Why Does New Hampshire Matter — Simultaneous v.s. Sequential Election with Multiple Candidates

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March, 2010

Abstract

I study and compare preference aggregation in a simultaneous and a sequential multicandidate election. Voters have perfect information about their own preference but do not know the median voter's preference. A voter has an incentive to vote for her second choice for fear that a tie between her second and third choice is more likely than she would like. Therefore, a voter may want to coordinate with supports of her second choice. I show that when voters' preference intensity for their first choice is moderate, in the limit as the electorate increases, there is a unique equilibrium in the voting game within one voting round exhibiting multi-candidate support. In such an equilibrium, the ex ante probability that a candidate wins increases in her supporters' preference intensity and decreases in her opponents' preference intensity. There is too much coordination with supporters of a voter's second choice in that sometimes the median voter's second choice wins the election. A sequential election allows later voters to coordinate with earlier voters. Therefore, in the last voting round, votes are split between the two front runners. The voting outcome in the first round affects the voting behavior of the second round. A victory of a voter's favorite candidate in the first round may change the outcome of the second round from the voter's second choice to her favorite candidate or from her last choice to her second choice. When preference intensity is moderate, voters vote more for their first choice if they vote first in a sequential election than in a simultaneous election, and the probability that the median voter's first choice does not win a voting round is smaller if voting takes place sequentially.

1 Introduction

The outcomes of early elections play an out-of-proportion role in the US Presidential primary. Adam (1987) reports that the 1984 New Hampshire primary got nearly 20% of the season's coverage in ABC,CBS, NBC and the New York Times, even though New Hampshire accounts for only 0.4% of the US population, and only four votes out of 538 electoral votes in the presidential election.

In the 1980 Republican primaries, George Bush and Ronald Reagan spent about 3/4 of their respective campaign budgets in early primary states, which account for much less than a fifth of the votes in the Republican convention in 1980 (Malbin, 1985). The emphasis on winning early primaries may come from the widely-held belief that early winners gain "momentum" due to the sequential nature of the election.

However, recent primaries have become more "front-loaded" into the early weeks. California has recently passed a legislation to move forward its primary to Feb. 5, 2008, only after 4 other primaries held in January. The media in general views this as "selfish" behavior on the part of those states. It has been argued that a more front-loaded primary system makes it more important for candidates to raise a lot of money early (William Schneider, 1997) and a more front-loaded 2008 primary gives well-established candidates an advantage. On what ground do these assertions stand? And if they are true, through what channel does the timing structure affect the voting outcomes?

Existing literature that study sequential elections has for the most part restricted attention to contests between two candidates. However, there are usually many candidates in a presidential primary. For example, Sen. Hillary Rodham Clinton of New York, Sen. Barack Obama of Illinois and former senator John Edwards of North Carolina, are all considered front runners in the 2008 primary for the Democratic party. With only two candidates, voters simply vote for their preferred candidate. In a multi-candidate contest, however, some voters have to vote strategically for their second choice if they believe their most preferred candidate has a smaller probability of being in a close race. Therefore, voters' beliefs about relative popularity of every candidate, and the relative likelihood of different pivotal events, play an important role in their decision.

Given this element of coordination in multicandidate contest under plurality rule, it is not surprising that with common knowledge assumption of the electoral situation, the voting outcome involves either a complete success or failure of coordination. Duverger's Law (see Riker 1982) asserts that "plurality rule brings about and maintains two-party competition", because only two candidates should be expected to get any vote. This represents complete success of coordination. Most of the literature focuses on these "Duvergerian" equilibria, but offers no formal theory as to which two candidates should be considered "serious" contenders. In addition, it cannot explain the incomplete coordination observed in many multicandidate election outcomes. For example, in the 1970 New York senatorial election, even the trailing candidate among the three got more than 24% of the votes, and the winner gets only 2% more votes than the second.

Moreover, common knowledge of the electoral situation seems a very strong assumption. The 1997 British Election Survey indicates that about two-third of voters who expected their preferred party to come second actually found that it came third (Fisher, 2000). There was clearly lack of common knowledge among voters as to the identities of the first and second place winner, which is inconsistent with that literature.

This paper presents a model of preference aggregation in a multi-candidate election that features a candidate who is "a common second choice" for supporters of the other two extreme candidates. Voters in the model only have imperfect information about the distribution of preferences in the electorate. Supporters of an extreme candidate have an incentive to coordinate with supporters of the "common second choice" against their least favorite candidate. Relaxing common knowledge assumption enables meaningful analysis of this coordination effect. I show that this coordination incentive among supporters of an extreme candidate is stronger when preference intensity for that candidate is smaller, when preference intensity for the opposing extreme candidate is higher, or when the prior belief of the share of supporters of the extreme candidate is smaller. In addition, in those situations, there is excess coordination in that the "common second choice" wins too often, i.e. sometimes "the common second choice" wins even though the median voter favors one of the extreme candidates. One interpretation of "the common second choice" is a candidate that's widely known and considered a "safe option".

I then study an election that involves voting in three states (electorates) in which the candidate winning the most states wins the election. This is close to a Republican primary system. I compare voting behavior and outcomes under simultaneous and sequential election. When preference intensity is not too big, in the last state, supporters of the extreme candidate that has not governed any victory always vote for the "common second choice". Thus the equilibrium exhibit winnowing down of front runners. In addition, a victory by one extreme candidate in the first state boosts the morale of her supporters in the second state and results in more aggressive voting behavior by her supporters and higher chance of winning in the second state. I show that when preference intensity is moderate, a sequential election reduces excess coordination motive in the first state as compared to the outcome under simultaneous election and reduces the ex ante probability that the candidate winning that state is not the median voter's first choice.

2 Literature Review

Dekel and Piccione (2000) and Ali and Kartik (2006) both study sequential elections between two candidates in which some voters have only imperfect information about their own preference over the two candidates. Dekel and Piccione (2000) show that any outcome of a voting equilibrium in a simultaneous election is also an equilibrium outcome of a sequential election with any timing structure. Ali and Kartik (2006), on the other hand, construct a Perfect Bayesian equilibrium in which "herding," i.e. voting according to the history of vote counts so far and disregarding one's own information, happens with positive probability. This suggests that in a race between two candidates, a simultaneous election can be (but is not necessarily due to multiplicity of equilibria) more efficient in gathering information than a sequential election.

Myerson and Weber (1993) and Myerson (2002) both assume common knowledge of the preference distribution of the electorate, and show that under plurality rule, for any pair of candidates in a "three-horse race", there exists an equilibrium in which only this pair are considered "serious" and get any vote. Myerson (2002) call these discriminary equilibria because labeling of the candidates matter as to whether they have positive probability of winning. They argue that "a large multiplicity of equilibria creates a wider scope for focal manipulation by political leaders."

Myerson and Weber (1993) also show via an example the existence of a "non-Duvergerian" equilibria in which a group of voters fail completely to coordinate to avoid the worst outcome, and the two losers exactly tie. They conjecture that some additional assumption of dynamic stability or persistence may be used to eliminate these "non-Duvergerian" equilibria.

This paper is most closely related to Myatt (2007), which studies simultaneous elections under plurality rule in which one candidate (the conservative status quo) has a commonly known fixed fraction ($< \frac{1}{2}$) of supporters, while the rest of the electorate share the distaste of the status quo but disagree on which of the other two (liberal) candidates is optimal. This assumption effectively reduces an election under plurality rule with three candidates to one under qualified-majority rule between two candidates. Essentially, the (liberal) voters have to coordinate behind the two (liberal) candidates. They relax the common knowledge assumption by assuming that each voter gets an imperfect signal about the preference distribution of the electorate (as evident in the UK General Election of 1997). They construct a uniqe symmetric equilibrium that is consistent with the 1970 New York Senatorial election, which displays limited strategic voting and incomplete coordination. However, the assumption of a fixed and commonly known support for one candidate does not seem to fit US Presidential primaries.

It is difficult to characterize equilibria in a large election because probability ratios of close-race events between different pairs of candidates can be quite intractable. Myatt (2007) develops the solution concept of *strategic-voting equilibrium* for large elections, which can be viewed as a Bayesian Nash equilibrium with a continuum of voters. It facilitates the calculation through law of large numbers arguments. Myerson (2000), on the other hand, tackels this issue by assuming population uncertainty. They assume that voter turnout follows a Poisson process with a commonly known preference distribution. The feature of Poisson process that an individual voter's belief about the behavior of the electorate does not depend on his own preference type facilitates comparison of limiting probabilities of different pivotal events as the size of the electorate goes to infinity.

On relaxing common knowledge assumptions in voting situations, Feddersen and Pesendorfer (1997, 1998) use a common value model for jury decision making. In their model, each juror decides on one of two votes based on a private signal about the defendant's guilt and aims to convict the guilty and acquit the innocent. Thus other jurors' information matters even for a juror's own preference over outcomes. Each juror infers about the merits of his two actions from an assessment of the information possessed by others conditional on his vote being pivotal. Therefore, if other jurors respond a lot to their signals, a juror may have an incentive to disregard his own signal because the information contained in the pivotal event outweighs his own information. This is why bandwagon effects may arise in sequential elections with two candidates in Ali and Kartik (2006). However, since there are only two outcomes, the coordination effect in multicandidate contests is not present in these models.

3 A Multicandidate Contest in One State

3.1 The Model

Three candidates L, M, R compete in a simultaneous election. There are n voters in the electorate where n follows a Poisson distribution with mean N. Each voter has to voter for exactly one candidate. A voter can be of three preference types: a right wing voter, r, prefers candidate R to M to L, a left-wing voter, l, prefers candidate L to M to R, while a moderate voter m prefers candidate M the most and is indifferent between R and L. A voter of preference type i receives payoff U_{ij} when candidate $j \in \{L, M, R\}$ wins the election. Write $\phi_r = \frac{U_{rR} - U_{rM}}{U_{rM} - U_{rL}}$. It represents the preference intensity of a right wing voter for her favorite candidate. Define $u_r = \log(2\phi_r)$. ϕ_l and u_l are defined analogously.

A voter in the electorate is right-wing with probability $F(\eta - \theta)$, left-wing with probability $F(-\eta - \theta)$ and moderate with probability $1 - F(\eta - \theta) - F(-\eta - \theta)$, where F is the cumulative distribution function for Laplace distribution with mean 0 and variance 2. θ is an exogenously given parameter of the model and in a way measures the size of the moderate population. Under this specification, the median voter is moderate if $\eta \in (-\theta, \theta)$, right-wing if $\eta > \theta$ and left-wing if $\eta < -\theta$.

A voter does not know the ideology of the median voter in her electorate. That is, a voter in the electorate does not know η. She believes that $\eta \sim \mathit{Laplace}(0, \alpha)$. Let $G(.)$ and $g(.)$ denote the cumulative distribution function and the probability density function of the prior. In addition to the common prior about η, voter i gets some additional information about the preference of the electorate. She obtains a signal $\hat{\eta}_i \in \mathit{Laplace}(\eta, 1)$ independent of her preference type. Based on her information and the prior, she then forms an updated belief about η. Denote by $f(.|\hat{\eta}_i)$ the probability density function of voter i's posterior given her signal $\hat{\eta}_i$.

3.2 Equilibria

3.2.1 Strategies and best responses

A voter's type is her ideology-information pair $(o_i, \hat{\eta}_i)$ where $o_i \in \{l, m, r\}$ and $\hat{\eta}_i \in \mathbb{R}$. A pure strategy for voter i is then a mapping from her type to the set of candidates $\{L, M, R\}$. A sincere voting strategy simply chooses the candidate that's most preferred according to voter i's ideology.

There are many equilibria in this game. For example, if every voter votes for candidate j, then a voter is never pivotal and thus she is indifferent between all candidates. Given any two candidates c_1, c_2, there is an equilibrium in which every voter votes for the one in $\{c_1, c_2\}$ that she prefers. In such an equilibrium, the election is reduced to a binary voting game. One can say that the two candidates c_1 and c_2 are the front-runners and the focal point of the election. However, the model cannot answer the question of how front runners are chosen.

For these reasons, we focus on Bayesian Nash equilibria in type-dependent strategies. In particular, we focus on equilibria in symmetric pure voting strategies where the same type-dependent voting strategy $s(o_i, \hat{\eta}_i)$ is used by every voter.

Consider a voter's payoff given that voting strategy s is adopted by all the other voters. Let x_j denote the number of votes candidate j gets from everyone other than voter 0. Then (x_R, x_M, x_L) is a vector of random variables whose distribution depend on the voting strategy v adopted by everyone else. If voter 0 is moderate, then it is her best response to vote for M regardless of her information because she is indifferent between R and L. It is a strict best response as long as $\Pr\{x_R = x_M \cup x_M = x_L|\hat{\eta}_i\} > 0$. If voter 0 is right-wing, then her best response is to vote for R if

$$\begin{aligned} & \left(\Pr\{x_R = x_M|\hat{\eta}_i\} + \frac{1}{2}\Pr\{|x_R - x_M| = 1|\hat{\eta}_i\} + \frac{1}{2}\Pr\{x_R = x_L|\hat{\eta}_i\}\right)(U_{rR} - U_{rM}) \\ \geq \; & \frac{1}{2}\left(\Pr\{x_M = x_L|\hat{\eta}_i\} + \Pr\{x_R = x_L|\hat{\eta}_i\}\right)(U_{rM} - U_{rL}), \end{aligned}$$

and to vote for M otherwise. A left-wing voter's strategy is analogous. Therefore, candidate R gets votes only from right-wing voters.

Denote by $p_j(\eta|v)$ the probability that a voter votes for candidate j conditional on η given that voting strategy v is adopted. Then

$$p_R(\eta|v) = F(\eta - \theta)\Pr\{\hat{\eta}_i : v(r, \hat{\eta}_i) = R|\eta\}.$$

3.2.2 Voting in Large Electorates

We assume that the turn-out, n, follows a Poisson distribution with mean N. Denote by $s_N(o_i, \hat{\eta}_i)$ an equilibrium voting strategy in such an electorate. We focus on the limit of the equilibrium voting strategy $s_N(o_i, \hat{\eta}_i)$ as $N \to \infty$.

Lemma 3.1 *If everyone else in the electorate adopts a voting strategy such that the probability that a voter votes for candidate c is equal to $p_c(\eta)$ when the state variable is η, and voter turn-out follows a Poisson process with mean N, then for any $d \in \{-1, 0, 1\}$,*

$$\begin{aligned} & \lim_{N\to\infty} \frac{\Pr\{|x_R - x_M| = d \text{ and } \min\{x_R, x_M\} > x_L|\hat{\eta}_i, p\}}{\Pr\{|x_L - x_M| = d \text{ and } \min\{x_M, x_L\} > x_R|\hat{\eta}_i, p\}} \\ = \; & \frac{f(\eta_R|\hat{\eta}_i)}{f(\eta_L|\hat{\eta}_i)} \frac{|p'_L(\eta_L) - p'_M(\eta_L)|}{|p'_R(\eta_R) - p'_M(\eta_R)|}, \end{aligned}$$

where η_R is the solution to $p_R(\eta) = p_M(\eta)$ and η_L is the solution to $p_L(\eta) = p_M(\eta)$. In addition, if $p_R(\hat{\eta}) < p_M(\hat{\eta})$ for all solution $\hat{\eta}$ to $p_R(\eta) = p_L(\eta)$, then

$$\lim_{N\to\infty} \frac{\Pr\{|x_R - x_L| = d \text{ and } \min\{x_R, x_L\} > x_M|\hat{\eta}_i, p\}}{\Pr\{|x_j - x_M| = d \text{ and } \min\{x_M, x_j\} > x_{-k}|\hat{\eta}_i, p\}} = 0$$

where $j, k \in \{R, L\}$ and $j \neq k$.

3.2.3 Equilibria Characterization

It follows that a right-wing voter votes for R if and only if

$$\log \frac{f(\eta_R|\hat{\eta}_i)}{f(\eta_L|\hat{\eta}_i)} \geq -\log 2\frac{U_{rR} - U_{rM}}{U_{rM} - U_{rL}} - \log \frac{|p'_R(\eta_R^*) - p'_M(\eta_R^*)|}{|p'_L(\eta_L^*) - p'_M(\eta_L^*)|}$$

where $f(.|\hat{\eta}_i)$ is a voter's posterior about η given her signal $\hat{\eta}_i$. Using Bayes update, we have

$$\log \frac{f(\eta_R|\hat{\eta}_i)}{f(\eta_L|\hat{\eta}_i)} = \begin{cases} (\eta_R - \eta_L) + 2\alpha\left(\eta_0 - \frac{\eta_R+\eta_L}{2}\right) & \text{if} \quad \hat{\eta}_i > \eta_R \\ 2\left(\hat{\eta}_i - \frac{\eta_R+\eta_L}{2}\right) + 2\alpha\left(\eta_0 - \frac{\eta_R+\eta_L}{2}\right) & \text{if} \quad \hat{\eta}_i \in (\eta_L, \eta_R) \\ -(\eta_R - \eta_L) + 2\alpha\left(\eta_0 - \frac{\eta_R+\eta_L}{2}\right) & \text{if} \quad \hat{\eta}_i < \eta_L \end{cases}.$$

Let $BR_N(v_N)(o_i, \hat{\eta}_i)$ be a voter's best response when everyone else adopts s_N when the mean of voter turnout is N. Write $u_R = \log 2\frac{U_{rR}-U_{rM}}{U_{rM}-U_{rL}}$, then $\lim_{N\to\infty} BR_N(v_N)(r, \hat{\eta}_i) = R$ if and only if

$$\min\{\hat{\eta}_i, \eta_R\} \geq (1+\alpha)\frac{\eta_R + \eta_L}{2} - \alpha\eta_0 - \frac{1}{2}\log\frac{|p'_R(\eta_R) - p'_M(\eta_R)|}{|p'_L(\eta_L) - p'_M(\eta_L)|} - \frac{1}{2}u_R$$

where η_c is the such that $\lim_{N\to\infty}(p_R(\eta_c|v_N) - p_M(\eta_c|v_N)) = 0$, for $c \in \{R, L\}$. Let s^* be the limit of s_N. Then by continuity, $\lim_N BR_N(s^*)(r, \hat{\eta}_i) = R$ if and only if

$$\min\{\hat{\eta}_i, \eta_R\} \geq (1+\alpha)\frac{\eta_R + \eta_L}{2} - \alpha\eta_0 - \frac{1}{2}\log\frac{|p'_R(\eta_R) - p'_M(\eta_R)|}{|p'_L(\eta_L) - p'_M(\eta_L)|} - \frac{1}{2}u_R$$

where η_c is the such that $p_c(\eta_c|s^*) = p_M(\eta_c|s^*) > p_{-c}(\eta_c|s^*)$, for $c \in \{R, L\}$. Therefore, if s^* is the limit of a symmetric equilibrium as $N \to \infty$, then it is a fixed point of the mapping $\lim_N BR_{N\to\infty}$.

A best response to any symmetric voting strategy profile is a cutoff strategy involving an information threshold: r votes for R if $\hat{\eta}_i \geq a - \frac{u_R}{2}$ and l votes for L if and only if $-\hat{\eta}_i \geq -a - \frac{u_L}{2}$. The information cutoff depends on the voter's preference intensity, but also on a systematic bias a. $a > 0$ represents a bias toward L because the information cutoff is higher than preference intensity for right wing voters, but lower than preference intensity for left-wing voters.

If everyone else adopts such a cutoff strategy indexed by a, then the probability that voter i votes for R is equal to

$$p_R(\eta; a) = F(\eta - \theta) F\left(\eta - a + \frac{u_R}{2}\right)$$

and the probability that voter i votes for L is

$$p_L(\eta; a) = F(-\eta - \theta) F\left(-\eta + a + \frac{u_L}{2}\right).$$

Because p_R is increasing in η and p_L is decreasing in η, there exists a unique solution $\tilde{\eta}$ to $p_R(\eta) = p_L(\eta)$. For $\theta > \frac{3}{2}$, $F(-\theta) < \frac{1}{3}$. Thus $p_R(\tilde{\eta}) = p_L(\tilde{\eta}) < p_M(\tilde{\eta})$. Define $\eta_R(a)$ to be the solution to $2p_R(\eta; a) + p_L(\eta; a) = 1$. Then if everyone adopts a cutoff strategy indexed by a, the probability that voter i votes for M is equal to the probability that voter i votes for R when $\eta = \eta_R(a)$. When the electorate is large, R ties with M for the winner at η near $\eta_R(a)$. Define

$$\hat{a}(a) = (1+\alpha)\frac{\eta_R(a) + \eta_L(a)}{2} - \alpha\eta_0 - \frac{1}{2}\log\frac{|p'_R(\eta_R(a)) - p'_M(\eta_R(a))|}{|p'_L(\eta_L(a)) - p'_M(\eta_L(a))|}.$$

Then if s_N is a symmetric equilibrium in an electorate with mean N, $\lim_{N\to\infty} s_N$ is a cutoff strategy indexed by a^* where a^* is a fixed point of $\hat{a}$.

We first solve for $p_R(\eta, a) = p_M(\eta, a)$.

Lemma 3.2 *If $u_R + u_L < 0$ and $\theta > \frac{3}{2}$, then*

$$\eta_R(a) = \log\frac{e^{\theta} + e^{a - \frac{u_R}{2}} + \sqrt{e^{2\theta} + e^{2a - u_R} - e^{-\theta + a + \frac{u_L}{2}}}}{2}.$$

Proposition 1 *If $u_R + u_L < 0$, $\alpha < \frac{1}{4}$ and $\theta > \min\left\{-\frac{u_R+u_L}{4}, -\frac{\max\{u_R,u_L\}}{2} + \log 2\right\}$, then there exists a unique fixed point a^* for the mapping $\hat{a}$. In addition, $a^* \cdot (u_R - u_L) < 0$ and $\eta_R^*(u_r, u_l, \theta, \alpha) := \eta_R(a^*, u_R, u_l, \theta, \alpha) > \theta$, $\eta_L^*(u_r, u_l, \theta, \alpha) := \eta_L(a^*, u_r, u_l, \theta, \alpha) < -\theta$.*

Therefore, the game has a unique symmetric equilibrium with multi-candidate support. The equilibrium involves threshold a^* such that a right wing voter votes for R if and only if her information is more optimistic than the threshold $a^* - \frac{u_r}{2}$. If voters in the opposing camp have higher preference intensity, then the threshold will be lower.

3.3 Comparative Statics

Let $\eta_c^*(u_r, u_l, \theta, \alpha) = \eta_c(a^*(u_r, u_l, \theta, \alpha))$ for $c \in \{R, L\}$.

Proposition 2 *If $u_r + u_l < 0$, $\alpha < \frac{1}{4}$, and $\theta > \min\left\{-\frac{u_r+u_l}{4}, -\frac{\max\{u_r,u_l\}}{2} + \log 2, \frac{3}{2}\right\}$, then $\frac{\partial\left|\eta_j^*(u_r,u_l,\theta,\alpha)\right|}{\partial u_j} < 0$ and $\frac{\partial\left|\eta_j^*(u_r,u_l,\theta,\alpha)\right|}{\partial u_k} > 0$ for $j \neq k$ and $j,k \in \{L, R\}$.*

η_R^* decreases with right wing voters' preference intensity u_r and increases with left-wing voters preference intensity u_l. In other words, the prior probability that R wins the election increases with u_r and decreases with u_l. This is true for preference intensities that are not very strong nor too weak.

When left-wing voters' preference intensity u_l goes up, there are two offsetting effects. First, this will increase the information threshold for right-wing voters and thus decrease the probability that a right wing voter votes for R by increasing the fixed point a^*. On the other hand, given the same a, this will decrease the information threshold for left-wing voters, and this will also decrease equilibrium a^*. A stronger left-wing force will eat into the voter base for M, and improves the prospect of R w.r.t. M. When u_l is not too big , the former force dominates.

Because $\eta_R^* > \theta$, strategic voting results in conservative voting behavior that favors candidate M, the common second choice or middle ground. When $\eta \in (\theta, \eta_R^*)$, the median voter prefers R, but M wins the election. On the other hand, if every voter votes sincerely, then the voting outcome exhibits miscoordination. There exists $\eta_R^N \in (0, \theta)$ where N stands for naive voting such that when $\eta \in \left(\eta_R^N, \theta\right)$, the median voter prefers M the most but R wins the election because left-wing voters do not coordinate with moderate voters.

$\eta_R^*(u, u, \theta, \alpha)$ is decreasing in u and increasing in θ.

4 Sequential v.s. Simultaneous Election

4.1 Model

The electorate consists of three states, state 1,2,3. The candidate that wins most states wins the election. In case of a tie between 2 or 3 candidates, the winner is determined by a random draw among those that tie for the first place. The winner within a state is determined also by plurality rule as described in the previous section. Voter i is state k is right-wing with probability $F(\eta_k - \theta)$ and left-wing with probability $F(-\eta_k - \theta)$. Every voter shares the same prior that η_k's follow *i.i.d.* $Laplace\,(0, \alpha)$. In addition to the common prior, voter i in state k obtains an additional signal $\hat{\eta}_i$ about η_k where $\hat{\eta}_i \sim Laplace\,(\eta_k, 1)$. The independence of η_k's across states implies that there is no learning when voting takes place sequentially. This allows me to focus on the coordination effect of sequential voting.

Let v_{oc} denote the payoff to voter of ideology type o when candidate c wins the election. We will look at the symmetric case where $v_{rR} = v_{lL} > v_{rM} = v_{lM} > v_{rL} = v_{lR}$ and $v_{mM} > v_{mL} = v_{mR}$. Define

$$\phi = \frac{v_{rR} - v_{rM}}{v_{rM} - v_{rL}}$$

and $u = \log 2\phi$. We call ϕ the extreme voters' preference intensity for their favorite candidate.

4.2 Sequential Election

This section analyzes equilibria in a sequential election and illustrate the coordination effect. We only look at the election where $\phi < \frac{1}{2}$. In such elections, coordination is important because the payoff difference between the second and the least favorite candidate is more than twice that of the first and the second favorite candidate.

4.2.1 Voting in the last state

The voting outcome in the last state may affect the election outcome if and only if the previous two states split between two candidates. It is weakly dominant for a moderate voter to vote for M. Given any voting strategy in which m always votes for M, the probability that candidate R ties with L vanishes more quickly than the probability that candidate L ties with M. Therefore, voter i only weighs between the probability of an $R-M$ tie and the probability of an $M-L$ tie.

When candidate L and candidate M each wins one state, then a right wing voter's payoff when candidate c wins the third state is given by $U_{rR} = \frac{v_{rR}+v_{rM}+v_{rL}}{3}$, $U_{rM} = v_{rM}$ and $U_{rL} = v_{rL}$. When $\phi < 1$, $U_{rM} - U_{rL} = \frac{(v_{rR}-v_{rM})-(v_{rM}-v_{rL})}{3} < 0$. Therefore, in both an $R-M$ tie and an $M-L$ tie, a right wing voter prefers to vote for M. Therefore, in all weakly undominated equilibria, a right wing voter votes for M. Thus the last state is a runoff between L and M. L wins the last state and the election if $\eta_3 < -\theta$ and M wins the last state and the election if $\eta_3 > -\theta$.

When candidate L and R each wins one state, $U_{rR}^{LR} - U_{rM}^{LR} = \frac{u_{rR}-u_{rM}+u_{rR}-u_{rL}}{3}$ and $U_{rM} - U_{rL} = \frac{u_{rR}-u_{rL}+u_{rM}-u_{rL}}{3}$. Therefore, the preference intensity for the last-state election, denoted by ϕ^{RL}, is equal to 1. Thus, the equilibrium in the subgame after R and L split the first two states gives rise to the two cutoff points $\eta_R^*(1,1,\alpha,\theta)$ and $\eta_L^*(1,1,\alpha,\theta)$. Because $\theta > \frac{1+1}{4}$, $\eta_R^*(1,1,\alpha,\theta) > \theta$.

4.2.2 Voting in the second state

In this section we will show how the cutoff points on η_2 for different voting outcomes in state 2 depends on the voting outcome of state 1. In particular, we will show that when preference intensity for the overall election is moderate, probability that candidate R wins the second state increases as the outcome of the state 1 changes from L to M to R. In particular, we will analyze how η_R^h changes with h, where $h \in \{R, M, L\}$ is the outcome of the first state and η_R^h is lower bound on η_2 for candidate R to win the second state.

Given the voting outcome $h \in \{R, M, L\}$ of state 1, the eventual election outcome depends on the voting outcome of state 2 and the electoral preference of state 3, η_3. Figure illustrates how the election outcome depends on the voting outcomes of the first two states and η_3.

Consider the voting game in state 2 after candidate R wins the first state. State 2's voting outcome is pivotal only when M will win state 3, i.e. $\eta_3 < \theta$. Therefore, we get $U_{rR} - U_{rM} = G(\theta)(v_{rR} - v_{rM})$, where G is the cumulative distribution function of the prior on η_3. But the payoff difference when M wins state 2 v.s. when L wins state 2 gets even smaller. Therefore, we get

$$\phi_r^R = \frac{G(\theta)\,\phi}{\frac{1}{2} - \frac{G(-\theta,\theta)}{2}\phi - \frac{G(-\theta,\theta)}{6}(1-\phi) - \frac{G\left(\theta,\eta_R^*(1,1)\right)}{3}(1-\phi)}.$$

So a win by R boosters the preference intensity of right-wing voters in the second state. The ratio $\frac{\phi_r^R}{\phi}$ is higher the weaker the general preference intensity is, and the less likely an extreme candidate will win state 3. Because the game is symmetric, $\phi_l^L = \phi_r^R$. ϕ_r^R is different from the payoff difference ratio in a simultaneous election conditional on one state being taken by candidate R. Conditional on one state being R, a R-win or an M-win makes a difference when state 3 is taken by either M or L. But in a sequential election, L never wins state 3 if R wins state 1 and state 2 is taken by either R or M. In other words, voting outcome in the first two states can change a left-wing state from being taken by L to being taken by M.

Consider the voting game in state 2 after L wins the first state. We get that

$$\phi_r^L = \frac{\left(\frac{1}{2} - \frac{G(-\theta,\theta)}{2}\right)\phi + \frac{G(-\theta,\theta)}{6}(1-\phi) + \frac{G(\theta,\eta_R^*(1,1))}{3}(1-\phi)}{G(\theta)}.$$

Given that M wins the first state, $\phi_r^M = \phi$.

Therefore, we see that ϕ_r^h increases as h changes from L to M to R. Right wing voters' preference intensity for the voting outcome in the second state is higher the closer the voting outcome in the first state is to their preferred choice.

Proposition 3 *If* $u < 0, \alpha < \frac{1}{4}$, *and* $\theta > \min\left\{-\frac{u}{2} + \log 2, \frac{3}{2}\right\}$, *then* $\eta_R^R(u,\theta,\alpha) < \eta_R^M(u,\theta,\alpha) < \eta_R^L(u,\theta,\alpha)$.

This follows immediately from Proposition 2 because right wing voters' preference intensity increases while left-wing voters' preference intensity decreases as the outcome of the first state changes from L to M to R.

4.2.3 Voting in the first state.

$$\phi_r^{\emptyset} = \frac{\phi - c^{\emptyset}}{1 - c^{\emptyset}\phi}$$

where

$$c^{\emptyset} = \frac{1}{2}\frac{\frac{2}{3} - \frac{2}{3}\frac{F(\theta,\eta_R^*(1,1))}{P(m)} - 2\frac{F\left(\eta_R^M,\eta_R^L\right)}{P^L(R)}\frac{P(r)}{P(m)}}{\frac{P^R(R)}{P^L(R)} + \frac{P(r)}{P^L(R)} + \frac{1}{2} - \frac{1}{6} - \frac{1}{3}\frac{F\left(\theta,\eta_R^*(1,1)\right)}{P(m)} + 2\frac{P(r)}{P^L(R)}\frac{P(r)}{P(m)} + \frac{F\left(\eta_R^R,\eta_R^M\right)}{P^L(R)}\frac{P(r)}{P(m)}}.$$

Because $\phi < 1, \phi_r^{\emptyset}$ is decreasing in $c^{\emptyset}$.

Outcome in the first state can change outcome in the second state and/or outcome in state 3. The reason a right-wing voter may strategically vote for M instead of her favorite candidate R is for fear of a tie between M and L and getting L elected instead of M in that situation. Roughly speaking, M and L tie in the overall election when one of the other two states is moderate and the other is left-wing. But when R wins the first state, the left state has to be very left for L to win, and once an $R-L$ split has formed, R may win a moderate state as well.

4.3 Simultaneous (Front-loaded) Election

The payoff difference to voter i in state k when candidate c wins state k v.s. candidate c' depends on how the voting outcome in state k affects the election outcome. We will focus on symmetric equilibria in which every voter in very state use the same voting strategy. Suppose voters in the other two states use voting strategy s such that R wins state k if $\eta_k > \tilde{\eta}$ and L wins state k if $\eta_k < -\tilde{\eta}$. Then the probability that R wins state k is $G(-\tilde{\eta})$. Denote by $p^F(c)$ the probability that candidate c wins a state. This vector of probabilities depend on the voting strategy s employed and is determined by $\tilde{\eta}$.

$$
\begin{aligned}
U_R^F - U_M^F &= P^F(R)\,P^F(M)\,(v_{cR}-v_{cM}) + P^F(R)\,P^F(L)\,\frac{(v_{cR}-v_{cM})+(v_{cR}-v_{cL})}{3} \\
&\quad + P^F(M)\,P^F(L)\,\frac{(v_{cR}-v_{cM})-(v_{cM}-v_{cL})}{3} \\
&= \left(P^F(R)\,P(M) + \frac{2}{3}P(R)\,P(L) + \frac{1}{3}P(M)\,P(L)\right)(v_{cR}-v_{cM}) \\
&\quad - P^F(L)\left(P^F(M)-P^F(R)\right)(v_{cM}-v_{cL}).
\end{aligned}
$$

Because the game is symmetric and we are looking for symmetric equilibria, $P^F(R) = P^F(L)$ and we get

$$
\begin{aligned}
\phi^F \quad &:= \frac{U_R^F - U_M^F}{U_M^F - U_L^F} \\
&= \frac{\phi - c^F}{1 - c^F\phi}
\end{aligned}
$$

where

$$
\begin{aligned}
c^F &= \frac{\frac{1}{3}\left(P^F(M)-P^F(R)\right)}{\frac{4}{3}P^F(M)+\frac{2}{3}P^F(L)} \\
&= \frac{1}{2}\frac{1-\frac{P^F(R)}{P^F(M)}}{2+\frac{P^F(L)}{P^F(M)}}.
\end{aligned}
$$

Note that c^F is a function of $\tilde{\eta}$, and thus u^F is a function of u and $\tilde{\eta}$.

Given that voters in the other two states use symmetric voting strategy v characterized by $\tilde{\eta}$, preference intensity for voting outcome of the state is given by $u^F(u,\tilde{\eta})$. Because the game within the state is symmetric, $a^* =$

0. In this equilibrium, an extreme voter votes for her favorite candidate if her signal $\hat{\eta}_i > -\frac{u^F(u,\tilde{\eta})}{2}$. Note that when $\phi < 1$, $\phi^F(\phi,\tilde{\eta}) < \phi$ if and only if $\frac{P^F(R)}{P^F(M)} < 1$. Therefore, in a symmetric equilibrium, the cutoff for R to win a state is $\eta_R^F(u,\theta,\alpha) > \eta_R^*(u,u,\theta,\alpha)$. Define $\eta^F(\tilde{\eta};u,\theta,\alpha) = \eta_R^*\left(u^F(u,\tilde{\eta}), u^F(u,\tilde{\eta}), \theta, \alpha\right)$. $\eta^F(\tilde{\eta})$ is increasing for $\tilde{\eta} \geq \eta_R^*(u,u,\theta,\alpha)$ and $\eta^F(\eta_R^*(u,u,\theta,\alpha)) > \eta_R^*$. Define the fixed point to be ∞ when $\eta^F(\tilde{\eta}) > \tilde{\eta}$ for all $\tilde{\eta} > \eta_R^*(u,u,\theta,\alpha)$. Then η_R^F is a fixed point of the function. $\eta_R^F = \infty$ is a simultaneous voting equilibrium in which all voters vote for M.

4.4 Voting Behavior in State 1 under sequential and front-loaded election

Comparing $\eta_R^\emptyset$ and η_R^F is equivalent to comparing $c^\emptyset$ and c^F. When $\phi < 1$, $\eta_R^\emptyset < \eta_R^F$ if and only if $c^\emptyset < c^F$.

Proposition 4 *For θ big enough, or u small enough, voters in state 1 behave more aggressively under a sequential election than under a simultaneous election.*

5 Conclusion

This paper studies preference aggregation in a multi-candidate contest when the preference of the electorate is not common knowledge. In a multi-candidate contest, voters have an incentive to coordinate with supporters of their second choice to avoid a victory by the least favorite candidate. I show that the coordination incentive is stronger when preference intensity is weaker. I then use this model as cornerstone to compare a simultaneous election in which several states vote at the same time and a sequential election in which each state votes one by one after observing outcomes of previous states. I show that when the prior probability of extreme voters is small or when the preference intensity of extreme voters is small, coordination incentives are stronger for extreme voters and thus they vote more aggressively in a sequential election than in a simultaneous election. As a result, the prior probability that the winner in a state is not the first choice of the median voter is smaller in a sequential election.

6 Appendix

6.1 Proof for lemma 3.1.

Proof. It suffices to show that

$$\lim_{N\to\infty} N \Pr\{V_R = V_M > V_L | \hat{\eta}_i, p\} = \frac{f(\eta_R|\hat{\eta}_i)}{|p'_R(\eta_R) - p'_M(\eta_R)|}.$$

Let

$$H^u = \{(V_R, V_M, V_L) \, | V_R = V_M > V_L \text{ where } V_c \geq 0 \text{ for } c = R, M, L\}$$

Then

$$\Pr\{V_R = V_M > V_L | \hat{\eta}_i, p\} = \int_{\eta=-\infty}^{\infty} P\left(H^u | N, p(\eta)\right) f\left(\eta | \hat{\eta}_i\right) d\eta.$$

Let

$$H = \{(V_R, V_M, V_L) \,| V_R = V_M \text{ where } V_c \geq 0 \text{ for } c = R, M, L\}$$

and $H^* = \{(V_R, V_M, V_L) \,| V_R = V_M \text{ where } V_c \geq 0 \text{ for } c = R, M, L\}$. Then H is a hyperplane in $(N \cup \{0\})^3$ spanned by $w_1 = (1, 1, 0)$ and $w_2 = (0, 0, 1)$.

Given η, we first show that $y_N := \left(\left[N\sqrt{p_R(\eta)\, p_M(\eta)}\right], \left[N\sqrt{p_R(\eta)\, p_M(\eta)}\right], [Np_L(\eta)]\right)$ is a near maximizer $\sum_c p_c \psi\left(\frac{x(c)}{Np_c}\right)$ over x in H^* where $\psi(\theta) = \theta(1 - \log\theta) - 1$. $H^* = \{\gamma(1, 1, 0) + j(0, 0, 1) \,| \gamma \geq 0 \text{ and } j \geq 0\}$. Let

$$(\gamma^*, j^*) \in \arg\max_{\gamma \geq 0, j \geq 0} \left(p_R \psi\left(\frac{\gamma}{Np_R}\right) + p_M \psi\left(\frac{\gamma}{Np_M}\right) + p_L \psi\left(\frac{j}{Np_L}\right)\right).$$

Because the derivative is ∞ for $\gamma = 0$ or $j = 0$ and the function goes to 0 as γ or $j \to \infty$, the solution must be interior of H^*. Thus γ^*, j^* satisfy the first order condition:

$$\begin{aligned}
0 &= -\log\frac{\gamma}{Np_R} - \log\frac{\gamma}{Np_M} \\
0 &= -\log\frac{j}{Np_L}.
\end{aligned}$$

So $\gamma^* = N\sqrt{p_R p_M}$ and $j^* = Np_L$. Then y_N as defined is a near maximizer.

$$\begin{aligned}
& p_R \psi\left(\frac{\gamma^*}{Np_R}\right) + p_M \psi\left(\frac{\gamma^*}{Np_M}\right) + p_L \psi\left(\frac{j^*}{Np_L}\right) \\
= \; & p_R\left(\frac{\gamma^*}{Np_R}\left(1 - \log\left(\frac{\gamma^*}{Np_R}\right)\right) - 1\right) \\
& + p_M\left(\frac{\gamma^*}{Np_M}\left(1 - \log\left(\frac{\gamma^*}{Np_M}\right)\right) - 1\right) \\
& + p_L\left(\frac{j^*}{Np_L}\left(1 - \log\left(\frac{j^*}{Np_L}\right)\right) - 1\right) \\
= \; & -1 + \frac{\gamma^*}{N}\left(1 - \log\left(\frac{\gamma^*}{Np_R}\right) + 1 - \log\left(\frac{\gamma^*}{Np_M}\right)\right) \\
& + \frac{j^*}{N}\left(1 - \log\left(\frac{j^*}{Np_L}\right)\right) \\
= \; & -1 + 2\frac{\gamma^*}{N} + \frac{j^*}{N} \\
= \; & 2\sqrt{p_R p_M} - (1 - p_L) \\
= \; & 2\sqrt{p_R p_M} - p_R - p_M \\
= \; & -\left(\sqrt{p_R} - \sqrt{p_M}\right)^2.
\end{aligned}$$

Then using theorem 3 in Myerson (2000),

$$\lim_{N\to\infty}\frac{\Pr\{H|Np(\eta)\}}{\Pr\{y_N|Np(\eta)\}(2\pi)(\det(M(y_N)))^{-0.5}}=1$$

where $M(y_N(\eta))=\begin{bmatrix}\frac{2}{\left[N\sqrt{p_R(\eta)p_M(\eta)}\right]} & 0\\ 0 & \frac{1}{[Np_L(\eta)]}\end{bmatrix}$ and $\lim_{N\to\infty}N*M(y_N)=\begin{bmatrix}\frac{2}{\sqrt{p_R(\eta)p_M(\eta)}} & 0\\ 0 & \frac{1}{p_L(\eta)}\end{bmatrix}$. By Myerson (2000),

$$\begin{aligned}\Pr\{y_N|Np(\eta)\} &\approx \frac{e^{N*\left(p_R\psi\left(\frac{\gamma^*}{Np_R}\right)+p_M\psi\left(\frac{\gamma^*}{Np_M}\right)+p_L\psi\left(\frac{j^*}{Np_L}\right)\right)}}{\Pi_{c\in\{R,M,L\}}\sqrt{2\pi y_N(c)}}\\ &= \frac{e^{-N\left(\sqrt{p_R}-\sqrt{p_M}\right)^2}}{(2\pi)^{\frac{3}{2}}\sqrt{(\gamma^*)^2 j^*}}\\ &= \frac{e^{-N\left(\sqrt{p_R}-\sqrt{p_M}\right)^2}}{(2N\pi)^{\frac{3}{2}}\sqrt{p_Rp_Mp_L}}.\end{aligned}$$

$$\begin{aligned}(\det(M(y_N)))^{-0.5} &\approx \left(\frac{1}{N^2\sqrt{p_Rp_M}p_L}\right)^{-0.5}\\ &= N\sqrt{\sqrt{p_Rp_M}p_L}.\end{aligned}$$

So

$$\begin{aligned}\Pr\{H^*|Np(\eta)\} &\approx \Pr\{y_N|Np(\eta)\}(2\pi)(\det(M(y_N)))^{-0.5}\\ &\approx N\sqrt{\sqrt{p_Rp_M}p_L}\,(2\pi)\,\frac{e^{-N\left(\sqrt{p_R}-\sqrt{p_M}\right)^2}}{(2N\pi)^{\frac{3}{2}}\sqrt{p_Rp_Mp_L}}\\ &= \frac{e^{-N\left(\sqrt{p_R}-\sqrt{p_M}\right)^2}}{\sqrt{2\pi N\sqrt{\sqrt{p_Rp_M}}}}.\end{aligned}$$

Given $\varepsilon>0$, let δ be such that $|p_R(\eta)-p_M(\eta)|\geq\varepsilon$ for all η such that $|\eta-\eta_R|\geq\delta$. Define $\Lambda_\delta:=\{\eta:|\eta-\eta_R|<\delta\}$. Then want to show that $\lim_{N\to\infty}\frac{\Pr\{H|Np(\eta)\}}{\Pr(H^*|Np(\eta))}=1$ for $\eta\in\Lambda_\delta$. Then show that $\lim_{N\to\infty}N\Pr\{H^*|Np(\eta)\}=0$ for $\eta\notin\Lambda_\delta$. Then

$$\begin{aligned}&\lim_{N\to\infty}N\Pr\{V_R=V_M>V_L|\hat\eta_i,p\}\\ =\;&\lim_{N\to\infty}N\int_\eta\Pr\{H|Np(\eta)\}f(\eta|\hat\eta_i)\,d\eta\\ =\;&\lim_{N\to\infty}\left(N\int_{\eta\in\Lambda_\delta}\Pr\{H|Np(\eta)\}f(\eta|\hat\eta_i)\,d\eta+N\int_{\eta\notin\Lambda_\delta}\Pr\{H|Np(\eta)\}f(\eta|\hat\eta_i)\,d\eta\right)\\ =\;&\lim_{N\to\infty}N\int_{\eta\in\Lambda_\delta}\Pr\{H|Np(\eta)\}f(\eta|\hat\eta_i)\,d\eta+\lim_{N\to\infty}N\int_{\eta\notin\Lambda_\delta}\Pr\{H|Np(\eta)\}f(\eta|\hat\eta_i)\,d\eta.\end{aligned}$$

$$\begin{aligned}
& \lim_{N\to\infty} N \int_{\eta\notin\Lambda_\delta} \Pr\left\{H|Np\left(\eta\right)\right\} f\left(\eta|\hat{\eta}_i\right) d\eta \\
\leq \quad & \lim_{N\to\infty} \int_{\eta\notin\Lambda_\delta} N \Pr\left\{H|Np\left(\eta\right)\right\} f\left(\eta|\hat{\eta}_i\right) d\eta \\
= \quad & 0.
\end{aligned}$$

And

$$\begin{aligned}
& N \int_{\eta\in\Lambda_\delta} \Pr\left\{H|Np\left(\eta\right)\right\} f\left(\eta|\hat{\eta}_i\right) d\eta \\
\in \quad & \left[\frac{\sqrt{2} f\left(\eta_R|\hat{\eta}_i\right)}{\sqrt{\sqrt{\frac{p_M(\eta_R)}{p_R(\eta_R)}}} p'_R\left(\eta_R\right) - \sqrt{\sqrt{\frac{p_R(\eta_R)}{p_M(\eta_R)}}} p'_M\left(\eta_R\right)} - \zeta, \frac{\sqrt{2} f\left(\eta_R|\hat{\eta}_i\right)}{\sqrt{\sqrt{\frac{p_M(\eta_R)}{p_R(\eta_R)}}} p'_R\left(\eta_R\right) - \sqrt{\sqrt{\frac{p_R(\eta_R)}{p_M(\eta_R)}}} p'_M\left(\eta_R\right)} + \zeta\right] \\
& * \int_{\eta\in\Lambda_\delta} N \left(\frac{\sqrt{2}}{\sqrt{\sqrt{\frac{p_M}{p_R}}} p'_R\left(\eta\right) - \sqrt{\sqrt{\frac{p_R}{p_M}}} p'_M\left(\eta\right)}\right)^{-1} \Pr\left\{H|Np\left(\eta\right)\right\} d\eta \\
= \quad & \left[\frac{\sqrt{2} f\left(\eta_R|\hat{\eta}_i\right)}{p'_R\left(\eta_R\right) - p'_M\left(\eta_R\right)} - \zeta, \frac{\sqrt{2} f\left(\eta_R|\hat{\eta}_i\right)}{p'_R\left(\eta_R\right) - p'_M\left(\eta_R\right)} + \zeta\right] \\
& * \int_{\eta\in\Lambda_\delta} N \left(\frac{\sqrt{2}}{\sqrt{\sqrt{\frac{p_M}{p_R}}} p'_R\left(\eta\right) - \sqrt{\sqrt{\frac{p_R}{p_M}}} p'_M\left(\eta\right)}\right)^{-1} \Pr\left\{H|Np\left(\eta\right)\right\} d\eta
\end{aligned}$$

$$\begin{aligned}
& \int_{\eta\in\Lambda_\delta} N \Pr\left\{H|Np\left(\eta\right)\right\} d\eta \\
= \quad & \int_{\eta=\eta_R-\varepsilon}^{\eta_R+\varepsilon} \frac{\sqrt{N} e^{-N\left(\sqrt{p_R(\eta)}-\sqrt{p_M(\eta)}\right)^2}}{\sqrt{2\pi}\sqrt{\sqrt{p_R\left(\eta\right) p_M\left(\eta\right)}}} d\eta.
\end{aligned}$$

Write $x = \sqrt{2N}\left(\sqrt{p_R\left(\eta\right)} - \sqrt{p_M\left(\eta\right)}\right)$. Then

$$\begin{aligned}
dx \quad = \quad & \sqrt{2N} \frac{\sqrt{p_M\left(\eta\right)} p'_R\left(\eta\right) - \sqrt{p_R\left(\eta\right)} p'_M\left(\eta\right)}{2\sqrt{p_R p_M}} d\eta \\
= \quad & \sqrt{N} \frac{\sqrt{\sqrt{\frac{p_M}{p_R}}} p'_R\left(\eta\right) - \sqrt{\sqrt{\frac{p_R}{p_M}}} p'_M\left(\eta\right)}{\sqrt{2}\sqrt{\sqrt{p_R p_M}}} d\eta
\end{aligned}$$

$$
\begin{aligned}
&\int_{\eta\in\Lambda_\delta} N \Pr\{H|Np(\eta)\}\,d\eta \\
= &\int_{x=\sqrt{2N}\left(\sqrt{p_R(\eta_R-\varepsilon)}-\sqrt{p_M(\eta_R-\varepsilon)}\right)}^{\sqrt{2N}\left(\sqrt{p_R(\eta_R+\varepsilon)}-\sqrt{p_M(\eta_R+\varepsilon)}\right)} \frac{\sqrt{2}}{\sqrt{\sqrt{\frac{p_M}{p_R}}p'_R(\eta)}-\sqrt{\sqrt{\frac{p_R}{p_M}}p'_M(\eta)}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\lim_{N\to\infty}\int_{\eta\in\Lambda_\delta} N\left(\frac{\sqrt{2}}{\sqrt{\sqrt{\frac{p_M}{p_R}}p'_R(\eta)}-\sqrt{\sqrt{\frac{p_R}{p_M}}p'_M(\eta)}}\right)^{-1} \Pr\{H|Np(\eta)\}\,d\eta \\
= &\lim_{N\to\infty}\int_{x=\sqrt{2N}\left(\sqrt{p_R(\eta_R-\varepsilon)}-\sqrt{p_M(\eta_R-\varepsilon)}\right)}^{\sqrt{2N}\left(\sqrt{p_R(\eta_R+\varepsilon)}-\sqrt{p_M(\eta_R+\varepsilon)}\right)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx \\
= &\ 1.
\end{aligned}
$$

Let $\zeta\to 0$. Then we get $\lim_{N\to\infty} N\Pr\{V_R=V_M>V_L|\hat{\eta}_i,p\} = \frac{\sqrt{2}f(\eta_R|\hat{\eta}_i)}{p'_R(\eta_R)-p'_M(\eta_R)}$. Need $\frac{\sqrt{2}f(\eta|\hat{\eta}_i)}{\sqrt{\sqrt{\frac{p_M(\eta)}{p_R(\eta)}}p'_R(\eta)}-\sqrt{\sqrt{\frac{p_R(\eta)}{p_M(\eta)}}p'_M(\eta)}}$ to be absolutely continuous. ■

6.2 Additional proofs and lemmas for Section3.2.3

Lemma 6.1 *If $\theta > \frac{u_R+u_L}{4}$, then $\eta_R(a) > \max\left\{\theta, a-\frac{u_R}{2}\right\}$ and $\eta_L(a) < \min\left\{-\theta, a+\frac{u_L}{2}\right\}$.*

Proof. We first observe that $2p_R(\eta,a)+p_L(\eta,a)$ is increasing in η. Suppose $\eta_R^*\in\left[\theta, a-\frac{\tilde{u}_R}{2}\right]$, then

$$
\begin{aligned}
0 &= 2p_R(\eta_R)+p_L(\eta_R)-1 \\
&= \left(1-\frac{1}{2}e^{-(\eta-\theta)}\right)e^{\left(\eta-a+\frac{\tilde{u}_R}{2}\right)}+\frac{1}{2}e^{-\eta-\theta}F\left(-\eta+a+\frac{\tilde{u}_L}{2}\right)-1 \\
&\le 2p_R\left(a-\frac{\tilde{u}_R}{2}\right)+p_L\left(a-\frac{\tilde{u}_R}{2}\right)-1 \\
&\le -\frac{1}{2}e^{\theta-a+\frac{\tilde{u}_R}{2}}+\frac{1}{2}e^{-\theta-a+\frac{\tilde{u}_R}{2}}<0,
\end{aligned}
$$

contradiction.

Otherwise,$\eta_R^*\in\left[a-\frac{\tilde{u}_R}{2},\theta\right]$. Because $\eta_R^*>0>-\theta$, then

$$
\begin{aligned}
0 &= 2p_R(\eta_R)+p_L(\eta_R)-1 \\
&= \left(1-\frac{1}{2}e^{-\eta+a-\frac{\tilde{u}_R}{2}}\right)e^{\eta-\theta}+\frac{1}{2}e^{-\eta_R-\theta}F\left(-\eta+a+\frac{\tilde{u}_L}{2}\right)-1 \\
&\le 2p_R(\theta)+p_L(\theta)-1 \\
&\le -\frac{1}{2}e^{-\theta+a-\frac{\tilde{u}_R}{2}}+\frac{1}{2}e^{-\theta-\theta} \\
&< 0
\end{aligned}
$$

contradiction. ■

Lemma 6.2 *For a such that* $\max\left\{\theta, a-\frac{u_R}{2}\right\} > a+\frac{u_L}{2}$ *and* $\max\left\{-\theta, -a-\frac{u_L}{2}\right\} > -a+\frac{u_R}{2}$,

$$\begin{aligned}
&\hat{a}\left(a\right)\\
= &\left(1+\frac{\alpha}{2}\right)\left[\begin{array}{c}\log\left(e^{\theta}+e^{a-\frac{1}{2}u_R}+\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}\right)\\ -\log\left(e^{\theta}+e^{-a-\frac{1}{2}u_L}+\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}\right)\end{array}\right]\\
&-\frac{1}{2}\left[\begin{array}{c}\log\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}\\ -\log\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}\end{array}\right]-\alpha\eta_0
\end{aligned}$$

Suppose $\eta_R > \max\left\{\theta, a-\frac{1}{2}\tilde{u}_R, a+\frac{\tilde{u}_L}{2}\right\}$ and $\eta_L < \min\left\{-\theta, a+\frac{u_L}{2}, -a-\frac{u_R}{2}\right\}$, then η_R is the solution to

$$1 = 2\left(1-\frac{1}{2}e^{-(\eta_R-\theta)}\right)\left(1-\frac{1}{2}e^{-\left(\eta_R-a+\frac{\tilde{u}_R}{2}\right)}\right)+\frac{1}{2}e^{-\eta_R-\theta}\frac{1}{2}e^{-\eta_R+a+\frac{\tilde{u}_L}{2}}.$$

So

$$e^{\eta_R} = \frac{1}{2}e^{\theta}+\frac{1}{2}e^{a-\frac{1}{2}u_R}+\frac{1}{2}\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}.$$

By symmetry,

$$e^{-\eta_L} = \frac{1}{2}e^{\theta}+\frac{1}{2}e^{-a-\frac{1}{2}u_L}+\frac{1}{2}\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}.$$

Lemma 6.3 *If* $\theta > \frac{u_R+u_L}{4}$, *then* $\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}} > \max\left\{\frac{e^{\theta}+e^{a-\frac{u_R}{2}}}{2}, \left|e^{\theta}-e^{a-\frac{u_R}{2}}\right|\right\}$

Proof.

$$\begin{aligned}
&4\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}^2-\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)^2\\
= &\ 3\left(e^{2\theta}+e^{2a-\tilde{u}_R}\right)-4e^{a-\theta+\frac{\tilde{u}_L}{2}}-2e^{a-\frac{u_R}{2}+\theta}\\
= &\ 3\left(e^{a-\frac{u_R}{2}}-e^{\theta}\right)^2+4\left(e^{a-\frac{u_R}{2}+\theta}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right)\\
\geq &\ 0
\end{aligned}$$

if $\frac{u_R+u_L}{2} < 2\theta$. ■

Lemma 6.4 *For a such that* $\max\left\{\theta, a-\frac{u_R}{2}\right\} > a+\frac{u_L}{2}$ *and* $\max\left\{\theta, -a-\frac{u_L}{2}\right\} > -a+\frac{u_R}{2}$,

$$\begin{aligned}
\hat{a}'(a) &= \frac{1}{2}+\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\left(1-\frac{\frac{e^{\theta}+e^{a-\frac{u_R}{2}}}{2}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\right)\\
&+\frac{1}{2}\frac{e^{-a-\frac{u_L}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\left(1-\frac{\frac{e^{\theta}+e^{-a-\frac{u_L}{2}}}{2}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right)\\
&+\frac{\alpha}{2}\left(1+\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}+\frac{1}{2}\frac{e^{-a-\frac{u_L}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right).
\end{aligned}$$

Proof. If $\theta > \frac{u_R+u_L}{4}$, then for $a < \theta - \frac{u_L}{2}$,

$$\begin{aligned}
\eta_R(a) &= \log\left(\frac{1}{2}e^{\theta}+\frac{1}{2}e^{a-\frac{1}{2}u_R}+\frac{1}{2}\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}\right)\\
&> \max\left\{\theta, a-\frac{u_R}{2}, a+\frac{u_L}{2}\right\}.
\end{aligned}$$

Suppose $\eta_R > \max\left\{\theta, a-\frac{1}{2\beta}\tilde{u}_R\right\}$. Then

$$\begin{aligned}
p_R'(\eta_R) &= (1-F(\eta_R-\theta))F\left(\eta_R-a+\frac{\tilde{u}_R}{2}\right)+F(\eta_R-\theta)\left(1-F\left(\eta_R-a+\frac{1}{2}\tilde{u}_R\right)\right)\\
&= F\left(\eta_R-a+\frac{\tilde{u}_R}{2}\right)+F(\eta_R-\theta)-2p_R(\eta_R).
\end{aligned}$$

If $\eta_R > \max\left\{-\theta, a+\frac{\tilde{u}_L}{2}\right\}$, then

$$p_L'(\eta_R) = -2p_L(\eta_R).$$

So if $\eta_R > \max\left\{\theta, a-\frac{1}{2}\tilde{u}_R, a+\frac{\tilde{u}_L}{2}\right\}$, then

$$\begin{aligned}
& p_R'(\eta_R)-p_M'(\eta_R)\\
&= 2p_R'(\eta_R)+p_L'(\eta_R)\\
&= 2F\left(\eta_R-a+\frac{\tilde{u}_R}{2}\right)+2F(\eta_R-\theta)-4p_R(\eta_R)-2p_L(\eta_R)\\
&= 2F\left(\eta_R-a+\frac{\tilde{u}_R}{2}\right)+2F(\eta_R-\theta)-2\\
&= 1-e^{-\left(\eta_R-a+\frac{\tilde{u}_R}{2}\right)}+1-e^{-(\eta_R-\theta)} \in (0,2)\\
&= e^{-\eta_R}\left(2e^{\eta_R}-e^{\theta}-e^{a-\frac{u_R}{2}}\right)\\
&= e^{-\eta_R}\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}.
\end{aligned}$$

So

$$\begin{aligned}
\frac{\left|p_R'\left(\eta_R^*\right)-p_M'\left(\eta_R^*\right)\right|}{\left|p_L'\left(\eta_L^*\right)-p_M'\left(\eta_L^*\right)\right|} &= \frac{1-e^{-\eta_R+a-\frac{\tilde{u}_R}{2}}+1-e^{-(\eta_R-\theta)}}{1-e^{\eta_L-a-\frac{1}{2}\tilde{u}_L}+1-e^{\eta_L-\theta}} \\
&= \frac{\frac{2e^{\eta_R}-\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)}{e^{\eta_R}}}{\frac{2e^{-\eta_L}-\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)}{e^{-\eta_L}}} \\
&= e^{-\eta_R-\eta_L}\frac{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}.
\end{aligned}$$

We thus get $\hat{a}\left(a\right)$ but substituting these expressions into

$$\hat{a}\left(a\right)=\left(1+\alpha\right)\frac{\eta_R+\eta_L}{2}-\alpha\eta_0-\frac{1}{2}\log\frac{\left|p_R'\left(\eta_R^*\right)-p_M'\left(\eta_R^*\right)\right|}{\left|p_L'\left(\eta_L^*\right)-p_M'\left(\eta_L^*\right)\right|}.$$

■

Therefore, for α sufficiently small, $\hat{a}'\left(a\right)<1$ if $u_R+u_L<0$ or for all $a\in\left(-\theta+\frac{u_R}{2},\theta-\frac{u_L}{2}\right)$. Let a^* denote a fixed point of $\hat{a}$.

Observation If $\theta<-\frac{u_R+u_L}{4}$, then $\left|\log\frac{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right|>\left|\log\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)}{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)}\right|$ and $\log\frac{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\log\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)}{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)}>0$. If $\theta>-\frac{u_R+u_L}{4}$, then $\left|\log\frac{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right|<\left|\log\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)}{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)}\right|$.

Proof.

$$\begin{aligned}
&\left(\frac{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right)^2-\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)^2}{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)^2} \\
=\ &\frac{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}-\frac{e^{2a-u_R}+e^{2\theta}+2e^{a-\frac{u_R}{2}+\theta}}{e^{-2a-u_L}+e^{2\theta}+2e^{-a-\frac{u_L}{2}+\theta}} \\
=\ &\frac{\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right)\left(1-e^{2\theta+\frac{u_R+u_L}{2}}\right)\left(2e^{\theta-\frac{u_R+u_L}{2}}+e^{-\theta}\right)}{\left(e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}\right)\left(e^{-2a-u_L}+e^{2\theta}+2e^{-a-\frac{u_L}{2}+\theta}\right)}.
\end{aligned}$$

■

Observation $\log\frac{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\log\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)}{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)}>0$ if $\theta>u_R+u_L$.

Proof.

$$
\begin{aligned}
& \sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}-\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}} \\
= \quad & \frac{e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}-\left(e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}\right)}{\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}+\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}} \\
= \quad & \frac{e^{2a-\tilde{u}_R}-e^{-2a-\tilde{u}_L}-\left(e^{a-\theta+\frac{\tilde{u}_L}{2}}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}\right)}{\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}+\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}} \\
= \quad & \frac{\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right)\left(e^{a-\frac{u_R}{2}}+e^{-a-\frac{u_L}{2}}\right)-\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right)e^{-\theta+\frac{u_R+u_L}{2}}}{\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}+\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}} \\
= \quad & \left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right)\frac{e^{a-\frac{u_R}{2}}+e^{-a-\frac{u_L}{2}}-e^{-\theta+\frac{u_R+u_L}{2}}}{\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}+\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}}
\end{aligned}
$$

■

Observation $\left(\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)^2}{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}-\frac{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)^2}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right)\left(e^{a-\frac{u_R}{2}}-e^{-a-\frac{u_L}{2}}\right)>0.$

Proof.

$$
\begin{aligned}
& \left(\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)^2}{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}}\right)^2-\left(\frac{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)^2}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right)^2 \\
= \quad & \frac{1}{\left(e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}\right)\left(e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}\right)} \\
& \times\left\{\begin{array}{l}\left(e^{2a-u_R}+e^{2\theta}+2e^{\theta+a-\frac{u_R}{2}}\right)^2\left(e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}\right) \\ -\left(e^{-2a-u_L}+e^{2\theta}+2e^{\theta-a-\frac{u_L}{2}}\right)^2\left(e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}\right)\end{array}\right\} \\
\geq \quad & 0
\end{aligned}
$$

after some algebra. ■

Lemma 6.5 $\hat{a}\,(a)\left(2a-\frac{u_R-u_L}{2}\right)<0$ *if* $\theta>u_R+u_L$.

Proof.

$$\hat{a}(a) = \alpha \log \frac{e^{a-\frac{u_R}{2}} + e^{\theta} + \sqrt{e^{2a-u_R} + e^{2\theta} - e^{a+\frac{u_L}{2}-\theta}}}{e^{-a-\frac{u_L}{2}} + e^{\theta} + \sqrt{e^{-2a-u_L} + e^{2\theta} - e^{-a+\frac{u_R}{2}-\theta}}}$$
$$+\frac{1}{2}\log \frac{\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)^2}{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}} + 2\left(e^{a-\frac{u_R}{2}} + e^{\theta}\right) + \sqrt{e^{2a-u_R} + e^{2\theta} - e^{a+\frac{u_L}{2}-\theta}}}{\frac{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)^2}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}} + 2\left(e^{-a-\frac{u_L}{2}} + e^{\theta}\right) + \sqrt{e^{-2a-u_L} + e^{2\theta} - e^{-a+\frac{u_R}{2}-\theta}}}.$$

Because $\left(\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}} - \sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}\right)\left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) >$ 0 and $\left(\frac{\left(e^{a-\frac{u_R}{2}}+e^{\theta}\right)^2}{\sqrt{e^{2a-u_R}+e^{2\theta}-e^{a+\frac{u_L}{2}-\theta}}} - \frac{\left(e^{-a-\frac{u_L}{2}}+e^{\theta}\right)^2}{\sqrt{e^{-2a-u_L}+e^{2\theta}-e^{-a+\frac{u_R}{2}-\theta}}}\right)\left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) >$ 0,

$$\hat{a}(a)\left(e^{a-\frac{u_R}{2}} - e^{-a-\frac{u_L}{2}}\right) > 0.$$

So $\hat{a}(0)(-u_R + u_L) > 0$. ■

Observation If $u_R + u_L < \theta < -\frac{u_R+u_L}{4}$, then

$$|\hat{a}(a)| < \frac{1+\alpha}{2}\left|\log \frac{e^{\theta} + e^{a-\frac{1}{2}u_R} + \sqrt{e^{2\theta} + e^{2a-u_R} - e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta} + e^{-a-\frac{1}{2}u_L} + \sqrt{e^{2\theta} + e^{-2a-u_L} - e^{-\theta-a+\frac{1}{2}u_R}}}\right|.$$

Proof. If $\theta > u_R + u_L$, then $\log \frac{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}$ has the same sign as $\log \frac{e^{\theta}+e^{a-\frac{1}{2}u_R}}{e^{\theta}+e^{-a-\frac{1}{2}u_L}}$ and hence $\log \frac{e^{\theta}+e^{a-\frac{1}{2}u_R}+\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta}+e^{-a-\frac{1}{2}u_L}+\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}$. If $\theta < -\frac{u_R+u_L}{4}$, then $\left|\log \frac{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right| > \log \frac{e^{\theta}+e^{a-\frac{1}{2}u_R}}{e^{\theta}+e^{-a-\frac{1}{2}u_L}}$, so $\left|\log \frac{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right| >$ $\left|\log \frac{e^{\theta}+e^{a-\frac{1}{2}u_R}+\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta}+e^{-a-\frac{1}{2}u_L}+\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right|$. We have

$$\begin{aligned}
&\hat{a}(a) \\
= \;& \frac{1+\alpha}{2}\log \frac{e^{\theta} + e^{a-\frac{1}{2}u_R} + \sqrt{e^{2\theta} + e^{2a-u_R} - e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta} + e^{-a-\frac{1}{2}u_L} + \sqrt{e^{2\theta} + e^{-2a-u_L} - e^{-\theta-a+\frac{1}{2}u_R}}} \\
&+\frac{1}{2}\left(\log \frac{e^{\theta} + e^{a-\frac{1}{2}u_R} + \sqrt{e^{2\theta} + e^{2a-u_R} - e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta} + e^{-a-\frac{1}{2}u_L} + \sqrt{e^{2\theta} + e^{-2a-u_L} - e^{-\theta-a+\frac{1}{2}u_R}}} - \log \frac{\sqrt{e^{2\theta} + e^{2a-u_R} - e^{-\theta+a+\frac{1}{2}u_L}}}{\sqrt{e^{2\theta} + e^{-2a-u_L} - e^{-\theta-a+\frac{1}{2}u_R}}}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
|\hat{a}(a)| &= \frac{1+\alpha}{2}\left|\log\frac{e^{\theta}+e^{a-\frac{1}{2}u_R}+\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta}+e^{-a-\frac{1}{2}u_L}+\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right| \\
&\quad +\frac{1}{2}\left(\begin{array}{c}\left|\log\frac{e^{\theta}+e^{a-\frac{1}{2}u_R}+\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta}+e^{-a-\frac{1}{2}u_L}+\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right| \\ -\left|\log\frac{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right|\end{array}\right) \\
&< \frac{1+\alpha}{2}\left|\log\frac{e^{\theta}+e^{a-\frac{1}{2}u_R}+\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta}+e^{-a-\frac{1}{2}u_L}+\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right|.
\end{aligned}$$

■

Observation $a^{**}\notin\left(0,\frac{u_R-u_L}{4}\right)$ if $u_R-u_L>0$ and $a^{**}\notin\left(\frac{u_R-u_L}{4},0\right)$ if $u_R-u_L<0$.

Proof. Because $\hat{a}(a)\left(a-\frac{u_R-u_L}{4}\right)>0$, If $0<a<\frac{u_R-u_L}{4}$, then $\hat{a}(a)<0<a$, so a cannot be a fixed point. Otherwise $-\frac{u_R-u_L}{4}<a<0$, but then $\hat{a}(a)>0>a$. ■

Lemma 6.6 *$\hat{a}'(a^{**})\in\left(0,\frac{3}{4}(1+\alpha)\right)$ if $\theta<u_R+u_L$ and either*

1. *$\theta>-\frac{u_R+u_L}{4}$, or*
2. *$\theta>-\frac{\max\{u_R,u_L\}}{2}+\log 2$.*

Proof.

$$\begin{aligned}
&\hat{a}'(a;u_R,u_L) \\
=\;& \frac{1}{2}+\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\left(1-\frac{\frac{e^{\theta}+e^{a-\frac{u_R}{2}}}{2}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\right) \\
&+\frac{1}{2}\frac{e^{-a-\frac{u_L}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\left(1-\frac{\frac{e^{\theta}+e^{-a-\frac{u_L}{2}}}{2}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right) \\
&+\frac{\alpha}{2}\left(1+\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}+\frac{1}{2}\frac{e^{-a-\frac{u_L}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right) \\
>\;& 0
\end{aligned}$$

because $1-\frac{\frac{e^{\theta}+e^{a-\frac{u_R}{2}}}{2}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\in\left(0,\frac{1}{2}\right)$ and $\frac{e^{a-\frac{u_R}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\in(-1,1)$ and $\left(1-\frac{\frac{e^{\theta}+e^{-a-\frac{u_L}{2}}}{2}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\right)\in\left(0,\frac{1}{2}\right)$ and $\frac{e^{-a-\frac{u_L}{2}}-e^{\theta}}{\sqrt{e^{2\theta}+e^{-2a-u_L}-e^{-\theta-a+\frac{1}{2}u_R}}}\in(-1,1)$.

If either $a-\frac{u_R}{2}<\theta$ or $-a-\frac{u_L}{2}<\theta$, then $\hat{a}'(a;u_R,u_L)<\frac{1}{2}+\frac{1}{4}+\frac{\alpha}{2}\left(1+\frac{1}{2}\right)=\frac{3}{4}(1+\alpha)$.

Case 1 $\theta > -\frac{u_R+u_L}{4}$

Proof. Then $a - \frac{u_R}{2} - \theta + \left(-a - \frac{u_L}{2} - \theta\right) < 0$, so either $a - \frac{u_R}{2} < \theta$ or $-a - \frac{u_L}{2} < \theta$ and $\hat{a}'(a) < \frac{3}{4}(1+\alpha)$. ■

Case 2 $\theta < -\frac{u_R+u_L}{4}$.

Proof. If $a \notin \left(\theta + \frac{u_R}{2}, -\theta - \frac{u_L}{2}\right)$, then either $a - \frac{u_R}{2} < \theta$ or $-a - \frac{u_L}{2} < \theta$. Therefore, the statement does not hold only if $a^{**} \in \left(\theta + \frac{u_R}{2}, -\theta - \frac{u_L}{2}\right)$. Because $\theta \in \left(u_R + u_L, -\frac{u_R+u_L}{4}\right)$,

$$\begin{aligned}
|\hat{a}(a)| &< \frac{1+\alpha}{2}\left|\log \frac{e^{\theta} + e^{a-\frac{1}{2}u_R} + \sqrt{e^{2\theta} + e^{2a-u_R} - e^{-\theta+a+\frac{1}{2}u_L}}}{e^{\theta} + e^{-a-\frac{1}{2}u_L} + \sqrt{e^{2\theta} + e^{-2a-u_L} - e^{-\theta-a+\frac{1}{2}u_R}}}\right| \\
&< \frac{1+\alpha}{2}\left(\log 2 + \left|\log e^{2a-\frac{u_R-u_L}{2}}\right|\right)
\end{aligned}$$

because $\sqrt{e^{2\theta} + e^{2a-u_R} - e^{-\theta+a+\frac{1}{2}u_L}} \in \left(0, e^{\theta} + e^{a-\frac{1}{2}u_R}\right)$ and $\sqrt{e^{2\theta} + e^{-2a-u_L} - e^{-\theta-a+\frac{1}{2}u_R}} \in \left(0, e^{\theta} + e^{-a-\frac{1}{2}u_L}\right)$. Consider $u_R > u_L$. By assumption, $\theta + \frac{u_R}{2} > 0$. Because $a^{**} \notin \left(0, \frac{u_R-u_L}{4}\right)$, if $a^{**} \in \left(\theta + \frac{u_R}{2}, -\theta - \frac{u_L}{2}\right)$ then $a^{**} > \frac{u_R-u_L}{4}$. Because $\hat{a}(a)\left(a - \frac{u_R-u_L}{4}\right) > 0$,

$$\begin{aligned}
&\hat{a}(a) \\
<\ & \frac{1+\alpha}{2}\left(\log 2 + 2a - \frac{u_R-u_L}{2}\right) \\
=\ & \frac{1+\alpha}{2}\left(\log 2 - \frac{u_R-u_L}{2}\right) + (1+\alpha)\,a
\end{aligned}$$

if $a \in \left(\frac{u_R-u_L}{4}, -\theta - \frac{u_L}{2}\right)$. If a fixed point a^{**} exists in $\left(\frac{u_R-u_L}{4}, -\theta - \frac{u_L}{2}\right)$, then

$$\begin{aligned}
a^{**} &= \hat{a}(a^{**}) \\
&< \frac{1+\alpha}{2}\left(\log 2 - \frac{u_R-u_L}{2}\right) + (1+\alpha)\,a^{**},
\end{aligned}$$

so

$$\begin{aligned}
a^{**} &> \left(1 + \frac{1}{\alpha}\right)\frac{1}{2}\left(\frac{u_R-u_L}{2} - \log 2\right) \\
&\geq \frac{u_R-u_L}{2} - \log 2 \text{ (because } \alpha \leq 1) \\
&= \frac{u_R}{2} - \log 2 - \frac{u_L}{2} \\
&> -\theta - \frac{u_L}{2} \text{ (because we assume that } \theta > -\frac{\max\{u_R, u_L\}}{2} + \log 2),
\end{aligned}$$

contradiction to the hypothesis that $a^{**} \in \left(\frac{u_R-u_L}{4}, -\theta - \frac{u_L}{2}\right)$. The case where $u_R < u_L$ is analogous. ■ ■

Lemma 6.7 *$a^{**}(u_R - u_L) < 0$ if $\theta < u_R + u_L$, $\alpha < \frac{1}{4}$ and either*

1. *$\theta > -\frac{u_R+u_L}{4}$, or*
2. *$\theta > -\frac{\max\{u_R,u_L\}}{2} + \log 2$.*

Proof. This follows because $\hat{a}'(a^{**}) < \frac{3}{4}(1+\alpha) < 1$ and $\hat{a}(0)\left(-\frac{u_R-u_L}{2}\right) > 0$. ∎

6.3 Proofs for Proposition 2

Lemma 6.8 *$\frac{\partial \eta_R(a;u_r,u_l)}{\partial a} \in (0,1)$ if $\theta > \frac{u_R+u_L}{4}$*

Lemma 6.9 *$\frac{\partial \eta_R(a;\tilde{u}_R,\tilde{u}_L)}{\partial \tilde{u}_R} < 0$ if $\theta > \frac{u_R+u_L}{4}$*

Proof.

$$\begin{aligned}
\frac{\partial \eta_R}{\partial u_R} &= -\frac{2\frac{\partial p_R(\eta_R;a)}{\partial u_R} + \frac{\partial p_L(\eta_R;a)}{\partial u_R}}{2p_R'(\eta_R) + p_L'(\eta_R)} \\
&= -\frac{2\left(F(\eta_R-\theta)\left(1-F\left(\eta_R - a + \frac{1}{2}\tilde{u}_R\right)\right)\right)\frac{1}{2}}{1-e^{-\eta_R+a-\frac{\tilde{u}_R}{2}} + 1 - e^{-(\eta_R-\theta)}} \\
&\quad \text{(this shows that it is negative)} \\
&= -\frac{\left(1-\frac{1}{2}e^{-\eta_R+\theta}\right)\frac{1}{2}e^{-\eta_R+a-\frac{u_R}{2}}}{1-e^{-\eta_R+a-\frac{\tilde{u}_R}{2}} + 1 - e^{-(\eta_R-\theta)}} \\
&= -\frac{\frac{1}{2}e^{a-\frac{u_R}{2}}\left(e^{a-\frac{u_R}{2}} + \sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}\right)}{e^{\eta_R}\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}} \\
&= -\frac{e^{a-\frac{u_R}{2}}}{e^{\theta}+e^{a-\frac{u_R}{2}}+\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\left(1+\frac{e^{a-\frac{u_R}{2}}}{\sqrt{e^{2\theta}+e^{2a-u_R}-e^{-\theta+a+\frac{1}{2}u_L}}}\right)
\end{aligned}$$

∎

Lemma 6.10 *$\frac{\partial \hat{a}(a;u_R,u_L)}{\partial \tilde{u}_R} < 0$ if $\theta > \frac{u_R+u_L}{4}$*

Proof.

$$
\begin{aligned}
\frac{\partial\alpha\left(a;u_R,u_L,\theta\right)}{\partial u_R} &= -\frac{1}{2}\left(1+\alpha\right)\frac{e^{a-\frac{u_R}{2}}}{e^{\eta_R}} \\
&\quad +\frac{1}{4}\frac{e^{2a-u_R}}{\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}}\left(\frac{1}{\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}}-\frac{1+\alpha}{e^{\eta_R}}\right) \\
&\quad -\frac{1}{4}\frac{\frac{1}{2}e^{-\theta-a+\frac{1}{2}u_R}}{\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}}\left(\frac{1}{\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}}-\frac{1+\alpha}{e^{-\eta_L}}\right) \\
&= -\frac{1}{2}\left(1+\alpha\right)\frac{e^{a-\frac{u_R}{2}}}{e^{\eta_R}} \\
&\quad +\frac{1}{4}\frac{e^{a-\frac{u_R}{2}}}{e^{\eta_R}}\frac{e^{a-\frac{u_R}{2}}\left(\begin{array}{c}\frac{1}{2}e^{\theta}+\frac{1}{2}e^{a-\frac{u_R}{2}} \\ -\frac{1}{2}\left(1+2\alpha\right)\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}\end{array}\right)}{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}} \\
&\quad -\frac{1}{4}\frac{\frac{1}{2}e^{-\theta-a+\frac{1}{2}u_R}}{\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}}\left(\frac{1}{\sqrt{e^{2\theta}+e^{-2a-\tilde{u}_L}-e^{-a-\theta+\frac{\tilde{u}_R}{2}}}}-\frac{1+\alpha}{e^{-\eta_L}}\right).
\end{aligned}
$$

So $\frac{\partial\alpha(a;u_R,u_L,\theta)}{\partial u_R}<0$ if

$$
\begin{aligned}
& e^{a-\frac{u_R}{2}}\left(e^{\theta}+e^{a-\frac{u_R}{2}}-\left(1+2\alpha\right)\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}\right) \\
<\; & 4\left(e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right).
\end{aligned}
$$

If $2\theta>\frac{u_R+u_L}{2}$, then

$$
\begin{aligned}
& e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}} \\
=\; & \left(e^{\theta}-e^{a-\frac{u_R}{2}}\right)^2+2e^{\theta+a-\frac{u_R}{2}}-e^{a-\theta+\frac{u_L}{2}} \\
\geq\; & \left(e^{\theta}-e^{a-\frac{u_R}{2}}\right)^2+e^{\theta+a-\frac{u_R}{2}}.
\end{aligned}
$$

Then

$$
\begin{aligned}
& e^{a-\frac{u_R}{2}}\left(e^{\theta}+e^{a-\frac{u_R}{2}}-\left(1+2\alpha\right)\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}\right) \\
\leq\; & e^{a-\frac{u_R}{2}}\left(e^{\theta}+e^{a-\frac{u_R}{2}}-\left|e^{\theta}-e^{a-\frac{u_R}{2}}\right|\right) \\
=\; & e^{a-\frac{u_R}{2}}2*\min\left\{e^{\theta},e^{a-\frac{u_R}{2}}\right\} \\
\leq\; & 2e^{\theta+a-\frac{u_R}{2}} \\
\leq\; & 2\left(\left(e^{\theta}-e^{a-\frac{u_R}{2}}\right)^2+e^{\theta+a-\frac{u_R}{2}}\right) \\
\leq\; & 4\left(e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right).
\end{aligned}
$$

So $2\theta > \frac{u_R+u_L}{2}$ is sufficient for $\alpha'(a) \in (0,1)$ for all a and $\frac{\partial\alpha(a;u_R,u_L,\theta)}{\partial u_R} < 0$.

In fact, because

$$\begin{aligned} & e^{a-\frac{u_R}{2}}\left(e^{\theta}+e^{a-\frac{u_R}{2}}-(1+2\alpha)\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}\right) \\ < \quad & 2\left(\left(e^{\theta}-e^{a-\frac{u_R}{2}}\right)^2+e^{\theta+a-\frac{u_R}{2}}\right) \end{aligned}$$

we get

$$\begin{aligned} \frac{\partial\alpha(a;u_R,u_L,\theta)}{\partial u_R} &< -\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}}{e^{\eta_R}}\left(1+\alpha-\frac{e^{a-\frac{u_R}{2}}\left(e^{\theta}+e^{a-\frac{u_R}{2}}-(1+2\alpha)\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}\right)}{4\left(e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right)}\right) \\ &< -\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}}{e^{\eta_R}}\frac{2\left(e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right)+2\left(e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right)-2\left(\left(e^{\theta}-e^{a-\frac{u_R}{2}}\right)^2+e^{\theta+a-\frac{u_R}{2}}\right)}{4\left(e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right)} \\ &= -\frac{1}{2}\frac{e^{a-\frac{u_R}{2}}}{e^{\eta_R}}\left(\frac{1}{2}+\frac{2\left(e^{\theta+a-\frac{u_R}{2}}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right)}{4\left(e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}\right)}\right) \\ &< -\frac{1}{4}\frac{e^{a-\frac{u_R}{2}}}{e^{\eta_R}} \end{aligned}$$

■

$$
\begin{aligned}
\frac{\partial \eta_R^{**}}{\partial u_L} &= \frac{\partial \eta_R(a;u_R,u_L)}{\partial a}\frac{\partial a^{**}}{\partial u_L} + \frac{\partial \eta_R(a;u_R,u_L)}{\partial u_L} \\
&= \frac{\partial \eta_R(a;u_R,u_L)}{\partial a}\frac{\frac{\partial \hat{a}(a;u_R,u_L)}{\partial u_L}}{1-\hat{a}'(a^{**})} + \frac{\partial \eta_R(a;u_R,u_L)}{\partial u_L} \\
&> \frac{1}{4}\frac{1}{e^{\eta_R^{**}}}\frac{1}{2\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}}\frac{1}{e^{-\eta_L}\left(1-\alpha'(a^{**})\right)} \\
&\quad \times \left[\begin{array}{c} \left(\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}e^{a-\frac{u_R}{2}}+e^{2a-u_R}-\frac{1}{2}e^{-\theta+a+\frac{u_L}{2}}\right)e^{-a-\frac{u_L}{2}} \\ -\left(1-\alpha'(a^{**})\right)e^{-\eta_L}e^{-\theta+a+\frac{1}{2}u_L} \end{array}\right] \\
&\propto \left[\begin{array}{c} \left(\sqrt{e^{2\theta}+e^{2a-\tilde{u}_R}-e^{a-\theta+\frac{\tilde{u}_L}{2}}}e^{a-\frac{u_R}{2}}+e^{2a-u_R}-\frac{1}{2}e^{-\theta+a+\frac{u_L}{2}}\right)e^{-a-\frac{u_L}{2}} \\ -\left(1-\alpha'(a^{**})\right)e^{-\eta_L}e^{-\theta+a+\frac{1}{2}u_L} \end{array}\right] \\
&> \frac{1}{2}e^{\theta-\frac{u_R+u_L}{2}}+\frac{3}{2}e^{a-u_R-\frac{u_L}{2}}-\frac{1}{2}e^{-\theta} \\
&\quad -\left(e^{\theta}+e^{-a-\frac{1}{2}u_L}\right)e^{-\theta+a+\frac{1}{2}u_L} \\
&> \frac{1}{2}e^{\theta-\frac{u_R+u_L}{2}}-\left(e^{-\theta}+e^{a+\frac{u_L}{2}}\right)+\frac{3}{2}e^{a-u_R-\frac{u_L}{2}}-\frac{1}{2}e^{-\theta} \\
&= \frac{1}{2}e^{\theta-\frac{u_R+u_L}{2}}-\frac{3}{2}e^{-\theta}+e^{a^{**}+\frac{u_L}{2}}\left(\frac{3}{2}e^{-(u_R+u_L)}-1\right).
\end{aligned}
$$

Therefore, $\frac{\partial \eta_R^{**}}{\partial u_L} > 0$ if

1. $u_R + u_L < \log\frac{3}{2}$ and $2\theta - \frac{u_R+u_L}{2} > \log 3$, or
2. $\frac{1}{2}e^{\theta-\frac{u_R+u_L}{2}} - \frac{3}{2}e^{-\theta} + e^{\frac{u_L}{2}}\left(\frac{3}{2}e^{-(u_R+u_L)}-1\right) > 0$ and $u_R > u_L$ because in that case, $a^{**} < 0$.

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