Ex-Post Individually Rational, Budget-Balanced Mechanisms and Allocation of Surplus*

Sergei Severinov
Department of Economics
University of British Columbia

Joon Song
Department of Economics
University of Essex

February, 2010

[Preliminary]

Abstract

We investigate the issue of implementation via ex-post individually rational budget-balanced Bayesian mechanisms and allocation of surplus. We set up an auxiliary problem for minimized information rent where an agent is committed to a certain mixed deviation strategy before the agent's own type is known to himself. This formulation allows us to provide necessary and sufficient implementability conditions. We also develop an algorithm to determine whether a decision rule is implementable or not and to compute the informational rents earned by the players. We provide full characterization of the optimal mechanisms and implementability conditions in a number of special, but common cases.

JEL Nos: C72, D82

Keywords: mechanism design, Bayesian implementation, ex-post individual rationality, budget balancing, surplus allocation.

*Acknowledgement: **[To be added]** contact: sseverinov@gmail.com (Sergei Severinov), joonsong.econ@gmail.com (Joon Song)

1 Introduction

Efficiency is an important issue in economics and mechanism design. A typical mechanism design problem tries to find the second-best efficient mechanism for an incentive compatibility constraint, an individual rationality constraint, a budget-balance constraint. However, there are possibilities that problems have additional constraints. These are often ignored or studied only in specific contexts (e.g., [REFERENCES]). Also, we often consider only a handful of alternatives as an implementable allocation in real life (e.g., political debate on finite number of alternatives, voting on binomial decision, and finite number of candidates during elections where the elected implement a specific policy upon winning), instead of considering all the feasible allocations. In this situation, we investigate each alternative whether it is feasible and to characterize how much of transfers (e.g. taxation/subsidization) are required for each.

We do not try to incorporate the further restrictions for implementation in a unified or specific framework of mechanism design problems. Instead, we consider a general problem where non-monetary allocation is given: for given non-monetary allocation, (i) we investigate when this allocation can be implemented with the three constraints (IC/IR/BB), and (ii) we characterize how much monetary transfer has to be for each realization. Thus we determine how much information rent should be given to each type of an agent.

The theory of Bayesian mechanism design provides a universally accepted implementation tool for a large variety of environments, such as contracting, auctions, and bargaining. For this reason, it is important to understand the scope and limits of Bayesian implementation. In this regard, it is reasonable to consider budget balance, ex-post individual rationality and efficiency as desirable properties of a mechanism. Examples of environments, where one would like these properties to hold jointly, include standard and double auctions, public good provision, various trading situations. Ex-post individual rational mechanisms secure the participation of all agents even when all relevant information is disclosed. In summary, our environment has interim incentive compatibility constraint, ex-post individual rationality constraint, and ex-ante budget balance.

Under the aforementioned environment, ex-ante budget balance, ex-post individual rationality and interim incentive compatibility conditions, we use several linear programmings to characterize information rent given to each agent, even for inefficient allocation rules. The environment includes interdependent values drawn from a correlated probability distribution. For any given allocation rule, social surplus from the allocation rule is "revenue" to the mechanism designer. The implementation of the allocation rule requires certain inputs: agents are "inputs" in the sense that they are the (only) resource for the "production process" of the allocation rule generating the revenue. The minimized cost of using each input factor (each agent) is minimal information rent given to the agent. If the sum of the minimal rent for each agent is smaller than the revenue, the mechanism is implementable with the ex-ante budget balance. Our paper's focus is more on the detailed characterization of minimal information rent for given allocation rule, than on the optimal choice of allocation rule.

In section 2, we introduce basic notations and mechanism design environment. In section 3, we set up an auxiliary problem for minimized information rent where an agent is committed to a certain mixed deviation strategy before the agent's own type is known to himself. The information rent given to the agent under the deviation strategy is weakly smaller than the actual information rent. We show that the information rent for a certain deviation strategy has the same value of the actual information rent. This observation is used to provide an implementation condition under which the information rent is finite. More importantly, such deviation strategy turns out to describe how information rent is accumulated across the types of an agent. We identify the deviation strategy achieving the minimized information rent by duality of linear programming. In section 4, we completely characterize the case where there are two types of an agent. In section 5, we introduce a few more linear programmings to fully characterize the case where there are three types of an agent. In section 6, we provide an algorithm to find information rent.

2 The Model

There are n agents, $N = \{1, 2, \dots, n\}$. Agent $i \in N$ has privately known type which belongs to the type space $\Theta_i \equiv \{\theta_i^1, ..., \theta_i^{m_i}\}$ of cardinality m_i, $2 \le m_i < \infty$. A generic element of Θ_i will be denoted by θ_i or θ_i'. A state of the world is characterized by a type profile $\theta = (\theta_1, ..., \theta_n)$. The set of type profiles is given by $\Theta \equiv \prod_{i=1,\dots,n} \Theta_i$, with cardinality $L \equiv \prod_{i=1,\dots,n} m_i$. When focussing on agent i, we will use the notation (θ_i, θ_{-i}) for the profile of agent-types, where θ_{-i} stands for the profile of types of agents other than i. Let $\Theta_{-i} = \prod_{l \neq i} \Theta_l$, $L_{-i} = \prod_{l \neq i} m_l$, $\Theta_{-i-j} = \prod_{l \notin \{i,j\}} \Theta_l$, and $L_{-i-j} = \prod_{l \notin \{i,j\}} m_l$. A generic element of Θ_{-i-j} is denoted by θ_{-i-j}.

The (true) probability distribution of the agents' type profile θ is denoted by $p(\theta)$, with $p_i(\theta_i)$ and $p_{i,j}(\theta_i, \theta_j)$ denoting the corresponding marginal probability distribution of agent i's type and the marginal probability distribution of types of agents i and j, respectively. We assume that $p(\theta)$ is common knowledge. We also assume that $p_{i,j}(\theta_i, \theta_j) > 0$ for any $\theta_i \in \Theta_i, \theta_j \in \Theta_j$ of any two agents i and j.[1] Further, let $p_{-i}(\theta_{-i}|\theta_i)$ denote the probability distribution of type profiles of agents other than i conditional on the type of agent i. We use a similar system of notation for other probability distributions over Θ that will be introduced below. The set of all probability distributions over Θ is denoted by $\mathcal{P}(\Theta)$.

A mechanism designer, who does not possess any private information, controls the set of public decisions X. Let x denote a generic element of X. Agent i's utility function is quasilinear in the decision x and transfer t_i that she receives from the mechanism and is given by $u_i(x, \theta) + t_i$. Without loss of generality, an agent's reservation utility is normalized to zero.[2] A (social) decision rule $x(\cdot)$ is a function mapping the type space Θ into the set of public decisions X.[3] Also, $t(\cdot) = (t_1(\cdot), ..., t_n(\cdot))$ is a collection of transfer functions to all agents, where $t_i(\cdot) : \Theta \mapsto \mathbf{R}$ is a transfer function to agent i. An allocation profile is a combination of a decision rule $x(\cdot)$ with a collection of transfer functions $t(\cdot)$.

By the Revelation Principle, we can restrict the analysis to direct mechanisms in which the mechanism designer offers an allocation profile to the agents. If the agents, informed of their types, decide to participate in this mechanism, they report their types to the mechanism designer, and the allocation corresponding to the reported type profile is implemented.

Our main goal is to provide necessary and sufficient conditions for the existence of ex-post individually rational and ex-ante budget-balanced Bayesian mechanisms implementing

[1] This condition is clearly generic.

[2] Suppose that agent i's utility from her outside option is equal to $w_i(\theta_i, \theta_{-i})$. Such environment is equivalent to the environment where i's utility function is given by $u_i(x, \theta) - w_i(\theta) + t_i$ and her outside option is 0. Note that the sets of ex-post efficient decision rules and the notions of social surplus are the same in both environments.

[3] Note that randomization in public decisions is implicitly allowed, since X can be regarded as a set of probability distributions over some set of "pure" outcomes.

desirable decision rules. Let us describe these properties formally.

We will say that the allocation profile $(x(\cdot), t(\cdot))$ is incentive compatible if the following *Interim Incentive Constraint* $IC_i(\theta_i, \theta_i')$ holds for all $i \in \{1, ..., n\}$ and $\theta_i, \theta_i' \in \Theta_i$:

$$\sum_{\theta_{-i}} \Big[u_i(x(\theta_{-i}, \theta_i), (\theta_{-i}, \theta_i)) + t_i(\theta_{-i}, \theta_i) - u_i(x(\theta_{-i}, \theta_i'), (\theta_{-i}, \theta_i)) - t_i(\theta_{-i}, \theta_i')\Big] p_{-i}(\theta_{-i}|\theta_i) \geq 0. \quad (1)$$

A decision rule $x(\cdot)$ is said to be *implementable* if there exists a profile of transfer functions $t(\cdot)$ such that $(x(\cdot), t(\cdot))$ is incentive compatible.

Ex-post Individual Rationality (EPIR) requires the following $EPIR_i(\theta)$ constraint to hold for all $i \in \{1, ..., n\}$ and $\theta \in \Theta$:

$$u_i(x(\theta), \theta) + t_i(\theta) \geq 0. \quad (2)$$

Ex-ante Budget Balancing (EABB) constraint can be written as follows:

$$\sum_{\theta \in \Theta} \sum_{i=1}^{n} t_i(\theta) p(\theta) = 0. \quad (3)$$

A decision rule $x(\cdot)$ is *ex-post efficient* if $x(\theta) \in \arg\max_{x \in X} \sum_{i=1}^{n} u_i(x, \theta)$ for all $\theta \in \Theta$, i.e. $x(\theta)$ maximizes ex-post *social surplus* $\sum_{i=1}^{n} u_i(x, \theta)$. Since the principal always has an option to disband the mechanism and cause the agents to take their outside options, we assume without loss of generality that $\max_{x \in X} \sum_{i=1}^{n} u_i(x, \theta) \geq 0$ for all $\theta \in \Theta$. Finally, *EPIR* and *EABB* together imply the following *Ex-Ante Social Rationality (EASR)* condition:

$$S \equiv \sum_{\theta \in \Theta} \sum_{i=1}^{n} u_i(x(\theta), \theta) p(\theta) \geq 0. \quad (4)$$

EASR simply says that a decision rule must generate a nonnegative (ex ante) expected surplus. Clearly, this is a very weak requirement. It is satisfied by a large variety of decision rules, including the ex-post efficient ones. Having established *EASR* as a necessary condition, in the next section we characterize necessary and sufficient conditions for *EPIR* and *EABB* implementation of *EASR* decision rules which include ex-post efficient ones.

3 Analysis

Under truth-telling agent i gets net utility

$$U_i(\theta) = u_i(x, \theta) + t_i(\theta). \quad (5)$$

Since function $U_i(\cdot)$ uniquely specifies transfers $t_i(\cdot)$ we can use them interchangeably. As it will be seen later the mechanism design in terms of $U_i(\cdot)$ is more convenient and intuitive. In terms of set of net utilities which the agents get under truth-telling, $\{U_i(\cdot), i \in \{1,...,n\}\}$, a decision rule $x(\cdot)$ is implementable via ex-post individual rational and ex-ante budget balanced mechanism if for every i the following conditions are satisfied:

$$IC_i(\theta_i, \theta_i') : \quad \sum_{\theta_{-i}\in\Theta_{-i}} p_{-i}(\theta_{-i}|\theta_i) \left\{U_i(\theta_{-i}, \theta_i) - U_i(\theta_{-i}, \theta_i')\right\}$$

$$\geq \sum_{\theta_{-i}\in\Theta_{-i}} \left\{u_i(x(\theta_{-i}, \theta_i'), (\theta_{-i}, \theta_i)) - u_i(x(\theta_{-i}, \theta_i'), (\theta_{-i}, \theta_i'))\right\} p_{-i}(\theta_{-i}|\theta_i), \quad (6)$$

$$EPIR_i(\theta) : \quad U_i(\theta) \geq 0, \quad (7)$$

$$EABB : \quad \sum_{\theta\in\Theta}\sum_{i=1}^{n} U_i(\theta)p(\theta) = \sum_{\theta\in\Theta}\sum_{i=1}^{n} u_i(x(\theta), \theta)p(\theta) = S. \quad (8)$$

The above inequalities are counterparts of (1), (2) and (3).

Next, consider the following problem of minimizing agent i's ax-ante expected surplus.

$$\min_{\{U_i(\theta)\geq 0:\theta\in\Theta\}} \sum_{\theta\in\Theta} U_i(\theta)p(\theta) \quad (9)$$

$$s.t. \quad U_i(\theta) \geq 0, \text{ for all } \theta \in \Theta \quad (10)$$

$$\sum_{\theta_{-i}\in\Theta_{-i}} p_{-i}(\theta_{-i}|\theta_i) \left\{U_i(\theta_{-i}, \theta_i) - U_i(\theta_{-i}, \theta_i')\right\} \quad (11)$$

$$\geq \sum_{\theta_{-i}\in\Theta_{-i}} \left\{u_i(x(\theta_{-i}, \theta_i'), (\theta_{-i}, \theta_i)) - u_i(x(\theta_{-i}, \theta_i'), (\theta_{-i}, \theta_i'))\right\} p_{-i}(\theta_{-i}|\theta_i), \text{ for all } \theta_i, \theta_i' \in \Theta_i$$

The solution to the linear programming problem (9)-(11) determines the minimal ex-ante surplus V_i necessary to ensure truth-telling by agent i and her voluntary participation in the mechanism. Specifically, if the constraint set of this problem is compatible i.e., there is a profile $\{U_i(\theta)\}_{\theta\in\Theta}$ s.t. all inequalities (10) and (11) hold, then there exists a solution $\{U_i^*(\theta)\}_{\theta\in\Theta}$ to (9)-(11), and

$$V_i = \sum_{\theta\in\Theta} U_i^*(\theta)p(\theta) < \infty. \quad (12)$$

Note that by by construction, $V_i \geq 0$.

If the constraint set is empty, i.e. there is no $\{U_i(\theta)\}_{\theta\in\Theta}$ satisfying (10) and (11), then we take the value of the problem (9) to be infinite i.e., $V_i = +\infty$.

Next, we establish the following important Lemma:

Lemma 1 *There exists a profile* $\{U_i(\theta)_{i\in N,\theta\in\Theta}\}$ *of players' net expected surpluses satisfying conditions (6)-(8) if and only if the solution to (9)-(11) for all* i *induces such* $V_1, ..., V_n$ *via (12) that*

$$\sum_{i=1}^{n} V_i \leq S. \tag{13}$$

Proof. **"If" part.** For each i, let $U_i^*(\theta)$ be a solution to (9)-(11) and suppose that (13) hold. Now consider a profile of players expected surpluses, $(\hat{U}_1(\theta), U_2^*(\cdot), ..., U_n^*(\cdot))$, where $\hat{U}_1(\theta) = U_1^*(\theta) + S - \sum_{i=1}^{n} V_i$ for all $\theta \in \Theta$. Obviously, this profile of expected surpluses they satisfies all constraints in (6)-(8).

"Only If" part. The proof is obvious, and is therefore omitted.

Lemma 1 establishes a very important decomposition property of our mechanism design problem that it is sufficient to analyze the incentive problem for each agent separately. This result will allow us to find minimal informational rents required for implementability. If social surplus is sufficient to cover these rents then the mechanism exist. From the proof of the lemma it can be seen that the excess of social surplus over the sum of minimal informational rents can be split among the agents in an arbitrary way.

The next subsection we will build on these insights.

3.1 The Minimal Informational Rent

Consider some agent i. As it will be seen later in this subsection, we need to elaborate the notion of *strategies* chosen by the agent *in a direct mechanism*. For this, we need some additional notation. Agent i's strategy s_i in a direct mechanism is a vector of size m_i^2 such that its entry $s_i(\theta_i, \theta_i')$ denotes the probability with which agent i of type θ_i reports type θ_i'. Note that it has to be $s_i(\theta_i, \theta_i') \in [0, 1]$ and $\sum_{\theta_i'\in\Theta_i} s_i(\theta_i, \theta_i') = 1$ for all $\theta_i \in \Theta_i$. Let S_i be the set of all such strategies s_i. A truthful strategy s_i^* of agent i is such that $s_i(\theta_i, \theta_i) = 1$ and as a result $s_i(\theta_i, \theta_i') = 0$ for all $\theta_i, \theta_i' \in \Theta_i$ s.t. $\theta_i \neq \theta_i'$. A strategy profile $\mathbf{s} \equiv (s_1, ..., s_n)$ is a collection of strategies followed by the agents. A strategy profile such that agent i follows strategy s_i and all other agents follow truthful strategies is denoted by (s_i, s_{-i}^*).

Definition 1 *Say that the strategy profile* $\mathbf{s} \equiv (s_1, ..., s_n)$ *induces the probability distribution over the reported type profiles* $q(.|\mathbf{s})$ *if type profile* $\theta' \in \Theta$ *is reported with probability* $q(\theta'|\mathbf{s})$ *when the agents follow strategies* $\mathbf{s} = (s_1, ..., s_n)$ *and the types are drawn from the prior* $p(\cdot)$*.*

To compute $q(.|\mathbf{s})$, note that

$$q(\theta'_1, ..., \theta'_n|\mathbf{s}) = \sum_{(\theta_1,...,\theta_n)\in\Theta} \Big(p(\theta_1, ..., \theta_n) \prod_{i=1}^{n} s_i(\theta_i, \theta'_i)\Big) \text{ for any } (\theta'_1, ..., \theta'_n) \in \Theta.$$

In terms of deviation strategies and induced probability distributions, the problem (9) can be rewritten as follows

$$V_i = \min_{U_i(\cdot)\geq 0} \sum_{\theta\in\Theta} U_i(\theta)p(\theta) \tag{14}$$

$$\text{s.t. for any } s_i \in \Theta_i,$$

$$\sum_{\theta\in\Theta} \{p(\theta) - q(\theta|s_i, s^*_{-i})\} U_i(\theta)$$

$$\geq \sum_{\theta_i,\theta'_i\in\Theta_i} \sum_{\theta_{-i}\in\Theta_{-i}} \big\{u_i(x(\theta_{-i}, \theta'_i), (\theta_{-i}, \theta_i)) - u_i(x(\theta_{-i}, \theta'_i), (\theta_{-i}, \theta'_i))\big\}\, s_i(\theta_i, \theta'_i)p(\theta_{-i}, \theta_i).$$

In the above problem an inequality has the following intuitive meaning. Individual rationality constraints secure the minimal transfer to the agent of size $-u_i(x(\theta_{-i}, \theta'_i), (\theta_{-i}, \theta'_i))$ given that the reported profile is (θ_{-i}, θ'_i). Meanwhile the utility she gets from misreport is $u_i(x(\theta_{-i}, \theta'_i), (\theta_{-i}, \theta_i))$. Hence the difference of utilities in (14) describes the "guaranteed" utility gain when agent-type θ_i misreports θ'_i in the environment where at all states of nature agent i gets zero net utility. The whole right-hand side is the ex-ante expected utility gains when the agent follows strategy s_i. Indeed, given types θ_{-i} agent-type θ_i misreports θ'_i with probability $s_i(\theta_i, \theta'_i)p(\theta_{-i}, \theta_i)$ so the summation yields the expected ex-ante value of the utility gain.

The left-hand side of the inequality reflects ex-ante potential losses (gains) when the probability distribution of the reported types changes given that the agent gets positive net utility at some states of nature.

In short, the equivalent problems (9) and (14) differ in the nature of their constraints. Problem (9) has restrictions on possible values of $U_i(\cdot)$ in terms of interim payoffs while problem (14) has the restrictions of the ex-ante nature.

Now rewrite the constraint in (14) as follows

$$s_i : \sum_{\theta \in \Theta} \{p(\theta) - q(\theta|s_i, s^*_{-i})\} U_i(\theta) \geq g_i(s_i) \tag{15}$$

where

$$g_i(s_i) = \sum_{\theta_i, \theta'_i \in \Theta_i} \sum_{\theta_{-i} \in \Theta_{-i}} [u_i(x(\theta_{-i}, \theta'_i), (\theta_{-i}, \theta_i)) - u_i(x(\theta_{-i}, \theta'_i), (\theta_{-i}, \theta'_i))] s_i(\theta_i, \theta'_i) p(\theta_{-i}, \theta_i). \tag{16}$$

To get a better understanding of problem (14), consider a simpler problem. Let us fix some strategy s_i and assume that there is only one constraint in (14) - the constraint which corresponds to the strategy s_i. The following are the simpler problem and its dual problem.

$$V_i(s_i) = \min_{U_i(\theta) \geq 0} \sum_{\theta} U_i(\theta) p(\theta) \quad s.t. \quad \sum_{\theta} (p(\theta) - q(\theta|s_i)) U_i(\theta) \geq g_i(s_i), \qquad [LP_P(s_i)]$$

$$V_i(s_i) = \max_{\alpha_i(s_i) \geq 0} g_i(s_i) \alpha_i(s_i) \quad s.t. \quad \alpha_i(s_i)[p(\theta) - q(\theta|s_i)] \leq p(\theta) \qquad [LP_D(s_i)]$$

where $\Delta u_i(\theta_i, \theta'_i) = \sum_{\theta_{-i} \in \Theta_{-i}} \{u_i(x(\theta_{-i}, \theta'_i), (\theta_{-i}, \theta_i)) - u_i(x(\theta_{-i}, \theta'_i), (\theta_{-i}, \theta'_i))\} p_{-i}(\theta_{-i}|\theta_i)$.

Note, $g_i(s_i) = \sum_{\theta_i, \theta'_i} \Delta u_i(\theta_i, \theta'_i) s_i(\theta_i, \theta'_i) p_i(\theta_i)$

The original problem and its dual are summarized as follows.

$$V_i = \min_{U_i(\theta) \geq 0} \sum_{\theta} U_i(\theta) p(\theta) \quad s.t. \quad \sum_{\theta_{-i}} p(\theta_i, \theta_{-i})[U_i(\theta) - U_i(\theta_{-i}, \theta'_i)] \geq \Delta u_i(\theta_i, \theta'_i) p_i(\theta_i) \qquad [LP_P]$$

$$V_i = \max_{\gamma_i(\theta_i, \theta'_i) \geq 0} \sum_{\theta_i, \theta'_i} \gamma_i(\theta_i, \theta'_i) \Delta u_i(\theta_i, \theta'_i) p_i(\theta_i)$$

$$s.t. \quad \sum_{\theta'_i} \gamma_i(\theta_i, \theta'_i) p_{-i}(\theta_{-i}, \theta_i) - \sum_{\theta'_i} \gamma_i(\theta'_i, \theta_i) p_{-i}(\theta_{-i}, \theta'_i) \leq p(\theta). \qquad [LP_D]$$

Note that the incentive compatibility constraints in $[LP_P]$ is written with respect to unconditional probabilities. Also note $\gamma_i(\theta'_i, \theta_i)$ is the dual value of the constraint in $[LP_P]$.

We reproduce the fundamental theorem of linear programming in the below.

Lemma 2 (Fundamental theorem of (finite) linear programming)

[1] The solution of the primal and dual linear programs coincides.

[2] If the shadow value of a constraint is positive, the constraint is binding.

[3] If a constraint is not binding, the shadow value of the constraint is zero.

3.2 Analysis of $V_i(s_i)$

The problem that we are considering can be viewed as a simple cost minimization problem. Indeed, suppose that $g_i(s_i) > 0$, and interpret $U_i(\theta)$ as an input, $p(\theta)$ as its price, $p(\theta) - q(\theta|s_i, s_{-i}^*)$ as marginal productivity of this input and, finally, $g_i(s_i)$ should be interpreted as an output.

We know that for cost minimization with linear production function we should use factors of production where marginal productivity to price ratio reaches the maximum or if $U_i^*(\theta) > 0$ then $\theta \in \arg\max_{\theta\in\Theta}\{(p(\theta) - q(\theta|s_i, s_{-i}^*))/p(\theta)\} = \arg\min_{\theta\in\Theta}\{q(\theta|s_i, s_{-i}^*))/p(\theta)\}$. As a result

$$V_i(s_i) = \frac{g_i(s_i)}{1 - \min_{\theta\in\Theta}\{q(\theta|s_i, s_{-i}^*))/p(\theta)\}}. \tag{17}$$

Formula (17) can also be interpreted as follows. Suppose that the mechanism designer provides 1 unit of ex-ante expected utility to player i. If this utility is provided by paying i when the state of the world is θ, then the payment to player i is equal to $U_i(\theta) = 1/p(\theta)$ for i and this will cause ex-ante losses from deviation of size $(p(\theta) - q(\theta|s_i, s_{-i}^*))/p(\theta)$. Thus the most efficient use of budget money is to put money at one of states where $(p(\theta) - q(\theta|s_i, s_{-i}^*))/p(\theta)$ $= \max_\theta\{(p(\theta) - q(\theta|s_i, s_{-i}^*))/p(\theta)\}$. And the total budget money required to create "ex-ante" punishment of size $g_i(s_i)$ is $g_i(s_i)/\max_\theta\{(p(\theta) - q(\theta|s_i, s_{-i}^*))/p(\theta)\}$ which is equal to (17).

We have derived the informational rent $V_i(s_i)$ of agent i when s_i is his only possible deviation strategy. Clearly, the informational rent V_i, which agent i earns when he could use any $s_i \in S_i$ exceeds (at least weakly) the maximum of of $V_i(s_i)$ over S_i i.e., $V_i \geq \sup_{s_i\in S_i} V_i(s_i)$.

Moreover, we $V_i \geq \sup_{s_i\in S_i} V_i(s_i)$ under the following condition.

Assumption 1 (Implementability Condition) *For any strategy s_i, if $q(\theta|s_i, s_{-i}^*) = p(\theta)$ for all θ, utility gain from this strategy $g_i(s_i)$ is not positive.*

If there is one θ such that $q(\theta|s_i) \neq p(\theta)$, there must be θ such that $q(\theta|s_i) < p(\theta)$ as $q(\cdot|s_i)$ and $p(\cdot)$ are probability distributions, i.e. $\sum_\theta p(\theta) = \sum_\theta q(\theta|s_i) = 1$. For $\theta = \operatorname{argmin}_{\tilde{\theta}} \left[\frac{q(\tilde{\theta}|s_i)}{p(\tilde{\theta})}\right]$, $\alpha(s_i) = \frac{1}{1 - \frac{q(\theta|s_i s)}{p(\theta)}}$ is an optimal solution for $LP_D(s_i)$. Thus we will not need any condition for $g_i(s_i)$ so that $V_i(s_i) < \infty$. However, if $q(\theta|s_i) = p(\theta)$ for all θ, we will need $g_i(s_i) \leq 0$: if that is not the case, $V_i = \infty$ as is clear from $LP_D(s_i)$. Therefore, the condition is a sufficient and necessary condition for $V_i < \infty$. This is why we call it Implementability Condition.

By proving $V_i = \max_{s_i} V_i(s_i)$, we can move between the analyses of V_i and $V_i(s_i)$ back and forth, whenever we want. We prove $V_i = \max_{s_i} V_i(s_i)$ by using the following lemma (Proofs for Lemma 3 and Proposition 1 are in Appendix A.1 and A.2).

Lemma 3 $\gamma_i(\theta_i, \theta_i') = s_i(\theta_i, \theta_i')\alpha_i$ *for an* $\alpha_i \geq 0$ *and* $s_i(\theta_i, \theta_i') \geq 0$ *such that* $\sum_{\theta_i'} s_i(\theta_i, \theta_i') = 1$.

Proposition 1 *If implementability condition holds, then the minimal informational rent is:*

$$V_i = \max_{s_i \in S_i} V_i(s_i) \quad \textit{where} \quad V_i(s_i) = \frac{g_i(s_i)}{1 - \min_{\theta \in \Theta}\{q(\theta|s_i, s_{-i}^*))/p(\theta)\}}.$$

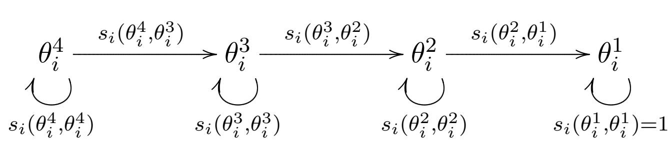
Remark: For optimal $s_i = \underset{\tilde{s}_i}{\operatorname{argmax}} V_i(\tilde{s}_i)$, it is useful to consider marginal benefit and cost in s_i. For example, a change in s_i may make $g_i(s_i)$ larger (marginal benefit), but may make $\frac{1}{1-\min \frac{q(\cdot|s_i)}{q(\cdot)}}$ smaller (marginal cost). We repeatedly use this kind of observation.

Remark: The existence of s_i such that $V_i(s_i) = V_i$ enables us to reduce $[LP_P]$ to a simpler problem $[LP_P(s_i)]$. This reduction makes it possible to simplify the characterization of how information rent is accumulated over types. More specifically, we can assume that a certain class of s_i achieves $V_i(s_i) = V_i$. Under the assumption, we can analyze the accumulation of information rent easily. After the analysis, we can derive the condition for the primitives (such as $p(\cdot)$ and $\Delta u_i(\theta_i, \theta_i')$) where s_i indeed achieves the maximum $V_i = V_i(s_i)$. This essentially generates a mapping from the set of primitives to the structure of information rent. We first examine an example in the following. Then, we provide full characterization for the cases where $|\Theta_i| = 2$ and $|\Theta_i| = 3$ in the next two sections.

Example 1: $\Theta_i = \{\theta_i^1, \theta_i^2, \theta_i^3, \theta_i^4\}$. The primitive has the characteristic of

$$\Delta u_i(\theta_i^4, \theta_i^3) > 0, \Delta u_i(\theta_i^3, \theta_i^2) > 0, \Delta u_i(\theta_i^2, \theta_i^1) > 0, \Delta u_i(\theta_i, \theta_i) = 0, \forall \theta_i, \Delta u_i(\cdot, \cdot) \ll 0 \text{ otherwise.}$$

In other words, it is prohibitively expensive for θ_i to mimic θ_i', except when (θ_i, θ_i') is one of (θ_i^4, θ_i^3), (θ_i^3, θ_i^2) and (θ_i^2, θ_i^1). Thus the following should be the picture of an optimal s_i.



Exact value of s_i is calculated by the following.

First, θ_i^4 achieves the minimum of $\frac{q(\theta|s_i)}{p(\theta)}$ with a certain θ_{-i}. Suppose not. if $s_i(\theta_i^4, \theta_i^4) = 0$, then $\min_\theta \frac{q(\theta|s_i)}{p(\theta)} = \frac{q(\theta_{-i}, \theta_i^4|s_i)}{p(\theta_{-i}, \theta_i^4)} = 0$. If $s_i(\theta_i^4, \theta_i^4) > 0$. Then a (marginal) change in $s_i(\theta_i^4, \cdot)$ does not change $\frac{1}{1-\min \frac{q(\cdot|s_i)}{p(\cdot)}}$, but only $g_i(s_i)$. If $s_i(\theta_i^4, \theta_i^4) > 0$, this s_i cannot be an optimum as decrease of $s_i(\theta_i^4, \theta_i^4)$ and increase of $s_i(\theta_i^4, \theta_i^3)$ increase $g_i(s_i)$ (hence, $V_i(s_i)$ too).

Second, we determine $s_i(\theta_i^3, \cdot)$ and $s_i(\theta_i^2, \cdot)$. Since $\min \frac{q(\cdot|s_i)}{p(\cdot)} = 1 - s_i(\theta_i^4, \theta_i^3)$, we get:

$$\frac{s_i(\theta_i^4, \theta_i^3)p(\theta_i^4, \theta_{-i}) + (1 - s_i(\theta_i^3, \theta_i^2))p(\theta_i^3, \theta_{-i})}{p(\theta_i^3, \theta_{-i})} \geq 1 - s_i(\theta_i^4, \theta_i^3) \tag{18}$$

$$\frac{s_i(\theta_i^3, \theta_i^2)p(\theta_i^3, \theta_{-i}) + (1 - s_i(\theta_i^2, \theta_i^1))p(\theta_i^2, \theta_{-i})}{p(\theta_i^2, \theta_{-i})} \geq 1 - s_i(\theta_i^4, \theta_i^3) \tag{19}$$

$$\frac{s_i(\theta_i^2, \theta_i^1)p(\theta_i^2, \theta_{-i}) + p(\theta_i^1, \theta_{-i})}{p(\theta_i^1, \theta_{-i})} = \frac{s_i(\theta_i^2, \theta_i^1)p(\theta_i^2, \theta_{-i})}{p(\theta_i^1, \theta_{-i})} + 1 \geq 1 - s_i(\theta_i^4, \theta_i^3) \tag{20}$$

We derive the following two inequalities from (18) and (19):

$$\Big(1 + \min_{\theta_{-i}} \frac{p(\theta_i^4, \theta_{-i})}{p(\theta_i^3, \theta_{-i})}\Big) s_i(\theta_i^4, \theta_i^3) \geq s_i(\theta_i^3, \theta_i^2), \tag{21}$$

$$\Big(1 + \min_{\theta_{-i}} \frac{p(\theta_i^3, \theta_{-i})}{p(\theta_i^2, \theta_{-i})}\Big(1 + \min_{\theta_{-i}} \frac{p(\theta_i^4, \theta_{-i})}{p(\theta_i^3, \theta_{-i})}\Big)\Big) s_i(\theta_i^4, \theta_i^3) \geq s_i(\theta_i^2, \theta_i^1), \tag{22}$$

Note that (22) cannot be an equality without (21) being an equality. Also note that $s_i(\theta_i^4, \theta_i^3)$ is strictly positive (otherwise, $s_i(\theta_i^3, \theta_i^2) = s_i(\theta_i^2, \theta_i^1) = 0$, which clearly cannot be an optimal solution). Thus, (20) does not bind. Note also that, when (21) and (22) are equalities, the relevant θ_i will achieve the minimum of $\frac{q(\cdot|s_i)}{p(\cdot)}$ with a certain θ_{-i}. For example, if $s_i(\theta_i^2, \theta_i^1) = \Big(1 + \min_{\theta_{-i}} \frac{p(\theta_i^3, \theta_{-i})}{p(\theta_i^2, \theta_{-i})}\Big(1 + \min_{\theta_{-i}} \frac{p(\theta_i^4, \theta_{-i})}{p(\theta_i^3, \theta_{-i})}\Big)\Big) s_i(\theta_i^4, \theta_i^3)$ in (22), $\frac{q(\cdot|s_i)}{p(\cdot)}$ achieves the minimum at $(\theta_i^2, \theta_{-i})$ where $\theta_{-i} = \text{argmin} \frac{p(\theta_i^3, \theta_{-i})}{p(\theta_i^2, \theta_{-i})}$.

Third, we determines$s_i(\theta_i^1, \cdot)$. Since the last inequality was strict, $s_i(\theta_i^1, \cdot)$ should not influence $\min \frac{q(\cdot|s_i)}{p(\cdot)}$. Thus $s_i(\theta_i^1, \theta_i^1) = 1$ should be optimal.

Fourth, we characterize $g_i(s_i)$. If (21) and (22) hold as equalities, i.e., if

$$\Big(1 + \min_{\theta_{-i}} \frac{p(\theta_i^3, \theta_{-i})}{p(\theta_i^2, \theta_{-i})}\Big(1 + \min_{\theta_{-i}} \frac{p(\theta_i^4, \theta_{-i})}{p(\theta_i^3, \theta_{-i})}\Big)\Big) s_i(\theta_i^4, \theta_i^3) \leq 1, \quad \Big(1 + \min_{\theta_{-i}} \frac{p(\theta_i^4, \theta_{-i})}{p(\theta_i^3, \theta_{-i})}\Big) s_i(\theta_i^4, \theta_i^3) \leq 1,$$

$g_i(s_i)$ is a linear function of $s_i(\theta_i^4, \theta_i^3)$, $g_i(s_i) = K s_i(\theta_i^4, \theta_i^3)$, where

$$\begin{aligned} K = &\Big[\Delta u_i(\theta_i^4, \theta_i^3)p_i(\theta_i^4) + \Delta u_i(\theta_i^3, \theta_i^2)\Big(1 + \min_{\theta_{-i}} \frac{p(\theta_i^4, \theta_{-i})}{p(\theta_i^3, \theta_{-i})}\Big)p_i(\theta_i^3) \\ &+ \Delta u_i(\theta_i^2, \theta_i^1)\Big(1 + \min_{\theta_{-i}} \frac{p(\theta_i^3, \theta_{-i})}{p(\theta_i^2, \theta_{-i})}\Big(1 + \min_{\theta_{-i}} \frac{p(\theta_i^4, \theta_{-i})}{p(\theta_i^3, \theta_{-i})}\Big)\Big)p_i(\theta_i^2)\Big]. \end{aligned}$$

Also $1-\min\frac{q(\cdot|s_i)}{p(\cdot)} = s_i(\theta_i^4,\theta_i^3)$. Thus, if (21) and (22) hold as equalities at an optimal solution, the value of $V_i(s_i)$ is $\frac{1}{1-\min\frac{q(\cdot|s_i)}{p(\cdot)}}g_i(s_i) = K$ for any $s_i(\theta_i^4,\theta_i^3)$ such that

$$0 \leq s_i(\theta_i^4,\theta_i^3) \leq \min\left\{\frac{1}{1+\min_{\theta_{-i}}\frac{p(\theta_i^4,\theta_{-i})}{p(\theta_i^3,\theta_{-i})}}, \frac{1}{1+\min_{\theta_{-i}}\frac{p(\theta_i^3,\theta_{-i})}{p(\theta_i^2,\theta_{-i})}\left(1+\min_{\theta_{-i}}\frac{p(\theta_i^4,\theta_{-i})}{p(\theta_i^3,\theta_{-i})}\right)}\right\}. \quad (23)$$

Finally, we show that V_i is maximized only if $s_i(\theta_i^4,\theta_i^3)$ is in the range given by (23). If $s_i(\theta_i^4,\theta_i^3)$ is not, $g_i(s_i)$ is no more a linear function, and $g_i(s_i) < Ks_i(\theta_i^4,\theta_i^3)$. Thus $\frac{1}{1-\min\frac{q(\cdot|s_i)}{p(\cdot)}}g_i(s_i) < K$. Thus, we have shown the result.

Note that V_i shows how information rent is accumulated across types. V_i is equivalent to:

$$\begin{aligned} V_i = {} & p_i(\theta_i^2)\Big[\Delta u_i(\theta_i^2,\theta_i^1)\Big] + p_i(\theta_i^3)\Big[\Delta u_i(\theta_i^3,\theta_i^2) + \min_{\theta_{-i}}\frac{p(\theta_{-i}|\theta_i^3)}{p(\theta_{-i}|\theta_i^2)}\Delta u_i(\theta_i^2,\theta_i^1)\Big] \\ & + p_i(\theta_i^4)\Big[\Delta u_i(\theta_i^4,\theta_i^3) + \min_{\theta_{-i}}\frac{p(\theta_{-i}|\theta_i^4)}{p(\theta_{-i}|\theta_i^3)}\Delta u_i(\theta_i^3,\theta_i^2) + \min_{\theta_{-i}}\frac{p(\theta_{-i}|\theta_i^4)}{p(\theta_{-i}|\theta_i^3)}\min_{\theta_{-i}}\frac{p(\theta_{-i}|\theta_i^3)}{p(\theta_{-i}|\theta_i^2)}\Delta u_i(\theta_i^2,\theta_i^1)\Big]. \end{aligned}$$

The term in the second bracket represents the information rent given to type θ_i^3. Note that the information rent given to type θ_i^2 is added to this information rent with discount factor $\min_{\theta_{-i}}\frac{p(\theta_{-i}|\theta_i^3)}{p(\theta_{-i}|\theta_i^2)}$. In the last line, a similar accumulation of information rent is seen for type θ_i^4.

Assuming that the support of $\min\frac{p(\cdot|\theta_i)}{p(\cdot|\tilde{\theta}_i)}$ is unique for any θ_i and $\tilde{\theta}_i$, it is straightforward that this accumulation of information rent can be achieved by the following transfer $U_i(\cdot)$:

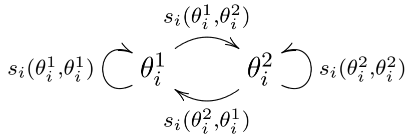
$$\begin{aligned} & U_i(\theta_i^1,\theta_{-i}) \equiv 0, \\ & U_i(\theta_i^2,\theta_{-i}) = \frac{\Delta u_i(\theta_i^2,\theta_i^1)}{p_i(\theta_i^2)} \text{ if } \theta_{-i} = \text{argmin}\frac{p(\theta_{-i}|\theta_i^3)}{p(\theta_{-i}|\theta_i^2)}, \quad 0 \text{ otherwise}, \\ & U_i(\theta_i^3,\theta_{-i}) = \frac{\Delta u_i(\theta_i^3,\theta_i^2)}{p_i(\theta_i^3)} \text{ if } \theta_{-i} = \text{argmin}\frac{p(\theta_{-i}|\theta_i^4)}{p(\theta_{-i}|\theta_i^3)}, \quad 0 \text{ otherwise}, \quad \text{and} \\ \text{any} \quad & U_i(\theta_i^4,\theta_{-i}) \text{s.t.} \quad \sum_{\theta_{-i}} U_i(\theta_i^4,\theta_{-i})p(\theta_{-i}|\theta_i^4) = \Delta u_i(\theta_i^4,\theta_i^3). \end{aligned}$$

Minimums of probability ratios determine the states when positive $U_i(\cdot)$ is given.

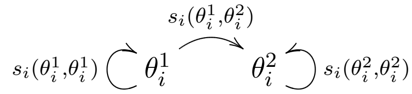
4 Full characterization of the case with $|\Theta_i| = 2$

Let $\Theta_i = \{\theta_i^1,\theta_i^2\}$. If $\Delta u_i(\theta_i^1,\theta_i^2) < 0$ and $\Delta u_i(\theta_i^2,\theta_i^1) < 0$, then $s_i(\theta_i,\theta_i) \equiv 1$ is optimal trivially.

Suppose $\Delta u_i(\theta_i^1,\theta_i^2) > 0$ and $\Delta u_i(\theta_i^1,\theta_i^2)p_i(\theta_i^1) \geq \Delta u_i(\theta_i^2,\theta_i^1)p_i(\theta_i^2)$ without loss of generality. There are two possibilities: $s_i(\theta_i^2,\theta_i^1) = 0$ or $s_i(\theta_i^2,\theta_i^1) > 0$ depicted below.



$Case : s_i(\theta_i^2, \theta_i^1) > 0$



$Case : s_i(\theta_i^2, \theta_i^1) = 0$

$s_i(\theta_i^l, \theta_i^k) > 0$ means that the incentive compatibility constraint for θ_i^l not to mimic θ_i^k is binding. The "arrows" in the diagrams represent the binding incentive compatibility constraints.

We characterize the first and the second cases and derive the condition separating them.

4.1 The case with $s_i(\theta_i^2, \theta_i^1) > 0$:

Lemma 4 *If $s_i(\theta_i^2, \theta_i^1) > 0$, then $(\theta_i^1, \theta_{-i})$, $(\theta_i^2, \theta'_{-i}) \in \operatorname{argmin} \frac{q(\cdot|s_i)}{p(\cdot)}$ for some θ_{-i}, $\theta'_{-i} \in \Theta_{-i}$.*

Proof. See Appendix A.3. ∎

From Lemma 4, both of θ_i^1 and θ_i^2 achieves the minimum. Hence, for certain θ_{-i} and θ'_{-i},

$$\min \frac{q(\cdot|s_i)}{p(\cdot)} = \frac{p(\theta_i^1, \theta_{-i}) s_i(\theta_i^1, \theta_i^1) + p(\theta_i^2, \theta_{-i}) s_i(\theta_i^2, \theta_i^1)}{p(\theta_i^1, \theta_{-i})} = \frac{p(\theta_i^2, \theta'_{-i}) s_i(\theta_i^2, \theta_i^2) + p(\theta_i^1, \theta'_{-i}) s_i(\theta_i^1, \theta_i^2)}{p(\theta_i^2, \theta'_{-i})}.$$

Plugging $s_i(\theta_i^1, \theta_i^1) = 1 - s_i(\theta_i^1, \theta_i^2)$ and $s_i(\theta_i^2, \theta_i^2) = 1 - s_i(\theta_i^2, \theta_i^1)$, we get

$$\Big(1 + \min_{\theta_{-i}} \frac{p(\theta_i^1, \theta_{-i})}{p(\theta_i^2, \theta_{-i})}\Big) s_i(\theta_i^1, \theta_i^2) = \Big(1 + \min_{\theta'_{-i}} \frac{p(\theta_i^2, \theta_{-i})}{p(\theta_i^1, \theta_{-i})}\Big) s_i(\theta_i^2, \theta_i^1). \tag{24}$$

Unless $\Delta u_i(\theta_i^1, \theta_i^2) p(\theta_i^1) + \Delta u_i(\theta_i^2, \theta_i^1) p(\theta_i^2) \leq 0$, the equality $\theta'_{-i} = \theta_{-i}$ implies that the information rent is infinity since

$$\min \frac{q(\cdot|s_i)}{p(\cdot)} = s_i(\theta_i^1, \theta_i^1) + \frac{p(\theta_i^2, \theta_{-i}) s_i(\theta_i^2, \theta_i^1)}{p(\theta_i^1, \theta_{-i})} = 1 - s_i(\theta_i^1, \theta_i^2) + \frac{p(\theta_i^1, \theta_{-i}) s_i(\theta_i^1, \theta_i^2)}{p(\theta_i^1, \theta_{-i})} = 1.$$

However, if $\theta'_{-i} \neq \theta_{-i}$, the value is not infinity. Firstly, we get the following from equality (24).

$$g_i(s_i) = \Big[\Delta u_i(\theta_i^1, \theta_i^2) p(\theta_i^1) + \Delta u_i(\theta_i^2, \theta_i^1) \frac{1 + \min_{\theta_{-i}} \frac{p(\theta_i^1, \theta_{-i})}{p(\theta_i^2, \theta_{-i})}}{1 + \min_{\theta_{-i}} \frac{p(\theta_i^2, \theta_{-i})}{p(\theta_i^1, \theta_{-i})}} p(\theta_i^2)\Big] s_i(\theta_i^1, \theta_i^2),$$

$$1 - \min \frac{q(\cdot|s_i)}{p(\cdot)} = s_i(\theta_i^1, \theta_i^2) - \min \frac{p(\theta_i^2, \cdot)}{p(\theta_i^1, \cdot)} s_i(\theta_i^2, \theta_i^1) = \left[1 - \min \frac{p(\theta_i^2, \cdot)}{p(\theta_i^1, \cdot)} \frac{1 + \min \frac{p(\theta_i^1, \cdot)}{p(\theta_i^2, \cdot)}}{1 + \min \frac{p(\theta_i^2, \cdot)}{p(\theta_i^1, \cdot)}}\right] s_i(\theta_i^1, \theta_i^2).$$

Then the optimal value V_i is

$$\Big[\Delta u_i(\theta_i^1,\theta_i^2)p(\theta_i^1)+\Delta u_i(\theta_i^2,\theta_i^1)\frac{1+\min\frac{p(\theta_i^1,\cdot)}{p(\theta_i^2,\cdot)}}{1+\min\frac{p(\theta_i^2,\cdot)}{p(\theta_i^1,\cdot)}}p(\theta_i^2)\Big]\Big/\Big[1-\min\frac{p(\theta_i^2,\cdot)}{p(\theta_i^1,\cdot)}\frac{1+\min\frac{p(\theta_i^1,\cdot)}{p(\theta_i^2,\cdot)}}{1+\min\frac{p(\theta_i^2,\cdot)}{p(\theta_i^1,\cdot)}}\Big]$$

$$=\frac{1}{1-\min\frac{p(\theta_i^2,\cdot)}{p(\theta_i^1,\cdot}\min\frac{p(\theta_i^1,\cdot)}{p(\theta_i^2,\cdot)}}\Big[\Delta u_i(\theta_i^1,\theta_i^2)\Big(1+\min\frac{p(\theta_i^2,\cdot)}{p(\theta_i^1,\cdot)}\Big)p_i(\theta_i^1)+\Delta u_i(\theta_i^2,\theta_i^1)\Big(1+\min\frac{p(\theta_i^1,\cdot)}{p(\theta_i^2,\cdot)}\Big)p_i(\theta_i^2)\Big]$$

$$=\Big[1+\min\frac{p(\theta_i^2,\cdot)}{p(\theta_i^1,\cdot)}\min\frac{p(\theta_i^1,\cdot)}{p(\theta_i^2,\cdot)}+\Big(\min\frac{p(\theta_i^2,\cdot)}{p(\theta_i^1,\cdot)}\min\frac{p(\theta_i^1,\cdot)}{p(\theta_i^2,\cdot)}\Big)^2+\cdots\Big]$$

$$\times\Big[\Delta u_i(\theta_i^1,\theta_i^2)\Big(1+\min\frac{p(\theta_i^2,\cdot)}{p(\theta_i^1,\cdot)}\Big)p_i(\theta_i^1)+\Delta u_i(\theta_i^2,\theta_i^1)\Big(1+\min\frac{p(\theta_i^1,\cdot)}{p(\theta_i^2,\cdot)}\Big)p_i(\theta_i^2)\Big]$$

$$=\Big[\Delta u_i(\theta_i^1,\theta_i^2)+\min\frac{p(\cdot|\theta_i^1)}{p(\cdot|\theta_i^2)}\Delta u_i(\theta_i^2,\theta_i^1)+\min\frac{p(\cdot|\theta_i^1)}{p(\cdot|\theta_i^2)}\min\frac{p(\cdot|\theta_i^2)}{p(\cdot|\theta_i^1)}\Delta u_i(\theta_i^1,\theta_i^2)$$

$$+\Big(\min\frac{p(\cdot|\theta_i^1)}{p(\cdot|\theta_i^2)}\Big)^2\min\frac{p(\cdot|\theta_i^2)}{p(\cdot|\theta_i^1)}\Delta u_i(\theta_i^2,\theta_i^1)+\Big(\min\frac{p(\cdot|\theta_i^1)}{p(\cdot|\theta_i^2)}\Big)^2\Big(\min\frac{p(\cdot|\theta_i^2)}{p(\cdot|\theta_i^1)}\Big)^2\Delta u_i(\theta_i^1,\theta_i^2)+\ldots\Big]p_i(\theta_i^1)$$

$$+\Big[\Delta u_i(\theta_i^2,\theta_i^1)+\min\frac{p(\cdot|\theta_i^2)}{p(\cdot|\theta_i^1)}\Delta u_i(\theta_i^1,\theta_i^2)+\min\frac{p(\cdot|\theta_i^2)}{p(\cdot|\theta_i^1)}\min\frac{p(\cdot|\theta_i^1)}{p(\cdot|\theta_i^2)}\Delta u_i(\theta_i^2,\theta_i^1)$$

$$+\Big(\min\frac{p(\cdot|\theta_i^2)}{p(\cdot|\theta_i^1)}\Big)^2\min\frac{p(\cdot|\theta_i^1)}{p(\cdot|\theta_i^2)}\Delta u_i(\theta_i^1,\theta_i^2)+\Big(\min\frac{p(\cdot|\theta_i^2)}{p(\cdot|\theta_i^1)}\Big)^2\Big(\min\frac{p(\cdot|\theta_i^1)}{p(\cdot|\theta_i^2)}\Big)^2\Delta u_i(\theta_i^2,\theta_i^1)+\ldots\Big]p_i(\theta_i^2).$$

The last four lines shows how the information rent is (infinitely) accumulated with discount rates $\min\frac{p(\cdot|\theta_i^2)}{p(\cdot|\theta_i^1)}$ and $\min\frac{p(\cdot|\theta_i^1)}{p(\cdot|\theta_i^2)}$.

4.2 The case with $s_i(\theta_i^2,\theta_i^1)=0$:

In this case, the minimum of $\frac{q(\cdot|s_i)}{p(\cdot)}$ is achieved only at (θ_i^1,θ_{-i}) for some $\theta_{-i}\in\Theta_{-i}$ since $\frac{q(\cdot|s_i)}{p(\cdot)}$ is already larger than unity at (θ_i^2,θ_{-i}) for any $\theta_{-i}\in\Theta_{-i}$. Thus $\min_{\theta_{-i}}\frac{q(\theta_i^1,\theta_{-i}|s_i)}{p(\theta_i^1,\theta_{-i})}=\frac{(1-s_i(\theta_i^1,\theta_i^2))p(\theta_i^1)}{p(\theta_i^1)}=1-s_i(\theta_i^1,\theta_i^2)$. So we derive the following:

$$V_i=\frac{1}{s_i(\theta_i^1,\theta_i^2)}\Big[s_i(\theta_i^1,\theta_i^2)\Delta u_i(\theta_i^1,\theta_i^2)p(\theta_i^1)\Big]=\Delta u_i(\theta_i^1,\theta_i^2)p(\theta_i^1).$$

4.3 Characterization

Let us compare the two optimal values of the two cases:

$$\Delta u_i(\theta_i^1,\theta_i^2)p(\theta_i^1) \lesseqgtr \frac{\Delta u_i(\theta_i^1,\theta_i^2)\Big(1+\min\frac{p(\theta_i^2,\cdot)}{p(\theta_i^1,\cdot)}\Big)p_i(\theta_i^1)+\Delta u_i(\theta_i^2,\theta_i^1)\Big(1+\min\frac{p(\theta_i^1,\cdot)}{p(\theta_i^2,\cdot)}\Big)p_i(\theta_i^2)}{1-\min\frac{p(\theta_i^2,\cdot)}{p(\theta_i^1,\cdot)}\min\frac{p(\theta_i^1,\cdot)}{p(\theta_i^2,\cdot)}}$$

$$\Leftrightarrow \quad 0 \lesseqgtr \Delta u_i(\theta_i^2,\theta_i^1)+\Delta u_i(\theta_i^1,\theta_i^2)\min_{\theta_{-i}}\frac{p(\theta_{-i}|\theta_i^2)}{p(\theta_{-i}|\theta_i^1)}. \tag{25}$$

In other words, if the θ_i^2's utility loss from mimicking θ_i^1, $-\Delta u_i(\theta_i^2,\theta_i^1)$, is smaller than the information rent accumulated to θ_i^1 discounted by $\min_{\theta_{-i}}\frac{p(\theta_{-i}|\theta_i^2)}{p(\theta_{-i}|\theta_i^1)}$, $\Big[\Delta u_i(\theta_i^1,\theta_i^2)\min_{\theta_{-i}}\frac{p(\theta_{-i}|\theta_i^2)}{p(\theta_{-i}|\theta_i^1)}\Big]$, then θ_i^2 will have incentive to mimic θ_i^1. Note that the discount factor $\min_{\theta_{-i}}\frac{p(\theta_{-i}|\theta_i^2)}{p(\theta_{-i}|\theta_i^1)}$ specifies the state of θ_{-i} in which type θ_i^1 receives positive $U_i(\theta_i,\theta_{-i})$.

Thus inequality (25) characterizes the shape of $s_i(\cdot,\cdot)$ (hence, $\gamma_i(\cdot,\cdot)$ as well).

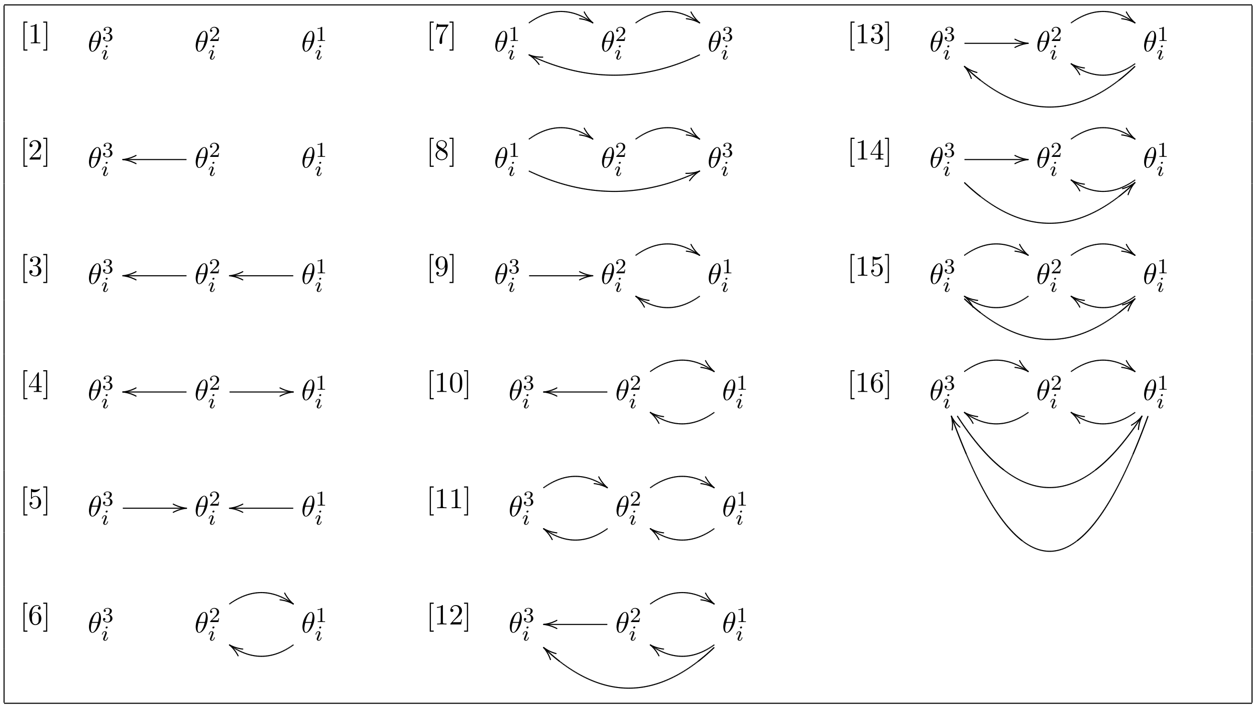
5 Full characterization of the case with $|\Theta_i| = 3$

There are three types of agent i, $\theta_i \in \{\theta_i^1,\theta_i^2,\theta_i^3\}$, and M types of agent $(-i)$, $\theta_{-i}\in\{\theta_{-i}^1,\ldots,\theta_{-i}^M\}$. For notational simplicity, we let the probability distribution function on Θ be $p^{jk} := p(\theta_i^j,\theta_{-i}^k)$, type j's marginal distribution of agent i is $p_j = \sum_{\theta_{-i}\in\Theta_{-i}} p(\theta_i^j,\theta_{-i})$, conditional probability of θ_{-i}^k given θ_i^j is $p_j^k = \frac{p^{jk}}{\sum_{1\le l\le M} p^{jl}}$, and we use $\gamma_{jk} := \gamma_i(\theta_i^j,\theta_i^k)$ and $\Delta_{jk} := \Delta u_i(\theta_i^j,\theta_i^k)$.

Each case is characterized by its unique set of binding incentive constraints. A binding incentive constraint between types θ_i^j to θ_i^k is represented by a directed edge from θ_i^j to θ_i^k in a graph with nodes of $\{\theta_i^1,\theta_i^2,\theta_i^3\}$. There are sixteen directed graphs to consider up to permutation of agent i's types (Figure 1). We will derive the conditions for each of the 16 cases to arise. The procedure in each case is as follows: (i) we provide the conditions for the corresponding incentive constraints to be binding and derive the optimal mechanism and the corresponding informational rents, then (ii) we provide the conditions under which all the other incentive constraints are non-binding.

Case 1: θ_i^3 $\quad\theta_i^2$ $\quad\theta_i^1$

In this case, information rents are trivially: $R_1 = 0$, $R_2 = 0$, $R_3 = 0$.



Six incentive compatibility constraints characterizing this case are:

$$R_1 = 0 \geq \Delta_{12}, R_1 = 0 \geq \Delta_{13}, R_2 = 0 \geq \Delta_{21}, R_2 = 0 \geq \Delta_{22}, R_3 = 0 \geq \Delta_{31}, R_3 = 0 \geq \Delta_{32}.$$

Case 2: $\theta_i^3 \longleftarrow \theta_i^2 \qquad \theta_i^1$

Informational rents are trivially: $R_1 = 0, \quad R_2 = \Delta_{23}, \quad R_3 = 0.$

Let $R_2(k)$ denotes the information rent given to θ_i^2 when $\theta_{-i} = \theta_{-i}^k$. Case 2 arises if and only if the following conditions are met:

$$R_2 = \Delta_{23} \geq 0, \quad R_1 = 0 \geq \Delta_{13}, \quad R_2 = \Delta_{23} \geq \Delta_{21}, \quad R_3 = 0 \geq \Delta_{31},$$
$$\exists (R_2(k) \geq 0)_{k \in \Theta_{-i}} \text{ s.t. } \Delta_{23} = \sum_k R_2(k) p_2^k, \quad 0 \geq \Delta_{12} + \sum_k R_2(k) p_1^k, \quad 0 \geq \Delta_{32} + \sum_k R_2(k) p_3^k.$$

The first constraint is non-negativity of type θ_i^2's information rent. The next three constraints are type θ_i^1's incentive compatibility constraint not to mimic θ_i^3, type θ_i^2's constraint not to mimic θ_i^3, and type θ_i^3's constraint not to mimic θ_i^1. In the second line, the first equality means that θ_i^2's expected information rent is Δ_{23}. The last two inequalities are type θ_i^1's incentive compatibility constraint not to mimic θ_i^2 and type θ_i^3's constraint not to mimic θ_i^2.

For notational simplicity, define the following vectors:

$$\mathcal{P}_1 = (p_1^k)_{k\in\Theta_{-i}}, \mathcal{P}_2 = (p_2^k)_{k\in\Theta_{-i}}, \mathcal{P}_3 = (p_3^k)_{k\in\Theta_{-i}},$$
$$\mathcal{R}_1 = (R_1(k))_{k\in\Theta_{-i}}, \mathcal{R}_2 = (R_2(k))_{k\in\Theta_{-i}}, \mathcal{R}_3 = (R_3(k))_{k\in\Theta_{-i}}.$$

For the existence of such $\mathcal{R}_2$, let us consider the following linear program:

$$\max_{\mathcal{R}_2\geq 0} \mathcal{R}_2 \cdot \mathbf{0} \;\text{ s.t. }\; \mathcal{P}_1 \cdot \mathcal{R}_2 \leq -\Delta_{12}, \;\; \mathcal{P}_3 \cdot \mathcal{R}_2 \leq -\Delta_{32}, \;\; \mathcal{P}_2 \cdot \mathcal{R}_2 = \Delta_{23}. \tag{26}$$

Note that the constraints are those in the second line of the above conditions.

Let ρ_1, ρ_3, and γ are the dual variables for constraints of (26), dual linear program is:

$$\min_{\rho\geq 0,\gamma} (-\Delta_{12}, -\Delta_{32}, \Delta_{23}) \cdot (\rho_1, \rho_3, \gamma) \;\text{ s.t. }\; p_1^k\rho_1 + p_3^k\rho_3 + p_2^k\gamma \geq 0 \;\text{ for all } k. \tag{27}$$

Note that the first two constraints for the primal LP are inequality constraints; thus, the dual variables for the constraints will be non-negative, i.e., $\rho_1 \geq 0$ and $\rho_3 \geq 0$. The last constraint for the primal LP is equality constraint; thus, the dual variable can be negative or positive, i.e., $\gamma \in \mathbb{R}$. Also the primal LP restricts that $R_2(k)$ is non-negative; thus the each dual constraint corresponding to each $R_2(k)$ is an inequality constraint.

Clearly, $\rho_1 = \rho_3 = \gamma = 0$ is a feasible solution, and the value of the dual linear program is zero at it. The optimal value of the primal linear program is zero, as long as the domain of the primal linear program is non-empty. Thus, $\mathcal{R}_2$ exists if and only if the optimal value of the dual linear program is zero, i.e., if and only if $\rho_1 = \rho_3 = \gamma = 0$ is an optimal solution.

Each constraint $p_1^k\rho_1 + p_3^k\rho_3 + p_2^k\gamma \geq 0$ (indexed by k) represents a half-space in three-dimensional Euclidean space $\mathbb{R}^3$ passing the origin. Let us it by H_k. Also $\rho_1, \rho_3 \in \mathbb{R}_+$ can be represented by the half planes, $1 \cdot \rho_1 + 0 \cdot \rho_3 + 0 \cdot \gamma \geq 0$ and $0 \cdot \rho_1 + 1 \cdot \rho_3 + 0 \cdot \gamma \geq 0$; let us denote the first half-space by Q_1, and the second half-space by Q_3. It is well known that the intersection of half-spaces passing the origin is a convex polyhedral cone. Thus the feasibility of the dual linear program is summarized by convex cone $\left(\bigcap_{k\in\Theta_{-i}} H_k\right) \cap Q_1 \cap Q_2$.

Using the aforementioned dual linear program, we can show the following proposition.

Proposition 2 *The following is a necessary and sufficient condition for case 2 to arise.*

$$\Delta_{23} \geq 0, 0 \geq \Delta_{13}, \Delta_{23} \geq \Delta_{21}, 0 \geq \Delta_{31},$$
$$\alpha\Delta_{12} + (1-\alpha)\Delta_{32} + \Delta_{23} \min_k \left[\alpha\frac{p_1^k}{p_2^k} + (1-\alpha)\frac{p_3^k}{p_2^k}\right] \leq 0, \ \forall\alpha \in [0,1].$$

Proof. See Appendix A.4 ∎

For $\alpha = 0$, the last inequality implies $\Delta_{32} + \Delta_{23} \min_k \left[\frac{p_3^k}{p_2^k}\right] \leq 0$, which means that the misrepresentation of θ_i^3 can be deterred. For $\alpha = 1$, the inequality implies $\Delta_{12} + \Delta_{23} \min_k \left[\frac{p_1^k}{p_2^k}\right] \leq 0$, which means that θ_i^1's misrepresentation is deterred.

Remark: The last inequality can be re-written as:

$$-\alpha\frac{\Delta_{12}}{\Delta_{23}} - (1-\alpha)\frac{\Delta_{32}}{\Delta_{23}} \geq \min_k \left[\alpha\frac{p_1^k}{p_2^k} + (1-\alpha)\frac{p_3^k}{p_2^k}\right] \leq 0, \ \forall\alpha \in [0,1].$$

For given α, the constraint is “easier” to satisfy as the structure of Θ_{-i} becomes “richer”.

Case 3: $\theta_i^3 \longleftarrow \theta_i^2 \longleftarrow \theta_i^1$

By the same way in Example 1, we calculate the informational rent as follows:

$$R_3 = 0, \quad R_2 = \Delta_{23}, \quad R_1 = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k} R_2 = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{23}$$

Let $k_2 = \operatorname*{argmin}_k \frac{p_1^k}{p_2^k}$, i.e., type θ_i^2 receives positive rent at state $\theta_{-i}^{k_2}$. The conditions for this case to arise are as follows:

$$R_1 = \mathcal{R}_1 \cdot \mathcal{P}_1 = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{23} \geq 0, \quad R_2 = \Delta_{23} \geq 0, \quad \Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{23} \geq \Delta_{13},$$
$$0 \geq \Delta_{32} + \frac{p_3^{k_2}}{p_2^{k_2}}\Delta_{23}, \quad \Delta_{23} \geq \Delta_{21} + \sum R_1(k)p_2^k, \quad 0 \geq \Delta_{31} + \sum R_1(k)p_2^k.$$

The first two are non-negativity conditions for R_1 and R_3. The next four conditions are the incentive compatibility constraints for θ_i^1 not to mimic θ_i^3, for θ_i^3 not to mimic θ_i^2, for θ_i^2 not to mimic θ_i^1, and for θ_i^3 not to mimic θ_i^1.

Similarly to case 2, we need to characterize $R_1(k)$ satisfying the following.

$$\mathcal{R}_1 \cdot \mathcal{P}_1 = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{23}, \quad \Delta_{23} = \Delta_{21} \geq \mathcal{R}_1 \cdot \mathcal{P}_2, \quad -\Delta_{31} \geq \mathcal{R}_1 \cdot \mathcal{P}_3$$

By a similar way to that of case 2, we can prove the following.

Proposition 3 *The following is a necessary and sufficient condition for case 3 to arise.*

$$\Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{23} \geq 0, \quad \Delta_{23} \geq 0, \quad \Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{23} \geq \Delta_{13}, \quad 0 \geq \Delta_{32} + \frac{p_3^{k_2}}{p_2^{k_2}}\Delta_{23}$$

$$\left(\Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{23}\right) \min_k \left[\alpha\frac{p_2^k}{p_1^k} + (1-\alpha)\frac{p_3^k}{p_1^k}\right] + \alpha(\Delta_{21} - \Delta_{23}) + (1-\alpha)\Delta_{31} \leq 0, \ \forall\alpha \in [0,1].$$

Case 4: $\theta_i^3 \longleftarrow \theta_i^2 \longrightarrow \theta_i^1$ (non-generic case)

Information rents are: $R_1 = 0, \quad R_2 = \Delta_{23} = \Delta_{32}, \quad R_3 = 0.$

Note that this case is non-generic because it requires $\Delta_{23} = \Delta_{32}$.

This case arises under the following conditions:

$$\Delta_{23} = \Delta_{32} = \mathcal{R}_2 \cdot \mathcal{P}_2 \geq 0, \quad 0 \geq \Delta_{12} + \mathcal{R}_2 \cdot \mathcal{P}_1, \quad 0 \geq \Delta_{13}, \quad 0 \geq \Delta_{31}, \quad 0 \geq \Delta_{32} + \mathcal{R}_2 \cdot \mathcal{P}_3.$$

Similarly to case 2, we can prove the following.

Proposition 4 *The following is a necessary and sufficient condition for case 4 to arise.*

$$\Delta_{23} = \Delta_{32} \geq 0, \quad 0 \geq \Delta_{13}, \quad 0 \geq \Delta_{31},$$

$$\Delta_{23} \min_k \left[\alpha\frac{p_1^k}{p_2^k} + (1-\alpha)\frac{p_3^k}{p_2^k}\right] + \alpha\Delta_{12} + (1-\alpha)\Delta_{32} \leq 0, \ \forall\alpha \in [0,1].$$

Case 5: $\theta_i^3 \longrightarrow \theta_i^2 \longleftarrow \theta_i^1$

The informational rents are: $R_1 = \Delta_{12}, \quad R_2 = 0, \quad R_3 = \Delta_{32}.$

This case is characterized by the following conditions:

$$\Delta_{12} \geq 0, \quad \Delta_{32} \geq 0, \quad \Delta_{12} \geq \Delta_{13} + \mathcal{R}_3 \cdot \mathcal{P}_1, \quad 0 \geq \Delta_{23} + \mathcal{R}_3 \cdot \mathcal{P}_2, \quad \Delta_{32} = \mathcal{R}_3 \cdot \mathcal{P}_3,$$

$$0 \geq \Delta_{21} + \mathcal{R}_1 \cdot \mathcal{P}_2, \quad \Delta_{32} \geq \Delta_{31} + \mathcal{R}_1 \cdot \mathcal{P}_3, \quad \Delta_{12} = \mathcal{R}_1 \cdot \mathcal{P}_1$$

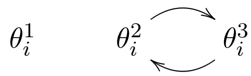
Similarly to case 2, we can prove the following.

Proposition 5 *The following is a necessary and sufficient condition for case 5 to arise.*

$$\Delta_{12} \geq 0, \quad \Delta_{32} \geq 0$$

$$\alpha(\Delta_{13} - \Delta_{12}) + (1-\alpha)\Delta_{23} + \Delta_{32} \min_k \left[\alpha\frac{p_1^k}{p_3^k} + (1-\alpha)\frac{p_2^k}{p_3^k}\right] \leq 0, \ \forall\alpha \in [0,1],$$

$$\alpha\Delta_{21} + (1-\alpha)(\Delta_{31} - \Delta_{32}) + \Delta_{12} \min_k \left[\alpha\frac{p_2^k}{p_1^k} + (1-\alpha)\frac{p_3^k}{p_1^k}\right] \leq 0, \ \forall\alpha \in [0,1].$$

Case 6: 

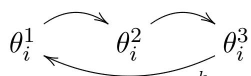
From $R_1 = 0$, $R_2 = \Delta_{23} + \min_k \frac{p_2^k}{p_3^k} R_3$, $R_3 = \Delta_{32} + \min_k \frac{p_3^k}{p_2^k} R_2$, we derive informational rent:

$$R_1 = 0, \quad R_2 = \frac{\Delta_{23} + \min_k \frac{p_2^k}{p_3^k}\Delta_{32}}{1 - \min_k \frac{p_2^k}{p_3^k} \min_k \frac{p_3^k}{p_2^k}}, \quad R_3 = \frac{\Delta_{32} + \min_k \frac{p_3^k}{p_2^k}\Delta_{23}}{1 - \min_k \frac{p_3^k}{p_2^k} \min_k \frac{p_2^k}{p_3^k}}.$$

Let $k_{23} = \underset{k}{\operatorname{argmin}} \frac{p_2^k}{p_3^k}$, and $k_{32} = \underset{k}{\operatorname{argmin}} \frac{p_3^k}{p_2^k}$. Type θ_i^2 receives positive rent at state $(\theta_i^2, \theta_{-i}^{k_{32}})$, and type θ_i^3 at $(\theta_i^3, \theta_{-i}^{k_{23}})$. For the information rent to be finite, k_{23} and k_{32} should be different.[4]

The necessary and sufficient conditions for this case are:

$$k_{23} \neq k_{32}, \quad R_2 \geq 0, \quad R_3 \geq 0, \quad R_1 \geq \Delta_{13} + \frac{p_1^{k_{23}}}{p_3^{k_{23}}} R_3, \quad R_1 \geq \Delta_{12} + \frac{p_1^{k_{32}}}{p_2^{k_{32}}} R_2, \quad R_2 \geq \Delta_{21}, \quad R_3 \geq \Delta_{32}.$$

Case 7: 

From $R_1 = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k} R_2$, $R_2 = \Delta_{23} + \min_k \frac{p_2^k}{p_3^k} R_3$, $R_3 = \Delta_{31} + \min_k \frac{p_3^k}{p_1^k} R_1$, we derive:

$$R_1 = \frac{\Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{23} + \min_k \frac{p_1^k}{p_2^k} \min_k \frac{p_2^k}{p_3^k}\Delta_{31}}{1 - \min_k \frac{p_1^k}{p_2^k} \min_k \frac{p_2^k}{p_3^k} \min_k \frac{p_3^k}{p_1^k}}, R_2 = \frac{\Delta_{23} + \min_k \frac{p_2^k}{p_3^k}\Delta_{31} + \min_k \frac{p_2^k}{p_3^k} \min_k \frac{p_3^k}{p_1^k}\Delta_{12}}{1 - \min_k \frac{p_1^k}{p_2^k} \min_k \frac{p_2^k}{p_3^k} \min_k \frac{p_3^k}{p_1^k}},$$

$$R_3 = \frac{\Delta_{31} + \min_k \frac{p_3^k}{p_1^k}\Delta_{12} + \min_k \frac{p_3^k}{p_1^k} \min_k \frac{p_1^k}{p_2^k}\Delta_{23}}{1 - \min_k \frac{p_1^k}{p_2^k} \min_k \frac{p_2^k}{p_3^k} \min_k \frac{p_3^k}{p_1^k}}.$$

Let $k_{12} = \underset{k}{\operatorname{argmin}} \frac{p_1^k}{p_2^k}$, $k_{23} = \underset{k}{\operatorname{argmin}} \frac{p_2^k}{p_3^k}$, and $k_{31} = \underset{k}{\operatorname{argmin}} \frac{p_3^k}{p_1^k}$. Type θ_i^1 receives positive rent at state $(\theta_i^1, \theta_{-i}^{k_{31}})$, type θ_i^2 at $(\theta_i^2, \theta_{-i}^{k_{12}})$, and type θ_i^3 at $(\theta_i^3, \theta_{-i}^{k_{23}})$. For finite information rent, at least two of k_{12}, k_{23}, and k_{31} should be different. The conditions for this case are:

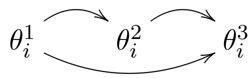
$$R_1 \geq 0, \quad R_2 \geq 0, \quad R_3 \geq 0, \quad R_1 \geq \Delta_{13} + \frac{p_1^{k_{23}}}{p_3^{k_{23}}} R_3, \quad R_2 \geq \Delta_{21} + \frac{p_2^{k_{31}}}{p_1^{k_{31}}} R_1, \quad R_3 \geq \Delta_{32} + \frac{p_3^{k_{12}}}{p_2^{k_{12}}} R_2$$

Assuming all of k_{12}, k_{23}, k_{31} are all different, The last three inequalities can be simplified into

$$R_1 \geq \frac{\Delta_{13} + \frac{p_1^{k_{23}}}{p_3^{k_{23}}}\Delta_{31}}{1 - \frac{p_1^{k_{23}}}{p_3^{k_{23}}}\frac{p_3^{k_{31}}}{p_1^{k_{31}}}}, \quad R_2 \geq \frac{\Delta_{21} + \frac{p_2^{k_{31}}}{p_1^{k_{31}}}\Delta_{12}}{1 - \frac{p_2^{k_{31}}}{p_1^{k_{31}}}\frac{p_1^{k_{12}}}{p_2^{k_{12}}}}, \quad R_3 \geq \frac{\Delta_{32} + \frac{p_3^{k_{12}}}{p_2^{k_{12}}}\Delta_{23}}{1 - \frac{p_3^{k_{12}}}{p_2^{k_{12}}}\frac{p_2^{k_{23}}}{p_3^{k_{23}}}}.$$

[4]Note $\min_k \frac{p_2^k}{p_3^k} \min_k \frac{p_3^k}{p_2^k} < 1$ if and only if $k_{23} \neq k_{32}$ since $\min_k \frac{p_l^k}{p_m^k} < 1$ unless $p(\cdot|\theta_i^m) \equiv p(\cdot|\theta_i^l)$.

These conditions mean that any deviation to form a local cycle (as in case 6) is never profitable.

Case 8: 

(non-generic case)

Information rents are:

$$R_3 = 0, \quad R_2 = \Delta_{23}, \quad R_1 = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k} R_2 = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k} \Delta_{23} = \Delta_{13}$$

Note that $\Delta_{12} + \min_k \frac{p_1^k}{p_2^k} \Delta_{23} = \Delta_{13}$ is a measure zero case. Thus this is a non-generic case.

For $k_2 = \text{argmin} \frac{p_1^k}{p_2^k}$, type θ_i^2 receives positive rent at $(\theta_i^2, \theta_{-i}^{k_2})$. This case is characterized by the following conditions on the primitive:

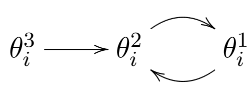
$$\Delta_{23} \geq 0, \quad \mathcal{R}_1 \cdot \mathcal{P}_1 = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k} \Delta_{23} = \Delta_{13} \geq 0$$

$$R_2 = \Delta_{23} \geq \Delta_{21} + \mathcal{R}_1 \cdot \mathcal{P}_2, \quad R_3 = 0 \geq \Delta_{31} + \mathcal{R}_1 \cdot \mathcal{P}_3, \quad R_3 = 0 \geq \Delta_{32} + \frac{p_3^{k_2}}{p_2^{k_2}} R_2 = \Delta_{32} + \frac{p_3^{k_2}}{p_2^{k_2}} \Delta_{23}$$

Similarly to case 2, we can prove the following.

Proposition 6 *The following is a necessary and sufficient condition for case 8 to arise.*

$$\Delta_{23} \geq 0, \quad \Delta_{12} + \min_k \frac{p_1^k}{p_2^k} \Delta_{23} = \Delta_{13} \geq 0, \quad 0 \geq \Delta_{32} + \frac{p_3^{k_2}}{p_2^{k_2}} \Delta_{23},$$
$$\Delta_{13} \min_k \left[\alpha \frac{p_2^k}{p_1^k} + (1-\alpha) \frac{p_3^k}{p_1^k} \right] + \alpha(\Delta_{21} - \Delta_{23}) + (1-\alpha)\Delta_{31} \leq 0, \ \forall \alpha \in [0,1].$$

Case 9: 

Information rents (Derivation is in Appendix A.5.) are:

$$R_1 = \frac{\Delta_{12} + \frac{p_1^{\bar{k}}}{p_2^{\bar{k}}} \Delta_{21}}{1 - \frac{p_1^{\bar{k}}}{p_2^{\bar{k}}} \min_k \frac{p_2^k}{p_1^k}}, \quad R_2 = \frac{\Delta_{21} + \min_k \frac{p_2^k}{p_1^k} \Delta_{12}}{1 - \frac{p_1^{\bar{k}}}{p_2^{\bar{k}}} \min_k \frac{p_2^k}{p_1^k}}, \quad R_3 = \Delta_{32} + \frac{p_3^{\bar{k}}}{p_2^{\bar{k}}} \frac{\Delta_{21} + \min_k \frac{p_2^k}{p_1^k} \Delta_{12}}{1 - \frac{p_1^{\bar{k}}}{p_2^{\bar{k}}} \min_k \frac{p_2^k}{p_1^k}}.$$

where $\bar{k} = \underset{\tilde{k}}{\text{argmin}} \left(1 + \frac{p^{1\tilde{k}} + p^{3\tilde{k}}}{p^{2\tilde{k}}}\right) \Big/ \left(1 - \frac{p_1^{\tilde{k}}}{p_2^{\tilde{k}}} \min_k \frac{p_2^k}{p_1^k}\right)$. The condition for finite information rent is $\bar{k} \neq \text{argmin} \frac{p_2^k}{p_1^k}$. $\bar{k}$ is such that type θ_i^2 receives positive rent at state $(\theta_i^2, \theta_{-i}^{\bar{k}})$, and $k_1 = \text{argmin} \frac{p_2^k}{p_1^k}$ is such that type θ_i^1 receives positive rent at state $(\theta_i^1, \theta_{-i}^{k_1})$.

This case is characterized by the following conditions on the primitive.

$$R_1 \geq 0, \quad R_2 \geq 0, \quad R_3 \geq 0, \quad R_1 \geq \Delta_{13} + \mathcal{R}_3 \cdot \mathcal{P}_1, \quad R_2 \geq \Delta_{23} + \mathcal{R}_3 \cdot \mathcal{P}_2, \quad R_3 \geq \Delta_{31} + R_1 \frac{p_3^{k_1}}{p_1^{k_1}}.$$

Similarly to case 2, we can prove the following.

Proposition 7 *The following is a necessary and sufficient condition for case 9 to arise.*

$$\frac{\Delta_{12} + \frac{p_1^{\bar{k}}}{p_2^{\bar{k}}}\Delta_{21}}{1 - \frac{p_1^{\bar{k}}}{p_2^{\bar{k}}} \min \frac{p_2^k}{p_1^k}} \geq 0, \ \frac{\Delta_{21} + \min \frac{p_2^k}{p_1^k}\Delta_{12}}{1 - \frac{p_1^{\bar{k}}}{p_2^{\bar{k}}} \min \frac{p_2^k}{p_1^k}} \geq 0, \ \Delta_{32} + \frac{p_3^{\bar{k}}}{p_2^{\bar{k}}} \frac{\Delta_{21} + \min \frac{p_2^k}{p_1^k}\Delta_{12}}{1 - \frac{p_1^{\bar{k}}}{p_2^{\bar{k}}} \min \frac{p_2^k}{p_1^k}} \geq 0, \ R_3 \geq \Delta_{31} + R_1 \frac{p_3^{k_1}}{p_1^{k_1}}$$

$$\alpha(\Delta_{13} - R_1) + (1-\alpha)(\Delta_{23} - R_2) + R_3 \min_k \left[\alpha \frac{p_1^k}{p_3^k} + (1-\alpha)\frac{p_2^k}{p_3^k}\right] \leq 0, \ \forall \alpha \in [0,1].$$

Case 10: $\theta_i^3 \longleftarrow \theta_i^2 \rightleftarrows \theta_i^1$ (non-generic case)

The following characterizes information rent:

$$R_2 = \Delta_{21} + R_1 \min_k \frac{p_2^k}{p_1^k} = \Delta_{23}, \quad R_1 = \Delta_{12} + R_2 \min_k \frac{p_1^k}{p_2^k}, \quad R_3 = 0$$

$$\Rightarrow R_1 = \frac{\Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{21}}{1 - \min_k \frac{p_2^k}{p_1^k} \min_k \frac{p_1^k}{p_2^k}}, \quad R_2 = \frac{\Delta_{21} + \min_k \frac{p_2^k}{p_1^k}\Delta_{12}}{1 - \min_k \frac{p_2^k}{p_1^k} \min_k \frac{p_1^k}{p_2^k}} = \Delta_{23}, \quad R_3 = 0$$

Note that $\Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{23} = \Delta_{13}$ is a measure zero case. Thus this is a non-generic case.

$k_1 = \text{argmin} \frac{p_1^k}{p_2^k}$ is the state when θ_i^1 is given positive rent, $k_2 = \text{argmin} \frac{p_2^k}{p_1^k}$ is when θ_i^2 is given positive rent, and $k_3 = \text{argmin} \frac{p_3^k}{p_2^k}$ is when θ_i^3 is given positive rent. $k_1 \neq k_2$ is required for finite information rent.

This case is characterized by the following conditions on the primitive.

$$R_1 \geq 0, \quad R_2 \geq 0, \quad R_1 \geq \Delta_{13}, \quad R_3 = 0 \geq \Delta_{31} + \frac{p_3^{k_1}}{p_1^{k_1}} R_1, \quad R_3 = 0 \geq \Delta_{32} + \frac{p_3^{k_2}}{p_2^{k_2}} R_2.$$

Case 11: $\theta_i^3 \rightleftarrows \theta_i^2 \rightleftarrows \theta_i^1$

Suppose $M = 2$, i.e., $\Theta_{-i} = \{\theta_{-i}^1, \theta_{-i}^2\}$. Information rents are decided by:

$$R_1 = \Delta_{12} + R_2(1)p_1^1 + R_2(2)p_1^2, \ R_3 = \Delta_{32} + R_2(1)p_3^1 + R_2(2)p_3^2$$

$$R_2(1)p_2^1 + R_2(2)p_2^2 = \Delta_{21} + \min_k \frac{p_2^k}{p_1^k} R_1 = \Delta_{23} + \min_k \frac{p_2^k}{p_3^k} R_3$$

Plugging in the first and the second constraints into the third and the fourth:

$$R_2(1)\left(p_2^1 - \min_k \frac{p_2^k}{p_1^k} p_1^1\right) + R_2(2)\left(p_2^2 - \min_k \frac{p_2^k}{p_1^k} p_1^2\right) = \Delta_{21} + \min_k \frac{p_2^k}{p_1^k}\Delta_{12}$$
$$R_2(1)\left(p_2^1 - \min_k \frac{p_2^k}{p_3^k} p_3^1\right) + R_2(2)\left(p_2^2 - \min_k \frac{p_2^k}{p_3^k} p_3^2\right) = \Delta_{23} + \min_k \frac{p_2^k}{p_3^k}\Delta_{32}$$

If $\operatorname{argmin}_k \frac{p_2^k}{p_1^k} = \operatorname{argmin}_k \frac{p_2^k}{p_3^k}$, the two equations above become:

$$\text{either}\quad R_2(2)\left(p_2^2 - \frac{p_2^1}{p_1^1} p_1^2\right) = \Delta_{21} + \frac{p_2^1}{p_1^1}\Delta_{12},\quad R_2(2)\left(p_2^2 - \frac{p_2^1}{p_3^1} p_3^2\right) = \Delta_{23} + \frac{p_2^1}{p_3^1}\Delta_{32}$$
$$\text{or}\quad R_2(1)\left(p_2^1 - \frac{p_2^2}{p_1^2} p_1^1\right) = \Delta_{21} + \frac{p_2^2}{p_1^2}\Delta_{12},\quad R_2(1)\left(p_2^1 - \frac{p_2^2}{p_3^2} p_3^1\right) = \Delta_{23} + \frac{p_2^2}{p_3^2}\Delta_{32}.$$

Thus there is no solution generically. On the other hand, if $\operatorname{argmin}_k \frac{p_2^k}{p_1^k} \neq \operatorname{argmin}_k \frac{p_2^k}{p_3^k}$, there exist a unique solution. For example, suppose $\operatorname{argmin}_k \frac{p_2^k}{p_1^k} = 1$ and $\operatorname{argmin}_k \frac{p_2^k}{p_3^k} = 2$. Define

$$Cycle(2,3) = \left(\Delta_{23} + \frac{p_2^2}{p_3^2}\Delta_{32}\right) \Big/ \left(1 - \frac{p_2^2}{p_3^2}\frac{p_3^1}{p_2^1}\right) \text{ and } Cycle(2,1) = \left(\Delta_{21} + \frac{p_2^1}{p_1^1}\Delta_{12}\right) \Big/ \left(1 - \frac{p_2^1}{p_1^1}\frac{p_1^2}{p_2^2}\right),$$

then

$$R_2(1) = \frac{1}{p_2^1} Cycle(2,3),\quad R_2(2) = \frac{1}{p_2^2} Cycle(2,1),\quad R_2 = Cycle(2,3) + Cycle(2,1),$$
$$R_1 = \Delta_{12} + \frac{p_1^1}{p_2^1} Cycle(2,3) + \frac{p_1^2}{p_2^2} Cycle(2,1),\quad R_3 = \Delta_{32} + \frac{p_3^1}{p_2^1} Cycle(2,3) + \frac{p_3^2}{p_2^2} Cycle(2,1).$$

This case is characterized by the following conditions on the primitive.

$$R_1 \geq 0,\quad R_2 \geq 0,\quad R_3 \geq 0,\quad R_3 \geq \Delta_{31} + \frac{p_3^{k_1}}{p_1^{k_1}} R_1,\quad R_1 \geq \Delta_{13} + \frac{p_1^{k_3}}{p_3^{k_3}} R_3,\quad k_1 \neq k_3$$

Consider a general case, i.e., $M > 2$. Information rents are decided by:

$$\min_{R_1, R_2(k), R_3} \; p_1 R_1 + p_2 \sum_k R_2(k) p_2^k + p_3 R_3$$
$$\text{s.t. } R_1 = \Delta_{12} + \sum_k R_2(k) p_1^k,\quad R_3 = \Delta_{32} + \sum_k R_2(k) p_3^k$$
$$\sum_k R_2(k) p_2^k = \Delta_{21} + \min_k \frac{p_2^k}{p_1^k} R_1 = \Delta_{23} + \min_k \frac{p_2^k}{p_3^k} R_3.$$

By eliminating R_1 and R_3, we get

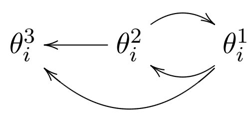
$$\min_{R_2(k)} \; p_1\Delta_{12} + p_3\Delta_{32} + \sum_k R_2(k)(p^{1k} + p^{2k} + p^{3k})$$
$$\text{s.t.} \;\; \sum_k R_2(k)\Big(p_2^k - \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}} p_1^k\Big) = \Delta_{21}, \quad \sum_k R_2(k)\Big(p_2^k - \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_3^{\tilde{k}}} p_3^k\Big) = \Delta_{23}$$

Note that even if $\underset{k}{\operatorname{argmin}} \frac{p_2^k}{p_1^k} = \underset{k}{\operatorname{argmin}} \frac{p_2^k}{p_3^k}$, the solution may exist. More precisely, the solution exists unless the following two vectors are parallel, but not identical.

$$\frac{1}{\Delta_{21}}\Big(p_2^k - \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}} p_1^k\Big)_{1\le k\le M} \quad \text{and} \quad \frac{1}{\Delta_{23}}\Big(p_2^k - \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_3^{\tilde{k}}} p_3^k\Big)_{1\le k\le M}.$$

Once we compute R_1, R_2 and R_3, the conditions characterizing this case are:

$$R_1 \ge 0, \quad R_2 \ge 0, \quad R_3 \ge 0, \quad R_3 \ge \Delta_{31} + \frac{p_3^{k_1}}{p_1^{k_1}} R_1, \quad R_1 \ge \Delta_{13} + \frac{p_1^{k_3}}{p_3^{k_3}} R_3, \quad k_1 \ne k_3.$$

Case 12:  (non-generic case)

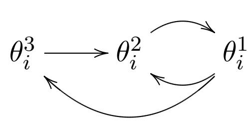
Information rents are:

$$R_1 = \Delta_{13} = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k} R_2, \quad R_2 = \Delta_{23} = \Delta_{21} + \min_k \frac{p_2^k}{p_1^k} R_1, \quad R_3 = 0$$
$$\Rightarrow \; R_1 = \Delta_{13} = \frac{\Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{21}}{1 - \min_k \frac{p_2^k}{p_1^k} \min_k \frac{p_1^k}{p_2^k}}, \quad R_2 = \Delta_{23} = \frac{\Delta_{21} + \min_k \frac{p_2^k}{p_1^k}\Delta_{12}}{1 - \min_k \frac{p_2^k}{p_1^k} \min_k \frac{p_1^k}{p_2^k}}, \quad R_3 = 0$$

Note that this is a measure zero case. For the information rent to be finite, $k_2 = \operatorname{argmin} \frac{p_2^k}{p_1^k} \ne k_1 = \operatorname{argmin} \frac{p_1^k}{p_2^k}$ is required.

This case is characterized by the following.

$$\Delta_{13} = \frac{\Delta_{12} + \frac{p_1^{k_1}}{p_2^{k_1}}\Delta_{21}}{1 - \frac{p_2^{k_2}}{p_1^{k_2}}\frac{p_1^{k_1}}{p_2^{k_1}}}, \; \Delta_{23} = \frac{\Delta_{21} + \frac{p_2^{k_2}}{p_1^{k_2}}\Delta_{12}}{1 - \frac{p_2^{k_2}}{p_1^{k_2}}\frac{p_1^{k_1}}{p_2^{k_1}}}, \; 0 \ge \Delta_{32} + \frac{p_3^{k_2}}{p_2^{k_2}} R_2, \; 0 \ge \Delta_{31} + \frac{p_3^{k_1}}{p_1^{k_1}} R_1, \; k_1 \ne k_2$$

Case 13: 

We can calculate information rent similarly to case 11. We omit the special case of $M = 2$. Information rents are calculated by:

$$\min_{R_1, R_2(k), R_3} \; p_1 R_1 + p_2 \sum_k R_2(k) p_2^k + p_3 R_3$$

$$R_1 = \Delta_{12} + \sum_k R_2(k) p_1^k = \Delta_{13} + \min_k \frac{p_1^k}{p_3^k} R_3,$$

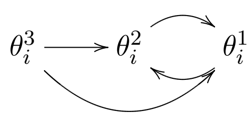
$$\sum_k R_2(k) p_2^k = \Delta_{21} + \min_k \frac{p_2^k}{p_1^k} R_1, \quad R_3 = \Delta_{32} + \sum_k R_2(k) p_3^k.$$

Similarly to case 11, we can show that the solution exists unless the following two vectors are parallel, but not identical.

$$\frac{1}{\Delta_{21}} \Big(p_2^k - \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}} p_1^k \Big)_{1 \le k \le M} \quad \text{and} \quad \frac{1}{\Delta_{23}} \Big(p_2^k - \min_{\tilde{k}} \frac{p_1^{\tilde{k}}}{p_3^{\tilde{k}}} p_3^k \Big)_{1 \le k \le M}.$$

For $k_1 = \operatorname{argmin} \frac{p_2^k}{p_1^k}$ and $k_3 = \operatorname{argmin} \frac{p_3^k}{p_1^k}$, this case is characterized by:

$$R_1 \ge 0, \quad R_2 \ge 0, \quad R_3 \ge 0, \quad R_3 \ge \Delta_{31} + \frac{p_3^{k_1}}{p_1^{k_1}} R_1, \quad R_2 \ge \Delta_{23} + \frac{p_2^{k_3}}{p_3^{k_3}} R_3.$$

Case 14: 

Suppose $M = 2$, i.e., $\Theta_{-i} = \{\theta_{-i}^1, \theta_{-i}^2\}$. Information rents are characterized by:

$$\min_{R_1(k), R_2(k), R_3} \; p_1(R_1(1)p_1^1 + R_1(2)p_1^2) + p_2(R_2(1)p_2^1 + R_2(2)p_2^2) + p_3 R_3$$

$$R_1(1)p_1^1 + R_1(2)p_1^2 = \Delta_{12} + R_2(1)p_1^1 + R_2(2)p_1^2$$

$$R_2(1)p_2^1 + R_2(2)p_2^2 = \Delta_{21} + R_1(1)p_2^1 + R_1(2)p_2^2$$

$$R_3 = \Delta_{31} + R_1(1)p_3^1 + R_1(2)p_3^2 = \Delta_{32} + R_2(1)p_3^1 + R_2(2)p_3^2$$

The constraints are simplified into

$$[R_1(1) - R_2(1)]p_1^1 + [R_1(2) - R_2(2)]p_1^2 = \Delta_{12}, \quad [R_1(1) - R_2(1)]p_2^1 + [R_1(2) - R_2(2)]p_2^2 = -\Delta_{21},$$
$$[R_1(1) - R_2(1)]p_3^1 + [R_1(2) - R_2(2)]p_3^2 = -\Delta_{31}.$$

Generically, a solution does not exist. However, if there are more states than two, it is not a non-generic case. For example, if there are three states, $\Theta_{-i} = \{\theta^1_{-i}, \theta^2_{-i}, \theta^3_{-i}\}$, we derive

$$\begin{aligned}
&[R_1(1) - R_2(1)]p_1^1 + [R_1(2) - R_2(2)]p_1^2 + [R_1(3) - R_2(3)]p_1^3 = \Delta_{12} \\
&[R_1(1) - R_2(1)]p_2^1 + [R_1(2) - R_2(2)]p_2^2 + [R_1(3) - R_2(3)]p_2^3 = -\Delta_{21} \\
&[R_1(1) - R_2(1)]p_3^1 + [R_1(2) - R_2(2)]p_3^2 + [R_1(3) - R_2(3)]p_3^3 = -\Delta_{31}.
\end{aligned}$$

The unique solution exists generically, which is

$$\begin{aligned}
[R_1(1) - R_2(1)] &= \frac{-\Delta_{31}p_2^2p_1^3 + \Delta_{21}p_3^2p_1^3 + \Delta_{31}p_1^2p_2^3 + \Delta_{12}p_3^2p_2^3 - \Delta_{21}p_1^2p_3^3 - \Delta_{12}p_2^2p_3^3}{p_3^1p_2^2p_1^3 - p_2^1p_3^2p_1^3 - p_3^1p_1^2p_2^3 + p_1^1p_3^2p_2^3 + p_2^1p_1^2p_3^3 - p_1^1p_2^2p_3^3}, \\
[R_1(2) - R_2(2)] &= \frac{-\Delta_{31}p_2^1p_1^3 + \Delta_{21}p_3^1p_1^3 + \Delta_{31}p_1^1p_2^3 + \Delta_{12}p_3^1p_2^3 - \Delta_{21}p_1^1p_3^3 - \Delta_{12}p_2^1p_3^3}{-p_3^1p_2^2p_1^3 + p_2^1p_3^2p_1^3 + p_3^1p_1^2p_2^3 - p_1^1p_3^2p_2^3 - p_2^1p_1^2p_3^3 + p_1^1p_2^2p_3^3}, \\
[R_1(3) - R_2(3)] &= \frac{-\Delta_{31}p_2^1p_1^2 + \Delta_{21}p_3^1p_1^2 + \Delta_{31}p_1^1p_2^2 + \Delta_{12}p_3^1p_2^2 - \Delta_{21}p_1^1p_3^2 - \Delta_{12}p_2^1p_3^2}{p_3^1p_2^2p_1^3 - p_2^1p_3^2p_1^3 - p_3^1p_1^2p_2^3 + p_1^1p_3^2p_2^3 + p_2^1p_1^2p_3^3 - p_1^1p_2^2p_3^3}
\end{aligned}$$

Plugging these into the objective function, $p_1(R_1(1)p_1^1 + R_1(2)p_1^2 + R_1(3)p_1^3) + p_2(R_2(1)p_2^1 + R_2(2)p_2^2 + R_2(3)p_2^3) + p_3R_3$, the information rent can be calculated.

The extension to the case of $M > 3$ is straightforward, so we omit it.

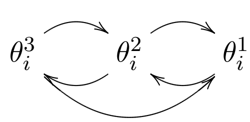
Once we calculate R_3, $\mathcal{R}_1$, and $\mathcal{R}_2$, we can derive the incentive compatibility constraints characterizing this case:

$$\begin{aligned}
&R_1 = \mathcal{R}_1 \cdot \mathcal{P}_1 \geq 0, \quad R_2 = \mathcal{R}_2 \cdot \mathcal{P}_2 \geq 0, \quad R_3 = \mathcal{R}_3 \cdot \mathcal{P}_3 \geq 0, \\
&\sum_k R_1(k)p_1^k \geq \Delta_{13} + \mathcal{R}_3 \cdot \mathcal{P}_1, \quad \sum_k R_2(k)p_2^k \geq \Delta_{23} + \mathcal{R}_3 \cdot \mathcal{P}_2.
\end{aligned}$$

By a similar way to that of case 2, we can prove the following.

Proposition 8 *The following is a necessary and sufficient condition for case 14 to arise.*

$$\begin{aligned}
&R_1 = \mathcal{R}_1 \cdot \mathcal{P}_1 \geq 0, \quad R_2 = \mathcal{R}_2 \cdot \mathcal{P}_2 \geq 0, \quad R_3 \geq 0 \\
&\alpha(\Delta_{13} - R_1) + (1-\alpha)(\Delta_{23} - R_2) + R_3 \min_k \left[\alpha\frac{p_1^k}{p_3^k} + (1-\alpha)\frac{p_2^k}{p_3^k}\right] \leq 0, \ \forall\alpha \in [0,1].
\end{aligned}$$

Case 15: 

Information rent is calculated by:

$$
\begin{aligned}
&R_1(1)p_1^1 + R_1(2)p_1^2 = \Delta_{12} + (R_2(1)p_1^1 + R_2(2)p_1^2)\\
&R_2(1)p_2^1 + R_2(2)p_2^2 = \Delta_{21} + (R_1(1)p_2^1 + R_1(2)p_2^2) = \Delta_{23} + \min_k \frac{p_2^k}{p_3^k} R_3\\
&R_3 = \Delta_{31} + (R_1(1)p_3^1 + R_1(2)p_3^2) = \Delta_{32} + (R_2(1)p_3^1 + R_2(2)p_3^2)
\end{aligned}
$$

By eliminating R_3, we get

$$
\begin{aligned}
&p_1^1[R_1(1) - R_2(1)] + p_1^2[R_1(2) - R_2(2)] = \Delta_{12}, \quad p_2^1[R_1(1) - R_2(1)] + p_2^2[R_1(2) - R_2(2)] = -\Delta_{21}\\
&R_1(1)\Big[p_2^1 - \min_k \frac{p_2^k}{p_3^k} p_3^1\Big] + R_1(2)\Big[p_2^2 - \min_k \frac{p_2^k}{p_3^k} p_3^2\Big] = \Delta_{23} + \min_k \frac{p_2^k}{p_3^k}\Delta_{31} - \Delta_{21}\\
&R_2(1)\Big[p_2^1 - \min_k \frac{p_2^k}{p_3^k} p_3^1\Big] + R_2(2)\Big[p_2^2 - \min_k \frac{p_2^k}{p_3^k} p_3^2\Big] = \Delta_{23} + \min_k \frac{p_2^k}{p_3^k}\Delta_{32}
\end{aligned}
$$

Without loss of generality, assume $\frac{p_2^1}{p_3^1} < \frac{p_2^2}{p_3^3}$. Then the constraints are:

$$
\begin{aligned}
&p_1^1[R_1(1) - R_2(1)] + p_1^2[R_1(2) - R_2(2)] = \Delta_{12}, \quad p_2^1[R_1(1) - R_2(1)] + p_2^2[R_1(2) - R_2(2)] = -\Delta_{21}\\
&R_1(2)\Big[p_2^2 - \frac{p_2^1}{p_3^1} p_3^2\Big] = \Delta_{23} + \frac{p_2^1}{p_3^1}\Delta_{31} - \Delta_{21}, \quad R_2(2)\Big[p_2^2 - \frac{p_2^1}{p_3^1} p_3^2\Big] = \Delta_{23} + \frac{p_2^1}{p_3^1}\Delta_{32}
\end{aligned}
$$

These are again reduced to:

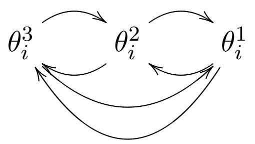
$$
\begin{aligned}
p_1^1[R_1(1) - R_2(1)] + \frac{p_1^2}{p_2^2}\frac{\frac{p_1^1}{p_3^1}(\Delta_{31} - \Delta_{32}) - \Delta_{21}}{1 - \frac{p_2^1}{p_3^1}\frac{p_3^2}{p_2^2}} &= \Delta_{12},\\
p_2^1[R_1(1) - R_2(1)] + \frac{\frac{p_1^1}{p_3^1}(\Delta_{31} - \Delta_{32}) - \Delta_{21}}{1 - \frac{p_2^1}{p_3^1}\frac{p_3^2}{p_2^2}} &= -\Delta_{21}
\end{aligned}
$$

Thus, this is a non-generic case. However, if there are more states than two, this non-genericity will disappear. Consider general case, i.e., $M > 2$. Information rent is calculated by:

$$
\begin{aligned}
\min_{\mathcal{R}_1, \mathcal{R}_2, R_3} \quad & \mathcal{R}_1 \cdot \mathcal{P}_2 + \mathcal{R}_1 \cdot \mathcal{P}_2 + R_3\\
s.t. \ & \mathcal{R}_1 \cdot \mathcal{P}_1 = \Delta_{12} + \mathcal{R}_2 \cdot \mathcal{P}_1, \quad \mathcal{R}_2 \cdot \mathcal{P}_2 = \Delta_{21} + \mathcal{R}_1 \cdot \mathcal{P}_2 = \Delta_{23} + \min_k \frac{p_2^k}{p_3^k} R_3\\
& R_3 = \Delta_{31} + \mathcal{R}_1 \cdot \mathcal{P}_3 = \Delta_{32} + \mathcal{R}_2 \cdot \mathcal{P}_3.
\end{aligned}
$$

Once we calculate the information rent, the following incentive compatibility constraints characterize case 15:

$$\mathcal{R}_1 \cdot \mathcal{P}_1 \geq 0, \quad \mathcal{R}_2 \cdot \mathcal{P}_2 \geq 0, \quad R_3 \geq 0, \quad \mathcal{R}_1 \cdot \mathcal{P}_1 \geq \Delta_{13} + \min_{\tilde{k}} \frac{p_1^{\bar{k}}}{p_3^{\bar{k}}} R_3 \quad \text{where} \quad \bar{k} = \text{argmin} \frac{p_3^k}{p_2^k}.$$

Case 16: 

Information rent is calculated by:

$$\begin{aligned}
R_1(1)p_1^1 + R_1(2)p_1^2 &= \Delta_{12} + (R_2(1)p_1^1 + R_2(2)p_1^2) = \Delta_{13} + (R_3(1)p_1^1 + R_3(2)p_1^2),\\
R_2(1)p_2^1 + R_2(2)p_2^2 &= \Delta_{21} + (R_1(1)p_2^1 + R_1(2)p_2^2) = \Delta_{23} + (R_3(1)p_2^1 + R_3(2)p_2^2),\\
R_3(1)p_3^1 + R_3(2)p_3^2 &= \Delta_{31} + (R_1(1)p_3^1 + R_1(2)p_3^2) = \Delta_{32} + (R_2(1)p_3^1 + R_2(2)p_3^2).
\end{aligned}$$

That is,

$$\begin{aligned}
&[R_1(1) - R_2(1)]p_1^1 + [R_1(2) - R_2(2)]p_1^2 = \Delta_{12}, \quad [R_1(1) - R_3(1)]p_1^1 + [R_1(2) - R_3(2)]p_1^2 = \Delta_{13},\\
&[R_2(1) - R_1(1)]p_2^1 + [R_2(2) - R_1(2)]p_2^2 = \Delta_{21}, \quad [R_2(1) - R_3(1)]p_2^1 + [R_2(2) - R_3(2)]p_2^2 = \Delta_{23},\\
&[R_3(1) - R_1(1)]p_3^1 + [R_3(2) - R_1(2)]p_3^2 = \Delta_{31}, \quad [R_3(1) - R_2(1)]p_3^1 + [R_3(2) - R_2(2)]p_3^2 = \Delta_{32}.
\end{aligned}$$

Generically, there will be no solution. However, if there are sufficient number of states, this will not be a non-generic case. In general, where $M > 2$, the information rents are derived by:

$$\begin{aligned}
\min_{\mathcal{R}_1,\mathcal{R}_2,\mathcal{R}_3} \quad & \mathcal{R}_1 \cdot \mathcal{P}_1 + \mathcal{R}_2 \cdot \mathcal{P}_2 + \mathcal{R}_3 \cdot \mathcal{P}_3\\
& \mathcal{R}_1 \cdot \mathcal{P}_1 = \Delta_{12} + \mathcal{R}_2 \cdot \mathcal{P}_1 = \Delta_{13} + \mathcal{R}_3 \cdot \mathcal{P}_1, \quad \mathcal{R}_2 \cdot \mathcal{P}_2 = \Delta_{21} + \mathcal{R}_1 \cdot \mathcal{P}_2 = \Delta_{23} + \mathcal{R}_3 \cdot \mathcal{P}_2,\\
& \mathcal{R}_3 \cdot \mathcal{P}_3 = \Delta_{31} + \mathcal{R}_1 \cdot \mathcal{P}_3 = \Delta_{32} + \mathcal{R}_2 \cdot \mathcal{P}_3.
\end{aligned}$$

The conditions characterizing case 16 are:

$$\mathcal{R}_1 \cdot \mathcal{P}_1 \geq 0, \quad \mathcal{R}_2 \cdot \mathcal{P}_2 \geq 0, \quad \mathcal{R}_3 \cdot \mathcal{P}_3 \geq 0$$

6 Algorithm to calculate information rent

We provide a method to calculate information rent.

Let $c_N = (\Delta u_i(\theta_i, \theta_i')p_i(\theta_i))_{\theta_i \neq \theta_i', (\theta_i, \theta_i') \in \Theta_i^2}$, $x_N = (\gamma_i(\theta_i, \theta_i'))_{\theta_i \neq \theta_i', (\theta_i, \theta_i') \in \Theta_i^2}$, $b = (p(\theta))_{\theta \in \Theta}$ and A_N be $|\Theta| \times |\Theta_i|(|\Theta_i| - 1)$ matrix where A_N's θ-th row and $(\tilde{\theta}_i, \tilde{\theta}_i')$-th column element is:

$$A_N[\theta, (\tilde{\theta}_i, \tilde{\theta}_i')] = \begin{cases} p(\theta) & \text{if } \tilde{\theta}_i = \theta_i \\ -p(\theta) & \text{if } \tilde{\theta}_i' = \theta_i \\ 0 & \text{otherwise.} \end{cases}$$

Then linear program LP_D is represented by a matrix form:

$$\max c_N \cdot x_N \quad \text{s.t.} \quad A_N x_N \leq b_N \tag{28}$$

(Step 1) Start from zero matrix $x_N = (\gamma_i(\theta_i, \theta_i')) = \mathbf{0}$. Choose (θ_i^j, θ_i^k) such that (θ_i^j, θ_i^k)-th column of c_N, $\Delta u_i(\theta_i^j, \theta_i^k)p_i(\theta_i^j)$, is the largest among all $\Delta u_i(\theta_i, \theta_i')p_i(\theta_i)$ for all θ_i, θ_i'. Increase (θ_i^j, θ_i^k)-th row of x_N, $\gamma_i(\theta_i^j, \theta_i^k)$, until one of the constraints in $A_N x_N \leq b$ binds.

Let B to be identity matrix of dimension $|\Theta| \times |\Theta|$. Then $A_N x_N \leq b$ is written as the following with auxiliary variable $x_B \geq 0$.

$$A_N x_N + B x_B = b \tag{29}$$

Trivially, $(x_N = \mathbf{0}, x_B = b)$ makes constraint (29) satisfied. As we increase $\gamma_i(\theta_i^j, \theta_i^k)$, x_B will change to satisfy constraint (29). Eventually, one row of x_B (say θ^1-th row) will become zero. In other words, the θ^1-th constraint of $A_N x_N \leq b$ becomes binding. (In this first step, there are many constraints becoming binding at $\gamma_i(\theta_i^j, \theta_i^k) = 1$, all θ such that $\theta = (\theta_i^j, \theta_{-i})$ with any $\theta_{-i} \in \Theta_{-i}$. Also note that $\gamma_i(\theta_i^j, \theta_i^k) = 1$ is not typical in other steps.)

In this procedure, (θ_i^j, θ_i^k)-th row of x_N became non-zero, and at least one row of x_B became zero. Interchange (θ_i^j, θ_i^k)-th row of x_N and the θ^1-th row of x_B (if there are more than one row that became zero, choose any arbitrary one). Also interchange (θ_i^j, θ_i^k)-th column of A_N and the θ^1-th column of B. Then the constraint still looks the same with the re-defined A_N, B, x_N and x_B:

$$A_N x_N + B x_B = b \quad \text{where} \quad x_N = 0.$$

Additionally, let c_B be zero matrix of dimension $1 \times |\Theta|$, and interchange (θ_i^j, θ_i^k)-th column of c_N and θ^1-th column of c_B. With the redefined c_N, c_B, x_N and x_B, the objective function is

$$c_N \cdot x_N + c_B \cdot x_B.$$

(Step 2) We want to increase another $\gamma_i(\tilde{\theta}_i, \tilde{\theta}'_i)$ to increase the value of the objective function.

The $(\theta_i^j, \theta_{-i})$-th constraint is already binding by Step 1, i.e.

$$\sum_{\theta'_i} \gamma_i(\theta_i^j, \theta'_i) p(\theta_i^j, \theta_{-i}) - \sum_{\theta'_i} \gamma_i(\theta'_i, \theta_i^j) p(\theta'_i, \theta_{-i}) = p(\theta_i^j, \theta_{-i}).$$

We cannot increase $\gamma_i(\theta_i^j, \tilde{\theta}_i)$ for $\tilde{\theta}_i \neq \theta_i^k$ without decreasing $\gamma_i(\theta_i^j, \theta_i^k)$. Since $\Delta u_i(\theta_i^j, \theta_i^k) p_i(\theta_i^j)$ is the largest, this is not an effective way to increase the objective function. In other words, there is a "negative externality" of increasing $\gamma_i(\theta_i^j, \tilde{\theta}_i)$ even if $\Delta u_i(\theta_i^j, \tilde{\theta}_i) p_i(\theta_i^j)$ is the second largest after $\Delta u_i(\theta_i^j, \theta_i^k) p_i(\theta_i^j)$. On the other hand, the increase of $\gamma_i(\tilde{\theta}_i, \theta_i^j)$ will make it possible to increase $\gamma_i(\theta_i^j, \theta_i^k)$ function; thus, there is a "positive externality". Even in this second step, it seems onerous to take all these concern on "externalities" into consideration. Moreover, if there are more than one $\gamma_i(\cdot, \cdot)$ that are positive during ongoing steps, our consideration of all these "externalities" seem to become more complicated.

However, matrix algebra can simply capture this concern on "externality" in the following way. The relevant columns and rows were interchanged between c_B and c_N, x_B and x_N, and A_N and B. These interchange set x_N to be zero vector again. Any increase in x_N will result in change in x_B from constraint (29). In detail, the change in x_N leads to change in $A_N x_N$. Accordingly $B x_B$ changes in the opposite direction as (29) is an equality constraint. Accordingly, the change in x_B is exactly measured by $B^{-1}(b - A_N x_N)$, and its effect on the objective function is measured by $c_B B^{-1}(b - A_N x_N)$. Note that B is and will be invertible in the ongoing steps (See Appendix A.6). Simply,

$$c_N \cdot x_N + c_B \cdot x_B = c_N \cdot x_N + c_B \cdot (B^{-1} b - B^{-1} A_N x_N) = c_B B^{-1} b + (c_N x_N - c_B B^{-1} A_N x_N).$$

The second to the last term, $c_N x_N$, measures the direct effect of increasing x_N. The last term $c_B B^{-1} A_N x_N$ captures the externality that we have concern about. In Step 1, $c_B = \mathbf{0}$. Thus the externality is zero. In Step 2, c_B is already a non-zero vector as there was an interchange of columns in c_B and c_N.

Increase $\gamma_i(\theta_i^l, \theta_i^m)$ when the (θ_i^l, θ_i^m)-th column of $(c_N - c_B B^{-1} A_N)$ is the largest positive number, i.e., when the direct benefit net of externality is the largest. If there is no positive column in $(c_N - c_B B^{-1} A_N)$, we end our procedure. When there is such positive column, we cannot increase $\gamma_i(\theta_i^l, \theta_i^m)$ infinitely. We can increase it only without breaking $x_B = B^{-1}(b - A_N x_N) \geq 0$, i.e., until a certain row in x_B becomes zero. Once a certain row of x_B (say, θ^2-th element) becomes zero, we exchange θ^2-th row of x_B and (θ_i^l, θ_i^m)-th row of x_N, θ^2-th column of c_B and (θ_i^l, θ_i^m)-th column of c_N, and θ^2-th column of B and (θ_i^l, θ_i^m)-th column of A_N, respectively. Note that the value of $\gamma_i(\theta_i^j, \theta_i^k)$ has also changed by increasing $\gamma_i(\theta_i^l, \theta_i^m)$. The changed value is the θ^1-th row of

$$B^{-1}(b - A_N \bar{x}_N)$$

where $\bar{x}_N$ is a zero vector with the exception of (θ_i^l, θ_i^m)-th row being the maximum of $\gamma_i(\theta_i^l, \theta_i^m)$.
(Step *n*) We repeat the same procedure given by Step 2 until $(c_N - c_B B^{-1} A_N)$ becomes a non-positive vector.

To summarize, a formal description of the algorithm is the following.

(Step 1) Start with $B = I$ and $c_B = \mathbf{0}$. Let $x_N = \mathbf{0}$ and $x_B = b$.
(Step 2) Choose (θ_i^j, θ_i^k) such that the (θ_i^j, θ_i^k)-th column of $(c_N - c_B B^{-1} A_N)$ is the maximal. Increase $\gamma_i(\theta_i^j, \theta_i^k)$ until one row in x_B (say, θ-th row) becomes zero where $A_N x_N + B x_B = b$.
(Step 3) Interchange the (θ_i^j, θ_i^k)-th column of c_N and the θ-th column of c_B. Accordingly, exchange the (θ_i^j, θ_i^k)-th row of x_N and θ-th row of x_B. Also exchange the (θ_i^j, θ_i^k)-th column of A_N and the θ-th column of B. Repeat Step 2.

Formally, an optimality condition is:

Proposition 9 *If $(c_N - c_B B^{-1} A_N) \leq \mathbf{0}$ after a certain number of steps, the basic feasible solution represented by $\left({x_B}^T, {x_N}^T\right) = \left((B^{-1}b)^T, 0\right)$ is optimal for LP_D.*

Proof. The algorithm is simplex method of linear programming. The proposition follows from the simplex method (see any LP TEXTBOOK). ∎

The algorithm finds an optimal solution in finite steps under the following condition.

Proposition 10 *If the basic feasible solution $B^{-1}b$ is a strictly positive vector in each iteration, the algorithm finishes in finite steps to find the optimal value with an optimal $\gamma_i(\cdot,\cdot)$.*

Proof. The algorithm is simplex method of linear programming. The proposition follows from the simplex method (see any LP TEXTBOOK). ∎

However, the basic feasible solution $B^{-1}b$ is often not strictly positive in our context. Thus the above algorithm may fail to finish in finite steps, and cycle (See Beale [1955] and Marshall and Suurballe [1969]). There are also a few additional tests preventing such possibility of a cycle, and the tests can be incorporated into the algorithm. We do not discuss the tests here as we do not see meaningful economic intuition behind them. Interested readers can refer to Dantzig, Ordern and Wolfe (1955), Bland (1977) and Hall and McKinnon (1996).

7 Conclusion

[To be added]

References

[1] Groves, T. (1973), "Incentive in Teams", Econometrica, 41: 617-631

[2] Legros, P. and H. Matsushima (1991), "Efficiency in Partnerships", Journal of Economic Theory, 55: 296-322

A Appendix

A.1 Proof for Lemma 3

Suppose not. Then, $\sum_{\theta_i'}\gamma_i(\theta_i,\theta_i')=f(\theta_i)$ where $f(\cdot)$ is not a constant function. Let $\alpha_i=\max_{\theta_i}f(\theta_i)$. Note that $\gamma_i(\theta_i,\theta_i)$ could be an arbitrary non-negative number since the incentive compatibility constraint associated with $\gamma_i(\theta_i,\theta_i)$ is a trivial constraint. Re-define $\gamma_i(\theta_i,\theta_i)$ as $\tilde{\gamma}_i(\theta_i,\theta_i)=\gamma_i(\theta_i,\theta_i)+(\alpha_i-f(\theta_i))$. Then we derive the condition $\sum_{\theta_i'}\gamma_i(\theta_i,\theta_i')=\alpha_i$ for all θ_i. Simply by defining $s_i(\theta_i,\theta_i'):=\gamma_i(\theta_i,\theta_i')/\alpha_i$, we prove the Lemma.

A.2 Proof for Proposition 1

First of all, $V_i \geq V_i(s_i)$. The result is trivial as there are more constraints for LP_P than for $LP_P(s_i)$.

We take $s_i(\theta_i, \theta_i')$ and α_i as in Lemma 3, i.e., $s_i(\theta_i, \theta_i') := \gamma_i(\theta_i, \theta_i')/\alpha_i$ and $\sum_{\theta_i'} s_i(\theta_i, \theta_i') = 1$. Then,

$$V_i = \max_{s_i(\theta_i,\theta_i')\geq 0: \sum_{\theta_i} s_i(\theta_i,\theta_i')=1} g_i(s_i)\alpha_i \quad \text{s.t. } \alpha_i[p(\theta) - q(\theta|s_i)] \leq p(\theta).$$

Since the domain for the above maximization problem is compact, there is $s_i \in S_i$ that maximizes the above program. For such s_i, we get

$$V_i \leq g_i(s_i)\frac{p(\theta)}{p(\theta) - q(\theta|s_i)} \quad \text{for all } \theta \text{ such that } q(\theta|s_i) < p(\theta).$$

If there is no such θ, $V_i = 0$ trivially. Take $\bar{\theta}$ such that $\bar{\theta} = \underset{\theta}{\operatorname{argmin}}\{\frac{q(\theta|s_i)}{p(\theta)}\}$. Then $V_i \leq g_i(s_i)\frac{p(\bar{\theta})}{p(\bar{\theta})-q(\bar{\theta}|s_i)} = V_i(s_i)$.

Thus we have shown $V_i = \max_{s_i} V(s_i)$ since we already know $V_i \geq V_i(s_i)$.

A.3 Proof for Lemma 4

Firstly, suppose that $\frac{q(\cdot|s_i)}{p(\cdot)}$ is minimized at $(\theta_i^2, \theta_{-i}')$ for some θ_{-i}', but not at $(\theta_i^1, \theta_{-i})$ for any θ_{-i}. A marginal change in $s_i(\theta_i^1, \cdot)$ does not affect $\frac{1}{1-\min \frac{q(\cdot|s_i)}{p(\cdot)}}$, but changes only $g_i(s_i)$. Note that $s_i(\theta_i^1, \theta_i^2) < 1$ cannot be the case as a marginal increase of $s_i(\theta_i^1, \theta_i^2)$ will increase $g_i(s_i)$ without changing $\frac{1}{1-\min \frac{q(\cdot|s_i)}{p(\cdot)}}$. Thus $s_i(\theta_i^1, \theta_i^2) = 1$. Also, $s_i(\theta_i^2, \theta_i^1) > 0$ since

$$\frac{s_i(\theta_i^2,\theta_i^1)p(\theta_i^2,\theta_{-i})}{p(\theta_i^1,\theta_{-i})} > \min_{\theta_{-i}} \frac{q(\theta_i^2,\theta_{-i}|s_i)}{p(\theta_i^2,\theta_{-i})} = \min_{\theta_{-i}} \frac{p(\theta_i^1,\theta_{-i}) + (1-s_i(\theta_i^2,\theta_i^1))p(\theta_i^2,\theta_{-i})}{p(\theta_i^2,\theta_{-i})}$$

$$\Rightarrow s_i(\theta_i^2,\theta_i^1)\min_{\theta_{-i}} \frac{p(\theta_i^2,\theta_{-i})}{p(\theta_i^1,\theta_{-i})} > \min_{\theta_{-i}} \frac{p(\theta_i^1,\theta_{-i})}{p(\theta_i^2,\theta_{-i})} + 1 - s_i(\theta_i^2,\theta_i^1) \Rightarrow s_i(\theta_i^2,\theta_i^1) > \frac{1+\min_{\theta_{-i}} \frac{p(\theta_i^1,\theta_{-i})}{p(\theta_i^2,\theta_{-i})}}{1+\min_{\theta_{-i}} \frac{p(\theta_i^2,\theta_{-i})}{p(\theta_i^1,\theta_{-i})}}.$$

Then $V_i(s_i)$ is

$$\begin{aligned} V_i(s_i) &= \frac{1}{1-\min_{\theta_{-i}} \frac{p(\theta_i^2,\theta_{-i})s_i(\theta_i^2,\theta_i^2)+p(\theta_i^1,\theta_{-i})}{p(\theta_i^2,\theta_{-i})}} \begin{bmatrix} & \Delta u_i(\theta_i^1,\theta_i^2)p_i(\theta_i^1) \\ + & s_i(\theta_i^2,\theta_i^1)\Delta u_i(\theta_i^2,\theta_i^1)p_i(\theta_i^2) \end{bmatrix} \\ &= \frac{1}{s_i(\theta_i^2,\theta_i^1) - \min_{\theta_{-i}} \frac{p(\theta_i^1,\theta_{-i})}{p(\theta_i^2,\theta_{-i})}} \Big[\Delta u_i(\theta_i^1,\theta_i^2)p_i(\theta_i) + s_i(\theta_i^2,\theta_i^1)\Delta u_i(\theta_i^2,\theta_i^1)p_i(\theta_i)\Big]. \end{aligned} \tag{30}$$

Since $V_i(s_i)$ is maximal, infinitesimal decrease in $s_i(\theta_i^2, \theta_i^1)$ (weakly) decreases $V_i(s_i)$, i.e.,

$$\frac{1}{\Big[s_i(\theta_i^2,\theta_i^1) - \min \frac{p(\theta_i^1,\cdot)}{p(\theta_i^2,\cdot)}\Big]^2} \begin{bmatrix} \Delta u_i(\theta_i^1,\theta_i^2)p_i(\theta_i) \\ +s_i(\theta_i^2,\theta_i^1)\Delta u_i(\theta_i^2,\theta_i^1)p_i(\theta_i) \end{bmatrix} - \frac{1}{s_i(\theta_i^2,\theta_i^1) - \min \frac{p(\theta_i^1,\cdot)}{p(\theta_i^2,\cdot)}} \Delta u_i(\theta_i^2,\theta_i^1)p_i(\theta_i^2) \leq 0$$

$$\Leftrightarrow V_i \leq \Delta u_i(\theta_i^2,\theta_i^1)p_i(\theta_i^2).$$

This means that $\Delta u_i(\theta_i^2, \theta_i^1)p_i(\theta_i^2) \geq 0$. Thus, from equality (30), we can clearly see $V_i(s_i) > \Delta u_i(\theta_i^1, \theta_i^2)p_i(\theta_i^1)$. This is a contradiction to $\Delta u_i(\theta_i^1, \theta_i^2)p_i(\theta_i^1) \geq \Delta u_i(\theta_i^2, \theta_i^1)p_i(\theta_i^2)$.

Secondly, suppose $\frac{q(\cdot|s_i)}{p(\cdot)}$ is minimized at $(\theta_i^1, \theta_{-i})$ for some θ_{-i}, but not at $(\theta_i^2, \theta'_{-i})$ for any θ'_{-i}. Also assume $\Delta u_i(\theta_i^2, \theta_i^1)p(\theta_i^2) > 0$. Then $s_i(\theta_i^2, \theta_i^1) = 1$ by a similar argument as in the above. Also, $s_i(\theta_i^1, \theta_i^2) > 0$ since

$$\min_{\theta_{-i}} \frac{p(\theta_i^1, \theta_{-i})(1 - s_i(\theta_i^1, \theta_i^2)) + p(\theta_i^2, \theta_{-i})}{p(\theta_i^1, \theta_{-i})} < \min_{\theta_{-i}} \frac{p(\theta_i^1, \theta_{-i})s_i(\theta_i^1, \theta_i^2)}{p(\theta_i^2, \theta_{-i})} \Rightarrow s_i(\theta_i^1, \theta_i^2) > \frac{1 + \min \frac{p(\theta_i^2, \cdot)}{p(\theta_i^1, \cdot)}}{1 + \min \frac{p(\theta_i^1, \cdot)}{p(\theta_i^2, \cdot)}}.$$

Similarly, we can calculate V_i as:

$$V_i = \frac{1}{s_i(\theta_i^1, \theta_i^2) - \min_{\theta_{-i}} \frac{p(\theta_i^2, \theta_{-i})}{p(\theta_i^1, \theta_{-i})}} \Big[s_i(\theta_i^1, \theta_i^2)\Delta u(\theta_i^1, \theta_i^2)p(\theta_i^1) + \Delta u(\theta_i^2, \theta_i^1)p(\theta_i^2) \Big].$$

Again, the decrease in $s_i(\theta_i^1, \theta_i^2)$ should not increase the value V_i; thus,

$$\frac{1}{\Big(s_i(\theta_i^1, \theta_i^2) - \min \frac{p(\theta_i^2, \cdot)}{p(\theta_i^1, \cdot)}\Big)} V_i - \frac{1}{s_i(\theta_i^1, \theta_i^2) - \min \frac{p(\theta_i^2, \cdot)}{p(\theta_i^1, \cdot)}} \Delta u_i(\theta_i^1, \theta_i^2)p(\theta_i^1) \leq 0 \Leftrightarrow V_i \leq \Delta u_i(\theta_i^1, \theta_i^2)p(\theta_i^1).$$

It is a contradiction since $V_i \geq \Delta u_i(\theta_i^1, \theta_i^2)p_i(\theta_i^1) + \Delta u_i(\theta_i^2, \theta_i^1)p_i(\theta_i^2)$ if $\gamma_i(\theta_i^1, \theta_i^2) = \gamma_i(\theta_i^2, \theta_i^1) = 1$.

Finally, suppose $\frac{q(\cdot|s_i)}{p(\cdot)}$ is minimized at θ_i^1, but not at θ_i^2, and $\Delta u_i(\theta_i^2, \theta_i^1)p(\theta_i^2) \leq 0$. Then $s_i(\theta_i^2, \theta_i^2) = 0$ by a similar reason above. However, this case falls in the second category.

A.4 Proof for Proposition 2

Since the domain is a convex cone, if the minimum is achieved at $(\rho_1 = 0, \rho_3 = 0, \gamma = 0)$, the value of the objective function will (weakly) decrease by moving from $(\rho_1 = 0, \rho_3 = 0, \gamma = 0)$ to some other point in the convex cone. For $\alpha \in [0, 1]$, consider the following point:

$$\Big(\rho_1 = \epsilon\alpha, \rho_3 = \epsilon(1 - \alpha), \gamma = -\epsilon \min_{\tilde{k}} \Big[\alpha \frac{p_1^k}{p_2^k} + (1 - \alpha)\frac{p_3^k}{p_2^k}\Big]\Big).$$

Point $(\rho_1 = \epsilon\alpha, \rho_3 = \epsilon(1 - \alpha))$ is away from $(0, 0)$ in the direction of $(\alpha, 1 - \alpha)$, and γ was minimally changed so that all the constraints in (27) are still satisfied, and at least one constraint is binding.[5] For $k \in \text{argmin} \Big[\alpha \frac{p_1^k}{p_2^k} + (1 - \alpha)\frac{p_3^k}{p_2^k}\Big]$, constraint k is still binding after this change.

From this change, the value of the objective function (see (27)) increases by

$$\epsilon \Big[-\alpha\Delta_{12} - (1 - \alpha)\Delta_{32} - \Delta_{23} \min_k \Big[\alpha \frac{p_1^k}{p_2^k} + (1 - \alpha)\frac{p_3^k}{p_2^k}\Big]\Big].$$

This change of the value is non-positive if and only if $(\rho_1 = 0, \rho_3 = 0, \gamma = 0)$ is the minimum of the dual linear program. Thus, the condition for the existence of $\mathcal{R}_2 \geq 0$ is:

$$\alpha\Delta_{12} + (1 - \alpha)\Delta_{32} + \Delta_{23} \min_k \Big[\alpha \frac{p_1^k}{p_2^k} + (1 - \alpha)\frac{p_3^k}{p_2^k}\Big] \leq 0, \ \forall \alpha \in [0, 1].$$

(Notice that a local minimum of the dual linear program is the global minimum.)

[5]Note $\rho_1 p_1^k + \rho_3 p_3^k + \gamma p_2^k \geq 0 \Leftrightarrow \gamma \geq -\big[\rho_1 p_1^k/p_2^k + \rho_3 p_3^k/p_2^k\big]$.

A.5 Derivation of information rent for case 9

Binding incentive compatibility constraints were enough to characterize information rent so far. However, concern on minimization is required in this case. Also, unlike Example 1, there could be more than one state where type θ_i^2 receives positive rent as there are two types trying to mimic type θ_i^2. Let $R_2(k)$ denotes the information rent given to θ_i^2 when $\theta_{-i} = \theta_{-i}^k$. The following minimization problem characterizes information rent.

$$\min_{R_1, R_2(k), R_3} \; p_1 R_1 + p_2 \sum_k R_2(k) p_2^k + p_3 R_3$$

$$\text{s.t.} \; \sum_k R_2(k) p_2^k = \Delta_{21} + R_1 \min_k \frac{p_2^k}{p_1^k}, \; R_1 = \Delta_{12} + \sum_k R_2(k) p_1^k, \; R_3 = \Delta_{32} + \sum_k R_2(k) p_3^k.$$

Plugging the second and the third constraints into the first and the objective function, we get:

$$\min_{R_2(k)} p_1\Delta_{12} + p_3\Delta_{32} + \sum_k R_2(k)(p^{1k} + p^{2k} + p^{3k}) \text{ s.t. } \sum_k R_2(k)\Big[p_2^k - \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}} p_1^k\Big] = \Delta_{21} + \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}} \Delta_{12}.$$

(Note $p^{1k} := p_1 \times p_1^k$.) The minimum is achieved when

$$R_2(k) = \Big(\Delta_{21} + \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}} \Delta_{12}\Big) \Big/ \Big(p_2^k - \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}} p_1^k\Big) \;\text{ if }\; k = \text{argmin} \frac{p^{1k} + p^{2k} + p^{3k}}{p_2^k - \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}} p_1^k}, \;\; 0 \;\text{ otherwise.}$$

Thus, for $k = \text{argmin}(p^{1k} + p^{2k} + p^{3k})/(p_2^k - \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}} p_1^k)$,

$$R_1 = \Delta_{12} + p_1^k \frac{\Delta_{21} + \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}} \Delta_{12}}{p_2^k - \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}} p_1^k}, \; R_2 = \frac{\Delta_{21} + \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}} \Delta_{12}}{1 - \frac{p_1^k}{p_2^k} \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}}}, \; R_3 = \Delta_{32} + p_3^k \frac{\Delta_{21} + \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}} \Delta_{12}}{p_2^k - \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}} p_1^k}.$$

A.6 Proof that B is invertible

Let $A_N^{(\theta_i, \theta_i')}$ be the (θ_i, θ_i')-th column of matrix A_N, B^θ be the θ-th column of matrix B, and $x_{B\theta}$ be the θ-th row of x_B. Since $x_N = \mathbf{0}$, (29) becomes

$$\sum_{\tilde{\theta}} B^{\tilde{\theta}} x_{B\tilde{\theta}} = b.$$

After increasing $\gamma_i(\theta_i^j, \theta_i^k)$ up to the maximum, we have

$$A_N^{(\theta_i^j, \theta_i^k)} \gamma_i(\theta_i^j, \theta_i^k) + \sum_{\tilde{\theta} \neq \theta} B^{\tilde{\theta}} \xi_{B\tilde{\theta}} = b$$

where ξ_B is the value changed from x_B by the change in $\gamma_i(\theta_i^j, \theta_i^k)$, and θ is such that the θ-th row of x_B hits zero the earliest.

From the two equations, we derive

$$A_N^{(\theta_i^j, \theta_i^k)} x_{N(\theta_i^j, \theta_i^k)} + \sum_{\tilde{\theta} \neq \theta} B^{\tilde{\theta}} (\xi_{B\tilde{\theta}} - x_{B\tilde{\theta}}) - B^\theta x_{B\theta} = 0.$$

If $\gamma_i(\theta_i^j, \theta_i^k) > 0$, $x_{B\theta} > 0$ (otherwise, $x_{B\theta}$ cannot be the row of x_B that becomes the zero the first). Thus, $A_N^{(\theta_i^j, \theta_i^k)}$ is spanned by $\{B^{\tilde{\theta}} : \tilde{\theta} \in \Theta\}$, and the coefficient for B^θ is non-zero. As long as $\{B^{\tilde{\theta}} : \tilde{\theta} \in \Theta\}$ is a basis, $\{B^{\tilde{\theta}} : \tilde{\theta} \neq \theta\} \cup \{A_N^{(\theta_i^j, \theta_i^k)}\}$ is a basis too. Therefore, the replacement of B^θ with $A_N{}^{(\theta_i^j, \theta_i^k)}$ makes matrix B invertible as long as B was invertible before the replacement. On the other hand, if $x_{N_{(\theta_i^j, \theta_i^k)}} = 0$, consider the situation of making $x_{N_{(\theta_i^j, \theta_i^k)}} = \epsilon > 0$. Then $\xi_{B\theta}$ will become a strictly negative number. Thus we can argue the same way to show that $A_N^{(\theta_i^j, \theta_i^k)}$ is spanned by $(B^{\tilde{\theta}})_{\tilde{\theta} \in \Theta}$, and the coefficient for B^θ is non-zero. Thus B is invertible after the replacement by the same reason.

Since the algorithm starts with $B = I$, B remains invertible along the ongoing steps.