathcal{O}}$.

For any $p \in R_{++}^{\ell_X}$ and $x \in \mathcal{X}$, we define $\bar{x}(p,x) = \arg\min\{p \cdot x' : x' \in \mathcal{X} \text{ and } x' \succeq_X x\}$. In other words, at price p, $\bar{x}(p,x)$ are the cheapest bundles in $\mathcal{X}$ that are weakly preferred to x. We define $\bar{y}(p',y)$ (for $p' \in R_{++}^{\ell_Y}$ and $y \in \mathcal{Y}$) in a similar way.

Lemma 2 *Suppose there exists a separable relation $\succeq$ on $\mathcal{C}$ that is consistent with revealed separability. Then there exists strongly monotone, continuous, and concave functions $U : R_+^{\ell_X} \to R_+$ and $V : R_+^{\ell_X} \to R_+$ with the following properties:*
(P1) U rationalizes $\mathcal{O}_X = \{(p_X^t, x^t)\}_{t\in T}$ and its restriction to $\mathcal{X}$ agrees with $\succeq_X$;
(P2) V rationalizes $\mathcal{O}_Y = \{(p_Y^t, y^t)\}_{t\in T}$ and its restriction to $\mathcal{Y}$ agrees with $\succeq_Y$;
(P3) If $p^t \cdot q^t > p_Z^t \cdot z^s$ but there does not exist $w_X \geqslant 0$ and $w_Y \geqslant 0$ such that $p^t \cdot q^t = p_Z^t \cdot z^s + w_X + w_Y$ with $p_X^t \cdot \bar{x}(p_X^t, x^s) \leqslant w_X$ and $p_Y^t \cdot \bar{y}(p_Y^t, y^s) \leqslant w_Y$, then there exists $w'_X \geqslant 0$ and $w'_Y \geqslant 0$ such that $p^t \cdot q^t = p_Z^t \cdot z^s + w'_X + w'_Y$, with $I_U(p_X^t, w'_X) < U(x^s)$ and $I_V(p_Y^t, w'_Y) < V(y^s)$
(P4) If $x \sim x^s$ for all $x \in \bar{x}(p_X^t, x^s)$ and $y \sim y^s$ for all $y \in \bar{y}(p_Y^t, y^s)$ and there is w'_X and w'_Y such that $p^t \cdot q^t = p_Z^t \cdot z^s + w'_X + w'_Y$ with $p_X^t \cdot \bar{x}(p_X^t, x^s) = w'_X$ and $p_Y^t \cdot \bar{y}(p_Y^t, y^s) = w'_Y$, then $I_U(p_X^t, w'_X) = U(x^s)$ and $I_V(p_Y^t, w'_Y) = V(y^s)$.

Proof: We say that $(t,s) \in H \subset T \times T$ if $p^t \cdot q^t > p_Z^t \cdot z^s$ but there does not exist $w_X > 0$ and $w_Y > 0$ such that $p^t \cdot q^t = p_Z^t \cdot z^s + w_X + w_Y$ with $p_X^t \cdot \bar{x}(p_X^t, x^s) \leqslant w_X$ and $p_Y^t \cdot \bar{y}(p_Y^t, y^s) \leqslant w_Y$. In this case, we can find $w_X(t,s) \geqslant 0$ and $w_Y(t,s) \geqslant 0$ such that $p^t \cdot q^t = p_Z^t \cdot z^s + w_X(t,s) + w_Y(t,s)$, with $w_X(t,s) < p_X^t \cdot \bar{x}(p_X^t, x^s)$ and $w_Y(t,s) < p_Y^t \cdot \bar{y}(p_Y^t, y^s)$. We say that $(t,s) \in H' \subset T \times T$ if $x \sim x^s$ for all $x \in \bar{x}(p_X^t, x^s)$ and $y \sim y^s$ for all $y \in \bar{y}(p_Y^t, y^s)$ and there is w'_X and w'_Y such that $p^t \cdot q^t = p_Z^t \cdot z^s + w'_X + w'_Y$, with $p_X^t \cdot \bar{x}(p_X^t, x^s) = w'_X$ and $p_Y^t \cdot \bar{y}(p_Y^t, y^s) = w'_Y$. In this case, we define $w_X(t,s) = w'_X$ and $w_Y(t,s) = w'_Y$.

Notice that, by our design, $x^s \succ_X \beta(p_X^t, w_X(t,s))$ for $(t,s) \in H$, so that $x^s \succeq_X \bar{\beta}(p_X^t, w_X(t,s))$. (Recall the definitions of β and $\bar{\beta}$ in (15) and (16).) For $(t,s) \in H'$, $x^s \in \bar{x}(p_X^t, x^s) = \beta(p_X^t, w_X(t,s))$ and $x^s \succ_X x$ for all $x \in \mathcal{X}$ such that $p_X^t \cdot x < w_X(t,s)$. Furthermore, $\succeq_X$ is consistent with the revealed relations of $\mathcal{O}_X$ ($\succeq_X^*$ and $\succ\!\!\succ_X^*$). By Corollary 1, there is U rationalizing $\mathcal{O}_X$ such that $I_U(p_X^t, w_X(t,s)) < U(x^s)$ for all $(t,s) \in H$ and $I_U(p_X^t, w_X(t,s)) = U(x^s)$ for all $(t,s) \in H'$. In a similar way, we can guarantee the existence of V obeying the prescribed properties. QED

Lemma 3 *Suppose there exists a separable relation $\succeq$ on $\mathcal{C}$ that is consistent with revealed separability and U and V are chosen to satisfy the conditions of Lemma 2. Then the relation $\widehat{\succsim}$ on $\widehat{\mathcal{Q}}$ defined by (25) is a preorder and it is consistent with the revealed preference relations of $\widehat{\mathcal{O}} = \{(K(p^t, p^t \cdot q^t), (u^t, v^t, z^t))\}_{t \in T}$.*

Proof: We have already shown that $\widehat{\succsim}$ is a preorder, so only consistency (the property (27)) needs to be checked. Suppose $\hat{q}^t = (u^t, v^t, z^t) \widehat{\succsim}^{**} \hat{q}^s = (u^s, v^s, z^s)$; then there is $\hat{w}_X \geqslant 0$ and $\hat{w}_Y \geqslant 0$ such that $p^t \cdot q^t = p_Z^t \cdot z^s + \hat{w}_X + \hat{w}_Y$ with $I_U(p_X^t, \hat{w}_X) \geqslant U(x^s)$ and $I_V(p_Y^t, \hat{w}_Y) \geqslant U(y^s)$. So there cannot be w'_X and w'_Y satisfying their required properties in (P3) (of Lemma 2). It follows from Lemma 2 that there are $w_X \geqslant 0$ and $w_Y \geqslant 0$ such that $p^t \cdot q^t = p_Z^t \cdot z^s + w_X + w_Y$, with $p_X^t \cdot \bar{x}(p_X^t, x^s) \leqslant w_X$ and $p_Y^t \cdot \bar{y}(p_Y^t, y^s) \leqslant w_Y$. Therefore, $(x^t, y^t, z^t) \succeq^{**} (x, y, z^s)$ for all $x \in \bar{x}(p_X^t, x^s)$ and all $y \in \bar{y}(p_Y^t, y^s)$ (by the (S1) condition on $\succeq^{**}$). The consistency of $\succeq$ then guarantees that

$$(x^t, y^t, z^t) \succeq (x, y, z^t). \tag{28}$$

We also have

$$(x, y, z^s) \succeq (x^s, y^s, z^s) \tag{29}$$

(see (23)). The transitivity of $\succeq$ guarantees that $(x^t, y^t, z^t) \succeq (x^s, y^s, z^s)$, so we have shown one half of (27).

Next we modify the argument to show the other half of property (27): if $\hat{q}^t = (u^t, v^t, z^t) \widehat{\succcurlyeq}^{**} \hat{q}^s = (u^s, v^s, z^s)$ then $(x^t, y^t, z^t) \succ (x^s, y^s, z^s)$. If $\hat{q}^t \widehat{\succcurlyeq}^{**} \hat{q}^s$, then there are $\hat{w}_X \geqslant 0$ and $\hat{w}_Y \geqslant 0$ such that $p^t \cdot q^t = p_Z^t \cdot z^s + \hat{w}_X + \hat{w}_Y$, with $I_U(p_X^t, \hat{w}_X) > U(x^s)$ and $I_V(p_Y^t, \hat{w}_Y) > U(y^s)$. It follows from (P3) in Lemma 3 that there are $w_X \geqslant 0$ and $w_Y \geqslant 0$ such that $p^t \cdot q^t = p_Z^t \cdot z^s + w_X + w_Y$, with $p_X^t \cdot \bar{x}(p_X^t, x^s) \leqslant w_X$ and $p_Y^t \cdot \bar{y}(p_Y^t, y^s) \leqslant w_Y$. It is *not* possible for the latter two inequalities to be equations and for $U(x) = U(x^s)$ for all $x \in \bar{x}(p_X^t, x^s)$ and $V(y) = V(y^s)$ for all $y \in V(\bar{y}(p_Y^t, y^s))$ because in that case (P4) in Lemma 3 guarantees $I_U(p_X^t, w_X) = U(x^s)$ and $I_V(p_Y^t, w_Y) = U(y^s)$ (and so it is impossible for $I_U(p_X^t, \hat{w}_X) > U(x^s)$ and $I_V(p_Y^t, \hat{w}_Y) > U(y^s)$). We are left with two cases: (i) either $p_X^t \cdot x < w_X$ for some $x \in \bar{x}(p_X^t, x^s)$ or $p_Y^t \cdot y < w_Y$ for some $y \in \bar{y}(p_Y^t, y^s)$; or (ii) either $x > x^s$ for some $x \in \bar{x}(p_X^t, x^s)$ or $y > y^s$ for some $y \in \bar{y}(p_Y^t, y^s)$. In case (i), we conclude that there is $(x, y) \in (\bar{x}(p_X^t, x^s), \bar{y}(p_Y^t, y^s))$ such that $(x^t, y^t, z^t) \succcurlyeq^{**} (x, y, z^s)$ (by (S1)), from which we obtain $(x^t, y^t, z^t) \succ (x, y, z^s)$. In case (ii), we conclude that there is $(x, y) \in (\bar{x}(p^t_X, x^s), \bar{y}(p^t_Y, y^s))$ such that $(x, y, z^s) \succ (x^s, y^s, z^s)$ (see (23)). Note that (28) and (29) continue to hold and so the transitivity of $\succeq$ guarantees that $(x^t, y^t, z^t) \succ (x^s, y^s, z^s)$. QED

Proof of Theorem 3: Choose U and V to satisfy the conditions of Lemma 2. In particular, U rationalizes $\mathcal{O}_X$ and the preorder induced by U on $\mathcal{X}$ coincides with $\succeq_X$; V has analogous properties. Lemma 3 guarantees that $\widehat{\succsim}$ is a preorder on $\widehat{\mathcal{Q}}$ consistent with the revealed preference relations of $\widehat{\mathcal{O}}$. Note also that $K(p^t, p^t \cdot q^t)$ is a regular constraint set (for all $t \in T$). By Theorem 2 there is a well-behaved function F that rationalizes $\widehat{\mathcal{O}}$ and such that the preorder induced by F coincides with $\widehat{\succsim}$ on $\widehat{\mathcal{Q}}$. Therefore, at price p^t, $x^t \in \arg\max\{U(x) : x \in B(p^t_X, p^t_X \cdot x^t)\}$, $y^t \in \arg\max\{V(y) : y \in B(p^t_Y, p^t_Y \cdot y^t)\}$, and

$$(U(x^t), V(y^t), z^t) \in \arg\max\{F(u, v, z) : (u, v, z) \in K(p^t, p^t \cdot (x^t, y^t, z^t))\}$$

By Proposition 3, $q^t = (x^t, y^t, z^t) \in \arg\max\{G(q) : q \in B(p^t, p^t \cdot q^t)\}$. We conclude that G rationalizes $\mathcal{O}$. Furthermore, by the definition of $\widehat{\succsim}$ (see (25)), if

$$q^t = (x^t, y^t, z^t) \succeq (\succ)\, (x^s, y^s, z^s) = q^s,$$

then $(u^t, v^t, z^t) \widehat{\succsim} (\widehat{\succ})\, (u^s, v^s, z^s)$. Since the preorder induced by F coincides with $\widehat{\succsim}$, we obtain

$$G(q^t) = F(u^t, v^t, z^t) \geqslant (>)\, F(u^s, v^s, z^s) = G(q^s).$$

So the preorder induced by G on $\mathcal{Q}$ coincides with the restriction of $\succeq$ to $\mathcal{Q}$.

If $\mathcal{C}$ is minimal, then for q, $q' \in \mathcal{C}$ such that $q \succeq q'$, there exists q^i $(i = 1, 2, ...(n-1))$ such that $q = q^0 \succeq q^1$, $q^1 \succeq q^2$, $q^1 \succeq q^2$,..., $q^{n-1} \succeq q^n = q'$, where either both q^i and q^{i+1} are in $\mathcal{Q}$ or the vectors q^i and q^{i+1} differ *only* in their X-subvectors or Y-subvectors. But in these cases, we also $G(q^i) \geqslant (>)\, G(q^{i+1})$ if $q^i \succeq (\succ)\, q^{i+1}$. Therefore, $G(q) \geqslant (>)\, G(q')$ if $q \succeq (\succ)\, q'$. QED

One may be tempted to think that so long as the revealed separability relations $\succeq^*$ and $\gg^*$ display no cycles on $\mathcal{C}$, then a consistent separable relation is admissible; in other words, that there is some analog to Proposition 2, which relates GARP with the admissibility of a consistent preorder on the finite data set. The next example shows that that is not the case.

Example 3: Consider a data set with the following four observations, drawn from an agent who is choosing a consumption bundle out of two X-goods and four Z-goods.

$p_X^1 = (1,2)$, $p_Z^1 = (2.5, 1, 100, 100)$, $x^1 = (1,0)$, $z^1 = (2,1,0,0)$, $w^1 = 7$
$p_X^2 = (2,1)$, $p_Z^2 = (1, 1.5, 100, 100)$, $x^2 = (0,1)$, $z^2 = (1,2,0,0)$, $w^2 = 5$
$p_X^3 = (2,1)$, $p_Z^3 = (100, 100, 2.5, 1)$, $x^3 = (0,1)$, $z^3 = (0,0,2,1)$, $w^3 = 7$
$p_X^4 = (1,2)$, $p_Z^4 = (100, 100, 1, 1.5)$, $x^4 = (1,0)$, $z^4 = (0,0,1,2)$, $w^4 = 5$.

The direct revealed separability relations on $\mathcal{C}$ are the following:
(i)$(x^1, z^1) \ggcurly^{**} (x^1, z^2)$ and $(x^1, z^1) \ggcurly^{**} (x^2, z^2)$
(ii) $(x^2, z^2) \ggcurly^{**} (x^2, z^1)$
(iii) $(x^3, z^3) \ggcurly^{**} (x^3, z^4)$ and $(x^3, z^3) \ggcurly^{**} (x^4, z^4)$
(iv) $(x^4, z^4) \ggcurly^{**} (x^4, z^3)$.
This is an exhaustive list of relations between distinct elements, there are no others.[11] In particular, there is no direct relation $\succeq_X^{**}$ between $x^1 = x^4$ and $x^2 = x^3$, so all the relationships on the list are obtained through criterion (S1). Since there are no cycles in the direct revealed preferences, GARP is satisfied, and the data can be rationalized by a monotone utility function. However, we claim that there is no consistent separable relation on $\mathcal{C}$, so the data cannot be rationalized by a separable utility function.

First, notice that it is not possible for $x^2 \succeq_X x^1$; if this were to hold,

$$(x^1, z^1) \succ (x^2, z^2) \succ (x^2, z^1) \succeq (x^1, z^1),$$

where the first relation follows from (i) and the second from (ii). Therefore, $(x^1, z^1) \succ (x^1, z^1)$, which is impossible since $\succeq$ is preorder. We conclude that $x^1 \succ_X x^2$; equivalently, $x^4 \succ_X x^3$. This in turn leads to

$$(x^3, z^3) \succ (x^4, z^4) \succ (x^4, z^3) \succ (x^3, z^3),$$

where the first relation follows from (iii) and the second from (iv). We obtain $(x^3, z^3) \succ (x^3, z^3)$, which is again impossible.

In Theorem 3, while the rationalizing utility functions U and V can be chosen to be concave, F was only only specified as well-behaved, i.e., strongly monotone and continuous. It is natural to ask whether, in the case where a data set is rationalizable with a separable utility function, then one could choose, not just U and V, but also F, to be concave. The following example shows that that is not the case.

[11] Notice that no bundle involving z^3 or z^4 is affordable at p^1 or p^2; similarly, no bundle involving z^1 or z^2 is affordable at p^3 or p^4.

Example 4: Consider a data set with the following six observations of an agent choosing from two X-goods and one Z-good. Note that w_X^t and w^t denote the total expenditure on the X-goods and the total expenditure over all goods respectively.
$p_X^1 = (1,2)$, $p_Z^1 = 1$, $x^1 = (1,1)$, $z^1 = 12$, $w_X^1 = 3$, $w^1 = 15$;
$p_X^2 = (1,2)$, $p_Z^2 = 1$, $x^2 = (4,4)$, $z^2 = 3$, $w_X^2 = 12$, $w^2 = 15$;
$p_X^3 = (1,2)$, $p_Z^3 = 1$, $x^3 = (4,0)$, $z^3 = 11$, $w_X^3 = 4$, $w^3 = 15$;
$p_X^4 = (2,1)$, $p_Z^4 = 1$, $x^4 = (1,1)$, $z^4 = 12$, $w_X^4 = 3$, $w^4 = 15$;
$p_X^5 = (2,1)$, $p_Z^5 = 1$, $x^5 = (4,4)$, $z^5 = 3$, $w_X^5 = 12$, $w^5 = 15$;
$p_X^6 = (2,1)$, $p_Z^6 = 1$, $x^6 = (4,0)$, $z^6 = 100$, $w_X^6 = 8$, $w^6 = 108$.

It is straightforward to check that the observations $\{(p_X^t, x^t)\}_{t=1}^6$ obey GARP. Let U be any strongly monotone, continuous, and concave utility function rationalizing the choice over the X-goods.

While there are six observations, there are only three distinct budget sets. Observations 1, 2, and 3 are all associated with the budget set $B((1,2,1),15)$, though the choices made are distinct. In z-u space (think of z and u on the horizontal and vertical axes respectively), the chosen bundles are $(12, U(1,1))$, $(3, U(4,4))$, and $(11, U(4,0))$ respectively. Since U rationalizes $\{(p_X^t, x^t)\}_{t=1}^6$, these points must lie on the upper boundary of $K^1 = K^2 = K^3$. Similarly, observations 4 and 5 are both associated with the budget set $B((2,1,1),15)$. In z-u space, the chosen bundles are $(12, U(1,1))$ and $(3, U(4,4))$; they lie on the upper boundary of $K^4 = K^5$. Notice that the upper boundaries of K^1 and K^4 intersect at two points at least. Observation 6 is associated with a distinct budget set, $B((2,1,1),106)$ and its chosen bundle in z-u space is $(100, U(4,0))$; the very high level of income guarantees that K^6 contains K^1 and K^4 in its interior.

We claim that the set $\{(K^t, (z^t, u^t)\}_{t=1}^6$ obeys GARP, so that there will be a strongly monotone and continuous function F rationalizing those observations (and hence $G(x,z) = F(U(x), z)$ will rationalize $\{(p^t, (x^t, z^t)\}_{t=1}^6$). To check this, we note that K^1 and K^4 have at least two intersections, at the chosen bundles $(12, U(1,1))$ and $(3, U(4,4))$; to check that GARP holds, all we need to do is check that $(11, U(4,0))$, which is also chosen at $K^3 = K^1$ is *not* in the interior of K^4. In fact, something stronger than this is true for any U because

$$I_U((2,1),4) < I_U((2,1),8) = U(4,0) = I_U((1,2),4),$$

where the strict inequality follows from the strong monotonicity of U, the first equality from observation 6, and the second equality from observation 3. So we obtain

$$(11, I_U((2,1),4)) < (11, U(4,0)) \tag{30}$$

where $(11, I_U((2,1),4))$ is on the upper boundary of K^4 and $(11, U(4,0))$ on the upper boundary of K^1.

Therefore there is a strongly monotone and continuous function F that rationalizes $\{(K^t, (z^t, u^t)\}_{t=1}^6$. However, such an F will *never* be quasiconcave, so long as U is concave. To see this, choose t so that $t3 + (1-t)12 = 11$. If F is quasiconcave, its indifference curves are convex; since $(3, U(4,4))$, $(12, U(1,1))$, and $(11, U(4,0))$ lie on the same convex indifference curve, we obtain

$$t(3, U(4,4)) + (1-t)(12, U(1,1)) = (11, tU(4,4) + (1-t)U(1,1)) \geqslant (11, U(4,0)).$$

It follows from (30) that

$$t(3, U(4,4)) + (1-t)(12, U(1,1)) = (11, tU(4,4) + (1-t)U(1,1)) > (11, I_U(2,1),4),$$

which is not possible since $(3, U(4,4))$, $(12, U(1,1))$ and $(11, I_U((2,1),4))$ are on the boundary of K^4 and K^4 is concave (because U is concave).[12]

Appendix

Proof of Proposition 2: Suppose that $\mathcal{O}$ admits a preorder $\geq$ that is consistent with its revealed relations. To establish that GARP holds, let there be observations satisfying (7). The consistency of $\geq$ (in particular, with $\geq^*$), guarantees that

$$x^1 \geq x^2 \geq x^3 \geq \dots x^{n-1} \geq x^n \geq x^1, \tag{31}$$

and hence $x^t \sim x^{t+1}$ for $t = 1, 2, ..., (n-1)$. Furthermore, $\geq$ is consistent with $\succ^*$ and so $x^t \not\succ^{**} x^{t+1}$ in (7), as required by GARP.

[12]In other words, since U is concave, both K_1 and K_4 are convex sets and in fact K_1 must be strictly convex because of (30). Therefore, the indifference curve of F, which must pass through three points on K_1 cannot be a convex curve.

To prove the "if" part of this proposition, we first define an equivalence relation on $\mathcal{X}$ in the following manner: the elements x and x' are related to each other if $x \succeq^* x'$ and $x' \succeq^* x$. This relation partitions $\mathcal{X}$ into equivalence classes; we denote a typical equivalence class by $[x]$. We shall prove this claim by induction on the number of equivalence classes in $\mathcal{X}$. If $\mathcal{X}$ consists of just one equivalence class, then we let $x \succeq x'$ for any two elements in $\mathcal{X}$. This must agree with the revealed relations unless, for some x^t and $x^{t'}$ in $\mathcal{X}$, we have $x^t \ggcurly^* x^{t'}$, which requires $x^t \succ x^{t'}$; but GARP says it is not possible for $x^t \ggcurly^* x^{t'}$ since we also have $x^{t'} \succeq^* x^t$.

Suppose the claim is true whenever $\mathcal{X}$ has K equivalence classes or less. We will show that the claim holds when $\mathcal{X}$ has $K+1$ equivalence classes. With an abuse notation, we define a relation $\succeq^*$ on the equivalence classes in $\mathcal{X}$ in the following manner: $[x] \succeq^* [x]'$ if there is $x^t \in [x]$ and $x^{t'} \in [x]'$ such that $x^t \succeq^* x^{t'}$. Notice that this relation is transitive and antisymmetric (the latter means that if $[x]$ and $[x]'$ are distinct and $[x] \succeq^* [x]'$ then $[x]' \not\succeq^* [x]$). It follows there must be an equivalence class $\underline{[x]}$ such that there is no other distinct equivalence class $[x]$ with $\underline{[x]} \succeq^* [x]$. Denote the set of observations associated with $\underline{[x]}$ by $\underline{\mathcal{O}}$, i.e., $(p^t, x^t) \in \underline{\mathcal{O}}$ if $x^t \in \underline{[x]}$.

Now consider the set of observations $\bar{\mathcal{O}} = \mathcal{O} \backslash \underline{\mathcal{O}}$, with the associated set of bundles $\bar{\mathcal{X}} = \mathcal{X} \backslash \underline{[x]}$. Notice that the removal of the observations $\underline{\mathcal{O}}$ does not affect the revealed preference relations; by this we mean that for x', $x \in \bar{\mathcal{X}}$, x' is revealed preferred (revealed strictly preferred) to x when the data set is $\mathcal{O}$ if and only if x' revealed preferred (revealed strictly preferred) to x when the data set is $\bar{\mathcal{O}}$. So the number of equivalence classes in $\bar{\mathcal{X}}$ is one less than the number in $\mathcal{X}$. By the induction hypothesis, there is a preorder $\succeq$ on $\bar{\mathcal{X}}$ that agrees with the revealed relations generated by $\bar{\mathcal{O}}$. We can extend $\succeq$ to $\mathcal{X}$ by defining $x^t \succ x^s$ for all $x^t \notin \underline{[x]}$ and $x^s \in \underline{[x]}$ and $x^s \sim x^{s'}$ for x^s, $x^{s'} \in \underline{[x]}$. This relation is a preorder on $\mathcal{X}$ and it agrees with the revealed relations of $\mathcal{O}$. QED

The proof of Corollary 1 requires the following two lemmas.

LEMMA 4 *Suppose $\mathcal{O}$ admits a preorder $\succeq$ on $\mathcal{X}$ that is consistent with its revealed relations. Given $(\hat{p}, \hat{w}) \in R^\ell_{++} \times R_+$, there exist $\hat{x} \in R^\ell_+$ and a preorder $\succeq'$ on $\mathcal{X}' = \mathcal{X} \cup \{\hat{x}\}$ such that (a) $\succeq'$ is consistent with the revealed relations of $\mathcal{O}' = \mathcal{O} \cup \{(\hat{p}, \hat{x})\}$; (b) $\succeq'$ is an extension of $\succeq$; and (c) $\overline{\beta}(\hat{p}, \hat{w}) \succ' \hat{x} \succeq' \beta(\hat{p}, \hat{w})$. We can choose $\hat{x} \in \beta(\hat{p}^m, \hat{w})$ if $\hat{p} \cdot \underline{x} = \hat{w}$ for all $\underline{x} \in \beta(\hat{p}, \hat{w})$; in this case, $\mathcal{X}' = \mathcal{X}$ and $\succeq' = \succeq$.*

Proof: We first consider the case where there is $x^{t'} \in \beta(\hat{p}, \hat{w})$ such that $\hat{p} \cdot x^{t'} < w$ and choose $\hat{x} > x^{t'}$. Let $\succeq'$ be the extension of $\succeq$ such tat $x^{t''} \succ' \hat{x}$ and $\hat{x} \succ' x^{t'}$ (where $x^{t''} \in \overline{\beta}(\hat{p}, \hat{w})$). We need to show the consistency of $\succeq'$ with the relations $\succeq^*$ and $\ggcurly^*$ induced by $\mathcal{O}'$.

It is clear from our construction that if $\hat{x} \succeq^{**} x^t$ then $\hat{x} \succ' x^t$, since $\hat{x} \succ' x^{t'}$ and $x^{t'} \succeq' x^t$. Suppose that for some s, $x^s \succeq^{**} \hat{x}$. Then $x^s \ggcurly^{**} x^{t'}$, since $\hat{x} > x^{t'}$. By the consistency of $\succeq$ (with respect to the revealed relations of $\mathcal{O}$), $x^s \succ x^{t'}$. Therefore, $x^s \notin B(\hat{p}, \hat{w})$ and we obtain $x^s \succeq x^{t''}$ (by the definition of $x^{t''}$). Since $\succeq'$ is an extension of $\succeq$, we also have $x^s \succeq' x^{t''}$ and, by construction of $\succeq'$, $x^{t''} \succ' \hat{x}$. By the transitivity of $\succeq'$, we obtain $x^s \succ' \hat{x}$.

We now turn to the case where $\hat{x} = x^{t'}$. We need to show that $\succeq$ remains consistent with the revealed relations of $\mathcal{O}'$. If $\hat{x} \succeq^{**} x^t$, then $\hat{x} = x^{t'} \succeq x^t$ by definition of $x^{t'}$. If $\hat{x} = x^{t'} \ggcurly^{**} x^t$ then $\hat{x} = x^{t'} \succ x^t$ since there does not exist $x^t \sim x^{t'}$ with $\hat{p} \cdot x^t < \hat{w}$. Suppose that for some s, $x^s \succeq^{**} \; (\ggcurly^{**})\, \hat{x} = x^{t'}$. Then $x^s \succeq \; (\succ)\, \hat{x} = x^{t'}$, by the consistency of $\succeq$ (with respect to the revealed relations of $\mathcal{O}$). QED

LEMMA 5 *Suppose $\mathcal{O}$ admits a preorder $\succeq$ on $\mathcal{X}$ that is consistent with its revealed relations. Given a finite set $\{(p^m, w^m)\}_{m \in M} \subset R^\ell_{++} \times R_+$, there exist $\{\hat{x}^m\}_{m \in M} \subset R^\ell_+$ and a preorder $\succeq'$ on $\mathcal{X}' = \mathcal{X} \cup \{\hat{x}^m\}_{m \in M}$ such that (a) $\succeq'$ is consistent with the revealed relations induced by $\mathcal{X}' = \mathcal{X} \cup \{\hat{x}^m\}_{m \in M}$; (b) $\succeq'$ is an extension of $\succeq$; (c)*

$$\overline{\beta}(p^m, w^m) \succ' \hat{x}^m \succeq' \beta(p^m, w^m) \text{ for all } m \in M, \tag{32}$$

where we can choose $\hat{x}^m \in \beta(p^m, w^m)$ if $p^m \cdot \underline{x} = w^m$ for all $\underline{x} \in \beta(p^m, w^m)$.[13]

Proof: Let $\widetilde{M} = \{m \in M : p^m \cdot \underline{x} = w^m \,\forall\, \underline{x} \in \beta(p^m, w^m)\}$. Choose $m' \in \widetilde{M}$ and let $\hat{x}^{m'} = x^{t'}$ for some $x^{t'} \in \beta(p^{m'}, w^{m'})$. By Lemma 4, $\succeq$ remains consistent with the revealed relations of $\mathcal{O}' = \mathcal{O} \cup \{(p^{m'}, \hat{x}^{m'})\}$. Now choose another element m' in $\widetilde{M}$ (if one exists), $\hat{x}^{m''} \in \beta(p^{m''}, w^{m''})$ and Lemma 4 again guarantees that $\succeq$ is consistent with the revealed relations of $\mathcal{O}'' = \mathcal{O}' \cup \{(p^{m''}, \hat{x}^{m''})\}$. Repeating this procedure until all the elements of $\widetilde{M}$ are exhausted, we conclude that $\succeq$ is consistent with the revealed relations of $\widetilde{\mathcal{O}} = \mathcal{O} \cup \{(p^m, \hat{x}^m)\}_{m \in \widetilde{M}}$.

Now choose an element $n \in M \backslash \widetilde{M}$. By Lemma 4, there is a bundle $\hat{x}^n$ and a preorder $\succeq'$ on $\mathcal{X}' = \mathcal{X} \cup \{\hat{x}^n\}$ such that $\succeq'$ extends $\succeq$, $\succeq'$ is consistent with the revealed relations of

[13] We deem $\overline{\beta}(p^m, w^m) \succ' \hat{x}^m$ to be satisfied if $\overline{\beta}(p^m, w^m)$ is empty. Note that β and $\overline{\beta}$ are defined with respect to $\mathcal{X}$.

$\widetilde{\mathcal{O}} \cup \{(p^n, \hat{x}^n)\}$, and the following holds:

$$\overline{\beta}(p^n, w^n) \succ' \hat{x}^n \succeq' \beta(p^n, w^n). \tag{33}$$

Suppose there exists $n' \neq n$ in $M \backslash \tilde{M}$, then by Lemma 4 again, there is $\hat{x}^{n'}$ and a preorder $\succeq''$ extending $\succeq'$ which is consistent with the revealed relations of $\widetilde{\mathcal{O}} \cup \{(p^n, \hat{x}^n), (p^{n'}, \hat{x}^{n'})\}$ and that satisfy

$$\overline{\beta}'(p^{n'}, w^{n'}) \succ'' \hat{x}^{n'} \succeq'' \beta'(p^{n'}, w^{n'}), \tag{34}$$

where, by definition,

$$\beta'(p, w) = \{x' \in B(p, w) \cap \mathcal{X}' : x' \succeq x \, \forall x' \in B(p, w) \cap \mathcal{X}'\}$$

and

$$\overline{\beta}'(p, w) = \{x'' \in \mathcal{X}' : x'' \succ \beta'(p, w) \text{ and if } y \in \mathcal{X}' \text{ obeys } y \succ \beta'(p, w) \text{ then } y \succeq x'' \ \}.$$

Significantly, since $\succeq''$ is an extension of $\succeq'$, we have $\overline{\beta}(p^{n'}, w^{n'}) \succeq'' \overline{\beta}'(p^{n'}, w^{n'})$ and also $\beta'(p^{n'}, w^{n'}) \succeq'' \beta(p^{n'}, w^{n'})$. It follows from (34) that

$$\overline{\beta}(p^{n'}, w^{n'}) \succ'' \hat{x}^{n'} \succeq'' \beta(p^{n'}, w^{n'}).$$

Since $\succeq''$ is an extension of $\succeq'$, (33) implies that

$$\overline{\beta}(p^n, w^n) \succ'' \hat{x}^n \succeq'' \beta(p^n, w^n).$$

Clearly, we can repeat the procedure for the other elements of $M \backslash \tilde{M}$; at each stage, we augment the set of 'observations' by an additional element and apply Lemma 4. QED

Proof of Corollary 1: By Lemma 5, there is a preorder $\succeq'$ on $\mathcal{X}' = \mathcal{X} \cup \{\hat{x}^m\}_{m \in M}$ satisfying properties (a), (b), and (c) in the lemma. By Theorem 1, there exists U concave, continuous, and strongly monotone that rationalizes U and the preorder induced by U on $\mathcal{X}'$ coincides with $\succeq'$. Therefore, (17) follows from (32) since $\hat{x}^m$ maximizes U in $B(p^m, w^m)$ and so $I_U(p^m, w^m) = U(\hat{x}^m)$. If $p^m \cdot \underline{x} = w^m$ for all $\underline{x} \in \beta(p^m, w^m)$, we can choose $\hat{x}^m \in \beta(p^m, w^m)$ and so $I_U(p^m, w^m) = U(\beta(p^m, w^m))$. QED

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