Optimal Mechanism Design without Money

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Abstract

We consider the standard mechanism design environment with linear utility but without monetary transfers. We first establish an equivalence between deterministic, dominant strategy incentive compatible mechanisms and generalized median voter schemes. We then use this equivalence to construct the constrained-efficient optimal mechanism for an utilitarian planner.

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1 Introduction

We study dominant strategy incentive compatible (DIC) and deterministic mechanisms[1] in a social choice setting where agents are privately informed and have linear utility functions over several alternatives, but where monetary transfers are not feasible. The absence of monetary transfers weakens the implication of Pareto efficiency: with monetary transfers, Pareto efficiency requires that the allocation rule maximize the sum of the agents' expected utilities; without monetary transfers, the set of Pareto efficient allocation is much bigger and therefore, in practice, one needs to choose among efficient mechanisms based on additional criteria.

Although deterministic, DIC mechanisms are described here by a function of several continuous variables satisfying complex constraints, our first main result shows how the problem of finding optimal mechanisms - that maximize some given social welfare functional that may depend on preference intensities - reduces to the problem of finding K non-negative constants adding up to $n-1$, where K is the number of alternatives and n is the number of agents. Our second main result focuses on social welfare maximization and offers, under standard assumptions on the distribution of types, precise formulae for these constants.

The present analysis combines insights from two important strands of the literature:

1. On the one hand, the private values model with quasi-linear utility and monetary transfers serves as the workhorse of a very large body of literature that focuses on trading mechanisms for the provision of public or private goods. Classical results in this literature include, for example, the characterization of value-maximizing mechanisms due to Vickrey-Clarke-Groves, and the characterization of revenue-maximizing auctions due to Myerson [1981]. Cardinal preference intensities are inherent in that underlying model, and play a main role in the formulation of both implementability and optimality results. In addition, monetary transfers are key to controlling the agents' incentives, and can be finely tuned to match the values obtained from physical allocations.

2. On the other hand, a distinct, very large body of work in the realm of social choice has focused on the implementation of desirable social choice rules in abstract frameworks with purely ordinal preferences, and without monetary transfers. Classical results include the Gibbard-Satterthwaite *Impossibility Theorem* (Gibbard [1973] and Satterthwaite [1975]) and the *Median Voter Theorem* for settings with single-peaked preferences (see Black [1948]). When a Pareto-efficient rule, say, is not implementable in a certain framework, that literature often remains silent about how to choose among implementable schemes because preference intensities are not part of the model, and because other goals are not easily formulated within it. For similar reasons, when multiple Pareto-efficient rules are implementable, this literature cannot meaningfully rank them.

[1] Some authors use the term "strategy proof" mechanisms.

While using the standard, cardinal, linear model of utilities, we assume that monetary transfers are not feasible. In particular, choice rules resembling the voting schemes often analyzed in the social choice literature come here to the forefront instead of, say, some kind of trading mechanisms.

Having private types and linear values allows us both the formulation of optimality criteria that involve preference intensities and the use of powerful characterization results about dominant strategy incentive compatibility that, basically, resemble those from the literature on trading mechanisms with money. A main difference is that the lack of monetary transfers puts restrictions on implementable mechanisms that do not easily reduce to some monotonicity condition. With one-dimensional private information and **with monetary transfers**, the DIC requirement translates into a monotonicity condition on the choice rule, and an envelope (or integrability) condition on equilibrium utility. Since it is always possible to augment a monotone rule with a transfer such that the envelope condition holds (this fact is behind the celebrated "payoff equivalence" result) monotonicity becomes the only relevant requirement. Although the characterization of DIC mechanism here is similar (see Lemma 1) **the lack of monetary transfers** means that not all monotone choice rules are implementable. For example, the welfare-maximizing rule in our framework is not implementable although it is monotone. Thus, the envelope condition is crucial, and its implications become rather subtle as soon as the number of social choice alternatives is strictly larger than two. In a sense, this is similar to the problems encountered in multi-dimensional mechanism design where it is also the case that not every monotone choice rule is integrable (see for example the exposition in Jehiel, Moldovanu and Stacchetti [1999]).

Another set of ideas comes from the second strand mentioned above: our linear framework yields a model with single-peaked preferences and therefore we can adapt and strengthen an elegant "converse" to the *Median Voter Theorem* due to Moulin [1980]. Roughly speaking, Moulin's result says that all anonymous DIC mechanisms that only depend on the agents' top alternatives (or peaks) can be described as schemes that choose the median among the n "real" peaks of actual voters and an additional, fixed number of "phantom" voters' peaks.[2]

Our first main result characterizes the set of DIC, Pareto efficient and anonymous mechanisms as median voter schemes. Importantly, we are here able to remove Moulin's crucial assumption that the allowed mechanisms only depend on peaks, while obtaining a result in the same spirit as his. Thus, no efficient, DIC mechanism can depend on preference intensities among alternatives, nor on the ranking of alternatives below the top. But the planner can choose a mechanism that optimizes a welfare functional that depends on all these features, on the distribution of types, etc... All the planner has to do is to appropriately determine K constants adding up to $n-1$, where each constant specifies the number of phantom peaks on a respective alternative.[3] For example, when there are only two alternatives, say a *status quo* and a *reform*, locating $m,\ 0 \leq m \leq n-1$, phantom peaks on the reform and $n-1-m$

[2] The precise result requires several technical conditions such as full domain, etc...

[3] The number of possible partitions can be calculated from the so called *Stirling numbers of the second kind.*

phantom peaks on the status-quo yields a choice rule where the reform is implemented if at least $n-m$ real voters are in its favor. The optimal m in the above case, and, more generally, the optimal numbers of phantom peaks on each of alternative, depend on the parameters of the social choice situation such as the slopes and intercepts of the utility functions or the distribution of real agents' types.

Our second main result focuses on maximization of social welfare under several standard assumptions on the distribution of types. It offers simple and intuitive formulae for the optimal number of phantom peaks that need to be placed on each alternative. The formulae are obtained by observing that shifting phantom peaks among adjacent alternatives has an effect only in cases where the median peak shifts as well. Just to give one simple example, if the real agents' types are independently and uniformly distributed, the number of phantom voters' peaks on alternative k is shown to be proportional to the share of real types whose top alternative is k. It is also interesting to note that, although the first-best mechanism is not implementable in our setting, the second-best (constrained efficient) mechanism obtained here approximates the first-best if the population is large. Optimal schemes for other criteria such as, say, a Rawlsian maximin, or maximax can be analogously obtained.

The paper is organized as follows: In Section 2 we describe the social choice model. In Section 3 we characterize DIC mechanisms via a monotonicity and an integrability condition. We also show that, in a DIC mechanism, two agents with the same ordinal preferences must be treated in the same way, although they may have different "types" that yield different cardinal preferences/intensities. In Section 4 we first show (Theorem 1) that, within the class of DIC and onto mechanisms (i.e., where every alternative is chosen at some profile) it is enough to restrict attention to mechanisms that only depend on reported peaks. In other words, within this class of mechanisms, the precise preference relation below the peak cannot matter for the choice rule. Theorem 2, our first main result, demonstrates that all Pareto-efficient and DIC mechanisms are medians of n real peaks and $n-1$ phantom ones. The proof of Theorem 2 is involved and, besides the result of Theorem 1, it uses several Lemmas and a result (Proposition 1 in the Appendix) that is an adaptation of Moulin's characterization of peaks-only mechanisms to our model. A new proof for Moulin's result is required here because his full-domain assumption is not satisfied in our linear model, i.e., not all possible single-peaked ordinal preferences arise for a given set of parameters of the utility functions. In Section 5 we use the above characterization result to precisely derive the formulae governing the location of phantom peaks in the social welfare maximizing (second-best) mechanism under standard assumptions about the distribution of types (Theorem 3, our second main result). We also discuss several intuitive implications of the resulting formulae. Extensions to nonlinear utilities and alternative welfare criteria are discussed in Section 6. All proofs are in the Appendix.

1.1 Related Literature

The idea of comparing voting rules in terms of the ex-ante expected utility they generate goes back to Rae [1969]. This paper and almost the entire following literature focus on settings with two social alternatives (a reform and a status quo, say) where a mechanism can be described by a single function, the probability that the reform is chosen given the agents' reports about their types. In this special case, the DIC constraint implies that deterministic mechanisms are, for any profile of others' reports, described by a step function with a unique jump. As a consequence of this simple structure, anonymous and constrained-efficient mechanisms can be represented by qualified majority rules where the reform is chosen if at least a certain number of agents votes in its favor. Schmitz and Tröger [2012] identify qualified majority rules as ex-ante welfare maximizing in the class of DIC mechanisms - as explained above this can be seen as an implication of our main result.[4] Azrieli and Kim [2011] nicely complement this analysis for two alternatives by showing that any interim Pareto efficient, Bayesian incentive compatible (BIC) choice rule must be a qualified majority rule.[5] The situation dramatically changes when there are three, or more alternatives: the DIC/BIC constraints and the mechanisms themselves are much more complex, and not much is known about them. Apesteguia, Ballester and Ferrer [2011] consider a social choice model where agents have cardinal utility and multidimensional "types", and evaluate mechanisms in terms of the ex-ante expected utility they generate.[6] But, their analysis completely abstracts from incentives constraints, i.e., strategic voting is not considered, and the scoring rules that emerge as optimal in their analysis are subject to strategic manipulation. Borgers and Postl [2009] study a setting with three alternatives: in their model it is common knowledge that the top alternative for one agent is the bottom for the other, and vice-versa. The agents also differ in the relative intensity of their preference for a middle alternative (the *compromise*) when compared to the top and bottom one, respectively. This intensity is private information. Besides a characterization in terms of monotonicity and envelope condition, Borgers and Postl mainly conduct numerical simulations and show that the efficiency loss from second-best rules is often small.

In a principal-agent model with quadratic utility functions, hidden information but without transfers, Kovac and Mylovanov [2009] find that the optimal mechanism is deterministic. Motivated by computer science applications, Hartline and Roughgarden [2008] study how the system designer can use service degradation (money burning) to align the private users' interests with the social objective.[7] Chakravarty and Kaplan [2013] and Condorelli [2012] analyze optimal allocation problems in private value environments without monetary transfers.

[4] These authors also perform an analysis for Bayesian mechanisms, which is not covered by our study.

[5] Again in a setting with two alternatives, Barbera and Jackson [2006] take the qualified majority rule as given, and derive the optimal weight that maximizes the total expected utilities of all agents.

[6] They also consider other goals such as maxmin, etc...

[7] See also McLean and Postlewaite [2002] who study Bayesian incentive compatibility in settings where monetary transfers are limited.

In their models agents can send costly and socially wasteful signals (these may be payments to outsiders). In contrast, Drexel and Klein [2013] allow the redistribution of the collected monetary payments among the agents. They confine attention to settings with two social alternatives and show that a principal who wishes to maximize the agents' welfare (i.e., welfare from the physical allocation minus potential transfers to outsiders) will use a mechanism that does not involve any monetary transfers! Thus, it must be the case that, for settings with two alternatives, their optimal mechanism coincides with the one derived in this paper, where monetary transfers are a priori ruled out.

A quite different line of study is pursued by Jackson and Sonnenschein [2007] who consider the linkage of many distinct social problems. Even if no monetary transfers within one problem are possible, the linkage with other decisions creates the possibility of fine-tuning incentives, which acts as having some "pseudo-transfers". Efficiency can be attained then in the limit, where the number of considered problems grows without bound.

As already mentioned above, the seminal paper in the social choice literature closely related to the present research is Moulin [1980]. Several authors have extended Moulin's characterization in terms of median choices and phantom voters by discarding the assumption that mechanisms can only depend on peaks.[8] Almost all these papers assume continuous spaces of alternatives, (continuous) ordinal preferences, and domain-richness assumptions on preferences, none of which are satisfied in our framework. Excellent examples in this strand are Barbera and Jackson [1994], Sprumont [1991], Ching [1997] and Schummer and Vohra [2007].[9] An exception is Chatterjee and Sen [2011] who do consider discrete domains with a finite number of alternatives. They establish a peaks-only result under a rather restrictive condition on the domain of preferences - unfortunately, their condition is not satisfied in our model as soon as there are at least 4 alternatives, and thus we cannot use here their analysis.

2 The Social Choice Model

We consider n agents who have to choose one out of K mutually exclusive alternatives. Let $\mathcal{K} = \{1, ..., K\}$ denote the set of alternatives. Agent $i \in \{1, ..., n\}$ has utility $u(k, x_i)$, where $k \in \mathcal{K}$ is the chosen alternative and x_i is a parameter (or type) privately known to agent i only. We assume that[10]

$$u(k, x_i) = a_k + b_k x_i.$$

[8] See Sprumont [1995] for an excellent survey. Recently, Ehlers, Peters and Storcken [2002] extend Moulin's characterization to probabilistic strategy-proof rules, and Nehring and Puppe [2007] extend it to a class of generalized single-peaked preference domains based on abstract betweenness relations.

[9] Schummer and Vohra [2002] and Dokow et al. [2012] study location choice on graphs, and also establish the equivalence between strategy-proof rules and generalized median voter schemes. In both models, however, agents' preferences are quadratic and thus parameterized solely by their peaks. Hence, they can directly focus on peaks-only mechanisms.

[10] We discuss nonlinear utility functions in Section 6.

The types $x_1, ..., x_n$ are distributed on the interval $[0,1]^n$ according to a joint cumulative distribution function Φ with density ϕ.[11] Each agent knows only his own type x_i. We assume that $b_k \geq 0$ for all $k \in \mathcal{K}$ and $b_k \neq b_l$ for all $l \neq k$. Without loss (by renaming alternatives if necessary), we assume that $b_K > b_{K-1} > ... > b_1 \geq 0$.

Note that we use here the one-dimensional, private values, linear utility specification – the most common one in the vast literature on optimal mechanism design with monetary transfers that followed Myerson's [1981] seminal contribution. But we assume that monetary transfers are not feasible in our framework.

The social planner's general goal is to reach, for any realization of types, a Pareto efficient allocation. We will primarily focus on the case of a utilitarian planner whose objective is to maximize the sum of the agents' expected utilities

$$\max_{k \in \mathcal{K}} E\left[\sum\nolimits_i u\left(k, x_i\right)\right].$$

Given any two different alternatives k and l with $b_k > b_l$, agent i is indifferent between them if and only if his type is

$$x^{l,k} \equiv \frac{a_l - a_k}{b_k - b_l}. \tag{1}$$

Types above $x^{l,k}$ prefer alternative k to l, while types below $x^{l,k}$ prefer alternative l to k. Denote by

$$x^k \equiv x^{k-1,k} = \frac{a_{k-1} - a_k}{b_k - b_{k-1}} \tag{2}$$

the cutoff type who is indifferent between two adjacent alternatives k and $k-1$. While there may be different cases induced by the parameters of the utility functions, we perform an explicit analysis for the most interesting case where

$$0 \equiv x^1 < ... < x^K < x^{K+1} \equiv 1.$$

Under these restrictions each alternative k is preferred by some agent types x_i with $x_i \in (x^k, x^{k+1}]$. These restrictions, together with the definition of $x^{l,k}$, imply that $x^{l,k} \in (x^{l+1}, x^k)$ for $k > l+1$, because

$$x^{l,k} = \frac{a_l - a_k}{b_k - b_l} = \frac{(a_l - a_{l+1}) + ... + (a_{k-1} - a_k)}{(b_{l+1} - b_l) + ... + (b_k - b_{k-1})}.$$

Remark 1 *The agents' preferences are here single-peaked. To see this, consider agent i with type $x_i \in \left(x^k, x^{k+1}\right)$. By definition of x^k, agent i prefers alternative k to any alternative $l < k$, and by definition of x^{k+1}, agent i prefers k over any $l > k$. Now consider two alternatives l and m with $l < m < k$. Since $x^l < x^m < x^k$, we have $x_i > x^{l,m}$ and agent i prefers m to l. Similarly, agent i prefers m to l if $k < m < l$. Therefore, agent i's preferences are single-peaked.*

[11] Here agents' types can be correlated. In Section 5 we shall assume independence between the agents' types.

*Our preference domain is a **strict subset** of the full single-peaked preference domain: not all single-peaked preferences are compatible with our linear environment. For instance, suppose there are 4 different alternatives* $(1, 2, 3$ *and* $4)$ *and* $x^{1,4} \in \left(x^{2,3}, x^{3,4}\right)$*, as shown in Figure 1.*

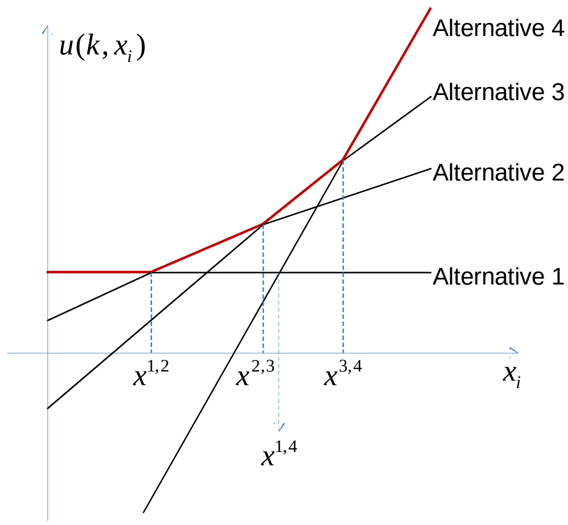

Figure 1: Not all single-peaked preferences are compatible with our linear structure.

The feasible single-peaked preferences that have alternative 2 on their top are $2 \succ 1 \succ 3 \succ 4$ *and* $2 \succ 3 \succ 1 \succ 4$*. In particular, the preference* $2 \succ 3 \succ 4 \succ 1$ *is not compatible with the linear environment. Similarly, if* $x^{1,4} \in \left(x^{1,2}, x^{2,3}\right)$*, the feasible single-peaked preferences that have alternative* 3 *on their top are* $3 \succ 2 \succ 4 \succ 1$ *and* $3 \succ 4 \succ 2 \succ 1$*. Here the preference profile* $3 \succ 2 \succ 1 \succ 4$ *is not compatible with our structure.*

3 Mechanisms and Implementation

We focus on deterministic, dominant strategy incentive compatible (DIC) mechanisms. If monetary transfers were available, the welfare-maximizing allocation would be easily achieved via the well-known Vickrey-Clarke-Groves mechanisms. But if transfers are not allowed, the first-best social choice rule need not be incentive compatible.

We restrict attention to direct, deterministic mechanisms where each agent reports his type and where, for each report profile, the mechanism chooses one alternative from the feasible set. Formally, a deterministic direct mechanism without transfers is a function $g : [0, 1]^n \to \mathcal{K}$.

Lemma 1 *A mechanism* $g(x_i, x_{-i})$ *is DIC if and only if*

1. *For all* x_{-i} *and for all* i*,* $g(x_i, x_{-i})$ *is increasing in* x_i*;*
2. *For any agent* i*, any* $x_i \in [0, 1]$ *and* $x_{-i} \in [0, 1]^{n-1}$*, the following condition holds:*

$$u(x_i, g(x_i, x_{-i})) = u(0, g(0, x_{-i})) + \int_0^{x_i} b_{g(z, x_{-i})} dz. \qquad (3)$$

This Lemma is analogous to a standard characterization result in mechanism design (see Myerson [1981]). When monetary transfers are feasible, any monotone decision rule $g(x_i, x_{-i})$ is incentive compatible since it is always possible to augment it with a transfer such that the equality required by (3) holds. Thus, with transfers, only monotonicity really matters for DIC. Since monetary transfers are not feasible here, equality (3) becomes crucial, and not all monotone decision rules $g(x_i, x_{-i})$ are implementable (see Example 1). The main difficulty in our analysis comes from the need to understand the implications of this condition.

Example 1 *The first-best mechanism that maximizes the sum of agents' expected utilities is monotone, but not DIC. To see this, consider the environment with two alternatives* $\{1, 2\}$ *and with two agents* $\{i, -i\}$*. The designer is indifferent between alternatives* 1 *and* 2 *if*

$$2a_1 + b_1 (x_i + x_{-i}) = 2a_2 + b_2 (x_i + x_{-i}).$$

The first-best rule conditions on the value of the average type, and is given by

$$g(x_i, x_{-i}) = \begin{cases} 1 & \text{if } \frac{1}{2}(x_i + x_{-i}) \in [0, x^2) \\ 2 & \text{if } \frac{1}{2}(x_i + x_{-i}) \in [x^2, 1] \end{cases}$$

where cutoff x^2 *is defined in (2):* $x^2 \equiv (a_1 - a_2) / (b_2 - b_1)$*. The first-best rule is increasing in both* x_i *and* x_{-i}*. But, for all* $x_{-i} \in [0, 2x^2)$ *and* $x_i \in [2x^2 - x_{-i}, 1)$*, we can rewrite the integral condition of Lemma 1 as*

$$a_2 + b_2 x_i = a_1 + \int_0^{2x^2 - x_{-i}} b_1 dz + \int_{2x^2 - x_{-i}}^{x_i} b_2 dz = a_1 + b_1 \left(2x^2 - x_{-i}\right) + b_2 \left(x_i - 2x^2 + x_{-i}\right),$$

which reduces to $x_{-i} = x^2$*. Therefore, the integral condition is violated for all* $x_{-i} \neq x^2$*.*

Following Barbera and Peleg [1990], we define agent i's *option set* $O_i(x_{-i})$ *given a mechanism* g as[12]

$$O_i(x_{-i}) = \{k \in \mathcal{K} : g(x_i, x_{-i}) = k \text{ for some } x_i \in [0, 1]\}.$$

That is, $O_i(x_{-i})$ is the set of alternatives that agent i can achieve when the other agents' preferences are fixed at x_{-i} given g. Denote by $J = |O_i(x_{-i})|$ the cardinality of the set $O_i(x_{-i})$. Denote by $O_i^1(x_{-i})$ the smallest element of $O_i(x_{-i})$, ..., and $O_i^J(x_{-i})$ the largest element of $O_i(x_{-i})$. The next Lemma presents an alternative characterization of deterministic DIC mechanisms.

Lemma 2 *A mechanism* $g(x_i, x_{-i})$ *is DIC if and only if, for any* i *and* x_{-i}*,* $g(x_i, x_{-i}) = O_i^k(x_{-i})$ *if and only if*

$$x_i \in (x^{O_i^{k-1}(x_{-i}), O_i^k(x_{-i})}, x^{O_i^k(x_{-i}), O_i^{k+1}(x_{-i})}],$$

where $x^{O_i^k(x_{-i}), O_i^{k+1}(x_{-i})}$ *is the cutoff type who is indifferent between alternatives* $O_i^k(x_{-i})$ *and* $O_i^{k+1}(x_{-i})$*.*

[12] We suppress notation of mechanism g in the definition of O_i, as it should not cause any confusion.

This Lemma shows that a DIC mechanism satisfies the following property: For any player i, the mechanism has to choose the most preferred alternative for that agent among the available alternatives, where the available alternatives depend on the reports of the other agents. It follows immediately from the definition of deterministic DIC mechanisms, and thus its proof is omitted. Both characterizations of DIC are valuable for our subsequent analysis.

4 Generalized Median Voter Schemes

This section characterizes implementable Pareto efficient mechanisms. We first show the equivalence between the class of deterministic DIC mechanisms and the class of mechanisms in which agents report only their most preferred alternative (*peaks-only mechanisms*). Thus, only the top alternatives (and not the entire ordinal rankings) play a role in determining the implemented outcome. Another implication is that, in order to satisfy incentive compatibility, the DIC mechanism must ignore the cardinal intensities of the agents' preferences. (Recall that the agents' signals affect the intensities of their preferences.) We then show that any Pareto efficient and anonymous mechanism is equivalent to a generalized median voter scheme with n real voters and $(n-1)$ phantom voters in which agents report only their most preferred alternative. We first need several definitions:

Definition 1 *1. A mechanism g is onto if, for every alternative $k \in \mathcal{K}$, there exists a type profile $(x_i, x_{-i}) \in [0,1]^n$ such that $g(x_i, x_{-i}) = k$.*

2. *A mechanism g is unanimous if $x_i \in \left(x^k, x^{k+1}\right)$ for all i implies $g(x_i, x_{-i}) = k$.*

3. *A mechanism g is Pareto efficient if for any profile of reports $x \in [0,1]^n$ there is no alternative $k \in \mathcal{K}$ such that $u_i(x_i, k) \geq u_i(x_i, g(x))$ for all i, with strict inequality for at least one agent.*

4. *A mechanism g is anonymous if for any profile of reports $x \in [0,1]^n$ $g(x_1, ..., x_n) = g\left(x_{\sigma(1)}, ..., x_{\sigma(n)}\right)$ where σ denotes any permutation of the set $\{1, ..., n\}$.*

Since in our environment every alternative is preferred by some agent's types, if a DIC mechanism is unanimous, it must be onto. The next Lemma shows that the reverse is also true.

Lemma 3 *Every onto and DIC mechanism g satisfies unanimity.*

Next we formally define peaks-only mechanisms and the equivalence criterion.

Definition 2 *A mechanism π is peaks-only if it has the form $\pi : \mathcal{K}^n \to \mathcal{K}$. We say that a DIC mechanism g is equivalent to a peaks-only mechanism π if*

$$g(x_1, ..., x_n) = \pi(k_1, ..., k_n),$$

for any type profile $(x_1, ..., x_n)$ *and for any alternative profile* $(k_1, ..., k_n)$ *such that* $x_i \in (x^{k_i}, x^{k_i+1}]$ *for all* i*.*

Note that if the original mechanism g is DIC, then in the equivalent mechanism π all agents truthfully report their peaks. In order to establish the equivalence, we first show that the option set $O_i(x_{-i})$ is "connected", i.e., it contains no gaps. We then recall Lemma 2 which states that every deterministic DIC mechanism can be characterized by intervals: for implementation purposes, it is enough to know the interval that contains the type without knowing the exact type (where the intervals are generated by cutoff types who are indifferent between two neighboring available alternatives). Therefore, in order to establish the equivalence, it is sufficient to show that these intervals can be "replaced" by the corresponding top alternatives. Since the option set has no gap, these intervals form a coarser partition than the one generated by the original cutoffs $\left(0, x^2, ..., x^K, 1\right)$. Hence, we are able to implement the same outcome even if all agents report their top alternatives instead of reporting the intervals containing their types.

Theorem 1 *For any onto and DIC mechanism* g *there exists an equivalent, peaks-only mechanism* π*.*

The onto requirement is crucial. To see this, consider the environment with three alternatives $(1, 2, 3)$ and two agents $(i, -i)$. Suppose only alternatives 1 and 3 can be chosen under mechanism g. Consider the following mechanism:

$$g(x_i, x_{-i}) = \begin{cases} 1 & \text{if } x_i \in [0, x^{1,3}] \text{ and } x_{-i} \in [0, x^{1,3}] \\ 3 & \text{otherwise} \end{cases} .$$

This mechanism is DIC, but there does not exist an equivalent peaks-only mechanism: knowing that alternative 2 is agent i's top alternative is not sufficient for inferring whether agent i's type is above or below $x^{1,3}$.

An influential paper by Moulin [1980] shows that, if each agent is **restricted** to report their top alternative only, then every DIC, efficient and anonymous voting scheme on the full domain of single-peaked preferences is equivalent to a generalized median voter scheme. That is, one can obtain each DIC, efficient and anonymous scheme by adding $(n-1)$ fixed ballots to the n voters' ballots and then choosing the median of this larger set of ballots. It turns out that Moulin's characterization also holds in our setting. Yet, the main difference between our characterization and that of Moulin is that in our environment the restriction to peak-only mechanisms is without loss of generality (as was shown in Theorem 1). We also need here a separate proof because our setting differs from Moulin's in several dimensions: 1) Moulin's proof requires that the preference domain contain all single-peaked preferences over the real line R, a rich domain assumption not satisfied here; 2) Our set of alternatives is finite; and 3) Our preferences may not be strict (there are ties for some types).

Theorem 2 *A Pareto efficient, anonymous mechanism g is DIC if and only if there exists $(n-1)$ numbers $\alpha_1, ..., \alpha_{n-1} \in \mathcal{K}$ such that for any type profile $(x_1, ..., x_n) \in [0,1]^n$ with $x_i \in (x^{k_i}, x^{k_i+1}]$ for all i, it holds that*

$$g(x_1, ..., x_n) = M(\alpha_1, ..., \alpha_{n-1}, k_1, ..., k_n),$$

where the function $M(\alpha_1, ..., \alpha_{n-1}, k_1, ..., k_n)$ returns the median of $(\alpha_1, ..., \alpha_{n-1}, k_1, ..., k_n)$.

As in Moulin [1980], we prove the above characterization by first establishing that any anonymous and DIC mechanism is equivalent to a generalized median voter scheme with n real voters and $n+1$ phantom voters. This result is formally stated in the Appendix as Proposition 1. Using Proposition 1 we can show that if a DIC mechanism is onto, then it is Pareto efficient. Therefore, for every onto and DIC mechanism, there exists an equivalent generalized median voter scheme with n real voters and $(n-1)$ phantom voters.

Lemma 4 *Every onto and DIC mechanism is Pareto efficient.*

5 Optimal Mechanisms

In this section we characterize socially optimal allocations that respect the incentive constraints (constrained efficiency, or "second-best"). Following the mechanism design literature, we shall primarily focus on the utilitarian welfare criterion: the social planner wants to maximize the sum of the agents' expected utilities. Since Pareto efficient mechanisms are necessarily unanimous, Lemma 3 and Lemma 4 imply that, for DIC mechanisms the three properties used above - onto, Pareto efficiency, and unanimity - are equivalent in our environment.[13] We confine attention below to mechanisms that satisfy these properties.

Given our earlier characterization, the set of onto and DIC mechanisms coincides with the set of generalized median voter schemes with n real voters and $(n-1)$ phantom voters. Therefore, the task of searching optimal mechanisms is reduced to finding the optimal position for the peaks of these $(n-1)$ phantom voters. In order to be able to offer simple, intuitive formulae for the number of phantom peaks on each alternative we make below several standard assumptions on the distribution of agents' signals. The first assumption yields the standard symmetric, independent private values model (SIPV) widely used in the literature on trading mechanisms with transfers.

Assumption A. *The agents' signals are distributed identically and independently of each other on the interval $[0,1]$ according to a cumulative distribution function F with density f.*

[13] See Schummer and Vohra [2007] for a similar observation in their environment with single-peaked preferences on an interval.

To introduce the second assumption and to simplify notation below, we now define two functions, $C(x)$ and $c(x)$, as follows:

$$C(x) = E[X|X > x] \text{ and } c(x) = E[X|X \leq x].$$

Assumption B. *Let X be the random variable representing the agents' type. The functions $x - C(x)$ and $x - c(x)$ are assumed to be strictly increasing.*

To better understand this assumption, let us recall a well known concept used in the theory of reliability.

Definition 3 *1. The mean residual life (MRL) of a random variable $X \in \left[0, \overline{\theta}\right]$ is defined as*

$$MRL(x) = \begin{cases} E[X - x|X \geq x] & \text{if} \quad x < \overline{\theta} \\ 0 & \text{if} \quad x = \overline{\theta} \end{cases}$$

2. A random variable X satisfies the decreasing mean residual life (DMRL) property if the function $MRL(x)$ is decreasing in x.

If we let X denote the life-time of a component, then $MRL(x)$ measures the expected remaining life of a component that has survived until time x.[14] Assuming that $x - C(x)$ is strictly increasing is equivalent to assuming a strictly decreasing mean residual life (DMRL). Assuming that $x - c(x)$ is strictly increasing is equivalent to assuming strict log-concavity of $\int_0^x F(s)\,ds$, because

$$x - c(x) = \frac{\int_0^x F(s)\,ds}{F(x)} \quad \text{and} \quad \frac{F(x)}{\int_0^x F(s)\,ds} = \frac{d}{dx} \log \left[\int_0^x F(s)\,ds\right]$$

A sufficient condition for $\int_0^x F(s)\,ds$ to be log-concave is that $F(x)$ is log-concave. A sufficient condition for both log-concavity of F and strict DMRL of F is that the density f is strictly log-concave.[15]

Consider now a situation with n real voters and let l_k denote the number of phantom voters with peak on alternative k in a generalized median voter scheme with $n-1$ phantom voters. Our analysis is based on a simple observation: if l_k is part of the optimal allocation of the $(n-1)$ phantoms and $l_k > 0$, then shifting one phantom voter from alternative k to

[14] The MRL function is related to the hazard rate (or failure rate) $\lambda(x) = f(x)/[1 - F(x)]$. The "increasing failure rate" (IFR) assumption is commonly made in the economics literature. DMRL is a weaker property, and it is implied by IFR.

[15] The log-concavity of density is stronger than (and implies) increasing failure rate (IFR) which is equivalent to log-concavity of the reliability function $(1-F)$. The family of log-concave densities is large and includes many commonly used distributions such as uniform, normal, exponential, logistic, extreme value etc. The power function distribution $\left(F(x) = x^k\right)$ has log-concave density if $k \geq 1$, but it does not if $k < 1$. However, one can easily verify the above two conditions hold for $F(x) = x^k$ even with $k < 1$. Therefore, log-concave density is not necessary. See Bagnoli and Bergstrom [2005] for an excellent discussion of log-concave distributions.

either alternative $k-1$ or $k+1$ weakly reduces the total expected utility.[16] For instance, shifting one phantom voter from alternative k to $k-1$ has an impact only if it changes the chosen alternative. However, the shift will change the chosen alternative only if there are $(n-1)$ voters (both "real" and "phantom") with values below x^k, in other words, only if there are exactly $\left(n-1-\sum_{m=1}^{k-1} l_m\right)$ real voters with values below x^k. These kind of arguments generate the following bounds on the cumulative distribution of phantom voters:

$$\sum_{m=1}^{k-1} l_m \geq n\frac{x^k - c\left(x^k\right)}{C\left(x^k\right) - c\left(x^k\right)} - 1, \text{ for all } k \geq 2, \tag{4}$$

$$\sum_{m=1}^{k} l_m \leq n\frac{x^{k+1} - c\left(x^{k+1}\right)}{C\left(x^{k+1}\right) - c\left(x^{k+1}\right)}, \text{ for all } k \leq K-1. \tag{5}$$

With these bounds, we can explicitly construct the essentially unique distribution of phantom peaks.

Theorem 3 *Suppose that Assumptions A and B hold, and let* $\lceil z \rceil$ *denote the largest integer that is below* z. *The optimal mechanism for* n *agents is a generalized median scheme with* $(n-1)$ *phantom voters' peaks distributed according to*

$$l_k^* = \begin{cases} \left\lceil n\frac{x^2-c(x^2)}{C(x^2)-c(x^2)} \right\rceil & \text{if} \quad k=1 \\ \left\lceil n\frac{x^{k+1}-c(x^{k+1})}{C(x^{k+1})-c(x^{k+1})} \right\rceil - \left\lceil n\frac{x^{k}-c(x^{k})}{C(x^{k})-c(x^{k})} \right\rceil & \text{if} \quad 1<k<K \\ n-1-\sum_{m=1}^{K-1} l_m^* & \text{if} \quad k=K \end{cases}.$$

The above theorem reveals that adding or eliminating an alternative has only a local effect. That is, if we add an alternative such that an interval $\left[x^k, x^{k+1}\right]$ is further divided into $\left[x^k, x^{k_1}\right]$ and $\left[x^{k_1}, x^{k+1}\right]$, the only effect on the optimal phantom allocation is that the original number of phantoms placed on alternatives k and $k+1$ are split between the original alternatives k, $k+1$, and the new alternative k_1. Similarly, if we eliminate alternative k, then the phantoms that were allocated on this alternative are now re-allocated to adjacent alternatives $k-1$ and $k+1$, without any effect on the other alternatives. This "locality-effect" follows from the single-peaked preferences: the social planner does not want to change the chosen alternative if the peak of the median voter does not change as a result of adding/eliminating the available alternatives.

Remark 2

1. *It can be easily shown that if the number of voters* n *is large enough, then the optimal number of phantom voters on every alternative is strictly positive.*

2. *It is interesting to note that, with a large number of voters, the optimal (second-best) mechanism approximates the welfare maximizing mechanism (first-best) which, as illustrated in Example 1, is not implementable in our setting. To see this, let*

[16] This is feasible only if $l_k > 0$. It turns out that the derived bounds (4) and (5) remain valid for alternatives with zero phantom voters. See Lemma 7 in the Appendix.

p_k denote the proportion of real voters who have their peak on alternative k, and let $q_k = l_k^* / (n-1)$ denote the proportion of phantom voters who have their peak on alternative k, $k = 1, ..., K$. When n is large, the optimal mechanism in Theorem 3 chooses the ***median*** *peak, i.e., the minimal alternative k such that*

$$\begin{aligned}
\sum_{m=1}^{k} p_m + \sum_{m=1}^{k} q_m &\geq 2 - \sum_{m=1}^{k} p_m - \sum_{m=1}^{k} q_m \Leftrightarrow \\
\sum_{m=1}^{k} p_m &\geq 1 - \sum_{m=1}^{k} q_m \Leftrightarrow \\
\sum_{m=1}^{k} p_m &\geq 1 - \frac{x^{k+1} - c(x^{k+1})}{C(x^{k+1}) - c(x^{k+1})} \Leftrightarrow \\
\sum_{m=1}^{k} p_m &\geq \frac{C(x^{k+1}) - x^{k+1}}{C(x^{k+1}) - c(x^{k+1})}
\end{aligned}$$

where, given the assumption of a large population, we abstract from integer constraints. By the same assumption, we can approximate $\sum_{m=1}^{k} p_m \approx F(x^{k+1})$, which yields the choice of the minimal alternative k such that

$$F(x^{k+1}) \geq \frac{C(x^{k+1}) - x^{k+1}}{C(x^{k+1}) - c(x^{k+1})}.$$

On the other hand, the first-best mechanism chooses the minimal alternative k such that ***the average type of real voters*** *is below x^{k+1}, i.e.,*

$$\sum_{m=1}^{K} p_m E\left[x_i | x_i \in \left[x^m, x^{m+1}\right]\right] \leq x^{k+1}.$$

Using the approximation $p_m \approx F\left(x^{m+1}\right) - F\left(x^m\right)$, and expanding the conditional expectations, we can translate the above condition into

$$\left(\sum_{m=1}^{k} p_m\right) c(x^{k+1}) + \left(1 - \sum_{m=1}^{k} p_m\right) C(x^{k+1}) \leq x^{k+1}.$$

Again, using the approximation $\sum_{m=1}^{k} p_m \approx F(x^{k+1})$, we can re-write the above inequality as

$$F(x^{k+1}) \geq \frac{C(x^{k+1}) - x^{k+1}}{C(x^{k+1}) - c(x^{k+1})}.$$

Thus, the two mechanisms are approximately the same when n is large.

Theorem 3 also yields immediate and intuitive comparative statics with respect to parameters of the utility function $\{a_k, b_k\}_{k=1}^{K}$. As part of the proof of Theorem 3, we show that the function $\frac{x - c(x)}{C(x) - c(x)}$ is strictly increasing. By the definition of the cutoffs x^k, increases in either a_k or b_k decrease x^k and increase x^{k+1}, which in turn increase l_k. That is, if the attractiveness of any alternative increases, the optimal number of the phantom voters with peaks on this alternative increases as well.

Example 2 *Suppose that the distribution of signals F is uniform on $[0,1]$. Then $C(x) = E[X|X > x] = (1+x)/2$ and $c(x) = E[X|X \leq x] = x/2$. Therefore, the optimal distribution of phantom voters' peaks is given by: $l_k^* = \lceil nx^{k+1} \rceil - \lceil nx^k \rceil$. Intuitively, here the number of phantom voters' peaks is proportional to the share of real types whose top alternative is k.*

To further illustrate Theorem 3, we now describe in more detail the optimal voting rules when there are either two agents or two alternatives.

If there are two agents (i and $-i$) and K alternatives, the set of peaks-only, onto, DIC and anonymous mechanisms contains exactly K generalized median voter schemes with one phantom voter (see Theorem 2). Therefore, we only need to find the optimal position for the one additional phantom voter.

Corollary 1 *Suppose there are only two agents. Under Assumptions A and B, the optimal mechanism is a generalized median voter scheme with one phantom voter whose peak is placed on*

$$k^* \equiv \min \left\{ \ k \in \mathcal{K} : x^{k+1} \geq \frac{1}{2} \left[C\left(x^{k+1}\right) + c\left(x^{k+1}\right) \right] \ \right\}.$$

Note that the condition for determining the optimal phantom voter peak can be rewritten as $k^* = k$ if $x^* \in \left[x^k, x^{k+1}\right]$, where $x^* = \frac{1}{2}\left[C\left(x^*\right) + c\left(x^*\right)\right]$. The critical value x^* has the same distance to the upper conditional mean $C\left(x^*\right)$ as to the lower conditional mean $c\left(x^*\right)$. In particular, if the distribution is symmetric, then x^* coincides with the mean of the distribution.

If $K = 3$, the choices of these mechanisms as a function of the agents' types can be described by the three tables in Figure 2 below.

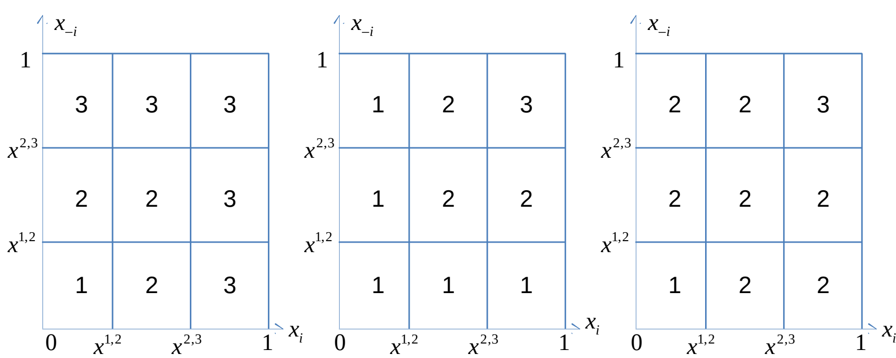

Figure 2: DIC mechanisms with two agents and three alternatives

The left table in Figure 2 corresponds to the case where the added phantom voter has a peak on $k^* = 3$. The middle table corresponds to the case where $k^* = 1$, while the right table corresponds to the case with $k^* = 2$.

If there are only two alternatives, Theorem 3 specifies the optimal qualified majority rule. That is, the optimal decision can also be implemented by voting with a properly chosen majority rule. Here are two examples: 1) Zero phantoms on one of the alternatives corresponds to the unanimity rule, and such a rule can be optimal only if the number of the real voters is relatively small; 2) For n odd, $(n-1)/2$ phantoms on each alternative corresponds to the simple majority rule, and such a rule is optimal in symmetric situations.

Corollary 2 *Suppose there are n agents and only two alternatives, $K = 2$. Under Assumptions A and B the optimal rule is implemented through a voting game in which alternative 1 is chosen if and only if at least $n - \left\lceil n\frac{x^2-c(x^2)}{C(x^2)-c(x^2)}\right\rceil$ voters voted in its favour.*

6 Extensions

6.1 Other Objective Functions

Other, non-utilitarian, objective functions can be considered as well. For example, if the designer's preferences are *maximin*, then the allocation the designer would like to implement is

$$g^{\min}(x_1, ..., x_n) = k^m$$

where k^m satisfies $x^m \in (x^{k^m}, x^{k^m+1}]$ with $x^m = \min\{x_1, ..., x_n\}$. That is, k^m is the most preferred alternative of the agent with the lowest signal. This rule is implementable through a peaks-only mechanism

$$\pi^{\min}(k_1, ..., k_n) = \min\{k_1, ..., k_n\}.$$

Similarly, if the designer's preferences are *maximax*, then the designer would like to implement allocation

$$g^{\max}(x_1, ..., x_n) = k^M$$

where k^M satisfies $x^M \in (x^{k^M}, x^{k^M+1}]$ with $x^M = \max\{x_1, ..., x_n\}$. That is, k^M is the most preferred alternative of the agent with the highest signal, and this rule is also implementable through a peaks-only mechanism

$$\pi^{\max}(k_1, ..., k_n) = \max\{k_1, ..., k_n\}.$$

6.2 Nonlinear Utilities

We have assumed that agents' utilities are linear. However, a careful inspection of the proofs for our characterization theorems (Theorem 1 and 2) reveals that the monotonicity of mechanism g and the single-peakedness of utilities $u(x_i, k)$ are crucial, but linearity of $u(x_i, k)$ is not. Therefore, as long as the utilities are single-peaked and the DIC mechanism g is monotone, our characterization of DIC mechanisms as generalized median voter schemes remains valid. But, with a different utility specification, the resulting optimal mechanism may differ from those we characterized earlier.

In order to extend our characterization to nonlinear utilities, we impose three restrictions on $u(x_i, k)$. First, for given x_i, $u(x_i, k)$ is assumed to be strictly concave in k, i.e., for all $k', k'' \in \mathcal{K}$ and for all $\alpha \in [0, 1]$ such that $\alpha k' + (1-\alpha)k'' \in \mathcal{K}$, it holds that

$u(x_i, \alpha k' + (1-\alpha) k'') > \alpha u(x_i, k') + (1-\alpha) u(x_i, k'')$. Therefore, all types of agents' preferences over $\mathcal{K}$ are single-peaked. Second, we assume that $u(x_i, k)$ has the *increasing difference property* (or supermodularity), i.e., for all $k, k' \in \mathcal{K}$ with $k > k'$,

$$u_1(x_i, k) - u_1\left(x_i, k'\right) > 0,$$

where $u_1(x_i, k)$ denotes the partial derivative with respect to x_i.[17] It is easy to verify that if a mechanism $g(x_i, x_{-i})$ is DIC and if $u(x_i, k)$ has increasing difference, then $g(x_i, x_{-i})$ must be increasing in x_i for all x_{-i} and for all i. Finally, we assume the function $u(x_i, k)$ is such that there exists

$$0 \equiv x^1 < x^2 < ... < x^K < x^{K+1} \equiv 1,$$

such that agent i's top alternative is k if $x_i \in \left[x^k, x^{k+1}\right]$.

The above specification nests both the linear utilities used above and the commonly used quadratic utilities. For example, suppose that $u(x_i, k)$ takes the following form:

$$u(x_i, k) = -\left(x_i - \frac{k}{K+1}\right)^2.$$

It is easy to see that $u(k, x_i)$ has increasing difference and is single-peaked. It is also easy to compute that

$$x^{l,k} = \frac{k+l}{2(K+1)} \text{ and } x^k = \frac{2k-1}{2(K+1)}.$$

Therefore, $0 \equiv x^1 < x^2 < ... < x^K < x^{K+1} \equiv 1$. More generally, for any Γ increasing and convex, the following utility function has the increasing difference property and is single-peaked:

$$u(x_i, k) = -\Gamma\left(\left(x_i - \frac{k}{K+1}\right)^2\right).$$

7 Concluding Remarks

We have characterized constrained efficient (i.e., second-best) dominant strategy incentive compatible and deterministic mechanisms in a setting where privately informed agents have linear utility functions, but where monetary transfers are not feasible. The analysis combines several insights from the mechanism design and from the social choice literatures. More generally, our approach allows a systematic choice among Pareto-efficient mechanisms based on the ex-ante utility they generate. Dominant strategy mechanisms are robust to variations in beliefs. In the standard setting with independent types, linear utility and monetary transfers, an equivalence result between dominant strategy incentive compatible and Bayes-Nash incentive compatible mechanisms has been established by Gershkov et al. [2013]. It is an open question whether using the more permissible Bayesian incentive compatibility concept can improve the performance of constrained efficient mechanisms in the present setting without monetary transfers.

[17] Note that if the alternative set is an interval and $u(x_i, k)$ is twice differentiable, then concavity requires $\partial^2 u(x_i, k) / \partial k^2 < 0$ while supermodularity requires $\partial^2 u(x_i, k) / \partial x_i \partial k > 0$.

8 Appendix: Proofs

Proof of Lemma 1. A mechanism g is dominant strategy incentive compatible (DIC) if for any player i, for any x_i, x_i' and x_{-i}:

$$u\left(x_i, g\left(x_i, x_{-i}\right)\right) \geq u\left(x_i, g\left(x_i', x_{-i}\right)\right). \tag{6}$$

We reverse the role of x_i and x_i' to obtain that

$$u\left(x_i', g\left(x_i', x_{-i}\right)\right) \geq u\left(x_i', g\left(x_i, x_{-i}\right)\right).$$

Adding the two inequalities together leads to

$$\left[u\left(x_i, g\left(x_i, x_{-i}\right)\right) - u\left(x_i', g\left(x_i, x_{-i}\right)\right)\right] - \left[u\left(x_i, g\left(x_i', x_{-i}\right)\right) - u\left(x_i', g\left(x_i', x_{-i}\right)\right)\right] \geq 0.$$

Since $u\left(x_i, k\right) - u\left(x_i', k\right) = b_k\left(x_i - x_i'\right)$, the above inequality reduces to

$$\left(x_i - x_i'\right)\left(b_{g\left(x_i, x_{-i}\right)} - b_{g\left(x_i', x_{-i}\right)}\right) \geq 0,$$

which implies that $g\left(x_i, x_{-i}\right)$ must be nondecreasing in x_i for all x_{-i}. DIC also implies that

$$u\left(x_i, g\left(x_i, x_{-i}\right)\right) = \max_{x_i' \in [0,1]} u\left(x_i, g\left(x_i', x_{-i}\right)\right).$$

We can apply the envelope theorem to obtain that

$$u\left(x_i, g\left(x_i, x_{-i}\right)\right) = u\left(0, g\left(0, x_{-i}\right)\right) + \int_0^{x_i} b_{g\left(z, x_{-i}\right)} dz.$$

We now show sufficiency: if monotonicity and the integral condition are satisfied, then the mechanism $g\left(x_i, x_{-i}\right)$ is DIC. First suppose $x_i > x_i'$. We can write the integral condition as

$$u\left(x_i, g\left(x_i, x_{-i}\right)\right) = u\left(x_i', g\left(x_i', x_{-i}\right)\right) + \int_{x_i'}^{x_i} b_{g\left(z, x_{-i}\right)} dz$$

By assumption, $g\left(z, x_{-i}\right) \geq g\left(x_i', x_{-i}\right)$ for all $z \geq x_i'$. Hence, we have

$$\int_{x_i'}^{x_i} b_{g\left(z, x_{-i}\right)} dz \geq \int_{x_i'}^{x_i} b_{g\left(x_i', x_{-i}\right)} dz,$$

and thus

$$\begin{aligned}
u\left(x_i, g\left(x_i, x_{-i}\right)\right) &\geq u\left(x_i', g\left(x_i', x_{-i}\right)\right) + \int_{x_i'}^{x_i} b_{g\left(x_i', x_{-i}\right)} dz \\
&= a_{g\left(x_i', x_{-i}\right)} + x_i' b_{g\left(x_i', x_{-i}\right)} + \left(x_i - x_i'\right) b_{g\left(x_i', x_{-i}\right)} \\
&= u\left(x_i, g\left(x_i', x_{-i}\right)\right).
\end{aligned}$$

The proof for the case of $x_i < x_i'$ is similar. Note that, if $x_i < x_i'$ we have

$$\begin{aligned}
u\left(x_i, g\left(x_i, x_{-i}\right)\right) &= u\left(x_i', g\left(x_i', x_{-i}\right)\right) - \int_{x_i}^{x_i'} b_{g\left(z, x_{-i}\right)} dz \\
&\geq u\left(x_i', g\left(x_i', x_{-i}\right)\right) - \int_{x_i}^{x_i'} b_{g\left(x_i', x_{-i}\right)} dz \\
&= u\left(x_i, g\left(x_i', x_{-i}\right)\right).
\end{aligned}$$

Hence, $g(x_i, x_{-i})$ is DIC. ■

Proof of Lemma 3. We prove the claim by contradiction. Suppose there exist an alternative k and a report profile $(\widehat{x}_1, ..., \widehat{x}_n)$ such that $\widehat{x}_i \in (x^k, x^{k+1})$ for all i but $g(\widehat{x}_1, ..., \widehat{x}_n) = l$ with $l \neq k$. Since the mechanism is onto, there exists some type profile $(x_1^*, ..., x_n^*)$ such that $g(x_1^*, ..., x_n^*) = k$. First suppose $x_i^* \in (x^k, x^{k+1})$ for all i. Consider agent 1 and fix the other agents' reports at $(x_2^*, ..., x_n^*)$. DIC implies that $g(\widehat{x}_1, x_2^*, ..., x_n^*) = k$, otherwise agent 1 could manipulate at $(\widehat{x}_1, x_2^* ..., x_n^*)$ via x_1^* to achieve his best alternative k. Next consider agent 2, and fix the other agents' reports at $(\widehat{x}_1, x_3^*, ..., x_n^*)$. Then, again we must have $g(\widehat{x}_1, \widehat{x}_2, x_3^*, ..., x_n^*) = k$, otherwise agent 2 could manipulate at $(\widehat{x}_1, \widehat{x}_2, x_3^*, ..., x_n^*)$ via x_2^*. Applying the same argument to the remaining agents, $3, ..., n$, we obtain that $g(\widehat{x}_1, ..., \widehat{x}_n) = k$, which is a contradiction. Therefore, there must exist at least one agent i such that $x_i^* \notin (x^k, x^{k+1})$ and $g(x_1^*, ..., x_i^{**}, ..., x_n^*) = m$ with $m \neq k$ and $x_i^{**} \in (x^k, x^{k+1})$. Fix the reports of all agents but i to $(x_1^*, ..., x_{i-1}^*, x_{i+1}^*, ..., x_n^*)$. This mechanism is not incentive compatible, because agent i with type x_i^{**} could manipulate at $(x_1^*, ..., x_{i-1}^*, x_i^{**}, x_{i+1}^*, ..., x_n^*)$ via x_i^* and achieve his best alternative k. ■

In order to prove Theorem 1 we first prove a lemma showing that for any player i and any x_{-i} the option set $O_i(x_{-i})$ associated with mechanism g is connected.

Lemma 5 *Consider a deterministic, onto and DIC mechanism g. For any i and any x_{-i}, if $k, l \in O_i(x_{-i})$ and $l < h < k$ then $h \in O_i(x_{-i})$.*

Proof of Lemma 5. Suppose the claim is false: there exist an agent, say agent 1, a report profile of other agents $(x_2^*, ..., x_n^*)$, and alternatives $l < h < k$ such that $k, l \in O_i(x_{-i})$ but $h \notin O_i(x_{-i})$. Assume for simplicity that $k = l + 2$. Since alternatives k and l are chosen, Lemma 2 implies that there exists a threshold $x^{l,k}$ such that l is chosen if $x_1 \in (x^l, x^{l,k})$ and k is chosen if $x_1 \in (x^{l,k}, x^{k+1})$. We know $x^{l,k} \in (x^{l+1}, x^k)$, and since $h = l+1$, we have $x^{l,k} \in (x^h, x^k)$. Therefore, there exist two types of agent 1, $x_1^{h'} \in (x^h, x^{l,k})$ and $x_1^{h''} \in (x^{l,k}, x^k)$ such that the DIC mechanism g chooses l if the report profile is $\left(x_1^{h'}, x_2^*, ..., x_n^*\right)$ and chooses k if the report profile is $\left(x_1^{h''}, x_2^*, ..., x_n^*\right)$. Note that, for both types of agent 1, alternative h is the best alternative. But, since h is not an option, type $x_1^{h'}$ prefers l among the available alternatives, while type $x_1^{h''}$ prefers k among the available alternatives.

Take another agent, say agent 2. We know that alternative h cannot be chosen if the type of 2 is x_2^*. We now show that, if g is DIC, then there are no types of agent 1 and 2 such that alternative h is chosen, keeping other agents' types fixed at $(x_3^*, ..., x_n^*)$. Assume this is not the case. Then there exists a type of agent 2 such that for some types of 1 alternative h is chosen. When h is chosen for this type of agent 2, the type of agent 1 must belong to (x^h, x^{h+1}) (if not, agent 1 with type $x_1 \in (x^h, x^{h+1})$ will misreport that type to obtain h). Consider two cases:

Case 1. Assume first that alternative h may be chosen for some type of agent 2 that is greater than x_2^*, i.e., $x_2 > x_2^*$. Then we get the following contradiction: fix the type of agent 1 to be $x_1^{h''}$, while fixing the types of all other agents $(3, ..., n)$ to be $(x_3^*, ..., x_n^*)$. Then increasing the type of agent 2 from x_2^* to x_2 leads to a change in the the social choice from alternative k to alternative h, which contradicts monotonicity of g.

Case 2. Next assume that alternative h may be chosen for some type of agent 2 that is smaller than x_2^*, i.e., $x_2 < x_2^*$. Then we get another contradiction: fix the type of agent 1 to be $x_1^{h'}$, while fixing the types of all other agents $(3, ..., n)$ to $(x_3^*, ..., x_n^*)$. Then decreasing the type of agent 2 from x_2^* to x_2 leads to a change from alternative l to alternative h, which again contradicts the monotonicity of g.

Therefore, alternative h is not chosen for any types of agents 1 and 2. Fix now the type of agent 2 to be in the interval $\left(x^h, x^{h+1}\right)$. That is, replace the type of agent 2 in the original profile $(x_1^*, x_2^*, x_3^*, ..., x_n^*)$ with $x_2' \in \left(x^h, x^{h+1}\right)$. From the previous step we know that there is no type of agent 1 and 2 such that alternative h is chosen. In order for our induction argument to work, we have to show that $k, l \in O_1\left(x_{-1-2}, x_2'\right)$.[18] If $x_2^* \in \left(x^h, x^{h+1}\right)$ we set $x_2' = x_2^*$. If $x_2^* < x^h$, then we set $x_2' \in \left(x^h, x^{l,k}\right)$. First we show that $l \in O_1\left(x_{-1-2}, x_2'\right)$. Since h is not available, alternative l is the second best alternative for $x_2' \in \left(x^h, x^{l,k}\right)$. Since $l \in O_1\left(x_{-1-2}, x_2^*\right)$ there exists a type of agent 1, x_1' such that l is chosen for profile (x_1', x_{-1-2}, x_2^*). Therefore, if $l \notin O_1\left(x_{-1-2}, x_2'\right)$ the mechanism is not DIC, since for type x_1' of agent 1, agent 2 of type x_2' prefers reporting type x_2^*. Assume now that $k \notin O_1\left(x_{-1-2}, x_2'\right)$. Since $k \in O_1\left(x_{-1-2}, x_2^*\right)$ there exists a type of agent 1, say x_1'', such that k is chosen if the type of the second agent is x_2^*. Since $x_2^* < x^h$, type x_2^* prefers alternative l over k. Therefore, if $k \notin O_1\left(x_{-1-2}, x_2'\right)$ and $l \in O_1\left(x_{-1-2}, x_2'\right)$ type x_2^* prefers reporting type x_2' instead. This yields a contradiction to DIC. If $x_2^* > x^{h+1}$ the procedure is similar, but with $x_2' \in \left(x^{l,k}, x^{h+1}\right)$.

Taking another agent, say agent 3, and using the same procedure we can show that there are no types of this agent such that alternative h is chosen. Again, we can fix agent 3's type to be in $\left(x^h, x^{h+1}\right)$. We can apply this argument to agents $4, ..., n$ and reach an contradiction to Lemma 3. Therefore, $h \in O_i\left(x_{-i}\right)$. ■

Proof of Theorem 1. Fix the reports of all agents other than i and consider agent i's option set $O_i\left(x_{-i}\right)$. If $O_i\left(x_{-i}\right) = \mathcal{K}$, then all alternatives are chosen for different types of agent i. DIC implies that we have cutoff types

$$0 \equiv x^1 < x^2 < ... < x^K < x^{K+1} \equiv 1.$$

The peaks-only result holds since knowing the top alternative is equivalent to knowing the interval. If $O_i\left(x_{-i}\right)$ is a strict subset of $\mathcal{K}$, then not all alternatives can be chosen. According

[18] Here we assume $l + 2 = h + 1 = k$. If $l + 2 < h + 1 < k$, then for our induction argument to work, it is sufficient to show that there exists k' and l' such that $l \leq l' < h$, $h < k' \leq k$ and $k', l' \in O_1\left(x_{-1-2}, x_2'\right)$. The proof below can be easily adapted to prove the existence of such k' and l'.

to Lemma 5, the alternatives that are not chosen must be "extreme" ones: either low alternatives $1, ..., s$ or high alternatives $d, ..., K$, or both high and low alternatives. In this case, the relevant cutoffs are just a subset of the original cutoffs given. That is, the cutoffs when all alternatives are chosen for different types of i, generate a finer partition of the interval of $[0,1]$ than the new cutoffs where some "extreme" alternatives are not chosen. Because we can infer from i's top alternative the interval (in terms of the original cutoffs) that contains the signal of i, we can also infer the new interval (in terms of new cutoffs) that contains i's type. Therefore, for any agent i and for any reports of agents other than i, any DIC, deterministic and onto mechanism can be replicated by another mechanism where i only reports only his top alternative. We can repeat this argument for all other agents, completing the proof. ■

In order to prove Theorem 2, we first prove a Lemma showing that for any player i and any k_{-i} the option set associated with a DIC, peaks-only mechanism is connected, and then a Proposition stating that any anonymous, DIC, and peaks-only mechanism is equivalent to a generalized median voter scheme with n real voters and $n+1$ phantom voters.

Lemma 6 *Consider a deterministic, DIC, and peaks-only mechanism π. Define the option set $\sigma_i(k_{-i})$ associated with π as*

$$\sigma_i(k_{-i}) = \{k \in \mathcal{K} : \pi(k_i, k_{-i}) = k \text{ for some } k_i \in \mathcal{K}\}.$$

For any i and any k_{-i}, if alternatives $l < h < k$ and $k, l \in \sigma_i(k_{-i})$, then $h \in \sigma_i(k_{-i})$.

Proof. Suppose the claim is false: there exist an agent (say agent 1), a report profile of other agents $(k_2^*, ..., k_n^*)$, and alternatives $l < h < k$ such that $k, l \in \sigma_i(k_{-i})$ but $h \notin \sigma_i(k_{-i})$. Since alternatives k and l are chosen, there exist two reports of agent 1, k_1^l and k_1^k such that

$$\pi\left(k_1^l, k_2^*, ..., k_n^*\right) = l \text{ and } \pi\left(k_1^k, k_2^*, ..., k_n^*\right) = k.$$

Note that $k_1^l \neq k_1^k$, because $l \neq k$. Therefore, either k_1^l or k_1^k is different from h. Now consider the following two types of agent 1: $x_1^{h'} \in \left(x^h, x^{l,k}\right)$ and $x_1^{h''} \in \left(x^{l,k}, x^k\right)$. Note that, for both types of agent 1, alternative h is the best alternative, but since h is not an option, type $x_1^{h'}$ prefers l among the available alternatives, while type $x_1^{h''}$ prefers k among the available alternatives. Therefore, under the peaks-only mechanism π, type $x_1^{h'}$ will report k_1^l and type $x_1^{h''}$ will report k_1^k. Since either k_1^l or k_1^k is different from h, mechanism π is not incentive compatible, yielding a contradiction. ■

Proposition 1 *A deterministic and peaks-only mechanism $\pi : \mathcal{K}^n \to \mathcal{K}$ is DIC and anonymous if and only if there exist $(n+1)$ integers $\alpha_1, ..., \alpha_{n+1} \in \mathcal{K}$ such that, for any $(k_1, ..., k_n) \in \mathcal{K}^n$,*

$$\pi(k_1, ..., k_n) = M(k_1, ..., k_n, \alpha_1, ..., \alpha_{n+1})$$

where the median function $M(k_1, ..., k_n, \alpha_1, ..., \alpha_{n+1})$ returns the median of $(k_1, ..., k_n, \alpha_1, ..., \alpha_{n+1})$.

Proof. The proof consists of four steps. Step 1 and 4 are identical to those in Moulin's proof. In Step 2 and 3, we need a slightly different logic.

Step 1. For each n, $n \geq 1$, define S_n as the following subset of $\mathcal{K}^{\mathcal{K}^n}$:

$$S_n = \{\pi : \mathcal{K}^n \to \mathcal{K} | \exists \alpha_1, ..., \alpha_{n+1} \in \mathcal{K} : \pi(k_1, ..., k_n) = M(k_1, ..., k_n, \alpha_1, ..., \alpha_{n+1})\}$$

It is easy to see that every element of S_n is DIC and anonymous. We now prove that, conversely, every DIC anonymous voting scheme belongs to S_n.

Step 2. We start with $n = 1$: one-agent voting schemes. We define

$$\alpha = \min_{k \in \mathcal{K}} \pi(k) \text{ and } \beta = \max_{k \in \mathcal{K}} \pi(k).$$

It is clear that $\alpha, \beta \in \mathcal{K}$ and $\alpha \leq \beta$. It is sufficient to show that for any DIC voting scheme $\pi(k)$ and for any $k \in \mathcal{K}$, we must have

$$\pi(k) = \begin{cases} \alpha & \text{if} \quad k \leq \alpha \\ k & \text{if} \quad \alpha \leq k \leq \beta \\ \beta & \text{if} \quad k \geq \beta \end{cases}$$

and therefore that $\pi(k) = M(k, \alpha, \beta)$ for all k.

Suppose $k \leq \alpha$, and assume that $\pi(k) > \alpha$. Then the agent would deviate and report $k_\alpha \in \arg\min_{k \in \mathcal{K}} \pi(k)$. Therefore, $\pi(k) = \alpha$ if $k \leq \alpha$. Next suppose $k \geq \beta$, and assume that $\pi(k) < \beta$. Then the agent could report $k_\beta \in \arg\max_{k \in \mathcal{K}} \pi(k)$ and be better off. Therefore, $\pi(k) = \beta$ if $k \geq \beta$. Finally, suppose $k \in [\alpha, \beta]$, and assume that $\pi(k) \neq k$. By Lemma 6, the option set of the agent given π is connected. Therefore, there exists k_k such that $\pi(k_k) = k$. Then, the agent could report k_k and be better off. Therefore, we have $\pi(k) = k$ if $k \in [\alpha, \beta]$. This proves that $\pi : \mathcal{K} \to \mathcal{K}$ belongs to S_1.

Step 3. Now we suppose that the claim holds for n, and we show that it holds also for $(n+1)$. Let $\pi(k_0, k_1, ..., k_n)$ be an anonymous DIC voting scheme among $(n+1)$ players. If we fix k_0, then

$$(k_1, ..., k_n) \to \pi(k_0, k_1, ..., k_n)$$

is an anonymous, DIC voting scheme among n players. By the induction assumption, it belongs to S_n. Therefore, there exist $(n+1)$ functions $\alpha_1, ..., \alpha_{n+1}$ mapping $\mathcal{K}$ to itself such that

$$\forall (k_0, k_1, ..., k_n) \in \mathcal{K}^{n+1}, \pi(k_0, k_1, ..., k_n) = M(k_1, ..., k_n, \alpha_1(k_0), ..., \alpha_{n+1}(k_0)).$$

Up to a possible relabelling of the α_i's, we can assume without loss of generality that

$$\forall k_0 \in \mathcal{K}, \ \ \alpha_1(k_0) \leq ... \leq \alpha_{n+1}(k_0) \tag{7}$$

We note that

$$M(\underbrace{..., 1, ...,}_{(n-\ell+1) \text{ times}} \underbrace{..., K, ...,}_{(\ell-1) \text{ times}} \alpha_1(k_0), ..., \alpha_{n+1}(k_0)) = \alpha_\ell(k_0)$$

and claim that $\alpha_\ell(k_0) \in S_1$, for all $\ell \in \{1, ..., n+1\}$. To prove this claim, we define

$$a_\ell = \min_{k_0 \in \mathcal{K}} \alpha_\ell(k_0) \text{ and } b_\ell = \max_{k_0 \in \mathcal{K}} \alpha_\ell(k_0).$$

Note that $\alpha_\ell(k_0)$ can be interpreted as agent ℓ's option set associated with π for given $(k_1, ..., k_n)$. By Lemma 6, the option set associated with π is connected. Therefore, we can follow the same procedure as in Step 2 to show that

$$\alpha_\ell(k_0) = M(k_0, a_\ell, b_\ell) \text{ where } 1 \le a_\ell \le b_\ell \le K, \tag{8}$$

That is, $\alpha_\ell(k_0) \in S_1$.

Now we can use (8) to reformulate our voting scheme π as

$$\pi(k_0, k_1, ..., k_n) = M(k_1, ..., k_n, M(k_0, a_1, b_1), ..., M(k_0, a_{n+1}, b_{n+1})).$$

We claim that

$$b_1 = a_2, ..., b_\ell = a_{\ell+1}, ..., b_n = a_{n+1} \tag{9}$$

and prove this by contradiction, using anonymity. The remaining proof in this step is very much the same as Moulin's. For completeness, we replicate it here.

We first note that (7) and (8) imply that, for all ℓ, $1 \le \ell \le K$,

$$\forall k_0 \in \mathcal{K}, \;\; M(k_0, a_\ell, b_\ell) \le M(k_0, a_{\ell+1}, b_{\ell+1})$$

which is equivalent to

$$a_\ell \le a_{\ell+1} \text{ and } b_\ell \le b_{\ell+1}. \tag{10}$$

To prove claim (9) it is sufficient to rule out both $a_{\ell+1} < b_\ell$ and $a_{\ell+1} > b_\ell$. First suppose $a_{\ell+1} < b_\ell$. We can then choose $(k_0, k_1, ..., k_n) \in \mathcal{K}^{n+1}$ such that

$$\begin{aligned} a_{\ell+1} &\le k_0 < k_n \le b_\ell \\ k_1 &= ... = k_{n-\ell} = 1 \le k_0 \\ k_{n-\ell+1} &= ... = k_{n-1} = K \ge k_n \end{aligned}$$

It follows from (8) and (10) that $\forall\; \ell' \le \ell - 1$ and $\ell'' \ge \ell + 1$

$$\alpha_{\ell'}(k_0) \le \alpha_\ell(k_0) = \alpha_{\ell+1}(k_0) = k_0 \le \alpha_{\ell''}(k_0).$$

Therefore,

$$\begin{aligned} & \pi(k_0, k_1, ..., k_n) \\ = \; & M(\underbrace{..., 1, ...,}_{(n-\ell)\text{ times}} \underbrace{..., K, ...,}_{(\ell-1)\text{ times}} k_n, \underbrace{..., \alpha_{\ell'}(k_0), ...,}_{(\ell-1)\text{ times}} k_0, k_0, \underbrace{..., \alpha_{\ell''}(k_0), ...}_{(n-\ell)\text{ times}}) \\ = \; & k_0. \end{aligned}$$

Similarly, it follows from (8) and (10) that $\forall\ \ell' \leq \ell - 1$ and $\ell'' \geq \ell + 1$

$$\alpha_{\ell'}(k_n) \leq \alpha_{\ell}(k_n) = \alpha_{\ell+1}(k_n) = k_n \leq \alpha_{\ell''}(k_n).$$

Therefore,

$$\begin{aligned} & \pi(k_n, k_1, ..., k_0) \\ = \ & M(\underbrace{..., 1, ...,}_{(n-\ell) \text{ times}} \underbrace{..., K, ...,}_{(\ell-1) \text{ times}} k_0, \underbrace{..., \alpha_{\ell'}(k_n), ...,}_{(\ell-1) \text{ times}} k_n, k_n, \underbrace{..., \alpha_{\ell''}(k_n), ...}_{(n-\ell) \text{ times}}) \\ = \ & k_n. \end{aligned}$$

But, given our assumption $k_0 < k_n$, this contradicts the anonymity of π. We have proved $b_\ell \leq a_{\ell+1}$. Suppose now $b_\ell < a_{\ell+1}$. We can choose $(k_0, k_1, ..., k_n) \in \mathcal{K}^{n+1}$ such that

$$\begin{aligned} b_\ell & \leq k_0 < k_n \leq a_{\ell+1} \\ k_1 & = ... = k_{n-\ell} = 1 \leq k_0 \\ k_{n-\ell+1} & = ... = k_{n-1} = K \geq k_n \end{aligned}$$

It follows from (8) and (10) that $\forall\ \ell' \leq \ell - 1$ and $\ell'' \geq \ell + 1$

$$\alpha_{\ell'}(k_0) \leq \alpha_{\ell}(k_0) = b_\ell < a_{\ell+1} = \alpha_{\ell+1}(k_0) \leq \alpha_{\ell''}(k_0).$$

Therefore,

$$\begin{aligned} & \pi(k_0, k_1, ..., k_n) \\ = \ & M(\underbrace{..., 1, ...,}_{(n-\ell) \text{ times}} \underbrace{..., K, ...,}_{(\ell-1) \text{ times}} k_n, \underbrace{..., \alpha_{\ell'}(k_0), ...,}_{(\ell-1) \text{ times}} b_\ell, a_{\ell+1}, \underbrace{..., \alpha_{\ell''}(k_0), ...}_{(n-\ell) \text{ times}}) \\ = \ & M(k_n, b_\ell, a_{\ell+1}) \\ = \ & k_n \end{aligned}$$

Similarly, it follows from (8) and (10) that $\forall\ \ell' \leq \ell - 1$ and $\ell'' \geq \ell + 1$

$$\alpha_{\ell'}(k_n) \leq \alpha_{\ell}(k_n) = b_\ell < a_{\ell+1} = \alpha_{\ell+1}(k_n) \leq \alpha_{\ell''}(k_n).$$

Therefore,

$$\begin{aligned} & \pi(k_n, k_1, ..., k_0) \\ = \ & M(\underbrace{..., 1, ...,}_{(n-\ell) \text{ times}} \underbrace{..., K, ...,}_{(\ell-1) \text{ times}} k_0, \underbrace{..., \alpha_{\ell'}(k_n), ...,}_{(\ell-1) \text{ times}} b_\ell, a_{\ell+1}, \underbrace{..., \alpha_{\ell''}(k_n), ...}_{(n-\ell) \text{ times}}) \\ = \ & M(k_0, b_\ell, a_{\ell+1}) \\ = \ & k_0 \end{aligned}$$

But this contradicts the anonymity of π since $k_0 < k_n$. Therefore, we must have $b_\ell = a_{\ell+1}$, which completes the proof for (9).

Now we can use (9) and set $b_{n+1} = a_{n+2}$ to obtain the following expression for π :

$$\pi(k_0, k_1, ..., k_n) = M(k_1, ..., k_n, M(k_0, a_1, a_2), ..., M(k_0, a_\ell, a_{\ell+1}), ..., M(k_0, a_{n+1}, a_{n+2})),$$

with $a_\ell \in \mathcal{K}$ for all ℓ, and

$$1 \leq a_1 \leq a_2 \leq ... \leq a_\ell \leq a_{\ell+1} \leq ... \leq a_{n+2} \leq K.$$

Step 4. Finally, we establish that, for any such increasing sequence of a_ℓ and for every $k_0, k_1, ..., k_n$:

$$\begin{aligned} & M(k_1, ..., k_n, M(k_0, a_1, a_2), ..., M(k_0, a_\ell, a_{\ell+1}), ..., M(k_0, a_{n+1}, a_{n+2})) \\ = \; & M(k_0, k_1, ..., k_n, a_1, ..., a_{n+2}). \end{aligned} \tag{11}$$

First suppose $k_0 \leq a_1$. We can rewrite the left-hand side in (11) as

$$M(k_1, ..., k_n, a_1, ..., a_{n+2}) = \theta$$

for some $\theta \in \mathcal{K}$. Since $(n+1)$ agents form a majority, we have $a_1 \leq \theta \leq a_{n+1}$. Thus, we have $k_0 \leq \theta \leq a_{n+2}$. We then use the following observation:

$$M(y_1, ..., y_p) = \theta \text{ and } y_{p+1} \leq \theta \leq y_{p+2} \;\Rightarrow M(y_1, ..., y_p, y_{p+1}, y_{p+2}) = \theta. \tag{12}$$

This implies that

$$M(k_0, k_1, ..., k_n, a_1, ..., a_{n+2}) = \theta.$$

The proof of formula (11) in the case $k_0 \geq a_{n+2}$ is similar.

Suppose now that for some ℓ, $1 \leq \ell \leq n+1$, $a_\ell \leq k_0 \leq a_{\ell+1}$. The left-hand side in (11) is reduced to

$$M(k_1, ..., k_n, a_2, ..., a_\ell, k_0, a_{\ell+1}, ..., a_{n+1}) = \theta'$$

for some $\theta' \in \mathcal{K}$. Since $a_2 \leq \theta' \leq a_{n+1}$, we obtain $a_1 \leq \theta' \leq a_{n+2}$. By observation (12):

$$M(k_0, k_1, ..., k_n, a_1, ..., a_{n+2}) = \theta'.$$

This concludes the proof. ■

Proof of Theorem 2. First recall that since any alternative is optimal for some types of the agents, Pareto efficiency implies that the mechanism must be onto. By Theorem 1, any deterministic, onto, and DIC mechanism g is equivalent to a peaks-only mechanism π. That is, for any report profile $(x_1, ..., x_n)$ and any alternative profile $(k_1, ..., k_n)$ such that $x_i \in (x^{k_i}, x^{k_i+1}]$ for all i, we have $g(x_1, ..., x_n) = \pi(k_1, ..., k_n)$. Second, by Proposition 1, for any deterministic, anonymous, and DIC mechanism π, there exist $(n+1)$ integers $\alpha_1, ..., \alpha_{n+1} \in \mathcal{K}$ such that, for any $(k_1, ..., k_n) \in \mathcal{K}^n$,

$$\pi(k_1, ..., k_n) = M(k_1, ..., k_n, \alpha_1, ..., \alpha_{n+1}).$$

Therefore, for any report profile $(x_1, ..., x_n)$ and any alternative profile $(k_1, ..., k_n)$ such that $x_i \in (x^{k_i}, x^{k_i+1}]$ for all i, there exist $(n+1)$ integers $\alpha_1, ..., \alpha_{n+1} \in \mathcal{K}$ such that

$$g(x_1, ..., x_n) = M(k_1, ..., k_n, \alpha_1, ..., \alpha_{n+1}).$$

Since g is Pareto efficient, for any $k \in \mathcal{K}$ and for any $z \in (x^k, x^{k+1}]$, we must have $g(z, ..., z) = k$. This implies that $\alpha_1, ..., \alpha_{n+1}$ cannot be all strictly higher than 1, and that $\alpha_1, ..., \alpha_{n+1}$ cannot be all strictly lower than K. That is, at least one of $\alpha_1, ..., \alpha_{n+1}$ is equal to 1 and one of them is equal to K. Therefore, we can drop these two phantoms and rewrite

$$g(x_1, ..., x_n) = M(k_1, ..., k_n, \alpha_1, ..., \alpha_{n-1}).$$

This completes the proof. ■

Proof of Lemma 4. Due to our equivalence result (Proposition 1), it is sufficient to prove the statement of the proposition for peaks-only mechanisms. Consider any deterministic DIC and onto mechanism $\pi(k_1, ..., k_n)$ where $k_1, ..., k_n$ are the reported peaks. The Pareto set given peaks $(k_1, ..., k_n)$ is

$$\{k \in \mathcal{K} : \min(k_1, ..., k_n) \leq k \leq \max(k_1, ..., k_n)\}.$$

Consider any profile of peaks $\left(\widehat{k}_1, ..., \widehat{k}_n\right)$. In order to show that π is Pareto efficient, it is sufficient to show that,

$$\min\left(\widehat{k}_1, ..., \widehat{k}_n\right) \leq \pi\left(\widehat{k}_1, ..., \widehat{k}_n\right) \leq \max\left(\widehat{k}_1, ..., \widehat{k}_n\right).$$

We prove the above by contradiction. First assume that $\pi\left(\widehat{k}_1, ..., \widehat{k}_n\right) > k \equiv \max\left(\widehat{k}_1, ..., \widehat{k}_n\right)$. Since the mechanism is onto, there exists a profile $(k_1^*, ..., k_n^*)$ such that

$$\pi(k_1^*, ..., k_n^*) = k.$$

Consider agent 1, and fix the types of other agents at $\widehat{k}_{-1} = \left(\widehat{k}_2, ..., \widehat{k}_n\right)$. Then DIC for agent 1 with type $\widehat{k}_1$ implies that, for all k_1,

$$\pi\left(k_1, \widehat{k}_2, ..., \widehat{k}_n\right) > k.$$

Now fix agent 1's type at k_1^*, and consider agent 2. Since

$$\pi\left(k_1^*, \widehat{k}_2, \widehat{k}_3, ..., \widehat{k}_n\right) > k.$$

then DIC for agent 2 with type $\widehat{k}_2$ implies that, for all k_2,

$$\pi\left(k_1^*, k_2, \widehat{k}_3, .., \widehat{k}_n\right) > k.$$

Now we fix agent 1 and 2's types at k_1^*, k_2^*, respectively. We can proceed as before and consider agent 3. We can argue that for all k_3, we have

$$\pi\left(k_1^*, k_2^*, k_3, \widehat{k}_4, .., \widehat{k}_n\right) > k.$$

Therefore for all k_n, we have

$$\pi\left(k_1^*, .., k_{n-1}^*, k_n\right) > k$$

But this contradicts the fact that $\pi\left(k_i^*, k_{-i}^*\right) = k$.

The proof of $\pi\left(\widehat{k}_1, ..., \widehat{k}_n\right) \geq \min\left(\widehat{k}_1, ..., \widehat{k}_n\right)$ is similar. Therefore, any deterministic DIC and onto mechanism must be Pareto efficient. ■

Proof of Theorem 3. Suppose that $l_k > 0$ for some alternative $k \geq 2$ is part of the optimal allocation of $(n-1)$ phantoms. By optimality, the social planner must prefer this allocation of phantoms over allocating $l_k - 1$ phantoms on alternative k and $l_{k-1}+1$ phantoms on alternative $k-1$. This change matters only if it affects the median among $n-1$ phantom and n real voters. For this to happen, it must be that the total number of voters ("real" and "phantom") with values below x^k is $(n-1)$: there are exactly $\left(n - 1 - \sum_{m=1}^{k-1} l_m\right)$ "real" voters with values below x^k and $\left(\sum_{m=1}^{k-1} l_m + 1\right)$ "real" voters with values above x^k. In this case, by moving a phantom from alternative k to alternative $k-1$, the planner changes the median from k to $k-1$. In this case, the total expected utility from alternative k is given by

$$na_k + \left(n - 1 - \sum_{m=1}^{k-1} l_m\right) b_k c(x^k) + \left(\sum_{m=1}^{k-1} l_m + 1\right) b_k C(x^k)$$

The total expected utility from alternative $k-1$ is given by

$$na_{k-1} + \left(n - 1 - \sum_{m=1}^{k-1} l_m\right) b_{k-1} c(x^k) + \left(\sum_{m=1}^{k-1} l_m + 1\right) b_{k-1} C(x^k).$$

Since the planner (weakly) prefers k to $k-1$, the total expected utility from alternative k must be higher than the total expected utility from alternative $k-1$. This gives us the following "first-order condition" for all $k \geq 2$ with $l_k > 0$:

$$\left(n - 1 - \sum_{m=1}^{k-1} l_m\right)\left(x^k - c(x^k)\right) + \left(\sum_{m=1}^{k-1} l_m + 1\right)\left(x^k - C(x^k)\right) \leq 0 \tag{13}$$

Similarly, if $l_k > 0$ with $k \leq K-1$ is part of the optimal allocation of $(n-1)$ phantoms, then the social planner must prefer this allocation of phantoms to allocating $l_k - 1$ phantoms on alternative k and $l_{k+1}+1$ phantoms on alternative $k+1$. This yields another "first-order condition" for all $k \leq K-1$ with $l_k > 0$:

$$\left(n - \sum_{m=1}^{k} l_m\right)\left(x^{k+1} - c(x^{k+1})\right) + \left(\sum_{m=1}^{k} l_m\right)\left(x^{k+1} - C(x^{k+1})\right) \geq 0. \tag{14}$$

This two first-order conditions can be rewritten as bounds (4) and (5) on phantom distributions, which are replicated here, for alternative k with $l_k > 0$:

$$\begin{aligned}
\sum_{m=1}^{k-1} l_m &\geq n\frac{x^k - c\left(x^k\right)}{C\left(x^k\right) - c\left(x^k\right)} - 1, \text{ for all } k \geq 2, \\
\sum_{m=1}^{k} l_m &\leq n\frac{x^{k+1} - c\left(x^{k+1}\right)}{C\left(x^{k+1}\right) - c\left(x^{k+1}\right)}, \text{ for all } k \leq K-1.
\end{aligned}$$

Lemma 7 below shows that the above two conditions hold strictly for alternative k with $l_k = 0$.

Therefore, we can construct the (unique) candidate distribution of phantom voters' peaks as follows. We first derive bounds for l_1^* by taking $k = 2$ in (4) and $k = 1$ in (5):

$$n\frac{x^2 - c\left(x^2\right)}{C\left(x^2\right) - c\left(x^2\right)} - 1 \leq l_1^* \leq n\frac{x^2 - c\left(x^2\right)}{C\left(x^2\right) - c\left(x^2\right)}.$$

Since the two bounds differ by 1 and l_1^* must be an integer, l_1^* is generically unique and must be equal to $\left\lceil n\frac{x^2-c\left(x^2\right)}{C(x^2)-c(x^2)}\right\rceil$, where $\lceil z \rceil$ denotes the largest integer that is below z.

Next note that, for all $2 \leq k \leq K-1$, conditions (4) and (5) imply that

$$n\frac{x^k - c\left(x^k\right)}{C\left(x^k\right) - c\left(x^k\right)} - 1 \leq \sum_{m=1}^{k} l_m^* \leq n\frac{x^{k+1} - c\left(x^{k+1}\right)}{C\left(x^{k+1}\right) - c\left(x^{k+1}\right)}.$$

Hence, $\sum_{m=1}^{k} l_m^*$ is also generically unique and must be equal to $\left\lceil n\frac{x^{k+1}-c\left(x^{k+1}\right)}{C(x^{k+1})-c(x^{k+1})}\right\rceil$. As a result, we can deduce l_2^* as

$$l_2^* = \sum_{m=1}^{2} l_m^* - l_1^* = \left\lceil n\frac{x^3 - c(x^3)}{C(x^3) - c(x^{k+1})}\right\rceil - \left\lceil n\frac{x^2 - c(x^2)}{C(x^2) - c(x^2)}\right\rceil.$$

Similarly, we can obtain recursively for all l_k^* with $2 \leq k \leq K-1$:

$$l_k^* = \left\lceil n\frac{x^{k+1} - c(x^{k+1})}{C(x^{k+1}) - c(x^{k+1})}\right\rceil - \left\lceil n\frac{x^k - c(x^k)}{C(x^k) - c(x^k)}\right\rceil.$$

Note that

$$\frac{C\left(x\right) - c\left(x\right)}{x - c\left(x\right)} = \frac{C\left(x\right) - x}{x - c\left(x\right)} + 1.$$

Since $C\left(x\right) - x$ is strictly decreasing in x while $x - c\left(x\right)$ is strictly increasing in x, the above expression is strictly decreasing in x. Therefore,

$$\frac{x - c\left(x\right)}{C\left(x\right) - c\left(x\right)}$$

is strictly increasing in x. Since $x^{k+1} > x^k$, we obtain that $l_k^* \geq 0$.

Finally, since there are $(n-1)$ phantom voters in total, we have

$$l_K^* = n - 1 - \sum_{m=1}^{K-1} l_m^* = n - 1 - \left\lceil n\frac{x^K - c(x^K)}{C(x^K) - c(x^K)}\right\rceil.$$

It is clear that $n\frac{x^K-c(x^K)}{C(x^K)-c(x^K)} < n$, so $l_K^* \geq 0$.

To complete the proof, we need to argue that the phantom distribution we constructed above is indeed optimal. Note that we are optimizing a bounded function over a discrete domain, so that the optimal solution always exists. Because the optimal solution has to satisfy the two necessary conditions (4) and (5), and because there is essentially unique distribution that satisfies these two conditions, our candidate distribution $\{l_k^*\}$ must be optimal. ■

Lemma 7 *The bounds (4) and (5) hold (with strict inequality) for all $k \in \mathcal{K}$ with $l_k = 0$.*

Proof. First let us define κ_1 and κ_2 as follows:

$$\begin{aligned}\kappa_1 &= \max\{m \in \mathcal{K} : l_k = 0 \text{ for all } k \leq m\}, \\ \kappa_2 &= \min\{m \in \mathcal{K} : l_k = 0 \text{ for all } k \geq m\}.\end{aligned}$$

We need to consider several cases.

Case 1: Both κ_1 and κ_2 exist. Then we have $l_1 = ... = l_{\kappa_1} = 0$, and $l_{\kappa_2} = ... = l_K = 0$. An alternative k with $l_k = 0$ could belong to one of the following three possible scenarios:

(i) $k \leq \kappa_1$. Since $l_1 = ... = l_{\kappa_1} = 0$, condition (5) holds trivially and we only need to prove condition (4). By definition of κ_1, $l_{\kappa_1+1} > 0$. Thus, we have

$$\sum_{m=1}^{\kappa_1} l_m \geq n\frac{x^{\kappa_1+1} - c\left(x^{\kappa_1+1}\right)}{C\left(x^{\kappa_1+1}\right) - c\left(x^{\kappa_1+1}\right)} - 1.$$

Since $l_1 = ... = l_{\kappa_1} = 0$, we have

$$\sum_{m=1}^{k-1} l_m = \sum_{m=1}^{\kappa_1} l_m \geq n\frac{x^{\kappa_1+1} - c\left(x^{\kappa_1+1}\right)}{C\left(x^{\kappa_1+1}\right) - c\left(x^{\kappa_1+1}\right)} - 1 > n\frac{x^k - c\left(x^k\right)}{C\left(x^k\right) - c\left(x^k\right)} - 1,$$

where the second inequality follows because $\frac{x-c(x)}{C(x)-c(x)}$ is strictly increasing in x and $x^{\kappa_1+1} > x^k$.

(ii) $k \geq \kappa_2$. Since $l_{\kappa_2} = ... = l_K = 0$, for all $k \geq \kappa_2$, we have

$$\sum_{m=1}^{k-1} l_m = n - 1 - \sum_{k}^{K} l_m = n - 1.$$

Hence, condition (4) is trivially satisfied, and we only need to prove condition (5). By definition of κ_2, $l_{\kappa_2-1} > 0$. So we have

$$\sum_{m=1}^{\kappa_2-1} l_m \leq n\frac{x^{\kappa_2} - c\left(x^{\kappa_2}\right)}{C\left(x^{\kappa_2}\right) - c\left(x^{\kappa_2}\right)}.$$

Therefore,

$$\sum_{m=1}^{k} l_m = n - 1 = \sum_{m=1}^{\kappa_2-1} l_m \leq n\frac{x^{\kappa_2} - c\left(x^{\kappa_2}\right)}{C\left(x^{\kappa_2}\right) - c\left(x^{\kappa_2}\right)} \leq n\frac{x^{k+1} - c\left(x^{k+1}\right)}{C\left(x^{k+1}\right) - c\left(x^{k+1}\right)}.$$

Again the last inequality follows from the monotonicity of $\frac{x-c(x)}{C(x)-c(x)}$ and the fact that $x^{\kappa_2} < x^{k+1}$.

(iii) $k \in (\kappa_1, \kappa_2)$. Define k_1 and k_2 as follows:

$$\begin{aligned}k_1 &= \max\{m \in \mathcal{K} : m < k \text{ and } l_m > 0\}, \\ k_2 &= \min\{m \in \mathcal{K} : m > k \text{ and } l_m > 0\}.\end{aligned}$$

Both k_1 and k_2 are well defined for all $k \in (\kappa_1, \kappa_2)$. By definition of k_1 and k_2, we have

$$\sum_{m=1}^{k} l_m = \sum_{m=1}^{k_1} l_m \text{ and } \sum_{m=1}^{k-1} l_m = \sum_{m=1}^{k_2-1} l_m,$$

and

$$\sum_{m=1}^{k_2-1} l_m \geq n\frac{x^{k_2} - c\left(x^{k_2}\right)}{C\left(x^{k_2}\right) - c\left(x^{k_2}\right)} - 1, \text{ and } \sum_{m=1}^{k_1} l_m \leq n\frac{x^{k_1+1} - c\left(x^{k_1+1}\right)}{C\left(x^{k_1+1}\right) - c\left(x^{k_1+1}\right)}.$$

Since $\frac{x-c(x)}{C(x)-c(x)}$ is increasing in x, and $x^{k_1+1} < x^{k+1} < x^{k_2+1}$, we have

$$\sum_{m=1}^{k-1} l_m > n\frac{x^{k} - c\left(x^{k}\right)}{C\left(x^{k}\right) - c\left(x^{k}\right)} - 1, \text{ and } \sum_{m=1}^{k} l_m < n\frac{x^{k+1} - c\left(x^{k+1}\right)}{C\left(x^{k+1}\right) - c\left(x^{k+1}\right)}.$$

Case 2: Neither κ_1 nor κ_2 exists. Then the argument of Case 1(iii) applies for all k with $l_k = 0$.

Case 3: κ_1 exists but κ_2 does not. Consider alternative k with $l_k = 0$. If $k \leq \kappa_1$, the argument of Case 1(i) applies. If $k > \kappa_1$, the argument of Case 1 (iii) applies.

Case 4: κ_2 exists but κ_1 does not. Consider alternative k with $l_k = 0$. If $k \geq \kappa_2$, the argument of Case 1(ii) applies. If $k < \kappa_2$, the argument of Case 1(iii) applies. ■

Proof of Corollary 1. Recall that the candidate position k^* is defined as

$$k^* \equiv \min\left\{k \in \mathcal{K} : x^{k+1} \geq (C(x^{k+1}) + c(x^{k+1}))/2\right\}.$$

By definition of k^*

$$x^{k+1} \geq (c\left(x^{k+1}\right) + C\left(x^{k+1}\right))/2 \text{ for all } k \geq k^*,$$

and

$$2x^{k+1} < (c\left(x^{k+1}\right) + C\left(x^{k+1}\right))/2 \text{ for all } k < k^*.$$

This implies that

$$\frac{x^{k+1} - c(x^{k+1})}{C(x^{k+1}) - c(x^{k+1})} \geq 1/2, \text{ for all } k \geq k^*,$$

and

$$\frac{x^{k+1} - c(x^{k+1})}{C(x^{k+1}) - c(x^{k+1})} < 1/2 \text{ for all } k < k^*.$$

Moreover we note that

$$\frac{x^{k+1} - c(x^{k+1})}{C(x^{k+1}) - c(x^{k+1})} < 1.$$

Therefore, by Theorem 3, in the optimal phantom distribution, $l^*_{k^*} = 1$, and $l^*_k = 0$ for all $k \neq k^*$. ■

References

[2011] Apesteguia, J, Ballester, M., and Ferrer, R. (2011), "On the Justice of Decision Rules," *Review of Economic Studies* **78**(1): 1-16.

[2011] Azrieli, Y. and Kim, S. (2011), "Pareto Efficiency and Weighted Majority Rules," discussion paper, Ohio State University.

[2005] Bagnoli, M. and Bergstrom, T. (2005), "Log-Concave Probability and Its Applications," *Economic Theory* **26**(2), 445-469.

[1994] Barbera, S., and Jackson, M. (1994), "A Characterization of Strategy Proof Mechanisms for Economies with Pure Public Goods," *Social Choice and Welfare* **11**, 241-252.

[2006] Barbera, S., and Jackson, M. (2006), "On the Weights of Nations: Assigning Voting Weights in a Heterogeneous Union," *Journal of Political Economy* **114**, 317-339.

[1990] Barbera, S. and Peleg, B. (1990), "Strategy-proof Voting Schemes with Continuous Preferences," *Social Choice and Welfare* **7**, 31-38.

[1948] Black, D. (1948), "On the Rationale of Group Decision-Making," *Journal of Political Economy* **56,** 23-34

[2009] Borgers, T., and Postl, P. (2009), "Efficient Compromising," *Journal of Economic Theory* **144**, 2057-2076.

[2013] Chakravarty, S. and Kaplan, T. (2013), "Optimal Allocation without Transfer Payments," *Games and Economic Behavior* **77**(1), 1-20.

[1997] Ching, S. (1997): "Strategy-Proofness and 'Median Voters', " *International Journal of Game Theory* **26,** 473-490.

[2011] Chatterjje, S. and Sen, A. (2011), "Tops-only Domains," *Economic Theory* **46**, 255-282.

[2013] Drexl, M. and Klein, A. (2013), "Collective Decision Making with Transferable Utilities," discussion paper, University of Bonn.

[2002] Ehlers, L., Peters, H., and Storcken, T. (2002), "Strategy-Proof Probabilistic Decision Schemes for One-Dimensional Single-Peaked Preferences," *Journal of Economic Theory* **105**, 408-434.

[2013] Gershkov, A., Goeree J., Kushnir, A., Moldovanu, B. and Shi, X. (2013), "On the Equivalence of Bayesian and Dominant Strategy Implementation," *Econometrica* **81**(1), 197–220.

[1973] Gibbard, A. (1973), "Manipulation of Voting Schemes: A General Result," *Econometrica* **41**(4), 587-601.

[2012] Condorelli, D. (2012), "What Money Can't Buy: Efficient Mechanism Design with Costly Signals," *Games and Economic Behavior* **75**(2), 613-624.

[2012] Dokow, E., Feldman, M., Meir, R. and Nehama, I. (2012), "Mechanism Design on Discrete Lines and Cycles," *EC'12*, June 4-8, 2012, Valencia, Spain.

[2008] Hartline, J. and Roughgarden, T. (2008), "Mechanism Design and Money Burning," *STOC*, May 17-20, 2008.

[2007] Jackson, M. and Sonnenschein, H. (2007), "Overcoming Incentive Constraints by Linking Decisions," *Econometrica* **75**, 241-258.

[1999] Jehiel, P., Moldovanu, B. and Stacchetti, E. (1999), "Multidimensional Mechanism Design for Auctions with Externalities," *Journal of Economic Theory* **85**(2), 1999, 258-294.

[2009] Kovac, E., and Mylovanov, T. (2009), "Stochastic Mechanisms in Settings without Monetary Transfers," *Journal of Economic Theory*, **144**(4), 2009, 1373-1395

[2002] McLean, R., and Postlewaite, A. (2002): "Informational Size and Incentive Compatibility," *Econometrica* **70**(6), 2421–2453.

[1980] Moulin, H. (1980), "On Strategy-Proofness and Single Peakedness," *Public Choice* **35**, 437-455.

[1981] Myerson, R. (1981), "Optimal Auction Design," *Mathematics of Operations Research* **5**, 58-73.

[2007] Nehring, K., and Puppe, C. (2007), "The Structure of Strategy-Proof Social Choice – Part I: General Characterization and Possibility Results on Median Spaces," *Journal of Economic Theory* **135**, 269-305.

[1969] Rae, D. (1969), "Decision Rules and Individual Values in Constitutional Choice," *American Political Science Review* **63**, 40-56.

[1975] Satterthwaite, M. (1975), "Strategy-Proofness and Arrow's Conditions. Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions," *Journal of Economic Theory* **10**, 187-217.

[2002] Schummer, J., and Vohra, R. (2002), "Strategy-proof Location on a Network," *Journal of Economic Theory* **104**, 405-428.

[2007] Schummer, J., and Vohra, R. (2007), "Mechanism Design without Money," in *Algorithmic Mechanism Design*, N. Nisan et al. (eds), Cambridge: Cambridge University Press.

[1991] Sprumont, Y. (1991), "The Division Problem with Single-Peaked Preferences: A Characterization of the Uniform Allocation Rule," *Econometrica* **59,** 509-519.

[1995] Sprumont, Y. (1995), "Strategy Proof Collective Choice in Economic and Political Environments," *Canadian Journal of Economics* **28**, 68-107.

[2012] Schmitz, P. and Tröger T. (2012), "The (Sub-) Optimality of the Majority Rule," *Games and Economic Behavior* **74**(2), 651-665.