Local Incentive Compatibility in Moral Hazard Problems: A Unifying Approach*

René Kirkegaard
Department of Economics and Finance
University of Guelph
rkirkega@uoguelph.ca

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Abstract

I suggest a unifying new approach to moral hazard. Once local incentive compatibility (L-IC) is satisfied, the problem of verifying global incentive compatibility (G-IC) is shown to be isomorphic to the well-understood problem of comparing two classes of distribution functions. The sufficient conditions for the validity of the first-order approach (FOA) provided by Rogerson and Jewitt are related to first and second order stochastic dominance, respectively. New conditions, among them one in the spirit of third order stochastic dominance, are presented. Conlon's multi-signal justifications can also be understood with this approach. New multi-signal conditions that rely on the more tractable orthant orders are provided. Even when the standard FOA is invalid, a modified FOA may be valid on the set of implementable actions. This resolves Mirrlees' famous "counterexample".

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1 Introduction

The principal-agent model of moral hazard is among the core models of microeconomic theory and central to the economics of information. The problem is conceptually simple; a principal must design a contract to induce the agent to take the desired action. From the agent's point of view the intended action must be made preferable to all other actions. Thus, a multitude of incentive compatibility constraints must be satisfied. Unfortunately, it is generally difficult to determine which constraints bind and to make robust predictions about the structure of optimal contracts.

In response, much of the literature has focused on environments where the only binding constraint is the "local" incentive compatibility constraint (L-IC). In such cases, ensuring the agent has no incentive to deviate marginally from the intended action guarantees global incentive compatibility (G-IC), i.e. larger deviations can be ruled out too. Indeed, the classic first-order approach (FOA) simply uses the agent's first-order condition to summarize G-IC. The optimal contract is then easily derived. The FOA has a long history, dating back to Holmström (1979) and Mirrlees (1976, 1999). Rogerson (1985) and Jewitt (1988) have provided sufficient conditions under which the FOA is valid. However, although there are similarities in the structure of their proofs, the techniques they use are quite different. Moreover, despite criticizing the stringency of his assumptions, most textbooks on the topic prove Rogerson's result, but, as Conlon (2009a) observes, none even state Jewitt's. In short, Jewitt's result may be underappreciated and there is little in the current literature to unify the two results. Similarly, Conlon (2009a) uses two different approaches to obtain his generalizations of Rogerson's and Jewitt's conditions to multi-signal environments.[1,2]

With these observations in mind, the objective of this paper is to propose an accessible and unifying approach to the moral hazard problem. From this methodological contribution flows two distinct sets of insights that enable previous results to be extended in several different directions. First, it provides a unified methodology to understand Rogerson's, Jewitt's, and Conlon's classic results on the validity of the FOA.[3] Indeed, several new justifications of the FOA are provided, for both the

[1] An earlier paper by Sinclair-Desgagné (1994) also extended Rogerson's conditions to the multi-signal model. However, Conlon (2009a) relaxes Sinclair-Desgagné's assumptions. Jewitt (1988) also offered two different multi-signal justifications of the FOA. Conlon further generalized one of these.

[2] Ke (2011a) proposes a fixed-point method for justifying the FOA. Araujo and Moreira (2001) propose a general Lagrangian approach to solve moral hazard problems when the FOA is not valid. See also Ke (2011b).

[3] Jewitt's (1988) original proof is made complicated by the fact that it relies on results in an unpublished working paper. The full proof is published in Conlon (2009b). In the existing literature, Conlon (2009a) comes closest to methodologically unifying Rogerson's and Jewitt's results. Specif-

one-signal and multi-signal models. Secondly, it is also possible to obtain insights into environments where the FOA is not valid and to establish a modified FOA that is valid in some cases.

The approach relies on "translating" the problem of verifying global incentive compatibility into a problem that is familiar to, and well-understood by, any economist. In particular, I will show that checking G-IC (once L-IC is satisfied) is isomorphic to the problem of comparing two classes of risky prospects, or two classes of distribution functions. Once this equivalence has been established, many of the results follow by simply calling upon well-known results from the literature on stochastic dominance. The remainder of this introduction outlines the main results.

Any contract translates into a distribution of wages (where the distribution is determined in part by the agent's action). For brevity, I will refer to a contract as nondecreasing or monotonic if the agent's utility is nondecreasing in the outcome or state. A contract is concave if the agent's utility is concave in the state. With this terminology, Rogerson's (1985) and Jewitt's (1988) proofs can be decomposed into two concise parts. In Rogerson's case, the first part is to identify conditions under which any monotonic and L-IC contract is also G-IC. The second part is then to identify additional conditions under which the candidate contract is in fact monotonic. In Jewitt's case, contracts are both monotonic and concave.

The two first columns in the top row of Table 1 summarize the conclusions in step 1 of Rogerson and Jewitt, respectively. For future reference, the third column identifies a natural extension. In comparison, the second row summarizes the notions of first, second, and third order stochastic dominance (FOSD, SOSD, and TOSD, respectively) between two lotteries, G and H.[4] Note that Jewitt weakens Rogerson's assumption on the distribution function, but in exchange has to strengthen the assumptions imposed on the shape of the contract. This trade-off is remarkably similar to the one encountered when FOSD and SOSD are compared. This is of course no coincide, and much can be gained from exploring the relationship between the two rows in Table 1. As the third column in Table 1 reveals, once the pattern is identified it becomes easy to develop a third set of conditions to validate the FOA.[5] Obviously,

ically, Conlon (2009a, footnote 7) observes that Rogerson's proof relies on integration by parts, and that a second round of integration by part can be used to prove Jewitt's result. He does not ask, for instance, what can be obtained from further rounds of integration by parts. As mentioned, Conlon's (2009a) multi-signal results rely on two different approaches.

[4]See Hadar and Russell (1969), Rothschild and Stiglitz (1970), Whitmore (1970), and Menezes et al (1980). For textbooks on stochastic orders, see Müller and Stoyan (2002) and Shaked and Shantikumar (2007).

[5]To make the second step in the proof work, assumptions on the agent's utility function and on the likelihood ratio are also needed. There is an appealing pattern in those assumptions as well.

infinitely many extensions to higher order stochastic dominance are possible. Moreover, by appealing to related stochastic orders (the increasing convex orders), it is possible to obtain another infinite sequence of justifications of the FOA in models where contracts are convex.

[TABLE 1 ABOUT HERE (SEE THE LAST PAGE)]

Conlon (2009a) generalizes Rogerson's and Jewitt's conditions to the multi-signal model. Both Jewitt and Conlon encounter obstacles in the multi-signal model, and they are unable to present succinct conditions in the standard Mirrlees formulation of the model. These difficulties can be explained by the direction in which they seek to extend the results into higher dimensions. Indeed, there are several ways in which FOSD and SOSD can be extended from one dimension to many dimensions. Some are more tractable than others. This simple insight allows a number of new multivariate justifications of the FOA to be derived. These are based on the so-called orthant orders.[6] Unlike Conlon's conditions, the new justifications presented here have the compelling property that the FOA remains valid as more and more independent signals are added, provided each signal satisfies Rogerson's or Jewitt's one-signal conditions.

It is well-known that the FOA is not always valid. The second contribution of the paper is to examine such environments. It is straightforward to characterize a subset of actions for which L-IC is guaranteed to be sufficient for G-IC among different subsets of contracts. In general, however, the set of actions for which L-IC implies G-IC is a subset of the set of implementable actions. However, I identify a model where the two sets coincide. Here, it is valid to apply the FOA on the "feasible set" of implementable actions. Specifically, this method of analysis is valid whenever Grossman and Hart's (1983) spanning condition is satisfied. Though this simple model was proposed three decades ago, no complete analysis has been offered until now. Again, the crucial step is an examination of the link between L-IC and G-IC. As a special case, the modified FOA is valid in textbook settings with two states (but a continuum of actions). The method also resolves Mirrlees' (1999) original "counterexample", the purpose of which was to demonstrate that the standard FOA may fail. Indeed, a conceptually much simpler "counterexample" can be constructed using the insights of this part of the paper. Finally, the leading example in Araujo and Moreira (2001) also succumbs to the modified FOA.

[6] Jewitt presents a second multi-signal justification for settings with two independent signals. This justification is in fact based on the lower orthant order, and is thus further generalized here.

2 Model and preliminaries

A risk averse agent takes a costly action that is not verifiable to others. The set of possible actions is some closed and bounded interval, $[\underline{a}, \overline{a}]$. The agent's action determines the joint distribution of $n \geq 1$ verifiable signals, denoted $\mathbf{x} = (x_1, ..., x_n)$. If the action is a, the cumulative distribution function is $F(\mathbf{x}|a)$, where it is assumed that the domain, $\mathcal{X} = \times_{i=1}^{n} [\underline{x}_i, \overline{x}_i]$, is convex, compact and independent of a. Define $\underline{\mathbf{x}} = (\underline{x}_1, ..., \underline{x}_n)$ and $\overline{\mathbf{x}} = (\overline{x}_1, ..., \overline{x}_n)$.[7] It is assumed that $F(\mathbf{x}|a)$ has no mass points and is continuously differentiable in $\mathbf{x}$ and a to the requisite degree, with $f(\mathbf{x}|a)$ denoting the density for fixed a. Assume that $f(\mathbf{x}|a)$ is strictly positive. Let $\overline{F}(\mathbf{x}|a)$ denote the survival function, i.e. the probability that the vector of signals is greater than $\mathbf{x}$. Generally, $\overline{F}(\mathbf{x}|a) \neq 1 - F(\mathbf{x}|a)$ when there are two or more signals.

The agent faces a contract that, to him, is fixed. He receives wage $w(\mathbf{x})$ if the outcome is $\mathbf{x}$, in which case utility is $v(w(\mathbf{x})) - a$.[8] The agent's expected utility (assuming it exists) given action a is then

$$EU(a) = \int v(w(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x} - a. \tag{1}$$

Evidently, costs are assumed to be linear in the action. For instance, think of the agent's action, a, as being his choice of what cost of effort to incur. The linearity is convenient since it implies that only the first term in (1) has curvature, which simplifies the search for necessary and sufficient conditions (which is pursued in Section 6). Incidentally, Rogerson (1985) chose this parameterization too, although he only pursued sufficient conditions. Conlon (2009a, footnote 3) also observes that curvature in the cost function can be important, and thus chooses the same parameterization.

The agent's utility function $v(w)$ is strictly increasing and differentiable to the requisite degree. Moreover, the agent is strictly risk averse, or $v''(\cdot) < 0$. The domain of the utility function is some interval which may or may not be the entire real line. Finally, utility is unbounded below and/or above. The latter assumption is invoked only in Section 6.

[7] The assumption that $\mathcal{X}$ is a hyperrectangle is for simplicity. If it is not a hyperrectangle, then let $\times_{i=1}^{n} [\underline{x}_i, \overline{x}_i]$ be the smallest hyperrectangle for which $\mathcal{X} \in \times_{i=1}^{n} [\underline{x}_i, \overline{x}_i]$. In the one-signal case, the support is simply denoted $[\underline{x}, \overline{x}]$.

[8] Additive separability is important. While it is a standard assumption in the literature, there are exceptions. Alvi (1997) and Fagart and Fluet (2012) provide conditions that justify the FOA without additive separability.

2.1 Incentive compatibility

If the principal wishes to induce action $a^* \in [\underline{a}, \overline{a}]$, this action must provide the agent with higher expected utility than any other action, or

$$EU(a^*) \geq EU(a) \text{ for all } a \in [\underline{a}, \overline{a}]\,, \tag{G-IC$_{a^*}$}$$

in which case the contract $w(\mathbf{x})$ is said to be globally incentive compatible. If $a^* \in (\underline{a}, \overline{a})$, a minimum requirement is that $EU(a)$ attains a stationary point at a^*, or

$$\int v(w(\mathbf{x})) f_a(\mathbf{x}|a) d\mathbf{x} - 1 = 0. \tag{L-IC$_{a^*}$}$$

Of course, the stationary point may in principle be a local minimum or a saddle-point. Nevertheless, I will refer to the condition $EU'(a^*) = 0$ as the local incentive compatibility condition.[9] Thus, any contract that satisfies $EU'(a^*) = 0$ will be termed L-IC$_{a^*}$ and any contract that satisfies $EU(a^*) \geq EU(a)$ for all $a \in [\underline{a}, \overline{a}]$ is G-IC$_{a^*}$. The implementation of $\underline{a}$ and $\overline{a}$ is discussed in Section 4.

In practice, a contract may need to satisfy a number of other constraints as well. Examples includes participation constraints, monotonicity of the principal's rewards, minimum wages, and the like. In the next section, any such constraints are simply ignored. The reason is that the question of when L-IC$_{a^*}$ implies G-IC$_{a^*}$ has little to do with these other constraints. Note, in particular, that the next section is not directly concerned with the design of optimal contracts. The participation constraint is added in Section 4, when optimal contracts and the FOA is analyzed.

3 From local to global incentive compatibility

This section focuses on the implications of local incentive compatibility. In particular, it will be argued that some valuable inferences can be drawn from the restrictions that L-IC on its own place on the contract.

The first step is to develop an alternative approach to the moral hazard problem. This device is useful because it provides a framework that not only conceptually unifies most results in the literature but which can also be used to guide the search for further generalizations.

[9] Note that I restrict attention to contracts that give the agent bounded utility. In principle, if v is unbounded above, any action could be implemented by specifying a contract that provides unbounded utility to the agent. Hence, contracts are assumed to yield bounded utility and to be integrable. Note that any monotonic contract must be bounded since $\mathcal{X}$ is compact,

3.1 An auxiliary problem

To develop the new approach, an auxiliary problem is introduced. Consider $a^* \in (\underline{a}, \overline{a})$ fixed. Think of this as the action the principal seeks to implement.

Next, fix $\mathbf{x}$ and think of a as a variable. Let

$$f^L(\mathbf{x}|a, a^*) = f(\mathbf{x}|a^*) + (a - a^*) f_a(\mathbf{x}|a^*) \tag{2}$$

and

$$F^L(\mathbf{x}|a, a^*) = F(\mathbf{x}|a^*) + (a - a^*) F_a(\mathbf{x}|a^*) \tag{3}$$

be the tangent lines to $f(\mathbf{x}|a)$ and $F(\mathbf{x}|a)$, respectively, at $a = a^*$.

Now switch the roles of $\mathbf{x}$ and a. Holding a (and a^*) fixed, consider the function $F^L(\mathbf{x}|a, a^*)$. Note that $F^L(\mathbf{x}|a, a^*)$ is not necessarily monotonic in $\mathbf{x}$, nor is it necessarily bounded between 0 and 1. Nevertheless, the following though experiment is proposed. Think of $f^L(\mathbf{x}|a, a^*)$ and $F^L(\mathbf{x}|a, a^*)$ as (admittedly odd) density and distribution functions, respectively. It is easy to see that F^L can be obtained by integrating f^L over $\mathbf{x}$. Now consider an artificial problem where the agent faces distribution function $F^L(\mathbf{x}|a, a^*)$ rather than $F(\mathbf{x}|a)$.

In defence of these unusual "distributions", note, for now, that F^L does in fact have the key properties that $F^L(\underline{\mathbf{x}}|a, a^*) = 0$ and $F^L(\overline{\mathbf{x}}|a, a^*) = 1$.[10] Recall, for instance, that the standard proof of the equivalence between the two definitions of univariate FOSD in Table 1 relies only on $G(\underline{x}) = H(\underline{x}) = 0$, $G(\overline{x}) = H(\overline{x}) = 1$, and the relative magnitudes of G and H, but not on monotonicity nor on the fact that proper distribution functions are bounded between 0 and 1. See also the discussion following Proposition 1, below.

"Expected utility" in the auxiliary problem is simply

$$EU^L(a|a^*) = \int v(w(\mathbf{x})) f^L(\mathbf{x}|a, a^*) d\mathbf{x} - a, \tag{4}$$

or

$$EU^L(a|a^*) = EU(a^*) + (a - a^*) \left[\int v(w(\mathbf{x})) f_a(\mathbf{x}|a^*) d\mathbf{x} - 1 \right]. \tag{5}$$

Evidently, the last term disappears if L-IC$_{a^*}$ is satisfied, in which case $EU^L(a|a^*) = EU(a^*)$ for all a. It now follows that G-IC$_{a^*}$ can equivalently be expressed as the requirement that

$$EU^L(a|a^*) \geq EU(a) \text{ for all } a \in [\underline{a}, \overline{a}]$$

[10] The claim follows from $F_a(\underline{\mathbf{x}}|a) = F_a(\overline{\mathbf{x}}|a) = 0$. Note also that $\int f_a(\mathbf{x}|a) d\mathbf{x} = 0$ because $\int f(\mathbf{x}|a) d\mathbf{x} = 1$ for all a.

or

$$\int v(w(\mathbf{x}))f^L(\mathbf{x}|a,a^*)d\mathbf{x} \geq \int v(w(\mathbf{x}))f(\mathbf{x}|a)d\mathbf{x} \text{ for all } a \in [\underline{a},\overline{a}]\,. \tag{6}$$

In essence, the continuum of incentive compatibility constraints in the original problem has been replaced with a continuum of comparisons of risky prospects. For instance, if $v(w(\mathbf{x}))$ is monotonic, it is fruitful to ask whether F^L first order stochastically dominates F. The point is that such comparisons are commonplace in economics, and that a large literature may now be accessed to inform the analysis. Proposition 1 records this conclusion.

Proposition 1 *Fix $a^* \in (\underline{a},\overline{a})$. Any L-IC$_{a^*}$ contract is G-IC$_{a^*}$ if and only if (6) holds.*

Note that (6) is satisfied if and only if

$$\int v(w(\mathbf{x}))\left[\kappa + \varepsilon f^L(\mathbf{x}|a,a^*)\right]d\mathbf{x} \geq \int v(w(\mathbf{x}))\left[\kappa + \varepsilon f(\mathbf{x}|a)\right]d\mathbf{x} \text{ for all } a \in [\underline{a},\overline{a}] \tag{7}$$

and all $\varepsilon > 0$ and all κ. It is trivial to select κ and $\varepsilon > 0$ in such a manner that both bracketed terms are proper densities, i.e. they are strictly positive and integrate to one. Now, all the stochastic orders invoked in this paper are *integral stochastic orders*, meaning that they can be expressed as follows: G dominates H if G is preferred to H for all utility functions in some class $\mathcal{U}$. For an introduction to integral stochastic orders, see Müller and Stoyan (2002). For these orders, positive affine transformations of the densities are obviously innocent; (6) and (7) are equivalent. The important implication is that even though f^L is not a proper density, stochastic dominance results can still be invoked. Thus, I will frequently abuse terminology and say that f^L dominates f in some (integral) stochastic order.

3.2 An illustration

To illustrate the approach, consider the one-signal case. Note that if $F(x|a)$ is convex in a, then its tangent line, $F^L(x|a,a^*)$, lies everywhere below the function itself. Thus, $F^L(\cdot|a,a^*)$ first order stochastically dominates $F(\cdot|a)$ for all a. It follows that any monotonic and L-IC$_{a^*}$ contract must be G-IC$_{a^*}$. Moreover, the argument holds regardless of a^*. If it can be established that the FOA candidate contract is in fact monotonic then the FOA is itself valid. Figure 1 exemplifies the auxiliary problem and the approach suggested here.

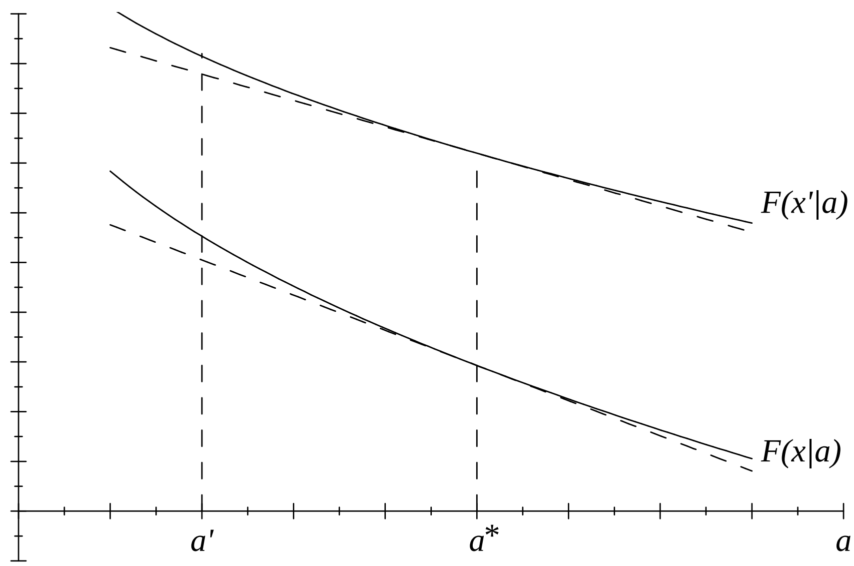

Figure 1: A new approach to moral hazard.

Note: Step 1: Fix a^*. For each x, construct the tangent line to $F(x|a)$ at a^* (moving horizontally). Step 2: For each $a \neq a^*$, like a', move vertically to trace out the cdf in the auxiliary and real problems. Here, F^L FOSD F (F^L lies always below F). Thus, any monotonic and L-IC$_{a^*}$ contract yields $EU(a^*) = EU^L(a^*) = EU^L(a') \geq EU(a')$. Step 3: To validate the FOA, the conclusion in step 2 must hold regardless of a^*.

For convenience, the following easy lemma notes necessary and sufficient conditions for F^L to ith order stochastically dominate F, $i = 1, 2, 3$, regardless of (a, a^*). Obviously, the characterization can be extended to higher stochastic orders.

Lemma 1 *Assume there is a single signal. $F^L(\cdot|a, a^*)$ ith order stochastically dominates $F(\cdot|a)$ for all $a \in [\underline{a}, \overline{a}]$ and all $a^* \in [\underline{a}, \overline{a}]$ if and only if the following conditions are satisfied for $i = 1, 2, 3$, respectively:*

1. *$F_{aa}(x|a) \geq 0$ for all $x \in [\underline{x}, \overline{x}]$ and all $a \in [\underline{a}, \overline{a}]$.*
2. *$\int_{\underline{x}}^{x} F_{aa}(y|a)dy \geq 0$ for all $x \in [\underline{x}, \overline{x}]$ and all $a \in [\underline{a}, \overline{a}]$.*
3. *$\int_{\underline{x}}^{x} \int_{\underline{x}}^{z} F(y|a)dydz \geq 0$ for all $x \in [\underline{x}, \overline{x}]$ and all $a \in [\underline{a}, \overline{a}]$ and $\int_{\underline{x}}^{\overline{x}} F_{aa}(y|a)dy \geq 0$ for all $a \in [\underline{a}, \overline{a}]$.*

Proof. The first part follows from the fact that a function is convex if and only if it lies everywhere above its tangent line. For the second part, note first that the tangent line to $\int_{\underline{x}}^{x} F(y|a)dy$ at $a = a^*$ is

$$\int_{\underline{x}}^{x} F(y|a^*)dy + (a - a^*) \int_{\underline{x}}^{x} F_a(y|a^*)dy = \int_{\underline{x}}^{x} F^L(y|a, a^*)dy.$$

The proof then concludes as in the first part. The proof for $i = 3$ is analogous. ■

Many results of the type presented in Lemma 1 are utilized in the analysis. Since the proofs are trivial and in any event analogous to the proof of Lemma 1, I will for the most part omit the formal proofs.

Of course, Rogerson's (1985) assumption is exactly that $F_{aa}(x|a) \geq 0$. His proof of the validity of the FOA is based on the observation that, in the one-signal case, integration by parts yields

$$EU(a) = v(w(\overline{x})) - \int_{\underline{x}}^{\overline{x}} F(x|a)dv(w(x)) - a, \tag{8}$$

and it follows that $EU(a)$ is concave when the contract is monotonic (or $dv \geq 0$). The condition $\int_{\underline{x}}^{x} F_{aa}(y|a)dy \geq 0$ is Jewitt's (1988) assumption (2.10a).[11] Conlon (2009a) points out that a second round of integration by parts can be used to prove concavity in Jewitt's model.

In fact, all the new justifications of the FOA that will be presented in Sections 4 and 5 can be shown to imply concavity. However, proving concavity in some cases requires repeated (and remarkably tedious) application of integration by parts. The method of proof I pursue is different and substantially less labor-intensive; the strategy is simply to invoke various stochastic orders. Indeed, the new results were discovered precisely by searching for usable stochastic orders, but it would be possible to rewrite the proofs in a more conventional manner by proving concavity directly.

Incidentally, note that Lemma 1 signifies that not only are Rogerson's and Jewitt's conditions sufficient, they are in fact the weakest conditions that can be imposed to ensure that L-IC implies G-IC for all a when the only characteristics of the contracts that are exploited are monotonicity or monotonicity and concavity. Thus, their results cannot be strengthened without imposing more structure on the contract (Section 6 contains a formal proof). In other words, the one-way implications ($\Downarrow$) in the first row of Table 1 can be converted into two-way implications ($\Updownarrow$), thereby

[11] Jewitt also imposes another assumption, (2.10b), but this assumption is redundant; see Conlon (2009a, 2009b). Assumptions (2.11) and (2.12) are used in the other step of his proof (see below).

cementing the analogy between the two rows.

Section 6 examine environments where L-IC$_{a^*}$ does not imply G-IC$_{a^*}$ for all a^*, or where the agent's expected utility is not necessarily concave in a. In such cases, the FOA may be invalid.

4 Justifying the first-order approach: One signal

Thus far, focus has been on interior a^*, where L-IC is necessary for utility maximization. However, boundary actions must be considered too, and so this section starts by clearing that technicality.

Thus, consider the corners, $\underline{a}$ and $\overline{a}$. With Rogerson's assumption, $EU'(\overline{a}) = 0$ (or L-IC$_{\overline{a}}$) is sufficient for G-IC$_{\overline{a}}$ among monotonic contracts. Indeed, if $EU'(\overline{a}) \geq 0$, it follows from (5) that $EU^L(\overline{a}|\overline{a}) = EU(\overline{a}) \geq EU^L(a|\overline{a})$ for all $a \in [\underline{a}, \overline{a}]$. Then, G-IC$_{\overline{a}}$ follows if $EU^L(a|\overline{a}) \geq EU(a)$. However, as long as the FOA contract is monotonic, (6) proves this is the case. Hence, at $\overline{a}$, any monotonic contract that satisfies $EU'(\overline{a}) \geq 0$ is G-IC$_{\overline{a}}$. Similarly, any monotonic contract that satisfies $EU'(\underline{a}) \leq 0$ is G-IC$_{\underline{a}}$ (a constant-wage contract is a special case).

Hence, given Rogerson's assumption, it is meaningful to replace the global incentive compatibility constraint in the principal's maximization problem with the more concise condition that

$$EU'(a^*) \begin{cases} \leq 0 \text{ if } a^* = \underline{a} \\ = 0 \text{ if } a^* \in (\underline{a}, \overline{a}) \\ \geq 0 \text{ if } a^* = \overline{a} \end{cases} . \tag{9}$$

A more general conclusion can be obtained. Specifically, if enough structure is imposed on F to ensure that L-IC$_{a^*}$ implies G-IC$_{a^*}$ for any interior a^* – among whatever subset of contracts is being considered – then that structure also implies that actions at the corners are easily handled too. One version of the FOA is then to replace G-IC$_{a^*}$ with L-IC$_{a^*}$, solve the principal's problem, and then compare the solution to the optimal implementation of $\underline{a}$ and $\overline{a}$ using $EU'(\underline{a}) \leq 0$ and $EU'(\overline{a}) \geq 0$, respectively. For expositional simplicity, I will assume the second best action is in the interior, but this assumption is evidently innocent and easily checked. Rogerson (1985) makes a similar assumption (see his Assumption A.10).

Returning to the main task at hand, justifying the FOA, recall the proof strategy. In the first step, sufficient conditions are given for L-IC$_a$ to imply G-IC$_a$ among a subset of contracts, for any a. In the second step, sufficient conditions are derived to ensure the FOA solution belongs to the relevant subset of contracts. Lemma 1 reveals the conditions required to invoke FOSD, SOSD, and TOSD, respectively. It remains to match these conditions with another set of assumptions that guarantees that the contract takes a form such that these stochastic orders are useful.

To this end, recall the equivalent definitions of these stochastic orders. Assuming differentiability, the distribution G is said to ith order stochastically dominate distribution H if the former is preferred to the latter for all utility functions $u(x)$ with the property that the first i derivatives of $-u(-x)$ are positive. I will refer to such functions as *i-antitone*. This terminology is inspired by a multivariate concept; see Section 5. Note that the derivatives of $u(x)$ alternates in sign, i.e. $(-1)^{s-1}u^{(s)} \geq 0$ for all $s = 1, 2, ..., i$, where $u^{(s)}$ denotes the sth derivative. Using difference operators, it is also possibly to extend the definition to utility functions that are not necessarily differentiable; see e.g. Müller and Stoyan (2002, Section 1.6). These stochastic orders are sometimes referred to as the *i-increasing concave* (i-icv) orders. That is, 1-icv, 2-icv, and 3-icv are just different names for FOSD, SOSD, and TOSD, respectively. For future reference, a related set of orders, the *i-increasing convex orders* (i-icx), apply to situations in which the first i derivatives of $u(x)$ are all positive. Such functions will be said to be *i-monotone.* The next step is to make sure that the endogenous function $v(w(x))$ falls within one of these classes of functions.

As in Jewitt (1988), assume the principal is risk neutral. Let $B(a)$ denote the expected gross benefit to the principal if the agent chooses action a. In many applications of the one-signal model, $B(a)$ is simply the expected value of x. Apart from incentive compatibility, the only other constraint is a participation constraint. Let $\overline{u}$ denote the agent's reservation utility. It will be assumed the constraint-set is non-empty, i.e. that there exists a contract that satisfies both the participation constraint and L-IC for some a.

The FOA relies on L-IC being sufficient for G-IC. If this is the case, the principal's problem can be written as follows:

$$\begin{aligned} &\max_{w,a} B(a) - \int_{\underline{x}}^{\overline{x}} w(x)f(x|a)dx \\ st. \quad &\int_{\underline{x}}^{\overline{x}} v(w(x))f(x|a)dx - a \geq \overline{u} \\ &\int_{\underline{x}}^{\overline{x}} v(w(x))f_a(x|a)dx - 1 = 0. \end{aligned}$$

Assume the likelihood-ratio

$$l(x|a) = \frac{f_a(x|a)}{f(x|a)}$$

is bounded below. As in Rogerson and Jewitt, assume that the monotone likelihood ratio property (MLRP) is satisfied, or $l_x(x|a) \geq 0$. This assumption in fact implies $F_a(x|a) \leq 0$, i.e. higher actions make low signals less likely. Finally, assume, in this section and the next, that it is optimal to offer a wage $w(x)$ in state x that is in the interior of the domain of $v(\cdot)$. For a fixed utility function, this assumption is typically satisfied if the agent's reservation utility is high enough.[12] In this case, $w(x)$ is characterized by a first order condition which can be written

$$\frac{1}{v'(w(x))} = \lambda + \mu l(x|a^*), \tag{10}$$

where $\lambda > 0$ is the multiplier of the participation constraint and $\mu \geq 0$ the multiplier of the local incentive compatibility constraint. If $a^* = \underline{a}$, a flat wage is optimal ($\mu = 0$). However, if $a^* > \underline{a}$ then $\mu > 0$ and so, by the MLRP, the wage schedule is monotonic.[13] These conclusions are due only to the assumptions that $v'(\cdot)$ is decreasing in w and $l(x|a^*)$ is increasing in x. Thus, the function $v(w(x))$ is both 1-isotone and 1-monotone. Hence, FOSD can be invoked.

Jewitt (1988) imposes more substantial joint conditions on the utility function and likelihood ratio. To aid the analysis, Jewitt defines the function

$$\omega(z) = v(v'^{-1}(1/z)),\ z > 0.^{14}$$

Note that $\omega'(z) > 0$ if and only if $v''(w) < 0$, which has already been assumed. Jewitt adds the assumption that $\omega''(z) \leq 0$ and $l_{xx}(x|a) \leq 0$. From (10),

$$v(w(x)) = \omega\left(\lambda + \mu l(x|a^*)\right).$$

Hence, Jewitt's assumptions imply that any contract that $v(w(x))$ is increasing and concave, or 2-isotone. SOSD can now be invoked. As the next lemma shows, it turns out that the pattern can be continued. Conditions are imposed on the inner function $l(x|a)$ and the outer function $\omega(z)$ to guarantee that the composite function $v(w(x)) = \omega\left(\lambda + \mu l(x|a^*)\right)$ has desirable properties

[12] See e.g. Jewitt et al (2008), and in particular Gutiérrez (2012) for a detailed discussion. As can be seen from (10), below, this also explains why $l(x|a)$ must be bounded.

[13] One of the contributions in Rogerson (1985) and Jewitt (1988) is to establish that $\mu > 0$. In fact, Jewitt's (1988) paper appears to be cited more often for this result (and its very elegant proof) than for his conditions justifying the FOA. As in Conlon (2009a), I omit the proof here. Rogerson (1985) allows the principal to be risk averse. It is considerable harder to allow a risk averse principal in Jewitt's framework; see Conlon (2009a).

[14] To clarify, $v'^{-1}(\cdot)$ refers to the inverse of $v'(\cdot)$.

Lemma 2 *(i)* $\omega(\lambda + \mu l(x|a))$ *is* i*-monotone in* x *if* ω *is* i*-monotone and* $l(x|a)$ *is* i*-monotone in* x*. (ii)* $\omega(\lambda + \mu l(x|a))$ *is* i*-isotone in* x *if* ω *is* i*-isotone and* $l(x|a)$ *is* i*-isotone in* x*.*

Proof. Repeated differentiation yields the result. ■

Thus, if $l_{xxx}(x|a) \geq 0$ and $\omega'''(z) \geq 0$, then $v(w(x))$ is 3-isotone and TOSD can be invoked.

Table 2 summarizes the main conclusions thus far. The first row identifies sufficient conditions for L-IC$_a$ (or rather (9)) to imply G-IC$_a$ among contracts that are 1-isotone, 2-isotone, and 3-isotone, respectively, for all a. The second row identifies sufficient conditions for the FOA candidate solution in (10) to be such a contract. The validity of the FOA follows by imposing both sets of assumptions.

[TABLE 2 ABOUT HERE (SEE THE LAST PAGE)]

Proposition 2 *Assume the second best action is in* $(\underline{a}, \overline{a})$*. Assume the joint conditions in one of the columns of Table 2 are satisfied. Then, the FOA is valid.*

Obviously, Table 2 and Proposition 2 can be extended to stochastic dominance of higher order (4-icv, 5-icv, etc.). In fact, there is a well-defined limit to the sequence of higher order stochastic dominance, namely the Laplace transform order. See e.g. Müller and Stoyan (2002).

Evidently, the assumptions in the first row of Table 2 become weaker as one moves rightward from one column to the next. As for the second row, consider the following possible utility functions:

$$v_1(w) = \ln w,\ v_2(w) = 1 - e^{-\alpha w},\ v_3(w) = w^{\beta},$$

where $\alpha, \beta > 0$. The domains of the functions are $(0, \infty)$, $(-\infty, \infty)$, and $[0, \infty)$, respectively (or convex subsets thereof). For these functions, $\omega(z)$ can be shown to be

$$\omega_1(z) = \ln z,\ \omega_2(z) = 1 - \frac{1}{\alpha z},\ \text{and } \omega_3(z) = (\beta z)^{\frac{\beta}{1-\beta}}$$

respectively. Thus, the first two functions are 3-isotone. The third function satisfies $\omega_3'(z) > 0$, $\omega_3''(z) \leq 0$ if and only if $\beta \in (0, .5]$, i.e. if the agent is sufficiently risk averse. However, for β in this range it is also the case that $\omega_3'''(z) \geq 0$. In fact, with this restriction, ω is i-isotone for *any* $i \geq 1$ in all three examples. Thus, in these examples, the assumptions on $\omega(z)$ in the third column of Table 1 are not any stronger than those in the second column. Hence, the main strengthening from Jewitt's conditions to the new conditions in the third column is in the added requirement that $l_{xxx}(x|a) \geq 0$. Incidentally, all Jewitt's (1988, page 1183) examples have the feature that $l(x|a)$ is i-isotone for any $i \geq 1$.

However, extensions in other directions beckon. Except for Rogerson's conditions, the conditions mentioned above assume that the composite function $\omega(\lambda + \mu l(x|a))$ is increasing and concave. Now consider the possibility that it is convex. Note that the outer function ω may be convex even if v is concave; indeed v must be concave for ω to be increasing. The i-icx orders, defined above, are relevant for such cases. Note that the utility function $v_3(w)$ mentioned above leads to an i-monotone $\omega(z)$ function if and only if $\beta \in [\frac{i-1}{i},)$.

For distribution functions G and H, an equivalent definition of 1-icx is that $\overline{G}(x) \geq \overline{H}(x)$ for all $x \in [\underline{x}, \overline{x}]$. An equivalent definition for 2-icx is that

$$\int_x^{\overline{x}} \overline{G}(z)dz \geq \int_x^{\overline{x}} \overline{H}(z)dz \text{ for all } x \in [\underline{x}, \overline{x}],$$

and so on for higher increasing-convex orders. The orders 1-icx, 2-icx, and 3-icx are the counterparts to FOSD, SOSD, and TOSD, respectively, for utility loving agents.[15] Note that 1-icx in fact coincides with FOSD (or 1-icv), meaning that Rogerson's conditions can also be seen as the starting point to the sequence of conditions developed next. The following proposition, and its proof, is analogous to Proposition 2. It can of course also be extended to higher icx orders.[16]

Proposition 3 *Assume the second best action is in $(\underline{a}, \overline{a})$. Assume the joint conditions in one of the columns of Table 3 are satisfied. Then, the FOA is valid.*

[TABLE 3 ABOUT HERE (SEE THE LAST PAGE)]

5 Multi-signal justifications of the FOA

Jewitt's (1988) Theorem 2 and Theorem 3 were the first attempts at providing multi-signal justifications for the FOA. These results assume there are exactly two signals,

[15] Note that the random variable X dominates the random variable Y in the s-icx order if and only if $-Y$ dominates $-X$ in the s-icv order.

[16] Jewitt offers a supremely convincing argument for his concavity assumption on $\int_{\underline{x}}^{x} F(z|a)dz$ in the special case where the signal x is production. Proposition 3 thus covers other cases.

and that the signals are independent. Moreover, as in Jewitt's one-signal model, ω is assumed to be concave. Sinclair-Desgagné (1994) generalized Rogerson's conditions to the case where there are multiple (not necessarily independent) signals. Finally, Conlon (2009a) further generalized Rogerson's conditions and offered an extension to Jewitt's Theorem 3, which he refers to as "Jewitt's (1988) main set of multisignal conditions".[17]

Here, I will verify that Conlon's results can be understood as appealing to multi-signal versions of FOSD and SOSD, respectively. Indeed, once the isomorphism in Section 3 has been established, it invites the search for other useful multi-variate stochastic orders. Thus, Jewitt's Theorem 2 can be resurrected and extended once the proper stochastic order, which turns out to be the lower orthant order, has been identified. Another related order, the upper orthant order, leads to complementary results. By appealing to higher orthant orders, it turns out to be possible to offer generalizations that are close in spirit to Jewitt's Theorem 3 as well.

5.1 Multivariate FOSD and related stochastic orders

Müller and Stoyan (2002) make the following very useful observation about extending the common stochastic orders from a univariate setting to a multivariate environment. Specifically, comparing two distribution functions, G and H, there are three equivalent definitions of FOSD in the univariate setting, namely: (i) G is preferred to H for all non-decreasing utility function, (ii) $G(x) \leq H(x)$ for all x, and (iii) $\overline{G}(x) \geq \overline{H}(x)$ for all x. The point is that none of these definitions are equivalent when there are multiple signals. Consequently, there are three plausible ways of extending FOSD, which leads to the following definitions:

1. *G first order stochastically dominates H* if G is preferred to H for all non-decreasing utility functions.

2. *G dominates H in the lower orthant order* if $G(\mathbf{x}) \leq H(\mathbf{x})$ for all $\mathbf{x}$.

3. *G dominates H in the upper orthant order* if $\overline{G}(\mathbf{x}) \geq \overline{H}(\mathbf{x})$ for all $\mathbf{x}$.

Using Conlon's (2009a) notation and terminology, let $\mathbf{E}$ be an increasing set. A set is increasing if $\mathbf{x} \in \mathbf{E}$ and $\mathbf{y} \geq \mathbf{x}$ implies $\mathbf{y} \in \mathbf{E}$. It is well-known that an equivalent definition of FOSD is that G has more probability mass in all increasing

[17] The results in Jewitt (1988), Sinclair-Desgagné (1994), and Conlon (2009) all rely on proving that the agent's expected payoff is concave in his action. Ke (2011a) takes an alternative approach to justifying the FOA. See Section 6.

sets than H does; see Müller and Stoyan (2002, Theorem 3.3.4). Thus, FOSD is stronger than the orthant orders. However, all three orders can be used to derive separate multi-signal justifications of the FOA.

Returning to the principal-agent model at hand, let

$$P(\mathbf{x} \in \mathbf{E}|a) = \int_{\mathbf{x}\in\mathbf{E}} f(\mathbf{y}|a)d\mathbf{y}$$

denote the probability that the vector of signals is in the increasing set $\mathbf{E}$, given action a. Let $P^L(\mathbf{x} \in \mathbf{E}|a, a^*) = P(\mathbf{x} \in \mathbf{E}|a^*) + (a - a^*)P_a(\mathbf{x} \in \mathbf{E}|a^*)$ denote the counterpart in the auxiliary problem. Now, Conlon (2009a) proposes a *concave increasing-set probability (CISP) condition*, specifically that $P_{aa}(\mathbf{x} \in \mathbf{E}|a) \leq 0$ for all increasing sets and all $a \in [\underline{a}, \overline{a}]$. Evidently, the CISP condition implies that $P^L(\mathbf{x} \in \mathbf{E}|a, a^*) \geq P(\mathbf{x} \in \mathbf{E}|a)$ for all all $a \in [\underline{a}, \overline{a}]$. In other words, $F^L(\mathbf{x}|a, a^*)$ first order stochastically dominates $F(\mathbf{x}|a)$. Hence, expected payoff in the auxiliary problem is greater than in the original problem as long as the FOA contract is monotonic, as continues to be the case as long as the (multivariate) MLRP holds. This explains Conlon's (2009a, Proposition 4) extension of Rogerson's conditions.

Conlon (2009a) devotes considerable effort to examining CISP and deriving sufficient conditions for its applicability. However, CISP can be weakened, even without moving to conditions that can be used to invoke SOSD. In particular, recall that the orthant orders are weaker than FOSD. They also have the desirable property that their "maximal generators" can be identified, which means that equivalent statements of these orders can be given in term of the class of utility functions for which one distribution is preferred to another. Specifically, it can be shown that G dominates H in the upper orthant order if and only if G is preferred to H for all Δ-*monotone* utility functions (Müller and Stoyan (2002, Theorem 3.3.15)). If the utility function $u(\mathbf{x})$ is n times differentiable, then it is Δ-monotone if and only if

$$\frac{\partial^{k_1+...+k_n} u(\mathbf{x})}{\partial x_1^{k_1} ... \partial x_n^{k_n}} \geq 0$$

for all $k_i \in \{0, 1\}$, $i = 1, ..., n$, with $k_1 + ... + k_n \geq 1$. In words, all the mixed partial derivatives must be non-negative. See Müller and Stoyan (2002) for a formal definition, in term of difference operators, that allows $u(\mathbf{x})$ to be non-differentiable. Similarly, G dominates H in the lower orthant order if and only if G is preferred to H for all utility functions with the property that $u(-\mathbf{x})$ is Δ-*isotone*, i.e. $-u(-\mathbf{x})$ is Δ-monotone. Note that the relationship between these orders is similar to the relationship between the i-icv and i-icx orders in the univariate case.

In the principal-agent model, the FOA implies that $v(w(\mathbf{x})) = \omega(\lambda + \mu l(\mathbf{x}|a))$. Jewitt (1988) and Conlon (2009a) observe that the multipliers remain positive in the multi-signal model. The next Lemma summarizes some pertinent observations about the composite function. The proof is straightforward and is thus omitted.

Lemma 3 *(i) $\omega(\lambda + \mu l(\mathbf{x}|a))$ is Δ-monotone in $\mathbf{x}$ if $l(\mathbf{x}|a)$ is Δ-monotone in $\mathbf{x}$ and ω is n-monotone. (ii) $\omega(\lambda + \mu l(\mathbf{x}|a))$ is Δ-isotone in $\mathbf{x}$ if $l(\mathbf{x}|a)$ is Δ-isotone in $\mathbf{x}$ and ω is n-isotone.*

Lemma 3 provides conditions under which the FOA candidate contract belongs to one of the classes of functions that are useful when one of the orthant orders apply. However, it remains to impose conditions on the distribution functions such that the orthant orders can indeed be invoked. To this end, note that if $F_{aa}(\mathbf{x}|a) \geq 0$ then $F^L(\mathbf{x}|a, a^*)$ dominates $F(\mathbf{x}|a)$ in the lower orthant order. Likewise, if $\overline{F}_{aa}(\mathbf{x}|a) \leq 0$ then $F^L(\mathbf{x}|a, a^*)$ dominates $F(\mathbf{x}|a)$ in the upper orthant order. These conditions coincide in the one-signal case, where they collapse to Rogerson's condition. Finally, Conlon's (2009a) CISP condition implies both $\overline{F}_{aa}(\mathbf{x}|a) \leq 0$ and $F_{aa}(\mathbf{x}|a) \geq 0$. Since Sinclair-Desgagné's (1994) condition is even stronger than CISP, it follows that his condition also imply $\overline{F}_{aa}(\mathbf{x}|a) \leq 0$ and $F_{aa}(\mathbf{x}|a) \geq 0$. New justifications of the FOA are now possible.

Proposition 4 *Assume the second best action is in $(\underline{a}, \overline{a})$. Then, the FOA is valid if either:*

1. *$\overline{F}_{aa}(\mathbf{x}|a) \leq 0$ for all $\mathbf{x}$ and all a, $l(\mathbf{x}|a)$ is Δ-monotone in $\mathbf{x}$ for all a, and ω is n-monotone, or*

2. *$F_{aa}(\mathbf{x}|a) \geq 0$ for all $\mathbf{x}$ and all a, $l(\mathbf{x}|a)$ is Δ-isotone in $\mathbf{x}$ for all a, and ω is n-isotone.*

Proof. For the first part, $\overline{F}_{aa}(\mathbf{x}|a) \leq 0$ implies that $F^L(\mathbf{x}|a, a^*)$ dominates $F(\mathbf{x}|a)$ in the upper orthant order. Hence, expected payoff in the auxiliary problem is higher than in the original problem as long as utility is Δ-monotone. The remaining conditions ensure this is the case, since they allow Lemma 3 to be invoked. The proof of the second part of the proposition is analogous. ■

Note the rather pleasing similarities between the conditions on ω and $l(\mathbf{x}|a)$, and their pattern, in the univariate case (Propositions 2 and 3) and the multivariate case (Proposition 4). Specifically, the conditions that must be added as another signal becomes available are similar to the conditions that must be added in the univariate case when the stochastic order is weakened by one degree (see also Corollary 1, below).

Conlon (2009a) makes the point that if the n signals are independent and each satisfies Rogerson's conditions, then the joint distribution function may nevertheless fail the CISP condition. In this sense, the CISP condition is a strong assumption. In contrast, the lower orthant order (though not the upper orthant order) is more amenable to such extensions.

Corollary 1 *Assume there are $n \geq 2$ independent signals, with distribution functions $F^i(x_i|a)$ and likelihood ratio $l^i(x|a)$, $i = 1, 2, ..., n$. Assume the second best action is in $(\underline{a}, \overline{a})$. Then, the FOA is valid if*

1. *Each signal satisfies Rogerson's condition; $F^i_{aa}(x_i|a) \geq 0$ and $l^i_x(x_i|a) \geq 0$ for all $i = 1, 2, ..., n$, and*

2. *ω is n-isotone.*

Proof. The MLRP implies that F^i is decreasing in a. Since F^i is also convex, it follows that the product $F(\mathbf{x}|a) = \Pi F^i(x_i|a)$ is also convex in a. When signals are independent, $l(\mathbf{x}|a) = \Sigma l^i(x_i|a)$. Hence, $l(\mathbf{x}|a)$ is Δ-isotone. The second part of Proposition 4 can now be invoked. ■

Jewitt (1988, Theorem 2) reports a special case of this corollary, with $n = 2$. In this case, the second condition requires ω to be increasing and concave, which is of course precisely Jewitt's one-signal condition. At first sight, Jewitt's result may seem peculiar because it combines Rogerson's and Jewitt's one-signal conditions. Indeed, Conlon (2009a) does not devote much attention to this result. However, he does supply the following generalization (with a proof in Conlon (2009b)), while attributing it to Jewitt.

Assume there are two signals, and that the likelihood ratio is decreasing and submodular in the two signals. Then, Conlon (2009b) proves the FOA is valid if $F_{aa}(\mathbf{x}|a) \geq 0$ (which he calls the lower quadrant convexity condition (LQCC)). Submodularity means that the cross-partial derivative is non-positive. Thus, with $n = 2$, $l(\mathbf{x}|a)$ is Δ-isotone, and so Proposition 4 in fact applies. Nevertheless, Conlon (2009a) concludes that "it is not clear how to extend this beyond the two-signal case." Note, however, that the submodular order and the lower orthant order coincide in the bivariate case. As Proposition 4 demonstrates, the latter is well suited for extensions to many signals.

Before proceeding to Jewitt's and Conlon's other results, it is worthwhile to comment on one aspect of the previous results. The stochastic orders underlying these results are all integral stochastic orders, as defined in Section 3. The set $\mathcal{F}$ is referred to as a generator of the stochastic order. For example, one generator for FOSD is the set of all increasing functions. Importantly, the integral stochastic orders invoked until now have well-defined "small" generators. In the case of univariate FOSD, this is the set of step-functions, which can be thought of as being at the "corner" of the set of increasing functions because any increasing function can be approximated by a combination of step-functions. The existence of a small generator is crucial in being able to obtain equivalent characterizations of an integral stochastic order. For instance, step-functions are used to prove the equivalence between the two definitions of univariate FOSD in Table 1. Unfortunately, not all integral stochastic orders have small generators. In particular, this problem arises when multi-variate SOSD is considered.

5.2 Multivariate SOSD and related stochastic orders

Among the ingredients in Jewitt's (1988, Theorem 3) second set of conditions and Conlon's (2009a, Proposition 2) extension thereof, are the assumptions that $l(\mathbf{x}|a)$ is increasing and concave in $\mathbf{x}$ and that ω is increasing and concave. These assumptions imply that $v(w(\mathbf{x}))$ is increasing and concave in $\mathbf{x}$. Naturally, this points in the direction of SOSD.

However, to close the proof, Jewitt and Conlon add conditions that on the surface appear different in nature from those in all previous results. In particular, they utilize the state-space formulation of the principal-agent model and assume that for each realization of the state, ϑ, each signal $x_i(a, \vartheta)$, is concave in a. The joint assumptions then ensure that the agent's problem is concave in a.

Conlon (2009a, p. 258) observes that "it is not immediately obvious how to express the condition, that $\mathbf{x}(a, \vartheta)$ is concave in a, using the Mirrlees notation [where everything is expressed in terms of $F(\mathbf{x}|a)$]." Indeed, Conlon (2009a, 2009b) goes to great lengths to illustrate these difficulties. However, the reason that it is not obvious how to translate the conditions into the Mirrlees notation is simple; it is impossible to do so. As Müller and Stoyan (2002, p. 98) succinctly put it, "there is no hope of finding a 'small' generator" for SOSD, and thus it is not possible to express SOSD with a set of conditions directly on $F(\mathbf{x}|a)$.[18]

There are, however, other stochastic orders that not only have a familiar flavor but that are also better suited for the Mirrlees formulation. Consider the following

[18] Another way to finish Jewitt's and Conlon's proofs would be to replace their assumption on $\mathbf{x}(a, \vartheta)$ with the (somewhat facetious) assumption that (6) holds true for all increasing and concave functions $v(w(\cdot))$ and all pairs (a, a^*), but that is hardly satisfying either.

orders, defined in Shaked and Shantikumar (2007):

1. *G dominates H in the lower orthant-concave order* if
$$\int_{\underline{x}_1}^{x_1} \cdots \int_{\underline{x}_n}^{x_n} G(y_1, ..., y_n) dy_n \cdots dy_1 \leq \int_{\underline{x}_1}^{x_1} \cdots \int_{\underline{x}_n}^{x_n} H(y_1, ..., y_n) dy_n \cdots dy_1 \text{ for all } \mathbf{x}.$$

2. *G dominates H in the upper orthant-convex order* if
$$\int_{x_1}^{\overline{x}_1} \cdots \int_{x_n}^{\overline{x}_n} \overline{G}(y_1, ..., y_n) dy_n \cdots dy_1 \geq \int_{x_1}^{\overline{x}_1} \cdots \int_{x_n}^{\overline{x}_n} \overline{H}(y_1, ..., y_n) dy_n \cdots dy_1 \text{ for all } \mathbf{x}.$$

Denuit and Mesfioui (2010) examine these and related stochastic orders. It can be shown that if G dominates H in the upper orthant-convex order then G is preferred to H for any utility function for which
$$\frac{\partial^{k_1+...+k_n} u(\mathbf{x})}{\partial x_1^{k_1} ... \partial x_n^{k_n}} \geq 0$$
for all $k_i \in \{0, 1, 2\}$, $i = 1, ..., n$, with $k_1 + ... + k_n \geq 1$. Similarly, if G dominates H in the lower orthant-concave order then G is preferred to H for any utility function for which $-u(-\mathbf{x})$ has the above property. In the univariate case, these orders obviously reduce to 2-icx and SOSD, respectively. The following proposition then follows from the usual logic.

Proposition 5 *Assume the second best action is in $(\underline{a}, \overline{a})$. Then, the FOA is valid if either:*

1. $\int_{\mathbf{y} \geq \mathbf{x}} \overline{F}_{aa}(\mathbf{y}|a) d\mathbf{y} \leq 0$ *for all* $\mathbf{x}$ *and all* a, ω *is* $2n$*-monotone, and*
$$\frac{\partial^{k_1+...+k_n} l(\mathbf{x}|a)}{\partial x_1^{k_1} ... \partial x_n^{k_n}} \geq 0$$
for all a *and for all* $k_i \in \{0, 1, 2\}$, $i = 1, ..., n$, *with* $k_1 + ... + k_n \geq 1$, *or*

2. $\int_{\mathbf{y} \leq \mathbf{x}} F_{aa}(\mathbf{y}|a) d\mathbf{y} \geq 0$ *for all* $\mathbf{x}$ *and all* a, ω *is* $2n$*-isotone, and*
$$\frac{\partial^{k_1+...+k_n} \left(-l(-\mathbf{x}|a)\right)}{\partial x_1^{k_1} ... \partial x_n^{k_n}} \geq 0$$
for all a *and for all* $k_i \in \{0, 1, 2\}$, $i = 1, ..., n$, *with* $k_1 + ... + k_n \geq 1$.

Jewitt's one-signal conditions imply the univariate function $v(w(x))$ has a negative second derivative. There are several ways in which this property can be extended into higher dimensions; requiring multivariate concavity is but one of them. Conlon's aim was precisely to include concavity in the sufficient conditions, but the Mirrlees formulation of the model was not up to the task. Thus, if the goal is sufficient conditions in Mirrlees notation then the most fruitful concept of "curvature" in the multi-signal model is not concavity. "Small" generators aside, to understand this result note that among the stochastic orders invoked in this paper, all but multivariate SOSD can be defined in terms only of the *sign* of certain derivatives. For multivariate SOSD, however, conditions must also be imposed upon the relative magnitude of various second derivatives; multivariate concavity is a messier concept. This is a significant difference, which on its own explains the difference in tractability.

Proposition 5 thus illustrates the price of escaping Conlon's conundrum. To recover sufficient conditions in the Mirrless notation, Conlon's implicit assumption about the relative magnitude of second derivatives must be replaced by conditions on the sign of higher-order derivatives.

A counterpart to Corollary 1 is also possible for the lower orthant-concave order. The proof is analogous to the proof of Corollary 1 and is thus omitted.

Corollary 2 *Assume there are $n \geq 2$ independent signals, with distribution functions $F^i(x_i|a)$ and likelihood ratio $l^i(x|a)$, $i = 1, 2, ..., n$. Assume the second best action is in $(\underline{a}, \overline{a})$. Then, the FOA is valid if*

1. *Each signal satisfies Jewitt's one-signal condition; $\int_{\underline{x}}^{x} F(y|a)dy \geq 0$, $l^i_x(x_i|a) \geq 0$, and $l^i_{xx}(x_i|a) \leq 0$ for all $i = 1, 2, ..., n$, and*

2. *ω is $2n$-isotone.*

Together, Corollary 1 and Corollary 2 offer an argument in favor of multi-signal conditions based on the orthant orders, like Propositions 4 and 5 in the current paper, over conditions based on the more demanding multivariate notions of FOSD and SOSD, like Jewitt's Theorem 3 or Conlon's (2009a) propositions. In practice, the orthant orders may also be easier to check. The second part of the corollaries captures the other side of the trade-off, namely that more conditions must be imposed on the underlying utility functions. However, the discussion following Proposition 2 reveals that this may be a small price to pay for a multi-signal extension.

Though it is not pursued here, there seems to be no conceptual obstacle to extending the result to higher orthant orders (i.e. imposing conditions on the antiderivatives of the antiderivatives, as in TOSD).

6 A modified first-order approach

As Mirrlees (1999) pointed out early on, the FOA is not always valid. For instance, it is possible that the FOA would identify a contract for which L-IC is not sufficient for G-IC. In this respect, note that the arguments in Section 3 can easily be modified to identify a subset of actions for which L-IC$_{a^*}$ implies G-IC$_{a^*}$ among e.g. monotonic or monotonic and concave contracts. For instance, in the one-signal case, if $F(x|a)$ coincides with its convex hull at a^* for all x then it is easily seen that any monotonic and L-IC$_{a^*}$ contract is G-IC$_{a^*}$.[19] Thus a "local" counterpart to Rogerson's condition is identified. Rogerson required that F is always convex in a, which is equivalent to requiring that F always coincides with its convex hull. For completeness, the following Lemma states a stronger version of this result.

Lemma 4 *Assume there is a single signal and that $F_a(x|a) < 0$ for all $x \in (\underline{x}, \overline{x})$ and all a.*[20] *Fix $a^* \in (\underline{a}, \overline{a})$. Then, any monotonic and L-IC$_{a^*}$ contract is G-IC$_{a^*}$ if and only if $F(x|a)$ coincides with its convex hull at a^* for all x.*

Proof. As mentioned above, the "if" part is trivial in light of the discussion in Section 3. For the other direction, assume there is some x such that $F(x|a)$ does not coincide with its convex hull at a^*. Note that such an x must necessarily be in $(\underline{x}, \overline{x})$. It suffices to find some monotonic and L-IC$_{a^*}$ contract that is not G-IC$_{a^*}$. Consider a step contract that delivers utility v_0 if the outcome is worse than x, and utility v_1 otherwise. The agent's expected utility is $EU(a) = v_1 + (v_0 - v_1)\, F(x|a)$ with $EU'(a^*) = (v_0 - v_1)\, F_a(x|a^*)$. Since $F_a(x|a^*) < 0$ and utility is assumed to be continuous and unbounded above and/or below, there exists a pair (v_0, v_1) that satisfies L-IC$_{a^*}$ and $v_1 > v_0$. However, because $F(x|a)$ does not coincide with its convex hull at a^* there is an alternative action that yields higher payoff for the agent. ■

It is of course possible to obtain similar local versions of the other results in this paper that can be characterized using the Mirrlees formulation.[21]

In an ambitious recent paper, Ke (2011a) notes that even when L-IC is not sufficient for G-IC for all actions, the FOA may nevertheless still identify the optimal

[19] Recall that the convex hull of a function $g(a)$ is the highest convex function that is always below $g(a)$; see Rockafellar (1970).

[20] The (strict) version of the MLRP implies $F_a(x|a) < 0$.

[21] Kadan and Swinkels (2012, Proposition 2) prove that any action can be implemented if there is a set of outcomes, S, such that the probability that $\mathbf{x} \in S$ is concave in the action (or equivalently that the complementary probability is convex). Their proof can be modified to establish that it is sufficient that there exists a set of outcomes, $S(a^*)$, for each action possible action, such that a^* is on the convex envelope of $\Pr(x \in S(a^*)|a)$.

contract. For instance, this occurs if the solution a^* lies in the set identified in the previous paragraph. It may also occur if L-IC$_{a^*}$ is not sufficient for G-IC$_{a^*}$ for all contracts but just happens to be sufficient with the specific contract identified by the FOA. Thus, Ke (2011a) proposes a fixed-point method designed to identify conditions under which the FOA produces the correct solution.

The purpose of the remainder of this section is to propose a modified FOA that works in one specific, but important, model in which the FOA is not generally valid and where none of the existing results solve the problem. I will also demonstrate that the modified FOA simplifies the analysis of some classic examples in the literature. As in the first part of the paper, the crucial step is to explore the link between L-IC and G-IC.

For notational simplicity, assume there is a single signal. However, the analysis does not rely on this assumption. The important assumption is that the distribution function can be written as

$$F(x|a) = p(a)G(x) + (1 - p(a))\,H(x), \tag{11}$$

where $p(a) \in [0,1]$ for all $a \in [\underline{a}, \overline{a}]$ and G and H are non-identical distribution functions with support $[\underline{x}, \overline{x}]$, and strictly positive densities $g(x)$ and $h(x)$, respectively. While this model is certainly too specialized to capture all principal-agent relationships, it should be stressed that it does have a compelling interpretation. For instance, $p(a)$ could be the proportion of time the parts-supplier (the agent) spends using the new and advanced technology G rather than the less reliable but more user-friendly old technology, H. Given such interpretations of the model, the most meaningful economic assumption is that $p(a)$ is monotonic. Thus, as is common in the literature, assume that $p'(a) > 0$ for all $a \in (\underline{a}, \overline{a}]$. The case where $p(a)$ is non-monotonic is not that much more difficult. It is discussed briefly later.

Distributions of this form have been studied extensively. Grossman and Hart (1983) say that the *spanning condition* is satisfied if $F(x|a)$ can be written as in (11). Since (11) is linear in p, Hart and Holmström (1987) refer to (11) as the Linear Distribution Function Condition (LDFC). The significance of the model and its place in the literature is discussed in detail after the formal analysis.

Typically, additional assumptions are imposed on the curvature of $p(a)$ as well as on the relationship between G and H. For instance, Sinclair-Desgagné (1994, 2009) points out that the FOA is valid if $p(a)$ is concave and $\frac{g(x)}{h(x)}$ is nondecreasing. The latter assumption implies the MLRP, while the former ensures concavity of the agent's objective function when he faces a monotonic contract. The second assumption also implies that G first order stochastically dominates H. Without assumptions on $p(a)$, Grossman and Hart (1983) prove that if $\frac{g(x)}{h(x)}$ is nondecreasing then any optimal contract must feature monotonic wages.[22] Ke (2011a, Proposition 7) shows that the FOA is valid if $p(a)$ is concave, even without the MLRP.

Here, I impose no such conditions on (11). For instance, $p(a)$ may be concave only locally, or not at all, and G and H may cross, as would be the case if H is a mean-preserving spread over G. No restrictions are placed on the shape of the contract either (apart from bounded utility).

Given the spanning condition, for any $a^* \in (\underline{a}, \overline{a})$, L-IC$_{a^*}$ is

$$p'(a^*) \int v(w(x)) \left(g(x) - h(x)\right) dx - 1 = 0. \tag{12}$$

Since $p'(a^*) > 0$, the integral must take the strictly positive value $\frac{1}{p'(a^*)}$ in order to satisfy (12). The agent's expected utility can be written

$$EU(a) = \frac{p(a)}{p'(a^*)} - a + \int v(w(x))h(x)dx,$$

should he take action a. It follows that

$$EU(a^*) - EU(a) = \frac{(-p(a)) - [-p(a^*) + (a - a^*)(-p'(a^*))]}{p'(a^*)} \tag{13}$$

for all $a \in [\underline{a}, \overline{a}]$. The term in the square brackets is the tangent line to $-p(a^*)$. Let A_p^C denote the set of actions in $(\underline{a}, \overline{a})$ for which $-p(a)$ coincides with its convex hull. By definition, $a^* \in A_p^C$ if and only if (13) is non-negative for any a.

Proposition 6 *Assume that $p'(a) > 0$ for all $a \in (\underline{a}, \overline{a}]$. Then, there exists a G-IC$_{a^*}$ contract (that yields bounded utility) if and only if $a^* \in A_p^C \cup \{\underline{a}, \overline{a}\}$.*

Proof. Assume $a^* \in (\underline{a}, \overline{a})$ and $a^* \notin A_p^C$. If there is a G-IC$_{a^*}$ contract, then that contract must necessarily be L-IC$_{a^*}$, and so (13) should apply. However, since $a^* \notin A_p^C$, there is some $a \in (\underline{a}, \overline{a})$ for which (13) is strictly negative, which contradicts G-IC$_{a^*}$.

For the other direction, assume $a^* \in A_p^C$. Since G and H are distinct, there is some $x \in (\underline{x}, \overline{x})$ for which $G(x) \neq H(x)$, or $F_a(x|a^*) \neq 0$. Now, as in the proof of Lemma 4, construct a step contract that satisfies L-IC$_{a^*}$ (contrary to Lemma 4,

[22] In their discrete model, Grossman and Hart (1983) allow multiple incentive compatibility constraints to bind. I will show, in the continuous model, that if a can be implemented then all but the local incentive compatibility constraint are redundant.

however, it is possible that $v_0 > v_1$). Since $a^* \in A_p^C$, (13) is everywhere non-negative. Hence, the contract is G-IC$_{a^*}$.

Now assume $a^* \in \{\underline{a}, \overline{a}\}$. By modifying the steps that led to (13), it is easy to see that a step contract that makes $EU'(\underline{a})$ sufficiently small or $EU'(\overline{a})$ sufficiently large is G-IC$_{\underline{a}}$ or G-IC$_{\overline{a}}$, respectively. ■

Thus, the spanning condition allows a succinct formulation of the "feasible set" of implementable actions.[23]. Moreover, it should be clear from the proof of Proposition 6 that L-IC$_{a^*}$ is in fact *necessary and sufficient* for G-IC$_{a^*}$, for any $a^* \in A_p^C$.

Proposition 7 *Assume that $p'(a) > 0$ for all $a \in (\underline{a}, \overline{a}]$. If $a^* \in A_p^C$ then L-IC$_{a^*}$ is necessary and sufficient for G-IC$_{a^*}$.*

Proof. Necessity is obvious. As in the proof of Proposition 6, sufficiency follows from the fact that (13) is everywhere positive if $a^* \in A_p^C$. ■

As a consequence of Propositions 6 and 7, a modified FOA suggests itself. In the first step, the feasible set is identified, $A_p^C \cup \{\underline{a}, \overline{a}\}$. The feasible set is closed (but not necessarily convex). In the second step, the FOA is applied to this set (i.e. with the constraint that $a \in A_p^C \cup \{\underline{a}, \overline{a}\}$). In a third step, the solution is compared to the payoff from optimally implementing $\underline{a}$ and $\overline{a}$. Whichever contract is superior is then chosen.

To find the optimal contract that implements $\underline{a}$ or $\overline{a}$, it turns out that the continuum of incentive compatibility constraints can again be summarized by one lone condition. For instance, consider implementing $\underline{a}$. Let $\underline{a}^c = \inf A_p^C$ if A_p^C is non-empty and let $\underline{a}^c = \overline{a}$ otherwise. If $\underline{a} = \underline{a}^c$, then $EU'(\underline{a}) \leq 0$ is sufficient for G-IC$_{\underline{a}}$. On the other hand, if $\underline{a} < \underline{a}^c$ then it can be shown that any contract that leaves no incentive for the agent to pick $\underline{a}^c$ over $\underline{a}$ is G-IC$_{\underline{a}}$. To implement $\overline{a}$, the relevant counterpart to $\underline{a}^c$ is $\overline{a}^c = \sup A_p^C$ when A_p^C is non-empty and $\overline{a}^c = \underline{a}$ otherwise.

Proposition 8 *Assume that $p'(a) > 0$ for all $a \in (\underline{a}, \overline{a}]$. Then, it is possible to implement the boundary actions, as follows:*

1. *If $\underline{a}^c = \underline{a}$ then $EU'(\underline{a}) \leq 0$ is necessary and sufficient for G-IC$_{\underline{a}}$. If $\underline{a}^c > \underline{a}$ then $EU(\underline{a}) \geq EU(\underline{a}^c)$ is necessary and sufficient for G-IC$_{\underline{a}}$.*

2. *If $\overline{a}^c = \overline{a}$ then $EU'(\overline{a}) \geq 0$ is necessary and sufficient for G-IC$_{\overline{a}}$. If $\overline{a}^c < \overline{a}$ then $EU(\overline{a}) \geq EU(\overline{a}^c)$ is necessary and sufficient for G-IC$_{\underline{a}}$.*

[23] Hermalin and Katz (1991) use tools from convex analysis to characterize the set of implementable actions in a model with a finite set of actions and a finite set of outcomes. Note that their analysis does not reveal when L-IC is sufficient for G-IC.

Proof. Necessity is obvious. For sufficiency in the first part of the proposition, consider first the "no-gap" case, $\underline{a}^c = \underline{a}$. Here, the slope of $-p(a)$ coincides with the slope of its convex hull at $\underline{a}$. As in the proof of Proposition 6, a modification of (13) then establishes that $EU'(\underline{a}) \leq 0$ is sufficient for G-IC$_{\underline{a}}$. However, this is not necessarily true in the "gap" case, where $\underline{a}^c > \underline{a}$. Note that

$$EU(\underline{a}) - EU(a) = (a - \underline{a}) \left[\frac{-p(a) - (-p(\underline{a}))}{a - \underline{a}} \int v(w(x)) \left(g(x) - h(x) \right) dx + 1 \right],$$

and so $EU(\underline{a}) \geq EU(\underline{a}^c)$ implies that the term in brackets must be non-negative when $a = \underline{a}^c$. If the integral is negative, then the term in brackets is positive for all a, or $EU(\underline{a}) \geq EU(a)$ for all a. That is, the contract is G-IC$_{\underline{a}}$. If the integral is positive, then the term in brackets is *minimized* at $a = \underline{a}^c$. This follows by definition of the convex hull, since the line from $(\underline{a}, -p(\underline{a}))$ to $(\underline{a}^c, -p(\underline{a}^c))$ is steeper than the line from $(\underline{a}, -p(\underline{a}))$ to any other point on $-p(\cdot)$. Hence, if $EU(\underline{a}) \geq EU(\underline{a}^c)$ then $EU(\underline{a}) \geq EU(a)$ for all $a \in [\underline{a}, \overline{a}]$, thus implying G-IC$_{\underline{a}}$. The proof of the second part of the proposition is analogous. ■

The assumption that $p(a)$ is monotonic seems justified on economic grounds. However, it is possible to allow $p(a)$ to be non-monotonic. First, note that the argument following (13) remains valid if $p'(a^*) > 0$ even if $p'(a^{**}) < 0$ for some $a^{**} \neq a^*$. That is, a^* can be implemented, and L-IC$_{a^*}$ is sufficient, if and only if a^* is on the convex hull of $-p(a^*)$. By similar reasoning, $a^{**} \in (\underline{a}, \overline{a})$ can be implemented, and L-IC$_{a^{**}}$ is sufficient, if and only if a^{**} is on the convex hull of $p(a^{**})$.[24] Thus, the set of implementable interior actions can be obtained by piecing together the sets of implementable actions with $p'(\cdot) > 0$ and $p'(\cdot) < 0$, respectively. Of course, if $a^* \in (\underline{a}, \overline{a})$ and $p'(a^*) = 0$ then no L-IC$_{a^*}$ contract exist (with bounded utility), as can be seen from (12). Similarly, if $p(a^*) = p(a')$, then a^* cannot be implemented if $a' < a^*$ because it would be cheaper for the agent to pick a' rather than a^*.[25] Note that such actions cannot be on the convex hull of either $-p(a)$ or $p(a)$ when $p'(a^*) \neq 0$.

The remainder of this section is devoted to demonstrating the significance of the spanning condition as well as illustrating some uses of the preceding characterization.

First, it is useful to recognize that the textbook case in which there are two outcomes (but a continuum of actions) is in fact a special case of (11). Specifically, this model corresponds to assuming that G and H are degenerate distributions, with

[24] This is easily seen by multiplying both numerator and denominator in (13) by -1.

[25] Consequently, once $p(a)$ is allowed to be non-monotonic, it is no longer necessarily the case that $\overline{a}$ can be implemented.

all mass concentrated at opposite ends of the support. Hence, Propositions 6 – 8 make it possible to reexamine some important examples in the literature.

Example 1 (Araujo and Moreira (2001)): Araujo and Moreira (2001) propose a general Lagrangian approach to the moral hazard problem that applies when the FOA is not valid. Their leading example is the following. There are two states, where state 1 is the bad state and state 2 is the good state. The agent picks an effort level, e, from $[\underline{e}, \overline{e}] \subseteq [0, 1]$. With effort e, the probability of the good state is $q(e) = e^3$. The cost of effort is $c(e) = e^2$. To reparameterize the model, let $a \equiv c(e) = e^2$ and $p(a) = q(c^{-1}(a)) = a^{\frac{3}{2}}$, $a \in [\underline{a}, \overline{a}] = [\underline{e}^2, \overline{e}^2]$. Note that $p(a)$ is increasing and convex. Thus, $-p(a)$ is concave and so A_p^C is empty. In other words, no interior action can be implemented. Moreover, the boundary actions can be implemented, and the only relevant incentive compatibility constraint is that the desired action be preferable to the action on the opposite end of the support. Consequently, this example essentially reduces to the textbook example with two outcomes and two actions, $\underline{a}$ and $\overline{a}$, and is therefore trivial to solve once a participation constraint is added. In contrast, to use their general approach to solve the example, Araujo and Moreira (2001) (having added assumptions on $v(w)$ and on the principal's payoff) construct an algorithm in Mathematica and use this to solve 20 non-linear systems of equations. As expected, they find the optimal action is at a corner. While their method is obviously powerful, using it on their leading example is overkill (not to mention labor intensive) and obscures the intuition. Ke (2011b) proposes another method to solve this problem. Though his method is simpler than that used by Araujo and Moreira (2001), it remains more complicated than the method suggested above. ▲

Mirrlees (1999) offers a famous example to illustrate how the FOA may fail. In their textbook, Bolton and Dewatripont (2005, p. 148) remark that: "This example is admittedly abstract, but this is the only one to our knowledge that addresses the technical issue." Next, I will show that Mirrlees' (1999) example can be analyzed using the techniques presented earlier in this section. In particular, the modified FOA correctly solves the problem. Thereafter, in the hope it will have some pedagogical value, I will provide a more straightforward example of how the FOA may fail. Again, the modified FOA allows the correct solution to be obtained.

Example 2 (Mirrlees (1999)): Consider an agent with payoff function

$$U(w, z) = we^{-(z+1)^2} - \left(-e^{-(z-1)^2}\right).$$

It may be helpful to think of this example as a special environment with two outcomes, where, for some reason, the wage in one state is exogenously fixed at 0. The principal controls the "bonus" w (which may be positive or negative) if the other state materializes. The agent's action is $z \in \mathbb{R}$. Think of $e^{-(z+1)^2}$ roughly as the probability of the state in which a bonus is paid out, and think of $-e^{-(z-1)^2}$ as the cost function. This example fits rather well with the model in (11). In particular, with the spanning condition and only two states, the agent's expected utility is separable in the action and the difference between utility in the two states (the bonus). Next, let $a = -e^{-(z-1)^2}$, and note that $a \in [-1, 0)$. Think of the agent as having a two-dimensional problem. First, he has to decide which cost level, a, to incur, and, second, whether to incur this cost with a z that is above or below 1 (since $z_- = 1 - \sqrt{-\ln(-a)}$ and $z_+ = 1 + \sqrt{-\ln(-a)}$ both yield the same a). Depending on whether $z < 1$ or $z > 1$, expected utility can be written as $V^-(w, a) = wp^-(a) - a$ or $V^+(w, a) = wp^+(a) - a$, respectively, where

$$p^-(a) = e^{-\left(2-\sqrt{-\ln(-a)}\right)^2}, \text{ and } p^+(a) = e^{-\left(2+\sqrt{-\ln(-a)}\right)^2}, \ a \in [-1, 0).$$

Clearly, $p^-(a) > p^+(a)$. Hence, $V^-(w, a) > V^+(w, a)$ if and only if w is strictly positive. It is now possible to split the problem into two entirely conventional problems. In one, the principal is constrained to $w \leq 0$ and the agent's payoff function is effectively $V^+(w, a)$. In the other, the constraint is $w \geq 0$ and the agent's payoff function is $V^-(w, a)$.

For the first problem, it can be shown that $p^+(a)$ is decreasing. Hence, negative wages are indeed necessary for L-IC. Moreover, $p^+(a)$ is convex and so coincides with its convex envelope. It follows from Propositions 6 and 7 that any interior action can be implemented and that the FOA is valid.

The second problem is more interesting. Here, $p^-(a)$ is increasing on $[-1, -e^{-4})$, and decreasing on $(-e^{-4}, 0)$. Since only non-negative wages can be used, there is no permissible contract that satisfies L-IC for any $a \geq -e^{-4} \approx -0.0183$. On the remaining support, $-p^-(a)$ coincides with its convex envelope if and only if $a \in [-1, -0.9982] \cup [-0.0217, -e^{-4})$. By Propositions 6 and 7, the modified FOA is valid on this set (and actions in $(-0.9982, -0.0217)$ cannot be implemented).

Next, Mirrlees specifies an objective function for the principal. There is no participation constraint. The principal seeks to maximize $-(z-1)^2 - (w-2)^2$ or, equivalently, $\ln(-a) - (w-2)^2$. The agent's first order condition yields $w = 1/p^{-\prime}(a)$ and $w = 1/p^{+\prime}(a)$, respectively. Substituting this into the principal's objective function and plotting the resulting functions reveals that positive bonuses are superior to negative bonuses and that the solution is at a corner of the feasible set, specifically at

$w = 1$ and $a = -0.9982$ (or $z_- = 0.957$). This of course coincides with the solution Mirrlees found, but not with the solution one would obtain from the standard FOA (which yields $a = -0.988\,97$ or $z_- = 0.895$, as demonstrated by Mirrlees). ▲

EXAMPLE 3 (SIMPLIFIED COUNTEREXAMPLE): There are two outcomes. Let v_1 be the agent's utility (from wages) if the outcome is bad and v_2 be his utility if the outcome is good. The outcomes are worth x_1 and x_2 to the principal, respectively. The probability of the good outcome is $p(a)$, with $p'(a) > 0$. The participation constraint and L-IC constraint yield the system

$$\begin{aligned} v_1 + p(a)(v_2 - v_1) - a &= \overline{u} \\ p'(a)(v_2 - v_1) - 1 &= 0 \end{aligned}$$

with solution

$$v_1 = \overline{u} + a - \frac{p(a)}{p'(a)},\ v_2 = \overline{u} + a + \frac{1 - p(a)}{p'(a)}.$$

If L-IC is sufficient, the risk-neutral principal's expected payoff is

$$\pi(a) = (1 - p(a))(x_1 - v^{-1}(v_1)) - p(a)(x_2 - v^{-1}(v_2)).$$

Assume $p(a) = a + \frac{1}{2}(a^2 - a^3)$, $a \in [0, \overline{a}]$, $\overline{a} \in (\frac{1}{2}, 1]$. Note that if $a = 0$ then $v_1 = v_2 = \overline{u} + a$, so in this special case, with $p(0) = 0$, $\pi(0)$ also describes the optimal way of implementing the lowest action.[26] Here, $p(a)$ is convex when $a < \frac{1}{3}$ and concave when $a > \frac{1}{3}$. However, the relevant set is A_p^C, which is $A_p^C = [\frac{1}{2}, \overline{a})$. Thus, the set of implementable actions is $\{0\} \cup [\frac{1}{2}, \overline{a}]$, with $\underline{a}^c = \frac{1}{2} > \underline{a}$ and $\overline{a}^c = \overline{a}$. Assume $\overline{u} = 2$, $v(w) = \sqrt{w}$ and $x_1 = 5$, $x_2 = 9.4$. Figure 2 plots $\pi(a)$ when $\overline{a} = \frac{2}{3}$.

There are two stationary points. The first, at $a^* = 0.114$, minimizes the principal's payoff and is not even implementable because the agent's payoff is convex whenever $a < \frac{1}{3}$. The second stationary point, at $a^{**} = 0.464$, is the global maximum of $\pi(a)$. However, a^{**} is not implementable either. Though the agent's payoff is locally concave at a^{**}, it is profitable for the agent to deviate to $\underline{a} = 0$. One way to see this is that $v_2 > v_1 > 2 = \overline{u}$ whenever a (futile) attempt is made at implementing an action in $\left(0, \frac{1}{2}\right)$. Given the feasible set is $\{0\} \cup [\frac{1}{2}, \overline{a}]$, it is clear from the figure that the optimal action to induce is $a = \frac{1}{2}$ (with $v_1 = 2$, $v_2 = \frac{26}{9}$). ▲

[26] In general, the cost of implementing a given action is discontinuous at $\underline{a}$. The highest action, $a = \overline{a}$, can be implemented with any contract for which $EU'(\overline{a}) \geq 0$. However, it is easy to see that any contract with $EU'(\overline{a}) > 0$ cannot be optimal. The reason is that such a contract unnecessarily imposes more risk on the agent ($v_2 - v_1$ is larger).

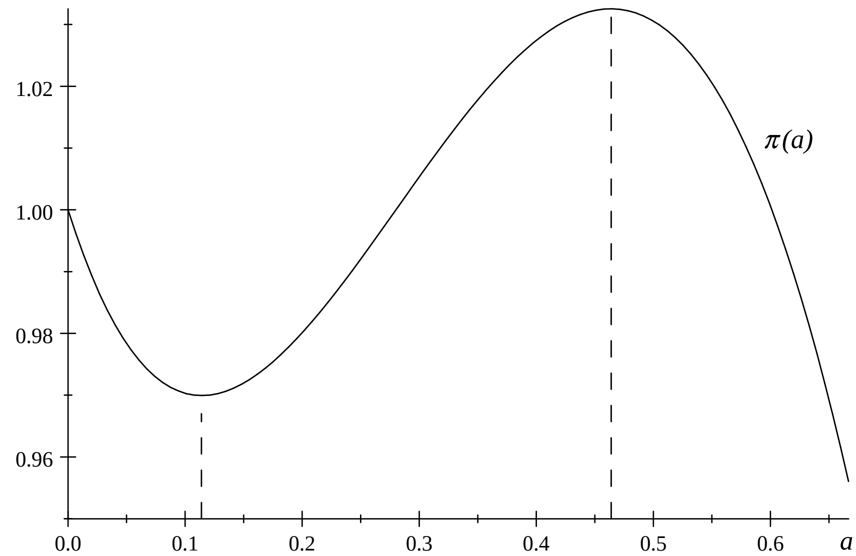

Figure 2: Simplified counterexample.

The spanning condition has often been implicitly imposed in papers with a continuum of actions. Perhaps the most significant example of this is in LiCalzi and Spaeter (2003) who provide two classes of distributions for which Rogerson's conditions are satisfied. Thus, this paper is customarily cited in papers that rely on the FOA. The first family of distributions is

$$F(x|a) = x + \beta(x)\gamma(a), \;\; x \in [0,1]. \tag{14}$$

Obviously, conditions must be imposed on $\beta(\cdot)$ and $\gamma(\cdot)$ to ensure that $F(x|a)$ is a proper distribution function. LiCalzi and Spaeter (2003) identify additional assumptions on both $\beta(\cdot)$ and $\gamma(\cdot)$ which ensure $F_{aa}(x|a) \geq 0$ and the MLRP. Note, however, that these distribution functions are separable in x and a. Thus, although it seems to not have been observed before, it should be clear that $F(x|a)$ could be stated as in (11). Hence, the modified FOA is always valid in this family of distributions, even without LiCalzi and Spaeter's (2003) additional assumptions.[27]

[27] Ke (2011a, Proposition 9) prove that LiCalzi and Spaeter's (2003) additional assumptions on $\beta(x)$ are not necessary for the validity of the FOA. Note the overlap between his Propositions 7 and 9.

Example 1 in Jewitt et al (2008) uses the Farlie-Gumbel-Morgenstern copula ($f(x|a) = 1 + \frac{1}{2}(1-2x)(1-2a)$, $x, a \in [0,1]$). This distribution is also separable in x and a and thus can be written as in (11). Finally, example 1 in Kadan and Swinkels (2012) can be written as

$$F(x|a) = p(a)x + (1 - p(a)) \left(x + x^2 - x^3\right),$$

where $p(a) = 2a^2 - a^3$, $a \in [\frac{2}{3}, 1]$, and $x \in [0,1]$

7 Conclusion

In this paper, a new approach to the moral hazard problem has been suggested. The approach is based on reformulating the problem in terms familiar to any economist. In particular, standard results from the theory of choice under uncertainty can be invoked to prove new and old results.

The new approach permits a unified proof of Rogerson's (1985) and Jewitt's (1988) one-signal justifications of the FOA. Indeed, the insights gained from reformulating the problem makes it possible to derive other sufficient conditions. Similarly, in the multi-signal model, the justifications provided by Jewitt (1988) and Conlon (2009a) can be explained with a common methodology. It is important to note that there are several different ways in which one-signal results can be extended into higher dimensions. The orthant orders form the basis of several tractable alternatives. One distinct advantage of the justifications based on the lower orthant order and the lower orthant-concave order is that they are robust to the inclusion of more independent signals.

In the second part of the paper, a more specific model was considered. Though the spanning condition looks simple and dates back to Grossman and Hart (1983), the first full characterization of its solution is given here. Mirrlees' (1999) famous counterexample can also be solved used the techniques presented here. As in the first part of the paper, the key step in the analysis is to exploit the information contained in the local incentive compatibility constraint.

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Rogerson	Jewitt	Third set of conditions
$F_{aa}(x\|a) \geq 0, \; \forall x, a$ $\Downarrow$ Any nondecreasing and L-IC contract is G-IC	$\int_{\underline{x}}^{x} F_{aa}(y\|a)dy \geq 0, \; \forall x, a$ $\Downarrow$ Any nondecreasing, concave, and L-IC contract is G-IC	$\int_{\underline{x}}^{x} \int_{\underline{x}}^{z} F_{aa}(y\|a)dydz \geq 0, \; \forall x, a$ and $\int_{\underline{x}}^{\overline{x}} F_{aa}(y\|a)dy \geq 0, \; \forall a$ $\Downarrow$ Any nondecreasing, concave, positively skewed, and L-IC contract is G-IC
FOSD	SOSD	TOSD
$G(x) \leq H(x), \; \forall x$ $\Updownarrow$ $E_G\left[u(x)\right] \geq E_H\left[u(x)\right]$ for any nondecreasing $u(x)$	$\int_{\underline{x}}^{x} G(y)dy \leq \int_{\underline{x}}^{x} H(y)dy, \; \forall x$ $\Updownarrow$ $E_G\left[u(x)\right] \geq E_H\left[u(x)\right]$ for any nondecreasing and concave $u(x)$	$\int_{\underline{x}}^{x} \int_{\underline{x}}^{z} G(y)dydz \leq \int_{\underline{x}}^{x} \int_{\underline{x}}^{z} H(y)dydz, \; \forall x$ and $\int_{\underline{x}}^{\overline{x}} G(y)dy \leq \int_{\underline{x}}^{x} H(y)dy$ $\Updownarrow$ $E_G\left[u(x)\right] \geq E_H\left[u(x)\right]$ for any nondecreasing, concave, and positively skewed $u(x)$.

Table 1: Rogerson, Jewitt and stochastic dominance.

Note: $F(\cdot|a)$ is the distribution over outcomes given action a. In the third column, a positively skewed utility function, $u(x)$, is one for which $u'(x)$ is non-negative, decreasing, and convex.

Rogerson	Jewitt	Third set of conditions
$F_{aa}(x\|a) \geq 0, \; \forall x, a.$	$\int_{\underline{x}}^{x} F_{aa}(y\|a)dy \geq 0, \; \forall x, a.$	$\int_{\underline{x}}^{x} \int_{\underline{x}}^{z} F_{aa}(y\|a)dydz \geq 0, \; \forall x, a$ and $\int_{\underline{x}}^{\overline{x}} F_{aa}(y\|a)dy \geq 0, \; \forall a.$
$\omega'(z) > 0$ $l_x(x\|a) \geq 0$	$\omega'(z) > 0, \; \omega''(z) \leq 0$ $l_x(x\|a) \geq 0, \; l_{xx}(x\|a) \leq 0$	$\omega'(z) > 0, \; \omega''(z) \leq 0, \; \omega'''(z) \geq 0$ $l_x(x\|a) \geq 0, \; l_{xx}(x\|a) \leq 0, \; l_{xxx}(x\|a) \geq 0$

Table 2: Justifying the first order approach, Part I.

Rogerson (1-icx)	2-icx	3-icx
$\overline{F}_{aa}(x\|a) \leq 0, \; \forall x, a.$	$\int_{x}^{\overline{x}} \overline{F}_{aa}(y\|a)dy \leq 0, \; \forall x, a.$	$\int_{x}^{\overline{x}} \int_{z}^{\overline{x}} \overline{F}_{aa}(y\|a)dydz \leq 0, \; \forall x, a$ and $\int_{\underline{x}}^{\overline{x}} \overline{F}_{aa}(y\|a)dy \leq 0, \; \forall a.$
$\omega'(z) > 0$ $l_x(x\|a) \geq 0$	$\omega'(z) > 0, \; \omega''(z) \geq 0$ $l_x(x\|a) \geq 0, \; l_{xx}(x\|a) \geq 0$	$\omega'(z) > 0, \; \omega''(z) \geq 0, \; \omega'''(z) \geq 0$ $l_x(x\|a) \geq 0, \; l_{xx}(x\|a) \geq 0, \; l_{xxx}(x\|a) \geq 0$

Table 3: Justifying the first order approach, Part II.