Revealed Preference Foundations of Expectations-Based Reference-Dependence

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Abstract

This paper provides revealed preference foundations for a model of expectations-based reference-dependence à la Kőszegi and Rabin (2006). Novel axioms provide distinguishing features of expectations-based reference-dependence under risk. The analysis completely characterizes the model's testable implications when expectations are unobservable.

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1 Introduction

Seminal work by Kahneman and Tversky introduced psychologically and experimentally motivated models of *reference-dependence* to economics. A limitation preventing the adoption of reference-dependent models is that reference points are not a directly observable economic variable. Kahneman and Tversky (1979) acknowledge that while it may be natural to assume that a decision-maker's status quo determines her reference point in their experiments, it is not appropriate in many interesting economic environments. The lack of a generally applicable model of reference point formation in economic environments has hindered applications of reference-dependence to economic settings.

Kőszegi and Rabin (2006) propose a model in which a decision-maker's recently-held expectations determine her reference point. Their solution concept for endogenously determined reference points has made their model convenient in numerous economic applications, including risk-taking and insurance decisions, consumption planning and informational preferences, firm pricing, short-run labour supply, labour market search, contracting under both moral hazard and adverse selection, and domestic violence.[1] In many of these applications, observed behaviour that appears impossible to explain using standard models naturally fits the intuition of expectations-based reference-dependence.

Little is known about the testable implications of expectations-based reference-dependence in more general settings in spite of the large number of applications. It has been suggested that models of expectations-based reference-dependence may have no meaningful revealed preference implications, and that their success comes from adding in an unobservable variable, the reference point, used at the modeller's discretion (Gul and Pesendorfer 2006). The results here confront this claim: models of expectations-based reference-dependence do have economically meaningful and testable implications for standard economic data. The revealed preference axioms of this paper completely summarize these implications.

[1] Kőszegi and Rabin (2007); Sydnor (2010); Kőszegi and Rabin (2009); Heidhues and Kőszegi (2008, 2012); Karle and Peitz (2012); Crawford and Meng (2011); Abeler et al. (2011); Pope and Schweitzer (2011); Eliaz and Spiegler (2012); Herweg et al. (2010); Carbajal and Ely (2012); Card and Dahl (2011)

The main contribution of this paper is to provide a set of revealed preference axioms that constitute necessary and sufficient conditions for a model of expectations-based reference-dependence. Commonly-used cases of Kőszegi and Rabin's model are special cases of the model studied here. The revealed preference axioms clarify how the model can be tested against both the standard rational model and against alternative behavioural theories.

As in existing models of reference-dependence, behaviour is consistent with maximizing preferences conditional on the decision-maker's reference point. The main challenge of the analysis is that expectations are not observed in standard economic data. Under expectations-based reference-dependence, the interaction between optimality given a reference point and the determination of the reference point as rational expectations can generate behaviour that appears unusual since expectations are not observed. The testable content of this unusual behaviour is revealed through axioms that rule out unusual behaviour that is not conceptually consistent with a behavioural influence of expectations.

1.1 Background: Expectations-Based Reference-Dependence

The logic of reference-dependence suggests that rather than using a single utility function, a reference-dependent decision-maker has a set of reference-dependent utility functions. The utility function $v(\cdot|r)$ defines the decision-maker's utility function given reference lottery r. When the reference lottery r is observable, as in the case where a decision-maker's status quo is her referent, standard techniques can be applied to study $v(\cdot|r)$. But when the reference lottery is determined endogenously and is unobserved, as in the case where the reference lottery is determined by the decision-maker's recent expectations, an additional modelling assumption is needed. To that end, Kőszegi and Rabin (2006) introduce two solution concepts - personal equilibrium and preferred personal equilibrium - that capture the endogenous determination of the reference lottery for models with expectations as the reference lottery.

In an environment in which a decision-maker faces a fully-anticipated choice set D, rational expectations require that the decision-maker's reference lottery corresponds with her actual choice from D. In such an environment, the set of *personal equilibria* of D provides a natural set of predictions of a decision-maker's choice from a set D:

$$PE(D) = \{p \in D : \ v(p|p) \geq v(q|p) \ \forall q \in D\} \tag{1}$$

The personal equilibrium concept has the following interpretation. When choosing from choice set D, a decision-maker uses her reference-dependent preferences $v(\cdot|r)$ given her reference lottery (r) and chooses $\arg\max_{p \in D} v(p|r)$. When forming expectations, the decision-maker recognizes that her expected choice p will determine the reference lottery that applies when she chooses from D. Thus, she would only expect a $p \in D$ if it would be chosen by the reference-dependent utility function $v(\cdot|p)$, that is, if $p \in \arg\max_{q \in D} v(q|p)$. The set of personal equilibria of D in (1) is the set of all such p.

There may be a multiplicity of personal equilibria for a given choice set. Indeed, if reference-dependence tends to bias a decision-maker towards her reference lottery, multiplicity is natural. At the time of forming her expectations, a decision-maker evaluates the lottery p according to $v(p|p)$, which reflects that she will evaluate outcomes of lottery as gains and losses relative to outcomes of p itself. The *preferred personal equilibrium* concept is a natural refinement of the set of personal equilibria based on a decision-maker picking her best personal equilibrium expectation according to $v(p|p)$:

$$PPE(D) = \arg\max_{p \in PE(D)} v(p|p) \tag{2}$$

Kőszegi and Rabin (2006) adopt a particular functional form for v. They assume that given probabilistic expectations summarized by the lottery r, a decision-maker ranks a lottery p according to:

$$v^{KR}(p|r) = \sum_k \sum_i p_i m^k(x_i^k) + \sum_k \sum_i \sum_j p_i r_j \mu \left(m^k(x_i^k) - m^k(x_j^k) \right) \tag{3}$$

In (3), m^k is a consumption utility function in "hedonic dimension" k; different hedonic dimensions are akin to different goods in a consumption bundle, but specified based on "psychological principles". The function μ is a gain-loss utility function which captures reference-dependent outcome evaluations.

The Kőszegi-Rabin model with the preferred personal equilibrium concept has been particularly amenable to applications, since the model's predictions are pinned down by (3) and (2). However, little is known about how the Kőszegi-Rabin model behaves except in very specific applications.

This paper focuses on expectations-based reference-dependent preferences with the preferred personal equilibrium concept as in (2). Theorem 1 provides a complete revealed preference characterization of the choice correspondence c that equals the set of all preferred personal equilibria of a choice set, $c(D) = PPE(D)$. The model of decision-making equivalent to the axioms does not restrict $v(\cdot|r)$ to the form in (3) but does require that v be jointly continuous in its arguments and that $v(\cdot|r)$ satisfy expected utility.

The tight characterization of expectations-based reference-dependence (EBRD) in Theorem 1 may come as a surprise relative to previous attempts (e.g. Gul and Pesendorfer 2006; Kőszegi 2010).[2] The analysis here also provides additional surprising connections. First, the EBRD representation is related to the shortlisting representation of Manzini and Mariotti (2007), a connection clarified in Proposition 1. Second, there is a tight connection between EBRD and failures of the Mixture Independence Axiom; violations of Independence of Irrelevant Alternatives (IIA) are sufficient but not necessary for expectations-dependent behaviour in the model (Proposition 3). Third, EBRD has an interpretation in terms of commitment preferences in which commitment is valuable when it affects the decision-maker's choice behaviour (Proposition 5).

1.2 Outline

Section 2 provides two examples that motivate EBRD, and a result that illustrates the difficulty in finding the model's testable implications. Section 3 provides revealed

[2] Gul and Pesendorfer (2006) show that with the personal equilibrium concept and without using any lottery structure, the reference-dependent preferences of Kőszegi and Rabin (2006) have no testable implications beyond an equivalence with a choice correspondence generated by a binary relation. Kőszegi (2010) initially proposed the personal equilibrium concept studied here but provides only a limited set of testable implications, and suggested that a complete revealed preference may not be possible: "I do not offer a revealed-preference foundation for the enriched preferences—it is not clear to what extent the decisionmaker's utility function can be extracted from her behavior."

axioms and a representation theorem, and suggests a way of defining expectations-dependence in terms of observable behaviour. Section 4 explores special cases of the model, including Kőszegi-Rabin and a new axiomatic model of expectations-based reference lottery bias. Section 5 applies two of these special cases to study how a consumer responds to a monopolist's use of sales.

2 Two examples and a motivating result

2.1 Mugs, pens, and expectations-based reference-dependence

The classic experimental motivation for loss-aversion in riskless choice comes from the *endowment effect*. An example of an endowment effect comes from the experimental finding that randomly-selected subjects given a mug have a median willingness-to-accept for a mug that is double the median willingness-to-pay of subjects who were not given a mug (Kahneman et al., 1990). This classic experiment provides no separation between status-quo-based and expectations-based theories of reference-dependence since subjects given a mug could expect to be able to keep it at the end of the experiment.

To separate expectations-based theories of reference-dependence with status-quo based theories, Ericson and Fuster (2011) design an experiment in which all subjects are endowed with a mug, but subjects are told that there is a p chance they will receive a mug, and a $1-p$ chance they will receive their choice between a mug and a pen; the conditional choice must be made before uncertainty is resolved. Subjects in a treatment with $p = .9$ expect to receive a mug with a high probability, and consistent with expectations-based reference-dependence, 77% of these subjects' conditionally choose the mug. In contrast, only 43% of subjects subjects in a treatment with $p = .1$ conditionally choose the mug.

Table 1: Larry's reference-dependent preferences

	$v(p\|\cdot)$	$v(q\|\cdot)$	$v(r\|\cdot)$
$v(\cdot\|p)$	1000	900	1050
$v(\cdot\|q)$	-1350	0	-75
$v(\cdot\|r)$	-1575	-450	-262.50

2.2 Loss-averse Larry and the Kőszegi-Rabin model

Larry is loss averse and defines losses relative to his recently held probabilistic expectations.[3] He has Kőszegi-Rabin preferences as in (3), with linear utility and linear loss aversion:[4]

$$m(x) = x, \quad \mu(x) = \begin{cases} x & \text{if } x \geq 0 \\ 3x & \text{if } x < 0 \end{cases}$$

When faced with a set of lotteries, Larry chooses his preferred personal equilibrium lottery as in (2). I take Loss-averse Larry as a protagonist for the novel predictions of the Kőszegi-Rabin model.

Consider the three lotteries p = (\$1000, 1), q = (\$0, .5; \$2900, .5), and r = (\$0, .5; \$2000, .25; \$4100, .25). As broken down in Table 1, Larry's choice correspondence is given by $p = c(\{p, q\})$, $q = c(\{q, r\})$, $r = c(\{p, r\})$, and $q = c(\{p, q, r\})$.

Choice from binary sets reveals an intransitive *cycle*. If we tried to apply standard revealed preference techniques to construct a preference relation to rationalize the data, pairwise choices would reveal a cyclic strict revealed preference relation. Since revealed strict preferences exhibit a cycle, no choice from $\{p, q, r\}$ is consistent with preference-maximization!

However, we do observe Larry make a choice from $\{p, q, r\}$. Applying the standard revealed preference construction again, choice from $\{p, q, r\}$ implies that q is strictly revealed-preferred to p, inconsistent with the opposite inference we draw from $\{p, q\}$.

[3] I would like to specially thank Matthew Rabin for suggesting this example.

[4] Linear loss aversion is used in most applications of Kőszegi-Rabin, and the chosen parameterization is broadly within the range implied by experimental studies.

In particular, adding the lottery r to the set $\{p, q\}$ generates a violation of IIA, since r is not chosen yet affects choice from the larger set. This illustrates that Larry exhibits behaviour that is inconsistent with standard revealed preference axioms and cannot be rationalizationed by a single complete and transitive preference relation.

Given fixed expectations r, Larry's behaviour would be consistent with the standard model: he would maximize $v(\cdot|r)$. Larry exhibits novel behaviour only because his expectations, and hence preferences, are determined endogenous to a choice set. However, the rational expectations combined with preferred personal equilibrium put quite a bit of structure on Larry's novel behaviour. The axiomatic analysis that follows will clarify the nature of such structure.

2.3 The testable implications of Kőszegi-Rabin: a negative result

Loss-averse Larry demonstrates that the Kőszegi-Rabin model generates choice behaviour that cannot be rationalized by a complete and transitive preference relation. Gul and Pesendorfer (2006) suggest that compared to the standard rational model, this may be the *only* revealed preference implication of the Kőszegi-Rabin model when paired with the personal equilibrium solution criteria in (1). Gul and Pesendorfer take as a starting point a finite set X of riskless elements, a reference-dependent utility $v : X \times X \to \Re$, and offer the following result:

Proposition. (Gul and Pesendorfer 2006). *The following are equivalent: (i) c is induced by a complete binary relation, (ii) there is a v such that $c(D) = PE(D)$ for any choice set D, (iii) there is a v that satifies (3) such that $c(D) = PE(D)$ for any choice set D.*

Proof. (partial sketch)

If $c(D) = \{x \in D : \, xRy \, \forall y \in D\}$ then define v by: $v(x|x) \geq v(y|x)$ if xRy, and $v(y|x) > v(x|x)$ otherwise. Then, $\{xRy \; \forall y \in D\} \iff \{v(x|x) \geq v(y|x) \; \forall y \in D\}$. By reversing the process, we could construct R from v. Thus (i) holds if and only if (ii) holds.

Gul and Pesendorfer cite Kőszegi and Rabin's (2006) argument that the set of hedonic dimensions in a given problem should be specified based on "psychological principles". Since X has no assumed structure, Gul and Pesendorfer infer hedonic dimensions from c and the structure imposed by (3). Their construction shows any v has a representation in terms of the functional form in (3).

$\square$

The analysis that follows uses two assumptions that allow for a rich set of testable implications of expectations-based reference-dependence. First, c is defined on a subsets of *lotteries* over a finite set. The structure of lotteries in choice sets places additional observable restrictions on expectations in a choice set and additional information on behaviour relative to expectations. The axioms make particular use of this lottery structure to trace the observable implications of expectations-based reference-dependence.

Second, the analysis looks for the revealed preference implications of *preferred* personal equilibrium. The sharper predictions of preferred personal equilibrium somewhat sharpen the testable implications of expectations-based reference-dependence, even in the absence of risk.

This choice space does not allow the analysis to say anything insightful about the set of hedonic dimensions of the problem. In light of Gul and Pesendorfer's (2006) result, the representation here does not seek any particular structure on the v that represents reference-dependent preferences. The analysis considers the particular structure imposed by the functional form (3) as a secondary issue for future work.

3 Revealed Preference Analysis

3.1 Technical prelude

Let Δ denote the set of all *lotteries* with support on a given finite set X, with typical elements $p, q, r \in \Delta$. Let $\mathcal{D}$ denote the set of all finite subsets of Δ, a typical $D \in \mathcal{D}$ is called a *choice set*. Define distance on lotteries using the Euclidean distance metric, $d^E(p,q) := \sqrt{\sum_i (p_i - q_i)^2}$, and the distance between choice sets using the Hausdorff metric, $d^H(D,D') := \max\left(\max_{p\in D}\left[\min_{q\in D'} d^E(p,q)\right], \max_{q\in D'}\left[\min_{p\in D} d^E(p,q)\right]\right)$.

The starting point for analysis is a *choice correspondence*, $c : \mathcal{D} \to \mathcal{D}$, which is taken as the set of elements we might observe DM choose from a set D. Assume $\emptyset \neq c(D) \subseteq D$, that is, DM always chooses something from her choice set.

It will be useful to offer a few definitions in advance of the analysis. Define the mixture operation $(1-\lambda)D+\lambda D' := \{p : \exists q \in D, r \in D' \text{ such that } p = (1-\lambda)q+\lambda r\}$. Define $c^U(D)$ as the upper hemicontinuous extension of c; that is, $c^U(D) := \{p \in D : \exists \{D^\epsilon\}_{\epsilon>0} \text{ such that } p^\epsilon \in c(D^\epsilon),\ p^\epsilon \to p,\ D^\epsilon \to D\}$. For $p \in \Delta$ and $\delta > 0$, let $N_p^\delta := \{p^\delta \in \Delta :\ d^E(p, p^\delta) < \delta\}$ denote a δ-neighbourhood of p. For any binary relation R, let cl$R$ denote its closure. For any finite set D and binary relation R, define $m(D,R) := \{p \in D :\ \nexists q \in D \text{ such that } qRp\}$ as the set of undominated elements in D according to binary relation R.

With the usual notational sloppiness, sometimes p will be used to denote the singleton menu $\{p\}$.

Say that a utility function u is *locally strict* if for every $p, q \in \Delta$ and $\epsilon > 0$, there are $p^\epsilon, q^\epsilon \in \Delta$ with $d^E(p, p^\epsilon) < \epsilon$ and $d^E(q, q^\epsilon) < \epsilon$ such that either $u(p^\epsilon) > u(q^\epsilon)$ or $u(q^\epsilon) > u(p^\epsilon)$.

Assume from here on that c satisfies a *richness property*: if $\{p, q\} = c(\{p, q\})$ then for any $\epsilon > 0$, there are $p^\epsilon, q^\epsilon \in \Delta$ with $d(p^\epsilon, p) < \epsilon$ and $d(q^\epsilon, q) < \epsilon$ such that $p^\epsilon = c^U(\{p^\epsilon, q^\epsilon\})$. The richness property imposes a form of local nonsatiation.

3.2 Revealed Preference Axioms

The classic IIA Axiom and the Mixture Independence Axiom provide the point of departure from standard models.

IIA. $D' \subset D$ and $c(D) \cap D' \neq \emptyset \implies c(D') = c(D) \cap D'$.

The IIA Axiom says that if D' is a subset of D, and some element in D' is chosen from D, then the set of chosen elements in D' equals the set of chosen elements in D that are also available in D'. The IIA Axiom as stated here is a necessary and sufficient condition for choice to be rationalizable by a complete and transitive preference relation (Arrow 1959). The example of Loss-averse Larry illustrates that an expectations-based reference-dependent preferences that capture the psychology of loss aversion can generate violations of IIA.

Mixture Independence. $(1-\alpha)c(D)+\alpha c(D')=c((1-\alpha)D+\alpha D')\ \forall\alpha\in(0,1)$

The Mixture Independence Axiom as stated is an adaptation of von-Neuman and Morgenstern's axiom to a choice correspondence. The data from Ericson and Fuster (2011) suggest an intuitive and empirically supported violation of Mixture Independence that is consistent with expectation-bias, and permitted by the axioms below.

The five axioms below weaken IIA and Mixture Independence to allow for the failures of these two behavioural properties that can arise from the endogenous determination of expectations and preferences in each choice set, but restrict violations of IIA that are not consistent with the behavioural influence of expectations.

The following Expansion axiom is due to Sen (1971).

Expansion. $p\in c(D)\cap c(D')\implies p\in c(D\cup D')$

Expansion says that if a lottery p is chosen in both D and D' then it is chosen in $D\cup D'$. This seems weak as both a normative and a descriptive property, and is an implication of variations on the Weak Axiom of Revealed Preference (see Sen (1971)). Expansion rules out the attraction and compromise effects, in which an agent chooses p over both q and r in pairwise choices, but chooses q from $\{p,q,r\}$ because r is similar to, but dominated by q.[5]

The Weak RARP (RARP for Richter's Axiom of Revealed Preference) is in the spirit of the classic axioms of revealed preference (like WARP, SARP, and GARP) albeit with an embedded continuity requirement. In particular, the axiom weakens (a suitably continuous version of) Richter's (1966) Axiom[6] of Revealed Preference.

Define $p\tilde{\tilde{R}}q$ if $p\in c(D)$ and $q\in c^U(\bar{D})$ for some $D,\bar{D}$ with $\{p,q\}\subseteq D\subseteq\bar{D}$. The relation $\tilde{\tilde{R}}$ is defined whenever sometimes p is chosen when q is available, and sometimes q is choosable (in the sense that $q\in c^U(\bar{D})$) when p is available. The statement $p\tilde{\tilde{R}}q$ holds when p is weakly chosen over q in a smaller set, but q is weakly chosen over p in a set that is larger in the sense of set inclusion. Define $p\tilde{\tilde{W}}q$ if there exist $p^0=p,\ p^1,...,p_{n-1},p_n=q$ such that $(p^{i-1},p^i)\in\mathrm{cl}\tilde{\tilde{R}}$ for $i=1,...,n$. That is, $\tilde{\tilde{W}}$ is the continuous and transitive extension of $\tilde{\tilde{R}}$.

[5]See Simonson (1989) for evidence on attraction and compromise effects. Ok et al. (2012) provide a model of the attraction effect that captures this phenomenon.

[6]Richter refers to his axiom as "Congruence". I use RARP to emphasize the close connection with WARP, SARP, GARP, etc. For more on the connection between these axioms, see Sen (1971).

Weak RARP. $p \in c(D),\ q \in c^U(\bar{D}),\ q \in D \subseteq \bar{D},$ and $q\bar{\tilde{W}}p \implies q \in c(D)$

The crucial implication of Weak RARP is captured by its main economic implication, *Weak WARP*: if $p = c(\{p,q\})$ and $p \in c(D)$ then $q \notin c(D')$ whenever $p \in D' \subset D$.[7] Manzini and Mariotti (2007) offer an interpretation in terms of constraining *reasons*: an agent might choose p over q in a smaller set, like $\{p,q\}$, yet might have a constraining reason against choosing p in a larger set D. However, if we observe p chosen from a large set D, then any D' that is a subset of D contains no constraining reason against choosing p. Thus, her choice in D' should be minimally consistent with her choice in $\{p,q\}$ and she should not choose q.

Weak RARP strengthens the logic of Weak WARP in two ways. First, Weak WARP allows only WARP violations consistent with the existence of constraining reasons, and takes choices from smaller sets - which can fewer constraining reasons - as the determinant of choice in the absence of constraining reasons. The main way Weak RARP strengthens Weak WARP is by imposing that choice among unconstrained options is determined by a transitive procedure.

Weak RARP as stated also strengthens a transitive version of Weak WARP by imposing continuity in two ways. Taking the topological closure of $\bar{\tilde{R}}$ and then taking the transitive closure imposes that choice among unconstrained options is determined by a rationale that is both transitive and continuous. This imposes a restriction that is economically natural relative to the topological structure of lotteries. The second continuity aspect of Weak RARP is that if $p \in c^U(D)$, p is seen as chooseable from D. That is, if it is revealed that there is no reason to reject p^ϵ from D^ϵ when p^ϵ and D^ϵ are 'arbitrarily close' to p and D respectively, then Weak RARP assumes that there is no reason revealed to reject p from D (even if p is not chosen at D). These two strengthenings in Weak RARP are natural given the topological structure of the space of lotteries.

IIA Independence weakens the Mixture Independence Axiom to a variation that only implies a restriction on behaviour in the presence of IIA violations, with an

[7] The following proof that Weak RARP implies Weak WARP may help clarify the connection. Suppose $p \in c(D)$, $p \in D' \subset D$, and $q \in c(D')$. Then $q\bar{\tilde{W}}p$, and so if $p \in c(\{p,q\})$, Weak RARP implies that $q \in c(\{p,q\})$ as well. Thus Weak RARP implies that if $p = c(\{p,q\})$ and $p \in c(D)$ hold, $q \in c(D')$ could not hold.

embedded continuity requirement.

IIA Independence. $p \in c(D)$, and $\exists \alpha \in (0,1]$ such that $\{p,\ (1-\alpha)p+\alpha q\} \cap c(D \cup (1-\alpha)p+\alpha q)) = \emptyset$, implies that $\exists \bar{\delta} > 0$ such that $\forall \alpha' \in (0,1]$, $\forall p^\delta \in N_p^{\bar{\delta}}$, $\forall q^\delta \in N_q^{\bar{\delta}}$ and $\forall D' \ni (1-\alpha')p^\delta + \alpha' q^\delta$, $p^\delta \notin c(D')$.

The choice pattern $p \in c(D)$ and $\{p,q\} \cap c(D \cup q) = \emptyset$ reveals an IIA violation. The IIA violation has an interpretation that adding q to the choice set prevents p from being chosen. The IIA Independence axiom requires that in this case, any mixture between q and p prevents p from being chosen from any choice set. A simple test of IIA Independence that could detect behaviour inconsistent with expectations-dependence would be to find p, q, α, D with $p \in c(D)$, $\{p,q\} \cap c(D \cup q) = \emptyset$ but $p \in c(D \cup ((1-\alpha)p+\alpha q))$; the first choice pattern reveals that when DM's expectations are p she would pick q over p. The logic of expectations-dependence then requires that the agent would not choose p when it involves a conditional choice of p over q.

The continuity requirement embedded in IIA Independence slightly strengthens restriction on c when adding q to the choice set prevents p from being conditionally chosen. The IIA Independence axiom requires that in this case, lotteries close to p prevent lotteries close to q from being conditionally chosen as well.

Say that q is a *strict conditional choice over r given p, qR_pr*, if $\exists \bar{\delta}, \bar{\epsilon} > 0$ such that $(1-\epsilon)p^\delta + \epsilon q^\delta = c((1-\epsilon)p^\delta + \epsilon\{q^\delta, r^\delta\})$ whenever $\epsilon \in (0, \bar{\epsilon})$ and $\max\{d^E(p^\delta, p), d^E(q^\delta, q), d^E(r^\delta, r)\} < \bar{\delta}$. A strict conditional involves a choice between q and r for a range of expectations that are close to p.

Transitive Limit. qR_pr and $rR_ps \implies qR_ps$.

If IIA violations are only driven by the behavioural influence of expectations and their endogenous determination, then the agent's behaviour should be consistent with the standard model when her expectations are fixed. The Transitive Limit axiom says that conditional choice behaviour should look like the standard model when expectations are almost fixed, although the axiom only imposes this restriction on strict conditional choices.

Limit Consistency. qR_pp implies $p \notin c(D)$ whenever $q \in D$.

The statement qR_pp says that q is always conditionally chosen over p when expectations are almost fixed at p. Limit Consistency requires that a decision-maker who always conditionally chooses q over p when her expectations are almost fixed at p would also never choose p when q is available. This is consistent with the logic of expectations-dependence. If instead qR_pp but p were chosen over q in some set D, then the decision-maker would choose p over q when her expectations are p even though she always conditionally chooses q over p when her expectations are almost fixed at p; such behaviour would be inconsistent with expectations-dependence and is ruled out.

3.3 Representation

Say c has a *Continuous EBRD representation* if there is a $v : \Delta \times \Delta \to \Re$ such that:

$$c(D) = \underset{p \in PE(D)}{\arg\max}\, v(p|p) \tag{4}$$

where v in (4) is jointly continuous, $v(\cdot|p)$ is locally strict, and $v(\cdot|p)$ is an expected utility function.

Theorem 1. *c satisfies Weak RARP, Expansion, IIA Independence, Transitive Limit, and Limit Consistency if and only if it has a Continuous EBRD representation.*

The full proof is in the appendix. In the analysis below, I sketch the proofs of two propositions. Proposition 1 studies the representation implied by the axioms that make no explicit reference to risk, Weak RARP and Expansion. Proposition 2 studies the additional structure on the representation that is implied by axioms that only have substance in environments with risk (IIA Independence, Transitive Limit, and Limit Consistency). Theorem 1 combines the results of Propositions 1 and 2.

Corollary 1. *Given a Continuous EBRD representation v for c, any other Continuous EBRD representation $\hat{v}$ for c satisfies $\hat{v}(q|p) \geq \hat{v}(r|p) \iff v(q|p) \geq v(r|p)$ and $\hat{v}(p|p) \geq \hat{v}(q|q)$ whenever $p\tilde{\bar{\bar{W}}}q$.*

Corollary 1 clarifies that a Continuous EBRD is unique in the sense that any v, $\hat{v}$

that represent the same c must represent the same reference-dependent preferences.[8] This definition of uniqueness captures that the underlying reference-dependent preferences are uniquely identified, but says nothing about the cardinal properties of reference-dependent utility functions. In an EBRD, v plays roles in both determining the set of of personal equilibria, and selecting from personal equilibria. The second part of Corollary 1 clarifies that this second role places a restriction that any v representing c must represent the same ranking of personal equilibria, at least when that ranking is revealed from choices.

3.4 Characterization without risk

Lotteries provide a rich structure. The IIA Independence, Transitive Limit, and Limit Consistency axioms place restrictions on c that are economically-sensible implications of expectations-dependence in environments with risk, but do not impose a restriction on behaviour in environments without risk. Moreover, these axioms would not be sensible if the mixture operation were defined, say, as taking convex combinations of consumption bundles. The Expansion and Weak RARP do not make any reference to the particular structure of risks. The following characterization sketches the observable implications of preferred personal equilibrium without making use of the rich structure provided by working with lotteries. The characterization of preferred personal equilibrium is in a similar vein.

Manzini and Mariotti (2007) characterize a *shortlisting representation*, $c(D) = m(m(D, P_1), P_2)$ for two binary relations P_1, P_2, in terms of two axioms, Expansion and Weak WARP.[9] If P_2 is transitive, say that P_1, P_2 is a *transitive shortlisting representation.*[10]

Say c has a *Very Weak EBRD representation* if there is a $v : \Delta \times \Delta \to \Re$ and a transitive binary relation P such that $c(D) = m(PE(D), P)$, where $PE(D)$ is defined

[8] A stronger uniqueness result is possible, since (i) each $v(\cdot|p)$ satisfies expected utility and thus has an affinely unique representation, (ii) joint continuity of v in the representation restricts the allowable class of transformations of v.

[9] Manzini and Mariotti (2007) and follow-up papers assume that c is a single-valued choice function, which simplifies analysis.

[10] This terminology is different from Au and Kawai (2011) and Horan (2012), who discuss shortlisting representations in which both P_1 and P_2 are transitive.

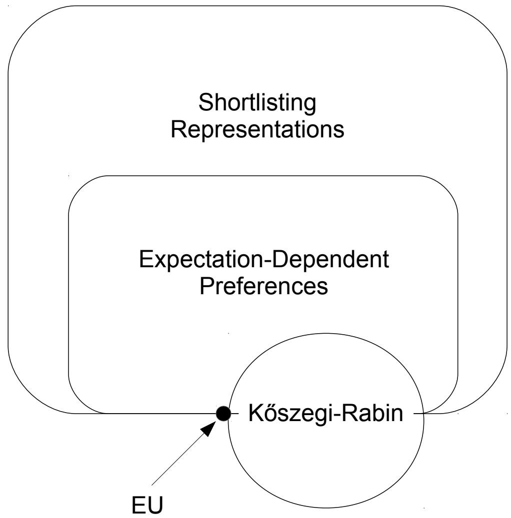

Figure 1: Relationship between representations

in (1). In a Very Weak EBRD representation, v is neither restricted to be continuous, nor is $v(\cdot|p)$ restricted to be an expected utility function, nor is P necessarily related to v. Say that c has a *Continuous Weak EBRD representation* if it has a EBRD representation as in (4) in which v is restricted to be jointly continuous but not necessarily EU nor locally strict.

Define $p\tilde{R}q$ if $p \in c(D)$ and $q \in c(\bar{D})$ for some $D, \bar{D}$ with $\{p, q\} \subseteq D \subseteq \bar{D}$. Define $p\tilde{W}q$ if there are $p^0 = p$, $p^1, ..., p_{n-1}, p_n = q$ such that $(p^{i-1}, p^i) \in \tilde{R}$ for $i = 1, ..., n$. Notice that $\tilde{R}$ and $\tilde{W}$ do not make use of the continuous structure of Δ (unlike $\tilde{\tilde{R}}$ and $\tilde{\tilde{W}}$ used in Weak RARP).

Weaker RARP. $p \in c(D)$, $q \in c(\bar{D})$, $q \in D \subseteq \bar{D}$ and $q\tilde{W}p \implies q \in c(D)$

Weaker RARP mimics Weak RARP, but makes no reference to the topological structure of Δ. That is, it does not impose continuity in any way.[11]

Retaining Weak RARP would use the topological but not economic properties of the space of lotteries. IIA Independence imposes both an economic restriction that is specific to the space of lotteries and an additional topological restriction that is not. IIA Continuity contains only the topological restriction in IIA Independence.

[11] Cherepanov et al. (2013) use the name "No Binary Chain Cycles" for Weaker RARP.

IIA Continuity. If $p \in c(D)$ and $\{p, q\} \cap c(D \cup q)) = \emptyset$, then $\exists \bar{\delta} > 0$ such that $\forall p^\delta \in N_p^\delta$, $\forall q^\delta \in N_q^\delta$ and $\forall D' \ni q^\delta$, $p^\delta \notin c(D')$.

While the core restriction in the IIA Independence axiom is specific to the economic properties of the space of lotteries, the continuity restriction that is additionally embedded in IIA Independence is not. IIA Continuity retains the continuity restriction of IIA Independence without making any reference to mixtures of lotteries.

Proposition 1. *Statements (i)-(iii) are equivalent: (i) c satisfies Weaker RARP and Expansion, (ii) c has a Very Weak EBRD representation, (iii) c has a transitive shortlisting representation. In addition, (iv) c satisfies Weak RARP, Expansion, and IIA Continuity if and only if c has a Continuous Weak EBRD representation.*

Proof. (sketch)

(i) $\Longrightarrow$ (iii):

Manzini and Mariotti define qP_1p if $\nexists D_{pq} \supset \{p, q\}$ with $p \in c(D_{pq})$. Define P_2 as the asymmetric part of $\tilde{W}$.

Suppose $p \in c(D)$. Then by the definition of P_1, $p \in m(D, P_1)$. If $q \in m(D, P_1)$ and $q\tilde{W}p$, then for each $r \in D$ there is a $D_{qr} \supset \{q, r\}$ with $q \in c(D_{qr})$. By repeatedly applying Expansion, $q \in c(\bigcup_{r \in D} D_{qr})$; also, $D \subseteq \bigcup_{r \in D} D_{qr}$. Then by Weaker RARP, $q \in c(D)$, thus $p\tilde{W}q$ by the definition of $\tilde{R}$, so it is not the case that qP_2p. By transitivity of P_2, $p \in m(m(D, P_1), P_2)$.

Now suppose $p \in m(m(D, P_1), P_2)$. By the definition of P_1, $p \in m(D, P_1)$ implies that for each $q \in D$, there is a $D_{pq} \supset \{p, q\}$ with $p \in c(D_{pq})$. By Expansion, $p \in c(\bigcup_{q \in D} D_{pq})$; also, $D \subseteq \bigcup_{q \in D} D_{pq}$. By the definition of P_2, if $q \in c(D)$ then $p\tilde{W}q$; by Weaker RARP it follows that $q \in c(D)$ as well (if there is no other $q \in c(D)$, then $p \in c(D)$ must hold by non-emptiness of c).

(ii) $\Longleftrightarrow$ (iii):

Suppose a transitive shortlisting representation P_1, P_2 represents c. Now take any $v : \Delta \times \Delta \to \Re$ that satisfies: $v(q|p) > v(p|p) \iff qP_1p$. Then from the initial representation, $c(D) = m(m(D, P_1), P_2) = m(\{p \in D : v(p|p) \geq v(q|p) \, \forall q \in D\}, P_2)$. Similarly, given v and P from a Very Weak EBRD, we can define qP_1p whenever $v(q|p) > v(p|p)$, and $P_1, P_2 = P$ defines an equivalent shortlisting representation.

Sketch of (iv):

Follow the previous step, but construct $P_2 = \tilde{\tilde{W}}$. By construction P_2 is continuous, thus has a utility representation; let u denote the corresponding utility function. Normalize $\hat{v}(q|p) = v(q|p)+u(p)-v(p|p)$; since $\hat{v}(\cdot|p)$ was derived from $v(\cdot|p)$ by adding a constant, these functions represent the same preferences, and now $\hat{v}(p|p) = u(p)$. Weak RARP and IIA Continuity imply the relevant continuity properties of v. Thus c has a Continuous Weak EBRD representation. $\square$

Remark. The actual proof of Theorem 1 requires care in defining v, since it will be pinned down uniquely given the other axioms; it may not be the same as v defined from the P_1 constructed in (i) $\Longrightarrow$ (ii) argument above.

Proposition 1 demonstrates that even without using the economic structure of environments with risk, preferred personal equilibrium does have novel testable implications: it is equivalent to a version of Weak RARP and Expansion. However, preferred personal equilibrium decision-making as captured by a Weak EBRD representation cannot be distinguished from an alternative model of transitive shortlisting, as shown in Figure 1. Additionally, v in a Weak EBRD representation characterized by Proposition 1 is highly non-unique: *any* $\hat{v}$ that satisfies $\hat{v}(q|p) > \hat{v}(p|p) \iff v(q|p) > v(p|p)$ and has $\hat{v}(p|p) = u(p)$ for *some* u that represents P_2 in the shortlisting representation also represents the same c.

Part (iv) of Proposition 1 shows that an element of continuity can be added by strengthening Weaker RARP to Weak RARP and adding in IIA Continuity. However, v in a Continuous Weak EBRD representation is highly non-unique, suggesting that this class of representations allows for behaviour that is conceptually distinct from expectations-dependence. This motivates the need to consider axioms specific to the structure of sets of lotteries to study the novel observable implications of expectations-based reference-dependence.

3.5 Elicitation of reference-dependent preferences with risk

There is a high level of non-uniqueness in the Weak EBRD representation, beyond the non-uniqueness inherent in shortlisting representations. Environments with risk contain additional structure. The IIA Independence, Transitive Limit, and Limit

Consistency axioms exploit this structure. Proposition 2 shows that given a Continuous Weak EBRD representation, IIA Independence, Transitive Limit, and Limit Consistency strengthen the representation to a Continuous EBRD representation.

Proposition 2. *Suppose c has a Continuous Weak EBRD representation. IIA Independence, Transitive Limit, and Limit Consistency hold if and only if c has a Continuous EBRD representation.*

Proof. *(very loose sketch)*

1. Start with $\hat{v}$ in a Continuous Weak EBRD representation. Argue that IIA Independence and Limit Consistency hold if and only if c has a Continuous Weak EBRD representation v that also satisifies: $v(q|p) > v(p|p) \iff v((1-\alpha)p + \alpha q|p) > v(p|p)$ $\forall \alpha \in [0, 1)$.

2. Show that $v(q|p) > v(r|p)$ if and only $qR_p r$.

3. Take R_p. If $qR_p r$ and $rR_p s$ then the Transitive Limit axiom implies $qR_p s$. By construction, R_p is continuous so has a continuous utility representation $v(\cdot|p)$.

4. Show that the construction of v from R_p is consistent with the requirements placed on v in the proof of Proposition 1 and Step 1.

5. Show that $v(\cdot|p)$ must satisfy expected utility given the definition of R_p.

□

3.6 A definition of expectations-dependence and its implications

Say that c exhibits *expectations-dependence* at D, α, p, q, r for $\alpha \in (0, 1)$ and $p, q, r \in \Delta$ if $(1-\alpha)p + \alpha r \in c((1-\alpha)p + \alpha D)$ but $(1-\alpha)q + \alpha r \notin c((1-\alpha)q + \alpha D)$. Interpret $(1-\alpha)p + \alpha r \in c((1-\alpha)p + \alpha D)$ as involving a *conditional choice* of r from D, conditional on fraction $1-\alpha$ of expectations being fixed by p. Say that *c exhibits strict expectations-dependence* at D, α, p, q, r for $D \in \mathcal{D}$, $\alpha \in (0, 1)$, and $p, q, r \in \Delta$ if there is a $\bar{\epsilon} > 0$ such that for all r^ϵ, D^ϵ pairs such that $r^\epsilon \in D^\epsilon$

Table 2: Two choice correspondences

	c	$\hat{c}$
$.9\{\text{pen}\} + .1\{\text{pen, mug}\}$	$[\text{pen}, 1]$	$[\text{pen}, 1]$
$.9\{\text{mug}\} + .1\{\text{pen, mug}\}$	$[\text{mug}, 1]$	$[\text{mug}, .9;\ \text{pen}, .1]$

and $\max\left[d^E(r^\epsilon, r), d^H(D^\epsilon, D)\right] < \epsilon$, $(1-\alpha)p + \alpha r^\epsilon \in c((1-\alpha)p + \alpha D^\epsilon)$ for all $\epsilon < \bar{\epsilon}$ but $(1-\alpha)q + \alpha r^\epsilon \notin c((1-\alpha)q + \alpha D^\epsilon)$ for all $\epsilon < \bar{\epsilon}$. This behavioural definition of expectations-dependence provides a tool for identifying and eliciting expectations-dependence, as illustrated by the example below which is close in spirit to, but different from, the experiment of Ericson and Fuster (2011).

Example (mugs and pens). Fix $\alpha = .1$, let $p = [\text{pen, } 1]$; $q = [\text{mug, } 1]$, $r = p$, and $D = \{p, q\}$.

Table 2 shows the values that two choice correspondences, c and $\hat{c}$, take on the menus $(1-\alpha)p + \alpha D = \{[\text{mug}, 1], [\text{mug}, .9\ ; \text{pen}, .1]\}$ and $(1-\alpha)q + \alpha D = \{[\text{mug}, .1; \text{pen}, .9], [\text{pen}, 1]\}$. Of these two choice correspondences, c exhibits expectations-dependence given D, α, p, q, r, while $\hat{c}$ does not.

$\diamond$

The definition of exhibiting expectations-dependence bears striking similarity to the Mixture Independence Axiom. Indeed, expectations-dependence as defined is a type of violation of Mixture Independence. The Proposition below clarifies the link between a exhibiting expectations-dependence, properties of a Continuous EBRD representation, and violations of the IIA axiom.

Proposition 3. *c with a Continuous EBRD strictly exhibits expectations-dependence if and only if $v(\cdot|p)$ is not ordinally equivalent to $v(\cdot|q)$ for some $p, q \in \Delta$. In addition, c with a Continuous EBRD that violates IIA exhibits strict expectations-dependence.*

The first part of Proposition 3 highlights how expectations-dependence in c is captured in an EBRD representation. There is a tight tie between expectations-dependence and failures of Mixture Independence in an EBRD representation, and the second part of Proposition 3 shows that a failure of IIA implies, but is not necessary for, expectations-dependence.

The mugs and pens example shows how one might study expectations-dependence based on the definition, and is distinct from Ericson and Fuster's (2011) closely related design. Proposition 3 shows that since Ericson and Fuster's data violate Mixture Independence in a way that an EBRD allows, any EBRD representation representing their median subject's behaviour must exhibit expectations-dependence.

3.7 Limited cycle property of an EBRD representation

The characterization of a Continuous EBRD representation in Theorem 1 is tight. However, it is possible that some structure already imposed on the problem implies additional structure on v. Proposition 4 shows that this is indeed the case.

Say that an EBRD representation satisfies the *limited cycle inequalities* if for any $p^0, p^1, ..., p^n \in \Delta$, $v(p^i|p^{i-1}) > v(p^{i-1}|p^{i-1})$ for $i = 1, ..., n$, then $v(p^n|p^n) \geq v(p^0|p^n)$.

Proposition 4. *Any EBRD representation satisfies the limited cycle inequalities. Moreover, if v is jointly continuous, satisfies the limited-cycle inequalities, and $v(\cdot|p)$ is EU for each $p \in \Delta$, then v defines a Continuous EBRD representation by (4).*

Proof. Take any $p^0, p^1, ..., p^n \in \Delta$, with $v(p^i|p^{i-1}) > v(p^{i-1}|p^{i-1})$. The i^{th} term in this sequence implies by the representation that $p^{i-1} \notin c(\{p^0, ..., p^n\})$; since $c(\{p^0, ..., p^n\}) \neq \emptyset$ by assumption it follows that $p^n = c(\{p^0, ..., p^n\})$. This implies, by the representation, that $v(p^n|p^n) \geq v(p^i|p^n)$ for all $i = 0, 1, ..., n-1$, which implies the desired result.

Conversely, for any v that satisfies the three given restrictions, the limited cycle inequalities imply that $PE(D)$ is non-empty for any $D \in \mathcal{D}$. Thus by Theorem 1, v defines a Continuous EBRD representation.

$\square$

Munro and Sugden (2003) mention the limited cycle inequalities (their Axiom C7), and defend the limited cycle inequalities based on a money-pump argument. In contrast, the limited cycle inequalities emerge here as a consequence of the assumption that $c(D)$ is always non-empty combined with the reference-dependent preference representation. If one considers a class of choice problems in which the agent always makes a choice, the limited cycle inequalities are a basic consequence of this and the agent's endogenous determination of her reference lottery, regardless of the normative interpretation of the inequalities.

3.8 EBRD and Commitment Preferences

Suppose that in addition to observing c, we also observe a complete and transitive ranking of choice sets, $\succeq$ defined on $\mathcal{D}$.

Non-Intrinsic Preference for Commitment. If $p \in c(D)$ and $p \in D' \subseteq D$, then $D' \succeq D$, and $D' \succ D$ if and only if $p \notin c(D')$.

The Non-Intrinsic Preference for Commitment Axiom says that an agent strictly prefers a subset D' of a choice set D if she does not choose one of her choices from D in D' in spite of it being available. Equivalently, an agent weakly prefers a smaller set to a larger one whenever the smaller set includes at least one of the elements chosen from the larger set. Non-Intrinsic Preference for Commitment implies that a choice set is indifferent to any of the elements chosen from it - that is, commitment is not intrinsically valuable.

Let $\succeq_L$ denote the preference on Δ induced by the restriction of $\succeq$ to singleton choice sets.

Proposition 5. *If $(c, \succeq)$ satisfis Non-Intrinsic Preference for Commitment, then c satisfies Weaker RARP with $\tilde{W} \subseteq \succeq_L$.*

Proof. Non-Intrinsic Preference for Commitment implies that if $p \in c(D)$, then $\{p\} \sim_L D$. Suppose $p\tilde{R}q$. Then there exist $p, q \in \Delta$ and $D, \bar{D} \in \mathcal{D}$ such that $p \in c(D)$, $q \in c(\bar{D})$, and $q \in D \subseteq \bar{D}$. Then Non-Intrinsic Preference Commitment requires $p \succeq_L q$. Thus by the construction of $\tilde{R}$, $\tilde{R} \subseteq \succeq_L$. Since $\succeq_L$ is transitive, $\tilde{W} \subseteq \succeq_L$ as well. Now take any $p, q \in \Delta$ and $D, \bar{D} \in \mathcal{D}$ such that $p \in c(D)$, $q \in c(\bar{D})$, and $q \in D \subseteq \bar{D}$. If $q \succeq_L p$, then Non-Intrinsic Preference for Commitment implies that $q \in c(D)$ as well. Thus Weaker RARP holds. $\square$

In light of Proposition 5, the Non-Intrinsic Preference for Commitment Axiom almost implies Weak RARP, and provides an interpretation of Weaker RARP in terms of commitment preferences, and in particular of $\tilde{W}$ in terms of an agent's ranking of lotteries when she first evaluates a choice set.

The Non-Intrinsic Preference for Commitment Axiom also sharply delineates the commitment preferences implied by a particular interpretation of the EBRD representation from theories of temptation and self-control. In particular, by requiring that $p \in c(D) \implies \{p\} \sim c(D)$, Non-Intrinsic Preference for Commitment is consistent with Dekel et al.'s (2009) *Desire for Commitment* axiom, which says that for any $D \in \mathcal{D}$, $\exists p \in D : \{p\} \succeq D$. However, the Non-Intrinsic Preference for Commitment axiom allows for commitment benefits inconsistent with Gul and Pesendorfer's (2001) *Set Betweenness* axiom, which says that $D \succeq D'$ implies that $D \succeq D \cup D' \succeq D'$.[12]

By Proposition 3, imposing the Mixture Independence Axiom on c with a Continuous EBRD representation would make rule out expectations-dependence and make the representation collapse to a standard Strotz (1955) representation. With only a handful of exceptations, models of commitment preferences following Dekel et al. (2009) and Gul and Pesendorfer (2001) impose a variant of the Mixture Independence Axiom on the agent's ranking of menus.

4 Special Cases

4.1 Kőszegi-Rabin Reference-Dependent Preferences

It may not be apparent at first glance whether Kőszegi-Rabin preferences in (3) satisfy the limited-cycle inequalities that an EBRD representation must satisfy to generate a non-empty choice correspondence. Indeed, Kőszegi and Rabin (2006) cite a result due to Kőszegi (2010, Theorem 1) that a personal equilibrium exists whenever D is convex, or equivalently, an agent is free to randomize among elements of any non-convex choice set. It is unclear whether or when this restriction is necessary to guarantee the existence of a non-empty choice correspondence.

Kőszegi and Rabin suggest restrictions on (3). In particular, applications of Kőszegi-Rabin have typically assumed *linear loss aversion*, which holds when there

[12]Dekel et al. (2009) decompose Set Betweenness into two parts: *Positive Set Betweenness* ($D \succeq D'$ implies $D \succeq D \cup D'$) and *Negative Set Betweenness* ($D \succeq D'$ implies $D \cup D' \succeq D'$). Non-Intrinsic Preference for Commitment implies Positive Set Betweenness but not Negative Set Betweenness.

are η and λ such that:

$$\mu(x) = \begin{cases} \eta x & \text{if } x \geq 0 \\ \eta\lambda x & \text{if } x < 0 \end{cases} \tag{5}$$

where $\lambda > 1$ captures loss aversion and $\eta \geq 0$ determines the relative weight on gain/loss utility. Proposition 6 shows that under linear loss aversion, Kőszegi-Rabin preferences with the PPE solution concept are a special case of the more general EBRD representation.[13]

Proposition 6. *Kőszegi-Rabin preferences that satisfy linear loss aversion satisfy the limited cycle inequalities.*

Proposition 6 here generalizes Kőszegi and Rabin's (2006) Proposition 1.3, and to my knowledge provides the first general proof that a personal equilibrium that does not involve randomization always exists in finite sets for this subclass of Kőszegi-Rabin preferences.

While commonly used versions of Kőszegi-Rabin preferences are special cases of the EBRD representation, there are (pathological?) cases of Kőszegi-Rabin preferences that are not.

Proposition 7. *Not all Kőszegi-Rabin preferences consistent with (3) satisfy the limited cycle inequalities.*

4.2 Expected Lottery Bias and Dynamically Consistent Non-expected Utility

Expectations-based reference-dependence is the central motivation to considering the EBRD representation. Now equipped with some understanding of the revealed preference implications of an EBRD representation v, we might take the preference relations $\succeq_L$ and $\{\succeq_p\}_{p\in\Delta}$ as a primitive, where $\succeq_p$ is the preference relation corresponding to $v(\cdot|p)$, and study axioms that seem to capture reference lottery bias. This is similar

[13]This result implies that Kőszegi-Rabin preferences that staisfy linear loss aversion, any finite choice set will have a personal equilibrium without the need to all the agent to randomize. I believe that this characterization is new to the literature.

to the standard exercise in the axiomatic literature on reference-dependent behaviour (e.g. Tversky and Kahneman (1991; 1992); Masatlioglu and Ok (2005; 2012); Neilson (2006)).[14] In that vein, consider the *Reference Lottery Bias* axiom.

Reference Lottery Bias. $p \succeq_L q \implies p \succeq_p q$

I offer three interpretations of Reference Lottery Bias. The first interprets $\succeq_L$ as representing the preferences that take into account that expecting to choose and then choosing lottery p leads to p being evaluated against itself as the reference lottery. Under this interpretation, if an agent would want to choose p over q, knowing that this choice would also determine the reference-lottery against which they would evaluate outcomes, then the agent would also choose p over q when p is the reference lottery. The second interpretation (along the lines of Masatlioglu and Ok (2005)) is that $\succeq_L$ captures reference-independent preferences; in this second interpretation, if p is preferred to q in a reference-independent comparison, then when p is the reference lottery, p is also preferred to q. According to either interpretation, Reference Lottery Bias imposes that $\succeq_p$ biases an agent towards p relative to $\succeq_L$. This seems like a natural generalization of the endowment effect under EBRD.

A third interpretation emphasizes $\succeq_L$ as the ranking of lotteries induced by the agent's ex-ante ranking of choice sets when restricted to singleton choice sets. Under this interpretation, an agent who wants to choose a lottery from a choice set according to her ex-ante ranking would also want to choose it from that choice set if she then expected that lottery, and it subsequently acted as her reference point.

What implications does the Reference Lottery Bias axiom have? Kőszegi-Rabin preferences do not satisfy Reference Lottery Bias; recall the example of Loss-averse Larry in Section 2.2 in which $v(p|p) > v(r|r)$ but $v(r|p) > v(p|p)$. This suggests a conflict between the psychology of reference-dependent loss aversion captured by the Kőszegi-Rabin model and the notion of Reference Lottery Bias defined in the axiom. No experimental evidence to my knowledge sheds light on this matter.

Proposition 8. *A Continuous EBRD representation satisfies Reference Lottery Bias if and only if* $c(D) = m(D, \succeq_L)$.

[14] $\succeq_L$ is from the section on commitment preferences; in a Continuous EBRD representation $q \succeq_L r$ if and only if $v(q|q) \geq v(r|r)$.

Proposition 8 implies (recalling Proposition 3) that under Reference Lottery Bias, reference-dependent behaviour in an EBRD representation is tightly connected to non-expected utility behaviour in $\succeq_L$.

The non-expected utility literature has provided numerous models of decision-making under risk based on complete and transitive preferences that, motivated by the Allais paradox, satisfy a relaxed version of the Mixture Independence Axiom (e.g. Quiggin (1982); Chew (1983); Dekel (1986); Gul (1991)). The model of expectations-based reference-dependence based on the Reference Lottery Bias axiom is based on a dynamically consistent implementation of non-expected utility preferences (as in Machina (1989)). I offer two examples of EBRD representations that satisfy Reference-Lottery Bias and capture expectations-based reference-dependence.

Example (Disappointment Aversion). Suppose $\succeq_L$ satisfies Gul's (1991) disappointment aversion. Then (letting $u(x)$ denote $u([x,1])$), dynamic consistency implies:

$$v^{DA}(p|r) = \frac{1}{1+\beta}\sum_i p_i \left(u(x_i) + \beta \min[u(x_i), u(r)]\right) \tag{6}$$

In cases of lotteries over multidimensional choice objects, it is not hard to see how to extend (6) via additive separability across dimensions. The resulting functional form captures loss aversion relative to past expectations (as in Kőszegi-Rabin) while retaining dynamic consistency.

5 Regular prices and sales in two EBRD-based models

Sales are a well-documented empirical regularity in the literature on firm pricing (e.g. Nakamura and Steinsson (2012)). The reasons why firms have sales is a matter of active debate. Heidhues and Kőszegi (2012) propose that a monopolist would use sales as a way of exploiting consumer loss aversion. In their application of the Kőszegi-Rabin model, firms use sales to get consumers to pay a higher average price than they would without sales. The example below revisits this explanation for sales and shows that the explanation is sensitive to how loss aversion is modelled.

Consider a consumer whose intrinsic willingness-to-pay for a pair of shoes is χ, that is, $\{\text{buy at price } P\} = c(\{\text{buy at price } P, \text{ don't buy}\})$ when $P < \chi$, but $\{\text{don't buy}\} = c(\{\text{buy at price } P, \text{ don't buy}\})$ when $P > \chi$. What would the consumer do if a shoe-selling monopolist puts shoes on sale at price P^L with probability q, but charges P^H normally?

Let $b^L \in \{0,1\}$ denote a consumer's buy/don't buy decision at price P^L, and $b^H \in \{0,1\}$ her buying decision at P^H. A buying plan $\{b^L, b^H\}$ induces a lottery $q_{\{b^L,b^H\}}$ over outcomes. The lottery $q_{\{b^L,b^H\}}$ can be broken down into the lotteries $q^{shoes}_{\{b^L,b^H\}} = [\chi b^L, q; \chi b^H, 1-q]$ over shoe outcomes and $q^{\$}_{\{b^L,b^H\}} = [-P^L b^L, q; P^H b^H, 1-q]$ over monetary outcomes.

Suppose futher that the consumer's utility is additively separable between money and shoes, as in the Kőszegi-Rabin preferences in (3) in which shoes and money and separate hedonic dimensions. Then, a consumer who expected to use buying plan $\{\hat{b}^L, \hat{b}^H\}$ evaluates an alternative buying plan $\{b^L, b^H\}$ according to:

$$v(q_{\{b^L,b^H\}}|q_{\{\hat{b}^L,\hat{b}^H\}}) = v^{shoes}(q^{shoes}_{\{b^L,b^H\}}|q^{shoes}_{\{\hat{b}^L,\hat{b}^H\}}) + v^{\$}(q^{\$}_{\{b^L,b^H\}}|q^{\$}_{\{\hat{b}^L,\hat{b}^H\}}) \tag{7}$$

What would expectations-based loss-aversion predict in this example? So far this paper has offered two existing ways of modelling expectations-based loss-aversion: Kőszegi-Rabin preferences as in (3) and disappointment averse preferences as in (6). Proposition 9 summarizes an interesting behavioural prediction of the Kőszegi-Rabin preferences in this setting: there is a range of values for regular prices, sale prices, and the frequency of sales that the monopolist could set under a which a loss-averse consumer would pay an average price of more than her intrinsic willingness-to-pay.

Proposition 9. (Heidhues and Kőszegi 2012). *Suppose c has a linear-loss averse Kőszegi-Rabin representation as in (3) and (5), and shoes and money are separate hedonic dimensions. (i) If $\eta > 0$, $\lambda > 1$, there exist q, P^H, P^L for which always buying is a preferred personal equilibrium outcome and $(1-q)P^H + qP^L > \chi$. (ii) If $qP^L + (1-q)P^H > \chi$, then never buying is the preferred personal equilibrium outcome when the decision-maker can only choose to always buy or never buy.*

Proof. (i) is from Heidhues and Kőszegi (2012, p. 10-13).

Proof of (ii):

$v(q_{\{0,0\}}|q_{\{0,0\}}) = 0$

$v(q_{\{1,1\}}|q_{\{0,0\}}) = (1+\eta)\left[\chi - (1-q)P^H - qP^L\right] - \eta(\lambda-1)\left[(1-q)P^H + qP^L\right] < 0$

$v(q_{\{1,1\}}|q_{\{1,1\}}) = \chi - (1-q)P^H - qP^L - \eta(\lambda-1)q(1-q)(P^H - P^L) < 0$

□

The logic behind Proposition 9(i) is that if P^L is sufficiently small, specifically, $P^L < \frac{1+\eta}{1+\eta\lambda}\chi$, then buying when the item is on sale is sufficiently attractive even when never expecting to buy that never buying cannot be a personal equilibrium. But expecting to buy only when the good is on sale creates an attachment to the good, which raises the consumer's willingness-to-pay, and makes the consumer experience a mixed feeling of a loss whenever she does not buy. Expecting to always buy raises the consumer's willingness to pay for shoes even further, making always buying a personal equilibrium for a range of prices including some with $(1-q)P^H + qP^L > \chi$. Always buying avoids this "painful loss" in the shoes dimension, and as a result is the preferred personal equilibrium for a range of P^H in which the consumer pays, on average, more than χ. Proposition 9(ii) clarifies that this novel result of Heidhues and Kőszegi (2012) is tightly tied to IIA violations in the Kőszegi-Rabin model.

Disappointment aversion as captured in (6) provides an alternative model of loss aversion relative to expectations. As in Kőszegi-Rabin, assume that utility is additively separable in money and shoes with disappointment and elation defined separately for each good, as in (7), and v^{shoes} and $v^{\$}$ take the disappointment averse functional form in (6). Proposition 10 shows disappointment aversion generates different predictions in this environment.

Proposition 10. *Suppose c has a representation that is additively separable in shoes and money as in (7), and each of $v^{\$}$, v^{shoes} has a disappointment averse representation as in (6) that is linear in money and shoes. If $\beta \geq 0$, then always buying is not a preferred personal equilibrium outcome whenever $qP^L + (1-q)P^H > \chi$.*

Proof. $(1+\beta)v(q_{\{1,1\}}|q_{\{1,1\}}) = (1+\beta)\chi - (1+\beta)(1-q)P^H + q\left(-P^L + \beta v^{\$}(a|a)\right) < 0$

$(1+\beta)v(q_{\{0,0\}}|q_{\{0,0\}}) = 0$

By Reference Lottery Bias, $q_{\{1,1\}} \notin c(\{q_{\{0,0\}}, q_{\{1,0\}}, q_{\{1,1\}}\})$, that is, always buying is not preferred personal equilibrium outcome. $\square$

Proposition 10 shows that a consumer with dimension-separable disappointment averse preferences as in (6) and (7) would never buy the shoes when their average price is higher than χ. Consumer disappointment aversion alone cannot explain why retailers have sales when modelled with the functional form in (6), though a retailer might still benefit from having sales for standard reasons. This contrasts with the Kőszegi-Rabin model, which provides a new explanation for why retailers have sales. The contrast between Propositions 9 and 10 clarifies that the prediction of Heidhues and Koszegi depends not just on loss/disappointment aversion, but on how it is modelled.

6 Conclusion

The Kőszegi-Rabin model of expectations-based reference-dependence has been extremely successful at explaining phenomena difficult to explain with previous models. Prior to this paper, it was unclear to what extent this success came from the model's flexibility in modelling unobserved reference points. The revealed preference analysis here makes clear that models of expectations-based reference-dependence do have testable implications which can separate the model's predictions from standard models and from alternative behavioural models.

In applied work, the Kőszegi-Rabin model of expectations-based reference-dependence has provided a useful way to study the impact of reference-dependence and loss aversion in economic environments. The analysis and examples in this paper show that it is not the only possible model of expectations-based reference-dependence, and provide a framework for developing alternative models. Examples of alternative models capture the main motivation behind expectations-based reference-dependence but have different testable implications in natural economic settings.

Appendix: Proofs

Proof of Theorem 1.

Preliminaries.

Let for $p, q \in \Delta$, let $D_{pq} \in \mathcal{D}$ denote an arbitrary choice set that contains p and q. With some notational slopiness, let p^δ denote a sequence converging to p, or a particular element of that sequence, where the meaning should be clear by the context, and where $d^E(p^\delta, p) < \delta$ for the element p^δ in the sequence.

Sufficiency.

1. Identified and unidentified blocking elements

Informally, let $B(p)$ denote a set of elements that "block" p. Define $\underline{B}(p) = \{q \in \Delta : \exists D \text{ s.t. } p \in c(D), \{p, q\} \cap c(D \cup q) = \emptyset\}$ and define $\bar{B}(p) = \{q \in \Delta : \forall D_{pq},\, p \notin c(D_{pq})\}$.

Lemma A1. $\underline{B}(p) \subseteq \bar{B}(p)\ \forall p \in \Delta$.

Proof of lemma.

Suppose $q \in \underline{B}(p)$. Then by the definition of $\underline{B}(p)$, $\exists D$ such that $p \in c(D)$ but $\{p, q\} \cap c(D \cup q) = \emptyset$. By contradiction, suppose $\exists D_{pq}$ such that $p \in c(D_{pq})$. Then by Expansion, $p \in c(D \cup D_{pq})$. Take $r \in c(D \cup q)$, and notice that by the definition of $\tilde{R}$, $p\tilde{R}r$, thus $p\bar{\tilde{W}}r$. Since $p \in c(D \cup D_{pq})$ as well, it follows by Weak RARP that $p \in c(D \cup q)$, a contradiction. Thus no such D_{pq} can exist, so it must be the case that $q \in \bar{B}(p)$. This proves that $\underline{B}(p) \subseteq \bar{B}(p)$. $\square$

2. Define v from R_p

Define W_p and $\hat{v}$. Approach the problem of constructing a utility representation for R_p by first seeking a reflexive completion of R_p.

Consider a revealed preference definition for the order W_p, defined by qW_pr if for each $\delta > 0$, there are sequences $p^\delta, q^\delta, r^\delta$ and $\bar{\epsilon}^\delta > 0$, such that $\forall \epsilon \in (0, \bar{\epsilon}^\delta)$ $(1-\epsilon)p^\delta + \epsilon q^\delta \in c((1-\epsilon)p^\delta + \epsilon\{q^\delta, r^\delta\})$.

Lemma A2. W_p is complete.

Proof of lemma.

Consider the procedure by which W_p is defined, given any p, q, r. Then by the non-emptiness of c, $\exists \bar{\epsilon} > 0$, $s \in \{q, r\}$, and a sequence $\hat{\epsilon}$ that is dense in $(0, \bar{\epsilon})$ such that $(1-\hat{\epsilon})p+\hat{\epsilon}s \in c((1-\hat{\epsilon})p+\hat{\epsilon}\{q,r\})$ for each $\hat{\epsilon}$. At an arbitrary $\epsilon \in (0, \bar{\epsilon})$, this implies that $(1-\epsilon)p+\epsilon s \in c^U((1-\epsilon)p+\epsilon\{q,r\})$ as well. Since $(1-\epsilon)p+\epsilon q \in c^U((1-\epsilon)p+\epsilon\{q,r\})$, Weak RARP and the definition of $\tilde{\bar{W}}$ imply that $(1-\epsilon)p+\epsilon s \in c((1-\epsilon)p+\epsilon\{q,r\})$ as well, thus W_p is complete. □

Lemma A3. W_p is jointly continuous.

Proof of lemma.

Consider a sequence $(p^\delta, q^\delta, r^\delta)$ with $q^\delta W_{p^\delta} r^\delta$ for each term in the sequence. By the definition of W_{p^δ}, this implies that for each fixed δ, there exists a sequence $(p^{\delta,\gamma}, q^{\delta,\gamma}, r^{\delta,\gamma})$ and a $\bar{\epsilon}^\gamma > 0$ for each γ such that $\forall \gamma > 0$ and for $\forall \epsilon \in (0, \bar{\epsilon}^\gamma)$, $(1-\epsilon)p^{\delta,\gamma}+\epsilon q^{\delta,\gamma} \in c((1-\epsilon)p^{\delta,\gamma}+\epsilon\{q^{\delta,\gamma}, r^{\delta,\gamma}\})$. Now define the sequence $(\hat{p}^\delta, \hat{q}^\delta, \hat{r}^\delta) := (p^{\frac{\delta}{2},\frac{\delta}{2}}, q^{\frac{\delta}{2},\frac{\delta}{2}}, r^{\frac{\delta}{2},\frac{\delta}{2}})$, noting that $d^E(p, \hat{p}^\delta) \leq d^E(p^{\frac{\delta}{2}}, p^{\frac{\delta}{2},\frac{\delta}{2}}) + d^E(p, p^{\frac{\delta}{2}}) \leq \delta$ by construction and the triangle inequality. By the definition of W_p and the construction of $(\hat{p}^\delta, \hat{q}^\delta, \hat{r}^\delta)$, this sequence establishes that qW_pr. □

Lemma A4. For any $p, q, r \in \Delta$ and any $\delta > 0$, $\exists (q^\delta, r^\delta) \in N_q^\delta \times N_r^\delta$ such that either $q^\delta R_p r^\delta$ or $r^\delta R_p q^\delta$.

Proof of lemma.

Suppose not (by contradiction).

Consider any $\delta > 0$ and any $(\bar{q}, \bar{r}) \in N_q^\delta \times N_r^\delta$. Suppose there exist sequences $(p^\gamma, \bar{q}^\gamma, \bar{r}^\gamma, \epsilon)$ and $(p^\zeta, \bar{q}^\zeta, \bar{r}^\zeta, \epsilon)$ such that $(1-\epsilon)p^\gamma + \epsilon\bar{q}^\gamma \in c((1-\epsilon)p^\gamma + \epsilon\{\bar{q}^\gamma, \bar{r}^\gamma\})$ and $(1-\epsilon)p^\zeta + \epsilon\bar{r}^\zeta \in c((1-\epsilon)p^\zeta + \epsilon\{\bar{q}^\zeta, \bar{r}^\zeta\})$ for each term in the respective sequences. Then, since such sequences exist for arbitrary $\delta > 0$ and $(\bar{q}, \bar{r}) \in N_q^\delta \times N_r^\delta$, it follows that for some $\hat{\delta}, \hat{\epsilon} > 0$ and all $\epsilon \in (0, \hat{\epsilon})$ and all $(\bar{q}, \bar{r}) \in N_q^{\hat{\delta}} \times N_r^{\hat{\delta}}$ that $(1-\epsilon)p+\epsilon\{\bar{q}, \bar{r}\} = c^U((1-\epsilon)p+\epsilon\{\bar{q}, \bar{r}\})$. This in turn contradicts richness. □

Lemma A5. W_p is transitive for each p.

Proof of lemma.

R_p is transitive by Transitive Limit. Now suppose qW_pr and rW_ps. Then applying Lemma A4, take sequences such that $q^\delta R_p r^\delta$ and $r^\delta R_p s^\delta$ for each term in the sequence. By Transitive Limit, $q^\delta R_p s^\delta$ for each term in the sequence. This implies that $q^\delta W_p s^\delta$ for each term in the sequence, which implies qW_ps since W_p in continuous. $\square$

The binary relation W_p so defined is the completion of R_p; by construction qR_pr if and only if it is not the case that rW_pq. By construction, W_p is continuous and continuous in p. By continuity and transitivity of W_p and Debreu's representation theorem (Ok 2012, Chapter 9), there exists a $\hat{v}(\cdot|p)$ that represents W_p.

Lemma A6. There exists a jointly continuous $\hat{v} : \Delta \times \Delta \to \Re$ that represents W_p.

Proof of lemma.

As cited earlier, standard results yield a (ordinal) $\hat{v}(\cdot|p)$ for each p. Consider a sequence $(p^\delta, q^\delta, r^\delta)$ with $\hat{v}(q^\delta|p^\delta) \geq \hat{v}(r^\delta|p^\delta)$ in each term along the sequence. Since $\hat{v}(\cdot|p)$ represents W_p, it follows that $q^\delta W_{p^\delta} r^\delta$ in each term along the sequence. Then by joint continuity of W_p, it follows that qW_pr, which implies $\hat{v}(q|p) \geq \hat{v}(r|p)$. $\square$

3. R_p satisfies the independence axiom

Lemma A7. qR_pr implies that for any $\gamma \in (0, 1)$ and any $s \in \Delta$, $((1-\gamma)s + \gamma q)R_p((1-\gamma)s + \gamma r)$.

Proof of lemma.

Suppose qR_pr. Then by the definition of R_p, there are $\bar{\delta}, \bar{\epsilon} > 0$ such that $\forall p^\delta \in N_p^{\bar{\delta}}$, $\forall q^\delta \in N_q^{\bar{\delta}}$, $\forall r^\delta \in N_r^{\bar{\delta}}$ $\forall \epsilon < \bar{\epsilon}$, $(1-\epsilon)p^\delta + \epsilon q^\delta = c((1-\epsilon)p^\delta + \epsilon\{q^\delta, r^\delta\})$. Pick an arbitrary $s \in \Delta$. Then pick $\bar{\alpha}, \hat{\delta} > 0$ such that $d^E((1-\bar{\alpha})p + \bar{\alpha}s, p) = \bar{\delta} - \hat{\delta}$. By the triangle inequality, $\forall \alpha \in (0, 1)$, $\forall p \in N_p^{\hat{\delta}}$, $\forall s \in N_s^{\hat{\delta}}$, $d^E((1-\alpha)p + \alpha s, (1-\alpha)p^\delta + \alpha s^\delta) < \hat{\delta}$. By the triangle inequality again, $\forall \alpha \in (0, \bar{\alpha})$, $\forall p \in N_p^{\hat{\delta}}$, $\forall s \in N_s^{\hat{\delta}}$, $(1-\alpha)p^\delta + \alpha s^\delta \in N_p^{\bar{\delta}}$. Then, $\forall \alpha \in (0, \bar{\alpha})$, $\forall \epsilon \in (0, \bar{\epsilon})$, $\forall p^\delta \in N_p^{\hat{\delta}}$, $q^\delta \in N_q^{\hat{\delta}}$, $r^\delta \in N_r^{\hat{\delta}}$, $\forall s \in N_s^{\hat{\delta}}$, $(1-\epsilon)\left((1-\alpha)p^\delta + \alpha s\right) + \epsilon q = c((1-\epsilon)\left((1-\alpha)p + \alpha s\right) + \epsilon\{q, r\})$. Notice that $(1-\epsilon)\left((1-\alpha)p + \alpha s^\delta\right) + \epsilon\{q^\delta, r^\delta\} = (1-\epsilon)(1-\alpha)p^\delta + (\alpha + \epsilon - \alpha\epsilon)\left(\frac{(1-\epsilon)\alpha}{\alpha+\epsilon-\alpha\epsilon}s^\delta + \frac{\epsilon}{\alpha+\epsilon-\alpha\epsilon}\{q^\delta, r^\delta\}\right)$. Observe that $\forall \gamma \in (0, 1)$, $\exists (\epsilon, \alpha) \in (0, \bar{\epsilon}) \times (0, \bar{\alpha})$ such that $\gamma = \frac{\epsilon}{\alpha+\epsilon-\alpha\epsilon}$. For any pair $(\epsilon, \alpha) \in (0, \bar{\epsilon}) \times (0, \bar{\alpha})$, define $\tilde{\epsilon} = \alpha + \epsilon - \alpha\epsilon$. By the preceding constuctions, $\forall \gamma \in (0, 1)$, $\forall \tilde{\epsilon} \in (0, \bar{\alpha} + \bar{\epsilon} - \bar{\alpha}\bar{\epsilon})$, $\forall p^\delta \in N_p^{\hat{\delta}}$, $q^\delta \in N_q^{\hat{\delta}}$, $r^\delta \in$

$N_r^{\hat{\delta}}$, $\forall s \in N_s^{\hat{\delta}}$, $(1-\tilde{\epsilon})p^\delta + \tilde{\epsilon}\left((1-\gamma)s^\delta + \gamma q^\delta\right) = c((1-\tilde{\epsilon})p^\delta + \tilde{\epsilon}\left((1-\gamma)s^\delta + \gamma\{q^\delta, r^\delta\}\right))$. Thus $\forall \gamma \in (0,1)$, $((1-\gamma)s + \gamma q)R_p((1-\gamma)s + \gamma r)$. $\square$

4. $v(\cdot|p)$ is consistent with $\underline{B}(p)$ and $\bar{B}(p)$

Lemma A8. $q \in \underline{B}(p)$ implies qR_pp. qR_pp implies $q \in \bar{B}(p)$.

Proof of lemma.

Suppose $q \in \underline{B}(p)$. By IIA Independence, this implies that there is a $\bar{\delta} > 0$ such that $\forall \alpha \in (0,1]$, $\forall p^\delta \in N_p^{\bar{\delta}}$, $\forall q \in N_q^{\bar{\delta}}$, $(1-\alpha)p^\delta + \alpha q^\delta \in \bar{B}(p^\delta)$. This implies that $(1-\alpha)p^\delta + \alpha q^\delta = c((1-\alpha)p^\delta + \alpha\{p^\delta, q^\delta\})$ $\forall \alpha \in (0,1]$, $\forall p^\delta \in N_p^{\bar{\delta}}$, $\forall q \in N_q^{\bar{\delta}}$, which implies that qR_pp.

Now suppose qR_pp. By Limit Consistency, there is no D_{pq} such that $p \in c(D_{pq})$. $\square$

5. Represent $\tilde{\bar{W}}$ and renormalize $\hat{v}$

Let u represent a continuous extension of $\tilde{\bar{W}}$, which we know exists (see Yi (1993)). Define $v(q|p) = \hat{v}(q|p) + u(p) - \hat{v}(p|p)$; thus $v(\cdot|p)$ is derived from $\hat{v}(\cdot|p)$ by adding a constant term that is continuous in p. Since u is continuous and $\hat{v}$ jointly continuous, v is jointly continuous. By the construction of v, $p\tilde{\bar{W}}q$ implies $v(p|p) \geq v(q|q)$.

6. Representation

Define $PE(D) = \{p \in D : \ v(p|p) \geq v(q|p)\,\forall q \in D\}$. If $p \in c(D)$, $q \in D$, and $\bar{B}(q) \cap D = \emptyset$, then (by repeated applying Expansion) there is a set $D_{pq} \supseteq D$ such that $q \in c(D_{pq})$. Thus $p\tilde{\bar{W}}q$. By the consistency between v and $\bar{B}$ and the definition of v from $\hat{v}$, $p \in c(D)$ thus implies $v(p|p) \geq v(q|q)$ for all $q \in PE(D)$. Thus the representation in (4) holds.

Necessity.

Expansion. Suppose c has a Continuous EBRD representation. If $p \in c(D) \cap c(D')$, then (i) $v(p|p) \geq v(q|p)\,\forall q \in D, \forall q \in D'$, and (ii) $v(p|p) \geq v(q|q)\,\forall q \in PE(D), \forall q \in PE(D')$. Thus: (i) implies $v(p|p) \geq v(q|p)\,\forall q \in D \cup D'$, which implies

$p \in PE(D \cup D')$, and (ii) implies $v(p|p) \geq v(q|q)\,\forall q \in PE(D) \cap PE(D')$. By the definition of $PE(\cdot)$, $PE(D \cup D') \subseteq PE(D) \cap PE(D')$; thus (ii) implies $v(p|p) \geq v(q|q)\,\forall q \in PE(D \cup D')$.

Weak RARP. Define $\hat{B}(p)$ from the representation by $\hat{B}(p) = \{q : v(q|p) > v(p|p)\}$. Continuity of v implies that $\hat{B}(p)$ is open. Since v is jointly continuous, the definitions of $\tilde{\bar{R}}$ and $\tilde{\bar{W}}$ imply that $(r, s) \in \mathrm{cl}\tilde{\bar{R}}$ implies $v(r|r) \geq v(s|s)$ and (since $v(\cdot|\cdot)$ represents a transitive binary relation) $r\bar{\tilde{\bar{W}}}s$ implies $v(r|r) \geq v(s|s)$. Thus $v(\cdot|\cdot)$ represents a continuous transitive extension of $\tilde{\bar{W}}$.

By definition, if $p\bar{\tilde{\bar{R}}}q$ then $p \notin \hat{B}(q)$ and $q \notin \hat{B}(p)$. Take $p, q \in \Delta$ and suppose there are sets $D_{pq} \subseteq \bar{D}_{pq}$ with $p \in c(D_{pq})$, $q \in c^U(\bar{D}_{pq})$. By the representation, $q \in c^U(\bar{D}_{pq})$ implies $\emptyset = \hat{B}(q) \cap \bar{D}_{pq} \supset \hat{B}(q) \cap D_{pq}$, thus $q \in PE(D_{pq})$ so $v(p|p) \geq v(q|q)$. Now if $q\bar{\tilde{\bar{W}}}p$, then since $p\bar{\tilde{\bar{R}}}q$ it follows that $v(p|p) = v(q|q)$, so by the representation $q \in c(D_{pq})$. Thus Weak RARP holds.

IIA Independence. Suppose $p \in c(D)$ but $\{p, (1-\alpha)p+\alpha q\} \cap c(D \cup ((1-\alpha)p+\alpha q)) = \emptyset$. Then the representation implies $v((1-\alpha)p + \alpha q|p) > v(p|p)$. Since v is jointly continuous and $v(\cdot|p)$ is EU, this further implies that there is a $\bar{\delta} > 0$ such that $\forall \alpha' \in (0, 1]$, $\forall p^\delta \in N_p^{\bar{\delta}}$, $\forall q^\delta \in N_q^{\bar{\delta}}$, $v((1-\alpha')p^\delta + \alpha' q^\delta|p^\delta) > v(p^\delta|p^\delta)$. Thus whenever $((1-\alpha')p^\delta + \alpha' q^\delta) \in D'$, $p^\delta \notin PE(D')$ thus $p^\delta \notin c(D')$. Thus IIA Independence holds.

Transitive Limit. First, I show that the antecedent of Transitive Limit has bite in the presence of, and only in the presence of, a strict preference. To be precise, suppose $(1-\epsilon)p^\delta + \epsilon q^\delta = c(\{(1-\epsilon)p^\delta + \epsilon q^\delta, (1-\epsilon)p^\delta + \epsilon r^\delta\})$ for all small ϵ, and $p^\delta, q^\delta, r^\delta$ sufficiently close to p, q, r. By the representation, this holds only if for all p^δ close to p, q^δ close to q, r^δ close to r, and ϵ close to zero, $v(q^\delta|(1-\epsilon)p^\delta + \epsilon q^\delta) \geq v(r^\delta|(1-\epsilon)p^\delta + \epsilon q^\delta)$, thus $v(q^\delta|p^\delta) \geq v(r^\delta|p^\delta)$ for all $p^\delta, q^\delta, r^\delta$. If $v(q|p) = v(r|p)$, then for every q^δ near q, $v(q^\delta|p) \geq v(q|p)$ and for every r^δ near r, $v(r|p) \geq v(r^\delta|p)$; this contradicts local strictness of $v(\cdot|p)$ in the representation. Thus when the antecedent of Transitive Limit holds, $v(q|p) > v(r|p)$ must hold.

Now take a Continuous EBRD representation and suppose $v(q|p) > v(r|p)$. Then, joint continuity implies that $v((1-\lambda)s + \lambda q^\delta|p^\delta) > v((1-\lambda)s + \lambda r^\delta|p^\delta)$ for any

$s \in \Delta$, $\lambda > 0$, and δ close to zero. It follows that $v((1-\epsilon)p^\delta + \epsilon q^\delta | (1-\epsilon)p^\delta + \epsilon r^\delta) > v((1-\epsilon)p^\delta + \epsilon r^\delta | (1-\epsilon)p^\delta + \epsilon r^\delta)$ for all δ, ϵ sufficiently small. Thus for sufficiently small δ, ϵ, $(1-\epsilon)p^\delta + \epsilon q^\delta = c(\{(1-\epsilon)p^\delta + \epsilon q^\delta,\ (1-\epsilon)p^\delta + \epsilon r^\delta\})$. Thus the antecedent of Transitive Limit holds when $v(q|p) > v(r|p)$.

Since $v(q|p) > v(r|p)$ and $v(r|p) > v(s|p)$ imply $v(q|p) > v(s|p)$, the analysis above implies that qR_pr and rR_ps implies qR_ps, so Transitive Limit must hold.

Limit Consistency. From above, qR_pp holds if and only if $v(q|p) > v(p|p)$; if that holds then $p \notin PE(D)$ whenever $q \in D$, hence $p \notin c(D)$ whenever $q \in D$.

□

Proof of Proposition 3.

Suppose that $v(\cdot|p)$ and $v(\cdot|q)$ are not ordinally equivalent. Then $\exists \bar{r}, \bar{s} \in \Delta$ such that $v(\bar{r}|p) > v(\bar{s}|p)$ but $v(\bar{r}|q) \leq v(\bar{s}|q)$. By local strictness, $\exists r, s \in \Delta$ that are close to $\bar{r}, \bar{s}$ such that $v(r|p) > v(s|p)$ but $v(r|q) < v(s|q)$. By EU of $v(\cdot|p)$ and continuity of v, this implies that $\exists \bar{\delta}, \bar{\epsilon} > 0$ such that $\forall \epsilon \in (0, \bar{\epsilon})$, $\forall r^\delta \in N_r^\delta$, $\forall s^\delta \in N_s^\delta$, $v((1-\epsilon)p + \epsilon r^\delta | (1-\epsilon)p + \epsilon s^\delta) > v((1-\epsilon)p + \epsilon s^\delta | (1-\epsilon)p + \epsilon s^\delta)$ but $v((1-\epsilon)q + \epsilon s^\delta | (1-\epsilon)q + \epsilon r^\delta) > v((1-\epsilon)q + \epsilon r^\delta | (1-\epsilon)q + \epsilon r^\delta)$. By the representation, this implies that for such $\epsilon, r^\delta, s^\delta$, (a) $(1-\epsilon)p + \epsilon r^\delta = c(\{(1-\epsilon)p + \epsilon r^\delta, (1-\epsilon)p + \epsilon s^\delta\})$ and (b) $(1-\epsilon)q + \epsilon s^\delta = c(\{(1-\epsilon)q + \epsilon r^\delta, (1-\epsilon)q + \epsilon s^\delta\})$. Thus if $v(\cdot|p)$ and $v(\cdot|q)$ are not ordinally equivalent, c strictly exhibits expectations-dependence.

Now suppose that c exhibits expectations-dependence at D, α, p, q, r. That is, $\exists \bar{\epsilon} > 0$ such that $\forall r^\epsilon \in N_r^\epsilon$, $\forall D^\epsilon \ni r^\epsilon$ such that $d^H(D^\epsilon, D) < \epsilon$, $(1-\alpha)p + \alpha r^\epsilon \in c((1-\alpha)p + \alpha D^\epsilon)$ but $(1-\alpha)q + \alpha r^\epsilon \notin c((1-\alpha)q + \alpha D^\epsilon)$. Since $(1-\alpha)q + \alpha r^\epsilon \notin c((1-\alpha)q + \alpha D^\epsilon)$, it follows that for each D^ϵ, $\exists \bar{s}^\epsilon \in D^\epsilon$, $v(\bar{s}^\epsilon | (1-\alpha)p + \alpha \bar{s}^\epsilon) \geq v(r^\epsilon | (1-\alpha)p + \alpha \bar{s}^\epsilon)$. Local strictness then implies that for each such $\bar{s}^\epsilon, r^\epsilon$ pair, there is an arbitrarily close pair $\hat{s}^\epsilon, \hat{r}^\epsilon$ such that $v(\hat{s}^\epsilon | (1-\alpha)p + \alpha \bar{s}^\epsilon) > v(\hat{r}^\epsilon | (1-\alpha)p + \alpha \bar{s}^\epsilon)$. By the representation, $(1-\alpha)p + \alpha r^\epsilon \in c((1-\alpha)p + \alpha D^\epsilon)$ implies that for each r^ϵ, $\forall s^\epsilon \in D^\epsilon$, $v(r^\epsilon | (1-\alpha)p + \alpha r^\epsilon) \geq v(s^\epsilon | (1-\alpha)p + \alpha r^\epsilon)$; thus $v(\hat{r}^\epsilon | (1-\alpha)p + \alpha \hat{r}^\epsilon) \geq v(\hat{s}^\epsilon | (1-\alpha)p + \alpha \hat{r}^\epsilon)$. Thus v exhibits strict expectations-dependence.

Proof of (iv) Suppose c violates IIA. Then there are D, D' such that $D' \subset D$ and $c(D) \cap D' \neq \emptyset$ but $c(D') \neq c(D) \cap D'$. This implies that either (a) or (b) holds:

(a) $\exists p \in c(D')$ such that $p \notin c(D)$. Then by the representation, this implies that $v(p|p) = v(q|q)$ for $q \in c(D')$, so for some $r \in D$, $v(r|p) > v(p|p) \geq v(q|p)$ but $v(q|q) \geq v(r|q)$

(b) $\exists p \in c(D) \cap D'$ with $p \notin c(D')$. Since $PE(D) \cap D' \subset PE(D')$, this implies that there is a $q \in c(D')$ with $v(q|q) > v(p|p)$. Thus $q \notin c(D) \implies q \notin PE(D)$, which implies that $\exists r \in D \backslash D'$ such that $v(r|q) > v(q|q) \geq v(p|q)$ but $v(p|p) \geq v(r|p)$.

In either case (a) or (b), by the (iii) implies (i) part of the proposition, c exhibits strict expectations-dependence.

□

Proof of Proposition 6

Start with a finite set X with $|X| = n + 1$ and assume (for now) that there is a single hedonic dimension. Without loss of generality, assume $m(x_1) > m(x_2) > ... > m(x_{n+1})$

Define the matrix V according to:

$$[V]_{ij} = m(x_i) + \eta[m(x_i) - m(x_j)] + \eta[\lambda - 1] \min[0,\ m(x_i) - m(x_j)] \tag{8}$$

Observe that $v(p|r) = p^T V r$. Let $\delta, \epsilon \in \Re^{n+1}$ denote vectors with $\sum_{i=1}^{n+1} \delta_i = \sum_{i=1}^{n+1} \epsilon_i = 0$. By matrix multiplication,

$$\begin{gathered} \delta^T V \epsilon = \eta[\lambda - 1] \times \\ [(m(x_1) - m(x_2))\delta_1 \epsilon_1 + (m(x_2) - m(x_3))(\delta_1 + \delta_2)(\epsilon_1 + \epsilon_2) + \\ ... + (m(x_n) - m(x_{n+1}))(\sum_{i=1}^{n} \delta_i)(\sum_{i=1}^{n} \epsilon_i)] \end{gathered} \tag{9}$$

Take a cycle $p^{i+1} = p^i + \epsilon^i$ with $v(p^{i+1}|p^i) > v(p^i|p^i)$ for $i = 0, ..., m$. Then:
$v(p^m|p^m) - v(p^0|p^m) = (p + \sum_{l=1}^m \epsilon^l)^T V (p + \sum_{l=1}^m \epsilon^l) - p^T V (p + \sum_{l=1}^m \epsilon^l)$
$= (\sum_{l=1}^m \epsilon^l)^T V (\sum_{l=1}^m \epsilon^l) + (\sum_{l=1}^m \epsilon^l)^T V p$

Rearranging the second term,

$= (\sum_{l=1}^{m}\epsilon^l)^T V(\sum_{l=1}^{m}\epsilon^l) + (\sum_{l=1}^{m-1}\epsilon^l)^T Vp + (\epsilon^m)^T V(p + \sum_{l=1}^{m-1}\epsilon^l) - (\epsilon^m)^T V(\sum_{l=1}^{m-1}\epsilon^l)$

$= (\sum_{l=1}^{m}\epsilon^l)^T V(\sum_{l=1}^{m}\epsilon^l) + (\sum_{l=1}^{m-2}\epsilon^l)^T Vp + (\epsilon^{m-1})^T V(p + \sum_{l=1}^{m-2}\epsilon^l) - (\epsilon^{m-1})^T V(\sum_{l=1}^{m-2}\epsilon^l) + (\epsilon^m)^T V(p + \sum_{l=1}^{m-1}\epsilon^l) - (\epsilon^m)^T V(\sum_{l=1}^{m-1}\epsilon^l)$

$= ... = (\sum_{l=1}^{m}\epsilon^l)^T V(\sum_{l=1}^{m}\epsilon^l) + \sum_i(\epsilon^i)^T V(p + \sum_{l=1}^{i-1}\epsilon^l) - \sum_{i=2}^{m}\epsilon^i V(\sum_{l=1}^{i-1}\epsilon^l)$

By the definition of the cycle, $(\epsilon^i)^T V(p + \sum_{l=1}^{i-1}\epsilon^l) > 0$ for each i, thus:

$> (\sum_{l=1}^{m}\epsilon^l)^T V(\sum_{l=1}^{m}\epsilon^l) - \sum_{i=2}^{m}\epsilon^i V(\sum_{l=1}^{i-1}\epsilon^l)$

By symmetry with respect to δ and ϵ in (9), it can be shown that $\sum_{i=2}^{m}\sum_{l=1}^{i-1}(\epsilon^i)^T V\epsilon^l = \sum_{j=1}^{m-1}\sum_{l=j+1}^{m}(\epsilon^j)^T V\epsilon^l$. Returning to the previous expression, more algebra establishes:

$= \sum_{l=1}^{m}(\epsilon^l)^T V\epsilon^l + \sum_{i=2}^{m}\sum_{l=1}^{i-1}(\epsilon^i)^T V\epsilon^l$

$= \frac{1}{2}\sum_{l=1}^{m}(\epsilon^l)^T V\epsilon^l + \frac{1}{2}(\sum_{l=1}^{m}\epsilon^l)^T V(\sum_{l=1}^{m}\epsilon^l)$

> 0

This completes the proof for the case with the case of one hedonic dimension.

To extend the argument to $K > 1$, break up a lottery p into marginals $\underset{k}{p}$ in each dimenion k, and define the matrix $\underset{k}{V}$ as the utility matrix corresponding to V in dimension k. we can write $v^{KR}(p|r) = \sum_k \underset{k}{p}^T \underset{k}{V}\underset{k}{r}$. Notice that all of the previously-proven properties of V apply to $\underset{k}{V}$; following through the previous steps yields the desired result.

□

Proof of Proposition 7

Gul and Pesendorfer (2006) prove that on a finite set X there is an assignment of hedonic dimensions such that any reference-dependent utility function $\hat{v}(x|y)$ can be written as a Kőszegi-Rabin preference as in (3). Extend $\hat{v}(x|y)$ to lotteries by setting $v(p|q) = \sum_i\sum_j p_i q_j \hat{v}(x|y)$. The resulting representation over Δ is thus consistent with (3).

Kőszegi (2010, Example 3 and footnote 6) provides an example of $v : \Delta \times \Delta \to \Re$ in which the only personal equilibrium involves randomization among elements of a choice set. Mapping the v from Kőszegi's example to a Kőszegi-Rabin preference as described provides an example of a Kőszegi-Rabin preference that does not satisfy the limited-cycle inequalities.

□

Proof of Proposition 8

Take a Continuous EBRD representation corresponding to $\succeq_L, \{\succeq_p\}_{p\in\Delta}$. Take $p \in D$. Reference Lottery Bias implies that if $p \succeq_L q \ \forall q \in D$ then $p \succeq_p q \ \forall q \in D$; thus, $p \in m(D, \succeq_L) \implies p \in PE(D)$, which jointly imply $p \in PPE(D) = c(D)$. Since $\succeq_L$ is continuous and D is finite, it has a maximizer in D, thus there is a $p \in m(D, \succeq_L)$; by the previous argument, for any other $q \in c(D)$ it follows from the representation that $q \succeq_L p$ thus $q \in m(D, \succeq_L)$ as well. It follows that if $\succeq_L, \{\succeq_p\}_{p\in\Delta}$ satisfies Reference Lottery Bias, that $c(D) = m(D, \succeq_L)$.

□

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