The Maximal Payoff and Coalition Formation in Coalitional Games*

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Summary: This paper studies the duality between the blocking power and producing ability of all sub-coalitions in coalitional TU and NTU games, and it establishes three main results: 1) the usual TU (NTU) core is empty if and only if (if) sub-coalitions can produce a higher payoff than the grand coalition's payoff; 2) the new core is always non-empty; and 3) optimal (efficient) coalitions will form in coalitional TU (NTU) games in a manner determined by their shadow values.

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1. Introduction

In the study of coalitional games with transferable or non-transferable utilities (TU or NTU), the previous literature has largely focused on the question of how to split the grand coalition's payoff. This paper explores whether players could obtain higher payoffs than the grand coalition's payoffs. Such exploration discovers each coalition's shadow value and the concept of the maximum of generated-payoffs (efficient generated-payoffs) in coalitional TU (NTU) games, which reveals the duality between sub-coalitions' blocking power and producing ability. In TU games, the duality says that the maximum of generated-payoffs (*mgp*) and the minimal worth of the grand coalition needed to guarantee no-blocking are exactly the same. In NTU games, the duality becomes that the set of efficient generated-payoffs (*EGP*) and the minimum no-blocking frontier have a non-empty intersection.

Such duality is perhaps the most salient feature of cooperation, because it leads to a new theory of coalition formation and the concept of the new core. Further, it implies not only the existing theorems on core existence (Bondareva [1962], Shapley [1967], and Scarf [1967b]) but also a new theorem on core existence: the usual TU (NTU) core is empty if and only if (if) players can produce a higher payoff than the grand coalition's payoff by forming a minimal balanced collection. Such payoffs generated by minimal balanced collections are a step forward beyond payoffs generated by partitions in previous studies such as Sun, Trockel, and Yang (2008) on market games, Zhou (1994) on bargaining set, and Guesnerie and Oddou (1979) on the c-core.

Finally, the paper defines the new TU (NTU) core as the splits of the maximal payoff (subset of the efficient payoffs) that are unblocked by all coalitions. It then shows that the new core is always non-empty and that the optimal (efficient) collections of coalitions will form in coalitional TU (NTU) games in a manner determined by their shadow values.

The rest of the paper is organized as follows. Section 2 studies the duality between sub-coalitions' blocking power and producing ability in TU games, section 3 studies the new TU core and the formation of optimal coalitions, and section 4 provides deeper results on the new NTU core and on the formation of efficient coalitions. Section 5 applies the new core to three known examples, section 6 concludes, and the appendix provides proofs.

2. The Maximum of Generated-payoffs and a New Theorem on the Usual Core

Let $N=\{1, 2, ..., n\}$ be the set of players, $\mathcal{N}=2^N$ be the set of all coalitions. A TU game in coalitional (or characteristic) form is a set function $v\colon \mathcal{N}\rightarrow \boldsymbol{R}_+$ with $v(\emptyset)=0$, given below,

(1) $$\Gamma = \{N, v(\cdot)\},$$

which specifies a joint payoff $v(S)$ for each coalition $S\in\mathcal{N}$. We use a lowercase v in $v(\cdot)$ to define the above TU game (1), and an uppercase V in $V(\cdot)$ to define NTU games in Section 4.

Let $X(v(N)) = \{x\in \boldsymbol{R}_+^n \,|\, \Sigma_{i\in N}x_i = v(N)\}$ denote the set of payoff vectors that are splits of $v(N)$, which is called the *preimputation space*. Given $S\in\mathcal{N}$, a split $x=(x_1,...,x_n)\in X(v(N))$ is unblocked by S if it gives S no less than $v(S)$ (i.e., $\Sigma_{i\in S}x_i \geq v(S)$), and it is in the usual core if it is unblocked by all $S\neq N$. Denote the usual core of (1) as

(2) $$c_0(\Gamma) = \{x\in X(v(N)) \,|\, \Sigma_{i\in S}x_i \geq v(S) \text{ for all } S \neq N\}.$$

We use a lowercase c in $c_0(\Gamma)$ to denote TU core and an uppercase C in $C_0(\Gamma)$ to denote NTU core. Before we define generated-payoffs, let us review the concept of balanced collections. A collection of coalitions $\mathcal{B} = \{T_1, ..., T_k\}$ is balanced if it has a balancing vector or a positive vector $w\in\boldsymbol{R}_{++}^k$ such that $\Sigma_{T\in\mathcal{B}(i)}w_T=1$ for each $i\in N$, where $\mathcal{B}(i)=\{T\in\mathcal{B} \,|\, i\in T\}$ is the subset of coalitions in $\mathcal{B}$ to which player i belongs. A balanced collection is minimal if no proper subcollection is balanced. It is known that a balanced collection is minimal if and only if its balancing vector is unique (Shapley, 1967).

Definition 1: *Given game (1) and a minimal balanced collection* $\mathcal{B}$ *with its unique balancing vector w, the payoff generated by* $\mathcal{B}$ *is given by* $gp(\mathcal{B}) = \Sigma_{T\in\mathcal{B}} w_T v(T)$. *The maximum of generated-payoffs (mgp) and the maximal payoff (mp) are given, respectively, by*

(3) $mgp = mgp(\Gamma) = Max\ \{gp(\mathcal{B}) \mid \mathcal{B} \in B\}$, *and*

(4) $mp = mp(\Gamma) = Max\ \{mgp, v(N)\}$, *where*

(5) $B = \{\mathcal{B} = \{T_1, ..., T_k\} \mid N \notin \mathcal{B}, \mathcal{B}$ *is a minimal balanced collection*$\}$

denotes the set of all minimal balanced collections, excluding the grand coalition.

The definition considers only minimal balanced collections because *mgp* is achieved among minimal balanced collections, this is similar to linear programming whose optimal value is achieved among its extreme points. If $\mathcal{B} = \Delta$ is a partition (i.e., $\cup T_i = \cup_{T\in\Delta} T = N$; $T_i \cap T_j = \varnothing, i \neq j$), it is clear $gp(\Delta) = \Sigma_{T\in\Delta} v(T)$. If $\mathcal{B}$ is not a partition, it might be unclear to some readers how $gp(\mathcal{B}) = \Sigma_{T\in\mathcal{B}} w_T v(T)$ is generated by the players. To ameliorate such conceptual difficulty, we treat game (1) as a production (or input allocation) problem described below.

Assume that each player *i* has one unit of inputs to produce a homogeneous output. For each coalition (or factory) *S*, it produces *v(S)* outputs if each of its members contributes one unit of inputs, and $w_S v(S)$ outputs ($0 \leq w_S \leq 1$) if each of its members contributes w_S of inputs. With such technology, a balanced collection $\mathcal{B}$ with its balancing vector *w* produce $gp(\mathcal{B}) = \Sigma_{T\in\mathcal{B}} w_T v(T)$ outputs, and the maximal outputs (excluding *v(N)*) are equal to *mgp* in (3).

Here, the balancing weights for a balanced collection $\mathcal{B}$ are the proportions of inputs allocated to the coalitions in $\mathcal{B}(i) = \{T \in \mathcal{B} \mid i \in T\}$ by each player *i*. Given such interpretation, it stands to reason that the grand coalition will not be formed in TU games with $mgp > v(N)$, because players could produce a higher payoff than grand coalition's payoff.[1]

To illustrate this point, consider the following Internet Game in which three students are asked to host an online-forum and will be paid by sponsors according to the number of visits they generate. Suppose each student has 100 minutes of connection time (i.e., one unit of input) and could simultaneously co-host two forums with each of the two others by logging onto two computers but the total connection time is limited to 100 minutes. An example of the payoffs for each coalition or forum T are given below:

Example 1 (Internet Game**)**: $n=3$, $v(1)=v(2)=v(3)=0$, $v(12)=v(13)=v(23)=v(123)=$ \$1000. The five minimal balanced collections (excluding $\{N\}$) are: $\mathcal{B}_1=\{1,2,3\}$, $\mathcal{B}_2=\{12,3\}$, $\mathcal{B}_3=\{13,2\}$, $\mathcal{B}_4=\{23,1\}$, and $\mathcal{B}_5=\{12,13,23\}$ with $w=\{0.5,0.5,0.5\}$. By $mgp=gp(\mathcal{B}_5)=1500>v(N)=1000$, students could obtain a higher payoff than the grand coalition's payoff by hosting each of the three two-member forums for 50 minutes.[2]

In this game, it is clear that the grand coalition will not be formed. By forming each of the three two-member coalitions for 50 minutes, students will earn \$500 more than $v(N)$. Such new payoff beyond $v(N)$ is what gives rise to our new theory. The following duality result provides a foundation for our new core results and coalition formation theory.

Proposition 1: *Given game (1), the maximization problem (3) for mgp is dual to the following minimization problem:*

(6) $$mnbp = Min\ \{\Sigma_{i\in N} x_i \mid x \in \mathbf{R}^n_+;\ \Sigma_{i\in S}\, x_i \geq v(S), \text{ all } S \neq N\},$$

so mgp = mnbp holds, where mnbp denoted the minimum no-blocking payoff.

By the above duality, the balancing weights or proportions of input allocations for

[1] Other interpretations are discussed in Remark 1 at the end of next section.

[2] We simplify $v(\{i\})$ as $v(i)$, $v(\{1,2\})$ as $v(12)$. Similar simplifications apply to other coalitions.

mgp in (3) are coalitions' shadow values[3] for *mnbp* in (6), so the maximum of generated-payoffs are produced by allocating inputs to coalitions in a minimal balanced collection according to their shadow values. Since *mgp* represents sub-coalitions' ability to produce outputs or payoffs that are different from $v(N)$ and *mnbp* represents their power to block the grand coalition's proposals (Zhao, 2001), the producing ability and blocking power of sub-coalitions are dual to each other. In other words, the duality says that the maximal outputs produced by sub-coalitions and the minimal worth of the grand coalition needed to guarantee no-blocking are exactly the same. Such dual relationship is perhaps the most salient feature of cooperation, as it also holds in NTU games (see Proposition 3), and it leads directly to both previously known theorems and a new theorem on the existence of the usual core as given below:

Corollary 1: *Given game (1), its usual core is non-empty if and only if each of the following three arguments holds:*

(i) the game is balanced (Bondareva, 1962; Shapley, 1967);

(ii) players can not produce a higher payoff than the grand coalition's payoff; and

(iii) the grand coalition has enough to guarantee no-blocking (Zhao, 2001).

Precisely, the above three core arguments are: (i) $\Sigma_{T\in\mathcal{B}} w_T v(T) \leq v(N)$ for each balanced $\mathcal{B}$ with a balancing vector w, (ii) $mgp \leq v(N)$, and (iii) $v(N) \geq mnbp$. Note that the precise statement of part (iii) or $v(N) \geq mnbp$ was only formulated by the author in 2001, although the intuition was well known since early 1950's when Shapley first used the term core in Princeton workshops. As readers will see, Proposition 1 not only implies the above three core arguments, but also answers four other questions in the next section: What payoffs will be

[3] The shadow value for the grand coalition N can be given as: $w_N=1$ if $v(N)\geq mgp$, and $w_N=0$ if $v(N)<mgp$.

split? How will the payoff be split? What coalitions will form? and How much inputs will each of the formed coalitions receive?

3. The New Core and its Optimal Coalitions

Part (ii) of Corollary 1 implies that players will not split the grand coalition's payoff in games with an empty usual core, because they could split a higher payoff than $v(N)$ by forming minimal balanced collections. Then, what payoffs will players split in games with an empty usual core? We postulate that they split the maximal payoff given by $mp = Max\ \{mgp, v(N)\}$ in (4). By $mp = v(N)$ if $c_0(\Gamma) \neq \varnothing$ and $mp = mgp > v(N)$ if $c_0(\Gamma) = \varnothing$, it stands to reason that players will always split mp. This answers the question of what payoffs will be split.

Next, consider the question of how to split the maximal payoff. The requirement that a solution be free of coalitional deviations leads to the following new core:

(7) $$c(\Gamma) = \{x \in X(mp) \mid \Sigma_{i\in S} x_i \geq v(S), \text{ all } S\} = \begin{cases} c_0(\Gamma) & \text{if } v(N) = mp \\ Y(\Gamma) & \text{if } v(N) < mp, \end{cases}$$

which is identical to the usual core in (2) if the usual core is non-empty, and the optimal set $Y(\Gamma)$ for $mnbp$ in (6) if the usual core is empty, where the optimal set $Y(\Gamma)$ is given by

(8) $$Y = Y(\Gamma) = Arg\text{-}Min\{\Sigma_{i\in N} x_i \mid x \in \mathbf{R}^+_n,\ \Sigma_{i\in S}\, x_i \geq v(S) \text{ for all } S \neq N\}.$$

Now, consider the question of what coalitions will form. Because players always split mp, they will form the optimal collections or optimal coalitions that generate mp, which will be either the grand coalition (if $v(N) > mgp$) or the optimal set for (3) (if $v(N) < mgp$) or their union (if $v(N) = mgp$). The set of optimal collections is denoted as $B^*(\Gamma)$ given below:

(9) $$B^* = B^*(\Gamma) = \begin{cases} \{N\} & \text{if } mgp(\Gamma) < v(N); \\ B_0(\Gamma) & \text{if } mgp(\Gamma) > v(N); \\ \{N\} \cup B_0(\Gamma) & \text{if } mgp(\Gamma) = v(N); \end{cases}$$

where $mgp(\Gamma)$ is given in (3), and $B_0(\Gamma)$ is its optimal set given by

(10) $$B_0 = B_0(\Gamma) = \{\mathcal{B}\in B \mid gp(\mathcal{B}) = mgp\} = Arg\text{-}Max\{gp(\mathcal{B}) \mid \mathcal{B}\in B\}.$$

Finally, the unique balancing vector for each optimal $\mathcal{B}$ answers the question of how much inputs will each of the formed coalitions receive. These answers are summarized below:

Proposition 2: *Given game (1), let* $mp(\Gamma)$, $c(\Gamma)\neq\varnothing$ *and* $B^*(\Gamma)$ *be given in (4), (7) and (9), respectively. Then, players will split* $mp(\Gamma)$ *within the new core* $c(\Gamma)$*; the optimal collections in* $B^*(\Gamma)$ *will form, and each coalition T in an optimal collection* $\mathcal{B}$ *with a unique balancing vector w will receive* w_T *of inputs from each of its members.*

In the following three remarks, we discuss new interpretations for the balancing weights, the relationship between the new core and the game's balanced cover, and the advantage of the new core and its policy implications, respectively.

Remark 1: Besides proportions of input allocations, the balancing weights can also be interpreted as: (i) length or percentage of time for which coalitions form, assuming the game lasts for one unit of time (see Example 6 in section 5); (ii) the frequency with which a player joins his coalitions in a collection, assuming the game is replicated/repeated for a finite number of times (such as k identical sets of the travelers in Example 5 in Section 5); and (iii) the probability with which each player joins the coalition, by introducing uncertainty.

Remark 2: To see the relationship between the new core and the core of the game's balanced cover, let $mp(S)$ denote the maximal payoff of the subgame obtained by restricting the game (1) to each $S\subseteq N$. Then, the game's balanced cover is the game $\Gamma_{bc}=\{N, mp(S)\}$, while the new core is the core of another game $\Gamma_{new}=\{N, u(S)\}$ in which $u(S)=v(S)$ for all $S\neq N$ and $u(N)= mp(N)=mp$. Therefore, the new core and the core of the game's balanced cover are the usual core of two different games, although they are identical.

Remark 3: The new core has three advantages: it is always non-empty, it advances coalition formation from grand coalition and partitions to optimal collections (see Myerson [1980] and Maskin [2003] for survey), and it achieves the maximal surplus with a superior policy (see the discussion after Example 5 for an example of such policy advantage).

4. The New NTU Core and its Efficient Coalitions

A coalitional NTU game, or an NTU game in characteristic form, is defined as

(11) $$\Gamma = \{N, V(\cdot)\},$$

which specifies a non-empty set of payoffs, $V(S)\subset \boldsymbol{R}^S$, for each $S\in\mathcal{N}$, where $\boldsymbol{R}^S$ is the Euclidean space whose dimension is the number of players in S and whose coordinates are the players in S. For each $S\in\mathcal{N}$, let the (weakly) efficient set of $V(S)$ be given as

$$\partial V(S) = \{ y\in V(S) \mid \text{there is no } x \in V(S) \text{ such that } x>>y\},$$

where vector inequalities are defined as below: $x \geq y \Leftrightarrow x_i \geq y_i$, all i; $x > y \Leftrightarrow x \geq y$ and $x \neq y$; and $x >> y \Leftrightarrow x_i > y_i$, all i.

Scarf (1967b) introduced the following two assumptions for (11): (i) each $V(S)$ is closed and comprehensive (i.e., $y\in V(S)$, $u\in \boldsymbol{R}^S$ and $u\leq y$ imply $u\in V(S)$); (ii) for each S, $\{y\in V(S)|y_i\geq\partial V(i)>0$, all $i\in S\}$ is non-empty and bounded, where $\partial V(i)=Max\{x_i|x_i\in V(i)\}$. Under these assumptions, each $\partial V(S)$ is closed, non-empty and bounded.

Given $S\in\mathcal{N}$, a payoff vector $u\in \boldsymbol{R}^n_+$ is blocked by S if S can obtain a higher payoff for each of its members than that given by u, or precisely if there is $y\in V(S)$ such that $y >> u_S = \{u_i|i\in S\}$ (i.e., $u_S\in V(S)\backslash\partial V(S)$). A payoff vector $u\in\partial V(N)$ is in the usual core if it is unblocked by all $S \neq N$, so the usual core of (11) can be given as

(12) $C_0(\Gamma) = \{ u \in \partial V(N) \mid u_S \notin V(S) \setminus \partial V(S), \text{ all } S \neq N \}.$

We now define a balanced NTU game (Scarf, 1967b) geometrically. For each $S \neq N$, let $\tilde{v}(S) = V(S) \times \boldsymbol{R}^{-S} \subset \boldsymbol{R}^n$, where $\boldsymbol{R}^{-S} = \prod_{i \in N \setminus S} \boldsymbol{R}^i$. The following concepts of efficient generated-payoffs and efficient payoffs are the NTU counterparts of *mgp* and *mp* in Definition 1:

Definition 2: *Given game (11), the payoffs generated by each* $\mathcal{B} \in B$ *are given by* $GP(\mathcal{B}) = \cap_{S \in \mathcal{B}} \tilde{v}(S) \subset \boldsymbol{R}^n$. *The sets of generated-payoffs (GP), efficient generated-payoffs (EGP) and efficient payoffs (EP) are given, respectively, as*

(13) $GP = \bigcup_{\mathcal{B} \in B} GP(\mathcal{B}),\ EGP = \partial GP = \{ y \in GP \mid \exists \text{ no } x \in GP \text{ such that } x >> y \}$, *and*

(14) $EP = EP(\Gamma) = \partial(GP \cup V(N)) = \{ y \in GP \cup V(N) \mid \exists \text{ no } x \in GP \cup V(N) \text{ with } x >> y \}$,

where B is the set of minimal balanced collections (excluding N) given in (5).

Note that $GP(\mathcal{B})$ is simplified to $GP(\mathcal{B}) = \prod_{S \in \mathcal{B}} V(S)$ when $\mathcal{B}$ is a partition. Similar to the TU case, we only need to consider minimal balanced collections because they determine *EGP*. Figure 1 illustrates such generated-payoffs in the following Example 2:

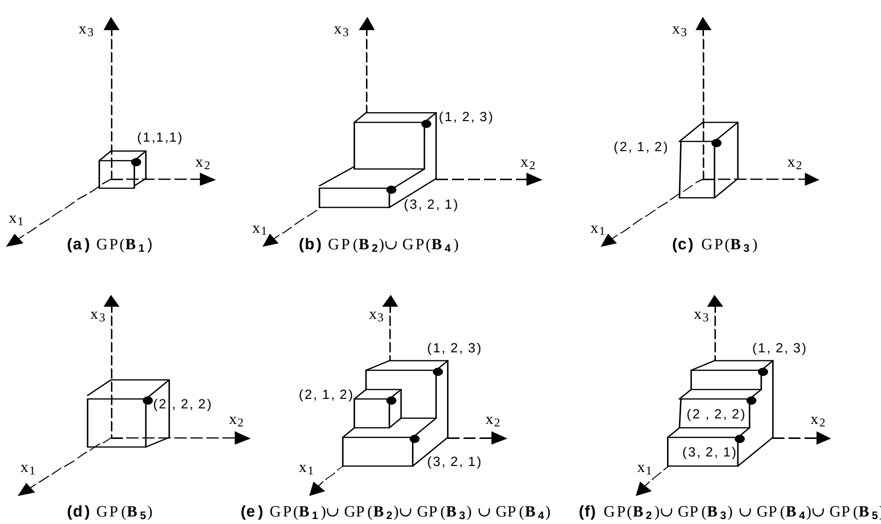


Figure 1. The generated payoffs in Example 2, where $B_1 = \{1, 2, 3\}$, $B_2 = \{12, 3\}$, $B_3 = \{13, 2\}$, $B_4 = \{23, 1\}$, and $B_5 = \{12, 13, 23\}$.

Example 2: $n = 3$, $V(i) = \{x_i \mid x_i \leq 1\}$, $i = 1, 2, 3$; $V(12) = \{(x_1, x_2) \mid (x_1, x_2) \leq (3,2)\}$, $V(13) = \{(x_1, x_3) \mid (x_1, x_3) \leq (2,2)\}$, $V(23) = \{(x_2, x_3) \mid (x_2, x_3) \leq (2, 3)\}$, $V(123) = \{x \mid x_1 + x_2 + x_3 \leq 5\}$. For each $\mathcal{B}_i \in \mathcal{B}$, one has: $GP(\mathcal{B}_1) = \{x \mid x \leq (1,1,1)\}$, $GP(\mathcal{B}_2) = \{x \mid x \leq (3,2,1)\}$, $GP(\mathcal{B}_3) = \{x \mid x \leq (2,1,2)\}$, $GP(\mathcal{B}_4) = \{x \mid x \leq (1,2,3)\}$, $GP(\mathcal{B}_5) = \{x \mid x \leq (2,2,2)\}$, where $\mathcal{B}_i$ are the same as in Example 1.

Now, the game (11) is balanced if $GP(\Gamma) \subset V(N)$ or if for each balanced $\mathcal{B}$, $u \in V(N)$ must hold if $u_S \in V(S)$ for all $S \in \mathcal{B}$. To see a balanced game geometrically, visualize that one is flying in a jet above the Rocky Mountains, and treat the generated-payoffs as peaks of the mountains and $V(N)$ as clouds. Then, a game is balanced if one sees only clouds and unbalanced if one sees at least one peak above the clouds. In Example 2, one sees three peaks above the clouds (see Figure 2b), so the game is unbalanced. In Example 3, one sees only clouds (see Figure 2a), so the game is now balanced.

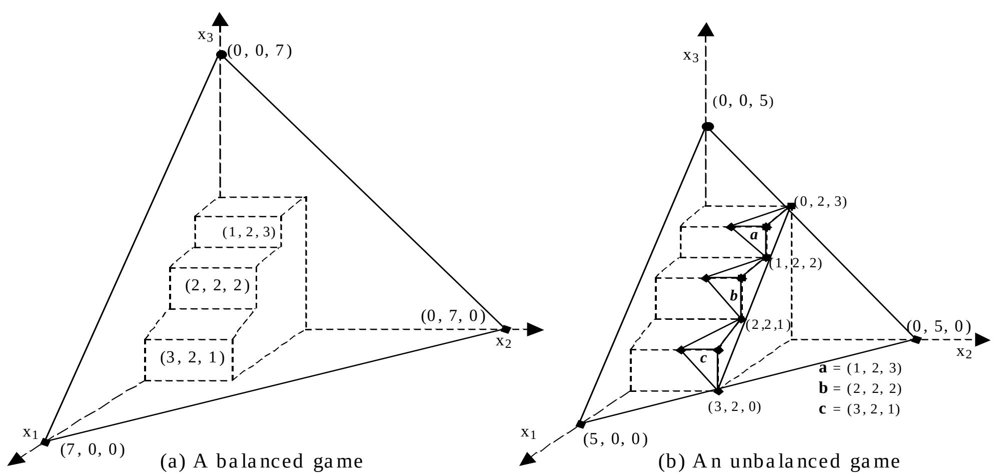


Figure 2. Balanced and unbalanced games.

Example 3: Same as Example 2 except $V(123) = \{x \mid x_1 + x_2 + x_3 \leq 7\}$.

Note that the collection $\mathcal{B}_5 = \{12, 13, 23\}$ in Example 2 generates new payoffs that are outside of those generated by the four partitions and are better than $v(N)$ (see point *b* in Figure 2b and the difference between [e] and [f] in Figure 1). Needless to say, it is the discovery of such new and better payoffs that gives rise to our new coalition formation theory.

Recall that a payoff vector u is unblocked by S if $u \in [V(S) \backslash \partial V(S)]^C \times \boldsymbol{R}^{-S} \subset \boldsymbol{R}^n$ or if $u_S \notin V(S) \backslash \partial V(S)$, where the superscript C denotes the complement of a set. The following concept of minimum no-blocking frontier is the NTU counterpart of *mnbp* in (6):

Definition 3: *Given game (11), the set of payoffs unblocked by all S≠N (UBP) and the minimum no-blocking frontier (MNBF) are given, respectively, as*

(15) $$UBP = UBP(\Gamma) = \bigcap_{S \neq N} \{[V(S) \backslash \partial V(S)]^C \times \boldsymbol{R}^{-S}\} \subset \boldsymbol{R}^n, \text{ and}$$

(16) $$MNBF = MNBF(\Gamma) = \partial UBP = \{y \in UBP \mid \exists \text{ no } x \in UBP \text{ such that } x << y\}.$$

It is easy to see that each payoff vector on or above *MNBF* is unblocked by all $S \neq N$, and the usual core can be given as $C_0(\Gamma) = UBP \cap \partial V(N) = MNBP \cap \partial V(N)$. Similar to the TU case, *MNBF* represents sub-coalitions' power to block the grand coalition's proposals. Proposition 3 below shows that sub-coalitions' blocking power and producing ability are also equivalent in coalitional NTU games, which is the NTU counterpart of Proposition 1.

Proposition 3: *Given game (11), its minimum no-blocking frontier and efficient generated-payoffs have a non-empty intersection.*

To put it differently, the NTU counterpart of *mnbp* = *mgp* in game (1) is

(17) $$Z = Z(\Gamma) = MNBF \cap EGP \neq \varnothing.$$

It is easy to verify $a, b, c \in Z$ in Example 2 (see Figure 2b), where $a = \{1, 2, 3\}$, $b = \{2, 2, 2\}$, and $c = \{3, 2, 1\}$, so $Z \neq \varnothing$ holds in the example.

Proposition 3 is proved by a version of Scarf's closed covering theorem (1967a) due to Zhou (1994). Recall that $EGP \subseteq V(N)$ holds in balanced games. Then, $MNBF \cap EGP \neq \varnothing$ implies $MNBF \cap \partial V(N) = C_0(\Gamma) \neq \varnothing$ in balanced games. Hence, our proof of Proposition 3 implies a new proof of Scarf's core theorem. Similar to the TU case, the duality between sub-coalitions' blocking power and producing ability leads directly to three (one previously known and two new) theorems on the existence of the usual NTU core as given below:

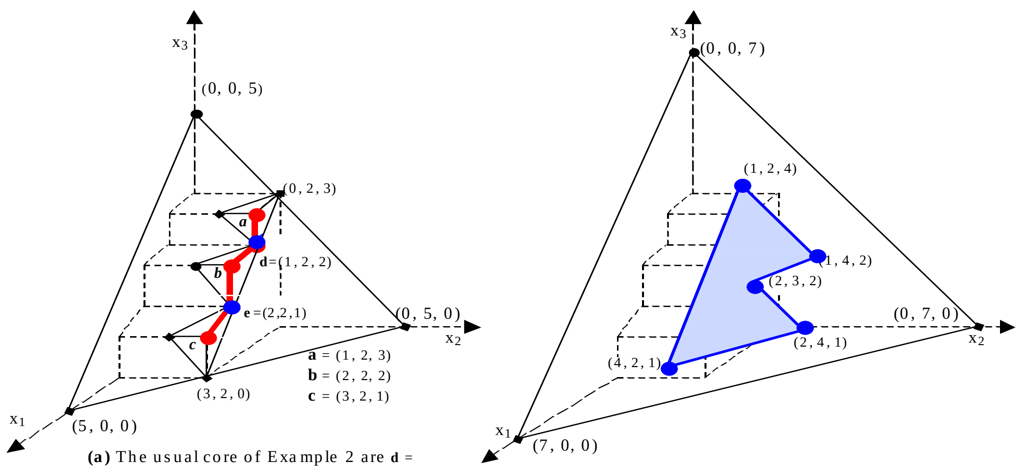
Proposition 4: *Given game (11), the following three claims hold:*

(i) its usual core is non-empty if it is balanced (Scarf, 1967b);

(ii) its usual core is empty if players can produce a better payoff than each of the grand coalition's payoff; and

(iii) its usual core is non-empty if and only if the grand coalition has enough to guarantee no blocking.

Precisely, the above three core results are: (i) $C_0(\Gamma)\neq\emptyset$ if $GP\subset V(N)$; (ii) $C_0(\Gamma)=\emptyset$ *if* $V(N)\subset GP\backslash\partial GP$; and (iii) $C_0(\Gamma)\neq\emptyset \Leftrightarrow$ there exists $x\in\partial V(N)$ and $y\in MNBF$ such that $x\geq y$.



(a) The usual core of Example 2 are **d** = (1,2,2) and **e** = (2,2,1), the new core is the thick red curve linking a, d, b, e, c.

(b) The usual core of Example 3.

Figure 3. The usual core and the new core: payoffs in the usual core are blue-colored, and payoffs in the new core are red-colored.

Due to the generality of non-transferable utilities, the usual NTU core is no longer convex (such as in Examples 2-3, see Figure 3), and the first two conditions in Proposition 4 now are only sufficient: balancedness is no longer necessary for a non-empty usual NTU core (e.g., the usual core is non-empty in the unbalanced Example 2, see Figure 3a), and "players can produce better payoffs than *V(N)*" is no longer necessary for an empty usual NTU core.

Similar to the TU case in which players will not split $v(N)$ if $v(N) < mgp$, players will not choose from $V(N)$ if they can produce better payoffs than $V(N)$ (i.e., if $V(N) \subset GP \backslash \partial GP$). This observation suggests that players will choose from the set of efficient payoffs (*EP*) given in (14). The requirement that a solution be unblocked further suggests that players will choose from a subset of *EP*, called the new NTU core, as given below:

$$(18) \quad C(\Gamma) = \{ u \in EP(\Gamma) \mid u_S \notin V(S) \backslash \partial V(S), \text{ all } S \subseteq N \}$$

$$= \begin{cases} C_0(\Gamma) & \text{if } GP(\Gamma) \subset V(N) \\ Z(\Gamma) & \text{if } V(N) \subset GP \backslash \partial GP \text{ or if } V(N) \not\subset GP \backslash \partial GP, GP \not\subset V(N), C_0(\Gamma) = \varnothing \\ C_0(\Gamma) \cup Z(\Gamma)^* & \text{if } V(N) \not\subset GP \backslash \partial GP, GP \not\subset V(N) \text{ and } C_0(\Gamma) \neq \varnothing, \end{cases}$$

where $C_0(\Gamma)$, $GP(\Gamma)$ and $Z(\Gamma)$ are given in (12), (13) and (17), respectively, and $Z(\Gamma)^* = Z(\Gamma) \cap [V(N) \backslash \partial V(N)]^C$. Note that the above new core is defined in three cases. It is equal to: (i) the usual core if the game is balanced; (ii) the set of unblocked and efficient generated-payoffs if players can produce better payoffs than $V(N)$ or if players can not produce better payoffs than $V(N)$ and the game is unbalanced with an empty usual core; and (iii) the union of the usual core and a subset of the second case if players can not produce better payoffs than $V(N)$ and the game is unbalanced with a non-empty usual core.

Figure 3a illustrates the difference between the new core and the usual core in Example 2, where the usual core has two points (i.e., *d* and *e*) and the new core is the curve connecting all three peaks. What makes the NTU core more general and deeper than the TU core is that unlike in TU games where either the game is balanced or players can produce a better payoff than $v(N)$ (i.e., either $v(N) \geq mgp$ or $v(N) < mgp$), there exist a large set of unbalanced NTU games in which players can not produce a better payoff than each $u \in V(N)$.

We are now ready to answer the question of what coalitions will form. Because players will choose payoffs within the new core, they will form the efficient collections or efficient coalitions that support the new core, which will be either the grand coalition or the set of minimal balanced collections supporting $Z(\Gamma)$ or a subset of their union. Such efficient collections is denoted as $D^*(\Gamma)$ and is given by

$$(19) \quad D^*(\Gamma) = \begin{cases} \{N\} & \text{if } GP(\Gamma) \subset V(N) \\ D_0(\Gamma) & \text{if } V(N) \subset GP \setminus \partial GP \text{ or if } V(N) \not\subset GP \setminus \partial GP, GP \not\subset V(N), C_0(\Gamma) = \emptyset \\ \{N\} \cup D_1(\Gamma) & \text{if } V(N) \not\subset GP \setminus \partial GP, GP \not\subset V(N) \text{ and } C_0(\Gamma) \neq \emptyset, \end{cases}$$

where $D_0(\Gamma) = \{\mathcal{B} \in B \mid GP(\mathcal{B}) \in Z(\Gamma)\}$ is the set of collections supporting $Z(\Gamma)$ in (17), which is the NTU counterpart of $B_0(\Gamma)$ in (10); $C_0(\Gamma)$ and $GP(\Gamma)$ are given in (12) and (13); $D_1(\Gamma) = \{\mathcal{B} \in D_0(\Gamma) \mid GP(\mathcal{B}) \in Z(\Gamma)^*\}$, where $Z(\Gamma)^* = Z(\Gamma) \cap [V(N) \setminus \partial V(N)]^C$ is the same as in (18).

It is useful to note that the efficient coalitions are defined according to the three cases of the core in (18). In Example 2, the efficient collections are: $\{N\}$, $\mathcal{B}_2 = \{12, 3\}$, $\mathcal{B}_3 = \{13, 2\}$, $\mathcal{B}_4 = \{23, 1\}$, and $\mathcal{B}_5 = \{12, 13, 23\}$.[4] Our last proposition summarizes the above results.

Proposition 5: *Given game (11), let $C(\Gamma) \neq \emptyset$ and $D^*(\Gamma)$ be given in (18) and (19). Then, players will choose efficient payoffs within the new core $C(\Gamma)$; the efficient collections in $D^*(\Gamma)$ will form, and each coalition T in an efficient collection $\mathcal{B}$ with a unique balancing vector w will receive w_T of inputs from each of its members.*

Analogous to the TU case, the advantage of the non-empty NTU core is the consequence of utilizing generated-payoffs: in the usual NTU core, players just choose from $\partial V(N)$; whereas in the new NTU core, players choose from the game's efficient payoffs, which in general includes $\partial V(N)$ as a proper subset.

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[4] Keep in mind that the set of efficient payoffs here is precisely the weakly efficient set. The payoff (2, 1, 2) is only weakly efficient because it is Pareto-dominated by (2, 2, 2).

5. More Examples

In this section, we apply the new core to three known examples in the literature. First, consider the following Garbage Game (Shapley and Shubik, 1969): each player has a bag of garbage which he must dump in someone's yard and the utility of having *b* bags dumped in one's yard is *-b*. It is known that its usual core is empty with three or more players.

Example 4 (Garbage Game, Shapley and Shubik [1969]**)**: $n\geq 3$, $v(k)=-(n-k)$ if $k<n$, and $v(n)=-n$, where $k\leq n$ is the size of each coalition. The emptiness of the usual core follows from $mp=gp(\mathcal{B}_0)=-n/(n-1)>v(N)=-n$, where $\mathcal{B}_0=\{N\backslash\{i\}|i\in N\}$ is the optimal collection with identical weight $w_T=1/(n-1)$. By Proposition 5, the new core is non-empty, and each optimal coalition $N\backslash\{i\}$ will be formed with $1/(n-1)$ of inputs (or for $[1/(n-1)]\%$ of time).

The new core suggests that a player *i* will dump $1/(n-1)$ of his garbage in the yard of each $j\,(\neq i)$ together with the $(n-2)$ players in $N\backslash\{i,j\}$.[5] This outcome is natural because it is most efficient for coalitions with $(n-1)$ players to do the job and one can not dump the garbage in one's own yard.

In the next Flight Game (Bejan and Gómez, 2009), three travelers are willing to pay \$700, \$1,000 and \$1,200, respectively, for a trip from New York to Los Angeles. A three-person jet charges \$1,000 per trip while a smaller two-person jet charges \$600.

Example 5 (Flight Game, Bejan and Gómez [2009]**)**: $n=3$, $v(1)=100$, $v(2)=400$, $v(3)=600$, $v(12)=1100$, $v(13)=1300$, $v(23)=1600$, $v(123)=1900$. By $mp=gp(\mathcal{B}_5)=2000>v(N)=1900$, the usual core is empty, the new core is unique with $x=\{400, 700, 900\}$, and the optimal collection is $\mathcal{B}_5=\{12, 13, 23\}$ with identical balancing weight $w_T=1/2$.

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[5] Here we interpret one bag of garbage as one unit of input. The frequency and percentage/length of time interpretations in Remark 1 can also be used to understand the new core for Garbage Game.

The new core recommends that a traveler flies equally often with each of the other two travelers in two-person jets (i.e., each of the three two-member coalitions be formed half of the time), while the state of art theory based the usual core recommends (see Bejan and Gómez [2009, page 4] for details) that all three fly in a three-person jet (i.e., the grand coalition be formed), where the modified usual core is non-empty with $x=\{380, 665, 855\}$, supported by $1,000 subsidy to the airline financed by a 5% surplus tax on travelers.

To see the advantage of the new core with frequency interpretation, consider a more practical version of the Flight Game, for example, in which each of the three travelers makes *2k* trips per year (or *2k* identical sets of them fly on the same day). The new core recommends flying the two-person jet $3k$ times with $300 tickets, achieving the maximal surplus of $2k\times mp$ $=\$4{,}000\times k$; while the modified usual core recommends flying the three-person jet $2k$ times with $300 ticket, achieving a surplus of only $2k\times v(N)=\$3{,}800\times k$;

Finally, consider the popular Voting Game or the game obtained by diving all payoffs by 1,000 in the Internet Game of Example 1, which has no solution in the previous literature because its usual core is empty.

Example 6 (Voting Game**):** $n=3, v(1)=v(2)=v(3)=0, v(12)=v(13)=v(23)=v(123)$ $=1$. By $mp=gp(\mathcal{B}_5)=1.5>v(N)=1$, the usual core is empty, the new core is $x=\{0.5, 0.5, 0.5\}$, and the optimal collection is $\mathcal{B}_5=\{12, 13, 23\}$ with identical weights $w_T=1/2$.

The new core predicts that a player will form an alliance with each of the other two for half of the time, which provides a natural interpretation for party-switching during a politician's career and multiple marriages in one's life. Such optimal and stable outcome can be achieved through a virtual process, such as in the Internet Game, in which a player is able to spend one half of his life before (or after) the game or two halves of his life simultaneously.[6]

6. Conclusion and Discussion

The above analysis has explored the possibility that players in a coalitional game sometimes could produce better payoffs than the grand coalition's payoffs by forming a minimal balanced collection, and it has established a duality or the equivalence between the blocking power and producing ability of all sub-coalitions. Such duality not only leads to three different understandings about the usual TU or NTU core, but also leads to a theory of coalition formation: players will end up with a payoff vector within the non-empty new core, and they will join or form the optimal or efficient coalitions for a length of time (with a quantity of inputs, or with a frequency) determined by the shadow values of the formed collections.

The new core has three advantages: it is always non-empty; it applies to situations in which minimal balanced collections, rather than a partition or the grand coalition, are formed; and it implies the first-best policy with maximal surplus. These advantages suggest a wide range of future applications of the new core, such as in partition function games, normal form games and general equilibrium models. Readers are encouraged to apply the new core to previous studies on values and core-enlargements in games with an empty usual core.

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[6] Although imaginative, such virtual processes are supported by heuristic empirical evidences. Pre- and post-season games in sports are examples of the pre- and post-game plays in a virtual process. Another example is China's three-kingdom period (220-280 A.D.) in which two players (Wei and Wu) engaged post-game plays (i.e., they lived after the famous three-kingdom game was finished).

Appendix

Proof of Proposition 1: For each $S \neq N$, let $e_S = (x_1, \ldots, x_n)' \in \mathbf{R}_n^+$ be its incidence vector or the column vector such that $x_i = 1$ if $i \in S$ and $x_i = 0$ if $i \notin S$, and $e = e_N = (1, \ldots, 1)'$ be a column vector of ones. Then, the dual problem for the minimization problem (6) is the following maximization problem:

(20) $$Max\ \{\Sigma_{S \neq N} y_S\, v(S) \mid y_S \geq 0 \text{ for all } S \neq N; \text{ and } \Sigma_{S \neq N} y_S\, e_S \leq e\}.$$

We will show that (20) is equivalent to the maximization problem (3). First, we show that the inequality constraints in (20) can be replaced by equation constraints.

Let $Ay \leq e$ and $y \geq 0$ denote the constraints in (20), where $A = A_{n \times (2^n-2)} = [e_S \mid S \neq N]$ is the constraint matrix, and y is the (2^n-2) dimensional vector whose indices are the proper coalitions. Let the rows of A be $a_1, \ldots, a_n$, and for each feasible y, let $T = T(y) = \{i \mid a_i \cdot y < 1\}$ be the set of loose constraints, so $N \backslash T = \{i \mid a_i \cdot y = 1\}$ is the set of binding constraints.

If $T(y) \neq \emptyset$, let z be defined as: $z_S = y_S + (1 - a_i \cdot y)$ if $S = \{i\}$, for each $i \in T$, and $z_S = y_S$ if $S \neq \{i\}$ for all $i \in T$. One sees that $z > y$ and $T(z) = \emptyset$. Hence, for any y with $T(y) \neq \emptyset$, there exists $z \geq 0$, $Az = e$ such that $\Sigma_{S \neq N} y_S\, v(S) \leq \Sigma_{S \neq N} z_S\, v(S)$. Hence, the feasible set of (20) can be reduced to $\{z \mid z \geq 0,\ Az = e\}$, without affecting the maximum value. So the maximization problem in (20) is equivalent to the following problem:

(21) $$Max\ \{\Sigma_{S \neq N} y_S\, v(S) \mid Ay = e, \text{ and } y \geq 0\}.$$

Next, we establish the one-to-one relationship between the extreme points of (21) and the minimal balanced collections. Note that for each feasible y in (21), $\mathcal{B}(y) = \{S \mid y_S > 0\}$ is a balanced collection. Let y be an extreme point of (21). We now show that $\mathcal{B}(y) = \{S \mid y_S > 0\}$ is a minimal balanced collection.

Assume by way of contradiction that $\mathcal{B}(y)$ is not minimal, then there exists a balanced subcollection $B' \subset \mathcal{B}(y)$ with balancing vector z. Note that $z_S > 0$ implies $y_S > 0$. Therefore, for a sufficiently small $t > 0$ (e.g., $0 < t \leq ½$, and $t \leq$ Min $\{y_S / |z_S - y_S| \mid$ all S with $y_S \neq z_S\}$), one has

$$w = y - t(y\text{-}z) \geq 0,\ w' = y + t(y\text{-}z) \geq 0.$$

$Ay = e$ and $Az = e$ lead to $Aw = e$ and $Aw' = e$. But $y = (w+w')/2$ and $w \neq w'$ contradict the assumption that y is an extreme point. So $\mathcal{B}(y)$ must be minimal.

Now, let $\mathcal{B} = \{T_1, \ldots, T_k\}$ be a minimal balanced collection with a balancing vector z. We need to show that z is an extreme point of (21). Assume again by way of contradiction that z is not an extreme point, so there exists $w \neq w'$ such that $z = (w+w')/2$. By $w \geq 0$, $w' \geq 0$, one has

$$\{S \mid w_S > 0\} \subseteq \mathcal{B} = \{S \mid z_S > 0\}, \text{ and } \{S \mid w'_S > 0\} \subseteq \mathcal{B} = \{S \mid z_S > 0\}.$$

The above two expressions show that both w and w' are balancing vectors for some subcollections of $\mathcal{B}$. Because $\mathcal{B}$ is minimal, one has $w = w' = z$, which contradicts $w \neq w'$. Therefore, z must be an extreme point of (21).

Finally, by the standard results in linear programming, the maximal value of (21) is achieved among the set of its extreme points, which are equivalent to the set of the minimal balanced collections, so (21) is equivalent to $Max\ \{\Sigma_{S \in \mathcal{B}} y_S v(S)\}$, subject to the requirements that $N \notin \mathcal{B}$ and $\mathcal{B}$ is a minimal balanced collection with the balancing vector y. This shows that (20) is equivalent to the maximization problem (3) for *mgp*, which completes the proof for Proposition 1. **Q.E.D**

Proof of Corollary 1: Given Proposition 1, it is straightforward to show parts (i-iii). Note that part (i) was first proved using Min$\{\Sigma_{i \in N} x_i \mid x \in X(v(N)),\ \Sigma_{i \in S} x_i \geq v(S)$, all $S \neq N\}$. **Q.E.D**

Proof of Proposition 2: Discussions before the proposition serve as a proof. **Q.E.D**

Our proof for Proposition 3 uses the following lemma on open covering of the simplex $\Delta^N = X(1) = \{x \in \boldsymbol{R}^+_n | \Sigma_{i\in N} x_i = 1\}$.

Lemma 1 (Scarf, 1967a; Zhou, 1994)**:** *Let $\{C_S\}$, $S \neq N$, be a family of open subsets of Δ^N that satisfy $\Delta^{N\backslash\{i\}} = \{x \in \Delta^N \mid x_i = 0\} \subset C_{\{i\}}$ for all $i \in N$, and $\cup_{S\neq N} C_S = \Delta^N$, then there exists a balanced collection of coalitions $\mathcal{B}$ such that $\cap_{S\in\mathcal{B}} C_S \neq \varnothing$.*

Proof of Proposition 3: Let *UBP* be the set of unblocked payoffs in (15), and *EGP* be the boundary or (weakly) efficient set of the generated payoff in (13). We shall first show that $UBP \cap EGP \neq \varnothing$.

For each coalition $S \neq N$, let $W_S = \{Int\, V(S) \times \boldsymbol{R}^{-S}\} \cap EGP$ be an open (relatively in *EGP*) subset of *EGP*, where $Int\, V(S) = V(S) \backslash \partial V(S)$ is the interior of $V(S)$. For each minimal balanced collection of coalitions $\mathcal{B}$, we claim that

(22) $$\cap_{S\in\mathcal{B}} W_S = \varnothing$$

holds. If (22) is false, there exists $y \in EGP$ and $y \in Int\, V(S) \times \boldsymbol{R}^{-S}$ for each $S \in \mathcal{B}$. We can now find a small $t > 0$ such that $y + te \in Int\, V(S) \times \boldsymbol{R}^{-S}$ for each $S \in \mathcal{B}$, where e is the vector of ones. By the definition of generated payoffs in (13), $y + te \in GP(\mathcal{B}) = \cap_{S\in\mathcal{B}} \{V(S) \times \boldsymbol{R}^{-S}\} \subset GP$, which contradicts $y \in EGP$. This proves (22).

Now, suppose by way of contradiction that $UBP \cap EGP = \varnothing$. Then, $EGP \subset UBP^C$, where superscript C denotes the complement of a set. The definition of W_S and

$$UBP^C = \{\cap_{S\neq N}\{[V(S)\backslash\partial V(S)]^C \times \boldsymbol{R}^{-S}\}\}^C = \cup_{S\neq N}\{Int\, V(S) \times \boldsymbol{R}^{-S}\}$$

together lead to $\cup_{S\neq N} W_S = EGP$, so $\{W_S\}$, $S \neq N$, is an open cover of *EGP*.

Because the set of generated payoffs is comprehensive and bounded from above, and the origin is in its interior (by $\partial V(i)>0$, all i), the following mapping from EGP to Δ^N:

$$f: x \to x/\Sigma x_i,$$

is a homeomorphism. Define $C_S = f(W_S)$ for all $S \subseteq N$, one sees that $\{C_S\}$, $S \neq N$, is an open cover of $\Delta^N = f(EGP)$.

For each $i \in N$, $\partial V(i)>0$ leads to $EGP \cap \{x \in \mathbf{R}^n \mid x_i = 0\} \subset W_{\{i\}}$, which in turn leads to $\Delta^{N \setminus \{i\}} = \{x \in \Delta^N \mid x_i = 0\} = f(EGP \cap \{x \in \mathbf{R}^n \mid x_i = 0\}) \subset C_{\{i\}} = f(W_{\{i\}})$. Therefore, $\{C_S\}$, $S \neq N$, is an open cover of Δ^N satisfying the conditions of Scarf-Zhou open covering theorem, so there exists a balanced collection of coalitions $\mathcal{B}_0$ such that

$$\cap_{S \in \mathcal{B}_0} C_S \neq \emptyset, \text{ or } \cap_{S \in \mathcal{B}_0} W_S \neq \emptyset,$$

which contradicts (22). Hence, $UBP \cap EGP \neq \emptyset$.

For each $x \in UBP \cap EGP$, we claim $x \in MNBF$. If this is false, we can find a small $\tau > 0$ such that $x - \tau e \in UBP$. Let $\mathcal{B} \in B$ be the minimal balanced collection of coalitions such that $x \in GP(\mathcal{B}) = \cap_{S \in \mathcal{B}} \{V(S) \times \mathbf{R}^{-S}\}$. Then, $x - \tau e \in Int\, V(S) \times \mathbf{R}^{-S}$ for each $S \in \mathcal{B}$, which contradicts $x - \tau e \in UBP$. Therefore, $MNBF \cap EGP = UBP \cap EGP \neq \emptyset$. **Q.E.D**

Proof of Propositions 4 and 5: The discussions preceding the propositions serve as their proofs. **Q.E.D**

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