Consistency requirements and pattern methods in cost sharing problems with technological cooperation

Eric Bahel* Christian Trudeau†

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Abstract

Using the discrete cost sharing model with technological cooperation, we investigate the implications of a number of consistency requirements. In a context where the enforcing authority cannot prevent agents (who seek to reduce their cost shares) from splitting or merging their demands, the methods used must make such manipulations unprofitable. The paper introduces a family of rules that are immune to these demand manipulations, the pattern methods. For each of these methods, the associated production pattern indicates how to use the different technologies in order to meet the agents' demands. Within this family, two rules stand out: the public Aumann-Shapley rule never rewards technological cooperation; and the private Aumann-Shapley rule generates the maximum technological rent for homogeneous problems. The paper also studies the sharing methods that are not affected by manipulations of the technology. A useful axiomatization of the public Aumann-Shapley rule ensues: it is the unique flow method that is immune to demand maneuvers and technology manipulations.

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*Department of Economics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA. Email: erbahel@vt.edu

†Department of Economics, University of Windsor, 401 Sunset Avenue, Windsor, Ontario, Canada. Email: trudeauc@uwindsor.ca

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1 Introduction

We examine discrete cost sharing problems where each coalition of agents is endowed with a specific technology. This setting, introduced by Bahel and Trudeau (2012), extends the traditional cost sharing model —Moulin (1995) and van den Nouweland *et al.* (1995)— by allowing the cost of meeting a given demand profile to vary depending on the agents who participate in the production process. In particular, some agents may demand zero while making their technology available to produce the others' demands. Minimum cost spanning trees —see for instance Bird (1976)— and network problems provide examples of situations where the cost of satisfying a fixed demand profile varies depending on the agents who cooperate. In such situations, it makes sense to allow for negative cost shares. Indeed, subsidizing an agent who improves the technology (and whose demand does not affect the cost) is acceptable, and even necessary, as soon as some basic fairness properties are required.

It is shown in Bahel and Trudeau (2012) that, provided three basic requirements are met, the same system of weights —or unit flow— can be used (for all cost functions) to compute the cost shares. The first requirement, the well-known additivity, states that the cost shares should be additive in the cost function. The second axiom, strong dummy, requires that the share of an agent whose technology (demand) does not affect the cost be nonnegative (nonpositive). The third property, monotonicity with respect to demand-increment costs, says that the cost share of an agent should not decrease as her demand becomes costlier. The representation in terms of a flow provides a counterpart to the result shown by Wang (1999) in the standard model.

The present paper studies cost sharing rules that are immune to manipulations of the demand or the technology by the agents involved. In a context where the authority assigning the shares cannot observe individual demands, agents have the option to merge or split their demands in order to reduce their cost shares. A consistent sharing method must therefore prevent such manipulations from being profitable. Following this line of idea, Sprumont (2005) provided an interesting axiomatization of the Aumann-Shapley cost sharing rule.[1] It is the only addititve and dummy method that is immune to merging or splitting maneuvers. Interestingly, under our more general framework, we find that an infinite number of flow methods are immune to manipulations of the demand. For these methods, the cost shares are computed using production patterns, which indicate how to mix the technologies available in order to produce the agents' demands. Two noticeable pattern methods are examined: the public Aumann-Shapley method, which never compensates technological cooperation, and the private Aumann-Shapley method.

Likewise, if the designer of the sharing mechanism cannot monitor the production

[1] Aumann-Shapley cost shares are computed using the Shapley value of the cooperative game where each unit demanded is viewed as a single player. Each agent then has to pay the sum of the prices assigned to the units they demand. This widely-used pricing rule was introduced in Aumann and Shapley (1974). One can consult Haimanko and Tauman (2002) for a survey of the results relating to the Aumann-Shapley method.

process, groups of agents will find profitable to understate some of their technologies in order to claim the rewards granted for technological improvements. Indeed, agents will sabotage their production if they get to pay lower cost shares as a result. It is well understood in the cost sharing literature that methods vulnerable to sabotage result in output loss and inflated production costs —see for example Moulin and Sprumont (2007). We show that the public Aumann-Shapley rule is the unique flow method that is immune to splitting and merging maneuvers involving either the demand or the technology. These results suggest that rewarding technological cooperation (while preventing manipulations) requires that the planner be able to perfectly monitor the production process. We do not consider manipulations where agents overstate their technology; it would be redundant to do so, since our characterization yields the public Aumann-Shapley rule (which is obviously immune to overstatements of the technology).

Focusing on the class of homogeneous problems (for which the cost is a multiplicatively separable function of the number of agents and the aggregate demand), we show that the public and private Aumann-Shapley rules are diametrically opposed within the family of pattern methods. The former minimizes the rent allocated to technological cooperation, whereas the latter maximizes it. Our results provide additional arguments in favor of the use of the Aumann-Shapley rule and its adaptations. Indeed, very few works offer axiomatizations of the discrete Aumann-Shapley rule. Besides Sprumont (2005), we know only of the axiomatization proposed by Santos and Calvo (2000), which is based on the concept of balanced contributions —introduced by Myerson (1980).

The paper is structured as follows. Section 2 presents the basic definitions and properties. The reader familiar with the model of Bahel and Trudeau (2012) may skip it and jump to Section 3, which formally introduces the consistency requirements needed to prevent profitable manipulations. Pattern methods and the related results are discussed in Section 4. In Section 5 we compare the public and private Aumann-Shapley methods in the light of homogeneous problems. Section 6 proposes an axiomatization of the public Aumann-Shapley rule. Finally, our concluding comments are made in the discussion of Section 7.

2 Basic definitions and properties

Throughout the paper, we use the following convention for vector inequalities:

- $\bar{x} = (\bar{x}_1, ..., \bar{x}_m) \leq \tilde{x} = (\tilde{x}_1, ..., \tilde{x}_m)$ iff $\bar{x}_i \leq \tilde{x}_i$, for every $i = 1, ..., m$;
- $\bar{x} = (\bar{x}_1, ..., \bar{x}_m) < \tilde{x} = (\tilde{x}_1, ..., \tilde{x}_m)$ iff $[\bar{x} \neq \tilde{x}$ and $\bar{x}_i \leq \tilde{x}_i,$ for every $i = 1, ..., m]$.

Moreover, for any vector $\bar{x} = (\bar{x}_1, ..., \bar{x}_m) \in \mathbb{R}^m_+$, let $N_0(\bar{x}) \equiv \{i \in \{1, ..., m\} | \ x_i = 0\}$ and $N(\bar{x}) \equiv \{1, ..., m\} \setminus N_0(\bar{x})$.

The framework we use is essentially that of Bahel and Trudeau (2012). In addition, since the present paper examines cost sharing problems with technological cooperation and variable population, we use the set of agents (beside the demand vector and the cost function) to describe any such problem. Let us denote by $\mathbb{N}$ (resp. $\mathbb{N}_{++}$) the set of nonnegative integers (resp. the set of natural numbers). Define $\mathcal{N}$ as the set containing all finite subsets of $\mathbb{N}_{++}$ that are constituted of at least two agents.

For any $N \in \mathcal{N}$, $x \in \mathbb{N}^N$ and $S \subseteq N$, let $x_S \in \mathbb{N}^S$ denote the vector whose coordinates are the demands of the agents in S. The cost function $C(S, \cdot)$ is defined over the set $\mathbb{N}^S$. For any $x_S \in \mathbb{N}^S$, $C(S, x_S) \in \mathbb{R}_+$ represents the cost of supplying the demand x_S when all the agents in S cooperate on its production. Note that some agents in S might demand zero while cooperating to produce the positive demands of the other agents in S. We make the following assumptions on these cost functions:

1. $C(S_1, x_{S_1}) \geq C\left(S_2, \left(x_{S_1}, 0_{S_2 \backslash S_1}\right)\right)$, where $S_1 \subseteq S_2 \subseteq N$, $x_{S_1} \in \mathbb{N}^{S_1}$ and $\left(x_{S_1}, 0_{S_2 \backslash S_1}\right) \in \mathbb{N}^{S_2}$ (the technology weakly improves as the number of active agents increases; that is, for a fixed demand profile to be produced, the cost cannot increase with the size of S).

2. $C(S, \bar{x}_S) \leq C(S, \tilde{x}_S)$ for any $\bar{x}_S$, $\bar{x}_S \in \mathbb{N}^S$ s.t. $\bar{x}_S \leq \bar{x}_S$ (for a fixed technology, the cost is nondecreasing in the demand profile).

3. $C(S, 0_S) = 0$ for any $S \subseteq N$ (for any technology, the cost of producing nothing is zero).

Note that we assume an agent has to make her technology available whenever her demand is positive. This is a natural assumption when looking for example at network problems, where an agent has to make her location available to be connected.

Let $\Gamma(N)$ be the set of cost functions $C(\cdot, \cdot)$ that satisfy conditions 1, 2 and 3 when the set of agents is N. A cost sharing problem is a triple (N, x, C) such that $N \in \mathcal{N}$, $x \in \mathbb{N}^N$ and $C \in \Gamma(N)$. We denote by $\mathcal{P}$ the set of such cost sharing problems.

Let us define $x(S) \equiv \sum_{i \in S} x_i$, for any $S \subseteq N$. The following example describes a sharing problem where the cost function varies depending on the agents who participate in the project.

Example 1 *Consider the network problem —described by Figure 1— where country C_0 (the source) owns a fixed pipeline that supplies the distribution centers D_1 and D_2.*[2] *At each distribution center D_k ($k \in \{1, 2\}$), there is a set of consumers I_k; and consumer i has the demand $x_i \in \mathbb{N}$. The demand profile to transport is thus (x_{I_1}, x_{I_2}). Each consumer has the option to use a transportation method other than the pipeline for her demand.*[3] *Let us assume that, for $k \in \{1, 2\}$, the cost of shipping the joint demand of the agents in I_k via this alternative means of transportation is*

[2]Pipelines are widely used for the transportation of crude oil and natural gas for example. A good illustration is the case of Russian natural gas (and oil) pipelines that go through Ukraine to supply many European countries.

[3]For instance, if pipeline facilities are not available, crude oil can be trucked or shipped.

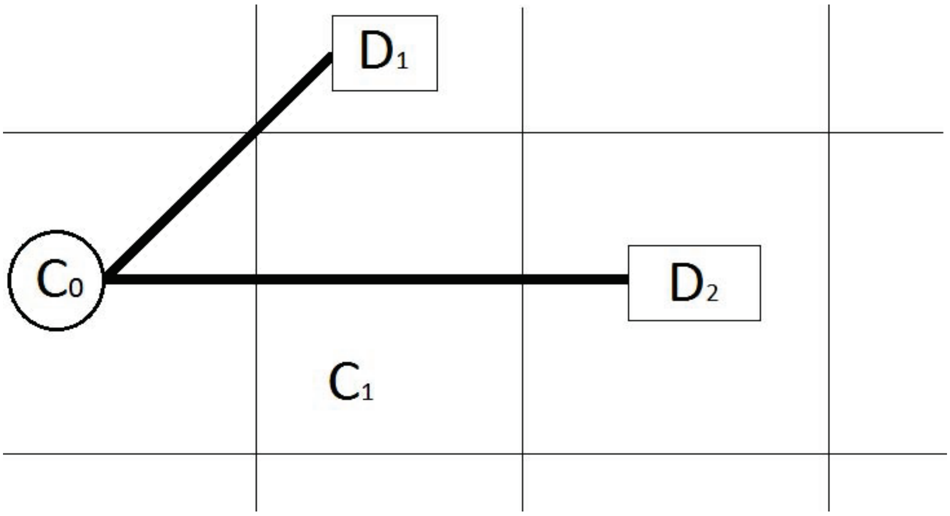

Figure 1: Map of countries and distribution centers

given by $\delta_k \sum_{i \in I_k} x_i$, *where* $\delta_k > 0$. *On the other hand, it costs* $\gamma_k \sum_{i \in I_k} x_i$ *to carry the aggregate demand of* D_i *through the pipeline (when it is available), with* $0 < \gamma_k < \delta_k$.

Note that although Country 1 hosts no distribution center, it does affect the cost function of the project. Indeed, the agents in I_2 *—but not those in* I_1*— have to use their alternative route, and thus pay the cost* $\delta_k \sum_{i \in I_k} x_i$, *if Country 1 does not agree to lend its territory. If* C_0 *(the owner of the pipeline) does not participate in the transportation project, then all agents in* $I_1 \cup I_2$ *must use alternative routes to ship their demands. The total cost of transporting the demand profile* x *depending on the agents* $S = I_1 \cup I_2 \cup T$ *(with* $T \subseteq \{C_0, C_1\}$*) who participate is:*[4]

$$C(S,x) = \begin{cases} \gamma_1 x(I_1) + \gamma_2 x(I_2), & \text{if } T = \{C_0, C_1\}; \\ \gamma_1 x(I_1) + \delta_2 x(I_2), & \text{if } T = \{C_0\}; \\ \delta_1 x(I_1) + \delta_2 x(I_2), & \text{if } C_0 \notin T. \end{cases}$$

This example will be recalled further on for illustration purposes. Let us now define the central notion of the paper.

Definition 1 *A cost sharing method (CSM)* y *is a mapping defined from* $\mathcal{P}$ *to* $\mathbb{R}^N$ *such that* $\sum_{i \in N} y_i(N,x,C) = C(N,x)$, *for any* $(N,x,C) \in \mathcal{P}$.

A CSM is thus a mechanism which, for every cost sharing problem, assigns a cost share to each of the agents, with the requirement that the shares sum up to the cost of producing the demand x when all the agents are cooperating. Next, we introduce the basic properties used in this model.

Additivity A CSM y meets Additivity if $y(N,x,C_1+C_2) = y(N,x,C_1)+y(N,x,C_2)$, for any $N \in \mathcal{N}$, $C_1, C_2 \in \Gamma(N)$ and $x \in \mathbb{N}^N$.

[4] Note that, just as S, the set I_k $(k = 1, 2)$ of agents located at D_k may vary.

Definition 2 *For the cost function $C \in \Gamma(N)$, we say that agent i is a*

- demand-dummy *if* $C(S \cup \{i\}, (t, a)) - C(S \cup \{i\}, (t, a-1)) = 0$ *for all* $S \subseteq N \setminus \{i\}$, $t \in N^S$ *and* $a = 1, 2, ...$;
- technology-dummy *if* $C(S \cup \{i\}, \{t, 0\}) - C(S, t) = 0$ *for all* $S \subseteq N \setminus \{i\}$ *and* $t \in N^S$*;*
- dummy *if she is both demand-dummy and technology-dummy.*

Strong Dummy: A CSM y meets Strong Dummy if, for any problem (N, x, C) and any $i \in N$, we have the following properties:
i) $y_i(N, x, C) \leq 0$, if agent i is demand-dummy for C;
ii) $y_i(N, x, C) \geq 0$, if agent i who is technology-dummy for C.

To allow for a unit-flow representation of cost sharing rules, let us re-express every cost function in $\Gamma(N)$ as follows. We define a vector z that encompasses both technological cooperation and demand: the first unit of z_i represents agent i's technological cooperation, the following units represent agent i's demand. One can interpret any $z \in \mathbb{N}^N \setminus \{0_N\}$ as follows: if $z_i > 0$, then agent i cooperates to the production; her demand (whether or not she cooperates) is $x_i = \max(0, z_i - 1)$.

Let $e^S \in \mathbb{N}^N$ be such that $e^S_j = 1$ if $j \in S$ and $e^S_j = 0$ otherwise. For $i \in N$, we will often write i instead of $\{i\}$. Note that the mapping Φ which, to any (S, x) s.t. $\emptyset \neq S \subseteq N$ and $x \in \mathbb{N}^S$, assigns $z = (x, 0_{N \setminus S}) + e^S$ is one-to-one. Indeed, recalling that $N(z) = \{i \in N \mid z_i > 0\}$, one can see that the (unique) inverse image of any $z \in \mathbb{N}^N \setminus \{0_N\}$ is given by (S, x) s.t. $S = N(z)$ and $x = (z - e^{N(z)})_{N(z)}$. For any $z \in \mathbb{N}^N \setminus \{0_N\}$, we define $N_1(z) \equiv \{i \in N | \ z_i = 1\}$ as the set of agents i who participate to the production process while demanding $x_i = z_i - 1 = 0$. In addition, for any demand profile $x \in \mathbb{N}^N$, we define $z^x \equiv \Phi(N, x)$. This implies that $N_1(z^x) = N_0(x)$.

The transformation Φ allows to have a cost function $C^* = C \circ \Phi^{-1}$ with a single argument z that accounts for both demand and technology, which proves to be convenient in the remainder of the paper. Observe that the domain of C^* is $\mathbb{N}^N \setminus \{0^N\}$.

Let $\Gamma^*(N)$ be the set of all cost functions that can be written as $C^* = C \circ \Phi^{-1}$, where $C \in \Gamma(N)$. Note that the properties 1, 2 and 3 that characterize the cost functions in $\Gamma(N)$ come down to:

$$C^*(a + e^i) \leq C^*(a) \text{ if } a_i = 0;$$
$$C^*(a + e^i) \geq C^*(a) \text{ if } a_i > 0;$$
$$C^*(a) = 0 \text{ if } a \in \left]0^N, e^N\right].$$

We may now state an additional axiom, which expresses the idea that an agent whose demand increments become costlier (ceteris paribus) should not pay less. For any $C^* \in \Gamma^*(N)$, $i \in N$ and $t \geq e^i$, let $\partial_i C^*(t) \equiv C^*(t) - C^*(t - e^i)$.

Monotonicity with respect to Demand-increment Costs (MDC)
A CSM y meets MDC if, $\forall x \in \mathbb{N}^N$, $\forall \hat{C}, \bar{C} \in \Gamma(N)$ such that $\hat{C}(N,x) \geq \bar{C}(N,x)$ and any agent i, we have

$$y_i(N,x,\hat{C}) \geq y_i(N,x,\bar{C}) \text{ if } \begin{cases} \partial_i \hat{C}^*(t) \geq \partial_i \bar{C}^*(t) \ \forall t \text{ s.t. } t_i \geq 2 \text{ and} \\ \partial_i \hat{C}^*(t) = \partial_i \bar{C}^*(t) \ \forall t \text{ s.t. } t_i = 1. \end{cases} \tag{1}$$

The transformation of (S,x) into z allows the use of flows methods (as in the traditional model) to compute the shares.

Definition 3
• *A (unit) flow to a demand profile $z \in \mathbb{N}^N$ is a mapping $f(z,.) : [0_N, z] \to \mathbb{R}_+^N$ that satisfies the following properties:*

$$\begin{array}{rl} (i) & f_i(z,t) = 0 \ \textit{if} \ t_i = 0 \\ (ii) & \sum_{i \in N} f_i(z,e_i) = 1 \\ (iii) & \sum_{i \in N} f_i(z,t) = \sum_{i \in \hat{N}(z,t)} f_i(z, t+e_i) \ \forall t \in]0_N, z], \end{array}$$

where $\hat{N}(z,t) = \{i \in N | \ t_i < z_i\}$.
• *A flow (system) is a list $f = \{f(z,.), \ z \in \cup_{N \in \mathcal{N}} \mathbb{N}^N\}$, where $f(z,.)$ is a flow to z.*

The following result, which is proved in Bahel and Trudeau (2012), shows that any CSM satisfying Additivity, Strong Dummy and MDC can be represented by a flow.

Theorem 1 (Bahel and Trudeau, 2012)
A CSM y satisfies Additivity, Strong Dummy and MDC if and only if there exists a (unique) flow f such that, for any $(N,x,C) \in \mathcal{P}$ and $i \in N$, we have

$$y_i(N,x,C) = \sum_{r \in [e^i, z^x]} f_i(z^x, r) \partial_i C^*(r).$$

In the remainder of the paper, we use the phrases "flow method" and "CSM satisfying Additivity, Strong Dummy and MDC" interchangeably.

3 Demand (technology) manipulations and consistency requirements

This section introduces the axioms that we examine in the present work. Let us consider the following example, which illustrates the fact that the cost sharing rule used by the enforcing authority must exhibit some consistency properties.

Example 2 *Consider the cost sharing problem* $P = (N, x, C)$*, where* $N = \{1, 2, 3\}, x = (2, 1, 0)$*, and* C *is the cost function defined by*

$$C(S, t_S) = \begin{cases} 2\sum_{i\in S} t_i^2, & \text{if } 3 \notin S \\ \sum_{i\in S\setminus\{3\}} t_i^2, & \text{if } 3 \in S. \end{cases}$$

In case agent 1 decides to split into two distinct agents (a and b) who each demand 1 and own her initial technology, we obtain a new cost sharing problem $P' = (N', x', C')$ *with 4 agents. Precisely, we have*

$$N' = \{a, b, 2, 3\}, x' = (x'_a, x'_b, x'_2, x'_3) = (1, 1, 1, 0),$$

$$C'(S, t_S) = \begin{cases} C(S, t_S) & \text{if } S \cap \{a, b\} = \emptyset \\ C\big(\{1\} \cup (S \cap \{2, 3\}), (t_a, t_{S\cap\{2,3\}})\big) & \text{if } S \cap \{a, b\} = \{a\} \\ C\big(\{1\} \cup (S \cap \{2, 3\}), (t_b, t_{S\cap\{2,3\}})\big) & \text{if } S \cap \{a, b\} = \{b\} \\ C\big(\{1\} \cup (S \cap \{2, 3\}), (t_a + t_b, t_{S\cap\{2,3\}})\big) & \text{if } S \cap \{a, b\} = \{a, b\}. \end{cases}$$

Note that the function C' *expresses the fact that the demands of agents a and b in* P' *are homogeneous, that is to say, only their sum matters in determining the cost.*

Suppose that the cost sharing rule $\hat{y}$ *adopted by the planner is such that*

$$\hat{y}_1(P) = 4, \hat{y}_2(P) = 2, \hat{y}_3(P) = -1$$

$$\hat{y}_a(P') = \hat{y}_b(P') = \hat{y}_2(P') = 5/3, \hat{y}_3(P) = 0$$

Since $\hat{y}_1(P) = 4 > \hat{y}_a(P') + \hat{y}_b(P') = 10/3$*, agent 1 will find profitable to split into two agents so as to transform* P *into* P'*.*

On the other hand, if we instead had $\hat{y}_1(P) < \hat{y}_a(P') + \hat{y}_b(P')$*, then the agents* a, b *would find profitable to merge into the single agent 1 so as to transform* P' *into* P*. In order to avoid these (merging or splitting) manipulations, a method* y *should hence satisfy* $y_1(P) = y_a(P') + y_b(P')$*.*

In what follows, we generalize the idea presented in Example 2 by adapting the axiom No Merging or Splitting, which was introduced in Sprumont (2005), to our framework. In the case where the enforcing authority can observe the technology, but not the individual demands, it has to foster a CSM that prevents manipulations of the demand. Let us introduce some terminology.

Definition 4 *Consider two cost sharing problems* $P = (N, x, C)$*,* $P' = (N', x', C') \in \mathcal{P}$*. We say that* P' *is a* ***demand-splitting manipulation of*** P ***available to agent*** $i \in N$ *if:*

1- $N' = (N \setminus i) \cup I$*, for some* $I \in \mathcal{N}$ *s.t.* $(N \setminus i) \cap I = \emptyset$*;*

2- $x'_{N\setminus i} = x_{N\setminus i}$ *and* $\sum_{i'\in I} x'_{i'} = x_i$ *(with* $x_{i'} \geq 1$*,* $\forall i' \in I$*);*

3- $C'(S, t_S) = C(S, t_S)$ *and* $C'(S \cup R, (t_S, t_R)) = C\left(S \cup i, \left(t_S, \sum_{i' \in R} t_{i'}\right)\right)$*, for all* $S \subseteq N \setminus i, \emptyset \neq R \subseteq I, t_S \in \mathbb{N}^S, t_R \in \mathbb{N}^R$.

Whenever two problems P, P' satisfy the three conditions in Definition 4, we equivalently say that P' is a *demand-merging manipulation available to the agents in* I. The agents in I have homogeneous demands in the sense that only the sum of their demands matters in determining the cost. Also note that in case one of the agents in I is active, she inherits the full technology of agent i (as opposed to a technology less efficient than that of i, which –as will be seen– may occur with an arbitrary splitting manipulation). The following axiom requires that such demand manipulations be unprofitable.

Weak No merging or splitting for demand (WNMSD)
A CSM y meets *weak no merging or splitting for demand* if, for any $P = (N, x, C), P' = (N', x', C') \in \mathcal{P}$ such that and P' is a demand-splitting manipulation of P available to agent $i \in N$ [with $N' = (N \setminus i) \cup I$], we have

$$y_i(N, x, C) = \sum_{i' \in I} y_{i'}(N', x', C').$$

With a sharing rule that satisfies WNMSD, agents do not directly benefit from splitting or merging manipulations. In the following example we show that, in a context where side payments are possible, demand-splitting manipulations could involve multiple agents.

Example 3 *Recall the problems* P, P' *of Example 2 and suppose now that the planner uses the CSM* $\bar{y}$ *such that*

$$\bar{y}_1(P) = 4, \bar{y}_2(P) = 2, \bar{y}_3(P) = -1$$

$$\bar{y}_a(P') + \bar{y}_b(P') = 4, \bar{y}_2(P') = 1, \bar{y}_3(P') = 0.$$

Although $\bar{y}$ *does not violate WNMSD, agent 1 could still benefit from splitting into the two agents* a, b *by coordinating with agent 2. Indeed, the splitting of agent 1 into* a *and* b *decreases the cost share of agent 2 from 2 to 1. Agent 1 could thus solicit a transfer of* $\tau \in (0, 1)$ *from agent 2 in order to split; and agent 2 would accept since the splitting of 1 would yield her the amount* $1 - \tau > 0$ *in net savings.*

The previous example shows that when transfers between agents are possible, a method is immune to demand manipulations if: (a) it does not change the overall share of the splitting agent, (b) it does not increase the share of some other agents. Combining the budget-balance condition with the requirements (a) and (b) suggests that all shares must be unaffected by merging or splitting manipulations for the CSM to be incentive compatible. This requirement is stated by the following.

No merging or splitting for demand (NMSD)
A CSM y meets *no merging or splitting for demand* if, for any $P = (N, x, C), P' = (N', x', C') \in \mathcal{P}$ such that P' is a demand-splitting manipulation of P available to agent $i \in N$ [with $N' = (N \setminus i) \cup I$], we have

$$y_{N\setminus i}(P) = y_{N\setminus i}(P') \text{ and } y_i(P) = \sum_{i' \in I} y_{i'}(P').$$

Obviously, no merging or splitting for demand is a stronger condition than WNMSD since it requires that all cost shares —as opposed to the mere share of the splitting agent— be unaffected by splitting or merging manipulations of the demand.

Note that a particular demand manipulation available to agent $i \in N$ is to change her name to $i' \in N \setminus i$. In this case, we have $I = \{i'\}$. Using this argument repeatedly, one can easily show that NMSD implies a symmetry property, which is stated by the following remark. Let σ_{ij} be the permutation that transposes the ith and jth coordinates. We say that agents i and j are symmetric for the cost function C if $C(S, \sigma_{ij}(x)) = C(S, x)$ for any $S \supseteq \{i, j\}$ and any $x \in \mathbb{N}^S$ (that is to say, if C is invariant under the transformation σ_{ij}).

Remark 1 *Any CSM y that satisfies no merging or splitting for demand necessarily meets the following symmetry property: for any problem (N, x, C) such that $i, j \in N$ are symmetric for C and $x_i = x_j > 0$, we have*

$$y_i(N, x, C) = y_j(N, x, C).$$

In a context where the enforcing authority cannot monitor the production process, some agents will sabotage their own technology if doing so turns out to be profitable. Note first that, because of the nonnegativity of the flow, it is never profitable for a single agent to pretend her technology is less efficient than it actually is. However, an individual who splits her demand between two distinct agents could benefit from understating the technology of one of them. This may occur if the method used does not exhibit some consistency properties as to the remuneration of technological cooperation. The following example illustrates this fact.

Example 4 *Consider two firms (providing the same good) that generate pollution due to their respective production activities. The firms (1 and 2) are required by law to clean up their emissions; and they are contemplating a cost sharing agreement. The emission levels are $(x_1, x_2) = (3, 2)$. Suppose that firm 1 owns a technology allowing to clean up emissions (generated by either firm) at a unit cost of 1. Firm 2 does not own such a technology: its only alternative is to hire an outside firm to clean up. The outside firm charges a price of 2 per unit of emissions. The resulting problem between 1 and 2 is therefore $P = (N, x, C)$ such that $N = \{1, 2\}$, $x = (x_1, x_2) = (3, 2)$; and $\forall S \subseteq N$(with $S \neq \emptyset$), $\forall t_S \in \mathbb{N}^S$,*

$$C(S, t_S) = \begin{cases} 2\sum_{i \in S} t_i = 2t_2, & \text{if } 1 \notin S \\ \sum_{i \in S} t_i, & \text{if } 1 \in S. \end{cases}$$

Firm 1 has the option to separate its production (firm a) and cleaning operations (firm b), either by splitting into two firms or by transferring its cleaning technology to a subsidiary. If firm 1 uses that option, the resulting problem is $P' = (N', x', C')$ *such that:* $N' = \{a, b, 2\}, x' = (x'_a, x'_b, x'_2) = (3, 0, 2)$*; and, for all* $S \subseteq N'$ *(with* $S \neq \emptyset$*) and* $t_S \in \mathbb{N}^S$,

$$C'(S, t_S) = \begin{cases} 2\sum_{i \in S} t_i, & \text{if } b \notin S \\ \sum_{i \in S} t_i, & \text{if } b \in S. \end{cases}$$

Suppose that the CSM y *used to compute the cost shares satisfies*

$$\begin{aligned} y_1(P) &= 2,\ y_2(P) = 3; \\ y_a(P') &= 5,\ y_b(P') = -4,\ y_2(P') = 4. \end{aligned}$$

One can then see that firm 1 *benefits from splitting into* a *and* b*, since it reduces its cost share by doing so:* $y_a(P') + y_b(P') = 1 < 2 = y_1(P)$.

If one instead starts from the problem P' *with a method* y *such that the shares satisfy* $y_a(P') + y_b(P') > y_2(P)$*, then firms* a *and* b *will find profitable to merge into firm* 1.

Let us now introduce some terminology allowing to generalize the manipulations depicted by Example 4.

Definition 5
Consider two cost sharing problems $P = (N, x, C)$, $P' = (N', x', C') \in \mathcal{P}$*. We say that* P' *is a* ***splitting manipulation of*** P ***available to agent*** $i \in N$ *if:*

1- $N' = (N \setminus i) \cup I$*, for some* $I \in \mathcal{N}$ *s.t.* $(N \setminus i) \cap I = \emptyset$;

2- $x'_{N \setminus i} = x_{N \setminus i}$ *and* $\sum\limits_{i' \in I} x_{i'} = x_i$;

3- *(a)* $C'(S, t_S) = C(S, t_S)$ *and (b)* $C'(S \cup R, (t_S, t_R)) \geq C\left(S \cup i, \left(t_S, \sum_{i' \in R} t_{i'}\right)\right)$*, for all* $S \subseteq N \setminus i, t_S \in \mathbb{N}^S, R \subseteq I, t_R \in \mathbb{N}^R$ *[where (b) is satisfied with equality whenever* $R = I$*].*

From the problem $P = (N, x, C)$ to the problem $P' = (N', x', C')$, the demand of agent i is split between the agents in I. As stated by 3-(a), the cost is the same (under C or C') when the participating agents are all in $N \setminus i$. Although the agents in I jointly own the same technology as i, they may choose to report a less efficient technology than i's, which is stated by 3-(b). To prevent such manipulations, the cost share assigned to agent i (for the problem P) should not exceed the sum of the shares paid by the agents in I (for the problem P').[5]

Equivalently, whenever the conditions $1, 2$ and 3 are satisfied for two problems P and P', we will say that P is a *merging manipulation of* P' *available to the agents*

[5] Observe from Definitions 4 and 5 that a demand-splitting manipulation is a particular type of splitting manipulation.

in I. Indeed, the agents in I will find it profitable to merge into the single agent i if the sum of their shares in the problem P' is greater than the share paid by agent i in the problem P. In this case, agent i's technology (in the problem P) is obtained by combining the respective technologies of the agents in I.

From what precedes, it is straightforward to see that, whenever the conditions 1, 2 and 3 hold, requiring agent i's cost share in P to be exactly equal to the sum of the shares of the agents in I is necessary to prevent both splitting and merging manipulations. The following axiom states this requirement.

No merging or splitting (NMS)
A CSM y meets *no merging or splitting* if, for any $P = (N, x, C), P' = (N', x', C') \in \mathcal{P}$ such that P' is a splitting manipulation of P available to agent $i \in N$ with $N' = (N \setminus i) \cup I$, we have

$$y_i(N, x, C) = \sum_{i' \in I} y_{i'}(N', x', C').$$

Note that, unlike what was done for the axiom NMSD, we do not require (under no merging or splitting) the shares of all agents other than i to be unaffected by i's splitting manipulations. Indeed, as will be seen in Section 6, adding this requirement would essentially lead to the same characterization result and would therefore be redundant.

Within our framework, there are (at least) two natural adaptations of the Aumann-Shapley method that satisfy the axiom NMSD; and one of them trivially satisfies NMS. The first one treats the most efficient technology as public; and therefore assigns cost shares by computing the Shapley price of each unit demanded when all agents participate in the production. We call this method the public Aumann-Shapley rule and use the notation AS^{pub}. Note that AS^{pub} trivially satisfies NMS, since it never rewards technological improvements.

The second adaptation considers the Shapley value of the game where each unit demanded, coupled with the technology of the agent who owns it, is viewed as a single player, as well as each agent who simply lends their technology (while demanding zero). Let us call this method the private Aumann-Shapley rule and use the notation AS^{pr}. It is useful to start with the following example illustrating these two rules, which are formally defined in the next section.

Example 5 *Recall the pipeline described by Example 1. Specifically, we have the cost sharing problem* $P = (N, x, C)$ *where* $N = \{C_0, C_1\} \cup I_1 \cup I_2$, $x = (0_{C_0}, 0_{C_1}, x_{I_1}, x_{I_2})$ *and , for all* $S = I_1 \cup I_2 \cup T$ *(with* $T \subseteq \{C_0, C_1\}$*),*

$$C(S, x) = \begin{cases} \gamma_1 x(I_1) + \gamma_2 x(I_2), & \text{if } T = \{C_0, C_1\}; \\ \gamma_1 x(I_1) + \delta_2 x(I_2), & \text{if } T = \{C_0\}; \\ \delta_1 x(I_1) + \delta_2 x(I_2), & \text{if } C_0 \notin T. \end{cases}$$

It is not difficult to see that technological manipulations are irrelevant in this context, since the countries (C_0) *cannot merge or split their territories (the pipeline). On* *the other hand, demand manipulations are very much a concern: the agents at the respective centers D_1 and D_2 will split or merge their demands (in order to reduce their overall share) in case they do not pay the same unit price.*

Recall that, under the public Aumann-Shapley rule, the pipeline and C_1's territory are made available (for free) to produce the positive demands (x_{I_1}, x_{I_2}). Therefore, the agents C_0 and C_1 each receive a technological rent of zero. That is to say, $AS^{pub}_{C_0}(P) = AS^{pub}_{C_1}(P) = 0$. It is then easy to see that —for any permutation of the $x(I_1) + x(I_2)$ units demanded— each unit transported to the center D_k will be charged a Shapley price of γ_k. Thus, each agent $i \in I_k$ pays the cost share $AS^{pub}_i(P) = \gamma_k x_i$ ($k \in \{1, 2\}$).

As one might expect, it is more complicated to compute the private Aumann-Shapley cost shares —which are obtained from the Shapley value of the game where C_0, C_1 and each of the $x(I_1) + x(I_2)$ units demanded are viewed as distinct players. Let $n_k = x(I_k)$, for $k = 1, 2$. Performing some combinatorial analysis and simplifying then gives:

$$\begin{aligned}
AS^{pr}_i(P) = &\quad x_i(\gamma_1 + \delta_1)/2, \forall i \in I_1;\\
AS^{pr}_j(P) = &\quad x_j \sum_{k=0}^{n_2} \frac{2(n_2 - k + 1)}{n_2(n_2 + 1)(n_2 + 2)} \left[k\gamma_2 + (n_2 - k)\delta_2\right], \forall j \in I_2;\\
AS^{pr}_{C_1}(P) = &\quad (\gamma_2 - \delta_2) \sum_{k=0}^{n_2} \frac{(n_2 - k + 1)(n_2 - k)}{(n_2 + 1)(n_2 + 2)} < 0;\\
AS^{pr}_{C_0}(P) = &\quad n_1 \frac{\gamma_1 - \delta_1}{2} + (\gamma_2 - \delta_2) \sum_{k=0}^{n_2} \frac{(n_2 - k + 1)(n_2 - k)}{(n_2 + 1)(n_2 + 2)} < 0.
\end{aligned}$$

In the above Example 5, it is easy to see that demand manipulations are unprofitable under both rules. Arguably, in this context where technological manipulations are irrelevant, the private Aumann-Shapley rule is the most appropriate method: it divides surpluses equitably between agents while preventing demand manipulations.

Finally, let us introduce another consistency property which relates to the axioms ordinality —discussed in Moulin (1995) and Moulin and Sprumont (2007)— and irrelevance of dummy units —Sprumont (2008). In the traditional model, these properties essentially state the idea that costless units may be eliminated from the problem without the cost shares being affected.

Irrelevant demand

A CSM y meets the axiom *irrelevant demand* if, for any $(N, x, C) \in \mathcal{P}$ and any $i \in N$ that is demand dummy for C (with $x_i > 0$), we have:

$$y(N, x, C) = y(N, (0_i, x_{N\setminus i}), C).$$

In essence, this axiom requires the demand of an agent who is demand dummy to be irrelevant to the determination of the cost shares. Indeed, since the units demanded by such an agent i are costless, the shares should not change if they are eliminated (which means that we have a new problem, where i cooperates to produce the others' demands). The axioms NMSD and irrelevant demand can be used to characterize the family of rules presented in the next section.

4 Pattern methods: definition and properties

In what follows we examine the set of methods that satisfy the axiom NMSD, that is to say, the set of cost sharing rules that are immune to demand manipulations with side payments between the agents. In addition, we require these CSM to satisfy irrelevant demand.

Let us first introduce some new concepts. Recall that a path γ to $z \in \mathbb{N}^N_{++}$ is a mapping from $\left\{0, 1, ..., \sum_{i \in N} z_i\right\}$ to $[0_N, z]$ such that: (i) $\gamma(\sum_{i \in N} z_i) = z$ and (ii) for each $k \in \left\{1, ..., \sum_{i \in N} z_i\right\}$, $\gamma(k)$ is identical to $\gamma(k-1)$ in all coordinates but one, say the ith, for which $\gamma_i(k) = \gamma_i(k-1) + 1$. The associated flow to z, $f^\gamma(z, .)$, is such that $f_i^\gamma(z, r) = 1$ if r and $r - e_i$ belong to γ (with $f_i^\gamma(z, r) = 0$ otherwise).

Definition 6 *(a) Let $N \in \mathcal{N}$ and consider $z \in \{1, 2\}^N$. We call* ***production pattern to*** z *any sequence $s : \{1, ..., \sum_{i \in N} z_i\} \to \{T, D\}$ that satisfies the following properties:*

- $|s^{-1}(D)| = |\{i \in N : z_i = 2\}|$;
- $|\{m : m < k$ *and* $s(m) = T\}| \geq 1 + |\{m : m < k$ *and* $s(m) = D\}|$, *for any* $k \in \{1, ..., \sum_{i \in N} z_i\}$ *such that* $s(k) = D$.

(b) We say that a path to $z \in \{1, 2\}^N$ follows the production pattern (to z) s if: $[\gamma(k) - \gamma(k-1) = e_i$ and $s(k) = T] \Leftrightarrow \gamma_i(k) = 1$, for any $k \in \{1, ..., \sum_{i \in N} z_i\}, i \in N$.

(c) For every production pattern s to $z = 2e_N$, let us define $f^s(z, .)$, ***the elementary flow to*** z ***associated with*** s*, as the average of all the path methods that follow the production pattern s to z.*

The first condition in Definition 6-(a) states the fact that each D in the pattern corresponds to a unit of demand. The second condition recalls that the technology of an agent must be activated before the production of her unit of demand, if any. Definition 6-(b) states the fact that two paths which follow the same production pattern to z always produce the same number of units before activating an additional technology. In the case where $z = 2e_N$, this means that the two paths to z are permutations of one another. Definition 6-(c) is illustrated by the following example.

Example 6 *As an illustration of Definition 6, when $z = (2, 2, 2)$, a particular production pattern to z is $TTDTDD$, and a path (to z) that follows this production pattern is $\gamma = (0_3, e_1, e_1 + e_2, 2e_1 + e_2, 2e_1 + e_2 + e_3, 2e_1 + 2e_2 + e_3, 2e_1 + 2e_2 + 2e_3)$. One can then see that the associated elementary flow to z is the one depicted by Figure*

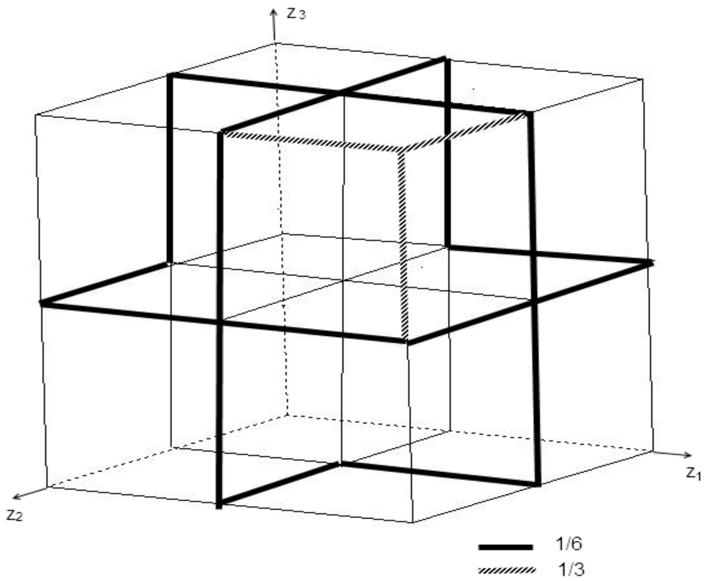

Figure 2: Elementary flow to (2,2,2) for the production pattern TTDTDD

2.[6] *The four other patterns to $z = (2, 2, 2)$ and the corresponding elementary flows are presented in Figure 3 (see Appendix).*[7]

It is useful to note the observation below.

Remark 2 *Every elementary flow is symmetric in the following sense. For all $z = 2e^N$, $t \in [0_N, z]$, and $i, j \in N$: $f_i^s(z,t) = f_j^s(z, \sigma_{ij}(t))$, where σ_{ij} is the permutation that transposes the ith and jth coordinates of any vector t.*

Together, Figures 2 and 3 illustrate the symmetry of the 5 elementary flows to $z = (2, 2, 2)$. Next, let us explain how patterns can be used to construct flow methods. Consider $N \in \mathcal{N}$; $z, z' \in \mathbb{N}^N_{++}$ (with $z \leq z'$); and $f(z', .)$ that is a flow to z'; and define the flow $f^{[z']}(z, .)$ to z by: for all $i \in N$,

$$f_i^{[z']}(z,r) = \begin{cases} f_i(z',r), & \text{if } J_i(r) = \emptyset; \\ \sum\limits_{t \in [z_{J_i(r)}, z'_{J_i(r)}]} f_i(z', (t, r_{N \setminus J_i(r)})), & \text{otherwise;} \end{cases} \tag{2}$$

where $J_i(r) = \{j \in N \setminus i : r_j = z_j\}$. In words,[8] $f^{[z']}(z, .)$ is the flow to z resulting from the projection of $f(z', .)$ on the box $[0_N, z]$.[9] Note that if $z = z'$, then $f^{[z']}(z, .) = f(z', .)$.

[6] In Figure 2 (and the following ones) we do not represent the part of the flow that is included in $[0_3, (1, 1, 1)]$. We adopt this convention to lighten the diagram —recall that these branches of the flow are irrelevant to the computation of the shares, since the cost is 0 for any $t \in [0_3, (1, 1, 1)]$.

[7] The interested reader can also consult Bahel (2011), where elementary (fixed) flows are studied.

[8] It is not difficult to see that $f^{[z']}(z, .)$ meets the properties of a flow to z.

[9] See for example Sprumont (2008), where this projection is used to define fixed flows.

Suppose now that the flow method f is such that, for any $z' = 2e_N$, the cost shares are computed according to $f^s(z,.)$, where s is some pattern to $2e_N$. Using the projection above, we extend the flow f to profiles $z \in \{1,2\}^N$ (that is, for any z such that $z_i \leq 2$ for all i) by requiring that $f(z,.) = f^{[2e_N]}(z,.)$, for any $z \in [0_N, 2e_N]$. This allows to define the flow to all demand profiles such that some agents i demand zero (that is, $z_i = 1$).

In order to fully define the cost sharing method represented by f, it remains to specify the flow to demand profiles where some agents i demand more than 1 unit (that is, some $z_i \geq 2$). For any $m \geq 1$, let

$$w(m) = \begin{cases} 1, & \text{if } m = 1; \\ (2, ..., 2) \in \mathbb{N}^{m-1}, & \text{if } m > 1. \end{cases}$$

Let $\eta(z) = |N_1(z)| + \sum_{i \in N}(z_i - 1)$ and $\Delta(z) = \{1, ..., \eta(z)\}$. We define the projection p by:

$$p(z) = (w(z_1), ..., w(z_{|N|})) \in \mathbb{N}^{\eta(z)}, \text{ for any } z \in \cup_{N \in \mathcal{N}} \mathbb{N}^N_{++}.$$

In words, when all demands are positive, $p(z)$ is the vector obtained by firstly splitting the demand $z_i - 1$ of each agent i into individual demands of 1, which are each combined with the technology of i (this explains the 2 in the expression of $w(m)$), and secondly putting the resulting vectors side by side.[10] In addition, observe that the set of indices in $\Delta(z) = \{1, ..., \eta(z)\}$ associated with agent i is given by the recursive formula

$$\tilde{I}(i,z) = \begin{cases} \{1, ..., \max(1, z_i - 1)\}, & \text{if } i = 1; \\ \{\max I(i-1) + 1, ..., \max I(i-1) + \max(1, z_i - 1)\}, & \text{if } i = 2, ..., |N|. \end{cases}$$

Next, for any given $z \in \mathbb{N}^N_{++}$, consider the multi-valued function $v = (v_i(z,.))_{i \in N}$ defined on $[0_{\eta(z)}, p(z)]$ by

$$v_i(z,t) = \begin{cases} 0, & \text{if } t_i = 0 \ \forall i \in \tilde{I}(i,z); \\ 1 + \sum\limits_{k \in \tilde{I}(i,z)} \max(t_k - 1, 0), & \text{otherwise.} \end{cases} \tag{3}$$

In essence, given $t \in [0_{\eta(z)}, p(z)]$, the vector $v(z,t) \in [0_N, z]$ generates the profile of unitary demands t following the splitting of the agents' demands.

Finally, for any $z \in \mathbb{N}^N_{++}$ (with $N \in \mathcal{N}$) and any $f(p(z),.)$ that is a flow to $p(z)$, let $\bar{f}(z,.)$ be the flow to z such that,[11] for all $r \in [0_N, z]$ and $i \in N$, we have

$$\bar{f}_i(z,r) = \sum_{t \in \Lambda(r)} \sum_{k \in \tilde{I}(i,z)} f_k(p(z), t), \tag{4}$$

[10] The mapping p is indeed a projection: it is not difficult to see that $p(p(z)) = p(z)$ is always satisfied, since we have $p(z') = z'$ as soon as $z' \in \cup_{N \in \mathcal{N}}\{1,2\}^N$.

[11] One can easily check that $\bar{f}(p(z),.)$ satisfies the properties of a flow to $p(z)$.

where $\Lambda(r) = v^{-1}(\{r\})$. That is to say: $r_i = v_i(z,t)$, for any $r \in [0_N, z]$, $t \in \Lambda(r)$ and $i \in N$.

The operator $\bar{\cdot}$ can be used to define the flow to any demand profile z, provided that the flow to $p(z)$ —the corresponding vector of unitary demands— is known. Figure 4 in the Appendix depicts an example where $\overline{f^s}(z,.)$ is derived from $f^s(p(z),.)$, given $z=(2,3)$ and $s = TTDTDD$.

We are now set to fully define the flow method generated by a pattern.

Definition 7 *We say that a flow method f is a **pattern method** if, for any $N \in \mathcal{N}$ and $z \in \mathbb{N}^N_{++}$, we have*

$$f(z,.) = \overline{(f^s)^{[p(z)+e^{N_1(z)}]}}(z,.) = \begin{cases} \overline{f^s}(z,.), & \textit{if } z \geq 2e^N; \\ \overline{(f^s)^{[2e^{\Delta(z)}]}}(z,.), & \textit{otherwise}; \end{cases}$$

for some s that is a production pattern to $p(z) + e^{N_1(z)} = 2e^{\Delta(z)}$.

From the above definition, every pattern method is uniquely characterized by a sequence of patterns of the form $\left(s(z), z \in \cup_{N \in \mathcal{N}} \left\{2e^N\right\}\right)$. Some noticeable patterns allow to define two interesting adaptations of the well-known Aumann-Shapley rule.

Definition 8 • *The public Aumann-Shapley rule, denoted by AS^{pub}, is the pattern method associated with the sequence of patterns*

$$\left(s(z) = \underbrace{T...T}_{|N| \ times} \overbrace{D...D}^{|N| \ times}, \ \textit{for all } z = 2e^N \ \textit{s.t. } N \in \mathcal{N} \right). \tag{5}$$

• *The private Aumann-Shapley rule, denoted by AS^{pr}, is the pattern method associated with the sequence of patterns*

$$\left(s(z) = \overbrace{TD...TD}^{|N| \ times}, \ \textit{for all } z = 2e^N \ \textit{s.t. } N \in \mathcal{N} \right). \tag{6}$$

The above patterns (5) and (6) formally defines the two rules introduced in Example 5. Note that AS^{pub} always activate all technological units before the production of any unit of demand (this means that technological cooperation is never remunerated). On the other hand, with unitary demands, AS^{pr} always activate a new technology in order to produce an additional unit of demand. Further on, an axiomatization is proposed for each of these distinguished methods.

More generally, pattern methods form a subclass of the family of flow methods, and their interpretation is quite intuitive: for every one of them, the associated flow is computed according to a specific production schedule which describes how to use the different technologies available in order to produce the agents' demands. It is shown in what follows that pattern methods possess some remarkable consistency properties.

For any cost sharing problem $P = (N, x, C)$, let $P_u = (N_u, x_u, C_u)$ denote the problem such that

$$\begin{aligned} N_u &= \Delta(z), z^{x_u} = p(z^x) + e^{N_0(x)} = 2e^{N_u} && (7) \\ C_u^*(t) &= C^* \left((\min(t_i, 1))_{i \in N_0(x)}, (v_i(z^x, t))_{i \in N(x)} \right), \forall t \in [0_{\eta(z^x)}, p(z^x + e^{N_0(x)})] && (8) \end{aligned}$$

The problem P_u is obtained from P by: (a) adding a costless unit to the demand of each agent in $N_0(x)$; and (b) splitting every agent's demand into separate (yet homogeneous) unitary demands that are each coupled with i's technology. Thus, the agents in P_u all have demands of 1 (that is $z_i = 2$). Using P_u, the following preliminary result can be shown.

Lemma 1 *Let y be a pattern method. Consider a cost sharing problem $P = (N, x, C)$ and $i \in N$ such that $x_i \geq 1$. Then we have:*

(i) $y_{i'}(P_u) = y_{i''}(P_u)$, for any $i', i'' \in \tilde{I}(i, z^x)$;

(ii) $y_i(P) = x_i y_{i'}(P_u)$, for any $i' \in \tilde{I}(i, z^x)$.

Proof. Suppose y is a pattern method represented by the flow system f; and consider $P = (N, x, C)$, $i \in N$ such that $x_i \geq 1$. Given that y is a pattern method and $z^{x_u} = 2e^{\Delta(z^x)}$, we have: (a) $f(z^{x_u}, .) = f^s(z^{x_u}, .)$, where $f^s(z^{x_u}, .)$ is the elementary flow associated with some s that is a production pattern to $z^{x_u} = 2e^{\Delta(z^x)}$; (b) $f(z^x, .) = \overline{(f^s)^{[z^{x_u}]}}(z^x, .)$.

(i) Recall from Remark 2 that $f^s(z^{x_u}, .)$ is symmetric. Hence, it follows from Theorem 1 that any two agents i', i'' who are symmetric for C_u —i.e., $\partial_{i'} C_u^*(t) = \partial_{i''} C_u^*(\sigma_{ij}(t))$, $\forall t \in [e_{i'}, z^{x_u}]$— have exactly the same cost share under y. Given that any $i', i'' \in \tilde{I}(i, z^x)$ are symmetric for C_u, we have the desired result: $y_{i'}(P_u) = y_{i''}(P_u)$.

(ii) Noting that $\partial_{i'} C_u^*(t) = \partial_i C^*(r)$ for any $r \in [e^i, z^x]$, $t \in \Lambda(r)$ and $i' \in \tilde{I}(i, z^x)$, one can write

$$\begin{aligned} y_i(P) = \sum_{r \in [e^i, z^x]} f_i(z^x, r) \partial_i C^*(r) &= \sum_{r \in [e^i, z^x]} \left(\overline{(f^s)^{[z^{x_u}]}} \right)_i (z^x, r) \partial_i C^*(r) \\ &= \sum_{r \in [e^i, z^x]} \sum_{t \in \Lambda(r)} \sum_{i' \in \tilde{I}(i, z^x)} (f_{i'}^s)^{[z^{x_u}]}(p(z^x), t) \partial_i C^*(r) \\ &= \sum_{i' \in \tilde{I}(i, z^x)} \left[\sum_{r \in [e^i, z^x]} \sum_{t \in \Lambda(r)} (f_{i'}^s)^{[z^{x_u}]}(p(z^x), t) \partial_i C^*(t) \right] \\ &= \sum_{i' \in \tilde{I}(i, z^x)} \sum_{t \in [e_{i'}, z^{x_u}]} f_{i'}^s(z^{x_u}, t) \partial_{i'} C_u^*(t) \\ &= \sum_{i' \in \tilde{I}(i, z^x)} y_{i'}(P_u) \end{aligned}$$

Plugging the result of statement (i) into this sum then gives the desired result:

$$y_i(P) = \sum_{i' \in \tilde{I}(i,z^x)} y_{i'}(P_u) = \underbrace{|\tilde{I}(i,z^x)|}_{x_i} y_{i'}(P_u) = x_i y_{i'}(P_u), \text{ for any } i' \in \tilde{I}(i,z^x). \blacksquare$$

This result allows to prove that pattern methods satisfy the axiom NMSD, which is stated by the following theorem.

Theorem 2 *Every pattern method satisfies the axiom no merging or splitting for demand.*

Proof. Suppose y is a pattern method represented by the flow system f, and consider $P = (N, x, C)$, $P' = (N', x', C') \in \mathcal{P}$ such that P' is a demand-splitting manipulation of P available to some agent $i \in N$, with $N' = (N \setminus i) \cup I$.

First, let us show that $P_u = P'_u$. To this end, observe that $p(z^x) = p(z^{x'})$ and $\tilde{I}(i, z^x) = \cup_{i' \in I} \tilde{I}(i', z^{x'})$. Indeed, recall that (a) $z^x_j = z^{x'}_j = x_j + 1$, $\forall j \in N \setminus i$, (b) $x_i = \sum_{i' \in I} x_{i'}$. Also note that $w(z^x_i) = (\overbrace{2, ..., 2}^{x_i \text{ times}}) = (\overbrace{w(z^{x'}_{i'})}^{(2,...,2) \in \mathbb{N}^{x'_{i'}}})_{i' \in I}$. Hence $N_u = N'_u$ and $x_u = x'_u$. In addition, using the third condition in the definition of a demand-splitting manipulation, one can see that $C_u = C'_u$.

Next, fixing $j \in \tilde{I}(i, z^x) = \cup_{i' \in I} \tilde{I}(i', z^{x'})$ and using Lemma 1-(i),[12] we have: $y_k(P'_u) = y_k(P_u) = y_j(P_u)$, for any $i' \in I$ and $k \in \tilde{I}(i', z^{x'}) \subset \tilde{I}(i, z^x)$. Invoking Lemma 1-(ii) repeatedly, we can write

$$\sum_{i' \in I} y_{i'}(P') = \sum_{i' \in I} x'_{i'} y_j(P_u) = \left(\sum_{i' \in I} x'_{i'} \right) y_j(P_u) = x_i y_j(P_u) = y_i(P).$$

In addition, given that $x_{N \setminus i} = x'_{N \setminus i}$, we have $\tilde{I}(l, z^x) = \tilde{I}(l, z^{x'})$, for any $l \in N \setminus i$. Finally, using the fact that $P_u = P'_u$ gives:

$$y_l(P) = \sum_{i' \in \tilde{I}(l,z^x)} y_{i'}(P_u) = \sum_{i' \in \tilde{I}(l,z^{x'})} y_l(P'_u) = y_{i'}(P'_u), \forall l \in N \setminus i.$$

This shows that an arbitrary splitting manipulation available to i does not change the share of the agents in $N \setminus i$ either. $\blacksquare$

Theorem 2 shows that the requirement NMSD is met by the class of pattern methods. An immediate corollary of this result is that all convex combinations of pattern methods also meet this property. The issue that naturally arises is to determine whether the set of rules satisfying the axiom NMSD is wider than the family we have just described. The next result shows that the property is satisfied by some other flow methods, which are more complex than mere convex combinations of pattern methods.

[12] Recall from the second condition in Definition 4 that $x_{i'} \geq 1, \forall i' \in I$ if P' is a demand-splitting manipulation of P.

Theorem 3 *A flow method f satisfies the axioms no merging or splitting for demand and irrelevant demand if and only if: for any $z \in \mathbb{N}^N$ (with $N \in \mathcal{N}$) we have*

$$f(z,.) = \sum_{k=1}^{K_z} \alpha_k(z) \overline{(f^{s_k})^{[2e^{\Delta(z)}]}}(z,.),$$

where each s_k is a production pattern to $p(z) + e^{N_1(z)} = 2e^{\Delta(z)}$, $K_z \in \mathbb{N}_{++}$ and $\alpha(z) \in \mathbb{R}_+^{K_z}$ is such that $\sum_{k=1}^{K_z} \alpha_k(z) = 1$.

Proof. See Appendix.

Theorem 3 states the fact that any flow method meeting NMSD can be written as a "pointwise" convex combination of pattern methods. Note that in the case where $z_i \geq 2\ \forall i \in N$, we have $\underline{p(z)} + e^{N_1(z)} = p(z)$ and the result of the theorem comes down to $f(z,.) = \sum_{k=1}^{K_z} \alpha_k(z) \overline{f^{s_k}}(z,.)$, with each s_k being a production pattern to $p(z)$. Also observe that both the number K_z of pattern methods in the decomposition and the vector of weights $\alpha(z)$ vary with the profile z. It follows that the class of pattern-generated methods described in Theorem 3 contains (as a proper subset) the family of convex combinations of patterns methods, for which the patterns and the weights used in the decomposition do not vary with z.

5 Homogeneous problems and the private Aumann-Shapley method

Let us now examine the compensation of technological cooperation within the family of pattern-generated methods (described by Theorem 3). We argue that, with regard to this criterion, the public Aumann-Shapley rule —see (5)— and the private Aumann-Shapley method —see (6)— lie at the two extremes of this family. As already pointed out, since it does not reward technological cooperation, AS^{pub} minimizes the rent paid to "technological agents", who demand zero while making their technology available to the group. In what follows we show that AS^{pr} is in some regard the best sharing rule for these agents because it generates the highest technological rent.

Definition 9 *We say that a problem $P = (N, x, C)$ is homogeneous if there exist a nonincreasing function $M : \mathbb{N}_{++} \to \mathbb{R}_+$ and a nondecreasing function $\tilde{C} : \mathbb{N} \to \mathbb{R}_+$ such that*

$$C(S,t) = M(|S|)\, \tilde{C}\left(\sum_{i \in S} t_i\right),$$

for any $S \subseteq N$ and $t \in [0_S, x_S]$.

Homogeneous problems are quite natural: the cost function is multiplicatively separable, and the production cost depends only on the sum of the demands and the number of agents who cooperate to the production.[13] Many authors have examined problems with homogeneous goods —see for instance Moulin and Shenker (1994), Friedman and Moulin (1999). In the traditional model, homogeneous cost functions depend only on the sum of the demands (since the technology does not vary at all): it is well known in that context that the (standard) Aumann-Shapley rule assigns cost shares in proportion to the agents' demands, which is referred to as *average cost pricing for homogeneous goods*. Under our more general framework, the counterpart of this property is stated by the following result.

Theorem 4 *Let $P = (N, x, C)$ be a homogeneous problem, and y be a CSM satisfying irrelevant demand and no merging or splitting for demand. Then, there exist two real numbers $\theta^0_y(P) \geq 0$ and $\theta^u_y(P) \geq 0$ such that:*

$$y_i(P) = \begin{cases} \theta^u_y(P) x_i, & \text{if } i \in N \setminus N_0(x); \\ -\theta^0_y(P), & \text{if } i \in N_0(x). \end{cases}$$

The proof is omitted. The result is easily derived from the combination of Theorem 3, Lemma 1, and the fact that all units of demand (technology) are symmetric in every homogeneous problem.

Theorem 4 shows that, for all pattern-generated methods, agents who demand homogeneous goods are charged in proportion to their demands. Furthermore, all technological agents receive the same rent $\theta^0_y(P)$. Observe that, for any fixed homogeneous problem P, the cost shares are entirely characterized by one of the two parameters $\theta^u_y(P), \theta^0_y(P)$. Indeed, the budget-balance requirement allows to derive one from the other:

$$\theta^u_y(P)\ x(N) - |N_0(x)|\ \theta^0_y(P) = \underbrace{M(|N|)\ \tilde{C}(x(N))}_{C(N,x)}. \tag{9}$$

Note in particular that the public Aumann-Shapley rule is characterized by $\theta^0_{pub}(P) = 0$ (for any problem P), since it never compensates technological improvements. On the other hand, performing some (tedious but straightforward) combinatorial analysis shows that in a homogeneous problem $P = (N, x, C)$, the technological rent assigned by AS^{pr} to every agent $i \in N_0(x)$ is

$$\theta^0_{pr}(P) = \sum_{S \subseteq N \setminus i} \sum_{t \in \left[e_S^{S \setminus N_0(x)}, x_S\right]} \beta(S,t) \left[M(|S|) - M\left(|S|+1\right)\right] \tilde{C}\left(t(S)\right), \tag{10}$$

where $\beta(S,t) = \frac{(|S \cap N_0(x)| + t(S))!\ (\eta(z^x) - |S \cap N_0(x)| - t(S) - 1)!}{(\eta(z^x))!} \prod_{i \in S \setminus N_0(x)} \frac{x_i!}{t_i!(x_i - t_i)!}$.

The next result claims that, within the class of rules described by Theorem 3, AS^{pr} maximizes (minimizes) the rent (cost share) allocated to technological contributions.

[13] We thank Hervé Moulin for drawing our attention to this class of cost functions.

Theorem 5 *On the set of flow methods satisfying the axioms irrelevant demand and no merging or splitting for demand, the private Aumann-Shapley rule is the unique method that maximizes the technological rent for all homogeneous problems. More precisely, for every CSM y:*

$$|AS_i^{pr}(N, x, C)| \geq |y_i(N, x, C)|,$$

for any problem $P = (N, x, C)$ that is homogeneous and any agent $i \in N_0(x)$ —with the strict inequality for at least one problem P if $y \neq AS^{pr}$.

Proof. See Appendix.

Thus, for homogeneous problems, AS^{pr} is the most preferred rule from the perspective of the agents who lend their technology to produce the others' demands. As such, it is diametrically opposed to AS^{pub}, which is obviously the least preferred rule of these technological agents.

6 No Merging and Splitting and the public Aumann-Shapley method

As already pointed out, NMS is a compelling requirement if the designer cannot monitor the production process and the agents' demands and technologies are prone to merging and splitting manipulations. In this section we investigate the implications of NMS. The following lemma shows that this requirement is not compatible with the remuneration of technological cooperation.

Lemma 2 *Suppose that the CSM y, which satisfies NMS, is represented by the flow system f. Then we have:*

(a) $f_i(z^x, 2e^i) = 0$, $\forall N \in \mathcal{N}, x \in \mathbb{N}^N, i \in N$ *such that* $z^x \geq 2e^i$ *and* $|N| \geq 3$*;*

(b) $f_i(z^x, e^S + 2e^i) = 0$, $\forall N \in \mathcal{N}, \emptyset \subseteq S \subsetneq N \setminus i, x \in \mathbb{N}^N, i \in N$ *such that* $z^x \geq 2e^i$ *and* $|N| \geq 3$*;*

(c) $f_i(z^x, 2e^i) = 0$ $\forall N \in \mathcal{N}, x \in \mathbb{N}^N, i \in N$ *such that* $z^x \geq 2e^i$ *and* $|N| = 2$.

Proof. See Appendix.

From the above Lemma 2, one can see that NMS prohibits any positive technological rent. In fact, as stated by the following result, NMS singles out a unique flow method: the public Aumann-Shapley rule.

Theorem 6 *The public Aumann-Shapley rule is the unique CSM that satisfies Additivity, Strong Dummy, MDC and NMS.*

Proof. It is not difficult to check that the public Aumann-Shapley rule satisfies each of the four axioms.[14] Conversely, we have to show that any method satisfying these four axioms necessarily coincides with the public Aumann-Shapley rule.

We know from Theorem 1 that Addivity, Strong Dummy and MDC characterize the family of flow methods. It is thus sufficient to show that any flow f that satisfies NMS is necessarily the flow associated with the public Aumann-Shapley method. We proceed in two steps.

Step 1: note that $f_i(z^x, r) = 0$, for any $r \in [0, z^x]$ s.t. $r_i = 1$ and $r_j \geq 2$ for some $i, j \in N$. This follows easily from the combination of flow conservation and Lemma 2. Hence, only "public" flow methods (which do not reward technological improvements) might satisfy NMS. Note that flow conservation then implies the following: $\sum_{j \in N} f_j(z^x, e^N) = 1$.

Step 2: show that $f_i(z^x, r) = \frac{z_i^x - r_i + 1}{z^x(N) - r(N) + 1} \sum_{j \in N(r - e^i)} f_j(z^x, r - e^i)$ for all $r \in \left[e^N + e^i, z\right]$ which, along with $\sum_{j \in N} f_j(z^x, e^N) = 1$, characterizes the public Aumann-Shapley method. We do not explicitly spell out the argument as it can be found in the literature. Indeed, since the technology of the grand coalition is available to every subgroup of agents at no cost (this is the result of Step 1), we may use the traditional cost sharing model.[15] One can then follow the steps described in the Appendix of Sprumont (2005) to show that, among public flow methods, only the flow associated with the Aumann-Shapley rule meets NMS.■

The result of Theorem 6 shows in particular that preventing all possible manipulations is incompatible with remunerating technological cooperation. In case the authority in charge of allocating the shares can observe neither the technology nor the individual demands of the agents, then the only incentive-compatible flow method is the public Aumann-Shapley rule, which prevents all manipulations but gives a weight of zero to any technological improvement.

Note from the result of Theorem 6 that it would be redundant (in the statement of the axiom NMS) to add the requirement that the shares of other agents be unaffected by splitting manipulations available to (any) agent i. Either formulation of the axiom leads to the same flow method (the public Aumann-Shapley rule), which is why we adopt the more economical way of stating the property. Likewise, it would be redundant to adopt the additional requirement that the shares be unaffected by manipulations where some agents overstate their technology, since that (stronger) version of the axiom would single out the same public Aumann-Shapley rule.

[14] Under public Aumann-Shapley pricing, manipulations of the technology are obviously unprofitable, since technological improvements are not rewarded. In addition, as a pattern method, AS^{pub} meets NMSD and, therefore, NMS. Finally, AS^{pub} satisfies the other three axioms because it is a flow method (see Theorem 1).

[15] Observe that in the case where the technology is the same for any coalition of agents, our version of NMS and the one in Sprumont (2005) coincide. If the technology instead varies depending on the agents who cooperates, our version of the axiom is stronger because it prevents manipulations of the technology (in addition to demand manipulations).

7 Discussion

The paper introduces and characterizes some remarkable cost sharing rules in the context where technological cooperation between agents is possible. These rules can be seen as extensions of the Aumann-Shapley method, which is the unique flow method that is immune to demand manipulations in the traditional model (see Sprumont, 2005). In our context we find an infinity of sharing methods satisfying this requirement, the pattern-generated methods. They differ from one another in the way they combine the multiple technologies available in order to produce the demanded output profile. It is also shown that pattern methods satisfy *irrelevant demand*, a property close to the axioms ordinality and independence of dummy changes discussed respectively in Moulin (1995) and Sprumont (2008): units of demand that do not affect the cost can be ignored. Irrelevant demand is important to the characterization result in Theorem 3. Indeed, removing this requirement from the statement would leave us with flows that are not fully defined in the case where some agents demand zero while making their technology available. Obviously, the axiom *no merging or splitting for demand* is the key ingredient to our characterization in Theorem 3. Without it we would be left with only the fixed-flow property allowing to compute the shares when some of the demands are null.

Within the class of pattern methods, two distinguished elements stand out. The first one, public Aumann-Shapley pricing, always considers the technology as public. As a consequence, it minimizes the remuneration of technological agents. In contrast, for the class of homogeneous problems, private Aumann-Shapley pricing maximizes the rent paid to technological agents. It is therefore the best rule in case the mechanism designer wants to promote technological cooperation and innovation. In addition, a useful formula —which allows to compute the private Aumann-Shapley shares in homogeneous problems— is proposed.

Our two adaptations of the standard Aumann-Shapley rule (and the pattern methods in general) are compelling in situations where the technological contribution of each agent is perfectly observable by the designer (see for instance Examples 1 and 5). In the case where the technology is not perfectly observable, the designer is left only with the public Aumann-Shapley rule —which does not reward technological improvements— as the unique flow method that prevents manipulations of the technology or the demand (see Theorem 6). Note that the four axioms in the characterization of Theorem 6 are independent. It is shown in Bahel and Trudeau (2012) that Additivity, Strong Dummy and MDC are independent in problems with fixed population. Using essentially the same arguments, one can show that any of these three axioms is not implied by the combination of the other two and *no merging or splitting* (when population is variable). In addition, given that some flow methods do not meet no merging or splitting,[16] it is not implied by the combination of Additivity, Strong Dummy and MDC.

[16]For example, flow methods that compute the cost shares according to a given path (that depends on the demand profile x) do not meet NMS.

As illustrated by some of the examples in the text, our framework applies to the study of distribution networks (such as pipelines) and the compensation of technological cooperation between countries or firms. Other possible applications of our results relate to the pricing of public utilities —see for example Billera *et al.* (1978) and de Frutos (1998), the design of patent licensing agreements —Tauman and Watanabe (2007), the allocation of pollution abatement costs between regions or countries —Petrosyan and Zaccour (2003), etc.

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Appendix

A Figures

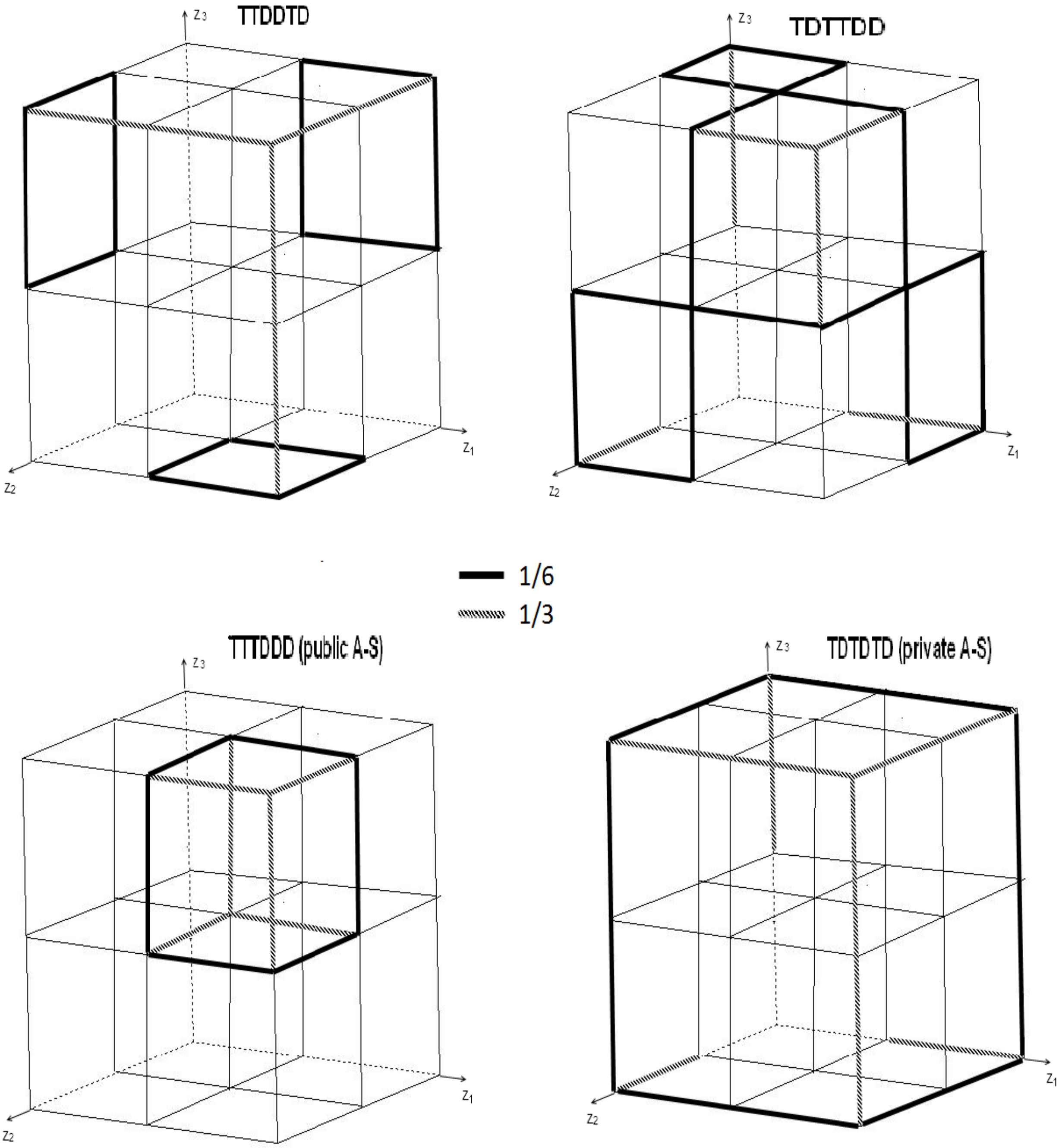

Figure 3: The 4 other elementary flows to (2,2,2)

In this figure we have for instance that $\Lambda(0,2) = \{(0,2,0),(0,0,2)\}$, $\Lambda(2,3) = \{(2,2,2)\}$; and $\tilde{I}(1,(2,2,2)) = \{1\}, \tilde{I}(2,(2,2,2)) = \{2,3\}$.

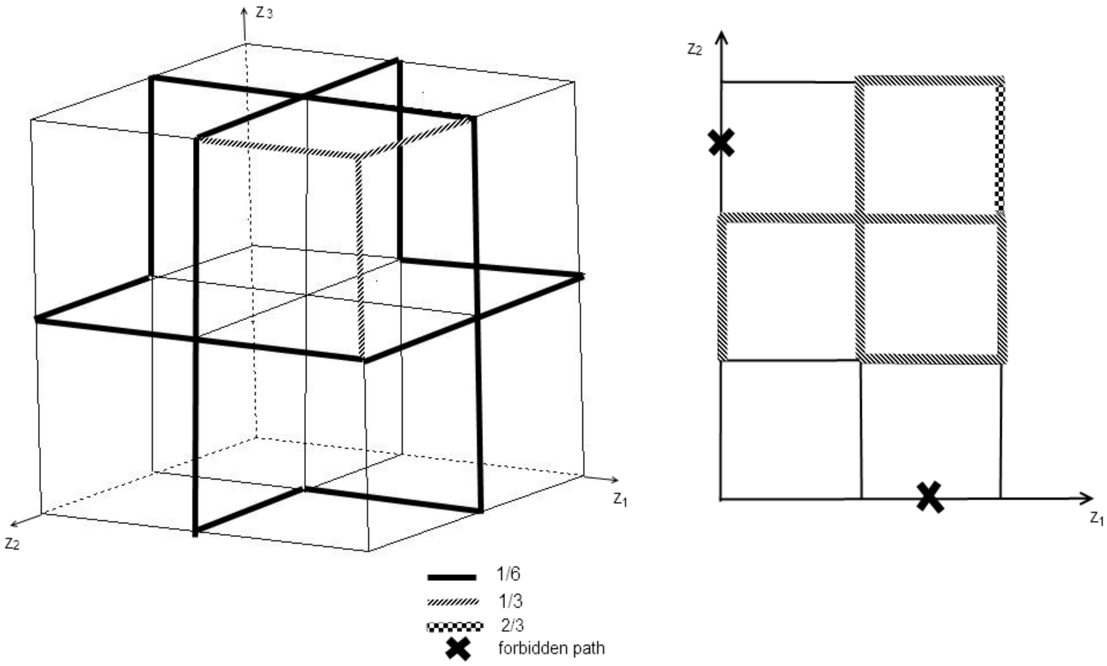

Figure 4: The flow $\overline{f^s}(z,.)$ to z=(2,3) generated by the production pattern s=TTDTDD to $p(2,3) = (2,2,2)$

B Proofs

B.1 Theorem 3

It is routine to check that all flow methods which can be written as "pointwise" convex combinations of pattern methods satisfy no merging splitting for demand and irrelevant demand. The proof of the necessity part of the statement unfolds in four steps.

Fix a cost sharing rule y (represented by the flow system f) and assume that it satisfies both no merging or splitting for demand and irrelevant demand.

Step 1: let us show that, for any $N \in \mathcal{N}$ and $x \in \mathbb{N}^N$, we have $f(z^x,.) = \overline{f(p(z^x),.)}$, where $\bar{\cdot}$ is the operator introduced in (4).

Consider an arbitrary problem $P = (N, x, C)$ and let $N \setminus N_0(x) = \{i \in N |\ x_i > 0\} = \{i_1, ..., i_m\}$. Since y meets no merging or splitting for demand, we can write

$$y_{i_1}(P) = \sum_{i' \in I_1} y_{i'}(P^{(1)}), \text{ and } y_{N \setminus i_1}(P) = y_{N \setminus i_1}(P^{(1)}), \tag{11}$$

where $P^{(1)}$ is the problem obtained from P by splitting the demand of agent i_1 into x_{i_1} distinct demands of 1. have: $P^{(1)} = (N^{(1)}, x^{(1)}, C^{(1)})$, where

$$\begin{aligned} N^{(1)} &= (N \setminus i_1) \cup I_1 \quad (\text{with } |I_1| = x_{i_1}); \\ x^{(1)} &= (\underbrace{2, ..., 2}_{I_1}, x_{N\setminus i_1}); \\ C^{(1)*}(t_{I_1}, t_{N\setminus i_1}) &= C^* \left(\sum_{i' \in I_1} t_{i'}, t_{N\setminus i_1} \right). \end{aligned}$$

By the same token, starting from the problem $P^{(1)}$ and splitting the demand of agent i_2 into x_{i_2} unitary demands gives:

$$y_{i_2}(P^{(1)}) = \sum_{i' \in I_2} y_{i'}(P^{(2)}), \text{ and } y_{N^{(1)}\setminus i_2}(P^{(1)}) = y_{N^{(1)}\setminus i_2}(P^{(2)}). \tag{12}$$

Combining (11) and (12), one can see that

$$y_{i_1}(P) = \sum_{i' \in I_1} y_{i'}(P^{(2)}), y_{i_2}(P) = \sum_{i' \in I_2} y_{i'}(P^{(2)}) \text{ and } y_{N\setminus\{i_1,i_2\}}(P) = y_{N\setminus\{i_1,i_2\}}(P^{(2)}). \tag{13}$$

Using the same argument repeatedly ($m = |\{i_1, ..., i_m\}|$ times, in fact), we get:

$$y_{i_1}(P) = \sum_{i' \in I_1} y_{i'}(P^{(m)}), ..., y_{i_m}(P) = \sum_{i' \in I_m} y_{i'}(P^{(m)}) \tag{14}$$

$$y_j(P) = y_j(P^{(m)}), \forall j \in N \setminus \{i_1, ..., i_m\}, \tag{15}$$

where $P^{(m)} = (N^{(m)}, x^{(m)}, C^{(m)})$ is such that

$$\begin{aligned} N^{(m)} &= \{1, ...\eta(z^x)\}; \\ z^{x^{(m)}} &= p(z^x); \\ C^{(m)*}(t_{N_0(x)}, t_{N_u\setminus N_0(x)}) &= C^* \left(t_{N_0(x)}, (v_i(z^x, t))_{i \in N\setminus N_0(x)} \right). \end{aligned}$$

Recall that the function $v_i(z, t)$ was introduced in (3).

Given that $\tilde{I}(i_1, z^x) = I_{i_1}, ..., \tilde{I}(i_m, z^x) = I_{i_m}$ and $\tilde{I}(j, z^x) = \{j\}$ for all $j \in N \setminus \{i_1, ..., i_m\}$, equations (14) and (15) come down to

$$y_i(P) = \sum_{i' \in \tilde{I}(i_1, z^x)} y_{i'}(P^{(m)}), \forall i \in N. \tag{16}$$

Since y is represented by the flow system f, we have $y_i(P) = \sum_{r \in [e^i, z^x]} f_i(z^x, r)\partial_i C^*(r)$, and $y_{i'}(P^{(m)}) = \sum_{t \in \left[e^{i'}, p(z^x)\right]} f_{i'}(p(z^x), t)\partial_{i'} C^{(m)*}(t)$, equation (16) can be rewritten as:

$$\begin{aligned}
\sum_{r\in[e^i,z^x]} f_i(z^x,r)\partial_i C^*(r) &= \sum_{i'\in\tilde{I}(i_1,z^x)} \sum_{t\in\left[e^{i'},p(z^x)\right]} f_{i'}(p(z^x),t)\partial_{i'}C^{(m)*}(t), \forall i\in N \\
&= \sum_{i'\in\tilde{I}(i,z^x)} \left[\sum_{r\in[e^i,z^x]} \sum_{t\in\Lambda(r)} f^s_{i'}(p(z^x),t)\partial_{i'}C^{(m)*}(t)\right] \\
&= \sum_{r\in[e^i,z^x]} \sum_{t\in\Lambda(r)} \sum_{i'\in\tilde{I}(i,z^x)} f^s_{i'}(p(z^x),t)\partial_i C^*(r),
\end{aligned}$$

where the last equality stems from the fact that $\partial_{i'}C^{(m)*}(t) = \partial_i C^*(r)$ for any $i\in N$, $r\in[e^i,z^x]$, $t\in\Lambda(r)$ and $i'\in\tilde{I}(i,z^x)$.

Therefore, we get the identity:[17]

$$\sum_{r\in[e^i,z^x]} f_i(z^x,r)\partial_i C^*(r) = \sum_{r\in[e^i,z^x]} \left[\overbrace{\sum_{t\in\Lambda(r)} \sum_{i'\in\tilde{I}(i,z^x)} f^s_{i'}(p(z^x),t)}^{\overline{f_i(p(z^x),r)}}\right] \partial_i C^*(r), \quad (17)$$

which gives the desired result, $f(z^x,.) = \overline{f(p(z^x),.)}$.

In case $N_0(x)=\emptyset$, we proceed to step 3. Otherwise, step 2 is needed.

<u>*Step 2:*</u> we show that, for any $N\in\mathcal{N}$ and $x\in\mathbb{N}^N$, $f(p(z^x),.) = f^{[p(z^x)+e^{N_0(x)}]}(p(z^x),.)$.

Fix $x\in\mathbb{N}^N$ and let C^* (defined on $\mathbb{N}^{N_u}$) represent an arbitrary cost function C. Consider a cost function $\bar{C}$ such that: (a) C^* and $\bar{C}^*$ coincide on the set $\{t\in\mathbb{N}^{N_u}|\ t_{N_0(x)}\le e_{N_0(x)}\}$; (b) all agents $i\in N_0(x)$ are demand-dummy for $\bar{C}$. Given that y meets irrelevant demand, we have $y(N_u,x_u,C) = y(N_u,x_u+e^{N_0(x)},\bar{C})$,[18] which (using the flow f) is equivalent to the following: for any $i\in N_u$,[19]

$$\sum_{r\in[e^i,p(z^x)]} f_i\left(p(z^x),r\right)\partial_i C^*(r) = \sum_{r\in\left[e^i,p(z^x)+e^{N_0(x)}\right]} f_i\left(p(z^x)+e^{N_0(x)},r\right)\partial_i\bar{C}^*(r), \quad (18)$$

Given that $p(z^x)+e^{N_0(x)} = 2e^{N_u}$, we have $J_i(r) = \{k\in N_0(x)\setminus i|r_k\ge 1\}$.[20]Moreover,

[17]Equation (17) is an identity because it holds true for any cost function C. Also recall that the flow $f(z,.)$ to z used to compute the shares under y is unique.

[18]We are implicitly using the well-known result that, for any flow method, the shares associated with the cost function $\bar{C}$ and the demand profile $p(z^x)$ depend only on the restriction of $\bar{C}^*$ to the box $[0,p(z^x)]$. This easily follows from Theorem 1.

[19]Remark that $z^{x_u+e^{N_0(x)}} = p(z^x)+e^{N_0(x)} = 2e^{N_u}$

[20]The notation $J_i(r)$ was introduced after Equation 2.

since all $i \in N_0(x)$ are demand-dummy for $\bar{C}$, one can then see that

$$\begin{aligned}
\partial_i \bar{C}^*(r) &= 0 && \text{if } i \in N_0(x), \text{ and } r_i = 2;\\
\partial_i \bar{C}^*(\underbrace{r}_{\in[0,p(z^x)]}) &= \partial_i C^*(r) && \text{if } J_i(r) = \emptyset \text{ and } (i \notin N_0(x) \text{ or } r_i \leq 1);\\
\partial_i \bar{C}^*(r) &= \partial_i C^*(\underbrace{e_{J_i(r)}, r_{N \setminus J_i(r)}}_{\in[e^i,p(z^x)]}) && \text{if } J_i(r) \neq \emptyset \text{ and } (i \notin N_0(x) \text{ or } r_i \leq 1).
\end{aligned}$$

Equation (18) then becomes

$$\sum_{r \in \left[e^i, p(z^x)\right]} f_i\left(p(z^x), r\right) \partial_i C^*(r) = \sum_{\substack{r \in \left[e^i, p(z^x)\right] \\ J_i(r) = \emptyset}} f_i\left(p(z^x) + e_{N_0(x)}, r\right) \partial_i C^*(r) \tag{19}$$

$$+ \sum_{\substack{r \in \left[e^i, p(z^x)\right] \\ J_i(r) \neq \emptyset}} \sum_{t \in \left[e_{J_i(t)}, 2e_{J_i(t)}\right]} f_i\left(p(z^x) + e_{N_0(x)}, (t, r_{N_u \setminus J_i(r)})\right) \partial_i C^*(r)$$

Since the identity (19) gives two expressions of the cost share (for every $i \in N_u$ and any C^* defined on $\mathbb{N}^{N_u}$), it follows from Theorem 1 that

$$f_i\left(p(z^x), r\right) = \begin{cases} f_i\left(p(z^x) + e_{N_0(x)}, r\right), & \text{if } J_i(r) = \emptyset; \\ \displaystyle\sum_{t \in \left[e_{J_i(t)}, 2e_{J_i(t)}\right]} f_i\left(p(z^x) + e_{N_0(x)}, (t, r_{N_u \setminus J_i(r)})\right), & \text{otherwise.} \end{cases}$$

This, by Equation (2), yields the desired result.

Step 3: next, observe that the flow $f(z^{x_u}, .)$ is necessarily symmetric. Indeed, given that y meets NMSD and $z^{x_u} = p(z^x) + e^{N_0(x)} = 2e_{N_u}$, we get from Remark 1 that

$$y_i(N_u, x_u, C) = y_j(N_u, x_u, C), \tag{20}$$

for any i, j in N_u and any C for which all agents in N_u are symmetric. It follows from (20) that $f(z^{x_u}, r) = f(z^{x_u}, \sigma_{ij}(r))$, for any $r \in [0_{N_u}, 2e_{N_u}]$ and $i, j \in N_u$.

Step 4: given that $f(z^{x_u}, .)$ is a symmetric flow to $z^{x_u} = 2e_{N_u}$, it can be written as a convex combination of elementary flows to z^{x_u}, i.e. $f(z^{x_u}, .) = \sum_{k=1}^{K_{z^x}} \alpha_k f^{s_k}(z^{x_u}, .)$, where each s_k is a pattern to $z^{x_u} = 2e_{N_u}$, $K_{z^x} \in \mathbb{N}_{++}$, and $\alpha \in \mathbb{R}^{K_{z^x}}$ satisfies $\sum_{k=1}^{K_{z^x}} \alpha_k(z^x) = 1$.

Combining the four steps above, one can finally write:

$$f(z^x, .) = \overline{\sum_{k=1}^{K_z^x} \alpha_k(z^x) (f^{s_k})^{[z^{x_u}]}(z^x, .)} = \sum_{k=1}^{K_z^x} \alpha_k(z^x) \overline{(f^{s_k})^{[2e^{\Delta(z^x)}]}}(z^x, .). \blacksquare$$

B.2 Theorem 5

Fix a homogeneous problem $P = (N, x, C)$, with $C(S, t_S) = M(|S|)\tilde{C}(t(S))$, and suppose $i \in N_0(x) \neq \emptyset$. Using Theorem 3, it is sufficient to show that $|y_i^s(P)| \leq |AS_i^{pr}(P)|$, for any pattern s to $2e^{\Delta(z^x)}$. To ease on notation, we will write z for z^x throughout this proof.

Note first that, by definition, any pattern s to $2e^{\Delta(z)}$ satisfies $|s^{-1}(\{T\})| = |\Delta(z)| = \eta(z)$.[21] Let then $s^{-1}(T) \equiv \{k_1^s, ..., k_{\eta(z)}^s\}$ be the ordered inverse image of T for the pattern s: that is to say, $1 = k_1^s < ... < k_{\eta(z)}^s \leq 2\eta(z) - 1$ and $s(k_j^s) = T$, for $j = 1, ..., \eta(z)$.[22]

Next, consider $i \in \Delta(z)$ and a path γ that follows the pattern s to $2e^{\Delta(z)}$. Let j_i^γ be the unique index in $\{1, ..., \eta(z)\}$ such that $\gamma(k_{j_i^\gamma}^s) - \gamma(k_{j_i^\gamma}^s - 1) = e_i$. In words, on the path γ, the technology of i is the j_i^γth to be activated. In addition, note that there exists a unique path $\bar{\gamma}$ to $2e^{\Delta(z)}$ such that: (a) $\bar{\gamma}$ follows the pattern $s^{pr}(z) = \underbrace{TD...TD}_{\eta(z) \text{ times}}$; (b) $j_i^{\bar{\gamma}} = j_i^\gamma$, for any $i \in \Delta(z)$. Essentially, $\bar{\gamma}$ is the unique path such that each unit demanded by an agent $j \in \Delta(z)$ is produced immediately after the activation of j's technology, with the order of activation of the technologies remaining the same as under γ.

Finally, for $i \in N_1(z) = N_0(x)$, define

$$\begin{aligned} q^\gamma(i) \equiv \quad & \left|\left\{l \in \Delta(z) \setminus N_1(z) : \gamma(k_{j_i^\gamma}^s) - 2e_l \geq 0_{\Delta(z)}\right\}\right|; \\ t^\gamma(i) \equiv \quad & \left|\left\{i' \in N \setminus i : \exists l \in \tilde{I}(i', z) \text{ s.t. } \gamma(k_{j_i^\gamma}^s) - e_l \geq 0_{\Delta(z)}\right\}\right|. \end{aligned}$$

The integer $q^\gamma(i)$ gives the number of costly units to produce prior to the activation of i's technology on the path γ. It is easy to see that: $q^\gamma(i) \leq q^{\bar{\gamma}}(i) = j_i^\gamma - 1$, for any path γ to $2e^{\Delta(z)}$ and any $i \in N_1(z) = N_0(x)$ [since $j_i^\gamma = j_i^{\bar{\gamma}}$ and $\bar{\gamma}$ follows s^{pr}]. As for $t^\gamma(i)$, it stands for the actual number of technologies that are available before that of agent i on the path γ.[23] Observe from the definitions of $\bar{\gamma}$ and $t^\gamma(i)$ that $t^\gamma(i) = t^{\bar{\gamma}}(i)$, for any $i \in N_0(x) = N_1(z)$.

Using the notation we have just introduced and the fact that $\tilde{C}$ is nondecreasing, one can write i's share for the path γ as:

$$\begin{aligned} y_i^\gamma(P) = \quad & [\overbrace{M(t^\gamma(i)) - M(t^\gamma(i) + 1)}^{\leq 0}]\tilde{C}(q^\gamma(i)) \\ \geq \quad & [M(t^\gamma(i)) - M(t^\gamma(i) + 1)]\tilde{C}(q^{\bar{\gamma}}(i)) \quad = [M(t^{\bar{\gamma}}(i)) - M(t^{\bar{\gamma}}(i) + 1)]\tilde{C}(q^{\bar{\gamma}}(i)) \\ & \qquad = y_i^{\bar{\gamma}}(P). \end{aligned}$$

[21] Recall that $\Delta(z) = \{1, ..., \eta(z)\}$.

[22] Observe that for any z and any pattern $s^{pr}(z) = \underbrace{TD...TD}_{\eta(z) \text{ times}}$ to $2e^{\Delta(z)}$, we have: $k_j^{pr} = 2(j-1)+1$, for $j = 1, ...\eta(z)$. The second condition of Definition 6-(a) then gives that $k_j^s \leq 2(j-1) + 1 = k_j^{pr}$, for any arbitrary pattern s.

[23] Recall that each unit $l \in \tilde{I}(i', z)$ possesses the full technology of agent i'. Therefore, activating two such units $l, l' \in \tilde{I}(i', z)$ gives the sole technology of i'.

Since the elementary flow f^s is defined as the average of all paths γ that follow the pattern s to $2e^{\Delta(z)}$, we have the desired result:

$$\begin{aligned} |y_i^s(P)| = & \quad -\frac{1}{(\eta(z))!} \sum_{\gamma \text{ that follow } s} y_i^{\gamma}(P) \\ \leq & \quad -\frac{1}{(\eta(z))!} \sum_{\gamma \text{ that follow } s} y_i^{\bar{\gamma}}(P) = -\frac{1}{(\eta(z))!} \sum_{\bar{\gamma} \text{ that follow } s^{pr}} y_i^{\bar{\gamma}}(P) = |AS_i^{pr}|\,. \end{aligned} \tag{21}$$

To conclude the proof, note that the inequality in (21) is strict whenever $\tilde{C}$ is increasing and $q^{\gamma}(i) < q^{\bar{\gamma}}(i)$, which will happen —for at least one homogeneous problem (N, x, C), one pattern s_k to $2e^{\Delta(z)}$ in the decomposition of Theorem 3, and one path γ (that follows s_k)— as soon as $y \neq AS^{pr}$. ■

B.3 Lemma 2

Statement (a). Let f be a flow method satisfying NMS. Fix $N \in \mathcal{N}, x \in \mathbb{N}^N, i \in N$ s.t. $|N| \geq 3, z^x \geq 2e^i$; and suppose that $f_i(z^x, 2e^i) > 0$. Let us consider the cost function C such that C^* is defined on $\mathbb{N}^N$ by

$$C^*(t) = \begin{cases} 1, & \text{if } t = ae^i \text{ with } a \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

We have $y_i(N, x, C) = f_i(z^x, 2e^i) > 0$ and $y_j(N, x, C) = -\sum_{a=2}^{z_i^x} f_j(z^x, ae^i + e_j)$, for any $j \in N \setminus i$.

First, we show that $y_j(N, x, C) < 0$, for any $j \in N \setminus i$. By way of contradiction, suppose $y_j(N, x, C) \geq 0$ for some $j \in N \setminus i$. Then, we necessarily have $y_j(N, x, C) = 0$, since j is demand-dummy for the cost function C. It follows that $y_i(N, x, C) + y_j(N, x, C) = f_i(z^x, 2e^i) > 0$. Let us now consider the merging manipulation (N', x', C') (available to $\{i, j\}$) such that: $N' = (N \setminus \{i, j\}) \cup i$, $x'_{N \setminus \{i,j\}} = x_{N \setminus \{i,j\}}$ and $x'_i = x_i + x_j$, $C'(t) = 0$ for any $t \in [0, z^{x'}]$. It is obvious that $y_i(N', x', C') = 0$. Hence, $y_i(N', x', C') < y_i(N, x, C) + y_j(N, x, C)$, which is a contradiction (since f satisfies NMS by assumption).

Second, note that flow conservation implies

$$f_i(z^x, 2e^i) = \sum_{j \in N \setminus i} \sum_{a=2}^{z_i^x} f_j(z^x, ae^i + e_j) = -\sum_{j \in N \setminus i} \underbrace{y_j(N, x, C)}_{<0}. \tag{22}$$

Since $|N| \geq 3$, we have $|N \setminus i| \geq 2$. Therefore, given a fixed $j_0 \in N \setminus i$, Equation (22) can be rewritten as

$$\underbrace{f_i(z^x, 2e^i)}_{=y_i(N,x,C)} + y_{j_0}(N, x, C) = -\sum_{j \in N \setminus \{i, j_0\}} \underbrace{y_j(N, x, C)}_{<0} > 0.$$

Examine the merging manipulation -available to $\{i,j_0\}$- (N',x',C') such that $N'=(N\setminus\{i,j_0\})\cup i$, $x'_{N\setminus\{i,j_0\}}=x_{N\setminus\{i,j\}}$ and $x'_i=x_i+x_{j_0}$, $C'(t)=0$ for any $t\in[0,z^{x'}]$. One can write $y_i(N',x',C')=0<y_i(N,x,C)+y_{j_0}(N,x,C)$, which once again violates NMS.

<u>**Statement (b)**</u>. We proceed by induction over S. Note that the statement (b) is satisfied when $S=\emptyset$ —since the statement (a) has been proved.[24] Fix $N\in\mathcal{N}, x\in\mathbb{N}^N, i\in N$ and S such that: $\emptyset\subseteq S\subsetneq N\setminus i$, $z^x\geq 2e^i$, and $|N|\geq 3$. Assume, by the induction hypothesis, that (b) is satisfied for all $S'\subsetneq S$.

Suppose that $f_i(z^x,e^S+2e^i)>0$ and consider the cost function C, with C^* defined on $\mathbb{N}^N$ by:[25]

$$C^*(t)=\begin{cases}1, & \text{if } t_k\geq 2 \text{ for some } k\in S;\\ 1, & \text{if } t^{N\setminus i}\leq e^S \text{ and } t_i\geq 2;\\ 0, & \text{otherwise.}\end{cases}$$

Observe that

$$y_i(N,x,C)=f_i(z^x,e^S+2e^i)+\sum_{S':\emptyset\subseteq S'\subsetneq S}\underbrace{f_i(z^x,e^{S'}+2e^i)}_{=0}=f_i(z^x,e^S+2e^i);$$

$$y_j(N,x,C)=-\sum_{S':\emptyset\subseteq S'\subseteq S}\sum_{a=2}^{z_i^x}f_j(z^x,e^{S'}+ae^i+e_j),\ \text{for any } j\in N\setminus(S\cup i).$$

However, since $f_i(z^x,e^{S'}+2e^i)=0$, flow conservation requires that

$$f_j(z^x,e^{S'}+ae^i+e_j)=0 \text{ for any } S'\subsetneq S \text{ and } j\notin S'\cup i.$$

Therefore,

$$y_j(N,x,C)=-\sum_{a=2}^{z_i^x}f_j(z^x,e^S+ae^i+e_j),\ \text{for any } j\in N\setminus(S\cup i). \tag{23}$$

Let us discuss the following two cases.
• <u>First case</u>: $|N\setminus(S\cup i)|\geq 2$
Note that we must have $y_j(N,x,C)<0$ for every $j\in N\setminus(S\cup i)$. Indeed, if $y_j(N,x,C)\geq 0$ for some $j\in N\setminus(S\cup i)$, then the agents i and j can reduce their positive joint share $f_i(z^x,e^S+2e^i)+y_j(N,x,C)$ to zero by merging according to the manipulation $N'=(N\setminus\{i,j\})\cup i'$, $x'_{N\setminus\{i,j\}}=x_{N\setminus\{i,j\}}$ and $x'_{i'}=x_i+x_j$, $C'(t)=0$ for any $t\in[0,z^{x'}]$.

[24] This is because $e^{\emptyset}=0^N$.
[25] C is such that producing any positive demand for some of the agents in $S\cup i$ entails a cost of 1, except maybe when agent i has the only positive demand (in which case the technological cooperation of any agent in $N\setminus(S\cup i)$ reduces the cost to zero).

Since $|N \setminus (S \cup i)| \geq 2$, for any fixed agent $j_0 \in N \setminus (S \cup i)$, we then have

$$\begin{aligned} y_{j_0}(N,x,C) &= -\sum_{a=2}^{z_i^x} f_{j_0}(z^x, e^S + ae^i + e_{j_0}) \\ &> -\sum_{j \in N\setminus(S\cup i)} \underbrace{\sum_{a=2}^{z_i^x} f_j(z^x, e^S + ae^i + e_j)}_{>0} \\ &\geq -\sum_{j \in N\setminus i} \sum_{a=2}^{z_i^x} \underbrace{f_j(z^x, e^S + ae^i + e_j)}_{\geq 0} = -f_i(z^x, e^S + 2e^i), \end{aligned}$$

where the last equality obtains from flow conservation. It follows that

$$\underbrace{y_i(N,x,C)}_{=f_i(z^x, e^S+2e^i)} + y_{j_0}(N,x,C) > 0,$$

and agents i and j_0, in violation of NMS, can decrease their positive joint cost share by merging according to the manipulation $P' = (N', x', C')$ such that $N' = (N \setminus \{i, j_0\}) \cup i'$; $x'_{i'} = x_i$, $x'_{N\setminus\{i,j_0\}} = x_{N\setminus\{i,j_0\}}$; $C'(t) = 1$ if $t_k \geq 2$ for some $k \in S$, and $C'(t) = 0$ otherwise. Indeed, since i' is a dummy agent in P', we have $y_{i'}(N', x', C') = 0 < y_i(N,x,C) + y_{j_0}(N,x,C)$. As a consequence, under this first case, we necessarily have $f_i(z^x, e^S + 2e^i) = 0$.

• <u>Second case</u>: $|N \setminus (S \cup i)| = 1$

Let j be the unique agent in $N \setminus (S \cup i)$; that is to say, $N = S \cup \{i, j\}$. Using the same argument as under the first case,[26] one can show that $y_j(N,x,C) < 0$ (otherwise, agent i and j have a profitable merging manipulation).

Next, consider the problem $P' = (N', x', C')$ such that $N' = (N \setminus j) \cup \{j, j'\}$; $x'_N = x_N$, $x'_{j'} = 0$; and

$$C'^*(t) = \begin{cases} 1, & \text{if } t_k \geq 2 \text{ for some } k \in S; \\ 1, & \text{if } t^{N'\setminus i} \leq e^S \text{ and } t_i \geq 2; \\ 0, & \text{otherwise.} \end{cases}$$

It is not difficult to see that $P = (N, x, C)$ is a merging manipulation of P' available to $\{j, j'\}$. Since $|N' \setminus (S \cup i)| = 2$, applying the result of the above first case to P' gives $f_i(z^{x'}, e^S + 2e^i) = 0$, and therefore $y_j(N', x', C') = y_{j'}(N', x', C') = 0$ (by flow conservation). It thus follows that $y_j(N', x', C') + y_{j'}(N', x', C') = 0 > y_j(N, x, C)$, which violates NMS.

<u>**Statement (c)**</u>. When $|N| = 2$, taking $S = \emptyset$ and replicating the argument of the second case in the proof of (b) proves statement (c). ■

[26]Recall that we still have $f_i(z^x, 2e^i) > 0$, which we have assumed by way of contradiction.