Optimal Deadlines for Agreements

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Abstract. Costly delay in negotiations can induce the negotiating parties to be more forthcoming with their information and improve the quality of the collective decision. Imposing a deadline may result in stalling, in which players at some point stop making concessions but switch back to conceding at the end, or a deadlock, in which concessions end permanently. Extending the deadline hurts the players in the first case but is beneficial in the second. When the initial conflict between the negotiating parties is intermediate, the optimal deadline is positive and finite, and is characterized by the shortest time that would allow efficient information aggregation in equilibrium.

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1. Introduction

When disagreements are resolved through negotiations, the time horizon of the negotiation process may influence the final outcome. In the classical finite-horizon, alternating-offer bargaining game of Stahl (1972), deadlines affect the way players make and accept bargaining demands through the logic of backward induction, even though the deadlines are never reached in equilibrium. In war of attrition games (e.g., Hendricks, Weiss and Wilson 1988), conflicts are gradually resolved with the passage of time. The presence of a deadline not only affects equilibrium behavior along the path, but can also determine the equilibrium outcome by imposing a default decision upon the arrival of the deadline. In both bargaining and war of attrition models, the negotiating parties disagree because they have opposing preferences over the outcome. In such situation of pure conflict, negotiation may determine the distribution of payoffs between the parties but not their sum. Thus protracted negotiation is invariably wasteful, as it introduces costly delay without any benefits. However, when disagreement is driven by different private information, and could be overcome after information-sharing, protracted negotiation can have positive welfare consequences by facilitating information aggregation. This paper studies the welfare effects of negotiation deadlines in an environment where the negotiating parties disagree both because of diverging preferences and because of different information, and characterizes the deadline that optimally balances the cost of strategic delay and the benefit of strategic information aggregation.

More specifically, our model of negotiation under a deadline has two central aspects. First, the underlying collective decision problem involves two proposed alternatives that have both a common value component and a private benefit component. Although the two sides can in principle reach a Pareto-preferred decision when the common value component dominates the private benefits, they each have private information about the value of their own proposed alternative. The presence of private information makes it difficult to separate narrow self-interests from the common interest. Not being sufficiently convinced that the opponent's proposal has a high common value, each side may want to push its own proposal for the private benefits despite knowing it has a low common value. At the same time, a seemingly self-serving alternative may be proposed by one side who knows that the alternative is good for both, but the question is how to convince the opponent when such private knowledge is unverifiable. The second aspect of our model is that the two sides commit to engaging each other repeatedly in reaching an agreement. The collective decision-making procedure does not allow side transfers, which might result in a failure to share private information if the decision needs to be made without delay. But delaying the decision is costly to both sides. The cost of delay can discourage them from exaggerating the value of their own proposals, and generate endogenous information that in equilibrium helps improve the quality of the collective decision.

The following examples illustrate a few negotiation problems that fit our theoretical framework.

Standard adoption. In an emerging industry two dominant firms try to establish a common standard or protocol. Both firms have an interest in adopting the standard which is technologically more versatile and efficient. At the same time, because of its head-start in development, each firm can obtain additional private benefits if its own standard is adopted as the common industry standard. Even though written documents of the proposed standards are shared in the negotiating stage, tacit knowledge about the strengths and weaknesses of a protocol obtained from the developmental stage is difficult to convey and easy to hide. Settling the issue through side transfers may not be a practical solution in a fast-changing industry. At the same time, delaying in adopting a common standard is costly to both firms regardless of the ultimate decision. Instead of an open-ended negotiation, the two firms may have an interest in imposing a binding deadline.

Workplace safety. Labor-management disputes are sometimes purely about private benefits, but there are also circumstances where the two sides would agree if only they could share their private information. A labor union may be knowledgeable about the presence of workplace hazards that would justify hiring additional workers in order to prevent major accidents that would hurt both workers and the firm. Management may be aware that market conditions and the firm's financial position require downsizing the company in order to prevent bankruptcy. Each side faces the problem of convincing the other side that its claim is not motivated by narrow self-interest of expanding the workforce or cutting costs, because the relevant information is not readily verifiable. Monetary transfers based on the decision may not be practical in such situation, and negotiation is costly to both the labor union and the management.

Recruiting. When deciding on departmental hires, recruiting committee members must often balance their personal research interest, which naturally biases them toward hiring candidate in their own field, with the value added to the department as a whole from hiring the candidate with the highest research potential. Each member might be willing to go along with a candidate in a field other than his own if the candidate has a high research productivity potential, but prefers one in his own field given two candidates with the same potential. The relative lacking of expertise in other committee members' field may make each member suspicious of the other's supposedly more informed assessments. Repeated recruiting committee meetings are costly, not just because they take valuable time from the members, but because delay in making a decision may lead to lost hiring opportunities. However, it is precisely this cost that may yield a better hiring decision than one made without delay.

Formally, we model negotiation under a deadline as a symmetric, continuous time, two-player war-of-attrition game.[1] There are two alternatives; each consists of a common value component, which represents its quality and is shared by both players, and a private value component, which benefits only one player. At any instant each player simultaneously chooses to persist with his favorite alternative, from which he alone draws the private benefit, or to concede to his rival for the latter's favorite alternative. The two players pay a flow cost of delay, until either they agree, at which point the agreement is implemented, or the deadline expires and a random decision is made. Each player is privately informed of whether the quality of his favorite alternative is high or low, but is unsure about the quality of his opponent's favorite alternative. We assume that the quality difference is greater than the private benefit, so when a high-quality type plays against a low-quality type, the two players would agree to adopt the high quality alternative if they could share their information. However, when two low-quality types play against each other, they

[1] See Farrell (1996) for a related model of standard adoption, with a continuous type space. He does not consider deadlines. In a follow-up paper, Farrell and Simcoe (2007) introduce welfare analysis of imposing a deadline. However, their analysis is restricted to the case that the presence of a deadline does not affect the equilibrium behavior.

would disagree even if they knew the true state due to the private benefit of choosing their own favorite.[2] The possibility of agreements is essential for deadlines to have interesting welfare effects, and the possibility of disagreements makes information sharing costly to achieve.

We show that generically there is a unique equilibrium in which the high-quality types always persist with their favorite alternative throughout the game. The low quality type's behavior depends on the time left before the expiration of the deadline and on his belief that the rival's favorite alternative also has low quality. If the time to deadline exceeds a certain critical horizon, which depends on the current belief, the low quality type concedes to the opponent's favorite alternative at some probability flow rate. This continuous-time version of randomization between conceding and persisting results because the deadline is too long for the low-quality type to persist all the way, but at the same time conceding with a strictly positive probability would give the opposing low-quality type incentives to persist just a little longer and reap the private benefit. Since the high-quality types always persist, in this concession phase of the game the Pareto-efficient agreement is reached with a positive probability. As the negotiation game continues during the concession phase, the low-quality type becomes less sure that his opponent also has a low quality alternative because, given the equilibrium strategies, his opponent's failure to concede is taken as evidence to the opposite. When the time to deadline reaches the critical horizon, the game enters a persistence phase in which the low types stop randomizing and persist until the deadline is reached. Interestingly, at the moment of the arrival of the deadline, the behavior of the low-quality types may change again. If they enter the persistence phase with a relatively high belief that their opponent also has a low quality alternative, they will keep persisting to the very end. This case may be interpreted as a deadlock. If their belief is low, however, they will switch to conceding just before the deadline expires. In this case, one can interpret the behavior of the players during the persistence phase as a stalling tactic.

[2] There would also be disagreement when two high-quality types meet each other. This possibility is assumed away in our model for simplicity.

Extending the deadline hurts both the high-quality and low-quality types if the starting point is shorter than the critical time horizon corresponding to the initial belief: it increases the delay without changing the equilibrium play when the deadline arrives. On the other hand, starting from any deadline beyond the critical time horizon, an extension does not change the welfare of the low-quality types, whose equilibrium payoff is pinned down by the payoff from concession and does not vary with the length of the deadline, but generally affects the welfare of the high-quality types. It turns out that extending the deadline is beneficial in the case of deadlock but is harmful in the case of stalling. By prolonging the concession phase of the negotiation, extending the deadline increases the chances that the high-quality type gets his favored decision at the cost of longer delay. In the case of a deadlock, such improvement in decision-making during the concession phase is relatively important because players have no chance to reach an agreement once the game enters the persistence phase. In case of stalling, on the other hand, players will eventually reach an agreement when the deadline expires. Therefore allowing more time for concession at the beginning of the game is relatively important. Besides deadlock and stalling, there is a third possibility in which low-quality types concede with a probability between zero and one when the deadline expires. We show that extending the deadline is also beneficial in this case. The contrasting marginal effects of lengthening the deadline for these different cases allow us to pin down the optimal deadline.

We provide a complete characterization of the optimal deadline that maximizes the ex ante payoffs to the players before they know their types. Naturally, the optimal deadline is zero when the low-quality types initially hold a sufficiently low belief that the rival also has low quality alternative, as the two players can reach the Pareto-efficient decision without delay. For intermediate initial beliefs, the optimal deadline is such that after the shortest concession phase the low-quality types persist until the deadline and then concedes with probability one. Thus, the optimal deadline is the shortest time length that achieves efficient information aggregation in equilibrium. This deadline effectively balances the trade-off between two conflicting goals—to avoid wasteful delay when disagreements are of fundamental nature, and to allow the players sufficient time to successfully reconcile disagreements driven by different information. When positive, the optimal deadline is necessarily finite, because given that the low-quality types concede with probability one at the deadline, extending it further would only hurt the high-quality types by unnecessarily prolonging the concession phase. Further, it cannot be arbitrarily short. Otherwise, the low-quality types would simply persist until the deadline and waste the delay cost. Finally, when positive, the optimal deadline is increasing in the low-types' initial belief that their rival also has a low quality alternative, because it takes longer to drive their belief down to a level at which they will be willing to concede upon the deadline. When the low-quality types have a sufficiently high belief, the optimal deadline is again zero. The positive welfare effects from information aggregation, obtained by extending the deadline beyond the critical horizon, are not sufficient to compensate the large payoff loss associated with the long deadline play.

The idea that endogenous delay can help separate one type from another type in bargaining with asymmetric information is not new (e.g., Admati and Perry 1987; Cramton 1992). We carry this idea further by studying how imposing negotiation deadlines may affect equilibrium behavior and outcome. Moreover, since the decision to be made has a common value component, there is a non-trivial welfare analysis of the trade-off between longer delay and better information sharing. This trade-off is the basis of our analysis of optimal deadlines.

There is a sizable theoretical literature on war of attrition and bargaining games concerning the "deadline effect," the idea that players make no attempt at reaching an agreement just before the deadline, but when the deadline arrives there are sudden attempts to resolve their differences.[3] Hendricks, Weiss and Wilson (1988) characterize mixed-strategy Nash equilibria of a continuous time, complete information war of attrition game, in which there is a mass point of concession at the deadline and no concession in a time interval preceding it. Spier (1992) shows that in pretrial negotiations with incomplete information, the settlement probability is U-shaped. Ma and Manove (1993) find strategic delay in bargaining games with complete information by assuming that there may be exogenous, random

[3] See also Roth, Murnighan and Shoumaker (1988) for an experimental investigation of eleventh hour agreements in bargaining. In the auction literature, "sniping" refers to bidding just before the auction closes. This has been analyzed by Roth and Ockenfels (2002).

delay in offer transmission. As early offers are rejected and the deadline approaches, there is an increasing risk of missing the deadline and negotiation activities pick up. Also in a bargaining game with complete information, Fershtman and Seidmann (1993) introduce the assumption that, by rejecting an offer, players commit to not accepting poorer offers in the future. They show that when players are sufficiently patient, there is a unique subgame perfect equilibrium in which players wait until the deadline to reach an agreement. Ponsati (1995) studies a war of attrition game in which each player has private information about his payoff loss incurred by conceding to the opponent and must choose the timing of concession. She shows that there is a unique pure strategy equilibrium in which both players never concede before the deadline is reached if their payoff losses are sufficiently large. Sandholm and Vulkan (1999) consider a bargaining game in which two players make offers continuously and an agreement is reached as soon as the offers are compatible with each other. The only private information a player has is the deadline he faces. They show that the only equilibrium is each player persisting by demanding the whole pie until the deadline and then switching to concede everything to his opponent. Finally, Yildiz (2004) shows that when players in a bargaining game are overly optimistic about their bargaining power at the deadline, it is an equilibrium to persist until close to the deadline to reach an agreement. However, when there is uncertainty about when the deadline arrives, the deadline effect disappears. Broadly consistent with the above papers, we offer a theory of the deadline effect in which there may be an eleventh-hour attempt at concession to reach an agreement before the deadline expires. But in addition to such stalling behavior, our model also allow for the possibility that deadlines may induce deadlock, in which disagreements persist through the end. More importantly, because our theory is based on asymmetric information about common values, we are also able to provide a welfare analysis of the optimal deadline.

2. A Repeated Proposal Game

We consider a symmetric model in which two players have to make a joint choice between two alternatives. Each alternative has a common value component that produces either a low value v_L or a high value v_H to both players, with $v_H > v_L > 0$. Regardless of its common value, each alternative also has a private value component which yields a benefit $\beta > 0$ to only one of the players.[4] We refer to a player's "favorite" alternative as the one that would give him the private benefit β. That is, the payoff to each player from implementing his favorite alternative is equal to its common value plus β, and the payoff from implementing his opponent's favorite alternative is just the common value of that alternative. To make our model interesting, we maintain the following assumption throughout this paper.

Assumption 1. $v_H - v_L > \beta$.

Each player is privately informed only about whether the common value of his own favorite alternative is high or low, referred to as "high type" and "low type" correspondingly. We assume that at most one of the two alternatives can be of high common value. Thus there are two symmetric "agreement states" and one "disagreement state:" in each agreement state, one player is high type and other is low type, so by Assumption 1 the two players would agree on the former's favorite alternative if they knew the state; in the disagreement state, both players are low type, so they would continue to disagree even if they knew the state.[5] That is, if a player is a high type, he knows that his opponent is a low type and it is an agreement state in which his favorite alternative should be implemented; and if he is a low type, he is unsure whether it is an agreement state for his opponent's favorite alternative or it is a disagreement state. Let $\gamma_0 < 1$ be the common belief of the low types that it is the disagreement state; we assume that it is common knowledge. The implied prior probability of the disagreement state is then $\gamma_0/(2-\gamma_0)$, and the prior probability of each agreement state is $(1-\gamma_0)/(2-\gamma_0)$.

The repeated proposal game is modeled in continuous time, running from $t = 0$ to the deadline T. We allow T to be infinite. The two players simultaneously propose one

[4] The assumption that the private benefit is independent of the common value is made purely for notational convenience. In the more general case, Assumption 1 below restricts only the private benefit to each player from implementing his favorite alternative when its common value is low. All our results hold without change so long as the private benefit is non-negative when the common value is high.

[5] We assume that there is no fourth state in which both alternatives have high common value. Allowing for such a possibility would not greatly change the equilibrium analysis of the model but would lower the advantages from using delay as a collective decision-making mechanism in the welfare analysis, because delay would be wasteful when two high types play against each another.

of the alternatives at each instant t, until the game ends. The game may end before the deadline if the two proposals by the two players agree, in which case the agreed alternative is implemented immediately. If instead the deadline T is reached, the game ends with the decision made by a fair coin flip. Until the game ends, each player incurs an additive payoff loss due to delay at a flow rate of δ.

The essential feature captured in the above configuration of preference and information structures, together with Assumption 1, is that players in a negotiation disagree over the joint decision based on their private information but might agree if their information were public. In particular, based on his own initial private information a low type player strictly prefers his favorite choice if

$$\gamma_0 > \gamma_* \equiv \frac{v_H - v_L - \beta}{v_H - v_L},$$

although it may be the agreement state for his opponent's favorite alternative. Note that by Assumption 1, γ_* is strictly between 0 and 1. An initial belief γ_0 higher than γ_* that it is the disagreement state means that there is a great degree of conflict between the two players. Another important feature of our model is that the high types have greater incentives to insist on their favorite alternative than do the low types. This is because the payoff gain for each player from implementing his favorite alternative over his opponent's favorite is larger in the corresponding agreement state (equal to $v_H - v_L + \beta$) than in the disagreement state (equal to β). This feature is helpful for equilibrium construction as it allows us to focus on the incentives of the low types.

Our modeling of the deadline amounts to specifying state-contingent default payoffs if the last attempt at an agreement fails. To see this, note that when $T = 0$ our model reduces to a static game in which each player can propose either alternative, and the outcome is that an agreement is implemented immediately and a disagreement results in a decision made by a coin flip. When the belief γ of the low types that it is the disagreement state is strictly higher than γ_*, this game has a unique equilibrium with each player proposing his favorite alternative. The equilibrium outcome is a coin flip, as the degree of conflict is too large to allow any information sharing.[6] For any belief of the low types $\gamma < \gamma_*$, there

[6] There is no mechanism that Pareto-improves on this outcome. More precisely, for any $\gamma > \gamma_*$, in any incentive compatible outcome of a direct mechanism without transfers the probability of implementing a

is a unique equilibrium in which the high types propose their own favorite and the low types propose the favorite alternative of their opponent. At $\gamma = \gamma_*$, there is a continuum of equilibria, in which the high types always propose their favorite while the low types propose their favorite with a probability between zero and one. Denoting as $U_L^0(\gamma)$ and $U_H^0(\gamma)$ the equilibrium payoffs of the low and high types respectively, we have

$$U_L^0(\gamma) = \begin{cases} \gamma(v_L + \beta/2) + (1-\gamma)v_H & \text{if } \gamma \in [0, \gamma_*) \\ \gamma(v_L + \beta/2) + (1-\gamma)(v_H + v_L + \beta)/2 & \text{if } \gamma \in (\gamma_*, 1] \end{cases} \tag{1}$$

with $U_L^0(\gamma_*) \in [\gamma_*(v_L + \beta/2) + (1-\gamma_*)(v_H + v_L + \beta)/2, \gamma_*(v_L + \beta/2) + (1-\gamma_*)v_H]$; and

$$U_H^0(\gamma) \begin{cases} = v_H + \beta & \text{if } \gamma \in [0, \gamma_*) \\ \in [(v_H + v_L + \beta)/2, v_H + \beta] & \text{if } \gamma = \gamma_* \\ = (v_H + v_L + \beta)/2 & \text{if } \gamma \in (\gamma_*, 1]. \end{cases}$$

Due to the symmetry of the model, any outcome in the disagreement state is Pareto-efficient. Thus, if $\gamma \in [0, \gamma_*)$, both the high and the low types receive their first best expected payoffs. However, when $\gamma \in (\gamma_*, 1]$, the equilibrium outcome is inefficient, as the expected payoffs for both types would increase if the low type agree to his opponent's favorite alternative instead of a coin flip.[7]

In our model of negotiation under a deadline, the deadline simply means deciding by a coin flip at a fixed future date T if no agreement has been reached. In practice, reaching the negotiation deadline without an agreement may instead trigger a binding arbitration process by an independent outside party that may involve activities such as presentations by each player or fact-finding by the arbitrator. We have taken a reduced-form approach by abstracting from such details of deadline implementation. The essential feature of the deadline we are trying to capture in this model is the ex ante two-part commitment: the negotiating parties commit to both not terminating the negotiation process before the

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fixed alternative is constant across the three states. See Damiano, Li and Suen (2009) for a formal argument.

[7] The specification of the default decision as a coin flip when the deadline expires implies stark payoff discontinuities in the no-delay game when the belief of the low types that it is the disagreement state is exactly γ_*. Our characterization of the optimal deadline turns out to be robust with respect to the payoff discontinuities. Section 5.2 presents an extension of the model with an alternative specification of the deadline default payoffs that eliminates the discontinuities. All our results are qualitatively unchanged.

fixed date T, and to not extending it beyond T. Although in reality both parts of this commitment are vulnerable to ex post renegotiation, we assume away the credibility issues in order to take the first step towards understanding welfare implications of deadlines.

3. Preliminary Analysis

We refer to as "persisting" the act of a player proposing his own favorite alternative, and "conceding" the act of proposing his opponent's favorite alternative. We will restrict our analysis to perfect Bayesian equilibria in which the high types persist with probability one throughout the game. We impose no restriction on the strategies of the low types and consider both strategies in which they mix between persisting and conceding at a given instant of time, as well as strategies where they mix continuously over an interval of time. For histories of "regular disagreement" where both players have persisted since the beginning of the game, such strategies can be described through two functions $y : [0, T] \to [0, 1]$ and $x : [0, T] \to [0, \infty)$, with the convention that $x(t) = 0$ whenever $y(t) > 0$. At any instant $t \in [0, T]$ reached by the game, $y(t)$ is the probability that the low type concedes upon reaching time t. When y is zero on a small time interval, $x(t)$ denotes the flow rate of concession at any t in the interval $[t, t + \mathrm{d}t)$. That is, upon reaching time t, the probability of a low type proposing his rival's alternative in the interval is $x(t)\mathrm{d}t$. How the game is played after a "reverse disagreement," where both players simultaneously concede, does not matter to our equilibrium construction and the welfare analysis.[8] For convenience, we assume that these are terminal histories where the decision is made by a coin toss.

3.1. Differential equations

In this section we derive some useful properties that hold in any symmetric equilibrium where the low types concede at flow rate $x(t) > 0$ for all t in some interval of time $[t_1, t_2)$, while the high types always persist. In any such equilibrium, by indifference the equilibrium expected payoff $\mathcal{U}_L(t)$ of a low type upon reaching $t \in [t_1, t_2)$ can be computed by assuming

[8] Reverse disagreements occur with probability zero both on the path in the equilibrium constructed below and after unilateral deviations. In proving that our equilibrium is generically unique, we require only that the continuation payoffs after a reverse disagreement are feasible.

that he concedes at t. Denoting as $\gamma(t)$ his belief at time t that it is the disagreement state, we have

$$\mathcal{U}_L(t) = \gamma(t)v_L + (1 - \gamma(t))v_H. \tag{2}$$

The above follows because by assumption $y(t) = 0$, and so even if his low type opponent's flow rate of concession is strictly positive, the probability that the latter concedes at the given time t is zero. Since $\mathcal{U}_L(t)$ depends on t only through $\gamma(t)$ in (2), we can define a payoff function

$$U_L(\gamma) = \gamma v_L + (1 - \gamma)v_H, \tag{3}$$

which is valid whenever $\gamma = \gamma(t)$ and $x(t) > 0$ for some $t \in [t_1, t_2)$.

Given that the equilibrium continuation payoff of the low type is pinned down by the belief $\gamma(t)$ for any t in the interval of time $[t_1, t_2)$, the indifference condition between conceding and persisting on the same interval then gives an equation that relates the rate of change of the belief γ to its current value $\gamma(t)$ and to the equilibrium flow rate of concession $x(t)$. Furthermore, the Bayesian updating rule provides another equation that relates the rate of change of $\gamma(t)$ to $x(t)$. These two equations can be combined to obtain a differential equation for the evolution of the belief of the low type in $[t_1, t_2)$. This result is stated in Lemma 1 below, and proved in Appendix A. An immediate implication of Lemma 1 is that the equilibrium belief of the low type $\gamma(t)$ and the equilibrium rate of concession $x(t)$ in the time interval (t_1, t_2) are functions of the starting belief $\gamma(t_1)$ only.

Lemma 1. *Let $(y(t), x(t))$ be the strategy and $\gamma(t)$ the belief of the low types in a symmetric equilibrium where the high types always persist. If $y(t) = 0$ and $x(t) > 0$ for all $t \in [t_1, t_2)$, then*

$$-\frac{\dot{\gamma}(t)}{1 - \gamma(t)} = \frac{\delta}{\beta}, \tag{4}$$

and

$$x(t) = \frac{1}{\gamma(t)}\frac{\delta}{\beta}.$$

Equation (4) represents the belief evolution for a low type who continuously randomizes and whose opponent has failed to concede so far. Since the high types persist with probability one, $\dot{\gamma}(t)$ is negative; that is, the low types attach a lower probability to the disagreement state as the negotiation game continues. The indifference condition between persisting and conceding then implies that the low types concede at an increasing flow rate as disagreement continues.

We can also use the equilibrium characterization of the flow rate of concession to pin down the evolution of the equilibrium continuation payoff for the high types. For any $t \in [t_1, t_2)$, let $\mathcal{U}_H(t)$ be their expected payoff at time t. Since the high types always persist, their payoff function satisfies the following Bellman equation:

$$\mathcal{U}_H(t) = x(t)\mathrm{d}t\ (\upsilon_H + \beta) + (1 - x(t)\mathrm{d}t)(-\delta\mathrm{d}t + \mathcal{U}_H(t + \mathrm{d}t)).$$

This can be written as a differential equation by taking dt to 0:

$$\dot{\mathcal{U}}_H(t) = \delta - x(t)(\upsilon_H + \beta - \mathcal{U}_H(t)). \tag{5}$$

Further, since $\gamma(t)$ is determined by an autonomous differential equation and $x(t)$ depends on t only through $\gamma(t)$ as given in Lemma 1, we can also describe the equilibrium continuation payoff of the high types as a function $U_H(\gamma)$. Using $\dot{\mathcal{U}}_H(t) = U'_H(\gamma(t))\dot{\gamma}(t)$, we can show that it satisfies the differential equation

$$U'_H(\gamma) = \frac{\upsilon_H + \beta - U_H(\gamma)}{\gamma(1-\gamma)} - \frac{\beta}{1-\gamma}. \tag{6}$$

Note that the equilibrium payoff to the high types is a function of the belief of the low types, even though the former know the state and always persist in equilibrium.

3.2. Equilibrium with no deadline

When there is no deadline to the negotiation process (i.e., $T = \infty$), the characterization result of Lemma 1 is sufficient for us to construct an equilibrium where the low types concede at a strictly positive flow rate until a time when they concede with probability one. The equilibrium strategy and the evolution of beliefs along the equilibrium path are entirely pinned down by the initial belief, and the atom of concession occurs when the low types become entirely convinced that it is an agreement state. Let $g(t; \gamma_0)$ be the unique solution to the differential equation (4) with the initial condition $g(0; \gamma_0) = \gamma_0$, given by

$$g(t; \gamma_0) = 1 - (1 - \gamma_0)e^{\delta t/\beta}. \tag{7}$$

Define the "terminal date" $D(\gamma_0)$ such that $g(D(\gamma_0);\gamma_0) = 0$, given explicitly by

$$D(\gamma_0) = -\frac{\beta \ln(1-\gamma_0)}{\delta}. \tag{8}$$

PROPOSITION 1. *Let $T = \infty$. There exists a symmetric equilibrium where the high types always persist, and where the strategy $(y(t), x(t))$ and the belief $\gamma(t)$ of the low types are such that:*

$$\begin{cases} y(t) = 0,\ x(t) = \delta/(\beta\gamma(t)),\ \text{and } \gamma(t) = g(t;\gamma_0) & \text{if } t < D(\gamma_0), \\ y(t) = 1,\ \text{and } \gamma(t) = 0 & \text{if } t \geq D(\gamma_0). \end{cases}$$

By construction, the low types are indifferent between conceding and persisting at any time $t < D(\gamma_0)$. Further, conceding is optimal at $t = D(\gamma_0)$ for them because their belief that it is the disagreement state becomes zero at that point.[9] For the high types, from the equilibrium strategies, their continuation payoff at the terminal date is the first best payoff $v_H + \beta$. In Appendix A, we use this boundary condition to explicitly solve the differential equation (6) for the high types' continuation payoff for any $t < D(\gamma_0)$, and verify that it is optimal for them to always persist.

In equilibrium, protracted negotiations make the low types increasingly convinced that it is the agreement state supporting the rival's favorite choice, and motivate them to concede at an increasing rate. This distinctive feature of "gradually increasing concessions," unique to our model of negotiation that combines preference-driven and information-driven disagreements, has implications for the duration of the negotiation process and its hazard rate function. Denote as τ_{HL} and τ_{LL} the random duration of the game conditional on it being an agreement state and the disagreement state respectively. In the former case one of the player is a high type, while in the latter case both are low. Since $x(t)\mathrm{d}t$ is the probability that the game ends in time interval $(t, t + \mathrm{d}t]$ conditional on it having survived up to time t, the hazard function of τ_{HL} is simply $x(t)$. When it is the disagreement state, independent randomization by the two players implies that the cumulative distribution functions $F_{HL}(t;\gamma_0)$ of τ_{HL} and the distribution function $F_{LL}(t;\gamma_0)$ of τ_{LL} satisfy

$$1 - F_{LL}(t;\gamma_0) = (1 - F_{HL}(t;\gamma_0))^2,$$

[9] The game ends with probability one before $t = D(\gamma_0)$. We specify the strategy and the belief of the low types after the terminal date to complete equilibrium description after unilateral deviations.

and thus the hazard function of τ_{LL} is $2x(t)$. The hazard rate is therefore increasing in time in both cases. From an outside observer's point of view, however, the more interesting object is the unconditional duration of the negotiation game. Let τ represent this random variable, and $F(t;\gamma_0)$ its distribution function. As the game continues, the conditional hazard rates for τ_{HL} and τ_{LL} both increase, but the probability that $\tau = \tau_{HL}$, which is associated with a lower hazard rate, also increases, so it is not obvious whether the unconditional hazard rate for τ will increase over time.[10] However, from the relationship

$$1 - F(t;\gamma_0) = \frac{\gamma_0}{2-\gamma_0}(1 - F_{LL}(t;\gamma_0)) + \frac{2(1-\gamma_0)}{2-\gamma_0}(1 - F_{HL}(t;\gamma_0))$$

we can obtain the hazard function of τ as

$$\frac{2}{g(t;\gamma_0)(2-g(t;\gamma_0))}\frac{\delta}{\beta},$$

which is decreasing in $g(t;\gamma_0)$.[11] Since in equilibrium the belief of the low types that it is the disagreement state decreases as disagreements continue, the unconditional hazard rate unambiguously increases in time. Combined with the fact that the belief $g(t;\gamma_0)$ is increasing in γ_0 for any t, an increase in the initial belief, representing a greater degree of conflict, reduces the unconditional hazard rate, and hence increases the unconditional expected duration of negotiation.

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[10] This is similar to the classic problem of duration dependence versus heterogeneity in the econometric analysis of duration data. See, for example, Heckman and Singer (1984).

[11] To derive the hazard function for τ, note that the conditional density functions $f_{HL}(t)$ and $f_{LL}(t)$ and the unconditional density function $f(t)$ satisfy

$$\frac{f(t)}{1-F(t)} = \frac{\gamma_0 f_{LL}(t) + 2(1-\gamma_0) f_{HL}(t)}{\gamma_0(1-F_{LL}(t)) + 2(1-\gamma_0)(1-F_{HL}(t))}.$$

The final result is obtained by using

$$1 - F_{HL}(t) = \frac{1-\gamma_0}{\gamma_0}\frac{g(t;\gamma_0)}{1-g(t;\gamma_0)}$$

and

$$f_{HL}(t) = \frac{1-F_{HL}(t;\gamma_0)}{g(t;\gamma_0)}\frac{\delta}{\beta},$$

and the corresponding expressions for F_{LL} and f_{LL}.

4. Finite Deadlines

We use the analysis in the previous section to construct a symmetric equilibrium in which the high types always persist, and the low types generally start by continuously randomizing between conceding and persisting when the time to the deadline is sufficiently long, stop and persist until just before the deadline is reached, and play an equilibrium of the no-delay game ($T = 0$) corresponding to the stopping belief. We later argue that this equilibrium is unique subject to the restriction that the high types always persist.

A remarkable feature of our construction is that the equilibrium randomization strategy of the low types is identical to the no-deadline case ($T = \infty$). That is, when the time to the deadline is sufficiently long, they behave as if there is no deadline. This feature is the main analytical advantage of a continuous time framework over a discrete time model. It follows from equation (3) in our preliminary analysis, because there is a unique equilibrium value function for a randomizing low type that depends on the time to deadline only through his belief.

4.1. Construction of an equilibrium

The necessity of having a persistence phase in equilibrium before the deadline is reached can be easily understood as follows. At any time t when the belief of a low type is $\gamma(t) = \gamma$ and he is conceding with a positive flow rate, his payoff is pinned down by the the function $U_L(\gamma)$ given in equation (3). For any $\gamma > 0$, this payoff is strictly lower than the payoff from the no-delay game $U_L^0(\gamma)$ as given in equation (1). If the time remaining to the deadline, $T - t$, is sufficiently short, persisting until the end and playing a no-delay equilibrium when the deadline arrives would constitute a profitable deviation for him. This deadline effect of having a persistence phase just before the deadline is robust with respect to our game specification. Whenever the default payoff at the deadline of a negotiation game yields an equilibrium payoff upon reaching the deadline that is larger than the payoff from concession, then in any equilibrium a period of inactivity always precedes the arrival of the deadline.[12]

[12] A similar deadline effect is present in existing models of war of attrition (e.g., Hendricks, Weiss and Wilson, 1988). The novel feature of our model as a war of attrition game is that endogenous information about the state is generated as the game continues, so that the deadline effect depends on the initial belief through the equilibrium belief evolution prior to stopping.

How long the persistence phase can last in equilibrium depends on the difference between the payoff from immediate concession $U_L(\gamma)$ and the payoff in the no-delay game $U^0_L(\gamma)$. To state our equilibrium characterization result in the next proposition, we define $B(\gamma)$ as the longest length of time from the deadline such that it is an equilibrium for a low type with belief γ to persist until the deadline and then play an equilibrium corresponding to the no-delay game associated with γ. In other words, the value of $B(\gamma)$ measures the maximum length of the persistence phase when the low types start with belief γ. For any belief $\gamma \neq \gamma_*$, this is uniquely given by

$$U^0_L(\gamma) - \delta B(\gamma) = U_L(\gamma).$$

Since $U^0_L(\gamma_*)$ assumes a continuum of values, corresponding to the probability of conceding ranging from zero to one, we choose the maximal value in the above equation to define $B(\gamma_*)$. Using the expressions for $U^0_L(\gamma)$ and $U_L(\gamma)$, we have

$$B(\gamma) = \begin{cases} \beta\gamma/(2\delta) & \text{if } \gamma \leq \gamma_*, \\ \beta(\gamma - \gamma_*)/(2\delta(1-\gamma_*)) & \text{if } \gamma > \gamma_*. \end{cases} \tag{9}$$

Note that $B(\gamma)$ is discontinuous at γ_*. Next, for an initial belief γ_0, we describe how long it takes, in equilibrium, before the persistence phase begins. To do so, we define $S(T;\gamma_0)$ as the earliest calendar time t such that the time-to-deadline is shorter than $B(\gamma(t))$ given that the belief $\gamma(t)$ of the low types evolves according to (7) starting with γ_0. That is,

$$S(T;\gamma_0) = \inf_{t \geq 0}\{t : T - t \leq B(g(t;\gamma_0))\}. \tag{10}$$

The two functions $S(T;\gamma_0)$ and $T - S(T;\gamma_0)$ describe the length of the concession and the persistence phases respectively in our equilibrium characterization. In other words, $S(T;\gamma_0)$ is the phase-switch time, or the time of stopping concessions, with the corresponding stopping belief of the low types, $g(S(T;\gamma_0);\gamma_0)$, remaining until the deadline T arrives. Note that by definition, $S(T;\gamma_0) = 0$ if $T \leq B(\gamma_0)$.

Proposition 2. *Let T be finite. There exists a symmetric equilibrium in which the high types always persist, and the strategy $(y(t), x(t))$ and the belief $\gamma(t)$ of the low types are* *such that (where $S = S(T;\gamma_0)$):*

$$\begin{cases} y(t)=0,\ x(t)=\delta/(\beta\gamma(t)),\ \gamma(t)=g(t;\gamma_0) & \text{if } T-t>B(g(t;\gamma_0)) \text{ and } t<D(\gamma_0), \\ y(t)=0,\ x(t)=0,\ \gamma(t)=g(S;\gamma_0) & \text{if } B(g(t;\gamma_0))\geq T-t>0 \text{ and } t<D(\gamma_0), \\ y(t)=1,\ \gamma(t)=0 & \text{if } T>t\geq D(\gamma_0); \end{cases}$$

$$\begin{cases} y(T)=0,\ \gamma(T)=g(S;\gamma_0) & \text{if } g(S;\gamma_0)>\gamma_*, \\ y(T)=2\delta(T-S)/(\beta\gamma_*),\ \gamma(T)=\gamma_* & \text{if } g(S;\gamma_0)=\gamma_*, \\ y(T)=1,\ \gamma(T)=g(S;\gamma_0) & \text{if } g(S;\gamma_0)<\gamma_*. \end{cases}$$

The logic of Proposition 2 is apparent from our construction of $B(\gamma)$ and $S(T;\gamma_0)$. For each belief γ of the low types, the equilibrium payoff function $U_L^0(\gamma)$ in the no-delay game gives a continuation equilibrium outcome at the instant when the deadline arrives, providing the starting point for backward induction. This continuation equilibrium outcome is unique if $\gamma \neq \gamma_*$, and so if the deadline T is short relative to the initial belief γ_0, i.e., if $T \leq B(\gamma_0)$, the equilibrium is for the low types to persist until the deadline and then play the continuation equilibrium corresponding to γ_0. By construction, when $T = B(\gamma_0)$, the equilibrium payoff to the low types is precisely $U_L(\gamma_0)$. If $\gamma_0 = \gamma_*$ and $T \leq B(\gamma_*)$, we choose a continuation equilibrium in the no-delay game, corresponding to a probability of concession $y(T) = 2\delta T/(\beta\gamma_*)$, such that the low types obtain the payoff of $U_L(\gamma_*)$ from this deadline play.[13] If the deadline T is sufficiently long relative to the initial belief γ_0, the low types start by conceding with a flow rate $x(t)$ given in Proposition 1 for the no-deadline game until $t = S(T;\gamma_0)$, when the belief becomes $g(S(T;\gamma_0);\gamma_0)$ and the payoff reaches $U_L(g(S(T;\gamma_0);\gamma_0))$, followed by the deadline play. Finally, if the deadline T is too long, with $T \geq D(\gamma_0)$, the equilibrium is identical to the one constructed in the no-deadline game.[14] Details of the proof of Proposition 2 (including the argument that the high types will indeed persist throughout) are presented in Appendix A.

The equilibrium behavior of the low types is illustrated in Figure 1. The horizontal axis represents both the deadline T and, for a fixed T, the time remaining before the deadline

[13] Since any $y(T)$ greater than $2\delta T/(\beta\gamma_*)$ preserves the incentives for the low types to persist, there is a continuum of equilibria when $\gamma_0 = \gamma_*$ and $T < B(\gamma_*)$.

[14] In this case, (10) implies that the phase-switch time $S(T;\gamma_0)$ is equal to $D(\gamma_0)$ and the corresponding belief $g(S(T;\gamma_0);\gamma_0)$ is zero.

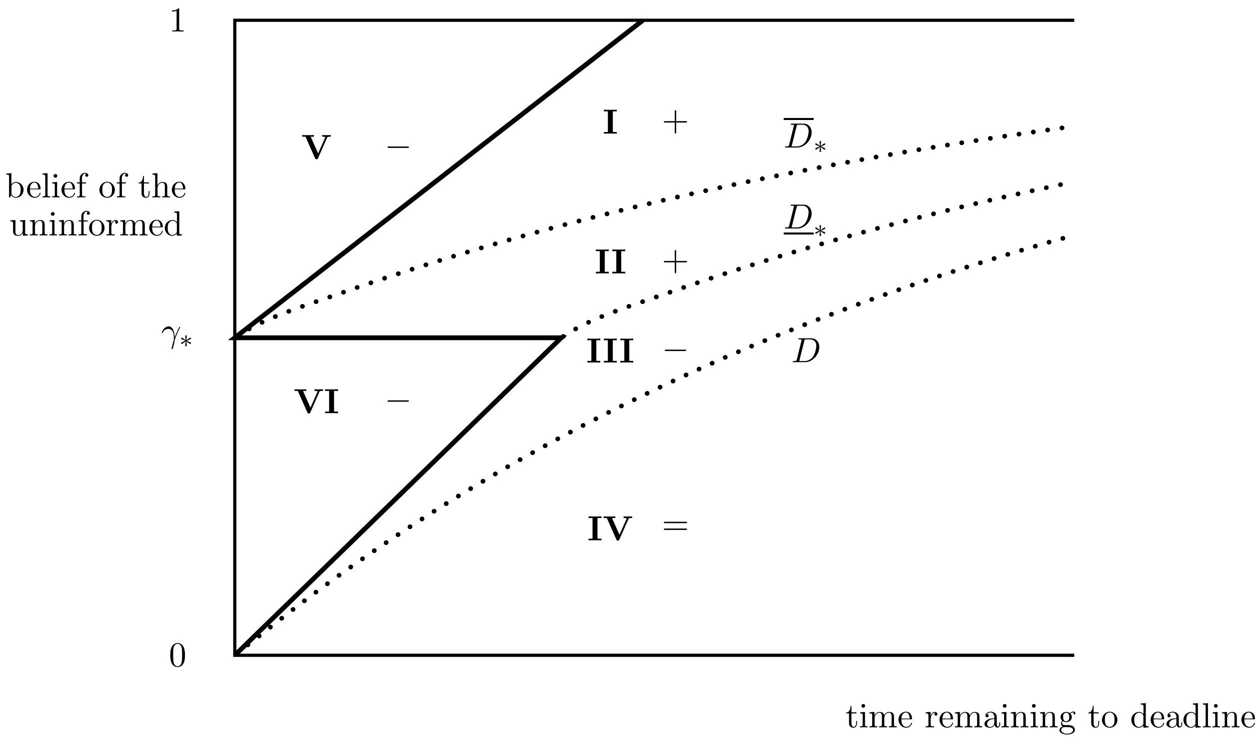

Figure 1

is reached. The vertical axis is the belief of the low types. For ease of interpretation, we have shown the discontinuous function $B(\gamma)$ as the thick piecewise-linear graph. It represents the boundary in the T-γ space between the persistence phase when the low types persist until the deadline and their belief does not change, and the concession phase when they concede with a positive and increasing flow rate and their belief continuously drops. The dotted curves in Figure 1 traces the equilibrium evolution of the belief $\gamma(t)$ until the phase-switch time, if such time exists. The curve D is given by the terminal date function in equation (8). For any deadline T and initial belief γ_0 on D, the equilibrium belief will reach zero at time T. For any deadline T and initial belief γ_0 on the dotted curve $\underline{D}_*$, the equilibrium belief will reach γ_* at time $T - B(\gamma_*)$, that is,

$$g(\underline{D}_*(\gamma_0) - B(\gamma_*); \gamma_0) = \gamma_*. \tag{11}$$

Similarly, for any deadline T and initial belief γ_0 on the curve $\overline{D}_*$, the equilibrium belief will reach γ_* at time T, that is,

$$g(\overline{D}_*(\gamma_0); \gamma_0) = \gamma_*.$$

Since the law of motion for equilibrium belief does not depend on the deadline T in the concession phase, the three dotted curves in Figure 1 are horizontal displacements of one another. Moreover, for any (T, γ_0) that lies above one of these curves, the trajectory of equilibrium belief will stay above the same curve throughout the concession phase. Therefore, we can summarize the equilibrium play of the low types by partitioning the T-γ space of Figure 1 into six regions:[15]

Region I. The low types concede with a flow rate $\delta/(\beta g(t;\gamma_0))$ for $t < S(T;\gamma_0)$, and persist for t larger.

Region II. The low types concede with a flow rate $\delta/(\beta g(t;\gamma_0))$ for $t < S(T;\gamma_0)$, persist for all $t \in [S(T;\gamma_0), T)$, and concede with probability $2\delta(T - S(T;\gamma_0))/(\beta\gamma_*)$ at $t = T$.

Region III. The low types concede with a flow rate $\delta/(\beta g(t;\gamma_0))$ for $t < S(T;\gamma_0)$, persist for all $t \in [S(T;\gamma_0), T)$, and concede with probability one at $t = T$.

Region IV. The low types concede with a flow rate $\delta/(\beta g(t;\gamma_0))$, with the game ending with probability one by the terminal date $D(\gamma_0)$ before the deadline expires.

Region V. The low types persist for all t.

Region VI. The low types persist for all $t < T$ and concede with probability one at $t = T$.

Each of the six regions above has its own distinctive features. Together they provide a rich set of negotiation dynamics that are possible in our model. In Region IV, the deadline is not binding. Gradual concessions are made at an increasing rate until an agreement is reached as if there is no deadline; the dynamics of endogenous information aggregation is already described in the previous section. In all other regions, the deadline is binding, with the effect of suspending the negotiations at some point of the process in anticipation of the arrival of the deadline. When the deadline is too short, in both Regions V and VI, and on the boundary between Regions VI and II, this effect takes hold at the very beginning so there is no attempt at resolving the differences before the deadline. The difference between the two regions is that V represents a "deadlock" with no hope of ever

[15] The boundary between Regions II and VI is formally part of Region II. We have $S(T;\gamma_0) = 0$ so there is no concession phase and the low types concede at $t = T$ with probability $2\delta T/(\beta\gamma_*)$. The assignment of other boundaries is immaterial.

reaching an agreement because the initial degree of conflict is too high, while the deadline effect in VI describes a “stalling” tactic before an eleventh-hour attempt at striking an agreement. When the deadline is sufficiently long relative to the initial degree of conflict, in Regions I, II and III, negotiations all start off with gradual and increasing concessions as in Region IV. The difference among the three regions lies in how much time and conflict remain when the deadline effect kicks in after the unsuccessful initial attempts. In Region I, too little time is left to overcome the residual conflict, so the negotiation becomes a deadlock. The opposite happens in Region III, as there is a complete change of position in the final attempt to reconcile the difference after a stalling period. In between we have Region II, where more time left when the deadline effect kicks in means a greater chance of reaching an agreement at the deadline.

4.2. Uniqueness of the equilibrium

The equilibrium constructed in Proposition 2 is generically unique in the class of perfect Bayesian equilibria with the high types always persisting. This is perhaps surprising, because the amount of endogenous information generated in equilibrium during the concession phase depends on the flow rate of concession of the low types, which in turn is determined by how much they learn in equilibrium about the state. One may wonder if it is possible to construct multiple equilibria by coordinating through calendar time the flow rate of concession of the low types. For example, after trying but failing to reach an agreement by conceding with a positive flow rate, the low types may persist for a fixed length of time before resuming a new concession phase. However, this and other possibilities for multiple equilibria are ruled out by the following proposition.

Proposition 3. *Given any deadline T and initial belief γ_0 of the low types, except for $T < B(\gamma_*)$ and $\gamma_0 = \gamma_*$, there is a unique equilibrium in which the high types always persist.*

When $T < B(\gamma_*)$ and $\gamma_0 = \gamma_*$, there is a continuum of equilibria in which the high types always persist and the low types persist for all $t < T$ followed by any probability of concession equal to or greater than $2\delta T/(\beta\gamma_*)$ at the deadline. This multiplicity of equilibria is due to the multiplicity in the no-delay game ($T = 0$) when the initial belief of the low types is γ_*. However, it is not generic, because for the same $T < B(\gamma_*)$ the equilibrium is unique when γ_0 is different from γ_*, no matter how small the difference is.[16] Moreover, since at $\gamma_0 = \gamma_*$ there is an equilibrium in the no-delay game with the first best payoffs, we argue that the optimal deadline for $\gamma_0 = \gamma_*$ is $T = 0$, and thus the particular multiplicity at γ_* does not affect our characterization of the optimal deadline.

The generic uniqueness of the equilibrium is important for our main objective in this paper, which is to characterize the ex ante optimal deadline. Moreover, Proposition 3 holds even in the case of $T = \infty$. The equilibrium described in Proposition 1 for the no-deadline case is a unique equilibrium in which the high types always persist. This implies that the equilibrium strategies in the game with finite deadline T cannot be supported as part of equilibrium in a no-deadline game, which means that deadlines are more than a mere coordinating device to select among multiple equilibria. In Appendix A we formally prove Proposition 3 by establishing a series of claims about the properties of any equilibrium. Here, we give intuitive explanations for some of the properties to highlight the underlying logic of why the equilibrium is unique.

A key step in establishing the generic uniqueness of the equilibrium is to show that in any equilibrium the low types cannot concede with a strictly positive probability before the deadline arrives. Intuitively, if a low type concedes with probability $y(t) > 0$ at some time $t < T$, then the opposing low type could persist at t and concede immediately after. The payoff gain relative to conceding would be strictly positive because $y(t) > 0$, while the loss from the extra delay would be arbitrarily small. An immediate implication is that at any time before the deadline, a low type must either persist with probability one or concede with a positive flow rate. In other words, the equilibrium play of a low type must either be in a persistence phase or in a concession phase.

In any equilibrium the persistence and concession phases of the two low types must be synchronized. That is, if the flow rate of concession $x(t)$ for one low type is positive in some interval period of time, then the same is true for his low type opponent. This is because

[16] In addition, the multiplicity of equilibria for $T < B(\gamma_*)$ and $\gamma_0 = \gamma_*$ is not robust with respect to the specification of the default payoffs in the no-delay game. In the model of Section 5.2 where we introduce a penalty that the players incur if they fail to reach an agreement when the deadline expires, the same argument for Proposition 3 can be used to establish that the equilibrium is unique for all T and γ_0.

if a low type is continuously indifferent between conceding and persisting in an interval period of time, his belief that it is a disagreement state must change over time. Otherwise, conceding at the beginning of the interval would give him the same expected payoff from the outcome but with a smaller delay cost. Synchronization then follows because his belief changes only if his opponent's flow rate of concession is positive. Conversely, if a low type persists in some time interval, then so does the opposing low type. Thus, the belief of neither player changes in a synchronized persistence phase. Since the payoff to a low type $U_L(\gamma(t))$ in a concession phase is pinned down by the corresponding belief $\gamma(t)$ and is computed with the opposing low type persisting at t, a persistence phase cannot be followed by a concession phase. Otherwise, each low type would strictly prefer to concede with probability one during the persistence phase to avoid the payoff loss from the delay, which we already argue cannot happen in an equilibrium.

Our game is symmetric. In any equilibrium the two low types not only synchronize their persistence and concession phases with the same phase-switch time, they also adopt identical strategies in the concession phase and in the deadline play. At the phase-switch time, both low types must be indifferent between immediate concession and continuation with the deadline play. Since their expected payoffs from both options are functions of their individual beliefs only, for the indifference conditions to hold at the same time, their beliefs must coincide. The symmetry of the equilibrium strategy of the low types in the concession phase then follows, because the uniqueness of the solution of the differential equation (4) implies that the beliefs of the two players coincide at the phase-switch time only if they are identical throughout the concession phase. Given the symmetry, the construction of the boundary $B(\gamma)$ and the phase-switch time $S(T;\gamma_0)$ is unique due to the indifference of the low types between an immediate concession with the corresponding payoff $U_L(g(S(T;\gamma_0);\gamma_0))$ and the deadline play with the corresponding payoff $U_L^0(g(S(T;\gamma_0);\gamma_0)) - \delta B(g(S(T;\gamma_0);\gamma_0))$, yielding the uniqueness of the equilibrium.

4.3. Optimal deadline

In this subsection we characterize the ex ante optimal deadline for the repeated proposal game. We start by studying the effects of marginally extending the deadline T on the equilibrium payoffs of the high and low types in the different regions of the T-γ_0 space. Refer to Figure 1.

In Regions V and VI of Figure 1, where $T < B(\gamma_0)$ and $\gamma_0 \neq \gamma_*$, the deadline is too short relative to the initial belief to allow a concession phase. The welfare effect of the deadline is clearly negative. Extending the deadline just makes the low types persist for a longer period of time without changing their behavior at the deadline. Consequently, both the high and low types are hurt by a longer deadline.

In Region IV, where $T \geq D(\gamma_0)$, the deadline is too long to allow a persistence phase. There is no welfare effect. Since the negotiation ends before the deadline with probability one, extending it further will not affect the equilibrium behavior or payoffs.

In Region II, where $T \in [\overline{D}_*(\gamma_0), \underline{D}_*(\gamma_0))$, the effect of lengthening the deadline is to make the low types persist longer after the phase switch, but concede with a larger probability when the deadline arrives. Since the behavior of the players during the concession phase does not depend on T, the phase-switch time $S(T;\gamma_0)$ is also independent of T. Once the negotiation enters the persistence phase, the low types persist from time $S(T;\gamma_0)$ through T, and then concede with probability $2\delta(T - S(T;\gamma_0))/(\beta\gamma_*)$. Lengthening the deadline increases the delay for the high types, but also increases their chance of getting their favorite decision rather than a coin toss. The net effect on the welfare of the high types is

$$\frac{\partial U_H(\gamma_0)}{\partial T} = -\delta + \frac{2\delta}{\beta\gamma_*}\frac{\upsilon_H - \upsilon_L + \beta}{2}, \tag{12}$$

which is positive by Assumption 1. There is no effect on the welfare of the low types, because their payoff is pinned down by $U_L(\gamma_0)$, which is independent of T. In sum, a longer deadline is beneficial for the ex ante welfare of the players in this region.[17]

Finally, let us consider Region I where $T \in [B(\gamma_0), \overline{D}_*(\gamma_0))$, and Region III where $T \in [B(\gamma_0), D(\gamma_0))$ for $\gamma_0 < \gamma_*$ or $T \in [\underline{D}_*(\gamma_0), D(\gamma_0))$ for $\gamma_0 \geq \gamma_*$. As in Region II, the equilibrium play of the low types in I or III consists of both a concession phase and a persistence phase. However, unlike in II, increasing the deadline in I or III lengthens the

[17] Under the selection of the continuation equilibrium given in Proposition 2, the same analysis and conclusion hold on the horizontal segment of the boundary B, with $\gamma_0 = \gamma_*$ and $T \leq B(\gamma_*)$.

concession phase while shortening the persistence phase, with no change in equilibrium play at the deadline ($y(T) = 0$ in Region I or $y(T) = 1$ in Region III). The welfare effect on the low types is again nil, since their payoff is fixed at $U_L(\gamma_0)$. The welfare effect on the high types can be studied by solving the differential equation (5) (or equivalently, 6) with appropriate boundary conditions obtained from the equilibrium deadline play of the low types.

Take Region I for example. The game enters the persistence phase from the concession phase at time $S(T;\gamma_0)$. From the deadline play of the low types, the payoff to the high types at $t = S(T;\gamma_0)$ is

$$\mathcal{U}_H(S(T;\gamma_0)) = \frac{1}{2}(v_H + v_L + \beta) - \delta(T - S(T;\gamma_0)).$$

Their payoff at the beginning of the game is

$$U_H(\gamma_0) = \mathcal{U}_H(0) = \mathcal{U}_H(S(T;\gamma_0)) - \int_0^{S(T;\gamma_0)} \dot{\mathcal{U}}_H(t)\mathrm{d}t,$$

where $\dot{\mathcal{U}}_H(t)$ is given by equation (5). Lengthening the deadline affects the welfare of the high types by changing the boundary value $\mathcal{U}_H(S(T;\gamma_0))$ directly and by prolonging the concession phase through increasing $S(T;\gamma_0)$. The overall effect is

$$\frac{\partial U_H(\gamma_0)}{\partial T} = -\delta + x(S(T;\gamma_0))(v_H + \beta - \mathcal{U}_H(S(T;\gamma_0)))\frac{\partial S(T;\gamma_0)}{\partial T}. \tag{13}$$

The loss of a longer deadline is δ, while the gain is the increased length of the concession phase times the flow rate of concession times the value of the resulting improvement in the decision. The analysis for Region III is similar, except that the boundary value becomes

$$\mathcal{U}_H(S(T;\gamma_0)) = v_H + \beta - \delta(T - S(T;\gamma_0)).$$

The welfare effect on the high type is given by the same expression (13).

Crucial to our characterization of the optimal deadline, we establish in the proof of Proposition 4 below that the welfare effect (13) is positive in Region I but negative in Region III. The intuition behind this result is quite simple. In Region I, the game will result in a "deadlock" if it survives past the phase-switch time. Because the low types will persist at the deadline, the quality of the decision is bad for the high types. Therefore a longer concession phase that allows more information aggregation in the beginning of the negotiation is highly valuable. In Region III, on the other hand, the game only leads to a "stalling" period past the phase-switch time. Since the low types will ultimately concede at the deadline, the high types will eventually obtain their favorite decision. Therefore a longer concession phase in the beginning is of less value. This explains the contrasting welfare effects for these two cases.

Figure 1 illustrates the welfare effects of a marginal extension of the deadline. A "+" sign indicates that a longer deadline improves the welfare of the high types, with no effect on the low types; a "−" sign indicates a negative welfare effect on the high types, together with either a negative effect (in Regions V and VI) or no effect (in Region III) on the low types; and a "=" sign indicates that the welfare effect is nil for both types. For $\gamma_0 \geq \gamma_*$, we can see that as the deadline T increases, the welfare effect is first negative in Region V, then positive in Regions I and II, and finally turning negative in Region III. Therefore the optimal deadline must be either zero, or $\underline{D}_*(\gamma_0)$, which is the boundary between Regions II and III. For $\gamma_0 < \gamma_*$, we see that the welfare effect is negative so long as the deadline is binding, and is nil when the deadline is too long. Therefore the optimal deadline must be $T = 0$. To state our main result on the optimal deadline, let

$$U^T(\gamma_0) = \frac{1}{2-\gamma_0} U_L^T(\gamma_0) + \frac{1-\gamma_0}{2-\gamma_0} U_H^T(\gamma_0) \tag{14}$$

denote the ex ante welfare of a player before he knows his type from the equilibrium under deadline T, where U_H^T and U_L^T are the corresponding payoffs for the high types and the low types derived from Proposition 2.

Proposition 4. *There exists a $\overline{\gamma} \in (\gamma_*, 1)$ such that the length of the deadline T that maximizes $U^T(\gamma_0)$ is $\underline{D}_*(\gamma_0)$ if $\gamma_0 \in (\gamma_*, \overline{\gamma})$, and is 0 otherwise.*

The proof of this proposition involves showing that the welfare effect (13) is positive in Region I and negative in Region III. Together with the result that the welfare effect (12) is positive in Region II, we establish that the local maxima of $U^T(\gamma_0)$ are at $T = 0$ and $T = \underline{D}_*(\gamma_0)$ when $\gamma > \gamma_*$. The remainder of the proof consists of comparing the values of $U^T(\gamma_0)$ at the two local maxima. The details are in Appendix A.

Proposition 4 shows that the optimal deadline is zero when γ_0 is either sufficiently small or sufficiently large. When $\gamma_0 \leq \gamma_*$, the equilibrium in the no-delay game is efficient, so that allowing the players to negotiate in a continuous-time game will only introduce unnecessary delay. At the other end, when γ_0 is sufficiently close to one, under a sufficiently long deadline the low types concede at a low rate and revise their belief slowly. Although the welfare effect of the deadline is locally positive, making the decision immediately by flipping a coin is even better from the ex ante perspective because the long delay is avoided in the first place.

For intermediate levels of γ_0, Proposition 4 shows that the optimal deadline is both finite and not arbitrarily close to zero. These two properties follow from the characterization of the optimal deadline by the condition that the remaining time for negotiation is $B(\gamma_*)$ when the belief of the low types drops to γ_* after an unsuccessful concession phase. Alternatively, since the low types in equilibrium concede with probability one if and only if the stopping belief is γ_* and the time remaining to the deadline is $B(\gamma_*)$, or the stopping belief is strictly lower than γ_*, the optimal deadline for the intermediate levels of initial belief γ_0 is such that the concession phase is the shortest, and correspondingly the persistence phase is longest, for there to be efficient information aggregation at the deadline. Thus, the optimal deadline is finite for $\gamma_0 \in (\gamma_*, \overline{\gamma})$, not because too long a deadline eventually becomes non-binding with no welfare effect, but because conditional on achieving efficient information aggregation at the deadline, the optimal deadline minimizes the length of the concession phase. That it is not arbitrarily close to zero implies that there are discontinuities in the optimal deadline, both at $\gamma_0 = \gamma_*$ and at $\gamma_0 = \overline{\gamma}$. These discontinuities are not a consequence of the equilibrium payoff discontinuity in the no-delay game.[18] Rather, they are due to the deadline effect: for deadlines sufficiently short, the low types will simply persist from the start all through the deadline, which means that the welfare effect is always negative for short deadlines. Put differently, when positive the optimal deadline cannot be too short because it has to allow a sufficiently long

[18] In Section 5.2, where we modify the non-delay game to eliminate the payoff discontinuity, the optimal deadline remains discontinuous.

delay to give incentives for the low types to change their deadline behavior and achieve efficient information aggregation.

Using the definition of $\underline{D}_*$ in equation (11), we can obtain an explicit formula for the optimal deadline when it is positive:

$$\underline{D}_*(\gamma_0) = \frac{\beta}{\delta}\Big(\frac{\gamma_*}{2} + \ln\frac{1-\gamma_*}{1-\gamma_0}\Big).$$

The above formula immediately reveals that the optimal deadline, when positive, is an increasing function of γ_0. This makes sense, because starting from a higher initial belief γ_0 it takes a longer time for the revised belief to reach γ_*. It is also straightforward to verify using the formula that the optimal deadline is longer the lower is the flow delay cost δ, the smaller is the common value difference $v_H - v_L$, or the greater is the low type's private benefit β. All these factors make the low types less willing to concede, therefore requiring a longer negotiation to achieve efficient information aggregation.

5. Extensions

In setting up the model we have abstracted from any detail in the deadline implementation to focus on the welfare effect of the deadline. In this section we briefly present two extensions of the model, both of which add greater detail and some degree of realism. However, this is not the main objective of these two extensions. Rather, we use them to gain more insight about the source of the welfare effect of the deadline, and to demonstrate its robustness.

5.1. Stochastic deadlines

Our analysis so far is confined to the case of pre-committed deterministic deadlines. We now study the repeated proposal game with exogenous but stochastic breakdowns, interpreted as stochastic deadlines. Let $\epsilon > 0$ be the constant rate of exogenous exit, so that upon reaching time t, the probability that the game ends exogenously in the next time interval $\mathrm{d}t$ is $\epsilon \mathrm{d}t$. In this event, we assume that the decision is made by a fair coin flip. For simplicity we assume that $T = \infty$. A smaller value of ϵ corresponds to a longer stochastic deadline, with $\epsilon = \infty$ corresponding to the no-delay game analyzed in Section 2, and $\epsilon = 0$ equivalent to the no deadline game analyzed in Section 3.

Following the same steps in deriving the differential equation for $\gamma(t)$ in the case of $\epsilon = 0$, we have

$$-\frac{\dot{\gamma}(t)}{1-\gamma(t)} = \frac{\delta}{\beta}\frac{\alpha-\gamma(t)}{\alpha-\gamma_*}, \tag{15}$$

where we have defined

$$\alpha \equiv \gamma_* + (1-\gamma_*)\frac{2\delta}{\beta\epsilon}.$$

The derivation of the above differential equation is in the proof of Proposition B1 in Appendix B. There are two cases to consider.

In the first case, $\gamma_0 < \min\{1, \alpha\}$, and the differential equation (15) gives the belief evolution of an equilibrium in which the high types always persist and the low types with belief γ concede with a flow rate $\epsilon(\alpha-\gamma)/(2(1-\gamma_*)\gamma)$.[19] In this case, the exogenous exit rate ϵ is sufficiently small, or equivalently the stochastic deadline is sufficiently long, relative to the initial belief γ_0 of the low types. Qualitatively, this case is similar to the no-deadline game of Section 3, or the non-binding deadline case of Section 4.

In the second case, with $\gamma_0 \in [\min\{1, \alpha\}, 1)$, in equilibrium the low types persist with probability one at any time t just as the high types, with the game ending by an exogenous exit. This case occurs when the exit rate ϵ is great and the initial belief γ_0 is high. Since flipping a coin gives a higher payoff to the low types than than $U_L(\gamma_0)$, and since the expected wait for the stochastic exit to occur is short when ϵ is large, they have no incentive to deviate to conceding. This case is qualitatively similar to the short deadline case in Section 4.

We are interested in the effect of the stochastic exit rate ϵ on players' welfare. The question we want to answer is whether in a game with no deterministic deadline, exogenous stochastic exit can be used to improve the ex ante welfare of the players in a way similar to the optimal finite deadline analyzed in Section 4. Since the equilibrium in the no-delay

[19] If $\epsilon \leq 2\delta/\beta$, this is the only possible case. Note that α approaches infinity as ϵ approaches 0, in which case (15) reduces to (4) for the no-deadline case.

game ($\epsilon = \infty$, or equivalently $T = 0$) is efficient for any initial belief γ_0 below γ_*, we are only interested in the question of the optimal exogenous exit rate for $\gamma_0 > \gamma_*$.

For the first case of $\gamma_0 < \min\{1, \alpha\}$, the payoff function for the low types $U_L(\gamma_0)$ is identical to $U_L(\gamma_0)$ given by in (3), and thus does not depend on ϵ. This is because a low type conceding with a positive rate is indifferent between persisting and conceding, and his payoff from conceding is computed with both the opposing low type conceding and the exogenous exit occurring at the instant with probability zero. For the high types, we show in the proof of Proposition B2 in Appendix B that the payoff function $U_H(\gamma_0)$ is decreasing in ϵ so long as $\gamma_0 > \gamma_*$. The intuition behind this result is that an increase in the exogenous exit rate directly reduces the probability that the high types receive their first best payoffs, which occurs only when the low types concede. Although an increase in ϵ generally has ambiguous effects on the equilibrium belief evolution and hence the equilibrium flow rate of concession by the low types, the negative direct effect dominates. The welfare effect of an increase in ϵ is negative in this case.

In the second case of $\gamma_0 \in [\min\{1, \alpha\}, 1)$, both $U_H(\gamma_0)$ and $U_L(\gamma_0)$ are increasing in ϵ, because a greater exogenous rate of exit reduces the expected duration of the equilibrium play without affecting the decision, which is always a coin flip. Therefore the welfare effect of an increase in ϵ is positive.

Thus, for any initial belief $\gamma_0 > \gamma_*$, as the exogenous exit rate ϵ increases, starting from $\epsilon = 0$ and α arbitrarily large, the welfare effect is negative for all ϵ such that $\alpha > \gamma_0$, and then positive for all greater ϵ. It follows that the optimal exogenous exit rate is either zero, which makes the game equivalent to the no-deadline game of $T = \infty$, or infinity, which is equivalent to ending the game by flipping a coin as in the equilibrium of the no-delay game of $T = 0$. In either case, we conclude that stochastic deadlines cannot be used to improve the ex ante welfare of the players.

The failure of stochastic deadlines illustrates the crucial role of the deadline play in improving the ex ante welfare of the players. Since the exogenous exit motivates the low types to either always concede with a positive flow rate, or always persist, stochastic deadlines cannot generate the kind of deadline effect under a finite deadline where the equilibrium play of the low types transits from an unsuccessful concession phase to a persistence phase when the time-to-deadline and the belief jointly reach some critical time horizon. The absence of such deadline effect under stochastic deadlines is the reason for its ineffectiveness in improving the ex ante welfare of the players.

5.2. Deadline penalties

A notable feature of the no-delay game with $T = 0$ in our model is that the equilibrium behavior of the low types, as well as the payoffs of both the low and high types, change discontinuously as γ increases from below γ_* to above. Corresponding to this discontinuity, there is a continuum of equilibria at $\gamma = \gamma_*$ when $T = 0$. This particular feature is not critical for our results. We demonstrate this robustness by modifying the model of Section 2 and introducing an additional payoff loss $\lambda > 0$ for the two players when they fail to reach an agreement by the end of the deadline. We assume that $\lambda < \beta/2$.[20]

The deadline penalty eliminates the payoff discontinuity and the multiplicity of equilibria at γ_* in the no-delay game, and redefines the boundary in the T-γ space that separates the concession and persistence phases. Define

$$\underline{\gamma}_* \equiv \frac{v_H - v_L - \beta + 2\lambda}{v_H - v_L + 4\lambda},$$

and

$$\overline{\gamma}_* \equiv \frac{v_H - v_L - \beta + 2\lambda}{v_H - v_L}.$$

There is now a unique equilibrium in the no-delay game ($T = 0$) for any belief γ. The high types always persist. The probability that the low types concede is zero for any $\gamma \geq \overline{\gamma}_*$, one for $\gamma \leq \underline{\gamma}_*$, and given by

$$Y(\gamma) = \frac{v_H - v_L - \beta + 2\lambda - (v_H - v_L)\gamma}{4\lambda\gamma} \tag{16}$$

for $\gamma \in (\underline{\gamma}_*, \overline{\gamma}_*)$. The new boundary $B(\gamma)$ is continuous, determined by the indifference condition of the low types between an immediate concession and the deadline play

$$U_L(\gamma) = -\delta B(\gamma) + U_L^0(\gamma),$$

[20] This assumption says that if it is known to be the disagreement state the low types still prefer the disagreement outcome of flipping a coin and paying the penalty λ to conceding to the other side and avoiding the penalty.

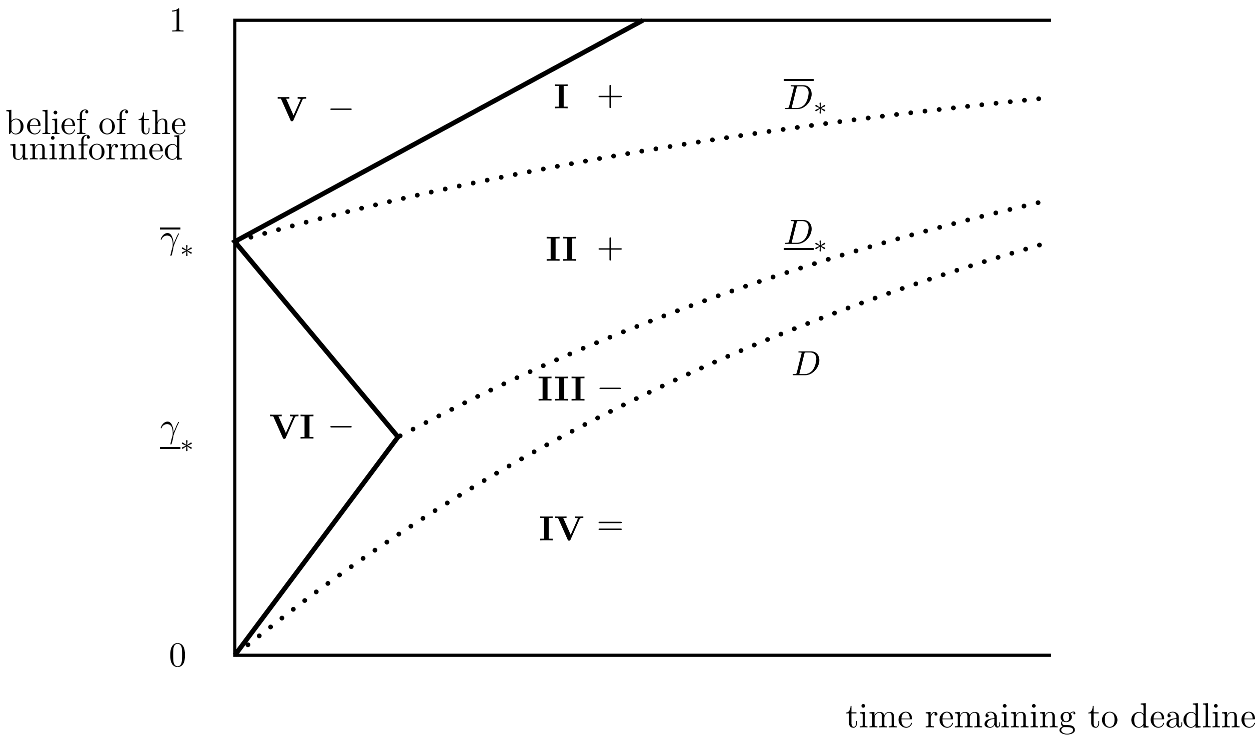

Figure 2

where $U_L(\gamma)$ remains the same as before and is given by (3), and $U_L^0(\gamma)$ is the unique continuation payoff in the no-delay game. The new boundary is shown as the thick piecewise linear graph in Figure 2. The main difference is that the horizontal segment corresponding to γ_* in Figure 1 is replaced by the downward sloping segment between $\overline{\gamma}_*$ and $\underline{\gamma}_*$ in Figure 2 when $\lambda > 0$.

Both the equilibrium characterization and the welfare analysis are quite similar to those for the case $\lambda = 0$, as can be seen in Figure 2 with two of the dotted curves indexed by λ. They are formally stated as Propositions C1 and C2 and proved in Appendix C. Here, we highlight the main difference that arises in this extension, which is the welfare analysis of the deadline in Region II in Figure 2. Let $S(T;\gamma_0)$ represent the phase-switch time when the updated belief hits the downward-sloping segment of the boundary B. The payoff $\mathcal{U}_H(S(T;\gamma_0))$ to the high types at $S(T;\gamma_0)$ is:

$$-\delta(T-S) + Y(g(S;\gamma_0))(v_H+\beta) + \Big(1 - Y(g(S;\gamma_0))\Big)\Big(\frac{v_H+v_L+\beta}{2} - \lambda\Big).$$

This is the boundary condition that determines the equilibrium payoff to the high types through the differential equation (5). Using the same argument as in the case without deadline penalty, we can decompose the welfare effect of the deadline $\partial U_H(\gamma_0)/\partial T$ in three terms as follows:

$$-\delta + Y'(g(S;\gamma_0))\dot{g}(S;\gamma_0)\Big(\frac{v_H - v_L + \beta}{2} + \lambda\Big)\frac{\partial S}{\partial T} + x(S)(v_H + \beta - \mathcal{U}_H(S))\frac{\partial S}{\partial T}. \qquad (17)$$

Lengthening the deadline prolongs the concession phase ($\partial S/\partial T > 0$). The loss is the additional delay, represented by the first term above, but there are two gains, represented by the second and third terms. The second term results because a prolonged concession phase means that the updated belief is lower when it hits the boundary ($\dot{g} < 0$), and thus the low types types concede with a higher probability at the deadline ($Y' < 0$ for $g(S;\gamma_0)$ between $\overline{\gamma}_*$ and $\underline{\gamma}_*$), reducing the chance of making the wrong decision and incurring the penalty. This term generalizes the second expression in (12) for Region II in the case of $\lambda = 0$. The third term is proportional to the flow rate of concession $x(S)$ by the low types times the relative gain to the high types of reaching an agreement during the concession phase. This term takes the form as in (13) for Regions I and III in the case of $\lambda = 0$, but is absent from (12) because the horizontal segment in Figure 1 means that $\partial S/\partial T = 0$ when $\lambda = 0$ in Region II. In spite of the differences, in the proof of Proposition C2 in Appendix C we show that the overall effect (17) is positive, as in Region II of Figure 1.

As in Section 4 where $\lambda = 0$, the optimal deadline is 0 for $\gamma_0 \leq \underline{\gamma}_*$, and is either 0 or $\underline{D}_*(\gamma_0)$ for $\gamma_0 > \underline{\gamma}_*$, where $\underline{D}_*(\gamma_0)$ is such that when the belief of the low types as determined by $g(t;\gamma_0)$ reaches $\underline{\gamma}_*$ the time remaining is $B(\underline{\gamma}_*)$. That is,

$$g(\underline{D}_*(\gamma_0) - B(\underline{\gamma}_*);\gamma_0) = \underline{\gamma}_*.$$

In the proof of Proposition C2 in Appendix C we compare the ex ante welfare at these two local maxima, and show that there exists an intermediate range of beliefs γ_0 above $\underline{\gamma}_*$ for which the optimal deadline is $\underline{D}_*(\gamma_0)$. Thus, the optimal deadline, when positive, is still characterized by the shortest concession phase that achieves efficient information aggregation at the deadline. The main properties of the optimal deadline established for the case of $\lambda = 0$—that it is finite, is not arbitrarily short, and is increasing in the degree of conflict—are all robust to the introduction of the deadline penalty.

6. Concluding Remarks

Damiano, Li and Suen (2009) use a discrete time model with more restrictive preference assumptions to show that costly delay can improve strategic information aggregation and hence ex ante welfare in a variety of environments with regard to deadlines. However, the discrete time framework is not suitable for studying the issue of optimal deadlines in strategic information aggregation, because an explicit characterization of equilibrium play is difficult to obtain.

In our model the positive welfare effects of extending the deadline are directly related to the deadline behavior of the low types stopping the concessions at some point and then conceding with a positive probability upon reaching the deadline. A longer deadline is beneficial for the high types even though the low types persist for a longer period of time during the deadline play, because the latter concede with a greater probability when the deadline is reached. We have argued that the failure in inducing this deadline behavior is the reason that stochastic deadlines, or exogenous negotiation breakdowns, are ineffective in raising ex ante welfare. However, an implicit assumption we have made in modeling stochastic deadlines is that exogenous breakdowns occur at a constant flow rate. We have not investigated either time-varying flow rates, or atoms in the flow rate. The latter case is perhaps more natural way of modeling stochastic deadlines, and is likely to generate some deadline behavior and positive welfare effects of increasing the breakdown rate. A related point is that we have assumed throughout that the two parties incur payoff losses from delay at a constant flow rate. It is possible that delay cost exhibits atoms in the flow rate that correspond to temporary suspensions of the negotiation process. We conjecture such atoms to generate some kind of deadline behavior and positive welfare effects.

In our framework of negotiation with a finite deadline, we have shown that there is a boundary such that there is a switch from a flow rate concession to full persistence by the low types. Modifications to the no-delay game change the equilibrium play only through changing the shape of the boundary. Although we have chosen the most natural no-delay game in our setup, it would be interesting to decouple the no-delay game and the no-deadline game by considering the no-delay payoffs in other reduced forms. Doing so may provide more general insights about the deadline effects and the optimal deadlines than given in the present model.

Our repeated proposal game is symmetric, and we have shown that there is a unique equilibrium and it is symmetric. Games with asymmetric preferences and delay costs are worth future research because asymmetry adds an interesting element to equilibrium dynamics of information aggregation. The restriction to equilibria in which the high types always persist is natural in our setup because they know what the mutually preferred choice is. Our Assumption 1, which implies that the payoff loss from making the wrong choice is greater for the high types than the payoff loss from conceding in the disagreement state for the low types, is sufficient for us to focus on equilibrium play of the low types and turn to the high types only for welfare analysis. In a more general setup, we may have one type better but not perfectly informed about the mutually preferred choice than other types. This would be more challenging as there is no longer the dichotomy between strategic analysis and welfare analysis, but the present paper is a starting point.

Our result that the optimal deadline is positive and increasing for intermediate levels of initial conflicts hinges on two implicit assumptions about the game that may be questioned in practice. First, the two parties in the joint decision situation are assumed to be able to commit to a precise deadline at the start of the negotiation process. According to our characterization of equilibrium play, before the process reaches the critical point when the parties are supposed to become inactive until the deadline arrives, they have no incentive to renegotiate the deadline. However, as soon as the critical point is reached, they would want to jump to the end-game play immediately. Of course if such renegotiation of the deadline is anticipated the equilibrium play before this critical point would be changed. It is potentially interesting to formalize this commitment issue and reexamine the optimal deadline. The other implicit assumption we have made is that the initial belief of the low types is common knowledge between the two parties when setting the deadline. We hasten to emphasize that our result that extending the deadline can have positive welfare effects is robust to slight perturbations to the initial belief of the low types. However, a perhaps more interesting issue is whether the two parties may find some way to communicate their knowledge about the initial degrees of conflict before jointly setting the deadline for negotiation. Such communication raises strategic issues that are worth further research.

Appendix A: Proofs

PROOF OF LEMMA 1.

For all time interval $[t, t + \mathrm{d}t)$ in $[t_1, t_2)$, a low type is indifferent between conceding, with the payoff $\mathcal{U}_L(t)$ given in (2), and persisting. Therefore,

$$\mathcal{U}_L(t) = \gamma(t)x(t)\mathrm{d}t\ (v_L + \beta) + \Big(\gamma(t)(1 - x(t)\mathrm{d}t) + (1 - \gamma(t))\Big)(-\delta\mathrm{d}t + \mathcal{U}_L(t + \mathrm{d}t)).$$

Subtracting $\mathcal{U}_L(t + \mathrm{d}t)$ from both sides of the equation, dividing by $\mathrm{d}t$, and taking the limit as $\mathrm{d}t$ goes to zero, we have a differential equation for the value function $\mathcal{U}_L(t)$. Using equation (2) for the value function, we can transform this differential equation for $\mathcal{U}_L(t)$ into a differential equation for $\gamma(t)$, given by

$$\dot{\gamma}(t) = \gamma(t)x(t)\left(\gamma(t) - \frac{v_H - v_L - \beta}{v_H - v_L}\right) - \frac{\delta}{v_H - v_L}.$$

By Bayes' rule, given the low type opponent is using the strategy represented by $x(t)$, the updated belief after persisting for the time interval $[t, t + \mathrm{d}t)$ is

$$\gamma(t + \mathrm{d}t) = \frac{\gamma(t)(1 - x(t)\mathrm{d}t)}{\gamma(t)(1 - x(t)\mathrm{d}t) + (1 - \gamma(t))}.$$

As $\mathrm{d}t$ goes to zero, the updating formula can be written as:

$$\dot{\gamma}(t) = -\gamma(t)(1 - \gamma(t))x(t).$$

The two equations for $\dot{\gamma}(t)$ and $x(t)$ reduce to (4). Using (4) and Bayes' rule, we also get

$$x(t) = \frac{1}{\gamma(t)}\frac{\delta}{\beta}.$$

PROOF OF PROPOSITION 1.

It suffices to show that it is optimal for the high types to always persist. This is clearly the case for $t \geq D(\gamma_0))$, as the continuation payoff for the high types is $v_H + \beta$ when the belief of the low types becomes zero. Using $U_H(0) = v_H + \beta$ as the boundary condition for the differential equation (6) and solving it, we have

$$U_H(\gamma) = v_H + \beta - \beta\left(1 + \frac{1 - \gamma}{\gamma}\ln(1 - \gamma)\right).$$

The above gives the equilibrium payoff of the high types for any $t < D(\gamma_0)$. Since $\gamma > 0$, it is immediate from the solution that this is greater than $v_H + \beta - \beta$, which by Assumption 1 is greater than v_L. Thus it is optimal for the high types to persist for any $t < D(\gamma_0)$.

Proof of Proposition 2.

Using the expressions (8) and (9), we can easily verify that $B(\gamma) \leq D(\gamma)$, with equality if and only if $\gamma = 0$. Thus, for T and γ_0 such that $T < D(\gamma_0)$, there is a unique phase-switch time $S = S(T; \gamma_0)$ given by (10). Further, $S > 0$ if and only if $T > B(\gamma_0)$. Finally, for T and γ_0 such that $T \in (B(\gamma_0), D(\gamma_0))$, by construction we have

$$U_L(g(S; \gamma_0)) = U_L^0(g(S; \gamma_0)) - \delta B(g(S; \gamma_0)),$$

so that the equilibrium payoff of the low types is continuous at $t = S$. We discuss three cases separately.

Case (i): $T \leq B(\gamma_0)$. The construction of B implies that it is optimal for the low types to persist for all $t < T$ and then concede with probability y at $t = T$, with $y = 1$ if $\gamma_0 < \gamma_*$, $y = 2\delta T/(\beta\gamma_*)$ if $\gamma_0 = \gamma_*$, and $y = 0$ if $\gamma_0 > \gamma_*$. For the high types, at any $t \leq T$, persisting all through the deadline yields

$$y(v_H + \beta) + (1 - y)\frac{v_H + v_L + \beta}{2} - \delta(T - t).$$

Conceding at any $t < T$ yields v_L, which by Assumption 1 is smaller than the above because

$$T - t < B(1) = \frac{\beta}{2\delta}.$$

Conceding at $t = T$ cannot be optimal either because it is not part of any equilibrium of the no-delay game.

Case (ii): $T \in (B(\gamma_0), D(\gamma_0))$. Case (i) already establishes that there is no incentive for any player to deviate at any $t \geq S$. Since the equilibrium payoff of the low types is continuous at $t = S$, there is no incentive for them to deviate at any $t < S$ either. For the high types, at any $t < S$ and corresponding belief $\gamma = g(t; \gamma_0)$ of the low types, the equilibrium payoff $U_H(\gamma)$ is given by the following solution to the differential equation (6):

$$U_H(\gamma) = v_H + \beta - \beta\frac{1 - \gamma}{\gamma}\ln(1 - \gamma) + \frac{1}{\gamma}\Big((1 - \gamma)(C + v_H + \beta) - \beta\Big),$$

where C is a constant determined by the boundary condition:

$$U_H(g(S;\gamma_0)) = y(\upsilon_H + \beta) + (1-y)\frac{\upsilon_H + \upsilon_L + \beta}{2} - \delta(T-S).$$

We already know from case (i) that $U_H(g(S;\gamma_0) \geq \upsilon_L$. For any $\gamma > g(S;\gamma_0)$, we have $U_H(\gamma) \geq \upsilon_L$ if

$$\frac{\upsilon_H - \upsilon_L}{1-\gamma} - \beta \ln(1-\gamma) + C \geq -\upsilon_L,$$

which is true because the left-hand-side is increasing in γ by Assumption 1. Thus, it is optimal for the high types to persist for all $t < S$.

Case (iii): $T \geq D(\gamma_0)$. The strategy and the belief given in the proposition form an equilibrium identical to the one in Proposition 1.

Proof of Proposition 3.

We first establish a series of claims. The proposition follows immediately after Claim 4 below.

Claim 1. In any equilibrium where the high types always persist, $y(t) = 0$ for all $t \in (0, T)$.

Proof. First, we show that $y(t) < 1$ for both low types at any time $t < T$. Suppose there is an equilibrium where a low type concedes for sure at some $t < T$. Then, for any $\eta > 0$, his low type opponent must concede with probability one before $t + \eta$. This is because, if the game continues past t, the opponent believes that it is the agreement state with probability one, and thus conceding immediately is optimal. Then, for η sufficiently small, persisting in the interval $[t, t+\eta)$ and then conceding at $t+\eta$ yields a strictly larger payoff than conceding with probability one at t for the initial player.

Next, we show that for any $t < T$ it cannot be the case that both low types concede with strictly positive probabilities. If $\gamma(t)$ is the belief of a low type upon reaching t, his equilibrium payoff is

$$\gamma(t)y(t)u_r + \gamma(t)(1-y(t))\upsilon_L + (1-\gamma(t))\upsilon_H,$$

where $y(t)$ is the probability that the opponent concedes at t and u_r the player's payoff in the continuation equilibrium after a reverse disagreement. For any small and positive

η, the payoff to the player from persisting in the interval $[t, t+\eta)$ and then conceding at $t+\eta$, is at least as large as

$$\gamma(t)y(t)(v_L+\beta)+\gamma(t)(1-y(t))v_L+(1-\gamma(t))v_H-\eta\delta.$$

Because the sum of the two players' payoffs in the continuation game after a reverse disagreement cannot exceed $2v_L+\beta$, for η sufficiently small at least one of the two low types has a profitable deviation.

Suppose now that one low type concedes with positive probability at some $t \in (0, T)$. His equilibrium payoff upon reaching t is $U_L(\gamma(t))$. Further, an argument similar to the above can be used to establish that for all η sufficiently small, his low type opponent must persist in the interval of time $[t-\eta, t]$. This implies that the player's belief γ does not change during the same interval. Then, conceding with probability one at $t-\eta$ is uniquely optimal because the player gets the same payoff from the decision but with a smaller delay cost. This is a contradiction because we have shown that conceding with probability one at any time but the deadline cannot be part of equilibrium strategies.

Claim 2. If in equilibrium the flow concession rate for one low type is $x(t)=0$ for all t in an open interval (t_1, t_2), then the same is true for the other low type. Further, $x(t)=0$ for all $t \in (t_1, T)$.

Proof. For the first part of the claim, suppose $x(t)=0$ for a low type in an interval (t_1, t_2). Then, his low type opponent's belief does not change over the same period. By the same argument as in the proof of Claim 1, if the opponent's equilibrium strategy is such that $\tilde{x}(t')>0$ for some $t' \in (t_1, t_2)$, it must also be the case that $\tilde{y}(t)=1$ for all $t \in (t_1, t')$, a contradiction to Claim 1. For the second part of the claim, the same argument as above implies that if $t'=\inf_{t\geq t_2}\{t : x(t)>0\}<T$, then the equilibrium strategy of this low type must prescribe that $y(t)=1$ for all $t \in (t_1, t')$, again a contradiction.

Claim 3. All equilibria where the high types always persist are symmetric.

Proof. First, we show that in any equilibrium if $t' \equiv \inf\{t : x(t)=0\}>0$, then the beliefs of the low types upon reaching t' are identical. By Claim 2, both low types stop conceding at the same time t' and persist until the deadline is reached. Thus, the belief of each low type upon reaching the deadline is the same as his belief upon reaching t'. Since t' is interior, Claim 1 implies that at t' both low types are indifferent between conceding and persisting until the deadline and then playing the equilibrium strategy in the no-delay game associated with their own beliefs at t'. Suppose that upon reaching t', the two low types hold beliefs γ' and γ'', and both are above γ_*. If B is the time left to the deadline at t' the indifference conditions of the two low types can be satisfied simultaneously $\gamma' = \gamma''$. If $\gamma' = \gamma'' = \gamma_*$, for the two indifference conditions to hold for the same B, the equilibrium strategy upon reaching the deadline must also be symmetric. The other cases are similar.

Claim 2 above has already established the symmetry of the equilibrium strategy for the low types when $t' = 0$. If instead $t' > 0$, the same claim also implies that both low types concede at a strictly positive flow rate in the interval $(0, t')$. Since the belief upon reaching t' is the same in equilibrium for the two low types, the uniqueness of the solution to the differential equation for the equilibrium evolution of the belief implies that both the beliefs and the flow rate of concession are identical for the two players for any $t \in (0, t')$. Since the initial belief γ_0 is identical for the two players, neither player can concede with a positive probability at $t = 0$ either, thus the equilibrium strategies are symmetric for all t.

Claim 4. For any initial belief γ_0 of the low types, equation (9) gives the unique value of $B(\gamma_0)$ such that in any equilibrium $x(0) = 0$ if $T < B(\gamma_0)$ and $x(0) > 0$ if $T > B(\gamma_0)$.

Proof. Fix any initial belief γ_0. Suppose that in some equilibrium $x(0) = 0$ for some deadline $T > B(\gamma_0)$. By Claim 2, we have $x(t) = 0$ for all $t \in [0, T)$. The payoff to the low types in this posited equilibrium is then $-T\delta + U_L^0(\gamma_0)$ where U_L^0 is the payoff function of the no-delay game (with the best equilibrium payoff corresponding to the low types conceding with probability one in the case of $\gamma_0 = \gamma_*$). By the construction of $B(\gamma_0)$, this payoff is strictly less than the payoff from conceding immediately at $t = 0$, a contradiction.

Now, suppose that in some equilibrium $x(0) > 0$ for some deadline $T < B(\gamma_0)$. The expected payoff to the low types from this posited equilibrium is equal to the payoff from conceding immediately, which is $U_L(\gamma_0)$. Consider the following deviation strategy for a low type: persist until the deadline, and then play the unique equilibrium strategy in the no-delay game corresponding to γ_0 if $\gamma_0 \neq \gamma_*$ and concede with probability one if $\gamma_0 = \gamma_*$. For $\gamma_0 \neq \gamma_*$, since the payoff to the low type increases whenever the low type opponent concedes, and in the no-delay game the equilibrium probability of concession is decreasing in the belief of the low types, the payoff from this deviation is at least as large as when the opposing low type follows the same deviation strategy. The same is true for $\gamma_0 = \gamma_*$, because if the low type opponent initially concedes with a positive flow rate for any arbitrarily small interval of time, his belief falls below γ_* in the posited equilibrium. It follows then from the construction of B that this is a profitable deviation, a contradiction.

Proof of Proposition 4.

First, we show that the welfare effect (13) is positive in Region I of Figure 1. The phase-switch time S is defined by the indifference condition:

$$\delta(T-S) = U_L^0(g(S;\gamma_0)) - U_L(g(S;\gamma_0)) = \frac{g(S;\gamma_0)-\gamma_*}{1-\gamma_*}\frac{\beta}{2}.$$

Taking derivative respect to T, and using the fact that $\dot{g} = -(1-g)\delta/\beta$, we obtain:

$$\frac{\partial S}{\partial T} = \frac{2(1-\gamma_*)}{1-2\gamma_* + g(S;\gamma_0)}.$$

Furthermore, by Assumption 1,

$$\upsilon_H + \beta - \mathcal{U}_H(S) = \frac{\upsilon_H - \upsilon_L + \beta}{2} + \delta(T-S) > \frac{\beta}{2}\left(1 + \frac{g(S;\gamma_0)-\gamma_*}{1-\gamma_*}\right).$$

Finally, since $x(S) = \delta/(\beta g(S;\gamma_0))$, we have

$$x(S)(\upsilon_H + \beta - \mathcal{U}_H(S))\frac{\partial S}{\partial T} > \frac{\delta}{g(S;\gamma_0)} > \delta.$$

Next, we show that the welfare effect (13) is negative in Region III. The phase-switch time S is defined by:

$$\delta(T-S) = g(S;\gamma_0)\frac{\beta}{2}.$$

Take derivative respect to T to get $\partial S/\partial T = 2/(1+g(S;\gamma_0))$. Furthermore,

$$\upsilon_H + \beta - \mathcal{U}_H(S) = \delta(T-S) = g(S;\gamma_0)\frac{\beta}{2}.$$

Therefore,

$$x(S)(\upsilon_H + \beta - \mathcal{U}_H(S))\frac{\partial S}{\partial T} = \frac{\delta}{1+g(S;\gamma_0)} < \delta.$$

The final part of the proof is to compare the value of $U^T(\gamma_0)$ at the two local maxima $T = 0$ and $T = \underline{D}_*(\gamma_0)$ for $\gamma_0 > \gamma_*$. The ex ante welfare $U^0(\gamma_0)$ for $T = 0$ is given by (14). Let $U_L^*(\gamma_0)$ and $U_H^*(\gamma_0)$ be the welfare of the low types and high types when $T = \underline{D}_*(\gamma_0)$, and let U^* be the weighted average of the two as in (14). We have $U_L^*(\gamma_0) = \gamma_0 \upsilon_L + (1-\gamma_0)\upsilon_H$ as given by (3). Solving the differential equation (6) for the payoff to the high types with the boundary condition $U_H^T(\gamma_*) = \upsilon_H + \beta - \delta B(\gamma_*)$, we obtain

$$U_H^*(\gamma_0) = \upsilon_H + \beta - \frac{1-\gamma_0}{\gamma_0}\left(\ln\left(\frac{1-\gamma_0}{1-\gamma_*}\right) + \frac{1}{1-\gamma_0} - \frac{2-\gamma_*^2}{2(1-\gamma_*)}\right)\beta.$$

The difference in ex ante welfare $U^*(\gamma_0) - U^0(\gamma_0)$ is equal to $\Delta(\gamma_0)/(2(2-\gamma_0))$, where

$$\Delta(\gamma_0) = 2(1-\gamma_0)(\upsilon_H - \upsilon_L) - \gamma_0\beta - \frac{2(1-\gamma_0)^2}{\gamma_0}\left(\ln\left(\frac{1-\gamma_0}{1-\gamma_*}\right) + \frac{1}{1-\gamma_0} - \frac{2-\gamma_*^2}{2(1-\gamma_*)}\right)\beta.$$

Take derivative of Δ with respect to γ_0 to obtain:

$$\Delta'(\gamma_0) = -2(\upsilon_H - \upsilon_L) - 3\beta + \frac{2(1-\gamma_0^2)}{\gamma_0^2}\left(\ln\left(\frac{1-\gamma_0}{1-\gamma_*}\right) + \frac{1}{1-\gamma_0} - \frac{2-\gamma_*^2}{2(1-\gamma_*)}\right)\beta.$$

The limit of the last term as γ_0 goes to one is equal to 4β. Further, it is increasing for all $\gamma_0 > \gamma_*$: the derivative has the same sign as

$$-1 - (1+\gamma_0)^2 - 2\ln\left(\frac{1-\gamma_0}{1-\gamma_*}\right) + \frac{2-\gamma_*^2}{1-\gamma_*},$$

which is an increasing function of γ_0; at $\gamma_0 = \gamma_*$, this derivative is equal to $\gamma_*^3/(1-\gamma_*)$, which is positive. Thus, $\Delta'(\gamma_0) \le -2(\upsilon_H - \upsilon_L) + \beta$, which is negative by Assumption 1. We have proved that $\Delta(\gamma_0) = 0$ implies $\Delta'(\gamma_0) < 0$ for all $\gamma_0 > \gamma_*$. Note that $\lim_{\gamma_0 \downarrow \gamma_*} \Delta(\gamma_*) = (1-\gamma_*)(\upsilon_H - \upsilon_L + \beta - \gamma_*\beta)$, which is positive by Assumption 1. Also, $\lim_{\gamma_0 \to 1} \Delta(\gamma_0) = -\beta < 0$. It follows from the intermediate value theorem that there exists a $\overline{\gamma} \in (\gamma_*, 1)$ such that $\Delta(\overline{\gamma}) = 0$. Moreover, the single-crossing property of Δ implies that such $\overline{\gamma}$ is unique, with $U^*(\gamma_0) > U^0(\gamma_0)$ if and only if $\gamma_0 \in (\gamma_*, \overline{\gamma})$.

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APPENDICES B AND C

(NOT INTENDED FOR PUBLICATION)

Appendix B. Stochastic Deadlines

PROPOSITION B1. *Suppose that $T = \infty$ and $\epsilon > 0$. There exists a symmetric equilibrium in which the high types always persist; the low types with belief γ concede with a flow rate equal to $\epsilon(\alpha - \gamma)/(2(1-\gamma_*)\gamma)$ if $\gamma(t) \in (0, \min\{1, \alpha\})$, concede with probability one if $\gamma = 0$ and persist if $\gamma \in [\min\{1, \alpha\}, 1)$; and the belief $\gamma(t)$ of the low types solves (15) with the initial value γ_0 if $\gamma_0 < \min\{1, \alpha\}$, and is equal to γ_0 if $\gamma_0 \in [\min\{1, \alpha\}, 1)$.*

PROOF. First, we derive the differential equation (15) for the equilibrium belief evolution. Note that the expected payoff of the low types from conceding is still given by (2). The payoff from persisting becomes

$$\gamma(t)x(t)\mathrm{d}t\ (\upsilon_L + \beta) + \Big(\gamma(t)(1 - x(t)\mathrm{d}t) + (1 - \gamma(t))\Big)(1 - \epsilon\mathrm{d}t)(-\delta\mathrm{d}t + \mathcal{U}_L(t + \mathrm{d}t))$$
$$+ \epsilon\mathrm{d}t\ \left((1 - \gamma(t))\frac{\upsilon_H + \upsilon_L + \beta}{2} + \gamma(t)(1 - x(t)\mathrm{d}t)\frac{2\upsilon_L + \beta}{2}\right),$$

where $x(t)$ denotes the flow rate of concession by the low types. Equating the two payoff expressions and using the same Bayes' rule as in the proof of Lemma 1 immediately give us (15). The corresponding flow rate of concession is

$$x(t) = \frac{\epsilon(\alpha - \gamma(t))}{2(1 - \gamma_*)\gamma(t)}.$$

For the case of $\gamma_0 \in (0, \min\{1, \alpha\})$, it suffices to verify that the equilibrium payoff of the high types is at least as large as the payoff from deviating to conceding, which is equal to υ_L regardless of ϵ. The differential equation for the value function of the high types is

$$U_H'(\gamma) = -\frac{(\alpha - \gamma_*)\beta + (1 - \gamma_*)(\upsilon_H - \upsilon_L + \beta)}{(1 - \gamma)(\alpha - \gamma)} + \frac{\alpha - \gamma + 2(1 - \gamma_*)\gamma}{\gamma(1 - \gamma)(\alpha - \gamma)}(\upsilon_H + \beta - U_H(\gamma)),$$

with the boundary condition $U_H(0) = \upsilon_H + \beta$. The solution to this differential equation is

$$U_H(\gamma) = \upsilon_H + \beta - \left(1 - \frac{1 - \gamma}{\gamma}\frac{K(\gamma)}{2(1 - \gamma_*)}\right)\frac{(\alpha - \gamma_*)\beta + (1 - \gamma_*)(\upsilon_H - \upsilon_L + \beta)}{(\alpha - \gamma_*) + (1 - \gamma_*)},$$

where

$$K(\gamma) \equiv \alpha - \alpha \left(\frac{\alpha(1-\gamma)}{\alpha - \gamma} \right)^{2\epsilon\beta/(2\delta - \epsilon\beta)}.$$

Note that $K(\gamma) > 0$ for all $\gamma \in (0, \alpha)$, regardless of whether α is greater or less than one. Since

$$\frac{(\alpha - \gamma_*)\beta + (1 - \gamma_*)(\upsilon_H - \upsilon_L + \beta)}{(\alpha - \gamma_*) + (1 - \gamma_*)} \leq \upsilon_H - \upsilon_L + \beta,$$

it follows immediately from Assumption 1 that $U_H(\gamma) \geq \underline{\upsilon}_L$ for all γ.

For the case of $\gamma_0 \in [\min\{1, \alpha\}, 1)$, in equilibrium the game ends with exogenous exit, with a terminal payoff of $(\upsilon_H + \upsilon_L + \beta)/2$ to the high types and

$$\gamma \frac{2\upsilon_L + \beta}{2} + (1 - \gamma) \frac{\upsilon_H + \upsilon_L + \beta}{2}$$

to the low types. Further, the exogenous exit time follows an exponential distribution with parameter ϵ, and hence the expected duration of the game is $1/\epsilon$. Thus, the equilibrium expected payoff loss from delay is δ/ϵ for both the high and low types. If the low types deviate to conceding, the expected payoff is

$$\gamma\beta + (1 - \gamma)\upsilon_H < \gamma \frac{2\upsilon_L + \beta}{2} + (1 - \gamma) \frac{\upsilon_H + \upsilon_L + \beta}{2} - \frac{\delta}{\epsilon},$$

because $\gamma < \alpha$. For the high types, the expected payoff from concession is υ_L, which is lower than the equilibrium payoff because $\upsilon_H - \upsilon_L + \beta > 2\delta/\epsilon$, by Assumption 1 and by the assumption that $\alpha < 1$.

Proposition B2. *Suppose that $T = \infty$. For any $\gamma_0 > \gamma_*$, the optimal exogenous exit rate is either zero or infinity.*

Proof. It suffices to establish that $U_H(\gamma_0)$ for the case $\gamma_0 < \min\{1, \alpha\}$ is decreasing in ϵ for $\gamma_0 > \gamma_*$.

It is convenient to use the fact that $\lim_{\gamma_0 \to 0} K(\gamma_0) = 0$ to write

$$K(\gamma_0) = \int_0^{\gamma_0} k(\gamma) \, \mathrm{d}\gamma,$$

where

$$k(\gamma) = \frac{2(1 - \gamma_*)}{(1 - \gamma)^2} \left(\frac{\alpha(1 - \gamma)}{\alpha - \gamma} \right)^{(2\delta + \epsilon\beta)/(2\delta - \epsilon\beta)}.$$

The term $K(\gamma_0)(1-\gamma_0)/\gamma_0$ is decreasing in γ_0 because its derivative is

$$\begin{aligned}
&\frac{1-\gamma_0}{\gamma_0}k(\gamma_0) - \frac{1}{\gamma_0^2}K(\gamma_0) \\
=&-\frac{\alpha}{\gamma_0^2}\left(1-\left(\frac{\alpha(1-\gamma_0)}{\alpha-\gamma_0}\right)^{2\epsilon\beta/(2\delta-\epsilon\beta)}\left(\frac{2(1-\gamma_*)\gamma_0}{\alpha-\gamma_0}+1\right)\right) \\
=&-\frac{\alpha}{\gamma_0^2}\int_0^{\gamma_0} 2(1-\gamma_*)\left(\frac{\alpha(1-\gamma)}{\alpha-\gamma}\right)^{2\epsilon\beta/(2\delta-\epsilon\beta)}\frac{\gamma((\alpha-\gamma_*)+(1-\gamma_*))}{(\alpha-\gamma)^2(1-\gamma)}\,\mathrm{d}\gamma,
\end{aligned}$$

which is negative as $\alpha > \gamma_*$. Now, since $\lim_{\gamma_0\to 0} K(\gamma_0) = 0$, and thus

$$\lim_{\gamma_0\to 0}\frac{K(\gamma_0)}{\gamma_0} = \lim_{\gamma_0\to 0} k(\gamma_0) = 2(1-\gamma_*),$$

we have

$$\frac{1-\gamma_0}{\gamma_0}\frac{K(\gamma_0)}{2(1-\gamma_*)} < 1$$

for all $\gamma_0 > 0$. Because the coefficient on $K(\gamma_0)$ in the $U_H(\gamma_0)$ function is increasing in ϵ, a sufficient condition for $U_H(\gamma_0)$ to be decreasing in ϵ is that $K(\gamma_0)$ is increasing in α. A sufficient condition for the latter is that $\ln k(\gamma_0)$ is increasing in α, or

$$-\ln\left(\frac{\alpha(1-\gamma_0)}{\alpha-\gamma_0}\right) + \frac{(\alpha-1)\gamma_0}{\alpha(1-\gamma_0)}\frac{(\alpha-\gamma_*)+(1-\gamma_*)}{2(1-\gamma_*)} > 0.$$

Since the above is equal to zero at $\gamma_0 = 0$, it is sufficient if its derivative with respect to γ_0 is strictly positive. This derivative is given by

$$\left(\frac{\alpha-1}{\alpha-\gamma_0}\right)^2\left(\frac{1}{1-\gamma_0}-\frac{1}{2(1-\gamma_*)}\right).$$

Therefore, $U_H(\gamma_0)$ decreases with ϵ so long as $\gamma_0 > \gamma_*$.

Appendix C. Deadline Penalties

PROPOSITION C1. *Suppose that $T < \infty$, and $\lambda \in (0, \beta/2]$. There is a symmetric equilibrium in which the high types always persist; the strategy of the low types at time t with any belief γ is such that: (i) if $t = T$, concede with probability one if $\gamma \le \underline{\gamma}_*$, with probability zero if $\gamma \ge \overline{\gamma}_*$, and with probability $Y(\gamma)$ if $\gamma \in (\underline{\gamma}_*, \overline{\gamma}_*)$; (ii) if $T - t \in (0, B(\gamma)]$, persist;*

and (iii) if $T - t > B(\gamma)$, concede with a flow rate $\delta/(\beta\gamma)$ if $\gamma > 0$ and with probability one if $\gamma = 0$.

PROOF. For case (i), we show that there is a unique equilibrium in the game without delay ($T = 0$). Fix a belief γ that it is the disagreement state for a low type. Suppose that the opposing low type concedes with probability y. Then the difference between the low type's payoff from conceding and his payoff from persisting is

$$-\gamma\Big(\frac{\beta}{2} - \lambda\Big) + (1-\gamma)\Big(\frac{v_H - v_L - \beta}{2} + \lambda\Big) - 2\gamma\lambda y.$$

The above is strictly decreasing in y, and therefore there is a unique equilibrium for any γ, given as follows. If $\gamma \leq \underline{\gamma}_*$, then the difference in payoffs is always non-negative, and thus the unique equilibrium is $y = 1$; if $\gamma \geq \overline{\gamma}_*$, the difference in payoffs is always non-positive and thus the unique equilibrium is $y = 0$; and if $\gamma \in (\underline{\gamma}_*, \overline{\gamma}_*)$, the unique equilibrium is $y = Y(\gamma)$, where $Y(\gamma)$ is given by (16).

For case (ii), the equilibrium payoff to the low types at any time $t' \in [t, T)$ from persisting throughout the game is given by

$$\gamma\Big(\tilde{Y}(\gamma)(v_L + \beta) + (1 - \tilde{Y}(\gamma))\Big(\frac{2v_L + \beta}{2} - \lambda\Big)\Big) + (1-\gamma)\Big(\frac{v_H + v_L + \beta}{2} - \lambda\Big) - \delta(T - t'),$$

where

$$\tilde{Y}(\gamma) = \begin{cases} 1 & \text{if } \gamma \leq \underline{\gamma}_* \\ Y(\gamma) & \text{if } \gamma_0 \in (\underline{\gamma}_*, \overline{\gamma}_*) \\ 0 & \text{if } \gamma \geq \overline{\gamma}_*. \end{cases}$$

It is straightforward to show that if $t' = t$ and $T - t = B(\gamma)$, the above is equal to $U_L(\gamma)$, the deviation payoff to a low type from conceding at time t' given the equilibrium strategy of the low type opponent. Thus, there is no incentive for the low types to deviate for any time $t' \in [t, T)$. For the high types, at any $t' \in [t, T]$ the equilibrium payoff from persisting is

$$\tilde{Y}(\gamma)(v_H + \beta) + (1 - \tilde{Y}(\gamma))\Big(\frac{v_H + v_L + \beta}{2} - \lambda\Big) - \delta(T - t').$$

The payoff from conceding right away is v_L. It is optimal for the high types to persist if

$$\tilde{Y}(\gamma)(v_H - v_L + \beta) + (1 - \tilde{Y}(\gamma))\Big(\frac{v_H - v_L + \beta}{2} - \lambda\Big) \geq \delta T.$$

We have just argued that the low types weakly prefer persisting until the deadline followed by conceding with probability $\tilde{Y}(\gamma)$ to conceding immediately. Since $\tilde{Y}(\gamma) > 0$ for $\gamma < \overline{\gamma}_*$, the equilibrium condition of the low types implies that

$$\gamma\tilde{Y}(\gamma)\Big(\frac{2\upsilon_L + \beta}{2} - \lambda\Big) + \gamma(1 - \tilde{Y}(\gamma))\upsilon_L + (1-\gamma)\upsilon_H - \delta(T-t) \geq U_L(\gamma),$$

or

$$\gamma\tilde{Y}(\gamma)\Big(\frac{\beta}{2} - \lambda\Big) \geq \delta(T-t).$$

By Assumption 1 and the assumption that $\lambda \leq \beta/2$, we have

$$\tilde{Y}(\gamma)(\upsilon_H - \upsilon_L + \beta) + (1 - \tilde{Y}(\gamma))\Big(\frac{\upsilon_H - \upsilon_L + \beta}{2} - \lambda\Big) > \frac{\upsilon_H - \upsilon_L + \beta}{2} - \lambda > \gamma\tilde{Y}(\gamma)\Big(\frac{\beta}{2} - \lambda\Big),$$

and thus the equilibrium condition of the high types is satisfied. For the case of $\gamma \geq \overline{\gamma}_*$ we have $\tilde{Y}(\gamma) = 0$, and the equilibrium condition of the low types is

$$\gamma\Big(\frac{2\upsilon_L + \beta}{2} - \lambda\Big) + (1-\gamma)\Big(\frac{\upsilon_H + \upsilon_L + \beta}{2} - \lambda\Big) - \delta(T-t) \geq \gamma\upsilon_L + (1-\gamma)\upsilon_H,$$

which implies

$$\gamma\Big(\frac{\beta}{2} - \lambda\Big) > \delta(T-t).$$

Thus, the equilibrium condition of the high types is satisfied.

For case (iii), for any initial belief γ_0, either $T > D(\gamma_0)$, in which case the proof is the same as the case of no deadlines in Section 3, or otherwise on the equilibrium path there is a unique time $S(T;\gamma_0) = S$ satisfying

$$T - S = B(g(S;\gamma_0)).$$

By construction, the low types are indifferent between conceding and persisting for all $t \in [0, S)$, so there is no profitable deviation before $t = S$. Further, by construction, the equilibrium payoff to the low types at $t = S$ is

$$\begin{aligned}\mathcal{U}_L(S) =& g(S;\gamma_0)\Big(\tilde{Y}(\gamma_0)(\upsilon_L + \beta) + (1 - \tilde{Y}(\gamma_0))\frac{2\upsilon_L + \beta}{2} - \lambda\Big) \\ &+ (1 - g(S;\gamma_0))\Big(\frac{\upsilon_H + \upsilon_L + \beta}{2} - \lambda\Big) - \delta(T-S).\end{aligned}$$

Thus, by the argument for cases (i) and (ii) above, there is no profitable deviation for the low types after $t = S$ either. For the high types, given the arguments for cases (i) and (ii), it suffices to show that there is no profitable deviation before $t = S$. The equilibrium payoff function $U_H(\gamma)$ at any $\gamma = g(t; \gamma_0)$ for $t < S$ is given by the solution to the differential equation (6) with the boundary condition that

$$\begin{aligned} U_H(g(S;\gamma_0)) = & \tilde{Y}(g(S;\gamma_0))(\upsilon_H + \beta) - \delta(T - S) \\ & + \Big(1 - \tilde{Y}(g(S;\gamma_0))\Big)\Big(\frac{\upsilon_H + \upsilon_L + \beta}{2} - \lambda\Big). \end{aligned}$$

The claim that it is optimal for the high types to persist at all $t < S$ follows from identical arguments as in the proof of Proposition 2.

PROPOSITION C2. *Suppose that $\lambda \in (0, \beta/2)$. There exist thresholds $\underline{\gamma}$ and $\overline{\gamma}$, with $\underline{\gamma}_* < \underline{\gamma} < \overline{\gamma}_* < \overline{\gamma} < 1$, such that the optimal deadline for any initial belief γ_0 is $\underline{D}_*(\gamma_0)$ if $\gamma_0 \in (\underline{\gamma}, \overline{\gamma})$, and is zero otherwise.*

PROOF. We first verify that the welfare effects are positive in Regions I and B but negative in Region III in Figure 2.

In Region II, the phase-switch time S is defined by the indifference condition for the low types at the boundary B:

$$\delta(T - S) = g(S;\gamma_0)Y(g(S;\gamma_0))\Big(\frac{\beta}{2} - \lambda\Big).$$

Taking derivative with respect to T, and using the definition of Y in equation (16), we obtain:

$$\frac{\partial S}{\partial T} = \frac{8\lambda\beta}{8\lambda\beta + (1 - g(S;\gamma_0))(\upsilon_H - \upsilon_L)(\beta - 2\lambda)}.$$

Now, an explicit calculation of $\partial U_H(\gamma_0)/\partial T$ given in equation (17) yields:

$$\begin{aligned} \frac{\partial U_H(\gamma_0)}{\partial T} = & \frac{\delta}{8\lambda\beta g(S;\gamma_0)}\Big((\beta + 2\lambda)(\upsilon_H - \upsilon_L + \beta + 2\lambda) \\ & + (\upsilon_H - \upsilon_L)(\beta - 2\lambda)(\overline{\gamma}_* - g(S;\gamma_0))\Big)\frac{\partial S}{\partial T} - \delta. \end{aligned}$$

Since $\partial S/\partial T > 0$, by Assumption 1 the above expression is greater than:

$$\delta \frac{(\beta + 2\lambda)^2 + (\upsilon_H - \upsilon_L)(\beta - 2\lambda)(\overline{\gamma}_* - g(S;\gamma_0))}{g(S;\gamma_0)\big(8\lambda\beta + (1 - g(S;\gamma_0))(\upsilon_H - \upsilon_L)(\beta - 2\lambda)\big)} - \delta,$$

which is equal to $\delta/g(S;\gamma_0) - \delta > 0$.

In Region I, the phase-switch time S is defined by the indifference condition:

$$\delta(T-S) = \frac{g(S;\gamma_0) - \gamma_*}{2(1-\gamma_*)}\beta - \lambda.$$

Take derivative respect to T to get

$$\frac{\partial S}{\partial T} = \frac{2(1-\gamma_*)}{1 - 2\gamma_* + g(S;\gamma_0)}.$$

Furthermore, by Assumption 1,

$$\upsilon_H + \beta - \mathcal{U}_H(S) = \frac{\upsilon_H - \upsilon_L + \beta}{2} + \delta(T-S) + \lambda > \frac{\beta}{2}\frac{1 - 2\gamma_* + g(S;\gamma_0)}{1-\gamma_*}.$$

Finally, since $x(S) = \delta/(\beta g(S;\gamma_0))$, we have

$$\frac{\partial U_H(\gamma_0)}{\partial T} = -\delta + x(S)(\upsilon_H + \beta - \mathcal{U}_H(S))\frac{\partial S}{\partial T} > 0.$$

In Region III, the phase-switch time S is defined by:

$$\delta(T-S) = g(S;\gamma_0)\Big(\frac{\beta}{2} - \lambda\Big).$$

Take derivative respect to T to get

$$\frac{\partial S}{\partial T} = \frac{2(1-\gamma_*)}{2(1-\gamma_*) - (1 - g(S;\gamma_0))(1-\overline{\gamma}_*)}.$$

Furthermore,

$$\upsilon_H + \beta - \mathcal{U}_H(S) = \delta(T-S) = \frac{g(S;\gamma_0)}{2}\frac{1-\overline{\gamma}_*}{1-\gamma_*}\beta.$$

Therefore,

$$\begin{aligned}\frac{\partial U_H(\gamma_0)}{\partial T} &= -\delta + x(S)(\upsilon_H + \beta - \mathcal{U}_H(S))\frac{\partial S}{\partial T} \\ &= -\delta + \frac{\delta(1-\overline{\gamma}_*)}{2(1-\gamma_*) - (1-g(S;\gamma_0))(1-\overline{\gamma}_*)} \\ &\leq \frac{2\delta(\gamma_* - \overline{\gamma}_*)}{2(1-\gamma_*) - (1-\overline{\gamma}_*)},\end{aligned}$$

which is negative.

The remainder of the proof is to compare the value of ex ante welfare $U_L(\gamma_0)$ at the two local maxima of zero and $\underline{D}_*(\gamma_0)$ for $\gamma_0 > \underline{\gamma}_*$.

The equilibrium payoff functions for the high and low types in the no-delay game are given by

$$U_L^0(\gamma_0) = \begin{cases} \gamma_0(\upsilon_L + \beta/2 - \lambda) + (1-\gamma_0)\upsilon_H & \text{if } \gamma_0 \in [0, \underline{\gamma}_*), \\ \gamma_0 \upsilon_L + (1-\gamma_0)\upsilon_H + \gamma_0 Y(\gamma_0)(\beta/2 - \lambda) & \text{if } \gamma_0 \in [\underline{\gamma}_*, \overline{\gamma}_*], \\ \gamma_0(\upsilon_L + \beta/2 - \lambda) + (1-\gamma_0)((\upsilon_H + \upsilon_L + \beta)/2 - \lambda) & \text{if } \gamma_0 \in (\overline{\gamma}_*, 1); \end{cases}$$

and

$$U_H^0(\gamma_0) = \begin{cases} \upsilon_H + \beta & \text{if } \gamma_0 \in [0, \underline{\gamma}_*), \\ Y(\gamma_0)(\upsilon_H + \beta) + (1 - Y(\gamma_0))((\upsilon_H + \upsilon_L + \beta)/2 - \lambda) & \text{if } \gamma_0 \in [\underline{\gamma}_*, \overline{\gamma}_*], \\ (\upsilon_H + \upsilon_L + \beta)/2 - \lambda & \text{if } \gamma_0 \in (\overline{\gamma}_*, 1). \end{cases}$$

Under the deadline $T = \underline{D}_*(\gamma_0)$, the payoff to the low types is simply $U_L^*(\gamma_0) = U_L(\gamma_0)$ as in (3). To compute the payoff to the high types, we solve the differential equation (6) with the boundary condition

$$U_H(\underline{\gamma}_*) = \upsilon_H + \beta - \delta B(\underline{\gamma}_*).$$

This gives the payoff to the high types when the deadline is $T = \underline{D}_*(\gamma_0)$:

$$\begin{aligned} U_H^*(\gamma_0) =& \upsilon_H + \beta - \frac{1-\gamma_0}{\gamma_0}\left(\ln\left(\frac{1-\gamma_0}{1-\underline{\gamma}_*}\right) + \frac{\gamma_0 - \underline{\gamma}_*}{(1-\gamma_0)(1-\underline{\gamma}_*)}\right)\beta \\ & - \frac{1-\gamma_0}{\gamma_0}\frac{\underline{\gamma}_*^2}{1-\underline{\gamma}_*}\left(\frac{\beta}{2} - \lambda\right). \end{aligned}$$

The difference in ex ante welfare $U_L^*(\gamma_0) - U_L^0(\gamma_0)$ is

$$\frac{1}{2-\gamma_0}(U_L^*(\gamma_0) - U_L^0(\gamma_0)) + \frac{1-\gamma_0}{2-\gamma_0}(U_H^*(\gamma_0) - U_H^0(\gamma_0)) \equiv \frac{1}{2(2-\gamma_0)}\Delta(\gamma_0).$$

Since $Y(\underline{\gamma}_*) = 1$, we have

$$\Delta(\underline{\gamma}_*) = -\underline{\gamma}_*(\beta - 2\lambda) - \underline{\gamma}_*(1-\underline{\gamma}_*)(\beta - 2\lambda) < 0.$$

Since $Y(\overline{\gamma}_*) = 0$, we have

$$\begin{aligned} \Delta(\overline{\gamma}_*) = (1-\overline{\gamma}_*)(\upsilon_H - \upsilon_L + \beta - 2\lambda) - \frac{2(1-\overline{\gamma}_*)^2}{\overline{\gamma}_*}\frac{\underline{\gamma}_*^2}{1-\underline{\gamma}_*}(\beta - 2\lambda) \\ - \frac{2(1-\overline{\gamma}_*)^2}{\overline{\gamma}_*}\left(\ln\left(\frac{1-\overline{\gamma}_*}{1-\underline{\gamma}_*}\right) + \frac{\overline{\gamma}_* - \underline{\gamma}_*}{(1-\overline{\gamma}_*)(1-\underline{\gamma}_*)}\right)\beta. \end{aligned}$$

Using Assumption 1, we can show that

$$\Delta(\overline{\gamma}_*) \geq \frac{1-\overline{\gamma}_*}{\beta+2\lambda}\Big((1-\underline{\gamma}_*)(\beta-2\lambda)^2+8(1-\overline{\gamma}_*)\lambda\beta\Big) > 0.$$

Thus, there exists a $\underline{\gamma} \in (\underline{\gamma}_*, \overline{\gamma}_*)$ such that $\Delta(\underline{\gamma}) = 0$. Taking derivatives of $\Delta(\gamma_0)$ with respect to $\gamma_0 \in (\underline{\gamma}_*, \overline{\gamma}_*)$ and evaluating at $\underline{\gamma}$ using $\Delta(\underline{\gamma}) = 0$ yield

$$\begin{aligned}\Delta'(\underline{\gamma}) =& \frac{\overline{\gamma}_*(1-\underline{\gamma}_*)}{\underline{\gamma}(\overline{\gamma}_*-\underline{\gamma}_*)}(v_H - v_L + \beta + 2\lambda) + \frac{\underline{\gamma}_*(2\underline{\gamma}-\overline{\gamma}_*(1+\underline{\gamma}))}{\underline{\gamma}(1-\underline{\gamma})(\overline{\gamma}_*-\underline{\gamma}_*)}(\beta-2\lambda) - 2\beta \\ >& \frac{1-\underline{\gamma}_*}{\overline{\gamma}_*-\underline{\gamma}_*}(v_H - v_L + \beta + 2\lambda) + \frac{2\underline{\gamma}_*-\overline{\gamma}_*(1+\underline{\gamma}_*)}{(1-\underline{\gamma}_*)(\overline{\gamma}_*-\underline{\gamma}_*)}(\beta-2\lambda) - 2\beta \\ >& \frac{(1-\underline{\gamma}_*)\overline{\gamma}_*}{\overline{\gamma}_*-\underline{\gamma}_*}(\beta+2\lambda) + \frac{2\underline{\gamma}_*-\overline{\gamma}_*(1+\underline{\gamma}_*)}{(1-\underline{\gamma}_*)(\overline{\gamma}_*-\underline{\gamma}_*)}(\beta-2\lambda) - 2\beta,\end{aligned}$$

where the first inequality follows because the first term in the expression is decreasing in $\underline{\gamma}$ while the second term is increasing in $\underline{\gamma}$, and the second inequality uses Assumption 1 and the assumption that $\lambda < \beta/2$. The above can be shown to be equal to

$$\frac{\beta-2\lambda}{2}\left(\frac{v_H-v_L-\beta}{\lambda}\left(\frac{v_H-v_L-\beta}{\beta+2\lambda}+\frac{3}{2}\right)+\frac{\beta-2\lambda}{\beta+2\lambda}+\frac{\beta-2\lambda}{\lambda}\right),$$

which is positive because $\lambda < \beta/2$. As a result, $\underline{\gamma}$ is unique, with $\Delta(\gamma_0) > 0$ if $\gamma_0 \in (\underline{\gamma}, \overline{\gamma}_*)$, and the opposite holding if $\gamma_0 \in (\underline{\gamma}_*, \underline{\gamma})$.

At the other end, we have

$$\lim_{\gamma_0 \to 1} \Delta(\gamma_0) = -(\beta - 2\lambda) < 0.$$

Thus, there exists a $\overline{\gamma} \in (\overline{\gamma}_*, 1)$ such that $\Delta(\overline{\gamma}) = 0$. The derivative of $\Delta(\gamma_0)$ with respect to $\gamma_0 \in (\overline{\gamma}_*, 1)$ is given by

$$\begin{aligned}\Delta'(\gamma_0) =& -2(v_H - v_L + 2\lambda) - 3\beta + \frac{(1-\gamma_0^2)\underline{\gamma}_*^2}{\gamma_0^2(1-\underline{\gamma}_*)}(\beta-2\lambda) \\ &+ \frac{2(1-\gamma_0^2)}{\gamma_0^2}\left(\ln\left(\frac{1-\gamma_0}{1-\underline{\gamma}_*}\right) + \frac{\gamma_0-\underline{\gamma}_*}{(1-\gamma_0)(1-\underline{\gamma}_*)}\right)\beta.\end{aligned}$$

As in the case of $\lambda = 0$, the sum of the last two terms in the above expression is increasing in γ_0 and approaches 4β as γ_0 approaches 1. Thus,

$$\Delta'(\gamma_0) < -2(v_H - v_L + \lambda) + \beta < 0,$$

because $\lambda < \beta/2$. It follows that $\overline{\gamma}$ is uniquely defined in $(\overline{\gamma}_*, 1)$, and $\Delta(\gamma_0) > 0$ for $\gamma_0 \in (\overline{\gamma}_*, \overline{\gamma})$ and the opposite holds for $\gamma_0 \in (\overline{\gamma}, 1)$.